UC Irvine
UC Irvine Electronic Theses and Dissertations

Title
Lyapunov exponents: continuity, positivity, and consequences for upper bounds in quantum dynamics and fractal spectrum

Permalink
https://escholarship.org/uc/item/0pb35302

Author
Powell, Matthew Taylor

Publication Date
2022

Copyright Information
This work is made available under the terms of a Creative Commons Attribution-NonCommercial-NoDerivatives License, availalbe at
https://creativecommons.org/licenses/by-nc-nd/4.0/

Peer reviewed|Thesis/dissertation

eScholarship.org Powered by the California Digital Library
University of California

UNIVERSITY OF CALIFORNIA,
IRVINE

Lyapunov exponents: continuity, positivity, and consequences for upper bounds in quantum dynamics and fractal spectrum

DISSERTATION

submitted in partial satisfaction of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

in Mathematics

by

Matthew Taylor Powell

Dissertation Committee:
Professor Svetlana Jitomirskaya, Chair
Professor Anton Gorodetski
Associate Professor Wencai Liu

2023

© 2023 Matthew Taylor Powell

TABLE OF CONTENTS

ACKNOWLEDGMENTS

First and foremost, I would like to express my gratitude to my PhD advisors, Svetlana Jitomirskaya and Wencai Liu, without whom this would not have been possible. Lana's guidance over the last five years as my PhD advisor, and the two additional years she spent guiding me as an undergraduate, have influenced how I look at mathematics. I would be half the mathematician I am now without it. I am also very thankful for the numerous and varied conversations I have had with Wencai both during his time at UCI and in the years since. He has helped me build confidence in myself and has been an amazing sounding board for various ideas, many of which influenced the results in this thesis.

I would also like to thank Professor Anton Gorodetski for our few but valuable discussions, for running an active learning seminar in Dynamical Systems, and for the various topics courses he has taught during my time at UCI. I learned Complex Analysis from him and Professor Abel Klein, and it is no exaggeration to say that this thesis would not exist without that.

I would also like to thank my friends at UCI, and the Mathematical Physics learning seminar participants, both faculty and student, past and present (in no particular order): Professor Abel Klein, Nishant Rangamani, Fernando Quintino, Xiaowen Zhu, Alberto Takase, Lingrui Ge, Xin Zhao, Christoph Fishbacher, Oluwadara Ogunkoya, Jiranan Kerdboon, Truong Vu, Lee Fisher, Omar Hurtado, Alexandro Luna, Gregorii Monakov.

Finally, last but certainly not least, I want to thank my family who gracefully put up with me and my studies!

This research was partially supported by various Fellowships and grants: the UCI Endowed Faculty Fellowship, the UCI Dissertation Fellowship, NSF DMS-2052572, DMS-2052899, DMS-2155211, DMS-2000345, and Simons 681675.

VITA

Matthew Taylor Powell

EDUCATION

Doctor of Philosophy in Mathematics **2023**
University of California, Irvine *Irvine, CA*

Masters of Science in Mathematics **2020**
University of California, Irvine *Irvine, CA*

Bachelor of Science in Mathematics **2018**
University of California, Irvine *Irvine, CA*

AWARDS, HONORS, AND FELLOWSHIPS

- UCI Dissertation Fellowship: 2022
- UCI Endowed Faculty Fellowship: 2022
- von Neumann Award for Outstanding Performance as a Graduate Student: 2022
- Connelly Award for Maintaining Outstanding Research and Teaching: 2021
- Phi Beta Kappa honor society: 2018
- UCI Chancellor's Award for Outstanding Undergraduate Research: 2018
- Pi Mu Epsilon honor society: 2016

RESEARCH

Interests

Mathematical physics, spectral theory, ergodic theory, harmonic analysis

Publications

- M. Landrigan and M. Powell. Fine dimensional properties of spectral measures. J. Spectr. Theory, to appear. 2022 arxiv:2107.10883

- S. Jitomirskaya and M. Powell. Logarithmic quantum dynamical bounds for arithmetically defined ergodic Schrodinger operators with smooth potentials. To appear in Analysis at Large: Dedicated to the Life and Work of Jean Bourgain, Springer (A. Avila, M. Rassias, Y. G. Sinai, eds.) November 2021. arxiv:2110.11883
- M. Powell. Fractal dimensions of spectral measures of rank one perturbations of a positive self-adjoint operator. J. Math. Anal. App., 475(2):1803?1817, July 2019.

Preprints

- M. Powell. Continuity of the Lyapunov exponent for analytic multi-frequency quasiperiodic cocycles. 2022. Preprint. arXiv:2210.09285
- M. Powell. Positivity of the Lyapunov exponent for analytic quasiperiodic operators with arbitrary finite-valued background. June 2022. arxiv:2206.04159

TALKS AND CONFERENCE PARTICIPATION

Invited talks and lectures

- AMS Special Session on Disordered and Periodic Quantum Graphs at Georgia Technical Institute (March 2023)
  - *Continuity of the Lyapuov exponent for analytic multifrequency quasi-periodic cocycles*
- UC Berkeley joint Mathematics and Physics seminar (Dec 2022)
  - *Continuity of the Lyapuov exponent for analytic multifrequency quasi-periodic cocycles*
- University of Houston Annual SIAM Chapter Meeting (Nov 2022)
  - *Continuity of the Lyapuov exponent for analytic multifrequency quasi-periodic cocycles*
- Rice University Spectral Theory Seminar (Oct 2022)
  - *Continuity of the Lyapuov exponent for analytic multifrequency quasi-periodic cocycles*
- Great Lakes Annual Mathematical Physics Meeting (June 2022)
  - *Positivity of Lyapunov exponents for analytic quasiperiodic operators with arbitrary finite-valued background*

- New directions in quantum systems (June 2022)
  - *Positivity of Lyapunov exponents for analytic quasiperiodic operators with arbitrary finite-valued background*
- BIRS meeting for almost periodic operators (April 2022)
  - *Positivity of Lyapunov exponents for analytic quasiperiodic operators with arbitrary finite-valued background*
- Texas A&M University Spectral Theory Seminar (January 2021 - March 2021)
  - *The Aubry-Andre conjecture and continuity of the spectrum for the almost Mathieu operator*
- Texas A&M University Spectral Theory Seminar (November 2021)
  - *Subordinacy theory for half-line Schrödinger operators*

Talks at UCI

- Mathematical Physics Seminar, Math Department (March 2022 - September 2022)
  - 15-week mini course on *Continuity of the Lyapunov exponent for multifrequency cocycles*
- Mathematical Physics Seminar, Math Department (November 2021)
- Mathematical Physics Seminar, Math Department (March 2020)
- Mathematical Physics Seminar, Math Department (February 2019)
- Mathematical Physics Seminar, Math Department (October 2018)

TEACHING EXPERIENCE

Instructor

Math 3D: Differential Equations **Winter 2022**
University of California, Irvine *Irvine, CA*

Math 5B: Calculus for Life Sciences **Summer 2021**
University of California, Irvine *Irvine, CA*

Teaching Assistant

Math 140A: Introduction to Analysis **Fall 2021**
University of California, Irvine *Irvine, CA*

Math 194: Problem Solving Seminar **Fall 2021**
University of California, Irvine *Irvine, CA*

Math 210B: Graduate Real Analysis **Winter 2021**
University of California, Irvine *Irvine, CA*

Math 210A: Graduate Real Analysis **Fall 2020**
University of California, Irvine *Irvine, CA*

Math 140B: Introduction to Analysis **Summer 2020**
University of California, Irvine *Irvine, CA*

Math 210C: Graduate Real Analysis **Spring 2020**
University of California, Irvine *Irvine, CA*

Math 140B: Introduction to Analysis **Winter 2020**
University of California, Irvine *Irvine, CA*

Math 3A: Linear Algebra **Winter 2020**
University of California, Irvine *Irvine, CA*

Math 2E: Multivariable Calculus II **Summer 2019**
University of California, Irvine *Irvine, CA*

Math 2D: Multivariable Calculus I **Spring 2019**
University of California, Irvine *Irvine, CA*

Math 3D: Differential Equations **Winter 2019**
University of California, Irvine *Irvine, CA*

Math 3A: Linear Algebra **Fall 2018**
University of California, Irvine *Irvine, CA*

SERVICE

- Conference Organizer: AMS Special Session on Quasiperiodic Operators and Quantum Graphs (March 2023)
- Referee: Journal of Spectral Theory (2022)
- Western States Mathematics Physics meeting (March 2020)

ABSTRACT OF THE DISSERTATION

Lyapunov exponents: continuity, positivity, and consequences for upper bounds in quantum dynamics and fractal spectrum

By

Matthew Taylor Powell

Doctor of Philosophy in Mathematics

University of California, Irvine, 2023

Professor Svetlana Jitomirskaya, Chair

We consider quasiperiodic Jacobi and Schrödinger operators of both a single- and multi-frequency. These operators appear very naturally in condensed matter physics, where they have seen applications in the study of Graphene and the Quantum Hall effect. The prototypical example of the single-frequency operator is the almost Mathieu operator (AMO). While much is known about the AMO, less may be said about the multifrequency analogues and even some perturbed single-frequency models. This thesis has one recurring theme: properties of the Lyapunov exponent (LE); moreover, this thesis may be split into two parts. First, we explore continuity of the LE for multifrequency analytic quasiperiodic cocycles and positivity of the LE for single-frequency analytic quasiperiodic Schrödinger operators with an additional background potential. We then derive upper bounds in quantum dynamics as a consequence of LE regularity, and explore the fractal properties of spectral measures.

In addressing the first part, we prove joint continuity of the LE for non-identically singular multifrequency analytic quasiperiodic cocycles in both cocycle and frequency, and we prove that the (lower) LE for single frequency analytic quasiperiodic Schrödinger operators with added background potential can be made uniformly positive by taking a sufficiently large coupling constant independent of the background.

The former is accomplished by adapting an inductive argument of Bourgain which was originally used to derive similar results for $SL(2,\mathbb{C})$-cocycles. The latter involves complexification in phase of the associated transfer matrix and appealing to various properties of analytic and subharmonic functions.

In the second part, we first derive a lite version of dynamical localization: under suitable assumptions on the frequency and LE, (time-averaged) moments of the position operator grow no faster than a power of the logarithm. The main achievement here is that our notion is stable under perturbations and holds for all values of the phase and an arithmetically defined set of frequencies of full measure.

We then extend the Jitomirskaya-Last power-law subordinacy theory and Last theory of quantum dynamics to encompass a more general version of Hausdorff dimension. This allows us to study fine dimensional properties of spectral measures, particularly 'zero-dimensional' measures.

Chapter 1

Introduction

Let $(\Omega, \mathcal{F}, \mu)$ denote a probability space, and let $T : \Omega \to \Omega$ denote an invertible measure preserving transformation. A 1-dimensional ergodic Jacobi operator $H_{T,x} : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ is given by

$$(H_{T,x}\psi)(n) = a(T^n x)\psi(n-1) + a(T^n x)\psi(n+1) + f(T^n(x))\psi(n), \quad n \in \mathbb{Z}, \tag{1.1}$$

where $f \in C(\Omega, \mathbb{R})$ is called the potential and $x \in \Omega$ is called the phase. Let $1 \leq d < \infty$. When $\Omega = \mathbb{T}^d \simeq (\mathbb{R}/\mathbb{Z})^d$ and $Tx = x + \omega$, for some $\omega \in \Omega$, we call such operators *quasiperiodic*, and denote them by $H_{\omega,x}$, and we call ω the frequency. When $a \equiv 1$, we call these operators *Schrödinger operators*. These will be our main focus, though we will occasionally consider more general objects.

The spectral theory of these operators is quite rich, and has been the subject of many works spanning the fields of mathematical physics, spectral theory, dynamical systems, and harmonic analysis, among others.

This work will focus on three properties of these operators (and their generalizations):

1. Regularity of the Lyapunov exponent;
2. Bounds in quantum dynamics;
3. Fractal properties of the associated spectral measures.

1.1 Motivation: some physics

Quasiperiodic operators arise very naturally in condensed matter physics. The effective tight-binding model in condensed matter physics allows us to construct a Hamiltonian for an electron in a 2D crystal lattice under the influence of a perpendicular magnetic field. A suitable gauge transform allows us to express this 2D problem as an effective 1D problem modeled by a quasiperiodic operator.

One of the most well-studied and well-understood examples which arises from such a situation is the almost Mathieu operator (AMO), which corresponds to $a \equiv 1$ and $f(x) = 2\lambda\cos(2\pi x)$. Introduced in the late 1970s, the AMO (also called Harper's model in physics literature) has been the source of many numerical studies, and a plot of its spectrum was the source of the first example of a fractal in physics; it has been the subject of intensive study, in both mathematics and physics literature, ever since.

A great deal is now known about the AMO, and it is a natural question whether methods used to study the AMO are applicable to more general operators. In the past two decades, this has been extensively studied for quasiperiodic operators of the form (1.1) with analytic f. Notable works include Avila's Global Theory for 1-frequency operators and its extension to Jacobi operators. However, there are have been efforts, both preceding (see e.g. [6, 7, 54]) and following (see e.g. [10, 11, 12, 23, 55, 59]) the development of Avila's theory, to consider more general objects which do not fall under the umbrella of Global Theory. Among other directions are the following: one route is to consider multi-frequency quasiperiodic operators and another is to consider 1-frequency quasiperiodic operators with a non-quasiperiodic background. Addressing these two situations is the main objective of this work.

We emphasize that there are two major sources of difficulty in these scenarios. For one, studying multi-frequency operators is fraught with issues arising from the interplay between the different components of the frequency. Moreover, Jacobi operators may contain zero off-diagonal entries, which introduces singularities when trying to use existing methods. Combined, these two notions have been shown to cause problems with certain arguments associated with Global Theory [25]. An important property of our general work on multi-frequency operators is that we neither require $a(x)$ to be bounded away from zero, nor do we restrict the frequencies ω we consider. Applications of our general work do impose certain restrictions on a and ω, but only when necessary. Secondly, adding non-quasiperiodic backgrounds destroys much of the dynamical systems behavior which underpins the Global Theory.

The central object of our study is the so-called Lyapunov exponent, which was used so fruitfully to study spectral problems associated to the AMO and related operators.

1.2 Transfer matrices and Lyapunov exponents

When one explores spectral theoretic questions about quasiperiodic operators, a key role is played by solutions of the eigenequation:

$$H_{\omega,x}\psi = E\psi, \tag{1.2}$$

where $E \in \mathbb{C}$. Any solution to the eigenequation can be reconstructed from the n-step transfer matrix

$$\prod_{k=n}^{1} \begin{pmatrix} f(T_\omega^k(x)) - E & -1 \\ 1 & 0 \end{pmatrix} \tag{1.3}$$

using the identity

$$\begin{pmatrix} \psi(n+1) \\ \psi(n) \end{pmatrix} = \prod_{k=n}^{1} \begin{pmatrix} f(T_\omega^k(x)) - E & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \psi(1) \\ \psi(0) \end{pmatrix}. \tag{1.4}$$

From a dynamical systems point of view, the transfer matrix is an example of a cocycle, and it is often useful to study properties of transfer matrices by studying more general cocycles instead.

A cocycle is a particular example of a dynamical system which arises in ergodic theory; cocycles are dynamical systems on vector bundles which preserve the linear bundle structure and induce a measure-preserving dynamical system on the base space. A particular class of examples are the quasiperiodic cocycles defined over $(\mathbb{C}^2, \mathbb{T}^d)$, with the underlying dynamical system given by a shift on the d-dimensional torus, $\mathbb{T}^d \simeq \mathbb{R}^d/\mathbb{Z}^d \simeq [0,1]^d$. More concretely, let $M(2,\mathbb{C})$ be the set of 2×2 matrices with complex entries, let $\mathbb{T}^d$ denote the d-dimensional torus, and for $\omega \in \mathbb{T}^d$, let $T_\omega : \mathbb{T}^d \to \mathbb{T}^d$ be given by $T_\omega x = x + \omega$. We call ω the frequency of the shift. A d-dimensional quasiperiodic cocycle is a pair $(A,\omega) \in C(\mathbb{T}^d, M(2,\mathbb{C})) \times \mathbb{T}^d$ when viewed as a linear skew product: (A,ω) acting on $\mathbb{C}^2 \times \mathbb{T}^d$ with

$$(A,\omega)(w,x) = (A(x)w, T_\omega x). \tag{1.5}$$

The cocycle iterates are given by

$$(A,\omega)^N = (A_N, N\omega), \tag{1.6}$$

where

$$A_N(x,\omega) = A(x+(N-1)\omega)\cdot A(x). \tag{1.7}$$

Throughout this work, we use the term cocycle to describe $A_N(x,\omega)$.

As we noted above, the study of quasiperiodic cocycles has immediate applications to the study of one-dimensional (and quasi-one-dimensional) quasiperiodic operators $H_{\omega,x}$: the associated transfer matrix:

$$\prod_{j=N}^{1}\begin{pmatrix} E-v(x+j\omega) & -1 \\ 1 & 0 \end{pmatrix}.$$

describes a cocycle, with

$$A(x) = \begin{pmatrix} E-v(x) & -1 \\ 1 & 0 \end{pmatrix}.$$

Cocycles of the above form are typically called Schrödinger cocycles.

Schrödinger cocycles are always $SL(2,\mathbb{R})$ cocycles. We may also consider quasiperiodic Jacobi operators on $L^2(\mathbb{Z})$ given by $h_{\omega,x}$, where

$$(h_{\omega,x}\psi)(n) = \overline{a(x+(n-1)\omega)}\psi(n-1) + a(x+n\omega)\psi(n+1) + v(x+n\omega)\psi(n), \tag{1.8}$$

where v is as in the Schrödinger case, and $a:\mathbb{T}^d\to\mathbb{C}$. We can define an analogous cocycle to study these operators:

$$A(x) = \begin{pmatrix} (E-v(x)) & -\overline{a(x-\omega)} \\ a(x) & 0 \end{pmatrix} \tag{1.9}$$

The fundamental difference between the Schrödinger and Jacobi case, though, is that, based on the choice of function a, the cocycle need not be $SL(2,\mathbb{R})$. In fact, it could have zero determinant somewhere. When a cocycle has zero determinant somewhere, we say it is *singular*, and we call the points where the determinant vanishes *singularities*.

The asymptotic behavior of a cocycle is captured by the Lyapunov exponent. We begin by defining

$$L'_N(A,\omega,x) = \frac{1}{N}\ln\|A_N(x,\omega)\|. \tag{1.10}$$

Denote

$$L'_N(A,\omega) = \int_{\mathbb{T}^d} L'_N(A,\omega,x). \tag{1.11}$$

It follows by subadditivity considerations that the limit

$$L'(A,\omega) = \lim_{N\to\infty}\int_{\mathbb{T}^d} L'_N(A,\omega,x)dx \tag{1.12}$$

exists, though it may be $-\infty$, depending on the behavior of the cocycle. We call $L'(A,\omega)$ the (upper) Lyapunov exponent of the cocycle (A,ω).

Remark 1. *It does not matter what norm we put on the space of* 2×2 *matrices for use in this definition, but it is often useful to consider the Hilbert-Schmidt norm:* $\|A\|^2_{HS} = tr(A^*A)$.

We typically ask two regularity questions about $L'(A,\omega)$, and each has consequences for the spectral properties of the underlying operator:

1. Is L' uniformly positive?

2. Is L' a continuous function of A and/or ω?

For Schrödinger operators where the potential f is an analytic function, uniform positivity of the Lyapunov exponent is typically indicative of Anderson localization (pure point spectrum with exponentially decaying eigenfunctions) and dynamical localization (boundedness in time of the moments of the position operator) for a.e. parameter. We expand on this below. Continuity, on the other hand, implies that spectral properties of an operator may be obtained by studying operators which *approximate* the desired operator in a suitable sense. For quasiperiodic operators, this is typically captured by varying the frequency; irrational frequencies may be approximated by rational frequencies, and rational frequencies create periodic operators, which are usually easier to understand.

1.3 Quantum dynamics

Given a (quasiperiodic) Schrödinger operator of the form (1.1), with $a \equiv 1$, an object often of physical interest is the time-evolution: $e^{-itH_{T,x}}$. Physically, this describes how an initial state, ψ evolves over time under the influence of the Hamiltonian $H_{T,x}$. One often wants to know if the operator $H_{T,x}$ is localized or not. Two related but different notions are Anderson localization and dynamical localization. An operator exhibits Anderson localization if its spectrum is pure point and the corresponding eigenfunctions decay exponentially. Dynamical localization (a stronger and more physically relevant notion) is characterized by boundedness in time of the moments of the position operator (defined below). It is well known that Anderson localization is highly unstable with respect to various perturbations. For quasiperiodic operators, it depends very sensitively on the arithmetics of the phase (a seemingly irrelevant parameter from the point of view of the physics of the problem), and doesn't hold generically [49]. It can also be destroyed by generic rank one perturbations [31, 22]. This instability is therefore also present for the - very physically relevant - notion of dynamical localization.

Let us consider dynamical localization more concretely. Consider the time-averaged quantity:

$$a(n,T) = \frac{2}{T}\int_0^\infty e^{2t/T}\frac{1}{2}\left(\left|\left\langle e^{-itH_{\omega,x}}\delta_0, \delta_n\right\rangle\right|^2 + \left|\left\langle e^{-itH_{\omega,x}}\delta_1, \delta_n\right\rangle\right|^2\right) dt, \tag{1.13}$$

where $\delta_n(m) = 1$ when $m = n$ and 0 otherwise. The moments of the position operator are given by

$$\langle |X|^p(T)\rangle = \sum_{n\in\mathbb{Z}}(1+|n|)^p a(n,T). \tag{1.14}$$

Moments of the position operator for generic rank one perturbations of many operators with a.e. (in phase) dynamical localization are unbounded in time. This bizarre situation is partially rescued by a result of [20, 19]: when eigenfunctions have an additional SULE (semi-uniform localization) property, the moments of the position operators of *all* rank-one perturbations grow at most power-logarithmically (see Chapter 5 for the definition of SULE and a further discussion). SULE has since been proved for all operators with localization that come from physically realizable models. From this point of view, power-logarithmic bounds of the moments are the stable - and therefore physically relevant - property, making it worthwhile to prove directly for operator families with (expected) a.e. (in phase) localization, bypassing the technically difficult localization proof. This, in particular, includes one-dimensional quasiperiodic operators $H_{\omega,x}$ with positive Lyapunov exponent.

1.4 Fractal behavior of spectral measures

Recall that, given a self-adjoint operator A on a Hilbert space $\mathcal{H}$, the spectrum of A is given by the set

$$\sigma(A) = \{\lambda \in \mathbb{R} : (A - \lambda Id) \text{ is not invertible}\}. \tag{1.15}$$

The functional calculus allows this set to be decomposed into three disjoint subsets: the pure point, singular continuous, and absolutely continuous components: σ_{pp}, σ_{sc}, and σ_{ac}, respectively. These may be studied via the operator's spectral measures. One of the most difficult objects to study has been the singular continuous spectrum, largely because singular continuous measures are defined by what they are not: neither pure point nor absolutely continuous. This rough definition, however, eliminates the fine structure of the singular continuous spectrum: for example, some measures are 'closer' to being pure point than others.

Classifying these measures goes back to work by Rogers and Taylor [63, 64], where measures were decomposed with respect to the classical Hausdorff measures. This theory has been applied to the spectral theory of Schrödinger operators by various authors (see [56, 57, 58, 37, 38, 32] among others). One limitation of this theory, however, is that the so-called zero-dimensional regime is still quite rich: zero-dimensional measures need not be pure point. This leads one to the natural question: can we stratify the zero-dimensional regime any further? As we will see, the answer is yes, and is, in fact, beneficial, with a wealth of consequences.

1.5 Outline

The results of this thesis are based on the four works [57, 46, 62, 61] and may best be summarized as a study of the regularity of Lyapunov exponents, along with consequences of that regularity. This work may be divided into three roughly interconnected pieces: regularity of the Lyapunov exponents (Chapters 2 and 3), consequences for the quantum dynamics of Schrödinger operators (Chapter 4), and the general fractal structure of spectral measures and its relation to the Lyapunov exponent and quantum dynamics (Chapter 5). Regularity (either positivity or continuity) of Lyapunov exponents makes an appearance in every chapter, and plays a critical role in our discussion of quantum dynamics in Chapter

4. As we will see, there is also strong interplay between the fractal structure of spectral measures and bounds in quantum dynamics (see Section 5.7 and Chapter 4). Moreover, positivity of the Lyapunov exponent has implications for this fractal structure.

The Lyapunov exponent plays a key role in this work, and we devote the first two chapters to exploring its regularity. We explore the continuity of the Lyapunov exponent for multi-frequency quasiperiodic cocycles in the abstract setting of non-identically singular cocycles in Chapter 2. To date, the only other abstract result as strong as ours for quasiperiodic operators with dynamics over $\mathbb{T}^d$ is due to Bourgain [6]. In Chapter 3, we explore a sufficient condition for the Lyapunov exponent of 'perturbed' operators to remain uniformly positive. Similar results for 'unperturbed' operators date back to Sorets and Spencer [69] and Bourgain [5].

We then turn our attention to a careful analysis of the quantum dynamics of quasiperiodic Schrödinger operators. As we have already discussed, there are various notions of localization, but they are typically unstable. In Chapter 4, we consider a lite version of localization which is, in fact, stable under perturbations. For quasiperiodic Schrödinger operators in the regime of positive Lyapunov exponent, various authors have shown that the moments of the position operator grow slower than any polynomial. We refine these results by employing techniques pioneered by Bourgain in order to show that positivity of the Lyapunov exponent actually implies logarithmic growth of the moments. These results are based on joint work with Jitomirskaya [46].

The spectrum of any self-adjoint operator may be decomposed into three parts: pure point, singular continuous, and absolutely continuous. From the standpoint of analysis, the first and third objects are fairly well-understood, while the second is not: it is typically defined as that which is neither pure point nor absolutely continuous. This odd definition presents a wealth of interesting questions. One method of studying the rich structure of singular continuous spectrum is by studying the inherent fractal properties. This is the subject of Chapter 5. We develop a general fractal dimension, based on the classical notion of Hausdorff dimension, and explore Schrödinger operators via this tool. In particular, we explore how positivity of the Lyapunov exponent implies singularity of spectral measures and how continuity of spectral measures implies lower bounds in quantum dynamics. This discussion is based on earlier work by Jitomirskaya and Last [37], Landrigan [56], Last [58], and our joint work with Landrigan [57].

Chapter 2

Lyapunov exponents I: continuity of Lyapunov exponents for $M(2, \mathbb{C})$ cocycles

2.1 Preliminaries

In this chapter, we study continuity properties of Lyapunov exponents related to analytic quasiperiodic cocycles. Such properties have been studied extensively for analytic Schrödinger cocycles both when $d = 1$ and $d > 1$, as well as non-identically singular $M(2, \mathbb{C})$ cocycles with when $d = 1$. Up until now, continuity for non-identically singular $M(2, \mathbb{C})$ cocycles when $d > 1$ is known only when the frequency satisfies a Diophantine condition. Here, we improve this to include all frequencies by extending Bourgain's multifrequency $SL(2, \mathbb{C})$ result to cover the general $M(2, \mathbb{C})$ case.

We now proceed to give (or recall) the formal definitions necessary to state our main theorem.

We are interested in *analytic quasiperiodic cocycles*. That is, quasiperiodic cocycles (A, ω), where A is taken to be an analytic $M(2, \mathbb{C})$-valued function on $\mathbb{T}^d$ with an analytic extension, continuous up to the boundary, to the complex strip $|\Im z_j| < \rho, \rho > 0$, for all $1 \leq j \leq d$. We denote the space of such A by $C_\rho(\mathbb{T}^d, M(2, \mathbb{C}))$. We put a natural metric on the space of these cocycles:

$$d((A, \omega), (B, \omega')) = \|A - B\|_\rho + \|\omega - \omega'\|_{\mathbb{T}^d}, \tag{2.1}$$

where

$$\|A - B\|_\rho = \sup_{z: |\Im z_j| < \rho} |A(z) - B(z)| \tag{2.2}$$

and $\|\omega - \omega'\|_{\mathbb{T}^d}$ is the usual norm on $\mathbb{R}^d/\mathbb{Z}^d = \mathbb{T}^d$. Note that any analytic function on $\mathbb{T}^d$ has an analytic extension to *some* complex strip $|\Im z_j| < \rho, \rho > 0$. This observation may be used to define an inductive topology on the space of all analytic cocycles: $\bigcup_{\rho > 0} C_\rho$. Moreover, we will assume $\det(A(x))$ is not identically zero.

Remark 2. Note that any $M(2, \mathbb{C})$ cocycle $A_N(x)$ for which $\det(A(x))$ is not identically zero can be renormalized to form an $SL(2, \mathbb{C})$ cocycle (see e.g.[35]), however the resulting cocycle may lose boundedness if $\det(A(x))$ has zeros. This is precisely the nature of difficulty when extending $SL(2, \mathbb{C})$ results to the $M(2, \mathbb{C})$ case.

The (upper) Lyapunov exponent is then defined as

$$L'(A, \omega) = \frac{1}{N} \int_{\mathbb{T}^d} \ln \|A_N(x, \omega)\| \, dx. \tag{2.3}$$

Note that, while $L'(A, \omega)$ need not be non-negative, the related object

$$L(A, \omega) = \lim_{N \to \infty} \int_{\mathbb{T}^d} L_N(\tilde{A}, \omega, x) dx, \tag{2.4}$$

is, where $\tilde{A} \in SL(2, \mathbb{C})$ is a renormalization of A :

$$\tilde{A} = \frac{1}{|\det A|^{1/2}} A. \tag{2.5}$$

Moreover, L_N and L'_N are related by the following relation:

$$L_N(A, \omega) = L'_N(A, \omega) - \frac{1}{2} \int_{\mathbb{T}^d} \ln|\det(A(x))| dx. \tag{2.6}$$

It follows that, when $\ln|\det(A(x))| \in L^1$, both L and L' share the same regularity properties. In particular, if one is continuous, in some sense, then so is the other. Throughout, we will occasionally write $L_N(A, x)$ or $L_N(x)$ in place of $L_N(A, \omega, x)$, when there can be no ambiguity. Similarly, we will occasionally write $L_N(A)$ in place of $L_N(A, \omega)$ when ω is clear.

Remark 3. We would like to make a note about a convention that we use. Throughout this paper, we use capital letters (e.g. C, C', etc.) to denote constants which are sufficiently large, and lower-case letters to denote constants which are sufficiently small (e.g. c, c', etc.). How large/small depends, unless otherwise specified, on the dimension, d, and uniform measurements of the cocycle, A.

We prove the following:

Theorem 2.1.1. *Suppose* $\omega = (\omega_1, ..., \omega_d) \in \mathbb{T}^d$. *Let* (A, ω) *be an analytic quasiperiodic* $M(2, \mathbb{C})$*-cocycle. Suppose, moreover, that* $\det(A) \not\equiv 0$. *Then* $L(A, \omega)$ *satisfies the following.*

(a) $L(A, \omega)$ *is continuous in* A *for any* $\omega \in \mathbb{T}^d$.

(b) $L(A, \omega)$ *is jointly continuous in* A *and* ω *for* ω *such that* $k \cdot \omega \neq 0$ *for any* $k \in \mathbb{Z}^d \setminus \{0\}$.

Remark 4. Analyticity is necessary for continuity of $L(A, \omega)$, in general. It is well-known that the Lyapunov exponent is discontinuous in C^0-topology at all non-uniformly hyperbolic cocycles. There are also examples in $C^r, 1 \leq r \leq \infty$. Wang-You [72] constructed $SL(2, \mathbb{C})$

cocycles which are C^∞ with $L(A,\omega) > 0$, yet may be approximated in C^∞ topology by cocycles with zero Lyapunov exponent. Similarly, Jitomirskaya-Marx [43] constructed examples of $M(2,\mathbb{C})$ cocycles which are discontinuous in C^∞-topology.

Remark 5. Part (b) of the above theorem is optimal, in the sense that there are examples of cocycles A_0 for which $L(A_0,\omega)$ is a discontinuous function of ω at frequencies such that $\|k\cdot\omega\| = 0$ for some $0 \neq k \in \mathbb{Z}^d$. Indeed, consider any $0 \neq k = (k_1, ..., k_d) \in \mathbb{Z}^d$. Let $\lambda(x) = e^{2\pi i k\cdot x} e^{-2\pi(k_1+\cdots+k_d)}$, and define

$$A_0(x) = \begin{pmatrix} e^{\lambda(x)} & 0 \\ 0 & e^{-\lambda(x)} \end{pmatrix}. \tag{2.7}$$

This generates an analytic quasiperiodic cocycle, (A_0,ω). We can easily verify the following:

1. If $\|k\cdot\omega\| = 0$, then $L(A_0,\omega) = \left(\frac{2}{\pi}\right) e^{-2\pi(k_1+\cdots+k_d)}$;
2. If $\|k\cdot\omega\| \neq 0$, then $L(A_0,\omega) = 0$.

Thus $L(A_0,\omega)$ is continuous at (A_0,ω) for any ω such that $\|k\cdot\omega\| \neq 0$ and is discontinuous at (A_0,ω) for all ω such that $\|k\cdot\omega\| = 0$.

The first result on continuity of L for analytic cocycles is a theorem of Goldstein and Schlag [30], who proved continuity in E (in fact, they proved Hölder continuity) for Schrödinger cocycles with $d = 1$, under the assumption that the frequency satisfies a strong Diophantine condition. The first result where (a) and (b) were established was [9], where it was done for 1-frequency $SL(2,\mathbb{C})$ cocycles. See the next remark for a brief summary of the relevant historical developments of (a) and (b).

Remark 6. 1. When $d = 1$, (a) and (b) were proved in [9] for $SL(2,\mathbb{C})$ cocycles, and later extended in [35] for non-identically singular $M(2,\mathbb{C})$ cocycles under a Diophantine frequency assumption.

2. When $d \geq 1$, (a) and (b) were proved by Bourgain for Schrödinger cocycles (though the argument clearly applies to $SL(2,\mathbb{C})$ cocycles) [6].

3. When $d = 1$, (a) was proven by Avila, Jitomirskaya, and Sadel for all analytic $M(n,\mathbb{C})$, with any n, cocycles [1] by a different method (see also [43]).

4. When $d \geq 1$, (a) was proven by Duarte and Klein for all analytic $M(n,\mathbb{C})$, with any n, cocycles, assuming a fixed Diophantine frequency [23, 26].

5. As far as we know, Theorem 2.1.1 is the first result establishing joint continuity for non-$SL(2,\mathbb{C})$ cocycles, and continuity in the cocycle for *all* frequencies.

This result applies to particular instances of the operator we will consider in Chapter 3, as we now describe. The multifrequency analytic quasiperiodic Jacobi operator is defined as

$$(h_{x,\omega}\psi)(n) = \overline{a(x+(n-1)\omega)}\psi(n-1) + a(x+n\omega)\psi(n+1) + v(x+n\omega)\psi(n), \tag{2.8}$$

where $v \in C^\omega_\rho(\mathbb{T}^d,\mathbb{R})$ and $a \in C^\omega_\rho(\mathbb{T}^d,\mathbb{C})$. Solutions to the eigenequation $H\psi = E\psi$ may be recovered using the transfer matrix:

$$\prod_{j=N}^{1} \begin{pmatrix} E - v(x+j\omega) & -\overline{a(x+(j-1)\omega)} \\ a(x+j\omega) & 0 \end{pmatrix}.$$

The transfer matrix may be realized as a cocycle by setting

$$A(x) = \begin{pmatrix} E - v(x+\omega) & -\overline{a(x-\omega)} \\ a(x) & 0 \end{pmatrix}.$$

These cocycles are singular precisely when $a(x)$ has zeros; we assume $a(x)$ does not vanish identically. The regularity of the Lyapunov exponent for such cocycles when $d = 1$ is already well-understood [35]. Consider, moreover, the quasiperiodic operator with a periodic background. That is, consider:

$$(\tilde{h}_{x,\omega}\psi)(n) = (h_{x,\omega}\psi)(n) + v_{per}(n)\psi(n), \tag{2.9}$$

where v_{per} is a q-periodic sequence of real numbers. Solutions to the eigenequation for this operator may be recovered from the new transfer matrix

$$\prod_{j=N}^{1} \begin{pmatrix} E - v(x+j\omega) - v_{per}(j) & -\overline{a(x+(j-1)\omega)} \\ a(x+j\omega) & 0 \end{pmatrix}.$$

This transfer matrix may be realized as a quasiperiodic cocycle by "regrouping along the period" and setting

$$A(x) = \prod_{j=q}^{1} \begin{pmatrix} E - v(x+j\omega) - v_{per}(j) & -\overline{a(x+(j-1)\omega)} \\ a(x+j\omega) & 0 \end{pmatrix}.$$

We can now define the Lyapunov exponent of this cocycle, $L(E, v, a, v_{per}, \omega)$, as usual, and it will, in fact, agree with the Lyapunov exponent associated with the transfer matrix. An immediate corollary of Theorem 2.1.1 is the following.

Corollary 2.1.1. *Consider the multifrequency quasiperiodic Schrödinger operator with periodic background given by* (2.9)*. Suppose $v \in C_\rho^\omega(\mathbb{T}^d, \mathbb{R})$ and $a \in C_\rho^\omega(\mathbb{T}^d, \mathbb{C})$ do not vanish identically, and suppose v_{per} is a q-periodic sequence of real numbers. Then we have the following.*

1. *$L(E, v, a, v_{per}, \omega)$ is continuous in E, v, a and v_{per} for any $\omega \in \mathbb{T}^d$.*

2. *$L(E, v, a, v_{per}, \omega)$ is jointly continuous in E, v, a, v_{per} and ω for ω such that $k \cdot \omega \neq 0$ for any $k \in \mathbb{Z}^d \setminus \{0\}$.*

Statements like Theorem 2.1.1, for both $d = 1$ and $d > 1$, have been studied extensively by many authors under suitable restrictions on the cocycle. Two particular methods have been used effectively in the past to establish continuity of $L(A, \omega)$ for quasiperiodic cocycles: one, used in [1] to prove continuity for arbitrary 1-frequency cocycles, is based on complexification of the cocycle and appealing to the notion of dominated splitting; recently, Duarte and Klein [25] showed that there are large classes of multifrequency quasiperiodic cocycles which do not have dominated splitting, thus showing that this method cannot be used to address the multifrequency case; the second method (c.f. [6, 9, 23, 35]) is an induction scheme using the so-called Avalanche Principle and statistical properties of quasiperiodic cocycles. In this paper, we adapt the second method.

Bourgain and Jitomirskaya [9] obtained joint continuity (as in (b)), a result that was essential for Avila's global theory, Ten Martini problem, and other important developments. It was observed by Jitomirskaya, Koslover, and Schulteis that this argument extends to the case of non-identically singular analytic cocycles which posses some analytic extension to a complex strip. The argument of these results relies on two ideas: first, a statistical property known as a large deviation theorem (LDT); and second, a general property of $SL(2, \mathbb{C})$ matrices with large norm, known as the Avalanche Principle (AP). The argument of [9] for $d = 1$ was based on the same basic ingredients as [30]: large deviation estimates LDT (i.e. statistical properties of the cocycle) and Avalanche Principle; however, [9] constructed a special inductive scheme to deal with arbitrary frequencies and joint continuity.

A large deviation estimate is an estimate of the form:

$$\left| \left\{ x \in X : |f(x) - \int_X f(x) d\mu(x)| > \eta \right\} \right| < \epsilon(\eta)$$

where, ideally, ϵ is exponentially small in η. Such estimates were first used by Bourgain and Goldstein [7] to establish Anderson localization for one-frequency quasiperiodic Schrödinger operators, and they have since been extended [59] and play an important role in the study of various properties of quasiperiodic Schrödinger operators. For example, they have been used recently to obtain estimates on quantum dynamics (c.f. [46, 41, 65] etc.).

The Avalanche principle was first introduced by Goldstein and Schlag [30] in their work on Hölder regularity of the integrated density of states, and variations of the original statement have been used in proofs of the continuity of $L(A, \omega)$ in various settings (c.f. [1, 9, 6, 35]).

The argument for $SL(2, \mathbb{C})$ cocycles with $d = 1$ developed in [9] roughly proceeds as follows. First, the frequency is assumed to be irrational, since rational frequencies are well-understood. Analyticity of the cocycle implies that $L_N(A, \omega, x)$ is subharmonic in x with a *bounded* subharmonic extension to the strip $|\Im z| < \rho$, for some $\rho > 0$. An analysis of bounded subharmonic functions on the strip leads to estimates on the decay of the Fourier coefficients of L_N, and this, in turn, leads to an LDT of the form

$$|\{x \in \mathbb{T} : |L_N(A, \omega, x) - L_N(A, \omega)| > \kappa\}| < e^{-c\kappa q},$$

where κ and q relate to properties of the frequency, ω. Combining this with the Avalanche Principle results in an estimate of the form

$$|L_{N_0}(A, \omega) - L_{N_1}(A, \omega)| < \kappa,$$

where N_0 is an initial scale which depends on measurements of the frequency, κ is an error which depends on measurements of the frequency and N_0, and N_1 is a multiple of N_0 which is not too large. This estimate is then used successively in an induction scheme to relate $L_{N_0}(A, \omega)$ to $L(A, \omega)$, and continuity of L follows from continuity of L_{N_0}.

When $d > 1$, serious technical issues arise which makes the arguments more complex. The only general result for arbitrary frequencies is Bourgain [6], where an exact analogue of Theorem 2.1.1 was established for Schrödinger cocycles (though the argument extends without issue to $SL(2,\mathbb{C})$ cocycles). One of the goals of this paper is to illustrate the power of the argument in [6] by extending it to a more difficult general $M(2,\mathbb{C})$ case, while also providing additional details and clarifications to the original argument.

The rest of this chapter is organized in the following way. In section 2.2 we briefly describe our argument. In Section 2.3 we recall the relevant facts about subharmonic and plurisubharmonic functions, and use them to prove two essential measure estimates, Lemma 2.3.8 and Theorem 2.3.2. In Section 2.4, we prove joint continuity of $L_N(A,\omega)$ for fixed N and arbitrary ω. In Section 2.5 we recall the Avalanche Principle and prove Theorem 2.5.3, which we use throughout our induction scheme. In Section 2.6, we establish estimates between $L_N(A,\omega)$ at different scales when ω satisfies a Liouville-type condition. In Section 2.7, we establish estimates between $L_N(A,\omega)$ at different scales when ω satisfies a mixed Liouville-Diophantine condition. In Section 2.8 we use induction to extend the conclusions of Sections 2.6 and 2.7 to larger length scales. Finally, in Section 2.9, we use our induction result and finite-scale continuity to prove Theorem 2.1.1. We also provide an appendix, where we provide proofs of the relevant plurisubharmonic function estimates from Section 2.3.

2.2 A brief description of our argument

In the $SL(2,\mathbb{C})$ case, the major differences between $d=1$ and $d>1$ are largely a result of the interactions between the different components of the frequency. In particular, $L_N(A,\omega,x)$ is a *bounded* plurisubharmonic function (i.e. a multivariable function which is subharmonic in each variable) which does not behave as well as a subharmonic function. This makes the analysis necessary to obtain an LDT more technical. Moreover, there is no longer a dichotomy between rational and irrational frequencies, but rather a trichotomy between frequencies whose components are purely Diophantine, purely Liouville (or rationally dependent), and those with some components which are Diophantine and some which are Liouville (or rationally dependent). This complicates the argument in two ways. First, an LDT can only be obtained for purely Diophantine ω, so an additional argument is needed to obtain some (weaker) measure-theoretic estimate which is applicable when the frequency is not purely Diophantine. Second, the inductive procedure necessary to relate L_{N_0} to L is different depending on what kind of ω we have.

Let us now take some time to briefly describe our argument. We follow the same general structure as in [6], and our argument can be viewed as an extension of Bourgain's. That is, the main scheme of our proof is adapted from [6]. However, while our result is significantly more general and more technically complex, our argument can be viewed as a clarification of Bourgain's main ideas, and hopefully, improves the readability of the argument. In particular, we provide missing details in the arguments in Section 2.6 and Section 2.8 and summarize the main ideas throughout. The original argument is not directly applicable in our general setting due to a few technical issues that arise while considering general cocycles. In particular, uniform (in N) pointwise boundedness and non-negativity of $L_N(x)$, as well as quantitative estimates on $|L_N(x) - L_N(x+\omega)|$, are used extensively in Bourgain's work, while they no longer hold if $\det(A(x))$ is allowed to vanish, as, say, in the case of Jacobi cocycles; these need to be dealt with uniformly in N.

Here, we give a brief description of Bourgain's scheme and the key difficulties in its adaptation.

The first step is to establish a large deviation estimate under suitable assumptions made on ω. As we noted above, this is typically arrived at by observing that $L_N(A, \omega, x)$ is plurisubharmonic with a *bounded* extension to a strip. Since we consider cocycles which may have singularities, $L_N(A, \omega, x)$ need not be bounded. Fortunately, following ideas introduced in [23], while we cannot say $L_N(A, \omega, x)$ is uniformly pointwise bounded, we can say it is uniformly L^2 bounded. It turns out that this is sufficient to perform the necessary analysis to obtain an LDT for ω which posses Diophantine-like properties up to suitably large scales (see Theorem 2.3.2).

The uniform large deviation estimate we establish here (see Theorem 2.3.2) is different from the uniform large deviation estimate established in [23] (c.f. [23] Theorem 6.6) in one crucial aspect: the result of Duarte and Klein requires an explicit Diophantine condition on the frequency, whereas our result requires a restricted Diophantine condition. In particular, every $\omega = (\omega_1, ..., \omega_d) \in \mathbb{T}^d$, where $\omega_1, ..., \omega_d$ are irrational and rationally independent, satisfies a restricted Diophantine condition but need not satisfy a Diophantine condition. Due to the generality of the frequencies we consider, we lose any control over the modulus of continuity. It is this difference, however, which allows us to establish continuity which does not require a Diophantine assumption.

Our second step is to establish quantitative estimates on $|L_N(A, \omega, x) - L_N(A, \omega, x + a)|$, which we use when our frequency is such that our LDT is not applicable. Our analysis of the plurisubharmonic function $L_N(A, \omega, x)$ allows us to say that, for any $a \in \mathbb{T}^d$, $L_N(A, \omega, x)$ and $L_N(A, \omega, x + a)$ are close, away from a set of small measure (see Lemma 2.3.8).

Next, we establish quantitative estimates on $|L_N(A, \omega, x) - L_N(A, \omega, x + \omega)|$, which we use throughout to relate $L_{N_0}(A, \omega, x)$ and $L_{N_1}(A, \omega, x)$. In the Schrödinger case (and, in fact, in the case of nowhere singular cocycles) this is a simple consequence of the everywhere invertibility of A; we consider cocycles which may be non-invertible somewhere. Once again, we are able to use our uniform L^2 boundedness, along with a uniform version of the Lojasiewicz inequality (see Lemma 2.3.3), to prove that this difference is small away from an exponentially small set (see Lemma 2.3.2). This is actually a recurring theme: any time Bourgain would have appealed to pointwise boundedness, we appeal to a cutoff argument which takes advantage of L^2 boundedness and the Lojasiewicz inequality.

Next, we turn our attention to something which, at first glance, might seem trivial. In the $SL(2,\mathbb{C})$ case, the entire argument relies on the well-understood fact that $L_N(A,\omega)$ is jointly continuous for any (A,ω) when N is fixed. The typical argument for this relies on boundedness of $L_N(A,\omega)$. Since we no longer have boundedness, it is not immediately obvious why continuity should still hold. In the not-identically singular 1-frequency case [35], joint continuity was proved using a cutoff argument and Diophantine considerations. In the multifrequency case with Diophantine frequency [23], continuity was proved using ergodicity considerations which required restrictive assumptions on the frequency. Neither method is wholly applicable in our setting, as we want a result for all frequencies. Using a uniform Lojasiewicz inequality, we are able to adapt the cutoff argument of Jitomirskaya, Koslover, and Schulteis and extend it to arbitrary frequencies.

Our next step is to establish our base estimate relating $L_{N_0}(A,\omega)$ to $L_{N_1}(A,\omega)$, for N_1 not too large, when ω is not Diophantine (see Theorem 2.6.1). We prove this as a consequence of the Avalanche Principle and Lemma 2.3.8. This argument is of critical importance, as it provides the framework for estimates whenever the frequency is not purely Diophantine, and we appeal to it again when we prove Theorem 2.7.1.

We then turn our attention to relating $L_{N_0}(A,\omega)$ to $L_{N_1}(A,\omega)$, for N_1 not too large, when some components of ω are Diophantine and other components are not (see Theorem 2.7.1). The case when the frequency is purely Diophantine is a special case of this. Our argument here relies on applying Lemma 2.3.8 in the variables corresponding to the non-Diophantine components of ω and applying Theorem 2.3.2 in those components which are Diophantine. This, eventually, leads us to a situation where the proof of Theorem 2.6.2 is applicable.

Remark 7. Our estimates in Theorems 2.6.1 and 2.7.1 differ from the corresponding estimates in the $SL(2,\mathbb{C})$ case (c.f. [6] Corollary 3.12 and Lemma 3.26) by a small power of κ, which is a consequence of using a uniform L^2 estimate for $L_N(A,x)$, rather than a uniform pointwise bound.

Our final step is an inductive argument allowing us to iterate our initial estimates to larger scales (see Theorem 2.8.1). This relies on a delicate argument where we alternate between applying a toral automorphism (i.e. a change of variables which does not change the value of $L_N(A, \omega)$) and applying Theorem 2.7.1.

The continuity of L then follows from Theorem 2.8.1 and continuity of $L_N(A, \omega)$.

2.3 Plurisubharmonic functions and related estimates

In this section, we present the relevant facts and estimates related to plurisubharmonic functions defined on complex strips in $\mathbb{C}^d$. The results here are based on results found in Chapter 6 of [23] and we apply them to recover results from Section 1 of [6]. We present the statements of the main results here, mostly without proof. We provide proofs of Lemma 2.3.7, Theorem 2.3.2, and Lemma 2.3.8, as we will make repeated use of these results in later sections. A detailed discussion and proofs of the remaining results is provided in **??**.

One of the major obstacles to extending results about Lyapunov exponents for Schrödinger cocycles to general $M(2, \mathbb{C})$ cocycles is the lack of uniform pointwise boundedness in the latter case. It turns out, however, that a uniform L^p estimate is sufficient for our argument. The following lemma establishes such a uniform estimate.

Lemma 2.3.1 ([23] Proposition 6.3). *Let $A \in C^\omega_\rho(\mathbb{T}^d, M(2, \mathbb{C}))$ with $\det(A)$ not identically 0. Then there are $\delta = \delta(A) > 0$ and $C = C(A) < \infty$ such that for any $B \in C^\omega_\rho(\mathbb{T}^d, M(2, \mathbb{C}))$, with $\|B - A\|_\rho < \delta$, then*

$$\|L'_n(B)\|_{L^2(\mathbb{T}^d)} \leq C \tag{2.10}$$

and

$$\|\ln|\det(B(x))|\|_{L^2(\mathbb{T}^d)} \le C. \tag{2.11}$$

By the definition of L_N, we have the following corollary.

Corollary 2.3.1. *Under the assumptions of Lemma 2.3.1, we have*

$$\|L_n(B)\|_{L^2(\mathbb{T}^d)} \le C. \tag{2.12}$$

The main application of this lemma will be in excising certain small sets of "bad" points (where the pointwise bound is large) and showing that the integral of $L_N(x)$ over these bad sets is small by Hölder's inequality.

Another major obstacle is relating $L_N(x)$ to $L_N(x+\omega)$. In the $SL(2,\mathbb{C})$ case, the well-known estimate

$$|L_N(x) - L_N(x+\omega)| < C\frac{1}{N}$$

holds. Such an estimate does not hold, in general, for non-invertible cocycles. However, it is possible to show that such an estimate holds for a large set of x.

Lemma 2.3.2 ([23] Proposition 6.4)**.** *Let $A \in C^\omega_\rho(\mathbb{T}^d, M(2,\mathbb{C}))$ with $\det(A)$ not identically 0. Then there are $\delta = \delta(A) > 0$ and $C = C(A) < \infty$ such that for any $0 < a < 1$, if $B \in C^\omega_\rho(\mathbb{T}^d, M(2,\mathbb{C}))$ with $\|B - A\|_\rho < \delta$, then*

$$|L'_N(B,x) - L'_N(B,x+\omega)| \le CN^{-a} \tag{2.13}$$

holds for all $N \ge 1$ and for all $x \notin F_N$, where $|F_N| < e^{-N^{1-a}}$. Moreover,

$$\ln|\det A_N(x)| - \ln|\det A_N(x+\omega)| \le CN^{-a} \tag{2.14}$$

in the same set F_N.

Again, the definition of L_N allows for an immediate corollary.

Corollary 2.3.2. *Under the assumptions of Lemma 2.3.2, we have*

$$|L_N(B,x) - L_N(B,x+\omega)| \leq CN^{-a} \tag{2.15}$$

holds for all $N \geq 1$ and for all $x \notin F_N$, where $|F_N| < e^{-N^{1-a}}$.

Both of these estimates relies on a uniform version of the Lojasiewicz inequality, which is of independent interest to us.

Lemma 2.3.3 ([23] Lemma 6.1)**.** *Let $f(x) \in C^{\omega}_{\rho}(\mathbb{T}^d, \mathbb{C})$ be such that $f(x)$ is not identically zero. Then ther are constants $\delta = \delta(f) > 0, S = S(f) < \infty$, and $b = b(f) > 0$ such that if $g(x) \in C^{\omega}_{\rho}(\mathbb{T}^d, \mathbb{C})$ with $\|g - f\|_{\rho} < \delta$, then*

$$\left|\left\{x \in \mathbb{T}^d : |g(x)| < t\right\}\right| < St^b \tag{2.16}$$

for all $t > 0$.

With these lemmas in hand, we may proceed with our analysis of $L_N(A,x)$. Our next goal is to obtain finer control over $L_N(x)$, with the eventual goal of obtaining a large deviation estimate. Large deviation estimates for quasiperiodic cocycles typically arise from suitable decay of the associated Fourier coefficients (and ergodicity), so we begin by controlling the behavior of the Fourier coefficients.

Remark 8. The following three lemmas may be recovered via a synthesis of the statements and proofs in Chapter 6 from [23], and are stated for L'_N. We will state corollaries of these results for L_N afterwards. For convenience, we provide a discussion of the proofs of these three results in the Appendix.

Lemma 2.3.4. *Let $A \in C^{\omega}_{\rho}(\mathbb{T}^d, M(2,\mathbb{C}))$ with $\det(A)$ not identically 0. Then there are $\delta = \delta(A) > 0$ and $C = C(A,\rho) < \infty$ such that for any $N \geq 1$ and $B \in C^{\omega}_{\rho}(\mathbb{T}^d, M(2,\mathbb{C}))$ with $\|B - A\|_{\rho} < \delta$,*

$$\sum_{k\in\mathbb{Z}^d, |k|>K_0} |\widehat{L'_N}(B,k)|^2 \leq C\frac{1}{K_0} \tag{2.17}$$

$$\sum_{k\in\mathbb{Z}^d, |k|>K_0} |\left(\ln|\det(B_N)|\right)^{\wedge}(k)|^2 \leq C\frac{1}{K_0} \tag{2.18}$$

Though we are interested in cocycles on $\mathbb{T}^d$, with $d > 1$, it is often possible to obtain results for $d > 1$ from the corresponding $d = 1$ result applied in each variable. We will occasionally use this technique when applying our Fourier coefficient estimate, so we also include the superior estimate we have when $d = 1$.

Lemma 2.3.5. *Let $A \in C^{\omega}_{\rho}(\mathbb{T}, M(2,\mathbb{C}))$ with $\det(A)$ not identically 0. Then there are $\delta = \delta(A) > 0$ and $C = C(A,\rho) < \infty$ such that for any $N \geq 1$ and $B \in C^{\omega}_{\rho}(\mathbb{T}, M(2,\mathbb{C}))$ with $\|B - A\|_{\rho} < \delta$,*

$$|\hat{L'}_N(B,k)| \leq C\frac{1}{k} \tag{2.19}$$

$$|\left(\ln|\det(B_N)|\right)^{\wedge}(k)| \leq C\frac{1}{k} \tag{2.20}$$

We are now in a position to discuss a large deviation estimate. The general strategy is to combine the estimates from Lemma 2.3.2 and Lemma 2.3.4 to obtain an L^1 estimate, which we then improve using the following fact about BMO (bounded mean oscillation) functions. We present this next estimate using $L'_N(x)$, but it, in fact, holds for plurisubharmonic functions defined on strips in $\mathbb{C}^d$ which obey certain a priori estimates.

Lemma 2.3.6. *Let $A \in C^{\omega}_{\rho}(\mathbb{T}^d, M(2,\mathbb{C}))$ with $\det(A)$ not identically 0. Moreover, suppose*

$$\left\| L'_N(A,x) - \int_{\mathbb{T}^d} L'_N(A,x)dx \right\|_{L^1} < \epsilon. \tag{2.21}$$

Then there is $c = c(d)$ such that

$$\left| \left\{ x \in \mathbb{T}^d : \left| L'_N(A,x) - \int_{\mathbb{T}^d} L'_N(A,x)dx \right| > \epsilon^c \right\} \right| < e^{\epsilon^{-c}}. \tag{2.22}$$

Remark 9. Note that if we assume (2.21) holds for $\ln|\det(A_N(x))|$ instead, then we may replace L'_N in the conclusion with $\ln|\det(A_N(x))|$.

These results allow us to obtain a uniform large deviation estimate for $L_N(A,x)$, which will be required in Section 2.7. We present the large deviation estimate in two steps, for clarity.

Lemma 2.3.7. *Let $A \in C^{\omega}_{\rho}(\mathbb{T}^d, M(2,\mathbb{C}))$ with $\det(A)$ not identically 0. Suppose $\omega \in \mathbb{T}^d$ is such that*

$$\|k \cdot \omega\| > \delta_0$$

for all $0 < |k| < K_0$. Moreover, suppose

$$R > \sqrt{K_0}\delta_0^{-1}.$$

Then there are $\delta = \delta(A) > 0$ and $C = C(A,\rho) < \infty$ such that for any $B \in C^{\omega}_{\rho}(\mathbb{T}^d, M(2,\mathbb{C}))$ with $\|B - A\|_{\rho} < \delta$,

$$\left| \left\{ x \in \mathbb{T}^d : \left| \frac{1}{R} \sum_{j=0}^{R-1} L'_N(B, x + j\omega) - \langle L'_N(B) \rangle \right| > C_{\rho} K_0^{-c} \right\} \right| < e^{-C_{\rho} K_0^{c}}. \tag{2.23}$$

Proof. Consider

$$\left| \frac{1}{R} \sum_{j=0}^{R-1} L'_N(B, x + j\omega) - \int_{\mathbb{T}^d} L'_N(B,x)dx \right|.$$

We have:

$$\frac{1}{R}\sum_{j=0}^{R-1} L'_N(B, x+j\omega) = \frac{1}{R}\sum_{j=0}^{R-1}\sum_{k\in\mathbb{Z}^d} \hat{L'}_N(k)(B)e^{2\pi i k\cdot(x+j\omega)} \tag{2.24}$$

$$= \frac{1}{R}\sum_{j=0}^{R-1} \hat{L'}_N(0)(B) + \frac{1}{R}\sum_{j=0}^{R-1}\sum_{0<|k|\le K_0} \hat{L'}_N(k)(B)e^{2\pi i k\cdot(x+j\omega)} \tag{2.25}$$

$$+ \frac{1}{R}\sum_{j=0}^{R-1}\sum_{|k|>K_0} \hat{L'}_N(k)(B)e^{2\pi i k\cdot(x+j\omega)} \tag{2.26}$$

$$= (I) + (II) + (III). \tag{2.27}$$

Observe that we have

$$(I) = \int_{\mathbb{T}^d} L'_N(B, x)dx.$$

Thus

$$(I) - \int_{\mathbb{T}^d} L'_N(B, x)dx = 0,$$

Next, for $0 < |k| \le K_0$ we may appeal to our condition on ω to conclude

$$\left|\frac{1}{R}\sum_{j=0}^{R-1} e^{2\pi i k\cdot j\omega}\right| \lesssim \frac{2}{R\,\|k\omega\|} \le 2K_0^{-1/2}.$$

Thus

$$\|(II)\|_{L^2} \le CK_0^{-1/2}. \tag{2.28}$$

Here C depends only on A.

Finally, for $|k| > K_0$, we know $\sum_{|k|>K_0} |\hat{L'}_N(k)(B)|^2 < C|K_0|^{-1}$, where C depends only on

A. Moreover, $|e^{2\pi ijk\cdot\omega}| = 1$, so

$$\|(III)\|_2 \leq \left(\sum_{|k|>K_0} |\hat{u}_n(k)|^2\right)^{1/2} \tag{2.29}$$

$$< CK_0^{-1/2}. \tag{2.30}$$

Hence

$$\left\|\frac{1}{R}\sum_{j=0}^{R-1} L'_N(B, x + j\omega) - \int_{\mathbb{T}^d} L'_N(B, x)dx\right\|_{L^1} < CK_0^{-1/2}.$$

Now we appeal to Lemma 2.3.6 applied to the function $\frac{1}{R}\sum_{j=0}^{R-1} L_N(B, x+j\omega) - \int_{\mathbb{T}^d} L_N(B, x)$, which completes our proof. $\square$

Remark 10. It is easy to see from the proof that this result also holds for $\ln|\det(B_N(x))|$ in place of $L'_N(B, x)$.

Theorem 2.3.1. *Let $A \in C^\omega_\rho(\mathbb{T}^d, M(2,\mathbb{C}))$ with $\det(A)$ not identically 0. Suppose $\omega \in \mathbb{T}^d$ is such that*

$$\|k \cdot \omega\| > \delta_0$$

for all $0 < |k| < K_0$. Moreover, suppose

$$N > K_0\delta_0^{-1}.$$

Then there are $\delta = \delta(A) > 0$ and $C = C(A, \rho) < \infty$ such that for any $B \in C^\omega_\rho(\mathbb{T}^d, M(2,\mathbb{C}))$ with $\|B - A\|_\rho < \delta$,

$$\left|\left\{x \in \mathbb{T}^d : |L'_N(B, x) - L'_N(B)| > C_\rho K_0^{-c}\right\}\right| < e^{-C_\rho K_0^c}. \tag{2.31}$$

Proof. Apply Lemma 2.3.7 with $R = \sqrt{N}$. We obtain

$$\left|\left\{x \in \mathbb{T}^d : \left|\frac{1}{\sqrt{N}} \sum_{j=0}^{\sqrt{N}-1} L'_N(B, x + j\omega) - \langle L'_N(B)\rangle\right| > C_\rho K_0^{-c}\right\}\right| < e^{-C_\rho K_0^c}. \tag{2.32}$$

Moreover, recalling

$$\left|\left\{x : |L'_N(B, x + j\omega) - L'_N(B, x)| < C|j|N^{-1/2}\right\}\right| < e^{-N^{1/2}},$$

where C depends only on A, away from a set of measure at most $R^2 e^{-N^{1/2}} < e^{-N^{1/3}}$, we have

$$\frac{1}{\sqrt{N}} \sum_{j=0}^{\sqrt{N}-1} L'_N(B, x + j\omega) = \frac{1}{\sqrt{N}} \sum_{j=0}^{\sqrt{N}-1} (L'_N(B, x) + O(|j|/N)) \tag{2.33}$$

$$= L'_N(B, x) + O(\sqrt{N}/N) \tag{2.34}$$

$$\leq L'_N(B, x) + CK_0^{-1/2}. \tag{2.35}$$

Triangle inequality thus yields

$$\left\|L_N(B, x) - \int_{\mathbb{T}^d} L_N(B, x)\right\|_{L^1} < CK_0^{-1/2}.$$

We now conclude in the same was as before. $\square$

Remark 11. The proof clearly works for $\ln|\det(B_N(x))|$ as well.

We can combine these results for L'_N and $\ln|\det(B_N(x))|$, and appeal to our definition of L_N, to obtain the desired result for L_N.

Theorem 2.3.2. *Let $A \in C^\omega_\rho(\mathbb{T}^d, M(2, \mathbb{C}))$ with $\det(A)$ not identically 0. Suppose $\omega \in \mathbb{T}^d$ is such that*

$$\|k \cdot \omega\| > \delta_0$$

for all $0 < |k| < K_0$*. Moreover, suppose*

$$N > K_0 \delta_0^{-1}.$$

Then there are $\delta = \delta(A) > 0$ *and* $C = C(A, \rho) < \infty$ *such that for any* $B \in C^\omega_\rho(\mathbb{T}^d, M(2, \mathbb{C}))$ *with* $\|B - A\|_\rho < \delta$*,*

$$\left|\left\{x \in \mathbb{T}^d : |L_N(B, x) - L_N(B)| > C_\rho K_0^{-c}\right\}\right| < e^{-C_\rho K_0^c}. \tag{2.36}$$

At this point, we have suitable estimates for frequencies which obey a diophantine estimate at certain length scales. We are interested, however, in general frequencies. The following estimate will be used in the absence of a diophantine frequency. This follows as a consequence of the Fourier coefficient decay estimate.

Lemma 2.3.8. *Let* $A \in C^\omega_\rho(\mathbb{T}^d, M(2, \mathbb{C}))$ *with* $\det(A)$ *not identically 0. Then there are* $\delta = \delta(A) > 0$ *and* $C = C(A, \rho) < \infty$ *such that for any* $B \in C^\omega_\rho(\mathbb{T}^d, M(2, \mathbb{C}))$ *with* $\|B - A\|_\rho < \delta$*, and for* $a \in \mathbb{T}$ *small and* $\kappa > 0$*, we have, uniformly in* N*,*

$$\left|\left\{x \in \mathbb{T}^d : |L_N(B, x) - L_N(B, x + a)| > \kappa\right\}\right| < C\kappa^{-3}|a| \tag{2.37}$$

Remark 12. We will need a to be small so that $C(A)^{-1}\kappa|a|^{-1} \geq 1$. This is necessary for us to define K_0 appropriately in our proof (see (2.46)).

Proof. We will prove this for $d = 1$. The general case follows from the $d = 1$ case and Fubini's theorem.

We have

$$L_N(B, x) = \sum_{k \in \mathbb{Z}} \hat{L}_n(B, k) e^{2\pi k \cdot x},$$

so

$$L_N(B,x) - L_N(B,x+a) = \sum_{k\in\mathbb{Z}} \hat{L}_n(B,k)e^{2\pi k\cdot x}(1-e^{2\pi k\cdot a}) \tag{2.38}$$

$$= \sum_{|k|<K_0} + \sum_{|k|\geq K_0} \tag{2.39}$$

$$= (I) + (II). \tag{2.40}$$

Recall that, when $d=1$, we have

$$\left|\hat{L}_n(B,k)\right| \leq C(A)(1+|k|)^{-1}.$$

It follows that

$$|(I)| \leq \sum_{|k|<K_0} \left|\hat{L}_n(B,k)\right| |(1-e^{2\pi k\cdot a})| \tag{2.41}$$

$$\leq \sum_{|k|<K_0} C(A)(1+|k|)^{-1}|k||a| \tag{2.42}$$

$$\leq C(A)K_0|a|. \tag{2.43}$$

For (II), we have

$$\|(II)\|_{L^2}^2 \leq \sum_{|k|\geq K_0} 2\left|\hat{L}_n(B,k)\right|^2 \tag{2.44}$$

$$\leq C(A)K_0^{-1}. \tag{2.45}$$

Taking

$$K_0 \sim C(A)^{-1}\kappa|a|^{-1}, \tag{2.46}$$

we have

$$|(I)| \leq \kappa \tag{2.47}$$

$$\|(II)\|_{L^2}^2 \leq C(A)\kappa^{-1}|a|. \tag{2.48}$$

Applying Chebyschev's inequality,

$$|\{x \in \mathbb{T} : |(II)| > \kappa\}| < C(A)\kappa^{-3}|a|. \tag{2.49}$$

It follows that

$$\big|\{x : |L_N(B,x) - L_N(B,x+a)| > \kappa\}\big| \tag{2.50}$$

$$\leq \left|\left\{x : |\sum_{|k|<K_0}| > \kappa\right\}\right| + \left|\left\{x : |\sum_{|k|\geq K_0}| > \kappa\right\}\right| \tag{2.51}$$

$$\leq C(A)\kappa^{-3}|a|. \tag{2.52}$$

□

2.4 Finite-scale continuity

In the Schrödinger cocycle case (and the $SL(2\mathbb{C})$ case more generally), one of the key observations is that, for fixed N, $L_N(A,\omega)$ is jointly continuous in A and ω for any ω. This is a simple consequence of the everywhere invertibility of the cocycle, A. The analogous result for $M(2,\mathbb{C})$ cocycles requires an argument.

In this section, we establish continuity of the finite-scale Lyapunov exponents, $L_N(A,\omega)$ jointly in A and ω for any not identically singular A and any frequency ω.

Lemma 2.4.1. *Let $(A, \omega) \in C_\rho(\mathbb{T}^d, M(2, \mathbb{C})) \times \mathbb{T}^d, A \not\equiv 0$, be an analytic quasiperiodic cocycle. Then for any $\epsilon > 0$, there exists constants $\delta = \delta(A, \epsilon), C = C(A, \epsilon)$, and $N_0 = N_0(A, \epsilon)$, such that for any $B \in C_\rho(\mathbb{T}^d, M(2, \mathbb{C}))$ with $\|A - B\|_\rho < \delta$ and $\|\omega - \omega'\| < \delta$ we have*

$$|L_N(A, \omega) - L_N(B, \omega')| < C_A \epsilon$$

for all $N > N_0$.

Proof. Continuity of L_N will follow from continuity of L'_N and $\int \ln|\det A_N(x)|dx$. We present the following argument for L'_N, but it is easy to see that it applies to $\int \ln|\det A(x)|dx$ as well.

Let $\delta_0 > 0$, fix $N > N_0$, and set

$$F_{A,\delta_0} := \left\{x \in \mathbb{T}^d : \|A_N(x, \omega)\| < e^{-N^{1+\delta_0}}\right\} \tag{2.53}$$

$$F_{B,\delta_0} := \left\{x \in \mathbb{T}^d : \|B_N(x, \omega')\| < e^{-N^{1+\delta_0}}\right\} \tag{2.54}$$

$$G := \left\{x \in \mathbb{T}^d : \|A_N(x, \omega)\| \le \|B_N(x, \omega')\|\right\}. \tag{2.55}$$

Note that

$$|L'_N(A, \omega) - L'_N(B, \omega')| = \left|\int_{\mathbb{T}^d} \frac{1}{N} \ln\left(\frac{\|A_N(x, \omega)\|}{\|B_N(x, \omega')\|}\right) dx\right| \tag{2.56}$$

$$= \left|\int_{F_A \cap F_B} + \int_{F_A^c \cap F_B} + \int_{F_A \cap F_B^c} + \int_{F_A^c \cap F_B^c}\right|. \tag{2.57}$$

Observe, for $x \in F_A^c \cap F_B^c$, we have

$$|L'_N(A, x, \omega) - L'_N(B, x, \omega')| = \begin{cases} \frac{1}{N} \ln\left(\frac{\|A_N(x,\omega)\|}{\|B_N(x,\omega')\|}\right) & x \notin G \\ \frac{1}{N} \ln\left(\frac{\|B_N(x,\omega')\|}{\|A_N(x,\omega)\|}\right) & x \in G \end{cases}. \tag{2.58}$$

Consider $x \in G$. The case $x \notin G$ will be the same. We have

$$\frac{1}{N}\ln\left(\frac{\|B_N(x,\omega')\|}{\|A_N(x,\omega)\|}\right) \le \frac{1}{N}\|A_N(x,\omega)\|^{-1} C_A^N \|A(x,\omega)-B(x,\omega')\|_\rho \tag{2.59}$$

$$\le e^{N^{1+\delta_0}} C_A^N \left(\|A(x,\omega)-B(x,\omega)\|_\rho + \|B(x,\omega)-B(x,\omega')\|\right). \tag{2.60}$$

Taking $\|A-B\|_\rho$ and $\|\omega-\omega'\|$ sufficiently small (dependent on A, N, and δ_0) ensures this is no more than $\epsilon/4$.

Next, consider $x \in F_A \cap F_B^c$. The case $x \in F_A^c \cap F_B$ is the same. We have, necessarily, $x \in G$, and thus

$$|L_N'(A,x,\omega) - L_N'(B,x,\omega')| = \frac{1}{N}\ln\left(\frac{\|B_N(x,\omega')\|}{\|A_N(x,\omega)\|}\right) \tag{2.61}$$

$$= \frac{1}{N}\left(\ln\|B_N(x,\omega')\| - \ln\|A_N(x,\omega)\|\right) \tag{2.62}$$

$$\le \frac{1}{N}\left(\ln\|B_N(x,\omega')\| - \sum_{j=1}^{N}\ln|\det A(x+j\omega)|\right) \tag{2.63}$$

$$\le \frac{1}{N}\left(NC_A - \sum_{j=1}^{N}\ln|\det A(x+j\omega)|\right). \tag{2.64}$$

Moreover,

$$F_A \cap F_B^c \subset F_A \subset \bigcup_{j=1}^{N}\left\{x : |\det(x+j\omega)| < e^{-N^{\delta_0}}\right\} =: \bigcup_{j=1}^{N} S_j.$$

Thus

$$\left|\int_{F_A\cap F_B^c}\right| \le \sum_{j=1}^{N}\frac{1}{N}\int_{S_j}\left(NC_A - \sum_{k=1}^{N}\ln|\det A(x+k\omega)|\right)dx \tag{2.65}$$

$$\le C_A\sum_{j=1}^{N}|S_j| + \frac{1}{N}\sum_{j=1}^{N}\sum_{k=1}^{N}\left|\int_{S_j}\ln|\det A(x+k\omega)|dx\right| \tag{2.66}$$

$$\le NC_A e^{-\sigma N^{\delta_0}} + \frac{1}{N}\sum_{j=1}^{N}\sum_{k=1}^{N}\left|\int_{S_j}\ln|\det A(x+k\omega)|dx\right|. \tag{2.67}$$

Now, we may use Lemmas 2.3.1 and 2.3.3 to bound

$$\frac{1}{N}\sum_{j=1}^{N}\sum_{k=1}^{N}\left|\int_{S_j}\ln|\det A(x+k\omega)|dx\right| \le Ce^{-\sigma N^{\delta_0}}N^{1+\delta_0}.$$

Putting all of this together and taking N sufficiently large guarantees this is no larger than $\epsilon/4$.

The case $x \in F_A \cap F_B$ is similar.

Thus $|L'_N(A,\omega) - L'_N(B,\omega')| < \epsilon$, so L'_N is continuous in A and ω.

Continuity of $\int \ln|\det A(x,\omega)|dx$ quickly follows by a similar argument. □

2.5 Avalanche principle and immediate consequences

Theorem 2.5.1 (Avalanche Principle)**.** *Suppose $A_1,...,A_n \in SL(2,\mathbb{C})$ are such that*

$$\min_{1\le j\le n}\|A_j\| \ge \mu > n \tag{2.68}$$

and

$$\max_{1\le j<n}|\ln\|A)_j\| + \|A_{j+1}\| - \ln\|A_{j+1}A_j\|| \le \frac{1}{2}\ln\mu. \tag{2.69}$$

Then

$$\left|\ln\|A_n\cdot A_1\| + \sum_{j=2}^{n-1}\ln\|A_j\| - \sum_{j=1}^{n-1}\ln\|A_{j+1}A_j\|\right| < C\frac{n}{\mu}. \tag{2.70}$$

The C above is an absolute constant. We include this result for completeness, but we will actually use a slight variation of this result which is due to Bourgain [6].

Theorem 2.5.2 (Avalanche Principle Variation)**.** *Suppose $A_1, ..., A_n \in SL_2(\mathbb{C})$ are such that*

$$\mu < \|A_j\| < \mu^C \tag{2.71}$$

for all $1 \leq j \leq n$ and some μ sufficiently large. Moreover, suppose

$$\max_{1\leq j<n} |\ln \|A_j\| + \|A_{j+1}\| - \ln \|A_{j+1}A_j\|| \leq \frac{1}{2}\ln \mu. \tag{2.72}$$

Then

$$\left|\ln \|A_n \cdot A_1\| + \sum_{j=1}^{n} \ln \|A_j\| - \sum_{j=1}^{n-1} \ln \|A_{j+1}A_j\|\right| < \frac{n}{\mu^{1/3}} + 4C\ln \mu. \tag{2.73}$$

We refer readers to [30] for the proof of Theorem 2.5.1 and to [6] for the proof of Theorem 2.5.2. Both of these references prove the Avalanche Principle for $SL(2,\mathbb{R})$ matrices, but the arguments clearly apply to $SL(2,\mathbb{C})$ matrices.

The Avalanche Principle allows us to relate Lyapunov exponents at some initial scale to Lyapunov exponents at a larger scale via the following.

Theorem 2.5.3. *Fix an analytic cocycle (A, ω). Fix $x \in \mathbb{T}^d$ and let $\delta > 0$ be a fixed constant. Let $N_0 > 0$ be sufficiently large (depending only on δ and measurements of $A(x)$). Take $N_1 \in \mathbb{Z}, N_1 \geq N_0, N_0|N_1$. Assume, moreover, that*

$$L_{N_0}(x) > \delta \tag{2.74}$$

$$|L_{N_0}(x) - L_{2N_0}(x)| < \frac{1}{100}L_{N_0}(x) \tag{2.75}$$

$$|L_N(x) - L_N(x + jN_0\omega)| < \frac{\delta}{100} \tag{2.76}$$

for $N = N_0, 2N_0,$ *and* $j \leq \frac{N_1}{N_0}$. *Then*

$$\begin{aligned} &\left| L_{N_1}(x) + \frac{1}{n}\sum_{j=0}^{n-1} L_{N_0}(x + jN_0\omega) - \frac{2}{n}\sum_{j=0}^{n-2} L_{2N_0}(x + jN_0\omega) \right| \\ &\qquad < \exp\left\{ -\frac{N_0}{4} L_{N_0}(x) \right\} + C|L_{N_0}(x)|\frac{N_0}{N_1}, \end{aligned} \tag{2.77}$$

where $n = N_1/N_0$. *Moreover,*

$$|L_{N_1}(x) + L_{N_0}(x) - 2L_{2N_0}(x)| < \frac{\delta}{20} + C|L_{N_0}(x)|\frac{N_0}{N_1}. \tag{2.78}$$

The C *here is an absolute constant.*

We will provide a proof of this result for convenience, though a proof for Schrödinger cocycles may be found in [6]. A key difference between our presentation and the presentation in [6] is the inclusion of $L_{N_0}(x)$ in the right hand side of (2.77) and (2.78). This is due to the absence of a uniform pointwise bound on $L_{N_0}(x)$ in the case of general cocycles. This will not pose a problem later, as we have Lemma 2.3.1 to deal with integrals of the right-hand side.

Proof. Fix a cocycle A. We will write $L_{N_0}(x)$ in place of $L_{N_0}(A, x)$. Define $M_j = A_{N_0}(x + jN_0\omega) \in SL(2, \mathbb{C})$. Our goal is to apply Theorem 2.5.2 to M_j. By (2.76) and (2.74), we obtain

$$\begin{aligned} \frac{99}{100} L_{N_0}(x) &\leq L_{N_0}(x) - \delta/100 \\ &\leq L_{N_0}(x + jN_0\omega) \\ &\leq L_{N_0}(x) + \delta/100 \\ &\leq \frac{101}{100} L_{N_0}(x). \end{aligned} \tag{2.79}$$

The definition of L_{N_0} now yields, for $j \leq N_1/N_0$,

$$\frac{99}{100}N_0 L_{N_0}(x) < \ln \|M_j\| < \frac{101}{100}N_0 L_{N_0}(x). \tag{2.80}$$

Setting

$$\mu = e^{\frac{99}{100}N_0 L_{N_0}(x)},$$

we have

$$\mu < \|M_j\| < \mu^{101/99} = \mu^C. \tag{2.81}$$

Moreover,

$$M_{j+1}M_j = A_{2N_0}(x + jN_0\omega).$$

Thus, for $j < N_1/N_0$,

$$\begin{aligned}
&\Big| \ln \|M_{j+1}\| + \ln \|M_j\| - \ln \|M_{j+1}M_j\| \Big| && (2.82)\\
&\quad = N_0 \left| L_{N_0}(x + (j+1)N_0\omega) + L_{N_0}(x + jN_0\omega) - 2L_{2N_0}(x + jN_0\omega) \right|. && (2.83)
\end{aligned}$$

Hence, by (2.76) and triangle inequality,

$$\begin{aligned}
&\Big| \ln \|M_{j+1}\| + \ln \|M_j\| - \ln \|M_{j+1}M_j\| \Big| && (2.84)\\
&\quad \leq 2N_0 \left| L_{N_0}(x) + L_{2N_0}(x) \right| + \frac{N_0\delta}{25} && (2.85)\\
&\quad \leq \frac{N_0\delta}{50} + \frac{N_0\delta}{25} && (2.86)\\
&\quad = \frac{3N_0\delta}{50} && (2.87)\\
&\quad < \frac{3N_0}{50}L_{N_0}(x) && (2.88)\\
&\quad < \frac{1}{10}\ln \mu. && (2.89)
\end{aligned}$$

The last inequality follows from the definition of μ. This, along with (2.81), means Theorem 2.5.2 is applicable, and we obtain, with $N = \frac{N_1}{N_0} - 1$,

$$\left|N_1 L_{N_1}(x) + N_0 \sum_{j=0}^{N} L_{N_0}(x + jN_0\omega) - 2N_0 \sum_{j=0}^{N-1} L_{2N_0}(x + jN_0\omega)\right| \tag{2.90}$$

$$< \frac{N_1}{N_0}\mu^{-1/3} + C \ln \mu \tag{2.91}$$

$$= \frac{N_1}{N_0}\mu^{-1/3} + C N_0 L_{N_0}(x). \tag{2.92}$$

Dividing by N_1, and using the definition of N, we have

$$\left|L_{N_1}(x) + \frac{1}{N}(1 - N_0/N_1) \sum_{j=0}^{N} L_{N_0}(x + jN_0\omega)\right. \tag{2.93}$$

$$\left. - 2\frac{1}{N}(1 - N_0/N_1) \sum_{j=0}^{N-1} L_{2N_0}(x + jN_0\omega)\right| \tag{2.94}$$

$$= \frac{1}{N_0}\mu^{-1/3} + C\frac{N_0}{N_1} L_{N_0}(x). \tag{2.95}$$

Now we note that

$$\frac{1}{N}(1 - N_0/N_1) = \frac{1}{N+1}$$

and

$$N_0^{-1}\mu^{-1/3} < e^{-\frac{1}{4}N_0 L_{N_0}(x)},$$

which together yield (2.77). Combining this with (2.76) and triangle inequality yields (2.78).

□

2.6 Comparing Lyapunov exponents at different scales: Liouville frequencies

Throughout this section, we establish estimates of the form $|L_{N_0}(A) - L_{N_1}(A)| < C'\kappa$. The constant C' depends on measurements of A and can thus be taken uniform for all B with $\|A - B\|$ sufficiently small. The uniformity of this constant will be essential to establishing continuity in Section 2.9. In what follows, the constant C will be taken to be sufficiently large so that Theorem 2.3.2 is applicable with $c = 1/C$.

Before considering the general case of arbitrary $\omega \in \mathbb{T}^d$, we will consider the special case where ω satisfies a Liouville-type condition.

Remark 13. We would like to note that we could just as easily have skipped the discussion in this section and simply proved Theorem 2.7.1. We choose to present the following special case to illustrate the main ideas of the proof in a simplified setting.

Lemma 2.6.1. *Fix $N_0 = a2^b q_0$, with $b \in \mathbb{N}$ large. Consider the set F consisting of all $x \in \mathbb{T}^d$ such that $|L_N(x) - L_N(x + jq_0\omega)| < \kappa$ for all $N = 2^{-s}N_0$, with $0 \le s \le -C_3 \ln \kappa = s_0$, and all $j \le N_1/q_0$ such that $N_0 2^{-s}/q_0$ divides j for some $0 \le s \le s_0$. Here C_3 is a sufficiently large constant. Then*

$$\left| \int_F L_{N_0}(x) - L_{N_1}(x) dx \right| < C\kappa.$$

Proof. Consider $x \in F$ such that $L_{N_0}(x) > 10^3\kappa$. Define $N_{0,1} = N_0/2$. Then $N_0 = 2N_{0,1}$.

Since $x \in F$, we have

$$
\begin{aligned}
L_{N_0}(x) &= \frac{1}{N_0} \ln \|A_{N_0}(x, \omega)\| && (2.96)\\
&= \frac{1}{2N_{0,1}} \ln \left\|A_{2N_{0,1}}(x, \omega)\right\| && (2.97)\\
&\leq \frac{1}{2N_{0,1}} \left(\ln \left\|A_{N_{0,1}}(x, \omega)\right\| + \ln \left\|A_{N_{0,1}}(x + N_{0,1}, \omega)\right\|\right) && (2.98)\\
&= \frac{1}{2} L_{N_{0,1}}(x) + \frac{1}{2} L_{N_{0,1}}(x + \frac{N_{0,1}}{q_0} q_0 \omega) && (2.99)\\
&\leq L_{N_{0,1}}(x) + \kappa. && (2.100)
\end{aligned}
$$

The last line follows from the definition of F. Thus

$$L_{N_0} \leq L_{N_0/2}(x) + \kappa.$$

We can obtain a similar estimate using $N_{0,s} = N_0/2^s$ instead:

$$L_{N_0 2^{-s}}(x) \leq L_{N_0 2^{-s'}}(x) + \kappa$$

for any $s \leq s' \leq s_0$.

Since $L_{N_0}(x) > 10^3 \kappa$, we have

$$999\kappa < L_{N_0 2^{-s}}(x). \tag{2.101}$$

Thus, for $0 \leq s \leq s_0$, the sequence $999\kappa < L_{2^{-s}N_0}(x)$ is increasing, up to modification by $O(\kappa)$.

At this point, we further restrict our allowable x to a suitable set

$$Z = \left\{ x \in \mathbb{T}^d : L_{2^{-s_0}N_0}(x) < \kappa^{-1} \right\}.$$

By Chebyschev's inequality, $|\mathbb{T}^d \backslash Z| < C(A)\kappa$, so

$$\left| \int_{\mathbb{T}^d \backslash Z} L_{N_1}(x) - L_{N_0}(x) dx \right| < C\kappa.$$

For $x \in F \cap Z$, we define $N_{00} = N_{00}(x) = \frac{N_0}{2^{s(x)}}$, where $0 \leq s(x) \leq s_0$ is chosen so that $N_{00} < \kappa N_0$, (which is possible because $s_0 = -C_3 \ln \kappa > -\ln \kappa$), and

$$\frac{99}{100} L_{N_{00}}(x) < L_{2N_{00}}(x) < L_{N_{00}}(x) + 10\kappa.$$

The right inequality is true because $L_{N_0 2^{-s}}(x)$ is increasing up to modification by $O(\kappa)$, and the left inequality is true by taking C_3 large. Indeed, if the left inequality fails for all choices of $N_{00}(x)$, then we have, for $x \in F \cap Z$,

$$\left(\frac{99}{100} \right)^{-C_3 \ln \kappa} \kappa^{-1} > \left(\frac{99}{100} \right)^{-C_3 \ln \kappa} L_{2^{-s_0}N_0}(x) > L_0(x) > 10^3 \kappa.$$

Taking a logarithm of the leftmost and rightmost terms, we quickly see that this is impossible for large C_3. (Say $C_3 > 10$).

Hence

$$|L_{N_{00}}(x) - L_{2N_{00}}(x)| < \frac{1}{100} L_{N_{00}}(x). \tag{2.102}$$

Now note that $x \in F$ and $N_{00}, 2N_{00}$ are of the form of the length scales included in the definition of F, so for $N = N_{00}$ and $2N_{00}$, and for all $j = n\frac{N_0}{2^s q_0}, j \leq N_1/q_0, 0 \leq s \leq s_0$, we have

$$|L_N(x + jq_0\omega) - L_N(x)| < \kappa. \tag{2.103}$$

In particular, if we take $j = n = N_{00}q_0^{-1}, n \leq N_1/N_{00}$, we have

$$|L_N(x + nN_{00}\omega) - L_N(x)| < \kappa. \tag{2.104}$$

Thus Theorem 2.5.3 is applicable, and we obtain

$$|L_{N_1}(x) + L_{N_{00}}(x) - 2L_{2N_{00}}(x)| < O(\kappa) + CL_{N_{00}}(x)\frac{N_{00}}{N_1} \tag{2.105}$$

$$|L_{N_0}(x) + L_{N_{00}}(x) - 2L_{2N_{00}}(x)| < O(\kappa) + CL_{N_{00}}(x)\frac{N_{00}}{N_0}. \tag{2.106}$$

Since $x \in F$, we have $L_{N_{00}}(x) \leq L_{2^{-s_0}N_0}(x) + \kappa$. Moreover, by construction, $\frac{N_{00}}{N_1}, \frac{N_{00}}{N_0} < \kappa$. Thus, for $x \in F$ such that $L_{N_0}(x) > 10^3\kappa$,

$$|L_{N_0}(x) - L_{N_1}(x)| < (C + L_{2^{-s_0}N_0}(x))\kappa. \tag{2.107}$$

Moreover, we know $\int_{\mathbb{T}^d} |L_{2^{-s_0}N_0}(x)|dx < C(A)$, so the integral of the above is bounded by $C(A)\kappa$. Finally, for $x \in F$ such that $L_{N_0}(x) < 10^3\kappa$, we have

$$L_{N_1}(x) \leq L_{N_0}(x) < 10^3\kappa.$$

Thus

$$\left| \int_F L_{N_1}(x) - L_{N_0}(x)dx \right| < C(A)\kappa. \tag{2.108}$$

$\square$

Lemma 2.6.2 (Liouville Frequencies). *Let $\kappa > 0$ be small, and let $N_0, q_0 \in \mathbb{N}$ be such that*

$$\|q_0\omega\| = \sum \|q_0\omega_j\| < \kappa^C \rho^4 \frac{q_0}{N_0} \tag{2.109}$$

and

$$N_0\kappa^C > q_0. \tag{2.110}$$

Then

$$|L_{N_1} - L_{N_0}| < C'\kappa^{1/2} \tag{2.111}$$

for all N_1 such that $N_0|N_1$ and

$$N_1 < \kappa^{C/2}\rho^2\sqrt{\frac{N_0 q_0}{\|q_0\omega\|}}. \tag{2.112}$$

Here $C = C(v, E)$ and can be taken uniform in E whenever E is restricted to a bounded set.

Remark 14. The following proof is essential to our overall argument, and elements of it will be used again to prove Theorem 2.7.1. In particular, the introduction of and restriction to the set F is essential. Once we restrict to F, the following argument goes through without issues. The key difficulty is showing that $\mathbb{T}^d \backslash F$ has small measure. This is accomplished here using Lemma 2.3.8. Later it will be accomplished using more involved estimates.

Proof. We will first prove this result for N_0 such that

$$N_0 = a2^b q_0, \quad a, b \in \mathbb{N}, b > -\frac{C}{2}\ln\kappa, \tag{2.113}$$

for some C sufficiently large (this is the same C which appears in the statement of the lemma). Note that this assumption on b allows for N_0 slightly smaller than those imposed by the condition (2.110). Then we will derive the general result for general N_0 satisfying (2.110).

Fix κ small. Consider the set of $x \in \mathbb{T}^d$ such that

$$|L_N(x + jq_0\omega) - L_N(x)| < \kappa$$

for all $2^{-s_0}N_0 \leq N \leq N_0$, where $s_0 = -C_3 \ln \kappa$, with $C_3 > 4$ and set $C > 0$ such that $C - C_3 > 4$, N is of the form $2^{-s}N_0$, and $j \leq N_1/q_0$ is such that $N_0/2^s q_0$ divides j for some $s \leq s_0$. In other words, we want to consider the set F defined as follows. Define

$$F_j^N = \left\{x \in \mathbb{T}^d : |L_N(x + jq_0\omega) - L_N(x)| < \kappa\right\} \tag{2.114}$$

and set

$$F = \bigcap_{n=0}^{s_0} \bigcap_{s=0}^{s_0} \bigcap_{m=1}^{N_1 2^s/N_0} F_{mN_0/2^s q_0}^{2^{-n}N_0} \tag{2.115}$$

We have

$$L_{N_0} - L_{N_1} = \int_F (L_{N_0}(x) - L_{N_1}(x))dx + \int_{\mathbb{T}^d \setminus F} (L_{N_0}(x) - L_{N_1}(x))dx.$$

Observe that Lemma 2.3.8 implies

$$\left|\mathbb{T}^d \setminus F_j^N\right| \leq C\kappa^{-3}\rho^{-3}|j| \, \|q_0\omega\| \, .$$

Thus, setting $\alpha(s) = N_1 2^s / N_0$, we have

$$
\begin{aligned}
|\mathbb{T}^d \backslash F| &\leq \sum_{n=0}^{s_0} \sum_{s=0}^{s_0} \sum_{m=1}^{\alpha(s)} C\kappa^{-3}\rho^{-3} \|q_0\omega\| \, mN_0/2^s q_0 && (2.116)\\
&\sim \sum_{n=0}^{s_0} \sum_{s=0}^{s_0} \kappa^{-3}\rho^{-3} \|q_0\omega\| \frac{N_1^2}{N_0 q_0} 2^s && (2.117)\\
&\sim \sum_{n=0}^{s_0} \kappa^{-3}\rho^{-3} \|q_0\omega\| \frac{N_1^2}{N_0 q_0} 2^{s_0} && (2.118)\\
&= \sum_{n=0}^{s_0} \kappa^{-3}\rho^{-3} \|q_0\omega\| \frac{N_1^2}{N_0 q_0} \kappa^{-C_3} && (2.119)\\
&\sim s_0 \kappa^{-3-C_3}\rho^{-3} \|q_0\omega\| \frac{N_1^2}{N_0 q_0} && (2.120)\\
&\leq \kappa^{-3-C_3}\rho^{-3} \|q_0\omega\| \frac{N_1^2}{N_0 q_0}. && (2.121)
\end{aligned}
$$

Thus, the set of x excluded from F has measure at most

$$
\kappa^{-3-C_3}\rho^{-3} \|q_0\omega\| \frac{N_1^2}{N_0 q_0} < \kappa,
$$

and we have

$$
\left| \int_{\mathbb{T}^d \backslash F} (L_{N_0}(x) - L_{N_1}(x)) dx \right| \leq C\kappa^{1/2}
$$

by Lemma 2.3.1.

It thus suffices to understand $\int_F (L_{N_0}(x) - L_{N_1}(x)) dx$.

Since the set F we consider here satisfies the conditions of the set from Lemma 2.6.1, we conclude that

$$
\left| \int_F (L_{N_0}(x) - L_{N_1}(x)) dx \right| < C\kappa.
$$

Our conclusion now follows, assuming

$$
N_0 = a 2^b q_0, \quad a, b \in \mathbb{N}, b > -C \ln \kappa. \tag{2.122}
$$

Now consider arbitrary N_0 such that (2.109) and (2.110) hold. Let $N_0' = 2^b q_0$ be of the form considered above such that $N_0' \leq \sqrt{N_0}$. Moreover, let $\alpha \in \mathbb{N}$ be such that $\alpha N_0' \leq N_0 \leq (\alpha+1)N_0'$. In particular, $N_0 = \alpha N_0' + O(N_0')$. We may now run the above argument using $\alpha N_0'$ as our initial scale. Let F be the same set considered before, with respect to the initial scale $\alpha N_0'$. Then $|\mathbb{T}^d \backslash F| < \kappa$, and thus

$$\left| \int_{\mathbb{T}^d \backslash F} L_{N_0}(x) - L_{N_1}(x) dx \right| < C\kappa$$

for all suitable $N_0 | N_1$. Furthermore, on F such that $L_{\alpha N_0'}(x) < 10^3 \kappa$ and

$$L_{O(N_0')}(A, x + \alpha N_0' \omega) < N_0 \kappa,$$

we also have $L_{N_0}(x) < C\kappa$ and $L_{N_1}(x) < C\kappa$. Since

$$\left| \left\{ x : L_{O(N_0')}(A, x + \alpha N_0' \omega) < \frac{N_0}{O(N_0')} \kappa \right\} \right| < C\kappa,$$

by Chebyschev's inequality and our choice of N_0', excluding this set only changes our final integral by a term of order κ.

On F such that $L_{N_0'}(x) > 10^3 \kappa$, we may define $N_{00}(x) < \kappa \alpha N_0' \leq \kappa N_0$ as before and note that there is $a \in \mathbb{N}$ such that $1/a < \kappa$ and

$$aN_{00} < N_0 < (a+1)N_{00}. \tag{2.123}$$

Moreover, we have, by (2.105),

$$|L_{\alpha N_0'}(x) - L_{(\alpha+1)N_0'}(x)| < C\kappa^{1/2}. \tag{2.124}$$

Next, we consider the set

$$G = \left\{ x \in \mathbb{T}^d : \left\| \tilde{A}_{O(N_0')}(x + \alpha N_0' \omega) \right\| < e^{N_0 \kappa} \right\}.$$

Clearly, since $N_0' < \sqrt{N_0}$, and $N_0 > \kappa^{-C}$, with $C > 4$, we may apply Chebyschev's inequality to obtain

$$|\mathbb{T}^d \backslash G| \leq C(A) \frac{N_0'}{N_0 \kappa} < C\kappa.$$

It follows that

$$\left| \int_{\mathbb{T}^d \backslash G} L_{N_0}(x) - L_{N_0'}(x) dx \right| < C\kappa.$$

On G, $L_{O(N_0')}(A, x + \alpha N_0' \omega) < \frac{N_0}{O(N_0')} \kappa$, so

$$L_{N_0}(x) \leq \frac{\alpha N_0'}{N_0} L_{\alpha N_0'}(x) + \frac{O(N_0')}{N_0} L_{O(N_0')}(x + \alpha N_0' \omega) \tag{2.125}$$

$$\leq L_{\alpha N_0'}(x) + \kappa. \tag{2.126}$$

We may similarly obtain

$$L_{(\alpha+1)N_0'}(x) \leq L_{N_0}(x) + \kappa \tag{2.127}$$

by considering a slightly different set G' with $|\mathbb{T}^d \backslash G'| < \kappa$ still. Thus

$$|L_{N_0}(x) - L_{\alpha N_0'}(x)| < C\kappa^{1/2}. \tag{2.128}$$

We can similarly obtain the bound

$$|L_{N_1}(x) - L_{\beta N_0'}(x)| < C\kappa^{1/2} \tag{2.129}$$

for $\beta = c\alpha$ such that $\beta N_0' < N_1 < (\beta+1)N_0'$ (away from a set of measure at most κ). Finally,

$$|L_{\beta N_0'}(x) - L_{\alpha N_0'}(x)| < C\kappa^{1/2}$$

by (2.107). Combining all of these with the triangle inequality, and Lemma 2.3.1 we obtain our desired result. $\square$

Our next step is to extend the conclusion of Lemma 2.6.2 to allow for larger scales N_1. This is most directly achieved by iterating Lemma 2.6.2: informally, we first apply the result with initial scale N_0 to obtain an estimate between scales N_0 and N_1; then apply the result with initial scale N_1 to obtain an estimate between scales N_1 and new scale N_2; repeating this iteration until we can no longer choose a suitable new initial scale. Unfortunately, as the result is stated now, this results in an error given by a multiple of κ, and that multiple could, potentially, be large enough to overcome the desired κ error. Therefore, it is necessary for us to rewrite the conclusion of Lemma 2.6.2 in a way which expresses the error in terms of N_0 and N_1. This is achieved in the following.

Lemma 2.6.3 (Liouville Frequencies)**.** *Let $N_0, q_0 \in \mathbb{N}$ and $c = C^{-1}$, with C as in Lemma 2.6.2. Suppose $N_0 > q_0$ and $\|q_0\omega\| < \rho^4 \frac{q_0}{N_0}$. Then for $N_0|N_1$ with $N_1 < \rho^2\sqrt{\frac{N_0 q_0}{\|q_0\omega\|}}$,*

$$|L_{N_1} - L_{N_0}| < C'\left(\left(\frac{q_0}{N_0}\right)^{c/2} + \left(\frac{N_1^2\,\|q_0\omega\|}{N_0 q_0 \rho^4}\right)^{c/2}\right). \tag{2.130}$$

Here $C' = C'(A)$ is a constant uniform in a neighborhood of the cocycle A.

Proof. Fix N_0, q_0, and ρ so that $N_0 > q_0$ and $\|q_0\omega\| < \rho^4 \frac{q_0}{N_0}$. Then let $1 > \kappa' > 0$ be the smallest possible κ such that Lemma 2.6.2 is applicable. That is, κ' is the smallest such number for which (2.109), (2.110), and (2.112) all hold. Then $|L_{N_0} - L_{N_1}| < C'\kappa'$. Moreover, equality must hold for one of (2.109), (2.110), or (2.112).

If equality holds in (2.109), then

$$\kappa' = \left(\frac{q_0}{N_0}\right)^c$$

and our conclusion follows.

If, on the other hand, equality does not hold in (2.109), then equality must hold in (2.112), since $N_1 > N_0$. Thus

$$\kappa' = \left(\frac{N_1^2 \|q_0\omega\|}{N_0 q_0 \rho^4}\right)^c$$

and our conclusion follows. $\square$

It is now possible to iterate Lemma 2.6.2 as we described to allow for larger scales N_1.

Theorem 2.6.1. *Suppose $N_0\kappa^C > q_0$ and $N_0 \|q_0\omega\| < \kappa^C \rho^4 q_0$. Then*

$$|L_{N'} - L_{N_0}| < C'\kappa^{1/6} \tag{2.131}$$

for all N' such that $N_0|N', N' = 2^j N_0$ for some $j \geq 0$, and

$$N' < \kappa^C \rho^4 \frac{q_0}{\|q_0\omega\|}. \tag{2.132}$$

Proof. Fix $N' < \kappa^C \rho^4 \frac{q_0}{\|q_0\omega\|}$. We will construct a sequence of scales, N_s, inductively. Starting with N_0 such that $q_0 < \kappa^C N_0$ and $\|q_0\omega\| < \kappa^C \rho^4 \frac{q_0}{N_0}$, we define

$$N_s \sim N_{s-1}^{2/3} \left(\frac{q_0 \rho^4}{\|q_0\omega\|}\right)^{1/3} \tag{2.133}$$

where $\sim$ here indicates that we take any value no larger than the right hand side such that $N_{s-1}|N_s$ and $N_{s-1} \neq N_s$. This last condition is possible because, by an inductive argument, $N_{s-1}^{2/3} \left(\frac{q_0\rho^4}{\|q_0\omega\|}\right)^{1/3} > N_{s-1}\kappa^{-C} \geq 2N_{s-1}$. Observe that this last condition ensures that N_s is an increasing sequence, and thus there is some $s_0 \geq 1$ such that

$$N_{s_0} < \kappa^C \rho^4 \frac{q_0}{\|q_0\omega\|}$$

and

$$N_{s_0+1} > \kappa^C \rho^4 \frac{q_0}{\|q_0\omega\|}.$$

Moreover, we have, for $0 \leq s \leq s_0$,

$$\begin{aligned}
N_s &\leq N_{s-1}^{2/3} \left(\frac{q_0\rho^4}{\|q_0\omega\|}\right)^{1/3} && (2.134)\\
&= N_{s-1}^{1/2} N_{s-1}^{1/6} \left(\frac{q_0\rho^4}{\|q_0\omega\|}\right)^{1/3} && (2.135)\\
&\leq N_{s-1}^{1/2} \kappa^{C/6} \left(\frac{q_0\rho^4}{\|q_0\omega\|}\right)^{1/6} \left(\frac{q_0\rho^4}{\|q_0\omega\|}\right)^{1/3} && (2.136)\\
&= \kappa^{C/6} \rho^2 \left(\frac{q_0 N_{s-1}}{\|q_0\omega\|}\right)^{1/2} && (2.137)\\
&\leq \rho^2 \left(\frac{q_0 N_{s-1}}{\|q_0\omega\|}\right)^{1/2}. && (2.138)
\end{aligned}$$

Thus Lemma 2.6.3 is applicable at scales N_s and N_{s-1} when $0 \leq s \leq s_0$. Applying Lemma 2.6.3, we obtain, for $s \leq s_0$,

$$\begin{aligned}
|L_{N_s} - L_{N_{s-1}}| &< C' \left(\left(\frac{q_0}{N_{s-1}}\right)^{c/2} + \left(\frac{N_s^2 \|q_0\omega\|}{N_{s-1} q_0 \rho^4}\right)^{c/2} \right) && (2.139)\\
&\leq C' \left(\left(\frac{q_0}{N_{s-1}}\right)^{c/2} + \left(\frac{N_{s-1} \|q_0\omega\|}{q_0 \rho^4}\right)^{c/6} \right). && (2.140)
\end{aligned}$$

Now we consider any $N' < \kappa^C \rho^4 \frac{q_0}{\|q_0\omega\|}$. There are two possibilities: (i) $N_s < N' \leq N_{s+1}, s < s_0$, or (ii) $N' > N_{s_0}$.

In the first case, we may simply redefine the appropriate $N_{s+1} = N'$. Then, we observe that

$2^{j-1}N_0 \leq N_{j-1} \leq 2^{-(s_0+1-j)}\kappa^C \rho^4 \frac{q_0}{\|q_0\omega\|}$, by construction. This yields

$$|L_{N_0} - L_{N'}| \leq \sum_{j=1}^{s} |L_{N_{j-1}} - L_{N_j}| \tag{2.141}$$

$$\leq \sum_{j=1}^{s} C' \left(\left(\frac{q_0}{N_{j-1}} \right)^{c/2} + \left(\frac{N_{j-1} \|q_0\omega\|}{q_0\rho^4} \right)^{c/6} \right) \tag{2.142}$$

$$\leq \sum_{j=1}^{s} C' \left(\left(\frac{q_0}{N_0} 2^{1-j} \right)^{c/2} + \left(2^{-(s_0+1-j)}\kappa^C \right)^{c/6} \right) \tag{2.143}$$

$$\leq C'' \left(\left(\frac{q_0}{N_0} \right)^{c/2} + \kappa^{1/6} \right) \tag{2.144}$$

$$\leq C''(\kappa^{1/2} + \kappa^{1/6}) \tag{2.145}$$

$$\leq C''\kappa^{1/6}. \tag{2.146}$$

For case (ii), we may repeat the same argument as above, plus the observation that

$$N' < N_{s_0+1} \tag{2.147}$$

$$\sim N_{s_0}^{2/3} \left(\frac{q_0\rho^4}{\|q_0\omega\|} \right)^{1/3} \tag{2.148}$$

$$< \kappa^{C/6} N_{s_0}^{1/2} (q_0\rho^4 / \|q_0\omega\|)^{1/2}. \tag{2.149}$$

Now, we may apply lemma 2.6.2 with $N_0 = N_{s_0}$ and $N_1 = N'$ and κ replaced by $\kappa^{1/3}$ to obtain the estimate

$$|L_{N_{s_0}} - L_{N'}| < C'\kappa^{1/6}.$$

This completes our proof. □

Note that this immediately implies continuity of $L(A, \omega)$ in A whenever the components of ω, are rational.

2.7 Comparing Lyapunov exponents at different scales: mixed Liouville-Diophantine frequencies

In this section, we turn out attention to obtaining estimates of the form $|L_{N_0}(A,\omega) - L_{N_1}(A,\omega)| < C\kappa$ for those ω which are not necessarily Liouville.

Theorem 2.7.1. *Assume $x = (x_1, x_2) \in \mathbb{T}^{d_1} \times \mathbb{T}^{d_2} = \mathbb{T}^d$. Suppose that $\omega = (\omega_1, \omega_2) \in \mathbb{T}^{d_1} \times \mathbb{T}^{d_2}$ is such that there is $\delta > 0, 0 < K_0 \in \mathbb{Z}$ and $0 < q_0 \in \mathbb{Z}$ such that*

$$\|q_0\omega_1\| < \kappa^C \rho^3 q_0 / N_0 \tag{2.150}$$

and

$$\|k \cdot \omega_2\| > \delta \tag{2.151}$$

for all $k \in \mathbb{Z}^{d_2}, 0 < |k| < K_0$, where

$$K_0 > (\rho^{1+c}\kappa)^{-C} q_0 \tag{2.152}$$

$$N_0 > \kappa^{-C}\delta^{-1}K_0. \tag{2.153}$$

Then

$$|L_N - L_{N_0}| < C'\kappa^{1/6} \tag{2.154}$$

when $N_0 | N, N = 2^j N_0$ for some $j \geq 0$, and

$$N < \min\left\{\kappa^C \rho^3 \frac{q_0}{\|q_0\omega_1\|}, N_0 e^{\left(\frac{K_0}{q_0}\right)^c}\right\}. \tag{2.155}$$

Here C is an absolute constant defined in the proof and $c = C^{-1}$ above.

We would like to make a note about our general approach to the proof, before we present the full details. As we remarked after Lemma 2.6.2, the argument we used to establish Lemma 2.6.2 applies once we restrict our attention to a suitable set F. Here, we will restrict to a suitable set F defined analogously, but we cannot just appeal to Lemma 2.3.8 to show $\mathbb{T}^d \backslash F$ has small measure, since Lemma 2.3.8 requires a small change in x, and we assume a Diophantine condition (corresponding to a large change in x) for part of ω. Therefore, the main difficulty we face here is showing $\mathbb{T}^d \backslash F$ has small measure. The idea we present here is to use Lemma 2.3.8 in the variables where we have suitable smallness of the frequency (namely x_1 with ω_1) and use Theorem 2.3.2 in the variables where we have a suitable Diophantine condition. Together, this will yield smallness of $\mathbb{T}^d \backslash F$.

Proof. As in Lemma 2.6.2, we will consider those N_0 of the form $a2^b q_0$, for suitably large b. The case of general N_0 will then follow in the same way as before.

Consider the set F consisting of points $x = (x_1, x_2) \in \mathbb{T}^{d_1} \times \mathbb{T}^{d_2} = \mathbb{T}^d$ such that

$$|L_N(x) - L_N(x + jq_0\omega)| < \kappa$$

for all $2^{-s_0} N_0 \leq N \leq N_0$, where $s_0 = -C_1 \ln \kappa$, $C_1 > 4$, with $C > 0$ such that $C > 6(3C_1 + 3) + 1$, N is of the form $2^{-s} N_0$, and $j \leq N_1/q_0$ is such that $N_0/2^s q_0$ divides j for some $s \leq s_0$. We are in precisely the setting of Lemma 2.6.1, so we conclude that

$$|\int_F L_{N_1}(x) - L_{N_0}(x) dx| < C'' \kappa. \tag{2.156}$$

It now suffices to understand $|\mathbb{T}^d \backslash F|$.

We will begin by restricting our attention to the set

$$H_N = \left\{ x \in \mathbb{T}^d : |L_N(x) - L_N(x + jq_0\omega)| < C(A)|j| q_0 N^{-1/2}, 1 \leq j \leq \kappa N/q_0 \right\},$$

which, by Lemma 2.3.2, has small complement

$$|\mathbb{T}^d \backslash H_N| < \kappa e^{-N^{1/3}}.$$

As in Lemma 2.6.1, we observe that Lemma 2.3.8 applies to $L_N(x_1, x_2)$ in the first variable, which yields

$$\begin{aligned} L_N(x + jq_0\omega) &= L_N(x_1 + jq_0\omega_1, x_2 + jq_0\omega_2) \\ &= L_N(x_1, x_2 + jq_0\omega_2) + O(\kappa), \end{aligned}$$

away from a set $G_{N,j} \subset \mathbb{T}^d$ of small measure:

$$|G_{N,j}| < C'\kappa^{-3}\rho^{-3}|j| \, \|q_0\omega_1\| \, .$$

Hence

$$L_N(x + jq_0\omega) = L_N(x_1, x_2 + jq_0\omega_2) + O(\kappa) + g_{N,j}(x), \tag{2.157}$$

where $g_{N,j}(x)$ is a function such that $\|g_{N,j}\|_{L^1(\mathbb{T}^d)} \leq C'\kappa^{-3}\rho^{-3}|j| \, \|q_0\omega_1\|$. In particular, $g_{N,j}(x)$ is the restriction of $L_N(x)$ to the set $G_{N,j}$. Moreover, for $x \in H_N \cap \bigcup_{j=0}^{R-1}(\mathbb{T}^d \cap G_{N,j})$ and $R < \kappa\frac{\sqrt{N}}{q_0}$,

$$L_N(x) = \frac{1}{R}\sum_{j=0}^{R-1} L_N(x + jq_0\omega) + C'Rq_0/\sqrt{N} \tag{2.158}$$

$$= \frac{1}{R}\sum_{j=0}^{R-1} L_N(x_1, x_2 + jq_0\omega_2) + O(\kappa) \tag{2.159}$$

and thus

$$L_N(x) = \frac{1}{R}\sum_{j=0}^{R-1} L_N(x_1, x_2 + jq_0\omega_2) + O(\kappa) + g_N(x), \tag{2.160}$$

where $g_N(x) = \frac{1}{R}\sum_{j=0}^{R-1} g_{N,j}(x)$ satisfies $\|g_N\|_{L^1(\mathbb{T}^d)} \leq C'\kappa^{-3}\rho^{-3}R\,\|q_0\omega_1\|$. Note that the $O(\kappa)$ term is no larger than 2κ.

Now, by defining $\omega_2' = q_0\omega_2$, we have $\|k \cdot \omega_2'\| > \delta$ for all $0 < |k| < K_0/q_0$. Thus we may apply Lemma 2.3.7 to $L_N(x_1, x_2)$ in the variable x_2 to obtain

$$\left|\left\{(x_1, x_2) \in \mathbb{T}^d : \left|\frac{1}{R}\sum_{j=0}^{R-1} L_N(x_1, x_2 + jq_0\omega_2) - \int L_N(x_1, x_2)dx_2\right| > \rho^{-1}(K_0/q_0)^c\right\}\right|$$
$$< e^{-\rho^{1+c}(K_0/q_0)^c}.$$

Denote by Γ_N the set on the left hand side. We have, $\rho^{-1}(K_0/q_0)^{-c} < \kappa$, so for $(x_1, x_2) \notin \Gamma_N$,

$$\frac{1}{R}\sum_{j=0}^{R-1} L_N(x_1, x_2 + jq_0\omega_2) = \int L_n(x_1, x_2)dx_2 + O(\kappa).$$

On the other hand, the integral of $\frac{1}{R}\sum_{j=0}^{R-1} L_N(x_1, x_2 + jq_0\omega_2)$ over Γ_N obeys

$$\int_{\Gamma_N}\left|\frac{1}{R}\sum_{j=0}^{R-1} L_N(x_1, x_2 + jq_0\omega_2)\right| dx_2 \leq C'e^{-\frac{1}{2}\rho^{1+c}(K_0/q_0)^c},$$

by Lemma 2.3.1. Hence

$$\frac{1}{R}\sum_{j=0}^{R-1} L_N(x_1, x_2 + jq_0\omega_2) = \int L_n(x_1, x_2)dx_2 + O(\kappa) + \gamma_N(x), \tag{2.161}$$

where $\gamma_N(x)$ is the restriction of $L_N(x)$ to Γ_N and satisfies $\|\gamma_N\|_{L^1(\mathbb{T}^d)} < e^{-\frac{1}{2}\rho^{1+c}(K_0/q_0)^c}$.

Combining this with (2.159), we have

$$L_N(x) = \int L_N(x_1, x_2)dx_2 + O(\kappa) + g_N(x) + \gamma_N(x). \tag{2.162}$$

We may apply Lemma 2.3.8 again in x_1 to $\int L_N(x_1, x_2)dx_2$ to obtain

$$\left| \int L_N(x_1, x_2)dx_2 - \int L_N(x_1 + jq_0\omega_1, x_2 + jq_0\omega_2)dx_2 \right| = O(\kappa) + h_{N,j}(x), \tag{2.163}$$

where $h_{N,j}(x)$ is a suitable restriction of $L_N(x)$ and satisfies $\|h_{N,j}(x)\|_{L^1} < \kappa^{-3}\rho^{-3}|j| \, \|q_0\omega_1\|$.

Altogether, we have

$$|L_N(x) - L_N(x + jq_0\omega)| = O(\kappa) + g_N(x) + \gamma_N(x) + h_{N,j}(x). \tag{2.164}$$

It now follows that

$$|\mathbb{T}^d \backslash F| < \kappa^{-C'}\rho^{-3}\frac{N_1^2 \|q_0\omega_1\|}{q_0 N_0} + \kappa^{-C'}\frac{N_1}{N_0}e^{-\rho^{1+c}\left(\frac{K_0}{q_0}\right)^c}.$$

Here c is a sufficiently small constant which is anything smaller than both the constant from Theorem 2.3.2 and $1/C$, and C' is such that $C/3 > C'$. In particular, this computation follows by observing that $\mathbb{T}^d \backslash F$ is contained in the union of the supports of g_N, γ_N, and $h_{N,j}$ over all relevant N and j.

It follows that

$$|L_{N_1} - L_{N_0}| < C''\kappa + \kappa^{-C'}\rho^{-3}\frac{N_1^2 \|q_0\omega_1\|}{q_0 N_0} + \kappa^{-C'}\frac{N_1}{N_0}e^{-\rho^{1+c}\left(\frac{K_0}{q_0}\right)^c}. \tag{2.165}$$

Now, if N_1 is such that

$$\rho^{-3}\frac{N_1^2\,\|q_0\omega_1\|}{q_0 N_0}+\frac{N_1}{N_0}e^{-\rho^{1+c}\left(\frac{K_0}{q_0}\right)^c}<\kappa^{1+C'},$$

then we have our desired bound. Moreover, if

$$\frac{N_1}{N_0}\le\min\left\{\left(\frac{\rho^3 q_0}{N_0\,\|q_0\omega_1\|}\right)^{1/3},e^{\frac{1}{2}\rho^{1+c}(K_0/q_0)^c}\right\},$$

then by direct computation (using (2.150)) such a bound is satisfied.

We can now use the exact same argument as was used in Theorem 2.6.1 to extend our allowable length scales N_1 to the desired range, since the scale

$$N_0\left(\frac{\rho^3 q_0}{N_0\,\|q_0\omega_1\|}\right)^{1/3}$$

is precisely the intermediate scale we used to extend Lemma 2.6.2 to Theorem 2.6.1. $\square$

2.8 Continuity of Lyapunov exponents

The main technical lemma which we will establish at the end of this section is the following.

Lemma 2.8.1. *Assume* $x=(x_1,x_2)\in\mathbb{T}^{d_1}\times\mathbb{T}^{d_2},\omega=(\omega_1,\omega_2)\in\mathbb{T}^{d_1}\times\mathbb{T}^{d_2}$ *such that*

$$\|q_0\omega_1\|=0$$

and

$$\|k\cdot\omega_2\|>\delta\quad\textit{for } k\in\mathbb{Z}^{d_2},0<|k|\le K,$$

where

$$q_0<K^{1/10}.$$

Then

$$|L_N - L| < \kappa$$

for all $N > K^2/\delta$.

This lemma will be sufficient to obtain continuity due to the following observation. Every $\omega = (\omega_1, ..., \omega_d) \in \mathbb{T}^d$ falls into one of three categories: (i) $k \cdot \omega \neq 0$ for every $k \in \mathbb{Z}^d \backslash \{0\}$, (ii) $\omega_j \in \mathbb{Q}$ for some j, or (iii) ω_j are all irrational but rationally dependent. The above lemma clearly applies to the ω in the first two of these possibilities. The last possibility can be "transformed" into the second possibility after an application of an appropriate toral automorphism (i.e. a suitable change of variables), $B \in SL(d, \mathbb{Z})$. Indeed, for any $B \in SL(d, \mathbb{Z})$, we have:

$$L_N(A, \omega) = \frac{1}{N} \int \ln \left\| \prod_{j=N-1}^{0} A(x + j\omega) \right\| dx \tag{2.166}$$

$$= \frac{1}{N} \int \ln \left\| \prod_{j=N-1}^{0} (A \circ B)(x + jB^{-1}\omega) \right\| dx \tag{2.167}$$

$$= L_N(A \circ B, B^{-1}\omega). \tag{2.168}$$

Since $(A \circ B, B^{-1}\omega)$ is still an analytic quasiperiodic cocycle, all of the results from the previous sections apply. The key idea is to now find an appropriate such B so that B^{-1} rearranges the components of ω into two pieces: one piece consisting of irrational and rationally independent components, and another piece consisting of rationals and rationally dependent components.

The main step towards achieving this is the following theorem.

Theorem 2.8.1. *Let* $(\omega_1, \omega_2) \in \mathbb{T}^{d_1} \times \mathbb{T}^{d_2}, d_1 + d_2 = d$. *Let* $1 \geq \delta_0 > 0, \epsilon_0 \geq 0$, *and suppose*

$q_0 \in \mathbb{N}$ and $K_0 \in \mathbb{N}$, with $q_0 < K_0^{1/10}$, satisfies

$$\|q_0\omega_1\| \leq \epsilon_0, \tag{2.169}$$

$$\|k\omega_2\| \geq \delta_0; \quad 0 < |k| \leq K_0. \tag{2.170}$$

Furthermore, suppose N_0 is such that

$$\frac{1}{2}K_0^2\delta_0^{-1} \leq N_0 < \epsilon_0^{-1}K_0^{-1}. \tag{2.171}$$

Finally, suppose $\rho > K_0^{-c}$. Then for $N_0|N_1, N_1 = 2^j N_0$ for some $j \geq 0$, and $N_1 \leq \epsilon^{-1}K_0^{-1}$, we have

$$|L_{N_0} - L_{N_1}| < K_0^{-c}. \tag{2.172}$$

We will prove this by induction, but before we present the proof, we will provide a lemma which reduces the above result to proving analogous bounds on shorter length scales.

Lemma 2.8.2. *In addition to the assumptions of Theorem 2.8.1, suppose, moreover, that*

$$|L_{N_0} - L_{N'}| < K_0^c \tag{2.173}$$

for $N_0|N', N' = 2^j N_0$ for any $j \geq 0$ such that

$$N' \leq \min\left\{\epsilon_0^{-1}K_0^{-1}, K_0^{40}\left(\min_{0<|k|\leq K_0^{20}} \|k \cdot \omega_2\|\right)^{-1} + N_0\right\}. \tag{2.174}$$

Then (2.172) holds for $N_0|N_1, N_1 = 2^j N_0$ for any $j \geq 0$ such that $N_1 \leq \epsilon^{-1}K_0^{-1}$.

Proof. Suppose (2.172) holds for N' as above. We will construct a sequence of length scales, $N_{0,s}$, and iterate the conclusion in a way that mirrors our proof of Theorem 2.6.1.

Note that, if

$$K_0^{40}\left(\min_{0\leq|k|\leq K_0^{20}}\|k\cdot\omega_2\|\right)^{-1}+N_0>\epsilon_0^{-1}K_0^{-1},$$

then there is nothing to prove, so we will assume

$$K_0^{40}\left(\min_{0\leq|k|\leq K_0^{20}}\|k\cdot\omega\|\right)^{-1}+N_0\leq\epsilon_0^{-1}K_0^{-1}$$

Set $K_1=K_0^{20},\delta_1=\min_{0<|k|\leq K_1}\|k\cdot\omega_2\|$. Then define

$$N_{0,1}\sim\frac{K_1^2}{\delta_1}+N_0, \tag{2.175}$$

where $\sim$ here means take the largest multiple of N_0 no larger than the right hand side of the form $N_{0,1}=2^jN_0$. By our assumptions, $N_{0,1}$ satisfies (2.174), and thus

$$|L_{N_0}-L_{N_{0,1}}|<K_0^{-c}.$$

Moreover, suppose $N_{0,1}<\epsilon_0^{-1}K_1^{-1}$. We will deal with the case where $N_{0,1}\geq\epsilon_0^{-1}K_1^{-1}$ at the end of the proof using Theorem 2.7.1.

Now we observe that we may replace δ_0,K_0, and N_0 in the hypothesis of our lemma with δ_1,K_1, and $N_{0,1}$, respectively. It then follows, by our assumptions, that

$$|L_{N'}-L_{N_{0,1}}|<K_1^{-c}$$

for all N' such that $N_{0,1}|N'$ and

$$N'\leq\min\left\{\epsilon_0^{-1}K_1^{-1},K_1^{40}\left(\min_{0<|k|\leq K_1^{20}}\|k\cdot\omega_2\|\right)^{-1}+N_{0,1}\right\}.$$

At this point, two possibilities arise. First, suppose

$$K_1^{40}\left(\min_{0<|k|\le K_1^{20}}\|k\cdot\omega_2\|\right)^{-1}+N_{0,1}\le\epsilon_0^{-1}K_1^{-20}. \tag{2.176}$$

We will see later that the case where (2.176) fails may be dealt with via Theorem 2.7.1. In fact, this assumption is analogous to the assumption $N_{0,1}<\epsilon_0^{-1}K_1^{-1}$ we made above. Set $K_2=K_1^{20}, \delta_2=\min_{0<|k|\le K_2}\|k\cdot\omega_2\|$. Then define

$$N_{0,2}\sim\frac{K_2^2}{\delta_2}+N_{0,1}, \tag{2.177}$$

where $\sim$ here means take the largest multiple of $N_{0,1}$ of the form $N_{0,2}=2^jN_{0,1}$ which is no larger than the right hand side. By our assumptions, $N_{0,2}$ satisfies (2.174), and thus

$$|L_{N_{0,1}}-L_{N_{0,2}}|<K_1^{-c}$$

and

$$|L_{N'}-L_{N_{0,2}}|<K_2^{-c}$$

for all N' such that $N_{0,2}|N'$ and

$$N'\le\min\left\{\epsilon_0^{-1}K_2^{-1},K_2^{40}\left(\min_{0<|k|\le K_2^{20}}\|k\cdot\omega_2\|\right)^{-1}+N_{0,2}\right\}.$$

We continue in this way to define $K_s=K_{s-1}^{20}, \delta_s=\min_{0<|k|\le K_s}\|k\cdot\omega_2\|$, and $N_{0,s}\sim\frac{K_s^2}{\delta_s}+N_{0,s-1}$ for $s\le s_0$, where s_0 is the first index for which

$$K_{s_0}^{40}\left(\min_{0<|k|\le K_{s_0}^{20}}\|k\cdot\omega_2\|\right)^{-1}+N_{0,s_0}>\epsilon_0^{-1}K_{s_0}^{-1}.$$

By construction, for all $s \le s_0$,

$$|L_{N_{0,s}} - L_{N_{0,s-1}}| < K_{s-1}^{-c} \tag{2.178}$$

and for $s < s_0$,

$$|L_{N_{0,s}} - L_{N'}| < K_{s-1}^{-c}, \tag{2.179}$$

for all $N' = 2^j N_{0,s}$, where

$$N' \le \min\left\{\epsilon_0^{-1} K_s^{-1}, K_{s+1}^2 \delta_{s+1} + N_{0,s}\right\}.$$

Now set $N'_{0,s_0} \sim \max\left\{N_{0,s_0-1}, \epsilon_0^{-1} K_{s_0}^{-1}\right\}$, where, again, $\sim$ denotes taking the largest value no larger than the right hand side such that $N_{0,s_0-1} | N'_{0,s_0}$. Thus, by construction,

$$|L_{N_{0,s_0-1}} - L'_{N_{0,s_0}}| < K_{s_0-1}^{-c}.$$

At this point, we may apply Theorem 2.7.1 to the scale N'_{0,s_0}, with K_{s_0-1} and δ_{s_0-1}. Indeed,

$$\begin{aligned}
N'_{0,s_0}\epsilon_0 &\le \max\left\{N_{0,s_0-1}\epsilon_0, K_{s_0}^{-1}\right\} && (2.180)\\
&\le \max\left\{K_{s_0-1}^{-1}, K_{s_0}^{-1}\right\} && (2.181)\\
&\le K_{s_0-1}^{-1} && (2.182)\\
&\le K_0^{-1} && (2.183)\\
&\le \kappa^C \rho^4 q_0, && (2.184)
\end{aligned}$$

where $\kappa = K_0^{-c}$, and C is chosen appropriately large, depending on c we used in the hypothesis of this theorem. Thus (2.150) holds. We also have

$$K_0 > \rho^{-1/C}\kappa^{-1/C}q_0, \tag{2.185}$$

so (2.152) holds. Finally,

$$\begin{aligned} N'_{0,s_0} &\geq N_{0,s_0-1} && (2.186)\\ &\geq K^2_{s_0-1}\delta^{-1}_{s_0-1} && (2.187)\\ &\geq K_{s_0-1}\delta^{-1}_{s_0-1}\kappa^{-C}, && (2.188) \end{aligned}$$

and thus (2.153) holds. It follows that Theorem 2.7.1 is applicable with our chosen parameters, and we obtain

$$|L_{N_{s_0}} - L_{N'}| < K_0^{-c} \tag{2.189}$$

for all N' of the form $2^j N_{0,s_0}$ such that

$$N' \leq \min\left\{\epsilon_0^{-1}K_0^{-1}, \epsilon_0^{-1}\right\}.$$

This now includes all $N' \leq \epsilon_0^{-1}K_0^{-1}$, as desired.

Now we fix any $N' = 2^j N_0$. Two possibilities arise. First, suppose $N_{0,s} < N_1 \leq N_{0,s+1}$ for some $0 \leq s \leq s_0$. By construction of $N_{0,s+1}$, we have

$$|L_{N_{0,s}} - L_{N'}| < K_s^{-c},$$

and thus

$$|L_{N_0} - L_{N'}| < |L_{N_0} - L_{N_{0,1}}| + \sum_{j=1}^{s-1} |L_{N_{0,j}} - L_{N_{0,j+1}}| \tag{2.190}$$

$$+ |L_{N_{0,s}} - L_{N'}| \tag{2.191}$$

$$\leq K_0^{-c} + \sum_{j=1}^{s-1} K_j^{-c} + K_s^{-c} \tag{2.192}$$

$$= K_0^{-c} + \sum_{j=1}^{s-1} K_0^{-c20^j} + K_0^{-c20^s} \tag{2.193}$$

$$\leq CK_0^{-c}, \tag{2.194}$$

where C is an absolute constant.

Now, suppose $N' > N_{0,s_0}$. As we observed above,

$$|L_{N_{s_0}} - L_{N'}| < K_0^{-c},$$

and the same argument as the previous case yields our desired result.

$\square$

At this point, we provide a lemma guaranteeing the existence of the special toral automorphisms we discussed at the beginning of this section.

Lemma 2.8.3 ([6] Lemma 4.41)**.** *Assume $k = (k_1, ..., k_d) \in \mathbb{Z}^d$ and $\gcd(k_1, ..., k_d) = 1$. Then there is $A \in SL_d(\mathbb{Z})$ satisfying*

$$A_{1j} = k_j; \quad 1 \leq j \leq d, \tag{2.195}$$

$$|A_{ij}| \leq |k| = \max |k_l|; \quad 1 \leq i, j \leq d. \tag{2.196}$$

We will now use this in an induction scheme to prove Theorem 2.8.1.

proof of Theorem 2.8.1. By Lemma 2.8.2, it suffices to prove our result for N_1 which satisfies (2.174). We will prove this by induction on d_2.

First, we consider the base case, $d_2 = 0$. Set $\kappa = K_0^{-c}$, with c a suitably small absolute constant. Set $C = c^{-1}$. Then

$$N_0 \kappa^C = N_0 K_0^{-1} > K_0 > q_0^{10} > q_0.$$

Moreover,

$$N_0 \|q_0 \omega_1\| \leq N_0 \epsilon_0 < K_0^{-1} < \kappa^C < \kappa^C \rho^4 q_0.$$

Thus Theorem 2.6.1 is applicable, and we obtain

$$|L_{N_0} - L_{N_1}| < K_0^{-c/3}$$

for

$$N_1 < K_0^{-1/2} \rho^4 q_0 / \|q_0 \omega_1\|.$$

Since $\epsilon^{-1} K_0^{-1} < K_0^{-1/2} \rho^4 q_0 / \|q_0 \omega_1\|$, our conclusion follows.

Now we turn our attention to the inductive step. Assume (2.172) holds for $\omega_2 \in \mathbb{T}^{d_2 - 1}$ and N_1 satisfying (2.174). We will show that (2.172) is true for $\omega_2 \in \mathbb{T}^{d_2}$.

Fix any suitable $\omega_2 \in \mathbb{T}^{d_2}$. Set $K_1 = K_0^{20}$ and $\delta_1 = \min_{0<|k|\leq K_1} \|k \cdot \omega_2\|$. Note that, if

$N_0 \geq K_1^2\delta_1^{-1}$, then any N_1 satisfying (2.174) also satisfies $N_1 < 2N_0$ and our result is vacuously true. Thus, we will assume $N_0 < K_1^2\delta_1^{-1}$.

Now, by the definition of δ_1, there is some $q' \in \mathbb{Z}^{d_2}$, with $0 < |q'| \leq K_1$, such that the minimum is achieved. That is, we can find $q' = (q'_1, \dots, q'_{d_2}) \in \mathbb{Z}^{d_2}$, with $0 < |q'| \leq K_1$, such that $\|q'\omega_2\| = \delta_1$. Write $q' = q_1 \cdot n_1$, where $q_1 \in \mathbb{Z}\setminus\{0\}, |q_1| \leq K_1$, and $n_1 = (n_{11}, ..., n_{1d_2})$ is such that $\gcd(n_{11}, ..., n_{1d_2}) = 1$.

At this point, our goal is to perform a suitable change of variables to reduce our situation to that covered in our induction hypothesis. We apply Lemma 2.8.3 with $k = n_1$ to construct $B \in SL_d(\mathbb{Z})$ with entries bounded by K_1 such that

$$B(\omega_1, \omega_2) = \omega' = (\omega'_1, \omega'_2) \in \mathbb{T}^{d_1+1} \times \mathbb{T}^{d_2-1}, \tag{2.197}$$

where

$$\|q'_1\omega'_1\| = \delta_1. \tag{2.198}$$

We may also assume that B fixes the first d_1 components of ω. Moreover, by Cramer's rule, the entries of B^{-1} are all bounded by K_1^{d-1}.

Now, since $B \in SL_d(\mathbb{Z})$, we have, for every N, ω,

$$\begin{aligned}
L_N(A,\omega) &= \frac{1}{N}\int_{\mathbb{T}^d} \ln\left\|\prod_{j=N}^{1} A(x+j\omega)\right\| dx && (2.199)\\
&= \frac{1}{N}\int_{\mathbb{T}^d} \ln\left\|\prod_{j=N}^{1} AB^{-1}(Bx+jB\omega)\right\| dx && (2.200)\\
&= \frac{1}{N}\int_{\mathbb{T}^d} \ln\left\|\prod_{j=N}^{1} AB^{-1}(x+jB\omega)\right\| dx && (2.201)\\
&= L_N(AB^{-1}, B\omega). && (2.202)
\end{aligned}$$

Here, the second to last equality is simply a change of variables, $Bx \mapsto u$, and we take advantage of the fact that $B \in SL_d(\mathbb{Z})$. Thus, to understand the Lyapunov exponent for the cocycle (A, ω), we may study the related cocycle $(AB^{-1}, B\omega) = (AB^{-1}, \omega')$. We now want to show that this new cocycle satisfies the induction hypothesis.

Recall that $L_N(A, \omega)$ has a plurisubharmonic extension to $|\Im z_j| < \rho = \rho_0$ which satisfies the conditions necessary to establish the results in the previous sections. It follows that $L_N(AB^{-1}, \omega')$ also has a plurisubharmonci extension to $|\Im z_j| < \|B^{-1}\| \rho_0$ for which the results of the previous section apply. Since $\|B^{-1}\| < K_1^{d-1}$, the extension can certainly be restricted to

$$|\Im z_j| < \rho_1 = K_1^{1-d} \rho_0.$$

Moreover, we know $\rho_0 > K_0^{-c}$, so

$$\rho_1 > K_0^{-c} K_1^{1-d} > K_1^{-d}. \tag{2.203}$$

Next, observe that

$$\|q_0 q_1 \omega_1'\| = \|q_0 q_1 (\omega_1, \tilde{\omega})\| \tag{2.204}$$

$$= \|q_0 q_1 \omega_1\| + \|q_0 q_1 \tilde{\omega}\| \tag{2.205}$$

$$< q_1 \epsilon_0 + q_0 \delta_1 \tag{2.206}$$

$$=: \epsilon_1. \tag{2.207}$$

Moreover,

$$\epsilon_1 \leq K_1(\epsilon_0 + \delta_1). \tag{2.208}$$

Now, define $K_2 = K_1^{C_1}$, where $C_1 = d/c$ and observe that (2.203) implies

$$\rho_1 > K_2^{-c}.$$

Finally, define

$$\delta_2 = \min_{0<|k|\leq K_2} \|k \cdot \omega_2'\|. \tag{2.209}$$

At this point there are two possible scenarios. Either

$$N_0 \geq K_2^2 \delta_2^{-1} \tag{2.210}$$

or

$$N_0 < K_2^2 \delta_2^{-1}. \tag{2.211}$$

We will consider (2.210) first.

Our strategy here is to appeal to Theorem 2.7.1. Suppose (2.210) holds. If $N_0 \leq \epsilon_1^{-1} K_2^{-1}$, then we may appeal to our induction hypothesis applied to ω' with K_0 and δ_0 replaced by K_2 and δ_2, respectively, to obtain

$$|L_{N_0} - L_{N_{0,2}}| < K_2^{-c} < K_0^{-c}, \tag{2.212}$$

for all $N_{0,2}$ such that $N_0 | N_{0,2}$ and $N_{0,2} \leq \epsilon_1^{-1} K_2^{-1}$. Now set $N_{0,2}$ as close as possible to $\epsilon_1^{-1} K_2^{-1}$. If $N_0 > \epsilon_1^{-1} K_2^{-1}$, set $N_{0,2} = N_0$.

Now that we have the new length scale $N_{0,2}$, we want to apply Theorem 2.7.1 to ω' with N_0 replaced by $N_{0,2}, \delta = \delta_0, K = K_0$, and $\kappa = K_0^{-c}$. It remains to verify the hypothesis of

Theorem 2.7.1.

First, if (2.150) fails, then

$$\epsilon_0 N_{0,2} > \kappa^C \rho^3 > K_0^{-1},$$

and thus $N_{0,2} > \epsilon_0^{-1} K_0^{-1}$ and we have reached our desired scale, in which case (2.212) is our desired conclusion. Thus, we may assume that (2.150) holds.

Next, recall that $N_{0,2} \geq N_0$, so

$$\begin{aligned}
\kappa^{-C}\delta_0^{-1}K_0 &= K_0^{1+cC}\delta_0^{-1} && (2.213)\\
&\leq K_0^2\delta_0^{-1} && (2.214)\\
&\leq N_0 && (2.215)\\
&\leq N_{0,2} && (2.216)
\end{aligned}$$

as long as $c \leq 1/C$. We also have

$$\begin{aligned}
(\rho^{1+c}\kappa)^{-C}q_0 &< (K_0^{-c-c^2}K_0^{-c})^{-C}q_0 && (2.217)\\
&= K_0^{(2+c)cC}q_0 && (2.218)\\
&< K_0^{(2+c)cC+1/10} && (2.219)\\
&< K_0 && (2.220)
\end{aligned}$$

for c sufficiently small (say $c < 1/3C$). Using such a c throughout only changes the exponent of κ in the conclusions in the previous sections by an amount proportional to the change we make to c here. It follows that (2.152) and (2.153) hold. Theorem 2.7.1 is thus applicable and we obtain

$$|L_{N_1} - L_{N_{0,2}}| < K_0^{-c}, \qquad (2.221)$$

and thus

$$|L_{N_0} - L_{N_1}| < 2K_0^{-c}, \tag{2.222}$$

for all $N_{0,2}|N_1$ such that

$$N_1 < \min\left\{\kappa^C \frac{\rho^3 q_0}{\|q_0\omega_1\|}, N_{0,2}e^{(K_0/q_0)^c}\right\}.$$

Note that

$$\kappa^C \frac{\rho^3 q_0}{\|q_0\omega_1\|} > K_0^{-1}\epsilon_0^{-1}$$

and, using (2.208),

$$\begin{aligned}
N_{0,2}e^{(K_0/q_0)^c} &> N_{0,2}e^{K_0^{c9/10}} && (2.223)\\
&\gtrsim K_1^{-1}\epsilon_1^{-1}e^{K_0^{c9/10}} && (2.224)\\
&= K_0^{-20C_1}\frac{1}{q_1\epsilon_0 + q_0\delta_1}e^{K_0^{c9/10}} && (2.225)\\
&> 2K_0^{40}(\delta_1 + \epsilon_0)^{-1} && (2.226)\\
&= 2K_1^{2}(\delta_1 + \epsilon_0)^{-1} && (2.227)
\end{aligned}$$

If $\delta_1 < \epsilon_0$, then the right hand side is no less than $K_1^2\epsilon_0^{-1} > \epsilon_0^{-1}K_0^{-1}$, and we have reached our desired scale. On the other hand, if $\delta_1 > \epsilon_0$, then the right hand side is no less that $K_1^2\delta_1^{-1}$. If

$$K_1^2\delta_1^{-1} > \epsilon_0^{-1}K_0^{-1},$$

then we have reached our desired scale length, and we are done. On the other hand, if

$$K_1^2\delta_1^{-1} < \epsilon_0^{-1}K_0^{-1},$$

then we may repeat our entire argument above with N_0 replaced by $N_1 \sim K_1^2\delta_1^{-1}$. This puts us in the situation where our conclusion is vacuously true, as described at the start of this proof. Thus, either situation leads to our desired conclusion.

It now remains to consider the case (2.211). We may perform another change of variables by applying another suitable matrix, $B_1 \in SL_d(\mathbb{Z})$, with entries bounded by K_2 so that

$$B_1\omega' = (\omega_1'', \omega_2'') \in \mathbb{T}^{d_1+2} \times \mathbb{T}^{d_2-2} \tag{2.228}$$

and

$$\|q_2\omega_1''\| = \delta_2 \tag{2.229}$$

for some $q_2 \in \mathbb{N}, q_2 \leq K_2$. This, as with the first change of variables, decreases the width of the strip for which we have a suitable subharmonic extension to

$$\rho_2 = \rho_1 K_2^{1-d} > K_2^{-d}.$$

We now define $K_3 = K_2^{C_1}$ so that $\rho_2 > K_3^{-c}$, and set

$$\delta_3 = \min_{0<|k|\leq K_3} \|k \cdot \omega_2''\| .$$

Now, we are once again in a situation where one of two things must hold. Either

$$N_0 \geq K_3^2/\delta_3 \tag{2.230}$$

or

$$N_0 < K_3^2/\delta_3. \tag{2.231}$$

We assume $N_0 \geq K_3^2/\delta_3$. Indeed, if not, we will perform another change of variables as above.

We now, as before, assume $N_0 < \epsilon_2^{-1}K_3^{-1}$ and apply our induction hypothesis to (ω_1'', ω_2'') with K_0 replaced with K_3, δ_0 replaced by δ_3, q_0 replaced by $q_0q_1q_2$, and ϵ_0 replaced by

$$\epsilon_2 = \|q_0q_1q_2\omega_1''\| \tag{2.232}$$

$$\leq q_2\epsilon_1 + q_0q_1\delta_2 \tag{2.233}$$

$$\leq q_2(q_0\delta_1 + q_1\epsilon_0) + q_0q_1\delta_2 \tag{2.234}$$

$$\leq K_2(\delta_1 + \delta_2 + \epsilon_0). \tag{2.235}$$

Thus

$$|L_{N_0} - L_{N_{0,3}}| < K_3^{-c} < K_0^{-c} \tag{2.236}$$

for all $N_{0,3}$ such that $N_0|N_{0,3}$ and $N_{0,3} < \epsilon_2^{-1}K_3^{-1}$. Now fix $N_{0,3}$ as close as possible to $\epsilon_2^{-1}K_3^{-1}$. If $N_0 > \epsilon_2^{-1}K_3^{-1}$, then we set $N_{0,3} = N_0$.

Now either $\epsilon_0 N_{0,3} > K_0^{-1}$, in which case we have reached our desired scale and there is nothing else to do, or we may apply Theorem 2.7.1, the hypotheses of which hold using the same argument as we used for $N_{0,2}$. In the latter case, we obtain

$$|L_N - L_{N_{0,3}}| < K_0^{-c} \tag{2.237}$$

$$|L_N - L_{N_0}| < K_0^{-c} \tag{2.238}$$

for all $N_{0,3}|N$ such that

$$N < \min\left\{\epsilon_0^{-1}K_0^{-1}, N_{0,3}e^{K_0^c}\right\}.$$

By our choice of $N_{0,3}$, we have

$$N_{0,3}e^{K_0^c} > K_2^4\epsilon_2^{-1} > \frac{K_2^2}{\epsilon_0 + \delta_1 + \delta_2}.$$

If $\epsilon_0 + \delta_1 > \delta_2$, then

$$N_{0,3} e^{K_0^c} > \frac{K_2^2}{2\epsilon_0 + 2\delta_1},$$

and we have reached the desired scale: either $\epsilon_0 < \delta_1$, in which case the right hand side is no less than $K_1^2 \delta_1^{-1} + N_0$, or $\epsilon_0 > \delta_1$, in which case the right hand side is no less than $\epsilon_0^{-1} K_0^{-1}$. On the other hand, if $\epsilon_0 + \delta_1 \leq \delta_2$, then

$$N_{0,3} e^{K_0^c} > \frac{K_2^2}{2\delta_2}.$$

In this case, we can repeat all of the preceding using $N_{0,3}$ instead of N_0. Since $N_{0,3} \geq \frac{K_2^2}{2\delta_2}$, we are in the first scenario we considered, and our conclusion follows.

If $N_0 < K_3^2/\delta_3$, then, as remarked above, we may perform another change of variables to define K_4, δ_4 and we may repeat the above procedure. Suppose, therefore, that for some $2 \leq j \leq d_2 + 1$, we may perform j changes of variables as above and obtain $N_0 \geq K_j^2/\delta_j$. We may set $\delta_{d_2+1} = 1$. The above procedure allows us to define a scale $N_{0,j}$ such that

$$|L_{N_0} - L_{N_{0,j}}| < K_0^{-c}$$

and

$$|L_{N_{0,j}} - L_N| < K_0^{-c}$$

for $N_{0,j}|N$ such that

$$N < \min\left\{\epsilon_0^{-1} K_0^{-1}, N_{0,j} e^{K_0^c}\right\}.$$

Either N has reached the desired scale, in which case we are done, or N satisfies $N \geq K_{j-1}^2/\delta_{j-1}$. We may now repeat the entire procedure starting at scale N instead of N_0, and we will reach our desired scale after at most d_2 iterations of this argument.

Finally, we must consider the case where, no matter how many changes of variables we use, we are never in the case where $N_0 \geq K_j^2/\delta_j$, where K_j and δ_j are defined inductively as above for $d_2 \geq j \geq 2$, and $\delta_{d_2+1} = 1$. In this case, we may apply Theorem 2.7.1 with N_0, δ_0, and K_0. This leads to

$$|L_{N_0} - L_{N_0'}| < K_0^{-c}$$

for $N_0 | N_0'$ and

$$N_0' \leq \min\left\{\epsilon_0^{-1} K_0^{-1}, N_0 e^{(K_0/q_0)^c}\right\}.$$

If $N_0 e^{(K_0/q_0)^c} > \epsilon_0^{-1} K_0^{-1}$, then we have reached our desired scale and we are done. Otherwise, set $N_0' \sim N_0 e^{(K_0/q_0)^c}$. At this point, we will suppose that $K_0 > K'(d_2)$ is large enough such that

$$N_0 e^{(K_0/q_0)^c} > K_{d_2+1}^2.$$

With this in hand, we may repeat the entire argument starting at scale N_0', and know that we are guaranteed to satisfy $N_0 \geq K_j^2/\delta_j$ for some $2 \leq j \leq d_2 + 1$.

This completes our induction argument.

□

We may now prove Lemma 2.8.1.

Proof of Lemma 2.8.1. Apply Theorem 2.8.1 with $\epsilon_0 = 0, \delta_0 = \delta, K_0 = K$, and $N_0 = N$. This yields

$$|L_N - L_{N'}| < K^{-c}$$

for all $N | N', N' < \infty$. Taking a limit, $N' \to \infty$, completes the proof. □

2.9 Proof of Continuity

We are now in a position to prove continuity of L.

Lemma 2.9.1. *Suppose $\omega = (\omega_1, ..., \omega_d) \in \mathbb{T}^d$. Let (A, ω) be an analytic quasiperiodic $M(2,\mathbb{C})$-cocycle which has an analytic extension to the strip $|\Im(z_j)| < \rho$. Suppose, moreover, that $\det(A) \not\equiv 0$. Then $L(A, \omega)$ is jointly continuous in A and ω for ω such that $k \cdot \omega \neq 0$ for any $k \in \mathbb{Z}^d \setminus \{0\}$.*

Proof. Fix a cocycle (A, ω), where $\omega = (\omega_1, ..., \omega_d)$ is such that $\|k \cdot \omega\| \neq 0$ for all $k \in \mathbb{Z}^d$, $|k| \neq 0$. Fix $\kappa > 0$ and let K_0 be large enough such that $K_0^{-c} < \kappa$. Set $\delta_0 = \min_{0<|k|\leq K_0} \{\|k \cdot \omega\|\} > 0$ and take $N > K_0^2/\delta_0$. We have, by Lemma 2.8.1 with $d_1 = 0$,

$$|L_N(A, \omega) - L(A, \omega)| < C(A)K_0^{-c} < C(A)\kappa.$$

Moreover, for fixed N, we know $L_N(A, \omega)$ is jointly continuous in A and ω, so for any cocycle (B, ω') such that $\|A - B\|$ and $\|\omega - \omega'\|$ are sufficiently small, we have

$$|L_N(A, \omega) - L_N(B, \omega')| < \kappa.$$

Finally, for ω' sufficiently close to ω, we have $\|k \cdot \omega'\| > \frac{1}{2}\delta_0$ for $0 < |k| \leq K_0$. Thus

$$|L_N(B, \omega') - L(B, \omega')| < C(A)\kappa.$$

Here we have $C(A)$ by taking B sufficiently close to A. Triangle inequality now yields our conclusion. $\square$

Lemma 2.9.2. *Suppose $\omega = (\omega_1, ..., \omega_d) \in \mathbb{T}^d$. Let (A, ω) be an analytic quasiperiodic $M(2,\mathbb{C})$-cocycle which has an analytic extension to the strip $|\Im(z_j)| < \rho$. Suppose, moreover, that $\det(A) \not\equiv 0$. Then $L(A, \omega)$ is continuous in A for any $\omega \in \mathbb{T}^d$.*

Proof. Fix $\omega \in \mathbb{T}^d$. If $k \cdot \omega \neq 0$ for any $k \in \mathbb{Z}^d \setminus \{0\}$, then continuity in A follows from joint continuity for at such ω. Thus, it suffices to suppose $k \cdot \omega = 0$ for some $k \in \mathbb{Z}^d \setminus \{0\}$. First, we claim that we may assume that $\omega = (\omega_1, \omega_2) \in \mathbb{T}^{d_1} \times \mathbb{T}^{d_2}$ is such that $\|q\omega_1\| = 0$ for some $q \in \mathbb{N}$ and $\|k \cdot \omega_2\| \neq 0$ for all $k \in \mathbb{Z}^{d_2}, |k| \neq 0$. Indeed, suppose $\|k \cdot \omega\| = 0$ for some $|k| \neq 0$. We may perform a change of variables, B_1, so that

$$B_1(\omega) = (\omega_1, \omega_2) \in \mathbb{T} \times \mathbb{T}^{d-1},$$

where $\omega_1 \in \mathbb{Q}$. If $\omega_2 \in \mathbb{T}^{d-1}$ is such that $\|k'\omega_2\| = 0$ for some $|k'| \neq 0$, then we may perform another change of variables, B_2, such that

$$B_2(\omega_1, \omega_2) = (\omega_1', \omega_2') \in \mathbb{T}^2 \times \mathbb{T}^{d-2},$$

where, for some q, $\|q\omega_1'\| = 0$. We may thus perform consecutive changes of variables until we reach $\omega' = (\omega_1', \omega_2') \in \mathbb{T}^{d_1} \times \mathbb{T}^{d_2}$ where $\|q\omega_1'\| = 0$ for some $q \in \mathbb{N}$ and $\|k \cdot \omega_2'\| \neq 0$ for all $k \in \mathbb{Z}^{d_2}, |k| \neq 0$. Since a change of variables will not change the regularity of the Lyapunov exponent, this proves our reduction claim.

Now, assuming

$$\omega = (\omega_1, \omega_2) \in \mathbb{T}^{d_1} \times \mathbb{T}^{d_2}$$

is such that $\|q\omega_1\| = 0$ for some $q \in \mathbb{N}$ and $\|k \cdot \omega_2\| \neq 0$ for all $k \in \mathbb{Z}^{d_2}, |k| \neq 0$, our conclusion follows from Lemma 2.8.1, continuity of L_N for fixed N, and triangle inequality. $\square$

Chapter 3

Lyapunov exponents II: positivity of Lyapunov exponents for operators with finite-valued background potentials

3.1 Preliminaries

Let us recall the usual 1-frequency quasiperiodic operator:

$$(H^v_{\lambda,\omega,x}u)(n) = u(n-1) + u(n+1) + \lambda v(x+n\omega)u(n). \tag{3.1}$$

In this chapter, we are interested in operators on $\ell^2(\mathbb{Z})$ of the form

$$\tilde{H}^{v,v_1}_{\lambda,\omega,x} = H^v_{\lambda,\omega,x} + v_1, \tag{3.2}$$

where v is real-analytic and $v_1 : \mathbb{Z} \to \mathbb{R}$ is a background sequence of real numbers. We will explore uniform positivity properties of the (lower) Lyapunov exponent. Combined with the results we will present in Chapter 5, the result in this chapter implies particular singularity of the operator's spectral measures.

Recently, many authors have turned their attention to the properties of Schrödinger operators of the above form. One of the commonly studied models is the mixed quasiperiodic-random potential (see [11, 12] and references therein for known results), which coincides to v_1 begin a realization of a random variable. Of particular relevance to this note is that Cai, Duarte, and Klein recently proved a criterion for positivity of the (maximal) Lyapunov exponent for mixed multifrequency quasiperiodic-random potentials, where the quasiperiodic potential is continuous [12].

It is also possible, however, to consider properties of operators with quasiperiodic plus a deterministic background, such as a periodic sequence. Recently, Damanik, Fillman, and Gohlke [14] studied, among other more general objects, such operators where the (one-frequency) quasiperiodic potential is a trigonometric polynomial and the deterministic background is q-periodic, and they showed that, for large coupling constant, λ, on the quasiperiodic potential, the Lyapunov exponent is positive. In particular, they showed that the Lyapunov exponent has an energy-independent lower bound of $\frac{1}{2}\ln(\lambda)$.

Liu has also considered models with low-complexity backgrounds and established large-deviation estimates and modulus of continuity for the integrated density of states associated with these models [59]; see also [10], where the low-complexity background was first incorporated in a localization-type argument.

We will focus on one-frequency quasiperiodic operators with analytic potential, along with a deterministic background consisting of a finite-range sequence—that is, a sequence which takes only finitely many values—and we prove that the (lower) Lyapunov exponent has an energy-independent (and, in a suitable sense, a background-independent) uniform lower bound when the coupling constant is sufficiently large. As we can see from the existing results on mixed-type potentials, such a result is not unexpected.

For one-frequency quasiperiodic operators with no background, positivity results go back to Herman [34] for trigonometric polynomial potentials, and to Sorets and Spencer [69] for analytic potentials; see also [5]. Such results originally took the form $L(E) \geq \frac{1}{2}\ln(\lambda)$. In the case of one-frequency analytic quasiperiodic operators with no background, this was improved to $L(E) \geq \ln(\lambda) - O(1)$ by Duarte and Klein [24] using a convexity argument for means of subharmonic functions, which bypasses the harmonic measure argument present in [5]. This lower bound is sharp, in the sense that $L(E) = \ln(\lambda)$ for the almost Mathieu operator.

In this chapter, we find that it is, in fact, possible to obtain analogous results as [24] by carefully modifying the harmonic measure argument of [5] without appealing to convexity. Moreover, our approach is robust enough to apply when a finite-valued background is present.

More precisely, we consider the quasiperiodic operator $\tilde{H}^{v,v_1}_{\lambda,\omega,x}$ where $\omega, x \in \mathbb{T}$ and $v : \mathbb{T} \to \mathbb{R}$ is an analytic function which is not identically zero. We are interested in the behavior when v_1 is a real-valued sequence on $\mathbb{Z}$ which takes only finitely many values. We consider lower limits of

$$L_N(E) = \frac{1}{N}\int_{\mathbb{T}} \ln \|M_N(x, E, \omega)\|\, dx \tag{3.3}$$

where

$$M_N(x, E, \omega) = \prod_{k=N}^{1} \begin{pmatrix} E - \lambda v(x + k\omega) - V_{per}(k) & -1 \\ 1 & 0 \end{pmatrix}. \tag{3.4}$$

We set

$$L(E) = \liminf_{N\to\infty} L_N(E).$$

We call $L(E)$ the (lower) Lyapunov exponent. It is important to note that the limit need not exist in general, however, if the background is periodic, then we can actually easily see that the limit exists. Moreover, if the background potential is described by some ergodic process, such as a sub-shift on a finite alphabet, then we may replace lim inf with lim and this definition will agree with the usual notion of Lyapunov exponent after integration by a suitable ergodic measure associated with the background.

The recent result by Damanik, Fillman, and Gohlke, $L(E) \geq \frac{1}{2}\ln(\lambda)$, (c.f. Theorem 4.2.5 of [14]) was established for potentials given by trigonometric polynomials with periodic backgrounds by appealing to Avila's global theory. The method of [14] does require periodicity of the background and does not easily extend to general analytic potentials. In contrast, we utilize properties of subharmonic functions to prove the sharp result $L(E) \geq \ln(\lambda) - O(1)$, which works for all analytic potentials with arbitrary finite-valued backgrounds.

Theorem 3.1.1. *Suppose $H = H^v_{\lambda,\omega,x} + v_1$, with $H^v_{\lambda,\omega,x}$ as above. Then for any $q \in \mathbb{N}$, there exists $\lambda_0 = \lambda_0(v, q)$, independent of the background, such that for any $\lambda > \lambda_0(v, q)$, and any sequence of q real numbers, v_1, we have $L(E) > \ln(\lambda) - O(1)$.*

Remark 15. The $O(1)$ term in Theorem 3.1.1 is independent of the background and may be written down explicitly in terms of λ_0 and properties of v.

The background potentials, v_1, we consider include periodic sequences, Sturmian sub-shifts of finite type (or, more generally, low-complexity sub-shifts over finite alphabets), and realizations of Bernoulli random variables.

3.2 Proof of positivity

Lemma 3.2.1. *Suppose v is a bounded 1-periodic non-constant analytic function on the complex strip $|\Im(z)| < \rho, \rho > 0$ Then for any $0 < \delta < \rho$ there is $\epsilon > 0$ depending only on $\delta, k,$ and v such that, for any k-tuple $(E_1, ..., E_k) \in \mathbb{R}^k$,*

$$\sup_{\frac{\delta}{2} \leq y \leq \delta} \min_{1 \leq j \leq k} \inf_{x \in [0,1]} |v(x+iy) - E_j| > \epsilon.$$

Proof. Fix $\delta < \rho$. Let $\sup_{|\Im(z)|<\rho} |v(z)| = C_v < \infty$. Observe that, if $|E_j| > 2C_v$, then the boundedness of v implies $|v(z) - E_j| > C_v$ for any $|\Im(z)| < \rho$ and $k \in \mathbb{Z}$. Thus, it just suffices to establish the claim for $|E_j| \leq 2C_v$.

Indeed, suppose not. Using compactness of $[-2C_v, 2C_v]^k$, we may suppose that there is some $(E_1, ..., E_k) \in [-2C_v, 2C_v]^k$ such that for any $\frac{\delta}{2} \leq y_0 \leq \delta$, we have

$$\inf_{x \in [0,1]} |v(x+iy_0) - E_j| = 0$$

for some $1 \leq j \leq k$.

Since there are infinitely many choices of y_0, but only finitely many choices of E_j, we must be able to find a fixed E_j, a sequence y_n in our desired interval, and a sequence $x_n \in [0,1]$ such that

$$v(x_n + iy_n) - E_j = 0.$$

Since the left hand side is an analytic function, and since we are taking $x_n + iy_n$ in a compact subset of $\mathbb{C}$, this analytic function must have an accumulation point of zeros in its domain, and thus it must be constant zero. This immediately implies $v(z)$ is constant, which is a contradiction. Thus, the claim holds. Uniformity of ϵ for any k-tuple follows from compactness of $[-2C_v, 2C_v]$. $\square$

With this lemma, we can now prove Theorem 3.1.1.

Proof of Theorem 3.1.1. Since v is 1-periodic and real analytic, it has a bounded complex-analytic extension to the strip $|\Im(z)| < \rho$. Say the extension is bounded by C_v. Moreover, if we add any real number to v, say α, then $v + \alpha$ still has a bounded complex-analytic extension to the same strip. Indeed, the bound has simply changed to $C_v + |\alpha|$.

We now consider the complex-analytic matrix-valued function

$$M_N(z,E) = \prod_{k=N}^{1} \begin{pmatrix} E - \lambda v(z + k\omega) - v_1(k) & -1 \\ 1 & 0 \end{pmatrix}. \tag{3.5}$$

Observe that

$$\|M_N(z,E)\| \leq (C_v|\lambda| + |E| + \max|v_1| + 1)^N,$$

and thus

$$u_N(z) = \frac{1}{N} \ln \|M_N(z,E)\|$$

is a subharmonic function on the strip $|\Im(z)| < \rho$ obeying

$$u_N(z) \leq \ln(C_v|\lambda| + |E| + \max|v_1| + 1).$$

Moreover, $M_N(z,E) \in SL_2(\mathbb{C})$, so $\|M_N(z,E)\| \geq 1$. Thus

$$0 \leq u_N(z) \leq \ln(C_v|\lambda| + |E| + \max|v_1| + 1).$$

Our conclusion will now follow if we can bound $\int_0^1 u_N(x)dx$ from below independent of N.

Let $k_1, \ldots, k_q$ denote the q points in $\mathbb{Z}$ which yield the q distinct values for v_1. Let us consider any fixed $E \in \mathbb{R}$ and let $E_j = E - v_1(k_j)$ so that

$$|v(x+iy) - E + v_1(k_j)| = |v(x+iy) - E_j|.$$

Observe that Lemma 3.2.1 is applicable in (q-tuple form) to the right hand side, so we can fix $0 < \delta = \frac{\rho}{100}$, and let $\epsilon > 0$ be the corresponding ϵ associated to this choice of δ in Lemma 3.2.1. We know that for any $E \in \mathbb{R}$, there is $\frac{\delta}{2} < y_0 < \delta$ such that for any $k \in \mathbb{Z}$

$$\inf_{x \in [0,1]} \left| v(x+iy_0) - \frac{E}{\lambda} + \frac{v_1(k)}{\lambda} \right| > \epsilon.$$

Since v is 1-periodic, we can extend this to the entire real line

$$\inf_{x \in \mathbb{R}} \left| v(x+iy_0) - \frac{E}{\lambda} + \frac{v_1(k)}{\lambda} \right| > \epsilon. \tag{3.6}$$

Define $\lambda_0 = \lambda_0(v) = 5C_v\epsilon^{-1}$ and fix $\lambda > \lambda_0$.

Consider the set $Z = \{j \in \{1, 2, \dots, N\} : |v_1(j) - E| \leq 5C_v\lambda\}$. Suppose $|Z| = k$, and let $\{j_1, \dots, j_{N-k}\} = \{1, \dots, N\} \setminus Z$. By induction on N, (3.6), and our definition of Z, we see that

$$\|M_N(a+iy_0, E)\| \geq \left| \left\langle M_N(a+iy_0, E) \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle \right| \tag{3.7}$$

$$\geq (\lambda\epsilon - 1)^k \prod_{n=1}^{N-k} (|E - \lambda v(a+iy_0+j_n\omega) - v_1(j_n)| - 1) \tag{3.8}$$

for any $a \in [0,1]$. Thus, we may improve our lower bound on u_N for any $a \in \mathbb{R}$

$$u_N(a+iy_0) > \frac{k}{N}\ln(|\lambda|\epsilon - 1) + \frac{1}{N}\sum_{n=1}^{N-k} \ln\left(|E - \lambda v(a+iy_0+j_n\omega) - v_1(j_n)| - 1\right) > 0.$$

Now, we let μ_{a+iy_0} denote the harmonic measure associated to $a+iy_0$ in the complex strip $0 \le iy \le i\rho/2$. In particular, μ_{a+iy_0} is a regular Borel measure on the two lines $y=0$ and $iy = i\rho/2$. The definition of the harmonic measure quickly yields

$$\mu_{a+iy_0}\{iy = i\rho/2\} = \frac{2y_0}{\rho} \tag{3.9}$$

$$\mu_{a+iy_0}\{iy = 0\} = 1 - \frac{2y_0}{\rho} \tag{3.10}$$

Since u_N is subharmonic, we have

$$\begin{aligned} u_N(a+iy_0) &\le \int_{iy=0} u_N(x) d\mu_{a+iy_0}(x) + \int_{iy=i\rho/2} u_N(x+iy) d\mu_{a+iy_0}(x) \\ &= \int_{iy=0} u_N(x+a) d\mu_{iy_0}(x) + \int_{iy=i\rho/2} u_N(x+a+iy) d\mu_{iy_0}(x). \end{aligned} \tag{3.11}$$

Here we used a change of variables and translation properties of the harmonic measure on a horizontal strip (see proof of Proposition 11.21 in [5]). Now we can integrate throughout in a over the unit interval, and appeal to periodicity of u in x to obtain

$$\begin{aligned} \int_0^1 u_N(x+iy_0)dx &< \int_0^1 u_N(x)dx \cdot \mu_{iy_0}\{iy=0\} + \int_0^1 u_N(x+i\frac{\rho}{2})dx \cdot \mu_{iy_0}\left\{iy = i\frac{\rho}{2}\right\} \\ &\le (1-2y_0/\rho)\int_0^1 u_N(x)dx + (2y_0/\rho)\int_0^1 u_N(x+i\rho/2)dx. \end{aligned} \tag{3.12}$$

Moreover,

$$u_N(x+i\rho/2) \le \frac{k}{N}\ln(2C_v|\lambda|) + \frac{1}{N}\sum_{n=1}^{N-k} \ln\left(|E - \lambda v(a+i\rho/2+j_n\omega) - v_1(j_n)| + 1\right),$$

so

$$\begin{aligned}\int_0^1 u_N(x+iy_0)dx \le \left(1-\frac{2y_0}{\rho}\right)&\int_0^1 u_N(x)dx \\ &+ \frac{2y_0}{\rho}\frac{k}{N}\ln(C_v|\lambda|) \\ &+ \frac{2y_0}{\rho}\frac{1}{N}\sum_{n=1}^{N-k}\int_0^1 \ln\left(|E-\lambda v(x+i\rho/2+j_n\omega)-v_1(j_n)|+1\right)dx.\end{aligned}$$

We thus have:

$$\begin{aligned}&\left(1-\frac{2y_0}{\rho}\right)\int_0^1 u_N(x)dx \\ &\qquad \ge \frac{k}{N}\ln(\lambda\epsilon-1) - \frac{2y_0}{\rho}\frac{k}{N}\ln(C_v|\lambda|) \\ &\qquad\quad + \frac{1}{N}\sum_{n=1}^{N-k}\int_0^1 \ln\left(|E-\lambda v(x+iy_0+j_n\omega)-v_1(j_n)|-1\right)dx \\ &\qquad\quad - \frac{2y_0}{\rho}\frac{1}{N}\sum_{n=1}^{N-k}\int_0^1 \ln\left(|E-\lambda v(x+i\rho/2+j_n\omega)-v_1(j_n)|+1\right)dx\end{aligned} \tag{3.13}$$

$$\begin{aligned}&\qquad \ge \frac{k}{N}\left(\ln(\lambda\epsilon-1)-\frac{2y_0}{\rho}\ln(C_v|\lambda|)\right) \\ &\qquad\quad + \frac{2y_0}{\rho}\frac{\rho}{2y_0}\frac{1}{N}\sum_{n=1}^{N-k}\int_0^1 \ln\left(|E-\lambda v(x+iy_0+j_n\omega)-v_1(j_n)|-1\right)dx \\ &\qquad\quad - \frac{2y_0}{\rho}\frac{1}{N}\sum_{n=1}^{N-k}\int_0^1 \ln\left(|E-\lambda v(x+i\rho/2+j_n\omega)-v_1(j_n)|+1\right)dx\end{aligned} \tag{3.14}$$

Now we have $v(x+i\rho/2) = v(x+iy_0)+\eta(x)$, where $|\eta(x)| \le C_v$, and $|E-\lambda v(x+iy_0+j_n\omega)-v_1(j_n)| \ge 4C_v\lambda$, so

$$\begin{aligned}&\ln(|E-\lambda v(x+i\rho/2+j_n\omega)-v_1(j_n)|+1) \\ &\qquad \le \ln(2|E-\lambda v(x+iy_0+j_n\omega)-\lambda\eta(x+j_n\omega)-v_1(j_n)|+1)\end{aligned}. \tag{3.15}$$

It now follows that we can bound (3.14) from below using (3.15), the definitions of $\eta(x)$ and

Z, and triangle inequality to obtain

$$\left(1-\frac{2y_0}{\rho}\right)\int_0^1 u_N(x)dx \tag{3.16}$$

$$\geq \frac{k}{N}\left(\ln(\lambda\epsilon-1)-\frac{2y_0}{\rho}\ln(C_v\lambda)\right) + \frac{2y_0}{\rho}\frac{1}{N}\sum_{n=1}^{N-k}\int_0^1 \ln\left((C_v\lambda)^{\rho/2y_0-1}\right)dx \tag{3.17}$$

$$\geq \frac{k}{N}\left(\ln(\lambda)+\ln(\epsilon)-\frac{2y_0}{\rho}\ln(\lambda)-\frac{2y_0}{\rho}\ln(C_v)\right) + \frac{2y_0}{\rho}\left(\frac{\rho}{2y_0}-1\right)\frac{N-k}{N}\ln(C_v\lambda) \tag{3.18}$$

$$= \frac{k}{N}\left(1-\frac{2y_0}{\rho}\right)\ln(\lambda)+\frac{k}{N}\left(\ln(\epsilon)-\frac{2y_0}{\rho}\ln(C_v)\right) + \frac{N-k}{N}\left(1-\frac{2y_0}{\rho}\right)\ln(\lambda)+C_{v,\rho} \tag{3.19}$$

Since $\delta/2 < y_0 < \delta = \rho/100$, the term $\ln(\epsilon)-\frac{2y_0}{\rho}\ln(C_v)$ is a constant which may be bounded by something depending only on v, λ_0, and ρ, dividing by $1-\frac{2y_0}{\rho}$ yields our result.

$\square$

Chapter 4

Further consequences for quantum dynamics: logarithmic upper bounds

In the Chapter 5, we will discuss and extension of Yoram Last's theory of quantum dynamics to establish lower bounds in quantum dynamics whenever spectral measures are sufficiently continuous. Here, we are interested in the opposite phenomenon: obtaining upper bounds in quantum dynamics; we will see that upper bounds in quantum dynamics imply singularity of the associated spectral measures.

Direct proofs of upper quantum dynamical bounds for quasiperiodic and other ergodic operators with positive Lyapunov exponents have been done, in increasing generality in [17, 45, 32]. In all of these cases, the results featured the desired stability in phase and were often arithmetic in frequency (in contrast with many localization proofs). All of the papers mentioned above obtain vanishing of the transport exponents $\beta(p)$ (see (4.3)), which implies *sub-polynomial* growth of the moments. Here we present a method that allows us to improve this to *power-logarithmic* bounds. We note that our results are also phase-stable and our frequency conditions are arithmetic. The only previous direct proof of power-logarithmic bounds was done for the Anderson model in [47] based on different considerations, but we note that for the Anderson model localization always holds ([13] or see a very simple recent argument in [50]). Thus, to the best of our knowledge, we present the first proof of power-logarithmic quantum dynamical bounds for models without localization.

Technically, our method goes back to [38] where the existence of transfer matrices growing appropriately along a subsequence was first used to prove zero Hausdorff dimension of spectral measures for one-frequency quasiperiodic operators, including in situations where localization cannot hold. The ideas of [38] were first applied in [17] to obtain vanishing transport exponents for those models, and then this was further modified and developed in [45] to allow very rough functions. These methods however required continued fraction techniques and did not extend naturally even to the case of shifts on higher-dimensional tori. This was tackled in [32] which developed a method allowing to handle general dynamics of zero topological entropy. Here, for our one-frequency result we go back to the approach of [38, 17, 45]. The method of [32] however is too rough for the logarithmic scale. It turns out that for higher-dimensional shifts and skew-shifts the basics of Bourgain's semi-algebraic/large deviations method [5] are ideally suited to obtain the desired power-logarithmic bounds on the moments.

The key estimate from Bourgain's method used here is the sublinear bound (4.22) on the number of hits of a semi-alebraic set by a shift ([5]) or skew-shift ([59]) trajectory. In fact, all we need is a much weaker statement: the existence of at least one miss in sublinear time, which of course follows from the sublinear bound. We make some explicit estimates on the power used in the sublinear bound ((4.22)) in Section 4.3. The sublinear bound was also fruitfully used in a recent work [41] to establish vanishing of transport exponents $\beta(p)$ (thus subpolynomial bounds on the moment growth) for long-range quasiperiodic operators, for which the authors of [41] developed a non-transfer-matrix based approach. It is an interesting question whether power-logarithmic bounds can be also obtained in that case.

We cover all scenarios where a.e. Anderson localization has been proved for one-dimensional operators with analytic quasiperiodic and skew-shift potentials as described in Bourgain's book [5] and with Gevrey extensions in [55, 54]. For all these models the a.e. dynamical localization was also shown to hold [8]. Essentially, what we demonstrate by this work is that power-logarithmic bounds on transport can be viewed as *dynamical localization-lite*, since the proof is considerably simpler than that of localization and in fact can be obtained in many known scenarios as a part of the latter proof. Yet the results are phase-stable and presumably optimal as far as phase-stable results go. Just as with Anderson localization, our theorems are non-perturbative (our results are obtained as a corollary of positive Lyapunov exponents, which holds independent of the frequency) for analytic potentials over shifts of higher-dimensional tori and Gevrey potentials for one-frequency shifts, while they require large coupling constants dependent on the frequency for the multifrequency Gevrey and skew shift cases. We note however, that all such dependence comes from the large deviation estimates that we use as a black box; we don't add any further "perturbative" components through our technique.

We proceed to formulate our main results. Consider the time-averaged quantity:

$$a(n,T) = \frac{2}{T}\int_0^\infty e^{2t/T}\frac{1}{2}\left(\left|\left\langle e^{-itH_{\omega,x}}\delta_0,\delta_n\right\rangle\right|^2 + \left|\left\langle e^{-itH_{\omega,x}}\delta_1,\delta_n\right\rangle\right|^2\right)dt, \tag{4.1}$$

where $\delta_n(m) = 1$ when $m = n$ and 0 otherwise.

Dynamical localization is characterized by boundedness in time of the moments of the position operator:

$$\langle |X|^p(T)\rangle = \sum_{n\in\mathbb{Z}}(1+|n|)^p a(n,T). \tag{4.2}$$

For simplicity, we are restricting our attention to time-averaged quantities, rather than considering $a(n,t) = \frac{1}{2}\left(\left|\left\langle e^{-itH_{\omega,x}}\delta_0,\delta_n\right\rangle\right|^2 + \left|\left\langle e^{-itH_{\omega,x}}\delta_1,\delta_n\right\rangle\right|^2\right)$, but our analysis can be carried through for non-time-averaged quantities as well, following the ideas in [18]. We only consider time-averaging for a small simplification.

Dynamical localization always implies Anderson localization, but is strictly stronger [20, 48]. When dynamical localization does not hold, the moments of the position are unbounded in time and a natural quantity of interest is how fast this growth is. Classically, this is captured by the upper and lower transport exponents:

$$\beta^+(p) = \limsup_{t\to\infty} \frac{\ln \langle |X|^p(t)\rangle}{p\ln t}; \quad \beta^-(p) = \liminf_{t\to\infty} \frac{\ln \langle |X|^p(t)\rangle}{p\ln t}, \tag{4.3}$$

which describe power-law bounds on the growth of the moments. It is known that, under very relaxed conditions (c.f. [32]), the transport exponents vanish when the Lyapunov exponent is positive. Let us refine the notion of transport exponents by defining the logarithmic transport exponents as

$$\beta^+_{\ln}(p) = \limsup_{t\to\infty} \frac{\ln \langle |X|^p(t)\rangle}{p\ln\ln t}; \quad \beta^-_{\ln}(p) = \liminf_{t\to\infty} \frac{\ln \langle |X|^p(t)\rangle}{p\ln\ln t}. \tag{4.4}$$

Our first result is that positivity of the Lyapunov exponent will imply that this exponent is finite for every p.

Let T_ω represent either the shift or the skew-shift on the torus, $\mathbb{T}^\nu$, $G^\sigma(\mathbb{T}^\nu)$ denote the Gevrey class, $L(E)$ denote the Lyapunov exponent, and $DC(A,c)$ and $SDC(A,c)$ denote Diophantine conditions (see Section 4.1 for the relevant definitions). In this regime, we have the following.

Theorem 4.0.1. *Let $H_{\omega,x}$ be an operator of the form (**??**) with T_ω given by the shift on $\mathbb{T}$, and either f is analytic or $f \in G^\sigma(\mathbb{T}), \sigma > 1$, and obeys the transversality condition* (4.11)*. Suppose that $L(E) > 0$ for every $E \in \mathbb{R}$. Then for any $x \in \mathbb{T}, \epsilon > 0$ and $m > 0$,*

1. *if $\omega \in \mathbb{R}\backslash\mathbb{Q}$, then $\liminf_{T\to\infty} \frac{\langle |X|^m(T)\rangle}{(\ln T)^{m(\sigma+1+\epsilon)}} < \infty$;*

2. *if* $\omega \in DC(A, c)$, *then* $\limsup_{T\to\infty} \frac{\langle |X|^m(T)\rangle}{(\ln T)^{m(\sigma+1+\epsilon)}} < \infty$.

Remark 16. We can rewrite the conclusions of Theorem 4.0.1 as follows:

1. if $\omega \in \mathbb{R}\backslash\mathbb{Q}$, then $\beta_{\ln}^{-}(p) \leq 1 + \sigma$ for every $p > 0$ and $x \in \mathbb{T}$.

2. if $\omega \in DC(A, c)$, then $\beta_{\ln}^{+}(p) \leq 1 + \sigma$ for every $p > 0$ and $x \in \mathbb{T}$.

Remark 17. For analytic f the conclusion holds with $\sigma = 1$.

We have similar logarithmic quantum-dynamical bounds for non-constant analytic potentials on higher-dimensional tori.

Theorem 4.0.2. *Let $H_{\omega,x}$ be an operator of the form* (**??**) *with T_ω given by the shift on $\mathbb{T}^\nu$ with $\nu > 1$. Suppose also that f is a non-constant analytic function on $\mathbb{T}^\nu$, $\omega \in DC(A, c)$, and that $L(E) > 0$ for every $E \in \mathbb{R}$. Then there exists $\gamma = \gamma(\nu, A)$ such that, for every $m > 0$,*

$$\beta_{\ln}^{\pm}(m) \leq \gamma. \tag{4.5}$$

for all $x \in \mathbb{T}^\nu$.

Remark 18. For analytic f, the condition $L(E) > 0$ for every $E \in \mathbb{R}$ is satisfied for λf, where $\lambda > \lambda_0(f)$. Also we have as an immediate corollary that there exists $\gamma(\nu)$ such that for a.e. $\omega \in \mathbb{T}^\nu, \beta_{\ln}^{\pm}(m) \leq \gamma(\nu)$ for every $m > 0$.

Things become a bit more technical when we consider the multi-frequency shift with potentials in the Gevrey class, or when considering the skew shift instead of the shift.

Theorem 4.0.3. *Let $x \in \mathbb{T}^\nu$. Let $H_{\omega,x}$ be an operator of the form* (**??**) *with T_ω given by the shift on $\mathbb{T}^\nu$ with $\nu > 1$. Suppose also that $f = \lambda f_0 \in G^\sigma(\mathbb{T}^\nu)$ such that f_0 obeys the* *transversality condition* (4.11), $\omega \in DC(A, c)$, *and that* $L(E) > 0$ *for every* $E \in \mathbb{R}$. *Then there exists* $\lambda_0 = \lambda_0(f_0, \omega) > 0$ *and* $\gamma = \gamma(\sigma, \nu, A)$ *such that, for every* $\lambda > \lambda_0$ *and* $m > 0$,

$$\beta_{\ln}^{\pm}(m) \leq \gamma. \tag{4.6}$$

Remark 19. The condition on λ_0 comes from [54] and is necessary to obtain and use a large deviation estimate which is critical to our proof. See Theorem 4.1.4.

Theorem 4.0.4. *Let* $H_{\omega,x}$ *be an operator of the form* (**??**) *with* T_ω *given by the skew-shift on* $\mathbb{T}^\nu$, *suppose* $f = \lambda f_0 \in G^\sigma(\mathbb{T}^\nu)$ *such that* f_0 *obeys* (4.11), *and* $\omega \in SDC(A, c)$, *for some* $A \leq 2$. *Suppose that* $L(E) > 0$ *for every* $E \in \mathbb{R}$. *Then there exists* $\lambda_0 = \lambda_0(f_0, \omega) > 0$ *and* $\gamma = \gamma(\sigma, \nu, A)$ *such that for every* $\lambda > \lambda_0$ *and* $m > 0$,

$$\beta_{\ln}^{\pm}(m) \leq \gamma. \tag{4.7}$$

for all $x \in \mathbb{T}^\nu$.

Remark 20. As mentioned earlier, the perturbative nature of Theorems 4.0.3 and 4.0.4 is fully captured in the ω-dependence of λ_0 that comes from [55, 54], while the bound γ that we prove to exist is constant for a.e. Diophantine ω.

Remark 21. We will see in our proof that the γ that appears in Theorems 4.0.3 and 4.0.4 has ω-dependence which appears precisely as the constant δ from (4.22). It is possible to explicitly compute $\gamma = C(\sigma\nu+1)\left(\frac{1}{\delta}\right)$. Here C is a universal constant $C = C(\nu)$. The constant δ is different for the shift and skew shift, and will be obtained by semialgebraic methods in section 4.3, where we obtain the explicit estimates $\delta \leq \frac{1}{A+\nu}$ for the shift and $\delta < \frac{1}{A\nu 2^{\nu-1}}$ for the skew-shift.

Remark 22. One of the only places where there is still room for improvement in this approach is the estimate on δ in Theorem 4.1.2. The closer δ is to 1, the smaller γ will be, and thus the better the localization result. Our estimate for the shift follows from a harmonic analysis approach given by Bourgain. For $\omega \in DC(A, c)$, other estimates have been obtained by other authors using alternative methods (c.f. [32] and [59]) but when $A \gg 1$, our localization result is stronger.

We note that the method in [32] while applicable to all our models and a lot more, is insufficient to obtain ln-type estimates which we are after here, largely because it allows to find the required exponential growth of the transfer matrix only on polynomially-large length scales, whereas the growth needs to be on logarithmic length scales to obtain ln-type estimates.

A corollary of these bounds may be obtained using the results we will present in Chapter 5 (see that chapter for the relevant definitions).

Corollary 4.0.1. *Under the assumptions of Theorem 4.0.1, with $\omega \in DC(A, c)$, we have $\dim_{\ln}^{+}(\mu) \leq 1 + \sigma$, where μ is the spectral measure related to δ_0 and $H_{\omega,x}$. Under the assumptions of Theorem 4.0.3, we have $\dim_{\ln}^{+}(\mu) \leq \gamma$.*

Other quantities have been proposed for studying dynamical localization-type estimates, see [4, 17], but one of the major advantages of $\beta_{\ln}^{\pm}(p)$ is that, similar to $\beta^{\pm}(p)$, it is stable under perturbations in certain circumstances. See Theorem 4.0.5 part (b) for a precise statement.

One transfer-matrix based way to approach upper dynamical bounds goes back to a scheme by Damanik and Tcheremchantsev [17] wherein the quantity $\beta^{\pm}(p)$ was related to suitable growth of the transfer matrices along suitable length scales (see also [47]) . In this paper, we refine this scheme to allow us to obtain finer dynamical estimates. Our contribution is the following theorem, which required us to address certain technical limitations in the original argument (see Section 4.1.2 for the relevant definitions and Section 4.2 for full details).

Theorem 4.0.5. *Suppose H_1 is of the form* (**??**) *with bounded potential v_1 and $\sigma(H_1) \subset [-K + 1, K - 1]$.*

(a) Suppose for all $\delta < \infty$ and $T > T_0$, we have

$$\int_{-K}^{K} \left(\min_{l=\pm 1} \max_{1 \le lj \le (\ln T)^\gamma} \left\| A_j^{v_1, E+i/T}(x) \right\|^2 \right)^{-1} dE = O(T^{-\delta}) \tag{4.8}$$

for some $\gamma > 1$. Then $\beta^+_{\ln,1}(p) \le \gamma$, where $\beta^+_{\ln,1}(p)$ is the transport exponent associated to H_1. If the above condition holds for a sequence $T_n \to \infty$, then $\beta^-_{\ln,1}(p) \le \gamma$.

*(b) In addition to the above, suppose also that H_2 is an operator of the form (**??**) with bounded potential v_2 such that $\sigma(H_2) \subset [-K+1, K-1]$ and suppose that there exists $B > 0$ such that for all $E \in [-K+1, K-1], 0 < \epsilon \le 1,$ and $|n| \le \ln(\epsilon^{-1})$,*

$$\epsilon^B \left\| A_n^{v_1, E+i\epsilon} \right\| \lesssim \left\| A_n^{v_2, E+i\epsilon} \right\| \lesssim \epsilon^{-B} \left\| A_n^{v_1, E+i\epsilon} \right\|. \tag{4.9}$$

Then $\beta^\pm_{\ln,2}(p) \le \gamma$ for every $p > 0$, where $\beta^\pm_{\ln,2}(p)$ is the transport exponent associated to H_2.

Remark 23. It is worth noting that Theorem 4.0.5 is a purely deterministic result, and thus holds for general operators of the form

$$(Hu)(n) = u(n-1) + u(n+1) + V(n)u(n),$$

where V is a bounded sequence of real numbers.

Theorem 4.0.5 is similar to Theorem 1 in [17], but there is a major issue with just repeating the proof of Theorem 1 in [17] using $(\ln T)^\gamma$ in place of T^γ. The problem is that the result in [17] a priori assumes that $\beta^\pm(p) < \infty$ for every $p > 0$. This is the well-known ballistic upper bound. We do not, unfortunately, have a similar a priori estimate on $\beta^\pm_{\ln}(p)$, even when $\beta^\pm(p) = 0$, which means the original argument is insufficient. Our main technical achievement on the way to a proof of Theorem 4.0.5 is a sufficient condition (Theorem 4.2.2) under which we can say $\beta^\pm_{\ln}(p) < C < \infty$ for every $p > 0$. Once we have this, we can use the ideas from [17] to obtain Theorem 4.0.5.

This essentially reduces the problem of bounding log-transport exponents to obtaining lower bounds on the growth of the transfer matrix along particular length scales. This will be done in a two-step process. First, we will demonstrate that, for a fixed energy and frequency, transfer matrix growth can be suboptimal only for a set of phases of small measure. This will be captured by so-called large deviation estimates. Then we will show that every phase will correspond to a transfer matrix with good growth after at most power-log many iterates of the transformation.

The rest of our paper is organized in the following way. In Section 4.1 we introduce the relevant definitions needed for our paper. Section 4.1.2 is devoted to those definitions needed for the proof of Theorem 4.0.5. Section 4.1.3 recalls facts about semialgebraic sets which will be necessary for the proof of Theorem 4.0.3. Section 4.1.4 recalls the large deviation theorems needed for measure estimates. We prove Theorem 4.0.5 in Section 4.2. We explicitly compute discrepancy bounds in Section 4.3. We prove two technical lemmas regarding the set of *good* phases in Section 4.4. Finally, we prove Theorem 4.0.1 in Section 4.5 and Theorem 4.0.3 in Section 4.6. Proofs of theorems 4.0.2 and 4.0.4 are essentially identical to that of theorem 4.0.3, however, we describe the small changes needed in, correspondingly, Section 4.7 and Section 4.8.

4.1 Preliminaries

4.1.1 Schrödinger operators and transfer matrices

We consider two particular types of Schrödinger operator, $H_{\omega,x} : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ given by

$$(H_{\omega,x}\psi)(n) = \psi(n-1) + \psi(n+1) + f(T_\omega^n(x))\psi(n), n \in \mathbb{Z}. \tag{4.10}$$

The first case we consider is where $x \in \mathbb{T}^\nu, T_\omega$ is the shift: $T_\omega x = x + \omega$, and $\omega = (\omega_1, ..., \omega_\nu)$ and $(\omega_1, ..., \omega_\nu, 1)$ are rationally independent. The second case we consider is where $x \in \mathbb{T}^\nu, T_\omega$ is the skew-shift: $T_\omega(x_1, ..., x_\nu) = (x_1 + \omega, x_2 + x_1, x_3 + x_2, ..., x_\nu + x_{\nu+1})$, and $\omega \in \mathbb{R} \backslash \mathbb{Q}$.

Additionally, we recall that $G^\sigma(\mathbb{T}^\nu)$ denotes the Gevrey class:

$$G^\sigma(\mathbb{T}^\nu) = \left\{ f : \mathbb{T}^\nu \to \mathbb{R} : \|D^\alpha f\|_\infty < C^{|\alpha|+1}(\alpha!)^\sigma \right\}.$$

An equivalent definition of G^σ which we will take advantage of is:

$$G^\sigma(\mathbb{T}^\nu) = \left\{ f : \mathbb{T}^\nu \to \mathbb{R} : |\hat{f}(n)| \le e^{-|n|^{1/\sigma}} \right\}.$$

In both of the cases, we will consider $f \in G^\sigma(\mathbb{T}^\nu)$ in (4.10).

For technical reasons, we will further restrict our attention to those Gevrey class functions that obey a transversality condition:

$$D^\alpha f(x) \neq 0 \quad \text{for any } x \in \mathbb{T}^\nu, \alpha \in \mathbb{N}^\nu. \tag{4.11}$$

From this point forward, when discussing $f \in G^\sigma(\mathbb{T}^\nu)$, we will mean those $f \in G^\sigma(\mathbb{T}^\nu)$ that satisfy (4.11). This transversality condition is a generalization of the "non-constant" assumption that is made when f is analytic. Recall that, for any $E \in \mathbb{C}$, any solution to the eigen-equation $H_{\omega,x}\psi = E\psi$ can be reconstructed from the n-step transfer matrix:

$$A_n^{f,E}(x) = \prod_{k=n}^{1} \begin{pmatrix} f(T_\omega^k(x)) - E & -1 \\ 1 & 0 \end{pmatrix} \tag{4.12}$$

by

$$\begin{pmatrix} \psi(n+1) \\ \psi(n) \end{pmatrix} = A_n^{f,E}(x) \begin{pmatrix} \psi(1) \\ \psi(0) \end{pmatrix}. \tag{4.13}$$

We can then define

$$L_n(E) = \frac{1}{n} \int \ln \left\| A_n^{f,E}(x) \right\| dx$$

and the Lyapunov exponent is given by

$$L(E) = \lim L_n(E) = \inf L_n(E).$$

We will also need a Diophantine condition. We say that $\omega \in DC(A,c)$ if $\|k \cdot \omega\| > c|k|^{-A}$ for every $k \in \mathbb{Z}^\nu \backslash \{0\}$. We say that $\omega \in SDC(A,c)$ if $\|k \cdot \omega\| > c\frac{1}{|k|(\ln|k|)^A}$. We will only consider $\omega \in SDC(A,c)$ for $A \leq 2$, which is a restriction imposed by Theorem 4.1.4. See [54] for details.

In what follows, C and c will denote finite constants and ϵ will denote a small constant, all of which can only depend on f, ν, ω, or E. Moreover, these constants may change throughout a proof, but ϵ will always denote a small constant, and boundedness of C and c will be unchanged.

4.1.2 Transport exponents

Recall that we have defined

$$\beta^+_{\ln}(p) = \limsup \frac{\ln \langle |X|^p(t)\rangle}{p \ln\ln t}; \quad \beta^-_{\ln}(p) = \liminf \frac{\ln \langle |X|^p_t\rangle}{p\ln\ln t}.$$

It is simple to verify via Hölder's inequality that $\beta^\pm_{\ln}(p)$ is non-decreasing in p, so obtaining a bound on $\beta^\pm_{\ln}(+\infty)$ is sufficient for bounding $\beta^\pm_{\ln}(p)$ for any $p > 0$.

To bound $\beta^\pm_{\ln}(+\infty)$, for general operators, we will need to define the so-called outside probabilities:

$$P_l(N,T) = \sum_{n<-N} a(n,T) \tag{4.14}$$

$$P_r(N,T) = \sum_{n>N} a(n,T) \tag{4.15}$$

$$P(N,T) = P_l(N,T) + P_r(N,T) \tag{4.16}$$

$$= \sum_{|n|>N} a(n,T) \tag{4.17}$$

along with associated log-transport quantities:

$$S^+_{\ln}(\alpha) = -\limsup \frac{\ln(P((\ln T)^\alpha - 2, T))}{\ln\ln T} \tag{4.18}$$

$$S^-_{\ln}(\alpha) = -\liminf \frac{\ln(P((\ln T)^\alpha - 2, T))}{\ln\ln T} \tag{4.19}$$

$$\alpha^\pm_{\ln} = \sup\left\{\alpha \geq 0 : S^\pm_{\ln}(\alpha) < \infty\right\}. \tag{4.20}$$

A quick note on our convention here; we use $(\ln T)^\alpha - 2$ so that $S^\pm_{\ln}(0) = 0$ as in [17].

Our goal in Section 4.2 will be to show that, under suitable conditions, $\beta^\pm_{\ln}(p) \leq \alpha^\pm_{\ln}$ for every $p > 0$, which will be used to establish Theorem 4.0.5.

4.1.3 Semialgebraic sets

Definition 4.1.1. We say that a set $\mathcal{S} \subset \mathbb{R}^n$ is semialgebraic if it can be written as a finite union of polynomial inequalities. More precisely, suppose $P = \{p_1, \ldots, p_s\} \subset \mathbb{R}[X_1, \ldots, X_n]$, is a finite collection of real polynomials in n variables, whose degrees are bounded by d. A closed semialgebraic set, $\mathcal{S} \subset \mathbb{R}^n$, is given by an expression of the form

$$\mathcal{S} = \bigcup_{j=1}^{k} \bigcap_{m \in Q_j} \{x \in \mathbb{R}^n : p_m s_{jm} 0\}, \tag{4.21}$$

where $Q_j \subset \{1, ..., s\}$ and $s_{jm} \in \{\leq, =, \geq\}$ are arbitrary. Moreover, we say that $\mathcal{S}$ has degree at most sd, and its degree is the infimum of sd over all representations as in (4.21).

Theorem 4.1.1 ([5] Corollary 9.6)**.** *Let $\mathcal{S} \subset [0,1]^n$ be semialgebraic of degree B. Let $\epsilon > 0$ be a small number and $|\mathcal{S}| < \epsilon^n$, where $|\cdot|$ represents Lebesgue measure. Then there exists $C = C(n)$ such that $\mathcal{S}$ may be covered by at most $B^C \epsilon^{1-n}$ ϵ-balls.*

Using these results for general semialgebraic sets, we can obtain sublinear bounds for the shift and skew-shift.

Theorem 4.1.2. *Let T_ω represent either the shift or the skew-shift. Let $\mathcal{S} \subset [0,1]^n$ be semialgebraic of degree B and $|\mathcal{S}| < \eta$. Let $\omega \in DC(A, c)$ (when considering the shift) or $\omega \in SDC(A, c)$ (when considering the skew-shift), and let N be an integer such that*

$$B \leq N < \frac{1}{\eta}.$$

Then there is $C = C(n)$ and $\delta = \delta(\omega)$ such that for any $x_0 \in \mathbb{T}^n$,

$$\#\left\{k = 1, ..., N : T_\omega^k(x_0) \in \mathcal{S}\right\} < N^{1-\delta} B^C. \tag{4.22}$$

Remark 24. While the above result holds for any $N \geq B$, the resulting bound, $N^{1-\delta} B^C$

will only be smaller than N when $\ln(N) > C\ln(B)$, where $C = C(n, \delta)$.

The case where T_ω is the shift is due to Bourgain [[5] Corollary 9.7] and the case for the skew-shift follows from Lemma 8.4 in [59]. The particular δ obtained differs between the shift and skew-shift, as we will show in Section 4.3.

Remark 25. Different authors obtain different values of δ for the shift (c.f. [59] and [32]) depending on what method they use. In Section 4.3 we explicitly estimate δ for the shift using the approach from [5], which turns out to be better than the values from [59] and [32] when $\omega \in DC(A, c), A \gg 1$ (i.e. very weak Diophantine conditions).

4.1.4 Large deviation theorems

Throughout the section, we will assume that the energy, E, is such that $L(E) > 0$.

The estimate we will obtain in section 4.3 will rely on estimates on the measure of semialgebraic sets. The particular semialgebraic sets we are interested in are the set of phases, x, for which $\frac{1}{n}\left\|A_n^{f,E}(x)\right\|$ converges to $L(E)$ slowly. To this end, we recall the following large deviation theorems, the first of which is due to Bourgain, Goldstein, and Schlag, and the second is due to S. Klein, which quantitatively measure the rate of convergence.

For the shift model with non-constant analytic potential, there is a well-known large deviation estimate.

Theorem 4.1.3 ([5] Theorem 5.5)**.** *Assume $\omega \in \mathbb{T}^\nu$ satisfies $\omega \in DC(A, c)$. Let f be a non-constant real analytic function on $\mathbb{T}^\nu$. Then there is $\alpha = \alpha(A) > 0$ such that*

$$\left|\left\{x \in \mathbb{T}^\nu : \left|\frac{1}{N}\ln\left\|A_N^{f,E}(x)\right\| - L_N(E)\right| < N^{-\alpha}\right\}\right| < e^{-N^\alpha}. \tag{4.23}$$

This can also be seen as a consequence of the more general large deviation estimate we obtain in Chapter 2.

For the shift model with Gevrey class potential and skew shift with analytic or Gevrey class potential satisfying a transversality condition, we have:

Theorem 4.1.4 ([54] Theorem 6.1)**.** *Assume $f \in G^{\sigma}(T^{\nu})$ satisfies a transversality condition, and suppose $f = \lambda f_0$, for some $\lambda \in \mathbb{R}$ and $f_0 \in G^{\sigma}$ fixed. Let $\omega \in DC(c, A)$ (for the shift) or $\omega \in SDC(A, c), A \leq 2$ (for the skew-shift). Then there exists $\lambda_0 = \lambda_0(f_0, A)$ such that for every fixed $|\lambda| > \lambda_0$ and for every energy E we have*

$$\left|\left\{x \in \mathbb{T}^{\nu} : \left|\frac{1}{N} \ln \left\|A_N^{f,E}(x)\right\| - L_N(E)\right| < N^{-\tau}\right\}\right| < e^{-N^{\alpha}}, \tag{4.24}$$

for some constants $\tau, \alpha > 0$ depending only on ν, and every $N > N_0(\lambda, c, f_0, \sigma, \nu)$.

4.2 Transport exponents

Our first goal in this section is to relate $\beta_{\ln}^{\pm}(p)$ to $S_{\ln}^{\pm}$. Observe that, if $S_{\ln}^{-}(\alpha) < +\infty$ we have:

$$P((\ln T)^{\alpha} - 2, T) > (\ln T)^{-S_{\ln}^{-}(\alpha)-} \tag{4.25}$$

and so

$$\langle |X|^p(T)\rangle = \sum_{n=-\infty}^{+\infty} (|n|+1)^p a(n,T) \tag{4.26}$$

$$\geq \sum_{|n|>(\ln T)^\alpha-2} (|n|+1)^p a(n,T) \tag{4.27}$$

$$\geq C(\ln T)^{\alpha p} P((\ln T)^\alpha - 2, T) \tag{4.28}$$

$$\geq C(\ln T)^{\alpha p} (\ln T)^{-S^-_{\ln}(\alpha)-} \tag{4.29}$$

$$= C(\ln T)^{\alpha p - S^-_{\ln}(\alpha)-} \tag{4.30}$$

and thus

$$\beta^-_{\ln}(p) \geq \alpha - \frac{S^-_{\ln}(\alpha)}{p}. \tag{4.31}$$

A similar analysis for $S^+_{\ln}(\alpha) < +\infty$ shows

$$\beta^+_{\ln}(p) \geq \alpha - \frac{S^+_{\ln}(\alpha)}{p}. \tag{4.32}$$

Together, this shows that

$$\beta^\pm_{\ln}(+\infty) \geq \alpha^\pm_{\ln}. \tag{4.33}$$

On the other hand, it is possible to use $\alpha^\pm_{\ln}$ to bound $\beta^\pm_{\ln}(+\infty)$ from above:

Theorem 4.2.1. *Let H be an operator of the form* (**??**) *with bounded potential and suppose that for some $\eta > 0$, and for all $p > 0$, we have*

$$\langle |X|^p(T)\rangle < C_p(\ln T)^{\eta p}. \tag{4.34}$$

Then $0 \leq \alpha_{\ln}^{\pm} \leq \eta$ *and*

$$\beta_{\ln}^{\pm}(+\infty) \leq \alpha_{\ln}^{\pm}. \tag{4.35}$$

Remark 26. We can replace (4.34) with the condition $\beta_{\ln}^{+}(p) < \eta$ for every $p > 0$.

Remark 27. The following proof uses the same ideas as the proof of Theorem 4.1 in [27].

Proof. The bound $0 \leq \alpha_{\ln}^{\pm} \leq \eta$ follows from the computation performed above, so we will focus on proving (4.35).

Fix $0 \leq \alpha \leq \alpha_{\ln}^{+}, \epsilon > 0$ and consider the following:

$$\langle |X|^p(T) \rangle = \sum_{n=-\infty}^{+\infty} (|n|+1)^p a(n,T) \tag{4.36}$$

$$= \sum_{|n| \leq (\ln T)^{\alpha} - 2} + \sum_{(\ln T)^{\alpha} - 2 < |n| \leq (\ln T)^{\alpha_{\ln}^{+} + \epsilon/2}} \tag{4.37}$$

$$+ \sum_{(\ln T)^{\alpha_{\ln}^{+} + \epsilon/2} < |n| \leq (\ln T)^{\eta + \epsilon}} + \sum_{(\ln T)^{\eta + \epsilon} < |n|}. \tag{4.38}$$

Let us label these sums 1 - 4. A few notes before we start bounding these sums. First, we will assume $\alpha > 0$. If $\alpha = 0$, then we may proceed by removing the second sum and replacing α with $\alpha_{\ln}^{+}$ in the first sum. Second, if $\alpha_{\ln}^{+} = \eta$, then the third sum is unnecessary.

We can bound sum 1 by

$$\sum_{|n| \leq (\ln T)^{\alpha} - 2} < C(\ln T)^{\alpha p}.$$

We can bound sum 2:

$$\sum_{(\ln T)^{\alpha} - 2 < |n| \leq (\ln T)^{\alpha_{\ln}^{+} + \epsilon/2}} \leq C(\ln T)^{p\alpha_{\ln}^{+} + p\epsilon/2} P((\ln T)^{\alpha} - 2, T).$$

If $\alpha_{\ln}^+ = \eta$, then sum 3 is unnecessary. If $\alpha_{\ln}^+ < \eta$, then we can bound sum 3 by

$$\sum_{(\ln T)^{\alpha_{\ln}^+ + \epsilon/2} < |n| \leq (\ln T)^{\eta+\epsilon}} \leq (\ln T)^{\eta p + p\epsilon} P((\ln T)^{\alpha_{\ln}^+ + \epsilon/2}, T),$$

and by definition of $\alpha_{\ln}^+$, the right hand side goes to 0, so it can be further bounded by some constant C.

Finally, we have the bound for sum 4. For any m,

$$\begin{aligned}\sum_{(\ln T)^{\eta+\epsilon} < |n|} &\leq (\ln T)^{-(\eta+\epsilon)m} \left\langle |X|^{p+m}(T) \right\rangle \\ &\leq C_{p+m} (\ln T)^{-(\eta+\epsilon)m} (\ln T)^{\eta(p+m)}.\end{aligned}$$

By taking $m > \eta p/\epsilon$, we have

$$\sum_{(\ln T)^{\eta+\epsilon} < |n|} < C.$$

Putting everything together, we have

$$\langle |X|^p(T) \rangle < C + C(\ln T)^{p\alpha} + C(\ln T)^{p\alpha_{\ln}^+ + p\epsilon/2} P((\ln T)^{\alpha} - 2, T). \tag{4.39}$$

Taking ln throughout, and letting

$$f(T, p, \alpha, \epsilon) = \max\left\{\alpha p \ln\ln(T), (p\alpha_{\ln}^+ + \frac{p\epsilon}{2}) \ln\ln(T) + \ln(P((\ln T)^{\alpha} - 2, T))\right\},$$

we have

$$\ln\left(\langle |X|^p(T) \rangle\right) < C + f(T, p, \alpha, \epsilon) \tag{4.40}$$

so

$$\beta_{\ln}^{+}(p) \leq \max\left\{\alpha, \alpha_{\ln}^{+} + \frac{\epsilon}{2} - \frac{S_{\ln}^{+}(\alpha)}{p}\right\}. \tag{4.41}$$

Taking $p \to \infty$ yields our result for $\beta_{\ln}^{+}(p)$. The proof for $\beta_{\ln}^{-}(p)$ is similar.

□

The major roadblock to using this result to obtain bounds on $\beta_{\ln}^{\pm}(p)$ is that it requires an a priori finite estimate on $\beta_{\ln}^{\pm}(p)$ for every $p > 0$, which we do not have in general. This differs from the situation arising when we merely want to bound $\beta^{\pm}(p)$, since in that case we usually have a trivial ballistic upper bound: $\beta^{\pm}(p) \leq 1$. To remedy this, we have the following, which provides a sufficient condition for $\beta^{\pm}(p) < C < \infty$ for every $p > 0$.

Theorem 4.2.2. *Let H be an operator of the form* (**??**) *with bounded potential and suppose that $\alpha_{\ln}^{\pm} < +\infty$. Moreover, suppose that, for some $\xi > 0$,*

$$P((\ln T)^{\xi}, T) = O(T^{-a}) \tag{4.42}$$

for every $a > 1$, and for some $\gamma < \infty$ we have

$$\langle |X|^{p}(T) \rangle < C_p T^{\gamma p}. \tag{4.43}$$

Then for some $\eta < \infty$ (4.34) *holds.*

Remark 28. As noted above, (4.43) always holds with $\gamma = 1$ when the potential is bounded.

Proof. The proof proceeds the same as before, expressing $\langle |X|^{p}(T) \rangle$ as a sum, and decomposing that sum into four further sums, except we take η to be ξ. With this modification, the bounds for sums 1 - 3 still hold, but we need to be more careful with the fourth sum.

We have:

$$\sum_{(\ln T)^{\xi+\epsilon}<|n|} = \sum_{(\ln T)^{\xi+\epsilon}<|n|\leq T^{\gamma+\epsilon}} + \sum_{T^{\gamma+\epsilon}<|n|}. \tag{4.44}$$

Let us denote the first sum by I and the second sum by II. We can bound sum I by

$$\sum_{(\ln T)^{\xi+\epsilon}<|n|\leq T^{\gamma+\epsilon}} \leq T^{(\gamma+\epsilon)p} P((\ln T)^{\xi+\epsilon}, T) \tag{4.45}$$

$$\leq T^{p(\gamma+\epsilon)-a} \tag{4.46}$$

for large T, where we can take any $a > 1$. Taking $a > p(\gamma+\epsilon)$, we have $\sum_{(\ln T)^{\xi+\epsilon}<|n|\leq T^{\gamma+\epsilon}} < C$. For sum II, we have

$$\sum_{T^{\gamma+\epsilon}<|n|} = T^{-m(\gamma+\epsilon)} \sum_{T^{\gamma+\epsilon}<|n|} (|n|+1)^{p+m} a(n,T) \tag{4.47}$$

$$\leq T^{-m(\gamma+\epsilon)} \left\langle |X|^{p+m}(T) \right\rangle \tag{4.48}$$

$$\leq C_{m+p} T^{(p+m)\gamma - m(\gamma+\epsilon)} < C. \tag{4.49}$$

for $m > \gamma p/\epsilon$. With these two bounds, we may proceed as before to conclude that $\beta^+_{\ln}(p) < C < +\infty$. □

We will now turn our attention to the proof of Theorem 4.0.5. We start with a lemma due to Damanik and Tcheremchantsev:

Lemma 4.2.1 ([17] Theorem 7). *Suppose H is of the form (**??**), where V is a bounded real-valued function, and $K \geq 4$ is such that $\sigma(H) \subset [-K+1, K-1]$. Then*

$$P_r(N,T) \lesssim e^{-cN} + T^3 \int_{-K}^{K} \left(\max_{1\leq n\leq N} \left\| A_n^{f,E+i/T} \right\|^2 \right)^{-1} dE \tag{4.50}$$

$$P_l(N,T) \lesssim e^{-cN} + T^3 \int_{-K}^{K} \left(\max_{1\leq n\leq N} \left\| A_{-n}^{f,E+i/T} \right\|^2 \right)^{-1} dE \tag{4.51}$$

With this lemma, and the preceding theorems, we will prove Theorem 4.0.5.

Proof of Theorem 4.0.5 (a). In light of Theorem 4.2.1, it suffices to show that $\alpha_{\ln}^{\pm} \leq \gamma$. We will do this for $\alpha_{\ln}^{+}$ and observe that the proof for $\alpha_{\ln}^{-}$ is the same.

Using (4.8) and Lemma 4.2.1, since $\gamma > 1$, we have

$$P((\ln T)^{\gamma}, T) = O(T^{-\delta}) \tag{4.52}$$

for every $\delta < \infty$. Thus

$$\frac{\ln\left(P((\ln T)^{\gamma}, T)\right)}{\ln\ln(T)} \leq \frac{-\delta \ln(T)}{\ln\ln(T)}. \tag{4.53}$$

We are left with

$$S_{\ln}^{+}(\gamma) = +\infty, \tag{4.54}$$

so $\alpha_{\ln}^{+} \leq \gamma$. □

We will now prove the second part.

Proof of Theorem 4.0.5(b). Fix H_1 and H_2 of the form (**??**) with bounded potentials, v_1 and v_2, and let $K \geq 4$ be such that $\sigma(H_i) \subset [-K+1, K-1]$ for $i = 1, 2$. Denote the corresponding transfer matrices by A^{v_1} and A^{v_2} and the corresponding transport exponents by $\beta_{\ln,1}^{\pm}(p), \beta_{\ln,2}^{\pm}(p)$. Suppose that there is $\gamma < \infty$ such that, for every $M > 0$ and $T > T_0(M)$,

$$\int_{-K}^{K} \left(\max_{0 \leq |n| \leq (\ln T)^{\gamma}} \|A_n^{v_1}(x, E + i/T)\|^2 \right)^{-1} dE \leq C T^{-M}.$$

Moreover, suppose that there exists $A > 0$ such that for all $E \in [-K+1, K-1], 0 < \epsilon \leq 1$, and $|n| \leq \ln(\epsilon^{-1})$,

$$\epsilon^{A} \left\| A_n^{v_1,E+i\epsilon} \right\| \lesssim \left\| A_n^{v_2,E+i\epsilon} \right\| \lesssim \epsilon^{-A} \left\| A_n^{v_1,E+i\epsilon} \right\|. \tag{4.55}$$

Let $P_1(N,T)$ and $P_2(N,T)$ be the corresponding outside probabilities.

Observe, by Lemma 4.2.1 and our assumptions above, that for any $M > 0$, and $T > T_0(M)$,

$$P_2((\ln T)^{\gamma}, T) \leq e^{-C(\ln T)^{\gamma}} + T^3 \int \int_{-K}^{K} \left(\max_{0 \leq |n| \leq (\ln T)^{\gamma}} \| A_n^{v_2}(x, E + i/T) \|^2 \right)^{-1} dE \tag{4.56}$$

$$\leq e^{-C(\ln T)^{\gamma}} + T^{3+A} \int \int_{-K}^{K} \left(\max_{0 \leq |n| \leq (\ln T)^{\gamma}} \| A_n^{v_1}(x, E + i/T) \|^2 \right)^{-1} dE \tag{4.57}$$

$$\leq CT^{-M}, \tag{4.58}$$

and thus

$$\frac{\ln(P_2((\ln T)^{\gamma}, T))}{\ln \ln(T)} \leq \frac{-M \ln(T) + \ln(C)}{\ln \ln(T)}. \tag{4.59}$$

We conclude as before.

□

4.3 Semialgebraic sets

Here we obtain an explicit estimate on the δ from Theorem 4.1.2.

Theorem 4.3.1. *When T_ω is the shift on $\mathbb{T}^n$, and $\omega \in DC(A,c)$, we can take $\delta \leq \frac{1}{A+n}$ in Theorem 4.1.2. When T_ω is the skew-shift on $\mathbb{T}^n$, and $\omega \in SDC(A,c)$, we can take $\delta < \frac{1}{n2^{n-1}(1+\epsilon)}$ for any $\epsilon > 0$.*

Remark 29. The general idea of the proof is the same in both cases. We first prove a bound of the form $\#\{k = 1, ..., N : T_\omega(x_0) \in B_\epsilon\} \leq N^{-\zeta}$, where B_ϵ is a ball of radius ϵ. Then we use the covering lemma for semialgebraic sets (Theorem 4.1.1) to cover the desired semialgebraic set by by ϵ-balls. Because of this similarity, we will only give a proof for the shift. The details for the skew-shift can be found in [59] (Lemma 8.4 and Theorem 8.7).

Proof. Fix $\epsilon = N^{-\delta}$ and let $\chi(x) = \chi_{B(0,\epsilon)}(x)$ be the characteristic function of the ball of radius ϵ centered at 0. Let $R = \frac{1}{10\epsilon}$ and let

$$F_R(x_j) = \frac{1}{R}\left(\frac{\sin(Rx/2)}{\sin(x/2)}\right)^2 = \sum_{|m|<R}\left(1 - \frac{|m|}{R}\right)e^{imx_j} = \sum_{|m|<R}\widehat{F_R}(m)e^{imx_j}$$

be the usual Fejer kenel on $\mathbb{R}$.

If $\chi(x) = 0$, then $\chi(x) \leq CR^{-n}\prod_{j=1}^n F_R(x_j)$ holds trivially. On the other hand, by our choice of ϵ and R, if $\chi(x) = 1$, then $F_R(x_j) \sim R$, since, for small x_j,

$$F_R(x_j) = \frac{1}{R}\left(\frac{\sin(Rx_j/2)}{\sin(x_j/2)}\right)^2 \sim \frac{1}{R}R^2 = R,$$

and we also have $\chi(x) \leq CR^{-n}\prod_{j=1}^n F_R(x_j)$. Thus we have

$$\begin{aligned}\prod_{j=1}^n F_R(x_j) &= \prod_{j=1}^n \sum_{|m|<R}\widehat{F_R}(m)e^{imx_j} \\ &= \sum_{|m|<R}\widehat{F_R}(m_1)\cdots\widehat{F_R}(m_n)e^{im\cdot x}.\end{aligned} \tag{4.60}$$

Hence, if we set $m = (m_1, ..., m_n)$, we have

$$\sum_{j=1}^{N} \chi(x_0 + j\omega) \leq \; CR^{-n} \sum_{j=1}^{N} \sum_{|m_k|<R; 1\leq k\leq n} \widehat{F_R}(m_1) \cdots \widehat{F_R}(m_n) e^{im\cdot(x_0+j\omega)} \tag{4.61}$$

$$\leq \; CR^{-n} \sum_{|m_k|<R; 1\leq k\leq n} \left(\widehat{F_R}(m_1) \cdots \widehat{F_R}(m_n) e^{im\cdot x} \left(\sum_{j=1}^{N} e^{ijm\cdot\omega} \right) \right) \tag{4.62}$$

$$\leq \; CR^{-n} \sum_{|m_k|<R; 1\leq k\leq n} \left(\widehat{F_R}(m_1) \cdots \widehat{F_R}(m_n) \left| \sum_{j=1}^{N} e^{ijm\cdot\omega} \right| \right). \tag{4.63}$$

At this point, we can split the sum into two parts: either $m_k = 0$ for all $1 \leq k \leq n$, or at least one $m_k \neq 0$. Thus we can write (4.63) = (4.64) + (4.65), where (4.64) and (4.65) are given by

$$CR^{-n} \widehat{F_R}(0)^n \left| \sum_{j=1}^{N} e^{ij0\cdot\omega} \right| \tag{4.64}$$

and

$$CR^{-n} \sum_{0\leq|m_k|<R; 1\leq k\leq n; \text{ some } m_k\neq 0} \left(\widehat{F_R}(m_1) \cdots \widehat{F_R}(m_n) \left| \sum_{j=1}^{N} e^{ijm\cdot\omega} \right| \right). \tag{4.65}$$

Since $0 < \widehat{F_R}(m) \leq 1$ and $\left| \sum_{j=1}^{N} e^{ijm\cdot\omega} \right| \leq N$, we have for any x_0

$$\begin{aligned}
\sum_{j=1}^{N} \chi(x_0 + j\omega) &\leq CR^{-n}N + CR^{-n} \sum_{0<|m|<R} \left| \sum_{j=1}^{N} e^{ijm\cdot\omega} \right| \\
&= CR^{-n}N + CR^{-n} \sum_{0<|m|<R} \left| \frac{1 - e^{iNm\cdot\omega}}{1 - e^{im\cdot\omega}} \right| \\
&\leq CR^{-n}N + CR^{-n} \sum_{0<|m|<R} 2|1 - e^{im\cdot\omega}|^{-1} \\
&\leq CR^{-n}N + C \max_{0<|m|<R} 2|1 - e^{im\cdot\omega}|^{-1}.
\end{aligned}$$

Since $\omega \in DC(c, A)$, we know $\|m \cdot \omega\| > c|m|^{-A}$, for every $m \neq 0$, so $|1 - e^{im\cdot\omega}|^{-1} \lesssim R^A$, and we conclude

$$\begin{aligned}\sum_{j=1}^{N} \chi(x_0 + j\omega) &\leq CR^{-n}N + CR^A \\ &\leq CN(R^{-n} + R^A N^{-1}) \\ &\leq CN(\epsilon^n + \epsilon^{-A}N^{-1}).\end{aligned}$$

Now, if we take $\delta = \frac{1}{n+A}$, then by our choice of ϵ we have

$$\begin{aligned}\epsilon^{-A}N^{-1} &= \epsilon^{-A}\epsilon^{A+n} \\ &= \epsilon^n,\end{aligned}$$

so

$$\sum_{j=1}^{N} \chi(x_0 + j\omega) \leq CN\epsilon^n.$$

We conclude the proof by observing that, by Theorem 4.1.1, it is possible to cover $\mathcal{S}$ using no more than $B^C\epsilon^{1-n}$ ϵ-balls, where $C = C(n)$. Thus the above computation shows that

$$\begin{aligned}\#\left\{k = 1, ..., N : x_0 + k\omega \in \mathcal{S}\right\} &\leq CN\epsilon^n B^C \epsilon^{1-n} \\ &= CNB^C\epsilon \\ &\leq N^{1-\delta}B^C.\end{aligned}$$

For the skew-shift, we have, by Lemma 8.3 and Theorem 8.7 from [59], that for any $\epsilon' > 0$,

$$\#\left\{k = 1, ..., N : T_\omega^k(x_0) \in B_\epsilon\right\} \leq CN^{-\frac{1}{2^{n-1}(1+\epsilon)}+\epsilon'}.$$

Applying Theorem 4.1.1, we have

$$\#\left\{k=1,...,N : T_\omega^k(x_0)\in\mathcal{S}\right\}\le CB^C\epsilon^{1-n}N^{-\frac{1}{2^{n-1}(1+\epsilon)}+\epsilon'}$$

$\square$

4.4 Technical lemmas

We will prove our results for right cocycles and observe that the exact same arguments establish the same results for left cocycles.

Let us define

$$V_k^f(E,a):=\left\{x\in\mathbb{T}^\nu : \frac{1}{k}\ln\left\|A_k^{f,E}(x)\right\|\ge a\right\}.$$

We will begin with the following lemma, which reduces everything to the study of semialgebraic sets. Fix $\tau<1$ and $1-\tau/16>a>c>d>1-\tau/8>1-\tau$.

Lemma 4.4.1. *Let $f\in G^\sigma(\mathbb{T}^\nu)$. There is some $k_\tau(E)<\infty$ so that for $k>k_\tau(E)$ and $|E-z|<e^{-\frac{k\tau L(E)}{\|f\|_\infty}}$, we can find $N_1<\infty$ so that we have the following sequence of inclusions:*

$$V_k^f(E,aL(E))\subset V_k^{\tilde f_{N_1}}(E,cL(E))\subset V_k^f(z,dL(E)) \tag{4.66}$$

where $\tilde f_{N_1}(x)$ is a certain polynomial of degree N_1, so $V_k^{\tilde f_{N_1}}(E,cL(E))$ is semialgebraic of degree at most kN_1.

Remark 30. We may take $N_1(k)\sim k^{\sigma\nu+}$ in the above lemma.

Proof. Let us fix $k \in \mathbb{N}$ large and $\epsilon > 0$ small. First, since $f \in G^\sigma(\mathbb{T}^\nu)$, we know that

$$|\hat{f}(n)| \leq C_1 e^{-|n|^{1/(\sigma+)}}. \tag{4.67}$$

Let $f_{N_0}(x) = \sum_{|n| \leq N_0} \hat{f}(n) e^{in \cdot x}$. For $N_0 \geq k^{\sigma+\epsilon}$, we have

$$|f(x) - f_{N_0}(x)| \leq e^{-k^{1+\epsilon}} \leq e^{-k(1-c)L(E)}.$$

Now for such N_0, there exists a polynomial $\tilde{f}_{N_1}(x)$ of degree N_1 with $N_1 = k^{\sigma\nu+\epsilon}$ so that

$$|f_{N_0}(x) - \tilde{f}_{N_1}(x)| \leq e^{-k(1-d)L(E)}.$$

This can be seen by approximating $e^{in_j x_j}$ by a Taylor polynomial of degree $k^{\sigma+}$ and then bounding the error as usual. Note that these two inequalities hold for k sufficiently large (dependent only on the dimension ν and ϵ).

By upper semicontinuity, compactness considerations, and a standard telescoping argument, we have

$$\left\| A_k^{f,E}(x) - A_k^{f_{N_0},E}(x) \right\| < e^{-k^{1+\epsilon}} \tag{4.68}$$

$$\left\| A_k^{f,E}(x) - A_k^{\tilde{f}_{N_1}(x),z} \right\| < e^{-k(1-d+\tau)L(E)} e^{k(L(E)+\epsilon)} < e^{k(L(E)/2+\epsilon)} \tag{4.69}$$

for k sufficiently large and $|E - z| < e^{-\frac{k\tau(L(E)+\epsilon)}{\|f\|_\infty}}$. The first inclusion can now be established by observing that, for $x \in V_k^f(E, aL(E))$, we have

$$\begin{aligned} \left\| A_k^{f_{N_0},E}(x) \right\| &\geq \left\| A_k^{f,E}(x) \right\| - \left\| A_k^{f,E}(x) - A_k^{f_{N_0},E}(x) \right\| \\ &\geq e^{ckL(E)}. \end{aligned}$$

The other inclusion is proved in the same way.

The semialgebraic bound on $V_k^{\tilde{f}_{N_1}}(E, cL(E))$ follows from the fact that $V_k^{\tilde{f}_{N_1}}(E, cL(E))$ is given by a single inequality involving a polynomial of degree kN_1. $\square$

Now we have

Lemma 4.4.2. *Let* k, E, z, d, *and* $V_k^f(z, dL(E))$ *be as in Lemma 4.4.1. Then* $|V_k^f(z, dL(E))| > 1/2$, *where* $|\cdot|$ *represents Lebesgue measure.*

Proof. By definition of $L(E)$ we have

$$\begin{aligned} L(E) &\leq \frac{1}{k} \int \ln \left\| A_k^{f,E}(x) \right\| dx \\ &\leq |V_k^f(E, aL(E))|(L(E) + \epsilon) + (1 - |V_k^f(E, aL(E))|)(aL(E)) \\ &\leq |V_k^f(E, aL(E))|((1 - a)L(E) + \epsilon) + aL(E). \end{aligned}$$

Thus, by choosing ϵ appropriately (which can be done by upper semicontinuity and taking $k > k_0(\epsilon)$ sufficiently large), and the fact that $a < 1$, we have

$$|V_k^f(E, aL(E))| \geq \frac{1}{2}. \tag{4.70}$$

The set inclusion proved above now yields the result. $\square$

Our next goal is to show that for T_ω either the shift or skew-shift, there is some $N_k < \infty$ such that, for every $x \in \mathbb{T}^\nu, T_\omega(x) \in V_k^f(z, dL(E))$ for some $1 \leq j \leq N_k$, and then obtain the required transfer matrix bounds. We will split the remaining argument up into three cases: the shift with $\nu = 1$, the shift with $\nu > 1$, and the skew shift with $\nu > 1$.

4.5 The case $\nu = 1$

Our goal is to first establish the following estimates. Let d be as in Lemma 4.4.1.

Theorem 4.5.1. *Let $f \in G^{\sigma}(\mathbb{T}), \omega \in \mathbb{R}\backslash\mathbb{Q}$, and $E \in \mathbb{C}$ such that $L(E) > 0$. For any $0 < \tau < 1$, there exist $k_\tau = k_\tau(E) < \infty$ such that for any $\epsilon > 0, k > k_\tau$, and $x \in \mathbb{T}$, there is $1 \leq j \leq Ck^{1+\sigma+\epsilon}$ so that for any $z \in \mathbb{C}$ with $|z - E| < e^{-\frac{\tau k L(E)}{\|f\|_\infty}}$ we have*

$$\left\| A_k^{f,z}(x + j\omega) \right\|^2 > e^{dkL(E)}. \tag{4.71}$$

Theorem 4.5.2. *Fix $\epsilon > 0$. Let $f \in G^{\sigma}(\mathbb{T}), \omega \in DC(A, c)$, and $L(E) > 0$. Then for any $\xi, \zeta > 1$, there is $C, c > 0$ and $T_E < \infty$ such that for $T > T_E$,*

$$\inf \left\{ \min_{\iota = \pm 1} \max_{1 \leq \iota m \leq C(\ln T)^{\zeta(1+\sigma+\epsilon)}} \left\| A_m^{f,z}(x) \right\|^2 T^{-\xi} \right\} > c \tag{4.72}$$

where the infimum is over all $x \in \mathbb{T}$ and $z \in \mathbb{C}$ with $|z - E| < T^{-\zeta}$. Moreover, T_E is uniformly bounded below for E in compact sets with positive $L(E)$.

In particular, for $E \in [-K, K]$, we have $\max_{1 \leq n \leq C(\ln T)^{\zeta(1+\sigma)}} \left\| A_n^{f,E+i/T} \right\|^2 \geq cT^{\xi}$ for every $\xi > 1$ and large T.

If $\omega \in \mathbb{R}\backslash\mathbb{Q}$, then the above holds for a sequence, T_n for $n > n_E$ for all E, and for $n > n_0$ for $E \in [-K, K]$.

When $\nu = 1$, we can write ω as a continued fraction. Let $\frac{p_n}{q_n}$ be the continued fraction approximation of ω. We then have the following lemma.

Lemma 4.5.1 (Lemma 9 from [38])**.** *Suppose $\Delta \subset \mathbb{T}$ is an interval with $|\Delta| > 1/q_n$. Then for every $x \in \mathbb{T}$, there exists $1 \leq j \leq q_n + q_{n-1} - 1$ such that $x + j\omega \in \Delta$.*

Lemmas 4.4.1 and 4.4.2, along with Remark 30, imply $V_k^f(z, dL(E))$ contains an open set,

Δ, of measure

$$\frac{1}{2k^{1+\sigma+\epsilon}} \lesssim |\Delta|.$$

Now if we take $k > Cq_n^{1/(1+\sigma+\epsilon)}$, we have $|\Delta| > 1/q_n$, and so, by Lemma 4.5.1,

Lemma 4.5.2. *Let f, E, z, and d be as in Lemma 4.4.1. For $k \sim q_n^{1/(1+\sigma+\epsilon)}$, there exists $1 \leq j \lesssim k^{1+\sigma+\epsilon}$ such that $x + j\omega \in V_k^f(z, dL(E))$.*

Theorem 4.5.1 now follows by the set inclusion we proved in the previous section.

Since the proof of Theorem 4.5.2 is identical to the proof of Theorem 4.6.2 in the next section, we omit it and refer readers to the next section for the details.

With Theorem 4.5.2, we can prove Theorem 4.0.1.

Proof of Theorem 4.0.1. Let us begin by fixing $x \in \mathbb{T}$ and $f \in G^\sigma(\mathbb{T})$. Moreover suppose that $L(E) > 0$ for every $E \in \mathbb{R}$. First, we will consider the case $\omega \in DC(A, c)$. Fix $\epsilon > 0$ and set $\gamma = 1 + \sigma$. The hypotheses of Theorem 4.5.2 are satisfied, and we can combine the conclusion of Theorem 4.5.2 with the conclusion of Lemma 4.2.1 to obtain

$$P((\ln T)^{\gamma+\epsilon} - 2, T) \leq e^{-C(\ln T)^{\zeta(\gamma+\epsilon)}} + CT^{-\delta}$$

for every $\zeta, \delta > 1$. Since $\gamma > 1$, we can further bound this by

$$P((\ln T)^{\gamma+\epsilon} - 2, T) \leq CT^{-\delta},$$

using a different constant C. As before, we obtain $\alpha_{\ln}^+ \leq 1 + \sigma < +\infty$.

We can now appeal to Theorem 4.2.2 to establish the hypotheses of Theorem 4.2.1, so $\beta_{\ln}^+(p) \leq \alpha_{\ln}^+ \leq 1 + \sigma$.

Now we turn to the case $\omega \in \mathbb{R}\backslash\mathbb{Q}$. We can appeal to Theorem 4.5.2 to obtain the above for a sequence $T_n \to \infty$. With a sequence, we have analagous statements as above, but for S^- and α^-. Thus we obtain $\beta^-_{\ln}(p) \le 1 + \sigma$.

□

4.6 The case $\nu > 1$

As in the case $\nu = 1$, our goal is to first establish the following estimates:

Theorem 4.6.1. *Let $f = \lambda f_0 \in G^\sigma(T^\nu), \nu > 1, \omega \in DC(A, c), \lambda > \lambda_0(f_0, \omega)$, and $E \in \mathbb{R}$ such that $L(E) > 0$. For any $0 < \tau < 1$, there exist $k_\tau = k_\tau(E) < \infty, \delta = \delta(\omega, \nu)$, and $\gamma = \gamma(\sigma, \nu, \delta)$ such that for any $\epsilon > 0, k > k_\tau$, and $x \in \mathbb{T}^\nu$, there is $1 \le j \le k^{\gamma+\epsilon}$ so that for any $z \in \mathbb{C}$ with $|z - E| < e^{-\frac{\tau k L(E)}{\|f\|_\infty}}$ we have*

$$\left\|A_k^{f,z}(x + j\omega)\right\| > e^{k(1-\tau)L(E)}. \tag{4.73}$$

Theorem 4.6.2. *Fix $\epsilon > 0$. Let $f = \lambda f_0 \in G^\sigma(\mathbb{T}^\nu), \nu > 1, \omega \in DC(c, A), \lambda > \lambda_0(f_0, \omega)$, and $L(E) > 0$. Then for any $\xi, \zeta > 1$, there is $c > 0$ and $T_E < \infty$ such that for $T > T_E$,*

$$\inf\left\{\min_{\iota=\pm1} \max_{1\le\iota m\le(\ln T)^{\zeta(\gamma+\epsilon)}} \left\|A_m^{f,z}(x)\right\|^2 T^{-\xi}\right\} > c \tag{4.74}$$

where γ and δ are as above, and the infimum is over all $x \in \mathbb{T}^\nu$ and $z \in \mathbb{C}$ with $|z-E| < T^{-\zeta}$. Moreover, the dependence of T_E on E is through $L(E)$, as in Theorem 4.5.2. Thus, as before, T_E is uniformly bounded below for E in compact sets with positive $L(E)$.

Remark 31. If we consider just $E \in [-K, K]$ in the above theorem, then continuity of $L(E)$, which was established for our situation in [54], and compactness of $[-K, K]$ yields the desired uniform lower bound on T.

When $\nu > 1$, we need to do a bit more work to obtain an analogue of Lemma 4.5.1.

We may appeal to Theorems 4.1.4 and 4.1.2 to obtain:

Lemma 4.6.1. *Let $\omega \in DC(A, c)$. For $f = \lambda f_0 \in G^\sigma(\mathbb{T}^\nu)$, there exists $\lambda_0(f_0, \omega)$ such that, for $\lambda > \lambda_0$ and every $x \in \mathbb{T}^\nu$ there exists $1 \leq j \leq k^{C(\nu+A)(\sigma\nu+1)+}$ such that $x + j\omega \in V_k^{\tilde{f}_{N_1}}(E, cL(E))$.*

Proof. Recall that by Theorem 4.1.4, combined with (4.68), with N_1 as in Lemma 4.4.1, there exists a λ_0 so that, for all $\lambda > \lambda_0$ and $f = \lambda f_0$, we have

$$\left|\left\{x \in \mathbb{T}^\nu : \left|\frac{1}{k} \ln \left\|A_k^{\tilde{f}_{N_1},E}(x)\right\| - L_k(E)\right| > 2k^{-\tau}\right\}\right| < e^{-k^\alpha}. \tag{4.75}$$

This implies

$$\left|\left\{x \in \mathbb{T}^\nu : \frac{1}{k} \ln \left\|A_k^{\tilde{f}_{N_1},E}(x)\right\| - L(E) < -2k^{-\tau}\right\}\right| < e^{-k^\alpha}, \tag{4.76}$$

since $L_k(E) \geq L(E)$. Thus, for k sufficiently large, and $N_1(k) \sim k^{\sigma\nu+}$, by Remark 30,

$$\left|\mathbb{T}^\nu \backslash V_k^{\tilde{f}_{N_1}}(E, cL(E))\right| < e^{-k^\alpha}. \tag{4.77}$$

Since the left hand side is the complement of a semialgebraic set of degree at most kN_1, it is itself semialgebraic of degree at most kN_1. By Theorem 4.3.1, for fixed $0 < \epsilon < \delta = \frac{1}{\nu+A}$, we can thus set $\mathcal{S} = \left(\mathbb{T}^\nu \backslash V_k^{\tilde{f}_{N_1}}(E, cL(E))\right), \eta = e^{-k^\alpha}$, $B = kN_1$, and $N = B^{C/(\delta-\epsilon)}$, and then appeal to Theorem 4.1.2 to obtain, for any $0 < \epsilon < \delta$,

$$\#\left\{1 \leq j \leq N : x + j\omega \in \mathcal{S}\right\} < B^{C\frac{1-\delta}{\delta-\epsilon}} B^C = B^{C\frac{1-\epsilon}{\delta-\epsilon}}. \tag{4.78}$$

Thus, for every $x \in \mathbb{T}^\nu$ there is a $1 \leq j \leq (kN_1)^{C\frac{1-\epsilon}{\delta-\epsilon}} < N^{1-\epsilon}$ so that $x + j\omega \in V_k^{\tilde{f}_{N_1}}(E, cL(E))$. The result now follows from our choice of $N_1 \sim k^{\sigma\nu+}$ in Lemma 4.4.1.

$\square$

Theorem 4.6.1 now follows from the fact that $V_k^{\tilde{f}_{N_1}}(E, cL(E)) \subset V_k^f(z, dL(E))$, and observing that $d > 1 - \tau$, just as in the case $\nu = 1$.

Theorem 4.6.2 can now be proved using Theorem 4.6.1.

Proof of Theorem 4.6.2. Fix $\xi, \zeta > 1$ and $0 < \tau < \frac{\zeta\|f\|_\infty}{\zeta\|f\|_\infty + \xi} < 1$. Consider any $M_k = M_k(\xi, \zeta)$ such that the following holds:

$$e^{k\tau L(E)/(\zeta\|f\|_\infty)} < M_k < e^{k(1-\tau)L(E)/\xi} \tag{4.79}$$

and

$$(\ln M_k)^{(\gamma+\epsilon)\zeta} > k^{\gamma+} + k. \tag{4.80}$$

Both conditions can be satisfied by taking k sufficiently large due to our choice of τ and $\zeta > 1$. Appealing to Theorem 4.6.1, for every $x \in \mathbb{T}^\nu$ there is $1 \leq j \leq (\ln M_k)^{(\gamma+\epsilon)\zeta} - k$ so that for $|z - E| < M_k^{-\zeta}$ we have

$$\left\|A_k^{f,z}(x + j\omega)\right\| \geq M_k^\xi. \tag{4.81}$$

Now recall that, by definition,

$$A_{k+j}^{f,z}(x) = A_k^{f,z}(x + j\omega) A_j^{f,z}(x). \tag{4.82}$$

Moreover, A is an $SL_2(\mathbb{R})$ cocycle, so $\|A_k\| = \left\|A_k^{-1}\right\|$, and thus

$$\left\|A_k^{f,z}(x + j\omega)\right\| \leq \left\|A_{k+j}^{f,z}(x)\right\| \left\|A_j^{f,z}(x)\right\|. \tag{4.83}$$

This together with (4.81) implies

$$\max_{1\leq j\leq(\ln M_k)^{(\gamma+\epsilon)\zeta}-k}\left\{\left\|A^{f,z}_{k+j}(x)\right\|,\left\|A^{f,z}_{j}(x)\right\|\right\}\geq M_k^{\xi}. \tag{4.84}$$

Thus we must have

$$\max_{1\leq j\leq(\ln M_k)^{(\gamma+\epsilon)\zeta}}\left\|A^{f,z}_{j}(x)\right\|^2\geq M_k^{\xi}. \tag{4.85}$$

It is not difficult to show that for some $T_0 = T_0(E) < \infty$, and any $T > T_0$, we can find $k < \infty$ and $M_k = T$ satisfying (4.79) and (4.80). Thus, we have, for any $\xi, \zeta > 1$,

$$\inf_{|z-E|<T^{-\zeta};x\in\mathbb{T}^\nu}\left\{\max_{1\leq \iota j\leq(\ln T)^{(\gamma+\epsilon)\zeta}}\left\|A^{f,z}_{j}(x)\right\|^2 T^{-\xi}\right\}>c>0. \tag{4.86}$$

It remains to show that we can also use the same M_k to obtain an analogous bound for the left transfer matrix. Note that for an ergodic invertible cocycle, the Lyapunov exponent of the forward cocycles and the Lyapunov exponent of the backward cocycles agree. Moreover, if $A_k(\omega, x)$ is the cocycle over rotations by ω, then $A_{-k}(\omega, x) = A_k(-\omega, x + \omega)$. Since ω and $-\omega$ obey the same Diophantine condition, Lemma 4.6.1 also holds for $A^{f,z}_{-k}(x)$, which means we can use the exact same M_k to obtain a bound as above.

□

Now we can turn to the proof of Theorem 4.0.3.

Proof of Theorem 4.0.3. We can follow the same idea as in the proof of Theorem 4.0.1, using Theorem 4.6.2 in place of Theorem 4.5.2. Let us fix $x \in \mathbb{T}^\nu, \omega \in DC(A, c) \subset \mathbb{T}^\nu$, and $f = \lambda f_0 \in G^\sigma(\mathbb{T}^\nu)$, where $\lambda > \lambda_0(f_0, \omega)$ so that we satisfy the conclusions of Theorem 4.1.4. Moreover, suppose that $L(E) > 0$ so that we may appeal to Theorem 4.6.2.

By Theorem 4.6.2, along with Theorem 4.2.1, we have

$$P((\ln T)^{\gamma+\epsilon}-2,T)\leq CT^{-\beta}$$

for some $\gamma=\gamma(A,c,\sigma,\nu)<+\infty$ and every $\beta>1$. Moreover, it is clear that

$$\frac{\ln(P((\ln T)^{\gamma+\epsilon}-2,T))}{\ln\ln(T)}\leq-\delta\frac{\ln(T)}{\ln\ln(T)}, \tag{4.87}$$

so by Theorems 4.2.2 and 4.2.1, $\beta^{\pm}_{\ln}(p)\leq\alpha^{\pm}_{\ln}\leq\gamma$.

□

4.7 The analytic case

The proofs of our main results in the case of an analytic potential are morally the same as those for Gevrey potentials. Indeed, we can quickly obtain the following using the same proofs as the analogous results above.

Theorem 4.7.1. *Let f be a non-constant analytic function on $\mathbb{T}^\nu,\nu\geq1,\omega\in DC(A,c)$, and $E\in\mathbb{R}$ such that $L(E)>0$. For any $0<\tau<1$, there exist $k_\tau=k_\tau(E)<\infty,\delta=\delta(\omega,\nu)$, and $\gamma=\gamma(\nu,\delta)$ such that for any $\epsilon>0,k>k_\tau$, and $x\in\mathbb{T}^\nu$, there is $1\leq j\leq k^{\gamma+\epsilon}$ so that for any $z\in\mathbb{C}$ with $|z-E|<e^{-\frac{\tau kL(E)}{\|f\|_\infty}}$ we have*

$$\left\|A_k^{f,z}(x+j\omega)\right\|>e^{k(1-\tau)L(E)}. \tag{4.88}$$

Theorem 4.7.2. *Fix $\epsilon>0$. Let f be a non-constant analytic function on $\mathbb{T}^\nu,\nu\geq1,\omega\in DC(c,A)$, and $L(E)>0$. Then for any $\xi,\zeta>1$, there is $c>0$ and $T_E<\infty$ such that for*

$T > T_E$,

$$\inf\left\{\min_{\iota=\pm 1}\max_{1\le \iota m\le (\ln T)^{\zeta(\gamma+\epsilon)}}\left\|A_m^{f,z}(x)\right\|^2 T^{-\xi}\right\} > c \tag{4.89}$$

where γ and δ are as before, and the infimum is over all $x \in \mathbb{T}^\nu$ and $z \in \mathbb{C}$ with $|z-E| < T^{-\zeta}$.

Moreover, the dependence of T_E on E is through $L(E)$, as in Theorem 4.5.2. Thus, as before, T_E is uniformly bounded below for E in compact sets with positive $L(E)$.

The main difference between these two results and the variants from Sections 4.5 and 4.6 is the assumption on f. Here, we do not need to assume $f = \lambda f_0$ for $\lambda > \lambda_0(f_0, \omega)$. Indeed, this condition is needed for the Gevrey case in order to use the large deviation estimate Theorem 4.1.4, but the analogous estimate for analytic potentials, Theorem 4.1.3, does not require such a condition. Once we have a large deviation estimate, the proofs proceed exactly as in the proof of Theorem 4.6.1, with (4.67) replaced by $|\hat{f}(n)| \le CE^{c|n|}$. Note that continuity of $L(E)$, which is required in the uniform minoration of T_E, was established in [5].

4.8 The skew-shift case, $\nu > 1$

Let T_ω denote the skew shift on $\mathbb{T}^\nu$. As in the shift case, our goal is to first establish the following estimates:

Theorem 4.8.1. *Let $f = \lambda f_0 \in G^\sigma(T^\nu), \nu > 1$, $\omega \in SDC(A, c), \lambda > \lambda_0(f_0, \omega)$ and $E \in \mathbb{R}$ such that $L(E) > 0$. For any $0 < \tau < 1$, there exist $k_\tau = k_\tau(E) < \infty, \delta = \delta(\omega, \nu)$, and $\gamma = \gamma(\sigma, \nu, \omega)$ such that for any $\epsilon > 0, k > k_\tau$, and $x \in \mathbb{T}^\nu$, there is $1 \le j \le k^{\gamma+\epsilon}$ so that for any $z \in \mathbb{C}$ with $|z - E| < e^{-\frac{\tau k L(E)}{\|f\|_\infty}}$ we have*

$$\left\|A_k^{f,z}(x + j\omega)\right\| > e^{k(1-\tau)L(E)}. \tag{4.90}$$

Theorem 4.8.2. *Fix $\epsilon > 0$. Let $f = \lambda f_0 \in G^\sigma(\mathbb{T}^\nu), \nu > 1, \omega \in SDC(c, A), \lambda > \lambda_0(f_0, \omega)$ and $L(E) > 0$. Then for any $\xi, \zeta > 1$, there is $c > 0$ and $T_E < \infty$ such that for $T > T_E$,*

$$\inf \left\{ \min_{\iota=\pm 1} \max_{1 \leq \iota m \leq (\ln T)^{\zeta(\gamma+\epsilon)}} \left\| A_m^{f,z}(x) \right\|^2 T^{-\xi} \right\} > c \tag{4.91}$$

where γ and δ are as above, and the infimum is over all $x \in \mathbb{T}^\nu$ and $z \in \mathbb{C}$ with $|z - E| < T^{-\zeta}$. Moreover, if we restrict our attention to E in some compact interval $[-K, K]$, we can take T_E uniformly bounded below.

In particular, for $E \in [-K, K]$, we have $\max_{1 \leq n \leq (\ln T)^{\zeta(\gamma+\epsilon)}} \left\| A_n^{f, E+i/T} \right\|^2 \geq CT^\xi$ for every $\xi > 1$ and T large.

An analogue of Lemma 4.5.1 follows using the same argument as in the multifrequency shift case. The proof is identical to the proof of Lemma 4.8.1, but we use the skew-shift bound from Theorem 4.1.2 instead of the shift bound.

Lemma 4.8.1. *Let δ be defined as above. For $f = \lambda f_0 \in G^\sigma(\mathbb{T}^\nu)$, there exists $\lambda_0(f_0, \omega)$ such that, for $\lambda > \lambda_0$, every $\epsilon > 0$ and $x \in \mathbb{T}^\nu$ there exists $1 \leq j \leq k^{C(1/\delta)(\sigma\nu+1)+\epsilon}$ such that $T_\omega(x) \in V_k^{\tilde{f}_{N_1}}(E, cL(E))$.*

Theorem 4.8.1 now follows from the fact that $V_k^{\tilde{f}_{N_1}}(E, cL(E)) \subset V_k^f(z, dL(E))$, and observing that $d > 1 - \tau$, just as in the case $\nu = 1$.

Theorem 4.8.2 can now be proved using Theorem 4.8.1 in the same way that Theorem 4.6.2 was proved using Theorem 4.6.1.

Proof of Theorem 4.0.4. We can use the same argument as the proof of Theorem 4.0.3, using the analogous results from this section rather than those from Section 4.6. $\square$

Chapter 5

Fractal properties of spectral measures: generalized Hausdorff dimension

5.1 Introduction

The classification of measures using the classical power-law Hausdorff measures and dimensions has found many applications within spectral theory ([16, 20, 33, 36, 37, 38, 42, 44, 70, 71] and others). While this classification theory has been very useful in many situations, notably when the Hausdorff dimension is positive, it has not been general enough to understand the differences between zero-dimensional spectra. This has been explored in recent papers by Mavi [60], who studied logarithmic dimension bounds for the disordered Holstein model, and Avila, Last, Shamis, and Zhou [2], who studied the modulus of continuity of the integrated density of states for the almost Mathieu operator using a logarithmic dimension.

Our primary purpose in this chapter is to develop general tools to study these and even finer spectral questions. Explicitly, we consider different kinds of singular-continuous measures based on a more general notion of Hausdorff measure and dimension. The relevant definitions are discussed in section 5.2.

One of the advantages of our approach is that it allows us to distinguish between measures that are classically termed "zero-dimensional". These broad questions are very relevant to the study of quantum dynamics, as we will explore in Section 5.7 and Chapter 4, where the fractal properties of spectral measures are usually connected with anomalous transport properties (e.g. [3, 58]), while "zero-dimensional" spectral measures naturally occur when studying ergodic Schrödinger operators with positive Lyapunov exponent (see e.g. [32, 37, 38, 44, 67]). In particular, a general result due to Simon [67] says that the spectral measures, μ_θ, associated to an ergodic family of Schrödinger operators, H_θ, on $l^2(\mathbb{Z})$ with positive Lyapunov exponent are supported on sets of logarithmic capacity 0. This implies that the spectral measures are zero-dimensional. We explored positivity of the Lyapunov exponent for special types of non-ergodic Schrödinger operators in Chapter 3.

Results pertaining to dynamics have always been closely tied to the dimensional characteristics of spectral measures, so a finer distinction between dimensions should also provide additional tools to strengthen dynamics results. Additionally, recent work by Jitomirskaya and Liu [40] leads us to expect that the existence of phase resonances in quasiperiodic models implies very deeply zero-dimensional spectral measures whose dimensional properties cannot be well understood with classical notions, or even the log-dimension which was developed by one of the authors in his thesis [56] and has been studied in recent papers [2, 60].

The generality of our analysis is only possible because of our development and exploration of a *complete family of Hausdorff dimension functions* (Definition 5.2.3). In particular, a key technical component of our theory is Theorem 5.2.2, which only discusses a single Hausdorff measure. This theorem becomes useful in practice once we restrict our attention to a suitable collection of Hausdorff measures, rather than all possible Hausdorff measures, since it is not possible to compare all Hausdorff measures to one another. This has been done in the past by considering powers of suitable gauge functions, such as $(\ln(1/t))^{-1}$, but this is not suitable for our fine analysis; for example, the dimension of a set with respect to the two families $\mathcal{F}_1 = \{(\ln(1/t))^{-\alpha} : \alpha > 0\}$ and $\mathcal{F}_2 = \{(\ln(1/t)\ln\ln(1/t))^{-\alpha} : \alpha > 0\}$ will always coincide. Determining which of the two is "closer" to the actual dimension requires a more general type of family.

The results in this chapter are motivated by two phenomena that we know yield zero-dimensional spectra:

1. Schrödinger operators with positive Lyapunov exponent;
2. Local perturbations of systems with exponentially localized eigenfunctions.

We also extend the theory of quantum dynamics to our more general setting, an application of which was presented in Chapter 4.

We are motivated by one particular question when considering the regime of positive upper Lyapunov exponent:

Question 1. *Does positive upper Lyapunov exponent imply an upper bound on how singular the spectral measure must be?*

In 1999, Jitomirskaya and Last [37] proved that spectral measures for half-line Schrödinger operators with phase boundary condition θ and positive upper Lyapunov exponent must be zero Hausdorff dimensional (in the classical sense of Hausdorff dimension) for every θ.

In 2001, one of the authors [56] introduced the notion of logarithmic dimension, and proved that spectral measures for half-line Schrödinger operators with phase boundary condition θ and positive upper Lyapunov exponent must have logarithmic dimension at most 1 for every θ. The results of the thesis [56] were never published previously and are incorporated here. In our current framework, the logarithmic dimension coincides with $\dim_{\mathcal{F}}$ when $\mathcal{F} = \{(\ln(1/t))^{-\alpha} : 0 < \alpha < \infty\}$.

In 2007, Simon [67] proved that, given a family of ergodic Schrödinger operators H_θ with positive Lyapunov exponent, the spectral measure μ_θ must be supported on a set with zero logarithmic capacity for a.e. θ. This immediately yields the upper bound on Hausdorff dimension obtained in [37] when Lyapunov exponent, rather than upper Lyapunov exponent, is considered. The ergodicity and positive Lyapunov exponent conditions in [67] are more restrictive than our requiring positive upper Lyapunov exponent. Hence, Simon's result still leaves us with the question of what (finer) upper bounds on the dimension are possible in the setting of positive upper Lyapunov exponent, as well as what happens on the excluded Lebesgue null set.

These results lead us to consider the following, more refined question:

Question 2. *Does positive upper Lyapunov exponent imply that the spectral measure is always singular with respect to the* $(\ln(1/t))^{-1}$*-Hausdorff measure?*

In this paper, we answer this in two ways for half-line operators with phase boundary condition θ. We prove that $\dim_{\mathcal{F}}(\mu_\theta) = 1$, when considering the family $\mathcal{F} = \{(\ln(1/t))^{-\alpha} : 0 < \alpha < \infty\}$, (Theorem 5.2.4) and that more generally, the spectral measure must be at least $(\ln(1/t)\ln\ln(1/t)^2)^{-1}$-singular (Theorem 5.2.3), where both results hold for every phase θ. Furthermore, we construct half-line operators with phase boundary condition θ and positive upper Lyapunov exponent for every θ such that $\dim_{\mathcal{F}}(\mu_\theta) = \alpha_0$ for Lebesgue a.e. θ and any complete family of Hausdorff dimension functions, $\mathcal{F} = \{\rho_\alpha : \alpha \in I\}$, such that $(\ln(1/t))^{-1} = \rho_{\alpha_0} \in \mathcal{F}$ (Theorem 5.2.4). These show that the ideal bound lies somewhere between $(\ln(1/t))^{-1}$ and $(\ln(1/t)(\ln\ln(1/t))^2)^{-1}$. These are the main results of our paper.

This, therefore, extends Corollary 4.2 from [37] in a natural way. Moreover, it shows that the bound is sharp for log-dimension, in the sense that we cannot do better in general. However, this sense of sharpness might not necessarily be true for our more refined notion of dimension. Furthermore, the recent work of Jitomirskaya and Liu [40] leads us to expect that phase resonances in quasiperiodic models implies very singular, but not necessarily pure-point, spectral measures. The existence of such phenomena would make obtaining lower bounds for our refined notion of dimension exceptionally difficult−perhaps even impossible−for general operators unless we consider additional assumptions on the potential, which is why we only obtain general results for upper bounds on the dimension. In comparing our results to Simon's [67], we are drawn to two major differences: (1) we do not assume ergodicity and (2) we do not exclude a Lebesgue null set.

The salient point here is that we arrive at a result similar to that in [67] without the assumption of ergodicity or positivity of the Lyapunov exponent; we just need positive upper Lyapunov exponent. It is possible to view an ergodic family of operators as similar to a family of operators with a phase boundary condition. This surface analogy would lead us to believe that the result in [67] should be the same as the result in our situation, but this is not the case. Moreover, if there were an analogy between the ergodic parameter and the phase boundary parameter, then Simon's result would lead one to believe that a Lebesgue null set of phases needs to be excluded; our result shows that this is not true. We are able to obtain a logarithmic bound for all boundary phases.

We are also interested in local perturbations of systems with exponentially localized eigenfunctions and the following question:

Question 3. *Suppose $A : l^2(\mathbb{Z}^\nu) \to l^2(\mathbb{Z}^\nu)$ is self-adjoint with semi-uniformly localized eigenfunctions, and let $\mu = \mu_0$ be the spectral measure for δ_0. If $A_\lambda = A + \lambda \langle \delta_0, \cdot \rangle \delta_0$ is a rank one perturbation at the origin, and if μ_λ is the spectral measure for δ_0 associated to A_λ, is there an upper bound on how singular μ_λ is?*

In 1996, del Rio, Jitomirskaya, Last, and Simon proved that μ_λ must be zero Hausdorff dimensional (in the classical sense of Hausdorff dimension) for every λ.

We refine this answer in the following way (Theorem 5.2.5): not only are the spectral measures zero-dimensional (see Definition 5.2.8), but the spectral measures μ_λ are in fact $(\ln(1/t))^{-\nu-\epsilon}$-singular for every λ and $\epsilon > 0$.

We also rigorously extend the quantum dynamic theory of Last from the power-law setting to the general Hausdorff dimension setting (section 5.7). A similar result extending quantum dynamics appears in Mavi [60]. We believe that these results, especially Theorem 5.2.6, can lead to a strengthening of existing dynamics results for quasiperiodic models (c.f. [32, 39, 42, 44]). Indeed, in [46], Jitomirskaya and one of the authors showed that, under the assumption of positive Lyapunov exponent, certain general quasiperiodic operators possess moments of the position operator which exhibit sub-logarithmic growth, which improved the classical results in [32, 39, 42, 44] which established sub-polynomial growth.

The starting point for our analysis is the decomposition theory of Rogers and Taylor [63, 64]. Classically, any σ-finite measure can be decomposed into pure point, singular continuous, and absolutely continuous parts via the Lebesgue decomposition theorem; Rogers and Taylor took this further and decomposed the singular continuous part into measures that are singular or continuous with respect to the power-law Hausdorff measures.

Briefly, a measure μ is said to have exact power-law dimension $\alpha \in [0, 1]$ if and only if $\mu(E) = 0$ for every set S with power-law Hausdorff dimension $\beta < \alpha$ and if μ is supported on a set a power-law Hausdorff dimension α. In the terms used in this paper, this is equivalent to the upper and lower dimensions with respect to the family $\mathcal{F} = \{t^\alpha : 0 < \alpha\}$ coinciding.

Measures with exact dimension 0 or 1 are often viewed as "close" to pure point or absolutely continuous measures, respectively, but they do not need to be pure point or absolutely continuous. Relevant examples include the spectral measures of 1D quasiperiodic Schrödinger operators. It is known that the spectral measure is 0-dimensional for every irrational frequency, yet there exist frequencies for which the measure is not pure point [38]. Much of the work applying this theory to spectral theory has been unable to address how close these measures are to these two extremes.

It is known, however, that more general Hausdorff measures can be defined by replacing t^α in the definition with a suitable gauge function $\rho(t)$.

One of the authors has explored a generalization using $(\ln(1/t))^{-\alpha}$ in the definition of Hausdorff measures to create a logarithmic dimension and used it to study spectral questions in his thesis [56]. This has already found applications in [15]. We take these concepts and generalize them even further using modern ideas into what we believe is the most natural general framework (complete families of Hausdorff measure functions). Using these general families of Hausdorff measures, a similar notion of dimension can be developed to address these more delicate situations. Full details are presented in section 5.2.

With this theory, we are able to extend the Gilbert-Pearson and Jitomirskaya-Last theories of power-law subordinacy (Theorem 5.2.2), and we prove that half-line operators with positive upper Lyapunov exponent have at most a logarithmic dimension (Theorem 5.2.3). Moreover, we are able to construct half-line operators that achieve any given dimension for Lebesgue a.e. boundary phase (Theorem 5.2.4). We have not extended this analysis to every boundary phase, but we believe that the removal of a null set of boundary phases is simply a limitation of our proof methods.

Moreover, in Theorem 5.2.3, we obtain a $(\ln\ln(1/t))^2$ correction term, which is lacking in all of the existing results that just use a log-dimension. The proof heavily relies on the fact that the spectral measure of an operator of the form (5.7) is supported on the set of energies, E, for which there exists a solution u_1 to $Hu_1 = Eu_1$ satisfying:

$$u_1(0) = 0 \text{ and } u_1(1) = 1$$

and for every $\delta > 0$,

$$\limsup_{L\to\infty} \frac{\|u_1\|_L^2}{L(\ln L)^{1+\delta}} < \infty, \tag{5.1}$$

where $\|\cdot\|_L$ is defined by (5.14) below. This is the origin of the $(\ln\ln(1/t))^{1+\delta}$ correction term, and reveals when we expect this correction term to be unnecessary: whenever we can improve (5.1) on certain length scales. This is precisely what we do in our proof of part 3 of Theorem 5.2.4. The potentials constructed in Theorem 5.2.4 do not exhibit this correction term for a.e. θ, so it would be of interest to know if there are potentials that yield spectral measures that achieve a dimension with this correction term.

The rest of our paper is organized in the following way: in section 5.2, we build a general Hausdorff dimension framework, and introduce the major definitions, models, and results of our paper. In section 5.3, we relate the Hausdorff dimension of a measure to tangential limits of its Borel transform. In section 5.4, we apply these notions to derive a subordinacy theory for half-line operators, proving Theorem 5.2.2 and the first part of Theorem 5.2.3. In section 5.5, we analyze the dimension of spectral measures associated to operators with sparse barrier potentials and prove Theorem 5.2.4. In section 5.6, we discuss the behavior of spectral measures under rank-one perturbations and show that, under local perturbations, the spectrum of systems with exponentially localized eigenfunctions remains at most a logarithmic-power dimension. Finally, in section 5.7 we use our general Hausdorff dimension framework to extend the quantum dynamics theory of Last [58] to encompass our more general notion of dimension.

5.2 Preliminaries: a generalized Hausdorff dimension

Now we will give an overview of the relevant definitions for a discussion of generalized Hausdorff dimension.

Our analysis begins with the decomposition theory of Rogers and Taylor [63, 64]. The classical Lebesgue decomposition theorem provides a way to decompose any measure into three pieces: an absolutely continuous piece, a singular continuous piece, and a pure point piece. Rogers and Taylor used Hausdorff measures to further decompose the singular continuous piece.

Definition 5.2.1. A Hausdorff dimension function, or gauge function, is a strictly increasing differentiable function $\rho : (0, \infty) \to (0, \infty)$ with

$$\lim_{t \to 0^+} \rho(t) = 0.$$

Definition 5.2.2. The ρ-dimensional Hausdorff measure, μ^ρ, is defined on the Borel σ-algebra as

$$\mu^\rho(F) := \lim_{\delta \to 0} \inf_{\delta\text{-covers}} \left\{ \sum_{i=1}^{\infty} \rho(|F_i|) \right\}.$$

Observe that if $\rho(t) = t^\alpha$ then we arrive at the usual α-dimensional Hausdorff measure.

Consider the family of all Hausdorff dimension functions, $\mathcal{H}$ and the partial order, $\prec$, on $\mathcal{H}$ given by $\rho \prec \xi$ if and only if

$$\lim_{t \to 0^+} \frac{\rho(t)}{\xi(t)} = \infty. \tag{5.2}$$

It is easy to see that if $\lim_{t \to 0^+} \frac{\rho(t)}{\xi(t)} = 0$ then $\xi \prec \rho$. Additionally, we will define an equivalence relation, $\sim$, on $\mathcal{H}$ by $\rho \sim \xi$ if and only if

$$0 < \lim_{t \to 0^+} \frac{\rho(t)}{\xi(t)} < \infty.$$

We say that $\rho \precsim \xi$ if and only if $\rho \prec \xi$ or $\rho \sim \xi$.

Definition 5.2.3. We say $\mathcal{F} \subset \mathcal{H}$ is a complete one-parameter family of comparable Hausdorff dimension functions if $\mathcal{F}$ is a totally ordered subset of $\mathcal{H}$ which is order isomorphic to a subinterval $I \subset \mathbb{R}$. That is, if every pair $\rho, \xi \in \mathcal{F}$ obeys either $\rho \prec \xi$ or $\xi \prec \rho$, and if there exists an interval $I \subset \mathbb{R}$ such that there is an order-preserving bijection from $\mathcal{F}$ to I. Particularly, we can write $\mathcal{F} = \{\rho_\alpha : \alpha \in I, \rho_\alpha \prec \rho_\beta \text{ iff } \alpha < \beta\}$.

For simplicity, since these are the only families we will work with in this paper, we will simply call these *comparable families* or *complete comparable families.*

Remark 32. We may relax the order isomorphism condition slightly to allow for order isomorphisms with boxes in $\mathbb{R}^n$, for $1 \leq n \leq \infty$, along with the lexicographical order. All of our applications, however, use $n = 1$.

From this point forwards, we will restrict our attention to such families. Typical examples include the usual functions $\{t^\alpha\}_{\alpha \in \mathbb{R}}$ used to define the usual Hausdorff dimension, and even more generally, families of the form $\{\rho(t)^\alpha\}_{\alpha > 0}$ for some fixed $\rho(t) \in \mathcal{H}$.

We can use the completely ordered family $(\mathcal{F}, \prec)$ to generalize the Hausdorff dimension of sets and measures in a way that reduces, in a sense, to the classical definition when $\mathcal{F} = \{t^\alpha : 0 < \alpha \leq 1\}$:

Definition 5.2.4. Let $\mathcal{F} = \{\rho_\alpha : \alpha \in I \subset (0, \infty), \rho_\alpha \prec \rho_\beta \text{ iff } \alpha < \beta\}$. The $\mathcal{F}$-dimension of a set S, denoted $\dim_{\mathcal{F}}(S)$, is given by

$$\dim_{\mathcal{F}}(S) = \begin{cases} \alpha' & \text{if } \alpha' \in I \\ 0 & \text{if } \alpha' \notin I \text{ and } \alpha' < \alpha \text{ for all } \alpha \in I \\ \infty & \text{if } \alpha' \notin I \text{ and } \alpha' > \alpha \text{ for all } \alpha \in I \end{cases} \tag{5.3}$$

where $\alpha' = \sup\{\alpha \in I : \mu^{\rho_\alpha}(S) = \infty\} = \inf\{\alpha \in I : \mu^{\rho_\alpha}(S) = 0\}$.

Remark 33. It is natural to ask that a "good" definition of the dimension of a set be well-defined for any Borel set and agree with the usual Hausdorff dimension when applicable. We require the order isomorphism to be with an interval (or more generally, a box) to avoid the pathological behavior caused by the presence of "gaps", which can be illustrated in two examples:

1. First, we have the case of $\mathcal{F} = \{t^\alpha : \alpha \in [0,1]\backslash\mathbb{Q}\}$, which is unable to describe the dimension of sets with usual Hausdorff dimension 1/2, since the notion of supremum and infimum are not defined on $\mathcal{F}$.

2. Second, we have the case of $\mathcal{F} = \{t^\alpha : \alpha \in (0,1/3] \cup [2/3,1]\}$, which is also unable to describe the dimension of sets with usual Hausdorff dimension 1/2, since the supremum and infimum will not agree.

Observe that this is a precise generalization of the normal Hausdorff dimension when $\mathcal{F}$ contains only functions of the form t^α.

Unlike sets, a measure need not have an $\mathcal{F}$-dimension, which motivates the following definitions:

Definition 5.2.5. We say that a measure μ is ρ-singular if there exists some set G such that $\mu(\mathbb{R}\backslash G) = 0$ and $\mu^\rho(G) = 0$. Similarly, we say that a measure μ is ρ-continuous if $\mu(S) = 0$ for every set S with $\mu^\rho(S) = 0$.

This leads us the the notion of upper and lower dimension:

Definition 5.2.6. The upper $\mathcal{F}$-dimension of a measure μ, denoted $\dim^+_{\mathcal{F}}(\mu)$, is given by

$$\dim^+_{\mathcal{F}}(\mu) = \begin{cases} \beta'; & \text{if } \beta' \in I \\ \infty; & \text{if } \beta' \notin I \text{ and } \beta' > \alpha \text{ for every } \alpha \in I \\ 0; & \text{if } \beta' \notin I \text{ and } \beta' < \alpha \text{ for every } \alpha \in I \end{cases} \tag{5.4}$$

where $\beta' = \inf \{\alpha \in I : \mu \text{ is } \rho_\alpha\text{-singular}\}$. Similarly, we define the lower $\mathcal{F}$-dimension of a measure μ, denoted $\dim^-_{\mathcal{F}}(\mu)$ is given by

$$\dim^-_{\mathcal{F}}(\mu) = \begin{cases} \gamma'; & \text{if } \gamma' \in I \\ 0; & \text{if } \gamma' \notin I \text{ and } \gamma' < \alpha \text{ for every } \alpha \in I \\ \infty; & \text{if } \gamma' \notin I \text{ and } \gamma' > \alpha \text{ for all } \alpha \in I \end{cases} \tag{5.5}$$

where $\gamma' = \sup \{\alpha \in I : \mu \text{ is } \rho_\alpha\text{-continuous}\}$.

We can now define the $\mathcal{F}$-dimension of a Borel measure μ.

Definition 5.2.7. The $\mathcal{F}$-dimension of a Borel measure μ, denoted $\dim_{\mathcal{F}}(\mu)$, is given by

$$\dim_{\mathcal{F}}(\mu) = \begin{cases} \dim^+_{\mathcal{F}}(\mu); & \text{if } \dim^+_{\mathcal{F}}(\mu) = \dim^-_{\mathcal{F}}(\mu) \\ \text{undefined}; & \text{if } \dim^+_{\mathcal{F}}(\mu) \neq \dim^-_{\mathcal{F}}(\mu) \end{cases}. \tag{5.6}$$

Related concepts we will occasionally use are the idea of zero-dimensional and positive-dimensional Hausdorff measure functions.

Definition 5.2.8. We say a function $\rho \in \mathcal{H}$ is a zero-dimensional Hausdorff dimension function if $\rho \prec t^\alpha$ for every $\alpha > 0$. Analogously, we say a function $\xi \in \mathcal{H}$ is a positive-dimensional Hausdorff dimension function if $t^\alpha \prec \xi$ for some $\alpha > 0$.

Our approach here is, as far as we know, novel. Past work in this direction has always dealt with studying the singularity and continuity of a measure with respect to families of the form $\{\rho^\alpha\}_{\alpha\in I}$, whereas our notion of a complete family of Hausdorff dimension functions allows us to consider more varied families, which allows us to gain sharper results.

5.2.1 Consequences for 1D Operators

First, we will examine dimensional properties of discrete Schrödinger operators on the half-line. We define

$$(H_\theta\psi)(n) = \psi(n-1) + \psi(n+1) + V(n)\psi(n), \tag{5.7}$$

along with a phase boundary condition

$$\psi(0)\cos\theta + \psi(1)\sin\theta = 0, \tag{5.8}$$

where $-\frac{\pi}{2} < \theta \le \frac{\pi}{2}$ and the potential $V = \{V(n)\}_{n=1}^\infty$ is a sequence of real numbers. The study of operators of the form (5.7) along with the boundary condition (5.8) is equivalent to the study of (5.7) with a Dirichlet boundary condition

$$\begin{aligned}\psi(0) &= 0\\ \psi(1) &= 1\end{aligned} \tag{5.9}$$

along with a rank-one perturbation at the origin

$$V(1) \mapsto V(1) - \tan\theta. \tag{5.10}$$

So, without loss of generality, we will confine our attention to operators of the form (5.7) on $l^2(\mathbb{Z}^+)$ along with the Dirichlet boundary condition (5.9) and interpret the boundary phase as applying the corresponding rank-one perturbation at the origin.

For these operators, it is known that the vector δ_1, which is 1 for $n = 1$ and 0 otherwise, is cyclic, so the spectral problem reduces to the study of the spectral measure $\mu = \mu_{\delta_1}$. The behavior of this spectral measure is related to the behavior of the Weyl-Titchmarsh m-function, which in our case coincides with the Borel transform of μ :

$$F_\mu(z) = \int_{\mathbb{R}} \frac{d\mu(x)}{x - z}. \tag{5.11}$$

When there is no ambiguity, we will usually omit the dependence on μ and express the Borel transform as $F(z)$. For a full discussion of this relationship, we refer the reader to Simon [66]. Of particular note is that the Borel transform of a measure μ exists whenever

$$\int \frac{d\mu(x)}{|x| + 1} < \infty. \tag{5.12}$$

We will assume all measures considered in this paper satisfy this condition.

Our first results will extend the Jitomirskaya-Last theory of power-law subordinacy [37], which is itself an extension of the Gilbert-Pearson theory [28, 29, 51], both of which relate spectral properties of the operator (5.7) to solutions of the corresponding Schrödinger equation

$$u(n-1) + u(n+1) + V(n)u(n) = Eu(n). \tag{5.13}$$

More specifically, we will let $\|u\|_L$ be the norm of u over the lattice interval of L. That is,

$$\|u\|_L = \left(\sum_{n=1}^{\lfloor L \rfloor} (|u(n)|^2 + (L - \lfloor L \rfloor)|u(\lfloor L \rfloor + 1)|^2) \right)^{1/2}, \tag{5.14}$$

where $\lfloor L \rfloor$ is the integer part of L. We say that a solution u of (5.13) is called subordinate if

$$\lim_{L\to\infty} \frac{\|u\|_L}{\|v\|_L} = 0 \tag{5.15}$$

for any other linearly independent solution v. The Gilbert-Pearson theory related the absolutely continuous part of the spectral measure μ to those energies E for which (5.13) has no subordinate solutions; likewise, the singular part of the spectral measure μ is supported on the set of energies for which the solutions to (5.13) with the Dirichlet boundary condition are subordinate. The Jitomirskaya-Last theory refined the treatment of the singular part of the spectral measure to consider different kinds of singular-continuous spectral measures based on the classification of those measures with respect to the usual power-law Hausdorff measures and dimensions using a decomposition theory developed by Rogers and Taylor [63, 64]. Our treatment goes further still and considers decompositions with respect to arbitrary families of Hausdorff measures, not just the usual power-law measures.

Given H_θ of the form (5.7), and $E \in \mathbb{R}$, we define u_1 to be the solution to (5.13) obeying the Dirichlet boundary condition:

$$\begin{aligned} u_1(0) &= 0 \\ u_1(1) &= 1 \end{aligned} \tag{5.16}$$

and let u_2 be the solution of (5.13) obeying the orthogonal boundary condition:

$$\begin{aligned} u_2(0) &= 1 \\ u_2(1) &= 0. \end{aligned} \tag{5.17}$$

Given any $\epsilon > 0$, we define the length scale $L(\epsilon) \in (0, \infty)$ as the length that yields the equality

$$\|u_1\|_{L(\epsilon)}^{-1} \|u_2\|_{L(\epsilon)}^{-1} = 2\epsilon. \tag{5.18}$$

Another useful tool in studying operators of the form (5.7) is the n-step *transfer matrix* $\Phi_n(\theta, E)$. This is the matrix

$$\Phi_n(\theta, E) = \begin{pmatrix} u_1(n+1) & u_2(n+1) \\ u_1(n) & u_2(n) \end{pmatrix}. \tag{5.19}$$

With this, we can define the *upper Lyapunov exponent*,

$$L^*(\theta, E) = \limsup_{n\to\infty} \frac{1}{n} \ln \|\Phi_n(\theta, E)\|. \tag{5.20}$$

We know of no explicit link between the operator H and the local scaling behavior of the spectral measure μ in this regime, so we begin with an important technical result relating the generalized Hausdorff dimension of a Borel measure to growth properties of its Borel transform:

Theorem 5.2.1. *Define $A_0 = \{0\}, A_1 = (0, \infty)$, and $A_2 = \{\infty\}$. Suppose ρ is a Hausdorff dimension function satisfying $\rho(t) \prec t^\alpha, \alpha < 1$. We have*

$$\limsup_{\epsilon\to 0^+} \frac{\mu((x-\epsilon, x+\epsilon))}{\rho(\epsilon)} \in A_i$$

if and only if

$$\limsup_{\epsilon\to 0^+} \frac{\epsilon}{\rho(\epsilon)} \operatorname{Im} F(x+i\epsilon) \in A_i,$$

with $i = 0, 1, 2$.

Our first core result is a subordinacy theory extending the work of Jitomirskaya-Last [37, 38], which links the generalized Hausdorff dimension of the spectral measure μ to growth properties of u_1 and u_2 :

Theorem 5.2.2. *Let u_1 and u_2 be solutions of the equation $Hu = Eu$ for $E \in \mathbb{R}$ obeying $u_1(0) = 0, u_1(1) = 1, u_2(0) = 1,$ and $u_2(1) = 0$. Let $\rho(t)$ be a Hausdorff measure function. We have*

$$\limsup_{\epsilon \to 0} \frac{\epsilon}{\rho(\epsilon)} F(E + i\epsilon) = \infty$$

if and only if

$$\liminf_{L \to \infty} \rho(\|u_1\|_L^{-1} \|u_2\|_L^{-1}) \|u_1\|_L^2 = 0.$$

Our second key result is a bound on the upper spectral dimension of a half-line operator with positive upper Lyapunov exponent:

Theorem 5.2.3. *Let $\mathcal{F} = \{\rho_\alpha : \alpha \in I\}$ be a family of comparable Hausdorff dimension functions such that for some $\delta, \epsilon > 0$, we have $\alpha_1, \alpha_2 \in I$ such that $f_\delta(t) = \rho_{\alpha_1}, g(t)^{1-\epsilon} = \rho_{\alpha_2} \in \mathcal{F}$, where*

$$f_\delta(t) = \frac{1}{\ln(1/t)(\ln(\ln(1/t)))^{1+\delta}} \tag{5.21}$$

and

$$g(t) = \frac{1}{\ln(1/t)}. \tag{5.22}$$

If the upper Lyapunov exponent is positive for every E in some Borel set A then $\dim^+_{\mathcal{F}}(\mu(A \cap \cdot)) \leq \alpha_1$. Moreover, there exists an operator of the form (5.7) with positive upper Lyapunov exponent whose spectral measure μ obeys $\dim^-_{\mathcal{F}}(\mu) \geq \alpha_2$.

This improves upon an earlier result from [37] which was only able to conclude that the power-law dimension was 0, and earlier results from Landrigan, which were only able to conclude that the log-dimension was at most 1. There are two immediate consequences of this result: (i) the lower dimension bound shows that Landrigan's log-dimension result is sharp, (ii) our upper dimension result reveals that there may be examples of operators with positive upper Lyapunov exponent that have a dimension strictly larger than α_2.

While we do not know which of these bounds is sharp, by considering an operator with a suitably sparse barrier potential, we are able to show that, with respect to a suitable family, $\mathcal{F}$, the lower dimension above can coincide with the actual dimension for Lebesgue a.e. boundary phase. Let $\beta(x)$ be a non-negative increasing convex function such that $\ln(\beta(x))$ is still convex. For example, we could take $\beta(x) = e^x$. Moreover, suppose that $G(t) = 1/\beta^{-1}(1/t^2)$ defines a zero-dimensional Hausdorff dimension function. Let f^j denote β^{-1} composed with itself j-times. We will consider any family of Hausdorff dimension functions, $\mathcal{F} = \{\rho_\alpha : \alpha \in I\}$, such that there are α_1, α_2, and $\alpha_3 \in I$ such that $\rho_{\alpha_1} = G(t)^{1/\eta} f^j(1/t^2)^{-1}$, $\rho_{\alpha_2} = G(t)^{(1-\epsilon)/\eta}$ and $\rho_{\alpha_3} = G(t)^{1/\eta}/(\ln(\beta^{-1}(1/t)))^{1+\delta}$. That is, $G(t)^{1/\eta} f^j(1/t^2)^{-1}$, $G(t)^{(1-\epsilon)/\eta}$ and $G(t)^{1/\eta}/(\ln(\beta^{-1}(1/t)))^{1+\delta} \in \mathcal{F}$ for some $j \geq 1$ and $\eta, \delta > 0$. Define length scales inductively by $L_1 = 2, L_{n+1} = \beta(L_n)^n$ and define a potential

$$V(n) = \begin{cases} \beta(L_k)^\eta & n = L_k \\ 0 & n \notin \{L_k\}_{k=1}^\infty \end{cases}. \tag{5.23}$$

A theorem of Simon and Spencer [68] ensures the resulting Schrödinger operator with potential as above has no absolutely continuous spectra, since $\lim_{n\to\infty} |V(n)| = \infty$. Moreover, this potential does not satisfy the criterion for presence of pure point spectra from [53]. These two observations make the following theorem particularly meaningful.

Theorem 5.2.4. *Let $\eta, \beta, G(t), \mathcal{F}, \rho_{\alpha_1}, \rho_{\alpha_2}, \rho_{\alpha_3}$ and $V(n)$ be as above. Let μ_θ be the spectral measure of the half-line operator $(H_\theta u)(n) = u(n+1) + u(n-1) + V(n)u(n)$ with boundary phase θ. Then* *(i) for every boundary phase* θ*, the spectrum of* H_θ *consists of the interval* $[-2, 2]$ *(which is the essential spectrum) along with some discrete point spectrum outside this interval;*

(ii) for every θ*,*

$$\alpha_2 \leq \dim_{\mathcal{F}}^{-}(\mu_\theta((-2,2) \cap \cdot)) \leq \dim_{\mathcal{F}}^{+}(\mu_\theta((-2,2) \cap \cdot)) \leq \alpha_3;$$

(iii) for Lebesgue a.e. θ*,* $\dim_{\mathcal{F}}^{+}(\mu) \leq \alpha_1$.

In particular, if we take $\beta(x) = e^x, \eta = 1$, then Theorem 5.2.4 proves the second part of Theorem 5.2.3.

One of the most interesting parts of this theorem is that we only prove an exact dimension result for a.e. boundary phase θ. This is a limitation of our proof, where we carefully study the existence of suitably decaying solutions in the case $\theta = 0$, and interpret different boundary phases as consequences of particular rank-one perturbations; by considering rank-one perturbations, we are able to deduce the existence of similarly decaying solutions for other boundary phases, but lose a Lebesgue null set in the process. A similar result is known to hold for every boundary phase when positive power-law Hausdorff dimensions are considered, but the only proof we are aware of requires more involved arguments involving quantum dynamics [71].

5.2.2 Consequences for systems with exponentially localized eigenfunctions

We then turn our attention to fractal properties of Schrödinger operators on the lattice $l^2(\mathbb{Z}^\nu), \nu \geq 1$.

First, we study what happens to the dimensional properties of spectral measures when we apply rank-one perturbations to operators with exponentially localized eigenfunctions. More specifically, by the spectral theorem it is known that every bounded self-adjoint operator on a Hilbert can be realized as $A : L^2(d\mu) \to L^2(d\mu)$, $\psi \mapsto \psi \cdot x$, for some suitable measure μ. If we let $\varphi \in L^2(d\mu)$ be a cyclic unit vector, then we can easily define the rank-one perturbation of A by φ as

$$A_\lambda = A + \lambda \langle \varphi, \cdot \rangle \varphi, \quad \lambda \in \mathbb{R}. \tag{5.24}$$

If we let μ_λ denote the spectral measure of A_λ associated to φ, and F_λ the Borel transform of μ_λ, then it is known that

$$F_\lambda(z) = \frac{F_0(z)}{1 + \lambda F_0(z)} \tag{5.25}$$

which, in conjunction with our work relating dimensional properties of a measure to growth properties of Borel transforms, allows us to study how the dimension of a spectral measure is affected when it is under the effect of a rank-one perturbation.

We say that a self-adjoint operator on $l^2(\mathbb{Z}^\nu)$ has semi-uniformly localized eigenfunctions (SULE) if and only if there is a complete set of orthonormal eigenfunctions, $\{\varphi_n\}_{n=1}^{\infty}$, there is $\alpha > 0$ and $m_n \in \mathbb{Z}^\nu, n \geq 1$ and for each $\delta > 0$, a $C_\delta > 0$ so that

$$|\varphi_n(m)| \leq C_\delta e^{\delta |m_n| - \alpha |m - m_n|} \tag{5.26}$$

for all $m \in \mathbb{Z}^\nu$, and $n \geq 1$. Here $|\cdot|$ on the r.h.s. denotes any $\mathbb{Z}^\nu$ norm.

It is known, [21], that if an operator $H : l^2(\mathbb{Z}^\nu) \to l^2(\mathbb{Z}^\nu)$ has SULE, if $H_\lambda = H + \lambda \langle \delta_0, \cdot \rangle \delta_0$, and if μ and μ_λ are the spectral measure for H and H_λ respectively associated to δ_0, then μ_λ is zero-dimensional. We are able to improve this into

Theorem 5.2.5. *Suppose $H : l^2(\mathbb{Z}^\nu) \to l^2(\mathbb{Z}^\nu)$ has SULE and let $\mathcal{F} = \{(\ln(1/t))^{-\alpha} : 0 < \alpha < \infty\}$.*

Let $H_\lambda = H + \lambda \langle \delta_0, \cdot \rangle \delta_0$. Let $d\mu$ be the spectral measure of H associated to δ_0, and let $d\mu_\lambda$ be the corresponding spectral measures for H_λ. Then for every λ, $\dim_{\mathcal{F}}(\mathrm{supp}(d\mu_\lambda)) \leq \nu$.

5.2.3 Consequences for quantum dynamics

We now turn our attention to dynamical properties of Schrödinger operators on the lattice $l^2(\mathbb{Z}^\nu), \nu \geq 1$. Our main interest in this setting is in dynamical properties of operators of the form

$$(H\psi)(n) = \sum_{|n-m|=1} \psi(m) + V(n)\psi(n), \tag{5.27}$$

though much of our discussion applies to any self-adjoint Hamiltonian. A theory based on the power-law dimension was developed by Last [58]. Notably, the theory establishes an extremely useful connection between the continuity of a spectral measure and the average growth of the moments of the corresponding position operator (see, e.g. [16], [52], [70], [71]). Our starting point is that the original theory of Rogers and Taylor was actually developed in the generality that we are using; in particular, the decomposition theory and the critical Theorem 4.2 in [58] exists in our general setting once a suitable notion of uniform Hölder continuity with respect to a general Hausdorff dimension function is realized. This allows us to proceed in much the same manner as Last.

A common application of Last's theory is the notion of a transport exponent, which relates to the average power-law growth of the p^{th} moment of the position operator. One of our most important results in this direction is

Theorem 5.2.6. *If H is self-adjoint on $l^2(\mathbb{Z}^\nu)$ and $P_{\rho c}\psi \neq 0$, where $P_{\rho c}$ is the orthogonal projection on $\mathscr{H}_{\rho c}$, then for each $m > 0$, there exists a constant $C = C(\psi, m)$ such that for* *every* $T > 1$

$$\langle\langle |X|^m \rangle\rangle_T > C\rho(1/T)^{-m/\nu}. \tag{5.28}$$

We refer readers to Section 5.7 for the definition of $\mathscr{H}_{\rho c}$ and the l.h.s. of (5.28). This may be used to define a more general notion of transport exponent than has previously been studied. Analysis of this transport exponent has been of central importance in many dynamical results (see e.g. [16], [52], [70], [71]).

5.3 General Hausdorff dimension of sets and measures

The following characterization dates back to the original work of Rogers and Taylor [63, 64]

Theorem 5.3.1. *Let A be a Borel set, and let μ be a Borel measure. Then*

1. *$\mu(\cdot \cap A)$ is ρ-singular if and only if*

$$\limsup_{\epsilon \to 0} \frac{\mu(x-\epsilon, x+\epsilon)}{\rho(\epsilon)} = \infty$$

   for μ-a.e. $x \in A$.

2. *$\mu(\cdot \cap A)$ is ξ-continuous if and only if*

$$\limsup_{\epsilon \to 0} \frac{\mu(x-\epsilon, x+\epsilon)}{\xi(\epsilon)} < \infty$$

   for μ-a.e. $x \in A$.

When μ is the spectral measure of some self-adjoint operator A, we know of no direct relation between the local scaling behavior of μ and spectral properties of A. To bridge the gap between the the two, we will need to introduce the Borel transform, as in [20]:

Definition 5.3.1. The Borel transform of a measure μ, denoted $F_\mu(z)$, is

$$F_\mu(z) = \int_{\mathbb{R}} \frac{d\mu(x)}{x - z}.$$

It is known that the Borel transform provides an alternate characterization to Theorem 5.3.1 for the usual Hausdorff dimension, but it in fact applies to our more general notion. Notably, we may now prove Theorem 5.2.1:

Proof of Theorem 5.2.1. Let $M_\mu^\delta(x_0) = \mu(x_0 - \delta, x_0 + \delta)$. By definition, we have

$$\operatorname{Im} F_\mu(x_0 + i\epsilon) = \epsilon \int_{-\infty}^{\infty} \frac{d\mu(y)}{(y - x_0)^2 + \epsilon^2} \geq \frac{1}{2\epsilon} M_\mu^\epsilon(x_0), \tag{5.29}$$

so

$$\frac{M_\mu^\epsilon(x_0)}{\rho(\epsilon)} \leq 2 \frac{\epsilon}{\rho(\epsilon)} \operatorname{Im} F_\mu(x_0 + i\epsilon). \tag{5.30}$$

Thus

$$\limsup_{\epsilon \to 0^+} \frac{M_\mu^\epsilon(x_0)}{\rho(\epsilon)} \leq 2 \limsup_{\epsilon \to 0^+} \frac{\epsilon}{\rho(\epsilon)} \operatorname{Im} F_\mu(x_0 + i\epsilon). \tag{5.31}$$

Hence, if the LHS $= \infty$, then so does the RHS. Analogously, if the RHS $= 0$, then so does the LHS.

On the other hand, suppose $\limsup_{\epsilon \to 0} \frac{M_\mu^\epsilon(x_0)}{\rho(\epsilon)} < \infty$. Then we know that

$$M_\mu^\delta(x_0) \leq C\rho(\delta), \tag{5.32}$$

for δ sufficiently small, so we have

$$\limsup_{\epsilon\to 0^+} \frac{\epsilon}{\rho(\epsilon)} \operatorname{Im} F_\mu(x_0+i\epsilon) \le \limsup \frac{\epsilon}{\rho(\epsilon)} |F_\mu(x_0+i\epsilon)| \tag{5.33}$$

$$\le \limsup \frac{\epsilon}{\rho(\epsilon)} \int_{-\infty}^{\infty} \frac{d\mu(y)}{[(x_0-y)^2+\epsilon^2]^{1/2}} \tag{5.34}$$

$$= \limsup \frac{\epsilon}{\rho(\epsilon)} \left(\int_{|y-x_0|>1} + \int_{|y-x_0|\le 1} \right) \tag{5.35}$$

$$= \limsup \frac{\epsilon}{\rho(\epsilon)} \int_{|y-x_0|\le 1} \frac{d\mu(y)}{[(x_0-y)^2+\epsilon^2]^{1/2}}. \tag{5.36}$$

Here (5.36) follows from the observation that

$$\frac{\epsilon}{\rho(\epsilon)} \int_{|y-x_0|>1} \frac{d\mu(y)}{[(x_0-y)^2+\epsilon^2]^{1/2}} \le \frac{\epsilon}{\rho(\epsilon)} C(x_0) \tag{5.37}$$

$$\to 0 \tag{5.38}$$

where the inequality is a consequence of (5.12) and the limit is a consequence of $\rho \prec t^\alpha \prec t$.

We can then evaluate the remaining integral by integrating by parts, and by observing that the boundary term at 0 vanishes:

$$\limsup \frac{\epsilon}{\rho(\epsilon)} \int_{|y-x_0|\le 1} \frac{d\mu(y)}{[(x_0-y)^2+\epsilon^2]^{1/2}} = \limsup \frac{\epsilon}{\rho(\epsilon)} \int_0^1 \frac{\delta}{(\epsilon^2+\delta^2)^{3/2}} M_\mu^\delta(x_0) d\delta \tag{5.39}$$

$$\le \limsup C \frac{\epsilon}{\rho(\epsilon)} \int_0^1 \frac{\delta\rho(\delta)}{(\epsilon^2+\delta^2)^{3/2}} d\delta. \tag{5.40}$$

Now we break the integral into two pieces: $\int_0^\epsilon + \int_\epsilon^1$ and observe that the first piece is uniformly bounded. The second piece can be bounded as

$$\limsup C \frac{\epsilon}{\rho(\epsilon)} \int_\epsilon^1 \frac{\delta\rho(\delta)}{(\epsilon^2+\delta^2)^{3/2}} d\delta \le C \frac{\epsilon}{\rho(\epsilon)} \int_\epsilon^1 \frac{\rho(\delta)}{\delta^2} d\delta. \tag{5.41}$$

Observe that either this is immediately obvious to be finite, or L'Hospital's rule applies to

$$\frac{\int_\epsilon^1 \frac{\rho(\delta)}{\delta^2} d\delta}{\rho(\epsilon)/\epsilon}.$$

Applying L'Hospital's rule, the limit will coincide with

$$\lim_{\epsilon \to 0} \frac{\rho(\epsilon)}{\rho(\epsilon)'\epsilon - \rho(\epsilon)}.$$

This will be finite as long as

$$\frac{\rho(\epsilon)}{\rho(\epsilon)'\epsilon}$$

is bounded away from 1. Now rewrite as

$$\frac{\rho'/\rho}{1/\epsilon}.$$

Since $\rho \prec t^\alpha$, we know that

$$\lim_{\epsilon \to 0} \ln(\rho(\epsilon))/\ln(\epsilon) \leq \alpha.$$

However, L'Hospital's rule also applies to this limit, which implies the desired boundedness property.

This finally implies that the two desired lim sup are either both finite or infinite. Moreover, if $\limsup_{\epsilon \to 0^+} \frac{M_\mu^\epsilon(x_0)}{\rho(\epsilon)} = 0$ then the constant C above may be taken to be arbitrarily small, ensuring that $\limsup_{\epsilon \to 0^+} \frac{\epsilon}{\rho(\epsilon)} \operatorname{Im} F_\mu(x_0 + i\epsilon) = 0$. This completes our proof.

$\square$

5.4 Half-line subordinacy

Let H_θ be the self-adjoint operator defined on $l^2(\mathbb{Z}^+)$ by

$$(H_\theta u)(n) = u(n-1) + u(n+1) + V(n)u(n), \tag{5.42}$$

where $\{V(n)\}_{n=1}^{\infty}$ is a sequence of real numbers along with the phase boundary condition

$$u(0)\cos\theta + u(1)\sin\theta = 0. \tag{5.43}$$

Definition 5.4.1. We define the length scale $L(\epsilon)$ as the length that yields the equality $\|u_1\|_{L(\epsilon)}^{-1} \|u_2\|_{L(\epsilon)}^{-1} = 2\epsilon$.

Theorem 5.4.1. *Let u_1 and u_2 be solutions of the equation $Hu = Eu$ for $E \in \mathbb{R}$ obeying $u_1(0) = 0, u_1(1) = 1, u_2(0) = 1,$ and $u_2(1) = 0$. Let $\rho(t)$ be a Hausdorff measure function. We have*

$$\limsup_{\epsilon\to 0} \frac{\epsilon}{\rho(\epsilon)} F(E + i\epsilon) = \infty$$

if and only if

$$\liminf_{L\to\infty} \rho(\|u_1\|_L^{-1} \|u_2\|_L^{-1}) \|u_1\|_L^2 = 0.$$

Proof. This follows from Theorem 1 of [37] and Theorem 5.2.1 above. □

We now have two applications of this theorem to zero dimensional Hausdorff dimension functions and positive dimension Hausdorff dimension functions.

Theorem 5.4.2. *Let $f(L)$ be a continuous, strictly increasing function such that (1) $f(0) \geq 0$ (2) $\lim_{L\to\infty} f(L) = \infty$ and (3) $\lim_{L\to\infty} \frac{L^\alpha}{f(L)} = 0$ for every $\alpha \geq 1$, and let $g(t) = \frac{1}{t(\ln t)^{1+\delta}}$. Suppose that for every E in some Borel set A, we can find a solution, $v = au_1 + bu_2$, to*

$Hv = Ev$ that satisfies

$$\limsup_{L\to\infty} \frac{\|v\|_L^2}{f(L)} \geq 1.$$

Let $\mathcal{F} = \{\rho_\alpha : \alpha \in I\}$ be any family of comparable Hausdorff dimension functions such that there is $\alpha_1 \in I$ such that $\rho_{\alpha_1} = g(f^{-1}(\frac{|b|^2}{1-|b|^2\epsilon}t^{-2}))$, for some constant $|b|$ and $0 < \epsilon < 1/|b|^2$. Then $\dim_{\mathcal{F}}^+(\mu(A \cap \cdot)) \leq \alpha_1$.

Proof. It is known that μ is supported on the set of energies E for which u_1 satisfies the inequality

$$\limsup_{L\to\infty} \frac{\|u_1\|_L^2}{L(\ln L)^{1+\delta}} < \infty, \tag{5.44}$$

for every $\delta > 0$, so we may restrict our attention to those energies. For every $E \in A, v$ must be a linear combination of u_1 and u_2, say $v = au_1 + bu_2$. Thus, for every L,

$$\|v\|_L \leq |a| \, \|u_1\|_L + |b| \, \|u_2\|_L \,.$$

By our choice of f, and our restriction on the energies, E, we see that we must have $b \neq 0$, so

$$\|u_2\|_L \geq \frac{\|v\|_L - |a| \, \|u_1\|_L}{|b|}.$$

Hence, we must also have

$$\limsup_{L\to\infty} \frac{\|u_2\|_L^2}{f(L)} \geq \frac{1}{|b|^2} \tag{5.45}$$

for all such E. Now (5.44) implies

$$\|u_1\|_L < CL^{1/2}(\ln L)^{(1+\delta)/2} \tag{5.46}$$

for some constant $C > 0$, and (5.45) implies

$$\|u_2\|_{L_n}^2 > \left(\frac{1}{|b|^2} - \epsilon\right) f(L_n) \tag{5.47}$$

for some sequence $L_n \to \infty$ and every $0 < \epsilon < \frac{1}{|b|^2}$.

Now consider ϵ_n such that $L_n = L(\epsilon_n)$. We have

$$\begin{aligned}
&g(f^{-1}(\frac{|b|^2}{1-|b|^2\epsilon} \|u_1\|_{L_n}^2 \|u_2\|_{L_n}^2)) \|u_1\|_{L_n}^2 && (5.48)\\
&\quad = \frac{\|u_1\|_{L_n}^2}{f^{-1}(\frac{|b|^2}{1-|b|^2\epsilon} \|u_1\|_{L_n}^2 \|u_2\|_{L_n}^2)(\ln(f^{-1})(\frac{|b|^2}{1-|b|^2\epsilon} \|u_1\|_{L_n}^2 \|u_2\|_{L_n}^2))^{1+\delta}} && (5.49)\\
&\quad \lesssim \frac{L_n(\ln L_n)^{1+\delta}}{f^{-1}(f(L_n)) \ln(f^{-1}(f(L_n)))^{1+\delta}} && (5.50)\\
&\quad = 1. && (5.51)
\end{aligned}$$

Thus, if $g(f^{-1}(\frac{|b|^2}{1-|b|^2\epsilon} t^{-2})) \prec \rho(t)$ it is easy to see that

$$\lim_{L_n \to \infty} \rho(\|u_1\|_{L_n}^2 \|u_2\|_{L_n}^2) \|u_1\|_{L_n}^2 = 0.$$

We now finish by appealing to Theorem 5.2.2. □

Corollary 5.4.1. *Let $f(t), g(t)$, and $\mathcal{F}$ be as in Theorem 5.4.2. Moreover, suppose ρ_{α_1} as before with $|b| = 1$. Let $\Phi_n(\theta, E)$ be the n-step transfer matrix associated to $Hu = Eu$ along with the boundary condition θ. Suppose*

$$\limsup_{L \to \infty} \frac{1}{f(L)} \sum_{n=1}^{L} \|\Phi_n(\theta, E)\|^2 \geq 2$$

for every E in some Borel set A. Then $\dim_{\mathcal{F}}^{+}(\mu(A \cap \cdot)) \leq \alpha_1$.

Proof. Recall that

$$\Phi_n(\theta, E) = \begin{pmatrix} u_1(n+1) & u_2(n+1) \\ u_1(n) & u_2(n) \end{pmatrix} \tag{5.52}$$

so

$$\|\Phi_n(\theta, E)\|^2 \leq |u_1(n+1)|^2 + |u_1(n)|^2 + |u_2(n+1)|^2 + |u_2(n)|^2, \tag{5.53}$$

and summing yields

$$\sum_{n=1}^{L} \|\Phi_n(\theta, E)\|^2 \leq 2(\|u_1\|_L^2 + \|u_2\|_L^2). \tag{5.54}$$

Thus we conclude that (5.45) holds, so Theorem 5.4.2 yields our result. □

Using Corollary 5.4.1, we can now prove Theorem 5.2.3:

Proof of Theorem 5.2.3. Positive upper Lyapunov exponent yields

$$\|\Phi_{n_k}(\theta, E)\| > e^{n_k(L^*(E)/2)}$$

for some subsequence n_k. Thus we may apply the corollary with $f(L) = e^{L(L^*(E)/2)}$ to see that

$$\dim_{\mathcal{F}}^{+}(\mu(A \cap \cdot)) \prec \frac{2}{L^*(E)\ln(\frac{1}{1-\epsilon}t^{-2})(\ln(2\ln(\frac{1}{1-\epsilon}t^{-2}))^{1+\delta}/L^*(E))}.$$

Note that

$$\ln(\frac{1}{1-\epsilon}t^{-2}) = \ln(t^{-2}) - \ln(1-\epsilon)$$

and

$$(\ln(2\ln(\frac{1}{1-\epsilon}t^{-2})))^{1+\delta} = (\ln 4 - \ln(\frac{1}{2}\ln(1-\epsilon) + \ln(t^{-1})))^{1+\delta}.$$

In both of these, the $\ln(1-\epsilon)$ and $\ln 4$ terms do not contribute meaningfully to the limit as $t \to 0$, so we can see that

$$\frac{2}{L^*(E)\ln(\frac{1}{1-\epsilon}t^{-2})(\ln(2\ln(\frac{1}{1-\epsilon}t^{-2}))^{1+\delta}/L^*(E))} \sim \frac{1}{\ln(1/t)(\ln(\ln(1/t)))^{1+\delta}},$$

where $\sim$ here is meant as the equivalence relation defined on the family of all Hausdorff dimension functions, $\mathcal{H}$. The second part of the theorem follows from Theorem 5.2.4, which is proved in the next section. $\square$

We also have

Theorem 5.4.3. *Let $f(L) = L^{g(L)}$ be a continuous, strictly increasing function such that (1) $f(0) \geq 0$ and (2) $\limsup_{L\to\infty} g(L) = \alpha \in (1,\infty)$. Let*

$$\rho_\beta(t) = \frac{t^{2/(1+\beta)}}{(\ln t)^{2\beta/(1+\beta)}} \tag{5.55}$$

and suppose $\mathcal{F}$ is a family of comparable Hausdorff dimension functions that contains ρ_β for some $\beta < \alpha$. Suppose that for every E in some Borel set A, we can find a solution, v, to $Hv = Ev$ that satisfies

$$\limsup_{L\to\infty} \frac{\|v\|_L^2}{f(L)} > 0.$$

Then $\dim_{\mathcal{F}}^+(\mu(A\cap\cdot)) \leq \beta$.

Proof. As before, we have the following bounds on $\|u_1\|$ and $\|u_2\|$:

$$\|u_1\|_L \lesssim L^{1/2}\ln L \tag{5.56}$$

and

$$\|u_2\|_{L_n}^2 \gtrsim f(L_n) \tag{5.57}$$

for some sequence $L_n \to \infty$. Taken together, we have

$$\rho_\beta(\|u_1\|_{L_n}^{-1}\|u_2\|_{L_n}^{-1})\|u_1\|_{L_n}^2 = \frac{\|u_1\|_{L_n}^{2\beta/(1+\beta)}}{\|u_2\|_{L_n}^{2/(1+\beta)}(\ln(\|u_1\|_{L_n}\|u_2\|_{L_n}))^{2\beta/(1+\beta)}2/(1+\beta)} \tag{5.58}$$

$$\lesssim \frac{L_n^{\beta/(1+\beta)}\ln(L_n)^{2\beta/(1+\beta)}}{L_n^{g(L_n)/(1+\beta)}(g(L_n)\ln(L_n))^{2\beta/(1+\beta)}} \tag{5.59}$$

$$= \frac{L_n^{\frac{\beta-g(L_n)}{1+\beta}}}{g(L_n)^{\frac{2\beta}{1+\beta}}} \to 0. \tag{5.60}$$

We now finish by appealing to Theorem 5.2.2. □

There is also an analogous version of Corollary 5.4.1.

5.5 Proof of Theorem 5.2.4

Let $\beta(x)$ be a non-negative increasing convex function such that $\ln(\beta(x))$ is still convex. For example, we could take $\beta(x) = e^x$. Moreover, suppose that $G(t) = 1/\beta^{-1}(1/t^2)$ defines a zero dimensional Hausdorff dimension function. Let f^j denote β^{-1} composed with itself j-times. We will consider any family of Hausdorff dimension functions, $\mathcal{F} = \{\rho_\alpha : \alpha \in I\}$, such that there are α_1, α_2, and $\alpha_3 \in I$ and $j \geq 1, \eta, \delta > 0$ such that $\rho_{\alpha_1} = G(t)^{1/\eta} f^j(1/t^2)^{-1}$, $\rho_{\alpha_2} = G(t)^{(1-\delta)/\eta}$ and $\rho_{\alpha_3} = G(t)^{1/\eta}/(\ln(\beta^{-1}(1/t)))^{1+\delta}$.

Define length scales inductively by $L_1 = 2, L_{n+1} = \beta(L_n)^n$ and define a potential

$$V(n) = \begin{cases} \beta(L_k)^\eta & n = L_k \\ 0 & n \notin \{L_k\}_{k=1}^\infty \end{cases}. \tag{5.61}$$

We will begin with an elementary lemma which will be useful:

Lemma 5.5.1. *Let $\beta(x)$ be defined as above. Then $\beta^{-1}(xy) \leq \beta^{-1}(x) + \beta^{-1}(y)$ for every*

$x, y \geq 0$.

Proof. Since β is increasing, this inequality is trivial if $x = 0$ or $y = 0$, so suppose $x, y > 0$. Then we can write $x = e^{x_1}$ and $y = e^{y_1}$ for some $x_1 \neq -y_1 \in \mathbb{R}$. Since $\ln(\beta(x))$ is convex, the inverse, $\beta^{-1}(e^x)$, is concave. Define $f(x) = \beta^{-1}(e^x)$. Then we have:

$$\beta^{-1}(xy) = f(x_1 + y_1). \tag{5.62}$$

Since positive concave functions are subadditive, we have

$$\begin{aligned} f(x_1 + x_2) &\leq f(x_1) + f(x_2) && (5.63) \\ &= \beta^{-1}(x) + \beta^{-1}(y). && (5.64) \end{aligned}$$

Hence $\beta^{-1}(xy) \leq \beta^{-1}(x) + \beta^{-1}(y)$. □

Now we may turn our attention to a proof of Theorem 5.2.4:

Proof of Theorem 5.2.4(i). It is well-known [37] that the essential spectrum is contained in the interval $[-2, 2]$, so it just remains to prove the dimension result. □

Proof of Theorem 5.2.4(ii). For simplicity, we will prove the theorem for $\eta = 1$. The proof of the general case is similar. Let $I = [a, b] \subset (-2, 2)$ It suffices to prove the theorem for $\mu_\theta(I \cap \cdot)$.

Let $\alpha(x)$ be defined such that $\beta(x) = x^{\alpha(x)}$.

First, we will prove that $\dim^+_{\mathcal{F}}(\mu_\theta(I \cap \cdot)) \leq \alpha_3$ for every every boundary phase θ. For every $E \in I, m > k \geq 0$, let

$$\Phi_{k,m}(E) = T_m(E)T_{m-1}(E) \cdots T_{k+1}(E) \tag{5.65}$$

where

$$T_n(E) = \begin{pmatrix} E - V(n) & -1 \\ 1 & 0 \end{pmatrix}. \tag{5.66}$$

Since $\det(\Phi_{k,m}(E)) = 1$, it follows that $\left\|\Phi_{k,m}^{-1}(E)\right\| = \left\|\Phi_{k,m}(E)\right\|^{-1}$. For any $n \in \mathbb{Z}^+$, if $L_n \leq k < m < L_{n+1}$, then $\Phi_{k,m}(E)$ is the same as the transfer matrix for the free Laplacian. In particular, there exists some constant C_I, depending only on the interval I, such that $1 \leq \|\Phi_{k,m}(E)\| \leq C_I$ for any such k, m and $E \in I$. Moreover, for any $n \in \mathbb{Z}^+$, we have

$$\Phi_{L_n-1,L_n}(E) = T_{L_n}(E) = \begin{pmatrix} E - V(L_n) & -1 \\ 1 & 0 \end{pmatrix}, \tag{5.67}$$

and so

$$\max\{1, V(L_n) - 2\} \leq \|T_{L_n}(E)\| \leq V(L_n) + 3. \tag{5.68}$$

If we consider some $n \in \mathbb{Z}^+$ and $L_n \leq m < L_{n+1}$, then we have

$$\Phi_{0,m}(E) = \Phi_{L_n,m} T_{L_n} \Phi_{L_{n-1},L_n-1} T_{L_{n-1}} \cdots \Phi_{L_1,L_2-1} T_{L_1} \Phi_{0,L_1-1}. \tag{5.69}$$

Thus we see that

$$\begin{aligned} \|\Phi_m(E)\| &\leq C_I^{n+1} \prod_{k=1}^{n} (V(L_k) + 3) \\ &\leq C_1^n \prod_{k=1}^{n} L_k^{\alpha(L_k)} \\ &\leq C_1^n \left(\prod_{k=1}^{n} L_k \right)^{\alpha(L_n)}, \end{aligned} \tag{5.70}$$

where C_1 is some constant. Similarly, for large n, we also have

$$\begin{aligned}\|\Phi_m(E)\| &\geq \left(C_I^{n+1}\prod_{k=1}^{n-1}(V(L_k)+3)\right)^{-1}(V(L_n)-2)\\ &\geq C_2^{-n}\left(\left(\prod_{k=1}^{n-1}L_k\right)^{-1}L_n\right)^{\alpha(L_n)},\end{aligned}\tag{5.71}$$

where C_2 is some constant. Since $L_{n+1}=\beta(L_n)^n$, we see that we can take n sufficiently large so that

$$\frac{L_n}{\beta^{-1}(L_n)} < \left(\prod_{k=1}^{n-1}L_k\right)^{-1}L_n < \prod_{k=1}^{n}L_k < L_n\beta^{-1}(L_n).\tag{5.72}$$

Similarly, for any $\epsilon>0$ and n large enough we have $C_1^n<\beta^{-1}(L_n)^\epsilon$ and $C_2^n<\beta^{-1}(L_n)^\epsilon$. Hence, for any $L_n\leq m<L_{n+1}$,

$$\left(\frac{L_n}{\beta^{-1}(L_n)}\right)^{\alpha(L_n)}\beta^{-1}(L_n)^{-\epsilon}\leq\|\Phi_m(E)\|\leq\left(L_n\beta^{-1}(L_n)\right)^{\alpha(L_n)}\beta^{-1}(L_n)^{\epsilon}.\tag{5.73}$$

Set $h(m)=\beta(m)$. By taking $m=L_n$, we have

$$\begin{aligned}\frac{1}{h(m)}\sum_{k=1}^{m}\|\Phi_k\|^2 &\geq L_n^{-\alpha(L_n)}\left(\frac{L_n}{\beta^{-1}(L_n)}\right)^{2\alpha(L_n)}\beta^{-1}(L_n)^{-2\epsilon}\\ &= L_n^{\alpha(L_n)}\beta^{-1}(L_n)^{-2\alpha(L_n)-2\epsilon}\end{aligned}\tag{5.74}$$

By our assumptions on β, (5.74) $\to\infty$. Corollary 5.4.1 now yields

$$\dim_{\mathcal{F}}^{+}(\mu_\theta(I\cap\cdot))\leq\alpha_3,$$

for every boundary phase θ as desired.

Now we will prove that $\dim_{\mathcal{F}}^{-}(\mu_\theta(I\cap\cdot))\geq\alpha_2$ for every every boundary phase θ. By Theorem

5.2.2 it suffices to prove that for every $E \in I$

$$\liminf_{L\to\infty}(\beta^{-1}(\|u_1\|_L^2 \|u_2\|_L^2))^{\delta-1} \|u_1\|_L^2 > 0. \tag{5.75}$$

First, note that we have $|u_1(m)|^2 + |u_2(m)|^2 \geq \|\Phi_m(E)\|^{-2}$. We see, by (5.73) and our choice of L_n, for sufficiently large n and $L_n \leq m < L_{n+1}$,

$$\|u_1\|_m^2 > \frac{1}{2}\left((L_n - L_{n-1})\left(\frac{L_{n-1}}{\beta^{-1}(L_{n-1})}\right)^{-2\alpha(L_{n-1})} + l\left(\frac{L_n}{\beta^{-1}(L_n)}\right)^{-2\alpha(L_n)}\right) \tag{5.76}$$

$$\geq \frac{L_n}{\beta^{-1}(L_n)} + l\left(\frac{L_n}{\beta^{-1}(L_n)}\right)^{-2\alpha(L_n)}, \tag{5.77}$$

where $l = m - L_n + 1$. Similarly, we have

$$\|u_2\|_m^2 < L_n\left(L_{n-1}\beta^{-1}(L_{n-1})\right)^{2\alpha(L_{n-1})} + l\left(L_n\beta^{-1}(L_n)\right)^{2\alpha(L_n)} \tag{5.78}$$

$$\leq L_n\beta^{-1}(L_n) + l\left(L_n\beta^{-1}(L_n)\right)^{2\alpha(L_n)}. \tag{5.79}$$

For simplicity, let

$$A_n = \frac{L_n}{\beta^{-1}(L_n)}$$

$$B_n = \left(\frac{L_n}{\beta^{-1}(L_n)}\right)^{-2\alpha(L_n)}$$

$$C_n = L_n\beta^{-1}(L_n)$$

$$D_n = \left(L_n\beta^{-1}(L_n)\right)^{2\alpha(L_n)}$$

so that (5.77) becomes $\|u_1\|_m^2 > A_n + lB_n$ and similarly (5.79) becomes $\|u_2\|_m^2 < C_n + lD_n$. By combining (5.77), (5.79) and (5.44), and letting $1 \leq l < L_{n+1} - L_n + 1$, we obtain

$$
\begin{aligned}
(\beta^{-1}(\|u_1\|_L^2 \|u_2\|_L^2))^{\delta-1} \|u_1\|_L^2 &\geq (\beta^{-1}((C_n + lD_n)(m(\ln(m))^2)))^{\delta-1}(A_n + lB_n) \\
&= \frac{A_n + lB_n}{(\beta^{-1}((C_n + lD_n)(l + L_n - 1)(\ln(l + L_n - 1))^2))^{1-\delta}} \qquad (5.80) \\
&\equiv F_{n,\delta}(l).
\end{aligned}
$$

We now need to analyze lower bounds for $F_{n,\delta}(l)$ for $1 \leq l < L_{n+1} - L_n + 1$, so consider the two cases:

Case 1: $lB_n \leq A_n$

Case 2: $lB_n \geq A_n$

We can see that case 1 is equivalent to the case where $l \leq D_n C_n L_n^{-2\epsilon}$ and case 2 is equivalent to the case where $l \geq D_n C_n L_n^{-2\epsilon}$.

Considering case 1, we have:

$$
\begin{aligned}
F_{n,\delta}(l) &\geq \frac{A_n}{(\beta^{-1}((C_n + C_n L_n^{-2\epsilon} D_n^2)(l + L_n - 1)(\ln(l + L_n - 1))^2))^{1-\delta}} && (5.81) \\
&\geq \frac{A_n}{(\beta^{-1}((2C_n L_n^{-2\epsilon} D_n^2)(2D_n C_n L_n^{-2\epsilon}) \ln(2D_n C_n L_n^{-2\epsilon})))^{1-\delta}} && (5.82) \\
&= \frac{A_n}{(\beta^{-1}(4D_n^3 C_n^2 L_n^{-4\epsilon} \ln(2D_n C_n L_n^{-2\epsilon}))^{1-\delta}} && (5.83) \\
&\geq \frac{A_n}{(\beta^{-1}(4D_n^{3+1/2} C_n^{2+1/2} L_n^{-4\epsilon-1/2}))^{1-\delta}}. && (5.84)
\end{aligned}
$$

We can now appeal to Lemma 5.5.1 and the fact that $D_n = \beta(L_n)^{2(1+\epsilon)} L_n^{\epsilon}$ to obtain:

$$
\frac{A_n}{\beta^{-1}(4D_n^{3+1/2} C_n^{2+1/2} L_n^{-4\epsilon-1/2}))^{1-\delta}} \geq \frac{L_n^{1-\epsilon}}{K L_n^{1-\delta}}, \qquad (5.85)
$$

for some constant $K > 0$. Since we may take ϵ arbitrarily small by taking n sufficiently large, we conclude that this limits to ∞ for every $\delta > 0$.

Now considering case 2, we have:

$$F_{n,\delta}(l) \geq \frac{A_n + lB_n}{(\beta^{-1}(KlD_n(2l)(\ln(2l))^2))^{1-\delta}} \tag{5.86}$$
$$\geq \frac{A_n + lB_n}{(\beta^{-1}(K) + 2\beta^{-1}(l) + \beta^{-1}(D_n) + \beta^{-1}((\ln(2l))^2))^{1-\delta}} \tag{5.87}$$
$$\to \infty. \tag{5.88}$$

Thus for every $\delta > 0$, $F_{n\delta}(l) \to \infty$ as $n, l \to \infty$, which completes our proof.

$\square$

Proof of Theorem 5.2.4(iii). Once again, we will consider $\eta = 1$. We will show that for Lebesgue a.e. E, and a.e. θ, the equation $Hu = Eu$ has solutions with appropriate decay properties.

Fix $\theta_0 = 0$, and let $H = H_{\theta_0}$. For each $m \in \mathbb{Z}^+$, let G_m be the operator on $l^2(\mathbb{Z}^+)$ given by

$$\langle \delta_i, G_m \delta_j \rangle = \delta_{i,m-1}\delta_{j,m} + \delta_{i-1,m}\delta_{j,m-1}. \tag{5.89}$$

For each $k \in Z^+$, define new operators $H'_k = H - G_{L_k}$ and $\hat{H}_k = H - G_{L_k} - G_{L_k+1}$, and for every $z \in \mathbb{C}$, define the resolvent operators $G(z) = (H - z)^{-1}, G'_k(z) = (H'_k - z)^{-1}$, and $\hat{G}_k(z) = (\hat{H}_k - z)^{-1}$. Moreover, for $i, j \in \mathbb{Z}^+$, let the corresponding Green's functions be given by

$$G(i,j,z) = \langle \delta_i, G(z)\delta_j \rangle \tag{5.90}$$
$$G'_k(i,j,z) = \langle \delta_i, G'_k(z)\delta_j \rangle \tag{5.91}$$
$$\hat{G}_k(i,j,z) = \left\langle \delta_i, \hat{G}_k(z)\delta_j \right\rangle. \tag{5.92}$$

Now considering some $n > L_k$, we can use the resolvent identity $G(z) = G'_k(z) - G(z)G_{L_k}G'_k(z)$ to obtain

$$G(1, n, z) = -G(1, L_k - 1, z)G'_k(L_k, n, z). \tag{5.93}$$

A similar computation with $G'_k(z)$, yields the identity

$$G'_k(z) = \hat{G}_k(z) - \hat{G}_k(z)G_{Lk+1}G'_k(z),$$

so we have

$$G'_k(Lk, n, z) = -\hat{G}_k(L_k, L_k, z)G'_k(L_k + 1, n, z) = \frac{-1}{V(L_k) - z}G'_k(L_k + 1, n, z). \tag{5.94}$$

Together, (5.93) and (5.94) yield

$$G(1, n, z) = G(1, L_k - 1, z)G'_k(L_k + 1, n, z)\frac{1}{V(L_k) - z}. \tag{5.95}$$

Whenever $z = E + i\epsilon, \epsilon > 0$, we see that $G(i, j, z)$ and $G'_k(i, j, z)$ have the form (5.11) and thus are Borel transforms of signed measures. We know (see e.g. [67] for details) that Borel transforms have finite non-tangential limits a.e. on the real axis: $|G(i, j, E)| = |G(i, j, E + i0)| < +\infty$ and $|G'_k(i, j, E)| = |G'_k(i, j, E + i0)| < +\infty$.

Let us also recall Boole's equality for Borel transforms of singular measures: if $F(z)$ is the Borel transform of a singular measure on $\mathbb{R}$ such that $\mu(\mathbb{R}) = 1$, then for any $\lambda > 0$, we have $|\{E : f(E) > \lambda\}| = 2/\lambda$. Since we have already shown that the spectral measures of H for any vector δ_i are singular (Theorem 5.2.4 (ii) above), we conclude that

$$|\{E : |G(i, j, E)| > \lambda\}| \leq 4/\lambda, \tag{5.96}$$

$$|\{E : |G'_k(i, j, E)| > \lambda\}| \leq 4/\lambda. \tag{5.97}$$

From this, we deduce that for any $j \geq 1, \gamma > 0$, $k > 1, n > L_k$ and $E \in (-2, 2)$,

$$\left|\left\{E : |G(1, n, E)| > \frac{f^j(L_k)^\gamma}{V(L_k) - 2}\right\}\right| \leq \frac{8}{f^j(L_k)^{\gamma/2}}, \tag{5.98}$$

where $f(x) = \beta^{-1}(x)$ and $f^j(x)$ is the j-fold composition of f with itself. Let us now fix $j \geq 1$. By our choice of L_k, $\sum_{k=2}^{\infty}(f^j(L_k))^{-\gamma/2} < \infty$ for every $j \geq 1$ and $\gamma > 0$. By the Borel-Cantelli lemma, for Lebesgue a.e. $E \in (-2, 2)$, there exists a $K(E, j)$ such that for any $k > K(E, j)$ and $n = L_k + 1$ or $L_k + 2$,

$$|G(1, n, e)| \leq \frac{f^j(L_k)^\gamma}{V(L_k) - 2}. \tag{5.99}$$

Now, if E is such that the sequence $u_n = \{G(1, n, E)\}_{n=1}^{\infty}$ exists, it necessarily solves the equation $Hu = Eu$ for $n > 2$. Thus for any $L_k + 2 < n \leq L_{k+1}$, we can recover $G(1, n, E)$ using $G(1, L_k + 1, E)$ and $G(1, L_k + 2, E)$ and the action of the free transfer matrix:

$$\begin{pmatrix} G(1, n+1, E) \\ G(1, n, E) \end{pmatrix} = \Phi_{L_k+2,n}(E) \begin{pmatrix} G(1, L_k + 2, E) \\ G(1, L_k + 1, E) \end{pmatrix}. \tag{5.100}$$

Since we know that the free transfer matrix is bounded, we have $\left\|\Phi_{L_k+2,n(E)}\right\| \leq C(E)$ for $L_k + 2 < n < L_{k+1}$ and $E \in (-2, 2)$. Hence for $L_k < n \leq L_{k+1}$, (5.99) holds for the same full measure set of E as above and $k > K(E)$.

It now follows that for Lebesgue a.e. $E \in (-2, 2)$, there exists a solution v of $Hu = Eu$ with $|v(0)|^2 + |v(1)|^2 = 1$ and a constant $C = C(E)$, such that for sufficiently large k and $n > L_k$,

$$|v(n)| < C\frac{f^j(L_k)^\gamma}{V(L_k)} = Cf^j(L_k)^\gamma\beta(L_k)^{-1} = Cf^j(L_k)L_{k+1}^{-1/k}. \tag{5.101}$$

Moreover, since there can be at most one subordinate solution of $Hu = Eu$ with the normalization property $|v(0)|^2 + |v(1)|^2 = 1$ which is decaying, v must be the unique subordinate solution of $Hu = Eu$. We also have, for $m \in \mathbb{Z}^+, L_n < m \leq L_{n+1}$ with n sufficiently large,

$$\|v\|_m^2 = \sum_{j=1}^{m} |v(j)|^2 \tag{5.102}$$

$$= \sum_{j=1}^{L_{k(E)}} |v(j)|^2 + \sum_{j=L_{k(E)}+1}^{m} |v(j)|^2 \tag{5.103}$$

$$\leq C(E, v) + C \sum_{i=k(E)}^{n} L_{i+1} f^j(L_i)^{2\gamma} L_{i+1}^{-2/i} \tag{5.104}$$

$$\leq C(E, v) + C f^j(L_n)^{2\gamma} L_{n+1}. \tag{5.105}$$

Now we return to considering H_θ, where θ can vary. Recall that we can view H_θ as H along with an appropriate rank-one perturbation at the origin. By the theory of rank-one perturbations (again, we refer readers to [67] for full details), it is known that for any set $A \subset \mathbb{R}$ with $|A| = 0$, we have $\mu(A) = 0$ for Lebesgue a.e. boundary phase θ. Since the set of energies for with the solution v above does not exist is a Lebesgue null set, we can conclude that for a.e. boundary phase θ, the associate spectral measure μ is supported on the set of E where the solution v above exists. Furthermore, since μ must also be supported on the set of energies for which u_1 is subordinate, it follows that for a.e. θ and a.e. E with respect to μ, u_1 must coincide with v above.

For this u_1 and $m = L_n + L_n L_{n+1}^{2/n}$, we have

$$\|u_1\|_m^2 = \sum_{j=1}^{L_n} |u_1(j)|^2 + \sum_{L_n+1}^{m} |u_1(j)|^2 \tag{5.106}$$

$$\leq C(E, v) + C f^j(L_{n-1})^{2\gamma} L_n + C(m - L_n) f^j(L_n)^{2\gamma} L_{n+1}^{-2/n} \tag{5.107}$$

$$= C(E, v) + C f^j(L_{n-1})^{2\gamma} L_n + C f^j(L_n) L_n \tag{5.108}$$

$$\leq C(1 + f^j(L_n)^{2\gamma} L_n). \tag{5.109}$$

On the other hand, a similar analysis yields

$$\|u_2\|_m^2 \geq L_n \beta(L_n)^2 \beta^{-1}(L_n)^{-1} \tag{5.110}$$

for $m = L_n + L_n L_{n+1}^{2/n}$.

Thus, if $g_k(x) = (\beta^{-1}(1/x^2) f^k(1/x^2))^{-1}$, and $m = L_n + L_n L_{n+1}^{2/n}$, then

$$g_k(\|u_1\|_m^{-1} \|u_2\|_m^{-1}) \|u_1\|_m^2 \leq \frac{C(1 + f^j(L_n)^{2\gamma} L_n)}{\beta^{-1}(L_n \beta(L_n)^2 \beta^{-1}(L_n)^{-1}) f^k(L_n \beta(L_n)^2 \beta^{-1}(L_n)^{-1})} \tag{5.111}$$

$$\leq \frac{C(1 + f^j(L_n)^{2\gamma} L_n)}{2 L_n f^{k-1}(L_n)}. \tag{5.112}$$

Since this limits to 0 whenever $k \leq j$, and since $j \geq 1$ was arbitrary, we conclude that $\dim_{\mathcal{F}}^{+}(\mu(A \cap \cdot)) \leq \alpha_1$. □

5.6 Rank one perturbations: general results

We will now consider a probability measure μ on $\mathbb{R}$ and the self-adjoint operator $H : L^2(d\mu) \to L^2(d\mu)$ given by multiplication by x. Let φ be any cyclic unit vector in $L^2(d\mu)$. We define the rank one perturbation of H by φ as

$$H_\lambda = H + \lambda \langle \varphi, \cdot \rangle \varphi, \quad \lambda \in \mathbb{R}. \tag{5.113}$$

We will let μ_λ denote the spectral measure associated to H_λ and φ. Let F_λ denote the Borel transform of μ_λ, and write $F_0 = F$. Then

$$F_\lambda(z) = \frac{F(z)}{1 + \lambda F(z)}, \tag{5.114}$$

$$\operatorname{Im} F_\lambda(z) = \frac{\operatorname{Im} F(z)}{|1 + \lambda F(z)|^2}, \tag{5.115}$$

$$d\mu_\lambda(x) = \lim_{x \to \epsilon^+} \frac{1}{\pi} \operatorname{Im} F_\lambda(x + i\epsilon) dx, \tag{5.116}$$

$$\mu_{\lambda,\text{sing}} \text{ is supported by } \{x : F(x + i0) = -1/\lambda\}. \tag{5.117}$$

In addition to the Borel transform, we define

$$G(x) = \int \frac{d\mu(y)}{(x - y)^2}. \tag{5.118}$$

It is well know that

$$\left\{x : G(x) < \infty, F(x + i0) = -\lambda^{-1}\right\} = \text{set of eigenvalues of } H_\lambda. \tag{5.119}$$

Lemma 5.6.1. *Let $\mathcal{F}$ be a family of comparable Hausdorff measure functions and let $f = \rho_{\alpha_1} \in \mathcal{F}$. Suppose that for a family of intervals A_n, we have*

$$|A_n| \leq f^{-1}(b_n)$$

where $b_n \geq 0$ is a summable sequence of real numbers. Then $\dim_{\mathcal{F}} (\limsup A_n) \leq \alpha_1$.

Proof. Fix $g \in \mathcal{F}$ such that $f \prec g$. That is, $\lim f(t)/g(t) = \infty$, so $g(t) \leq f(t)$. Thus, $g^{-1}(t) \geq f^{-1}(t)$. Since f is a Hausdorff dimension function, $f^{-1}(t) \to 0$ as $t \to 0$. Hence $|A_n| \to 0$, so given δ, we can choose N_δ so that $|A_n| \leq \delta$ for $n \geq N_\delta$. Then for $m \geq N_\delta$,

$\bigcup_{n=m}^{\infty} A_n$ is a δ-cover of $\limsup A_n$. Thus,

$$\sum_{n=m}^{\infty} g(|A_n|) \leq \sum_{n=m}^{\infty} g(f^{-1}(b_n)) \leq \sum_{n=1}^{\infty} g(g^{-1}(b_n)) < \infty. \tag{5.120}$$

Thus, as $m \to \infty$, we have $\sum_{n=m}^{\infty} g(|A_n|) \to 0$. We conclude that

$$\lim_{\delta \to 0} \inf_{\delta\text{-covers}} \left\{ \sum_{i=1}^{\infty} g(|F_i|) \right\} = 0. \tag{5.121}$$

Thus $\dim_{\mathcal{F}}(\limsup A_n) < \rho_\beta$ for every $\beta > \alpha_1$, so $\dim_{\mathcal{F}}(\limsup A_n) \leq \alpha_1$. $\square$

Theorem 5.6.1. *Let $f(t)$ be a zero dimensional Hausdorff dimension function, let $\mathcal{F} = \{\rho_\alpha = f(t^\alpha) : \alpha > 0\}$, and suppose $d\mu(E) = \sum_{n=1}^{\infty} a_n d\delta_{E_n}(E)$ where a_n obeys the condition that*

$$|a_n| \leq f^{-1}(b_n),$$

where b_n is a summable sequence of positive real numbers. Then for every λ we have $\dim_{\mathcal{F}}(\operatorname{supp}(d\mu_\lambda)) \leq 2$.

Proof. Let $G(x)$ be defined as above and let $S = \{x : G(x) = \infty, x \notin \{E_i\}_{i=1}^{\infty}\}$. Then the Aronszajn-Donoghue theory [66] says that for any $\lambda \neq 0$, $d\mu_\lambda^{sc}$ is supported by S, Thus, the spectral measure $d\mu_\lambda$ is supported by $S \cup \{\text{eigenvalues of } H_\lambda\}$. Since the set of eigenvalues is countable, it will not contribute to the dimension of $\operatorname{supp}(d\mu_\lambda)$, so it suffices to prove that S has $\dim_{\mathcal{F}}(S) \leq 2$.

Fix $\epsilon > 0$, let $c_{n,\epsilon} = \frac{1}{2}|a_n|^{1/2-\epsilon}$ and let $A_n^\epsilon = [E_n - c_{n,\epsilon}, E_n + c_{n,\epsilon}]$. Then

$$|A_n^\epsilon| = 2c_{n,\epsilon} = |a_n|^{1/2-\epsilon} \leq f^{-1}(b_n)^{1/2-\epsilon}.$$

Now by Lemma 5.6.1, for every $\epsilon > 0$ and every $f(t^{2/(1-2\epsilon)}) \prec g(t)$ we have $\mu^g(\limsup A_n^\epsilon) = 0$.

Now it remains to show that $S \subset \limsup A_n^\epsilon$ for every ϵ. That is, it remains to show that if $x \notin \limsup A_n^\epsilon$ and $x \notin \{E_n\}_{n=1}^\infty$, then $G(x) < \infty$. If $x \notin \limsup A_n^\epsilon$ then for some N_0, we must have $x \notin \bigcup_{n=N_o}^\infty A_n^\epsilon$. Now observe that

$$G(x) = \sum_{n=1}^{\infty} \frac{a_n}{|x - E_n|^2} \tag{5.122}$$

$$= \sum_{n=1}^{N_0} \frac{a_n}{|x - E_n|^2} + \sum_{n=N_0}^{N\infty} \frac{a_n}{|x - E_n|^2} \tag{5.123}$$

$$\leq C + \sum_{n=N_0}^{\infty} \frac{a_n}{c_{n,\epsilon}^2} \tag{5.124}$$

$$\leq C + \sum_{n=N_0}^{\infty} 2a_n^{2\epsilon} \tag{5.125}$$

$$\leq C + \sum_{n=N_0}^{\infty} 2f^{-1}(b_n)^{2\epsilon}. \tag{5.126}$$

The first sum is bounded because $x \notin \{E_n\}_{n=1}^\infty$. Since f is a zero dimensional Hausdorff dimension function, $t^\alpha \prec f^{-1}(t)$ for every $\alpha > 0$, so $f^{-1}(b_n)^{2\epsilon}$ is summable, so we have $G(x) < \infty$. Thus $S \subset \limsup A_n^\epsilon$ for every ϵ. Thus $\dim_{\mathcal{F}}(S) \leq 2/(1-2\epsilon)$ for every $\epsilon > 0$. By our definition of $\mathcal{F}$, it follows that $\dim_{\mathcal{F}}(S) \leq 2$. $\square$

By considering the larger family $\mathcal{G} = \left\{ f(t^\alpha)^\beta : \alpha, \beta > 0 \right\}$, we can actually take $b_n = 1/n$ in the above theorem and conclude with the same result, for a suitably redefined choice of indices ρ_γ.

Definition 5.6.1. Let H be a self-adjoint operator on $l^2(\mathbb{Z}^\nu)$. We say that H has semi-uniformly localized eigenfunctions (SULE) if and only if H has a complete set $\{\varphi_n\}_{n=1}^\infty$ of orthnormal eigenfunctions, there is $\alpha > 0$ and $m_n \in \mathbb{Z}^\nu, n = 1, ...,$ and for each $\delta > 0$, a C_δ so that

$$|\varphi_n(m)| \leq C_\delta e^{\delta |m_n| - \alpha |m - m_n|} \tag{5.127}$$

for all $m \in \mathbb{Z}^\nu$ and $n = 1, 2, ...$

Lemma 5.6.2 ([20])**.** *Suppose that H has SULE. Then there are C and D and a labeling of eigenfunctions so that*

$$|\varphi_n(0)| \le C \exp(-Dn^{1/\nu}). \tag{5.128}$$

Theorem 5.6.2 ((c.f. Theorem 5.2.5))**.** *Suppose H has SULE and let $\mathcal{F} = \{(\ln(1/t))^{-\alpha} : 0 < \alpha < \infty\}$. Let $H_\lambda = H + \lambda \langle \delta_0, \cdot \rangle \delta_0$. Let $d\mu$ be the spectral measure of H associated to δ_0, and let $d\mu_\lambda$ be the corresponding spectral measures for H_λ. Then for every λ, $\dim_{\mathcal{F}}(\mathrm{supp}(d\mu_\lambda)) \le \nu$.*

Proof. Let μ be the spectral measure associated to H and δ_0, and μ_λ the spectral measures of H_λ. Observe that we have

$$\delta_0 = \sum_{n=1}^{\infty} \varphi_n(0)\varphi_n. \tag{5.129}$$

Set $a_n = \varphi_n(0)$. We can see that

$$d\mu(E) = \sum_{n=1}^{\infty} a_n d\delta_{E_n}, \tag{5.130}$$

where E_n is the eigenvalue associated to the eigenfunction φ_n. By Lemma 5.6.2, we have $|a_n| \le C \exp(-Dn^{1/\nu}) = C/f^{-1}(n)$. We can see that $f(n) = \left(\frac{-\ln(n)}{D}\right)^\nu$, so by Theorem 5.6.1 we conclude that, for every λ, $\dim_{\mathcal{F}}(\mathrm{supp}(d\mu_\lambda)) \le \nu$. $\square$

5.7 Dynamical bounds

Consider a separable Hilbert space $\mathscr{H}$ and $H : \mathscr{H} \to \mathscr{H}$ a self adjoint operator. Let us fix a vector $\psi \in \mathscr{H}$ with $\|\psi\| = 1$. The time evolution of ψ is given by

$$\psi(t) = e^{-iHt}\psi. \tag{5.131}$$

We now introduce the following notation:

$$\langle A \rangle (t) = \langle \psi(t), A\psi(t) \rangle \tag{5.132}$$

for any operator A on $\mathscr{H}$, and

$$\langle f \rangle_T = \langle f(t) \rangle_T = \frac{1}{T}\int_0^T f(t)dt \tag{5.133}$$

for any measurable function f.

We also have the moments of the position operator in $l^2(\mathbb{Z}^\nu)$:

$$|X|^m = \sum_{n\in\mathbb{Z}^\nu} |n|^m \langle \delta_n, \cdot \rangle \delta_n. \tag{5.134}$$

Definition 5.7.1. Let μ be a finite Borel measure, and let ρ be a Hausdorff dimension function. We say the measure μ is uniformly ρ-Hölder continuous (UρH) if there exists a constant $C > 0$ such that $\mu(I) < C\rho(|I|)$ for sufficiently small intervals I.

Definition 5.7.2. Let H be a self-adjoint operator on a Hilbert space $\mathscr{H}$. We denote the the ρ-continuous subspace as

$$\mathscr{H}_{\rho c} := \{\psi \in \mathscr{H} : \mu_\psi \text{ is } \rho\text{-continuous}\}. \tag{5.135}$$

Theorem 5.7.1 (Rogers and Taylor [63])**.** *Let μ be a finite Borel measure on $\mathbb{R}$ and let ρ be a Hausdorff dimension function. Then μ is ρ-continuous if and only if for each $\epsilon > 0$ there are mutually singular Borel measures $\mu_1^\epsilon, \mu_2^\epsilon$, such that $d\mu = d\mu_1^\epsilon + d\mu_2^\epsilon$, μ_1^ϵ is UρH, and $\mu_2^\epsilon(\mathbb{R}) < \epsilon$.*

Theorem 5.7.2. *Let ρ be a Hausdorff dimension function and $\mathscr{H}_{uh}(\rho) = \{\psi : \mu_\psi \text{ is } U\rho H\}$. Then $\mathscr{H}_{uh}(\rho)$ is a vector space and*

$$\overline{\mathscr{H}_{uh}(\rho)} = \mathscr{H}_{\rho c}. \tag{5.136}$$

Proof. The only non-trivial vector space property is that $\mathscr{H}_{uh}(\rho)$ is closed under linear combinations, so that is all we will prove here. Let $\psi_1, \psi_2 \in \mathscr{H}_{uh}(\rho)$, and let $\varphi = a\psi_1 + b\psi_2$. By assumption, there are constants C_1 and C_2 and $\delta > 0$ such that $\mu_{\psi_1}(I) < C_1\rho(|I|)$ and $\mu_{\psi_2}(I) < C_2\rho(|I|)$ for all intervals I with $|I| < \delta$. For such I, let P_I denote the spectral projection on I. Then

$$\begin{aligned} \mu_\varphi(I) &= \langle \varphi, P_I\varphi \rangle \\ &= \langle a\psi_1 + b\psi_2, aP_I\psi_1 + bP_I\psi_2 \rangle \\ &\leq |a|^2 \langle \psi_1, P_I\psi_1 \rangle + |b|^2 \langle \psi_2, P_i\psi_2 \rangle + 2|a||b| | \langle \psi_1, P_I\psi_2 \rangle |. \end{aligned} \tag{5.137}$$

Now

$$\begin{aligned} | \langle \psi_1, P_I\psi_2 \rangle | &\leq \sqrt{\langle \psi_1, P_I\psi_1 \rangle \langle \psi_2, P_I\psi_2 \rangle} \\ &\leq \frac{1}{2}(\langle \psi_1, P_i\psi_1 \rangle + \langle \psi_2, P_I\psi_2 \rangle), \end{aligned} \tag{5.138}$$

so we have

$$\begin{aligned}
\mu_\varphi(I) &\le |a|^2 \langle \psi_1, P_I\psi_1\rangle + |b|^2 \langle \psi_2, P_i\psi_2\rangle + 2|a||b| |\langle \psi_1, P_I\psi_2\rangle | \\
&\le (|a|^2 + |a||b|) \langle \psi_1, P_I\psi_1\rangle + (|b|^2 + |a||b|) \langle \psi_2, P_I\psi_2\rangle \\
&= (|a|^2 + |a||b|)\mu_{\psi_1}(I) + (|b|^2 + |a||b|)\mu_{\psi_2}(I) \\
&\le C_1(|a|^2 + |a||b|)\rho(|I|) + C_2(|b|^2 + |a||b|)\rho(|I|) \\
&= C\rho(|I|).
\end{aligned} \tag{5.139}$$

Thus $\mathscr{H}_{uh}(\rho)$ is a vector space.

Since Theorem 5.7.1 implies that $\mathscr{H}_{uh}(\rho) \subset \mathscr{H}_{\rho c}$, we have $\overline{\mathscr{H}_{uh}(\rho)} \subset \overline{\mathscr{H}_{\rho c}}$. Since $\mathscr{H}_{\rho c}$ is closed, we have $\overline{\mathscr{H}_{uh}(\rho)} \subset \mathscr{H}_{\rho c}$. By Theorem 5.7.1, we can decompose $d\mu_\varphi, \varphi \in \mathscr{H}_{\rho c}$, into a sum of mutually singular measures: $d\mu_\varphi = d\mu_1^\epsilon + d\mu_2^\epsilon$, where $d\mu_1^\epsilon$ is UρH and $d\mu_2^\epsilon(\mathbb{R}) < \epsilon$. Let S_ϵ be a Borel set that supports μ_2^ϵ such that $\mu_1^\epsilon(S_\epsilon) = 0$, and let P_{S_ϵ} denote the spectral projection on S_ϵ. We have

$$\varphi = P_{S_\epsilon}\varphi + (1 - P_{S_\epsilon})\varphi$$

with $P_{S_\epsilon}\varphi \in \mathscr{H}_{uh}(\rho)$ and $\|(1 - P_{S_\epsilon})\varphi\|^2 < \epsilon$. Thus φ is the norm-limit of vectors in $\mathscr{H}_{uh}(\rho)$, so $\mathscr{H}_{\rho c} \subset \overline{\mathscr{H}_{uh}(\rho)}$ □

Lemma 5.7.1. *If μ_ψ is UρH, then there exists a constant $C = C(\psi)$ such that for any $\varphi \in \mathscr{H}$ with $\|\varphi\| \le 1$, we have*

$$\langle | \langle \varphi, \psi(t)\rangle |^2\rangle_T < C\rho(1/T). \tag{5.140}$$

Remark 34. This is simply Lemma 3.2 from [58] converted into the general gauge function setting. The proof is identical, but we will provide it below for convenience.

Proof. The spectral theorem implies that H restricted to the cyclic subspace spanned by ψ is unitarily equivalent to multiplication by x on $L^2(\mathbb{R}, d\mu_\psi)$. Thus, for each $\varphi \in \mathscr{H}$ there exists $f_\varphi \in L^2(\mathbb{R}, d\mu_\psi)$ with $\|f_\varphi\|_2 \le \|\varphi\|$ such that

$$\langle \varphi, \psi(t) \rangle = \langle \varphi, e^{-iHt}\psi \rangle = \int e^{-ixt} f_\varphi(x) d\mu_\psi(x).$$

By definition, in conjunction with the above,

$$\begin{aligned}
\langle | \langle \varphi, \psi(t) \rangle |^2 \rangle_T &= \int_{\mathbb{R}} \int_{\mathbb{R}} f_\varphi(x) \overline{f_\varphi(y)} e^{-(x-y)^2T^2/4} d\mu_\psi(y) d\mu_\psi(x) && (5.141)\\
&\le \int_{\mathbb{R}} \int_{\mathbb{R}} |f_\varphi(x)||f_\varphi(y)| e^{-(x-y)^2T^2/8} e^{-(x-y)^2T^2/8} d\mu_\psi(y) d\mu_\psi(x). && (5.142)
\end{aligned}$$

Applying the Cauchy-Schwartz inequality in the y variable yields

$$| \langle \varphi, \psi(t) \rangle |^2 \rangle_T \le \int_{\mathbb{R}} \int_{\mathbb{R}} |f_\varphi(x)|^2 e^{-(x-y)^2T^2/4} d\mu_\psi(x) d\mu_\psi(y). \tag{5.143}$$

Since μ_ψ is UρH, we have

$$\begin{aligned}
\int_{\mathbb{R}} e^{-(x-y)^2T^2/4} d\mu_\psi(y) &= \sum_{n=0}^{\infty} \int_{n/2T \le |x-y| < (n+1)/2T} e^{-(x-y)^2T^2/4} d\mu_\psi(y)\\
&\le C\rho(1/T).
\end{aligned}$$

Thus

$$\begin{aligned}
| \langle \varphi, \psi(t) \rangle |^2 \rangle_T &\le C\rho(1/T) \int |f_\varphi(x)|^2 d\mu_\psi(x) && (5.144)\\
&\le C \|f_\varphi\|_2 \rho(1/T) && (5.145)\\
&\le C\rho(1/T). && (5.146)
\end{aligned}$$

□

Theorem 5.7.3. *Suppose μ_ψ is UρH. Then there exists a constant $C = C(\psi)$ such that for* *any compact operator* $A, p \in \mathbb{N}$, *and* $T > 0$:

$$\langle | \langle A \rangle | \rangle_T < C^{1/p} \|A\|_p \rho(1/T)^{1/p}. \tag{5.147}$$

Proof. Since A is compact, the spectral theorem guarantees the existence of orthonormal bases $\{\psi_n\}_{n=1}^{\infty}, \{\varphi_n\}_{n=1}^{\infty}$, and a monotonely decreasing sequence $\{E_n\}_{n=1}^{\infty},, E_n \geq 0$, such that A is given by the norm-convergent sum

$$A = \sum_{n=1}^{\infty} E_n \langle \varphi_n, \cdot \rangle \psi_n. \tag{5.148}$$

Moreover, $\|A\|_p = \|E_n\|_{l^p}$. Thus we have

$$\langle |\langle A \rangle| \rangle_T = \left\langle \left| \sum_{n=1}^{\infty} E_n \langle \varphi_n, \psi(t) \rangle \langle \psi(t), \psi_n \rangle \right| \right\rangle_T \tag{5.149}$$

$$\leq \sum_{n=1}^{\infty} E_n \langle | \langle \varphi_n, \psi(t) \rangle \langle \psi(t), \psi_n \rangle | \rangle_T \tag{5.150}$$

$$\leq \sum_{n=1}^{\infty} E_n (\langle | \langle \varphi_n, \psi(t) \rangle |^2 \rangle_T)^{1/2} (\langle | \langle \psi(t), \psi_n \rangle |^2 \rangle_T)^{1/2}. \tag{5.151}$$

If we let $p, q \in \mathbb{N}$ be such that $1/p + 1/q = 1$, then we may apply Hölder's inequality to obtain

$$\langle |\langle A \rangle| \rangle_T \leq \|E_n\|_{l^p} \left\| (\langle | \langle \varphi_n, \psi(t) \rangle |^2 \rangle_T)^{1/2} (\langle | \langle \psi(t), \psi_n \rangle |^2 \rangle_T)^{1/2} \right\|_{l^q} \tag{5.152}$$

$$\leq \|A\|_p \left\| \langle | \langle \varphi_n, \psi(t) \rangle |^2 \rangle_T \right\|_{l^q}^{1/2} \left\| \langle | \langle \psi(t), \psi_n \rangle |^2 \rangle_T \right\|_{l^q}^{1/2}. \tag{5.153}$$

Moreover, by Lemma 5.7.1, we have

$$\langle | \langle \varphi_n, \psi(t) \rangle |^2 \rangle_T < C(\psi) \rho(1/T)$$

$$\langle | \langle \psi(t), \psi_n \rangle |^2 \rangle_T < C(\psi) \rho(1/T).$$

Since the ψ_n and φ_n form orthonormal bases, and since e^{-iHt} is unitary, we have

$$\sum_{n=1}^{\infty} \langle | \langle \varphi_n, \psi(t) \rangle |^2 \rangle_T = \sum_{n=1}^{\infty} \langle | \langle \psi(t), \psi_n \rangle |^2 \rangle_T = \|\psi\|^2 = 1. \tag{5.154}$$

Thus

$$\begin{aligned} \left\| \langle | \langle \varphi_n, \psi(t) \rangle |^2 \rangle_T \right\|_{l^q}^q &< (C(\psi)\rho(1/T))^{q-1} \\ \left\| \langle | \langle \psi(t), \psi_n \rangle |^2 \rangle_T \right\|_{l^q}^q &< (C(\psi)\rho(1/T))^{q-1}. \end{aligned} \tag{5.155}$$

Putting (5.155) and (5.153) together, we have

$$\langle | \langle A \rangle | \rangle_T < \|A\|_p \, (C(\psi)\rho(1/T))^{(q-1)/q} = C(\psi)^{1/p} \, \|A\|_p \, \rho(1/T)^{1/p}, \tag{5.156}$$

which completes our proof.

$\square$

Now we can prove Theorem 5.2.6:

Proof of Theorem 5.2.6. Let $\psi_{\rho c} = P_{\rho c}\psi, \psi_{\rho s} = (1 - P_{\rho c})\psi$. By Theorem 5.7.1, there exist mutually singular Borel measures, μ_1, μ_2 such that $d\mu_{\psi_{\rho c}} = d\mu_1 + d\mu_2$, where μ_1 is UρH and $\mu_2(\mathbb{R}) < \frac{1}{2} \|\psi_{\rho c}\|^2$. Let S_1 be a Borel set that supports μ_1 and $\mu_2(S_1) = 0$. Let P_{S_1} denote the spectral projection on S_1 and set $\psi_1 = P_{S_1}\psi_{\rho c}$ and $\psi_2 = (1 - P_{S_1})\psi_{\rho c} + \psi_{\rho s}$. Clearly $\psi = \psi_1 + \psi_2$ and $\psi_1 \perp \psi_2$. Moreover, we have $d\mu_{\psi_1} = d\mu_1$, so ψ_1 is UρH and

$$\|\psi_1\|^2 = \int d\mu_{\psi_1} = \int d\mu_{\psi_{\rho c}} - \int d\mu_2 \geq \frac{1}{2} \|\psi_{\rho c}\|^2 \tag{5.157}$$

and

$$1 = \|\psi_1\|^2 + \|\psi_2\|^2 . \tag{5.158}$$

Let P_N be the projection on the sphere of radius $N \in [0, \infty)$, defined by

$$P_N = \sum_{|n| \leq N} \langle \delta_n, \cdot \rangle \, \delta_n. \tag{5.159}$$

We can see that

$$\begin{aligned} Tr(P_N) &= \sum_{n \in \mathbb{Z}^\nu} \langle \delta_n, P_N \delta_n \rangle \\ &= \sum_{n \in \mathbb{Z}^\nu} \left\langle \delta_n, \sum_{|k| \leq N} \langle \delta_k, \delta_n \rangle \, \delta_k \right\rangle \\ &= \sum_{n \in \mathbb{Z}^\nu} \sum_{|k| \leq N} \langle \delta_n, \delta_k \rangle \, \langle \delta_k, \delta_n \rangle \\ &= \sum_{|n| \leq N} 1 \\ &= c_\nu N^\nu. \end{aligned}$$

Where c_ν depends only on the space dimension ν. Thus P_N is compact and it follows from Theorem 5.7.3 that there exists a constant C_1, which depends only on ψ_1, such that for T sufficiently large and $N > 0$,

$$\begin{aligned} \langle \|P_N \psi_1(t)\|^2 \rangle_T &= \langle \langle \psi_1(t), P_N \psi_1(t) \rangle \rangle_T \\ &< C_1 Tr(P_N) \rho(1/T) \\ &< c_\nu C_\psi N^\nu \rho(1/T). \end{aligned} \tag{5.160}$$

Moreover, we have

$$\begin{aligned} \langle \|P_N \psi(t)\|^2 \rangle_T &\leq \langle (\|P_N \psi_1(t)\| + \|P_N \psi_2(t)\|)^2 \rangle_T \\ &\leq \langle (\|P_N \psi_1(t)\| + \|\psi_2\|)^2 \rangle_T \\ &\leq (\sqrt{\langle \|P_N \psi_1(t)\|^2 \rangle_T} + \|\psi_2\|)^2. \end{aligned}$$

Now if we set

$$N_T = \left(\frac{\|\psi_1\|^4}{64 C_1 c_\nu \rho(1/T)} \right)^{1/\nu}$$

then we have

$$\begin{aligned}
\langle \|P_N \psi(t)\|^2 \rangle_T &< \left(\frac{\|\psi_1\|^2}{8} + \|\psi_2\| \right)^2 \\
&= \frac{\|\psi_1\|^4}{64} + \|\psi_2\|^2 + \frac{1}{4} \|\psi_1\|^2 \|\psi_2\| \\
&< \|\psi_2\|^2 + \frac{1}{2} \|\psi_1\|^2 \\
&= 1 - \frac{1}{2} \|\psi_1\|^2 .
\end{aligned}$$

Since

$$\langle \|P_{N_T} \psi(t)\|^2 \rangle_T + \langle \|(1 - P_{N_T})\psi(t)\|^2 \rangle_T = 1, \tag{5.161}$$

we have

$$\langle \|(1 - P_{N_T})\psi(t)\|^2 \rangle_T > \frac{1}{2} \|\psi_1\|^2 . \tag{5.162}$$

Hence

$$\begin{aligned}
\langle \langle |X|^m \rangle \rangle_T &= \left\langle \left\langle \psi(t), \sum_{n \in \mathbb{Z}^\nu} |n|^m \langle \delta_n, \psi(t) \rangle \delta_n \right\rangle \right\rangle_T && (5.163) \\
&\geq \left\langle \left\langle \psi(t), \sum_{|n| \geq N_T} N_T^m \langle \delta_n, \psi(t) \rangle \delta_n \right\rangle \right\rangle_T && (5.164) \\
&= N_T^m \langle \langle \psi(t), (1 - P_{N_T})\psi(t) \rangle \rangle_T && (5.165) \\
&\geq \frac{1}{2} \|\psi_1\|^2 N_T^m && (5.166) \\
&= \frac{1}{2} \|\psi_1\|^2 \left(\frac{\|\psi_1\|^4}{64 C_1 c \nu} \right)^{m/\nu} \rho(1/T)^{-m/\nu}. && (5.167)
\end{aligned}$$

This completes our proof.

□

Bibliography

[1] A. Avila, S. Jitomirskaya, and C. Sadel. Complex one-frequency cocycles. *Journal of the European Mathematical Society, Volume 16, Issue 9, 2014, pp. 1915-1935*, 06 2013.

[2] A. Avila, Y. Last, M. Shamis, and Q. Zhou. On the abominable properties of the almost Mathieu operator with Lioville frequencies. In preparation.

[3] J.-M. Barbaroux, F. Germine, and S. Tcheremchantsev. Fractal dimension and the phenomenon of intermittency in quantum dynamics. *Duke Mathematical Journal*, 110(1):161–194, 2001.

[4] J.-M. Barbaroux, F. Germinet, and S. Tcheremchantsev. Fractal dimensions and the phenomenon of intermittency in quantum dynamics. *Duke Mathematical Journal*, 1:161 – 193, 2001.

[5] J. Bourgain. *Green's function estimates for lattice Schrödinger operators and applications*. Princeton University Press, 2005.

[6] J. Bourgain. Positivity and continuity of the Lyapounov exponent for shifts on $\mathbb{T}^d$ with arbitrary frequency vector and real analytic potential. *Journal d'Analyse Mathématique*, 96(313 - 355), 2005.

[7] J. Bourgain and M. Goldstein. On nonperturbative localization with quasi-periodic potential. *Ann. of Math.*, 2(152):835 0 879, 2000.

[8] J. Bourgain and S. Jitomirskaya. Anderson localization for the band model. *Geometric Aspects of Functional Analysis. Lecture Notes in Mathematics*, 1745(67 - 79), 2000.

[9] J. Bourgain and S. Jitomirskaya. Continuity of the Lyapunov exponent for quasiperiodic operators with analytic potential. *Journal of Statistical Physics*, 108(516):1203 – 1218, 2002.

[10] J. Bourgain and I. Kachkovski. Anderson localization for two interacting quasiperiodic particles. *Geom. Funct. Anal.*, 29:3 – 43, 2019.

[11] A. Cai, P. Duarte, and S. Klein. Mixed random-quasiperiodic cocycles. *Preprint*, 09 2021.

[12] A. Cai, P. Duarte, and S. Klein. Furstenberg theory of mixed random-quasiperiodic cocycles. *Preprint*, 01 2022.

[13] R. Carmona, A. Klein, and F. Martinelli. Anderson localization for Bernoulli and other singular potentials. *Comm. Math. Phys.*, 188:41 – 66, 1987.

[14] D. Damanik, J. Fillman, and P. Gohlke. Spectral characteristics of schrödinger operators generated by product systems. *Preprint*, 03 2022.

[15] D. Damanik and M. Landragin. Log-dimensional spectral properties of one-dimensional quasicrystals. *Proc. Amer. Math. Soc.*, 131, 2003.

[16] D. Damanik and S. Tcheremchantsev. Power-law bounds on transfer matrices and quantum dynamics in one dimension. *Comm. Math. Phys.*, 236:513–534, 2003.

[17] D. Damanik and S. Tcheremchantsev. Upper bounds in quantum dynamics. *Journal of the AMS*, 20(3):799 – 827, 2007.

[18] D. Damanik and S. Tcheremchantsev. Quantum dynamics via complex analysis methods: general upper bounds wihtout time-averaging and tight lower bounds for the strongly coupled Fibonacci Hamiltonian. *Journal of Functional Analysis*, 255(10):2872 – 2887, 2008.

[19] R. del Rio, S. Jitomirskaya, Y. Last, and B. Simon. What is localization. *Phys. Rev. Lett*, 75:117, 1995.

[20] R. del Rio, S. Jitomirskaya, Y. Last, and B. Simon. Operators with singular continuous spectrum, IV. Hausdorff dimensions, rank one perturbations, and localization. *Journal d'Analyse Mathématique*, 69(1):153–200, 1996.

[21] R. del Rio, M. Kudryavtsev, and L. Silva. Rank one perturbations of Jacobi matrices with mixed spectra. *Math. Nachr.*, 279(5-6):502–512, 2006.

[22] R. del Rio, M. Makarov, and B. Simon. Operators with singular continuous spectrum. II. Rank one operators. *Comm. Math. Phys.*, 165(1):59–67, 1994.

[23] P. Duarte and S. Klein. *Lyapunov exponents of linear cocycles: continuity via large deviations*, volume 3 of *Atlantis studies in dynamical systems*. Atlantis press, 2016.

[24] P. Duarte and S. Klein. *Continuity of the Lyapunov exponents of linear cocycles*. IMPA, 2017.

[25] P. Duarte and S. Klein. Topological obstructions to dominated splitting for egodic translations on the higher dimension torus. *Discrete & continuous dynamical systems*, 38(11):5379 – 5387, 2018.

[26] P. Duarte and S. Klein. Continuity, positivity and simplicity of the lyapunov exponents for quasi-periodic cocycles. *Journal of the European Math. Soc.*, page 56 pages, 2019.

[27] F. Germinet, A. Kiselev, and S. Tcheremchantsev. Transfer matrices and transport for Schrödinger operators. *Annales de L'Institut Fourier*, 54(3):787 – 830, 2004.

[28] D. J. Gilbert. On subordinace and analysis of the spectrum of Schrödinger operators with two singular endpoints. *Proc. Roy. Soc. Edinburgh Sect. A*, 112:213–229, 1989.

[29] D. J. Gilbert and D. B. Pearson. On subordinacy and analysis of the spectrum of one-dimensional Schrödinger operators. *Journal of Mathematical Analysis and Applications*, 128:30–56, 1987.

[30] M. Goldstein and W. Schlag. Hölder continuity of the integrated density of states for quasi-periodic schrödinger equations and averages of shifts of subharmonic functions. *Ann. Math.*, 154:155–203, 2001.

[31] A. Gordon. The point spectrum of the one-dimensional Schrödinger operator. *Uspehi Mat. Nauk*, 31:257 – 258, 1976.

[32] R. Han and S. Jitomirskaya. Quantum dynamical bounds for ergodic potentials with underlying dynamics of zero topological entropy. *Analysis and PDE*, 12(4):867–902, 2019.

[33] B. Helffer, Q. Liu, Y. Qu, and Q. Zhou. Positive Hausdorff dimensional spectrum for the critical almost Mathieu operator. *Comm. Math. Phys.*, 368:369–382, 2019.

[34] M. Herman. Une methode pour minorer les exposants de Lyapounov et quelques exemples montrant le caractère local d'un théorème d'Arnold et de Moser sur le tore de dimension 2. *Comment. Math. Helv.*, 58:453 – 562, 1983.

[35] S. Jitomirskaya, D. A. Koslover, and M. S. Schulteis. Continuity of the Lyapunov exponent for analytic quasiperiodc cocycles. *Ergod. Th. & Dynam. Sys.*, 29:1881 – 1905, 2009.

[36] S. Jitomirskaya and I. Krasovsky. Critical almost Mathieu operator: hidden singularity, gap continuity, and the Hausdorff dimension of the spectrum. 2019.

[37] S. Jitomirskaya and Y. Last. Power-law subordinacy and singular spectra I. half-line operators. *Acta. Math.*, 183:171–189, 1999.

[38] S. Jitomirskaya and Y. Last. Power-law subordinacy and singular spectra. II. Line operators. *Comm. Math. Phys.*, 211:643–658, 2000.

[39] S. Jitomirskaya and W. Liu. Arithmetic sepctral transitions for the Maryland model. *Comm. Math. Phys.*, (6):1025–1051, 2017.

[40] S. Jitomirskaya and W. Liu. Universal reflective-hierarchical structure of quasiperiodic eigenfunctions and sharp spectral transition in phase. https://arxiv.org/abs/1802.00781, 2018.

[41] S. Jitomirskaya and W. Liu. Upper bounds on transport exponents for long range operators. *J. Math. Phys.*, 62(7), 2021. Special issue celebrating the work of Jean Bourgain.

[42] S. Jitomirskaya, W. Liu, and S. Tcheremchantsev. Wavepacket spreading and fractal spectral dimension of quasiperiodic operators with singular continuous spectrum. *Preprint*, 2015.

[43] S. Jitomirskaya and C. A. Marx. Analytic quasi-periodic cocycles with singularities and the Lyapunov exponent of extended Harper's model. *Comm. Math. Phys.*, 316:237 – 267, 2012.

[44] S. Jitomirskaya and R. Mavi. Continuity of the measure of the spectrum for quasiperiodic Schrödinger operators with rough potentials. *Comm. Math. Phys.*, 325:585–601, 2014.

[45] S. Jitomirskaya and R. Mavi. Dynamical bounds for quasiperiodic schrödinger operators with rough potentials. *International Mathematics Research Notices*, 2016.

[46] S. Jitomirskaya and M. Powell. Logarithmic quantum dynamical bounds for arithmetically defined ergodic schrödinger operators with smooth potentials. In *Analysis at large: dedicated to the life and work of Jean Bourgain*. Springer. To appear, November 2021.

[47] S. Jitomirskaya and H. Schulz-Baldes. Upper bounds on wavepacket spreading for random Jacobi matrices. *Comm. Math. Phys.*, 273:601 – 618, 2007.

[48] S. Jitomirskaya, H. Schulz-Baldes, and G. Stolz. Delocalization in random polymer models. *Communications in Mathematical Physics*, 233(1):27–48, Feb 2003.

[49] S. Jitomirskaya and B. Simon. Operators with singular continuous spectrum, III. almost periodic Schrödinger operators. *Comm. Math. Phys.*, 165:201 – 205, 1994.

[50] S. Jitomirskaya and X. Zhu. Large deviations of the Lyapunov exponent and localization for the 1D Anderson model. *Comm. Math. Phys.*, 370(3):311 – 324, 2019.

[51] S. Kahn and D. B. Pearson. Subordinacy and spectral theory for infinite matrices. *Helv. Phys. Acta*, 65:505–527, 1992.

[52] R. Killip, A. Kiselev, and Y. Last. Dynamical upper bounds on wavepacket spreading. *American Journal of Mathematics*, 125(5):1165–1198, October 2003.

[53] W. Kirsch, S. A. Molchanov, and L. A. Pastur. The one-dimensional Schrödinger operator with unbounded potential: the pure point spectrum. *Funct. Anal. Appl.*, 24:176 – 186, 1990.

[54] S. Klein. Anderson localization for the discrete one-dimensional quasi-periodic Schrödinger operator with potential defined by a Gevrey-class function. *Journal of Functional Analysis*, 218:255–292, 2005.

[55] S. Klein. Localization for quasiperiodic Schrödinger operators with multivariable Gevrey potential functions. *J. Spectr. Theory*, 4(3):431 – 484, 2014.

[56] M. Landrigan. *Log-dimensional properties of spectral measures.* PhD thesis, UC Irvine, 2001.

[57] M. Landrigan and M. Powell. Fine dimensional properties of spectral measures. *In Preparation*, 2021.

[58] Y. Last. Quantum dynamics. *Journal of Functional Analysis*, 142(0155):406–445, 1996.

[59] W. Liu. Quantitative inductive estimates for Green's functions of non-self-adjoin matrices. *Analysis and PDE*, 2022. To appear.

[60] R. Mavi. Log-dimensional bounds for the spectral measure of the disordered Holstein model. https://arxiv.org/abs/2009.03182v1.

[61] M. Powell. Continuity of the lyapunov exponent for analytic multi-frequency quasiperiodic cocycles. In preparation.

[62] M. Powell. Positivity of the Lyapunov exponent for analytic quasiperiodic operators with arbitrary finite valued background. Preprint. Arxiv, June 2022.

[63] C. A. Rogers and S. J. Taylor. The analysis of additive set functions in Euclidean space. *Acta. Math.*, 101:273–302, 1959.

[64] C. A. Rogers and S. J. Taylor. Additive set functions in Euclidean space. II. *Acta. Math.*, 109:207 – 240, 1963.

[65] M. Shamis and S. Sodin. Upper bounds on quantum dynamics in arbitrary dimension. Preprint, arXiv 2111.10902, November 2021.

[66] B. Simon. Spectral analysis of rank one perturbations and applications. In *CRM Proc. Lecture Notes*, volume 8, pages 109–149, 1995.

[67] B. Simon. Equilibrium measures and capacities in spectral theory. *Inverse Problems Imaging*, 1:713–772, 2007.

[68] B. Simon and T. Spencer. Trace class perturbations and the absence of absolutely continuous spectra. *Comm. Math. Phys.*, 125:113 – 125, 1989.

[69] E. Soretz and T. Spencer. Positive Lyapounov exponents for schrödinger operators with quasi-periodic potentials. *Comm. Math. Phys.*, 142(3):543 – 566, 1991.

[70] S. Tcheremchantsev. Mixed lower bounds for quantum transport. *Journal of Functional Analysis*, 197:247–282, 2003.

[71] S. Tcheremchantsev. Dynamical analysis of Schrödinger operators with growing sparse potentials. *Comm. Math. Phys.*, 253:221 – 252, 2005.

[72] Y. Wang and J. You. Examples of discontinuity of lyapunov exponent in smooth quasi-periodic cocycles. *Duke Mathematical Journal*, 162(13):2363 – 2412, 2013.

Appendix A

Plurisubharmonic function facts and estimates: the proofs

Our goal here is to obtain estimates on the decay of the Fourier coefficients for subharmonic and plurisubharmonic functions, and then establish a boosting inequality which we use in Chapter 5 to establish a large deviation estimate.

When $d = 1$, (i.e. subharmonic case) Fourier coefficient decay follows from an application of the following result from [23].

Lemma A.0.1 ([23] Lemma 6.7)**.** *Suppose $u : \mathbb{T} \to \mathbb{R}$ is a subharmonic function with a subharmonic extension to $|\Im z| < \rho$ such that*

$$\sup_{|\Im z| < \rho/4} u(z) + \|u\|_{L^2} \leq C. \tag{A.1}$$

Then there exists a constant C', dependent only on C and ρ, such that

$$|\hat{u}(k)| < C('|k| + 1)^{-1}. \tag{A.2}$$

The multifrequency (i.e. plurisubharmonic) estimate follows from the 1-frequency estimate.

Lemma A.0.2. *Suppose* $u : \mathbb{T}^d \to \mathbb{R}$ *is a plurisubharmonic function with a plurisubharmonic extension to* $|\Im z_j| < \rho$ *such that*

$$\sup_{|\Im z_j|<\rho/4} u(z) + \max_{1\le j\le d} \sup_{x_i\in\mathbb{T}, i\neq j} \|u(x_1, ..., x_j, ..., x_d)\|_{L^2(dx_j)} < C. \tag{A.3}$$

Then there exists a constant C'*, dependent only on* C *and* ρ*, such that*

$$\sum_{|k|>K_0} |\hat{u}(k)|^2 \le C' K_0^{-1}. \tag{A.4}$$

Proof. Observe that, for any fixed $x_1, ..., x_{j-1}, x_{j+1}, ..., x_d$, we may define $u_j(x_j) = u(x_1, ..., x_j, ..., x_d)$ and, by assumption, we have

$$\sup_{|\Im z|<\rho/4} u_j(z) + \|u_j(x_j)\|_{L^2(dx_j)} \le C.$$

Thus Lemma A.0.1 applies to u_j and we have, for every j,

$$|\hat{u}_j(k_j)| \le C'(|k_j| + 1)^{-1}.$$

Moreover, the constant C' is independent of j and $x_i, i \neq j$. It follows that

$$\sum_{|k_j|>K_0} |\hat{u}_j(k_j)|^2 \le (C')^2 K_0^{-1}.$$

This may be rewritten as

$$\sum_{|k_j|>K_0} |\hat{u}(x_1, ..., x_{j-1}, x_{j+1}, ..., x_d)(k_j)|^2 \le (C')^2 K_0^{-1}.$$

We may now integrate the left hand side in the variables $x_i, i \neq j$, and apply Parseval's identity to obtain

$$\sum_{|k_j|>K_0, k_i\in\mathbb{Z}, i\neq j} |\hat{u}(k_1, ..., k_{j-1}, k_j, k_{j+1}, ..., k_d)|^2 \leq (C')^2 K_0^{-1}.$$

Since this holds for all $1 \leq j \leq d$, we have

$$\sum_{k\in\mathbb{Z}^d, |k|>K_0} |\hat{u}(k)|^2 < (C')^2 K_0^{-1}$$

as desired. $\square$

It now suffices to show that the hypothesis of these previous lemmas hold in the setting of Chapter 5.

Lemma A.0.3. *Suppose* (A, ω) *is an analytic* $M(2, \mathbb{C})$ *cocycle such that* $\det(A)$ *does not vanish everywhere. Moreover, suppose*

$$\max_{1\leq j\leq d} \sup_{x_i\in\mathbb{T}, i\neq j} \|\ln|\det A(x_1, ..., x_j, ..., x_d)|\|_{L^2(dx_j)} < C. \tag{A.5}$$

Then the previous two lemmas apply with $u(x) = L_N(A, x)$ *with* $C' = C(A)$ *independent of* N.

Proof. Set $u(x) = L_N(A, x)$. It suffices to verify that $u(x)$ satisfies

$$\sup_{|\Im z_j|<\rho/4} u(z) + \max_{1\leq j\leq d} \sup_{x_i\in\mathbb{T}, i\neq j} \|u(x_1, ..., x_j, ..., x_d)\|_{L^2(x_j)} < C. \tag{A.6}$$

Indeed, recall that, for any $A \in M(2, \mathbb{C})$ with $\det(A) \neq 0$, we have

$$\|A\|^2 \geq |\det(A)|.$$

Since we assume $\det(A(x))$ does not vanish everywhere, we know

$$\|A(x)\|^2 \geq |\det(A(x))|$$

for a.e. $x \in \mathbb{T}^d$. Moreover, $A(x)$ has an analytic extension, continuous up to the boundary, to $|\Im z_j| < \rho$ for some $\rho > 0$, such that, for some $M > 0, \|A\|_\rho < M$. Thus

$$u(x) = L_N(A, x)$$

is a plurisubharmonic function with a plurisubharmonic extension to $|\Im z_j| < \rho$ such that

$$\max_{|\Im z_j| < \rho/4} u(z) < |\ln M|.$$

Finally, we have

$$u(x) \geq \frac{1}{2N} \sum_{j=0}^{N-1} \ln |\det(A(x + j\omega))|$$

by properties of det. Hence

$$\|u(x_1, ..., x_j, ..., x_d)\|_{L^2(dx_j)} \leq \max \left\{ |\ln M|, \left\| \frac{1}{2N} \sum_{k=0}^{N-1} \ln |\det(A(x + k\omega))| \right\|_{L^2(dx_j)} \right\} \tag{A.7}$$

$$\leq \max \left\{ |\ln M|, \frac{1}{2N} \sum_{k=0}^{N-1} \|\ln |\det(A(x + k\omega))|\|_{L^2(dx_j)} \right\} \tag{A.8}$$

$$\leq \left\{ |\ln M|, \frac{1}{2} |\ln C| \right\}. \tag{A.9}$$

It follows that, for some C, depending only on properties of A,

$$\max_{|\Im z_j| < \rho/4} u(z) + \max_{1 \leq j \leq d} \sup_{x_i \in \mathbb{T}, i \neq j} \int_{\mathbb{T}} |u(x_1, ..., x_j, ..., x_d)|^2 dx_j < C,$$

and we are done. $\square$

It now suffices to ensure that (A.5) holds. We first recall a useful lemma due to Duarte and Klein.

Lemma A.0.4 ([23] Theorem 6.3)**.** *Suppose $A \in C^{\omega}_{\rho}(\mathbb{T}^d, \mathbb{C})$ is such that $\det(A)$ does not vanish identically. Then there exist $\delta = \delta(A) > 0, C = C(A) < \infty$, and a linear change of coordinates matrix, $M \in SL(d, \mathbb{Z})$, such that for any $B \in C^{\omega}_{\rho}(\mathbb{T}^d, \mathbb{C})$ such that $\|A - B\|_{\rho} < \delta$, we have $f(x) = \det(B \circ M(x))$ satisfies*

$$\max_{1\leq j\leq d} \sup_{x_i\in\mathbb{T}, i\neq j} \|f(x_1, ..., x_j, ..., x_d)\|_{L^2(dx_j)} < C.$$

This ensures Lemma A.0.3 is applicable to the cocycle $A \circ M(x)$. This implies that the argument in our paper actually applies to $A \circ M$. However, since $M \in SL(d, \mathbb{Z})$ is a constant matrix, the Lyapunov exponent for $A \circ M$ and A are the same. We may assume, therefore, that M is the identity matrix. This establishes the desired decay of the Fourier coefficients.

Now we turn our attention to the boosting inequality. We will derive the $d > 1$ estimate from the $d = 1$ estimate. First, we recall a known BMO estimate.

Lemma A.0.5 ([23] Lemma 6.8)**.** *Suppose $u : \mathbb{T} \to \mathbb{R}$ is a subharmonic function with a subharmonic extension to $|\Im z| < \rho$ such that*

$$\sup_{|\Im z|<\rho/4} u(z) + \|u\|_{L^2} \leq C. \tag{A.10}$$

Moreover, suppose

$$\left|\left\{x \in \mathbb{T} : |u(x) - \int u(x)dx| > \epsilon_0\right\}\right| < \epsilon_1. \tag{A.11}$$

Then there exists a constant C', dependent only on C and ρ, such that

$$\|u\|_{BMO(\mathbb{T})} \leq C'(\epsilon_0 + \epsilon_1^{1/2}). \tag{A.12}$$

We may combine the BMO estimate and the Fourier coefficient decay to obtain the next lemma. The argument we use was originally used by Bourgain (see [6] Corollary 1.24) for *bounded* plurisubharmonic functions. The main difference is that we must use (A.13) in lieu of boundedness, and Lemma A.0.5 in lieu of Bourgain's BMO estimate (see [6] Lemma 1.16).

Lemma A.0.6. *Suppose $u : \mathbb{T} \to \mathbb{R}$ is a subharmonic function with a subharmonic extension to $|\Im z| < \rho$ such that*

$$\sup_{|\Im z|<\rho/4} u(z) + \|u\|_{L^2} \leq C. \tag{A.13}$$

Moreover, suppose

$$\left\| u - \int u dx \right\|_{L^1(\mathbb{T})} < \epsilon. \tag{A.14}$$

Then there exists a constant c, dependent only on C and ρ, such that

$$\left| \left\{ x \in \mathbb{T} : \left| u(x) - \int_{\mathbb{T}} u \right| > \epsilon^{1/6} \right\} \right| < e^{c\epsilon^{-1/6}}. \tag{A.15}$$

Proof. Let $\epsilon_0 = \epsilon^{1/3}$ and $\epsilon_1 = \epsilon^{2/3}$. Then

$$\left| \left\{ x \in \mathbb{T} : |u(x) - \int u(x)dx| > \epsilon_0 \right\} \right| < \epsilon_1,$$

and Lemma A.0.5 is applicable. We obtain

$$\|u\|_{BMO(\mathbb{T})} \leq C'\epsilon^{1/3}. \tag{A.16}$$

Now recall the John-Nirenberg inequality:

$$\left|\left\{x \in \mathbb{T} : \left|f - \int_{\mathbb{T}} f\right| > \lambda\right\}\right| < c_1 e^{-c_2 \frac{\lambda}{\|f\|_{BMO}}}. \tag{A.17}$$

Setting $f = u$ and $\lambda = \epsilon^{1/6}$ completes our proof. □

We can now apply this in each variable to deduce an analogue for $d > 1$ (c.f. Lemma 1.27 [6]).

Lemma A.0.7. *Suppose $u : \mathbb{T}^d \to \mathbb{R}$ is a plurisubharmonic function with a plurisubharmonic extension to $|\Im z_j| < \rho$ such that*

$$\sup_{|\Im z_j| < \rho/4} u(z) + \max_{1 \leq j \leq d} \sup_{x_i \in \mathbb{T}, i \neq j} \|u(x_1, ..., x_j, ..., x_d)\|_{L^2(dx_j)} < C. \tag{A.18}$$

Assume, moreover, that

$$\left\|u - \int_{\mathbb{T}^d} u\right\|_{L^1(\mathbb{T}^d)} < \epsilon.$$

Then there exist constants c, dependent only on C and ρ, and $a = a(d)$, such that

$$\left|\left\{x \in \mathbb{T} : \left|u(x) - \int_{\mathbb{T}^d} u\right| > \epsilon^a\right\}\right| < e^{c\epsilon^{-a}}. \tag{A.19}$$