Development in the Estimation of Degree Measure: Integrating Analog and Discrete Representations

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Abstract

We examined adult and child performance on two numerical, geometric estimation tasks. In both tasks adults demonstrated greater accuracy than children as well as more mature representations, in general. Furthermore, evidence from mouse tracking data demonstrates that adult strategy includes the application of discrete landmark values while child strategy, generally, does not. This evidence suggests that adults construct mental representations of landmark values and successfully integrate them into analog tasks. Implications for future intervention studies are discussed.

Keywords: numerical estimation, embodied cognition, mathematical development, cognitive assessment

Introduction

Numerical estimation tasks provide researchers with a powerful means of assessing individuals' mental representation for number. Evidence from brain scans demonstrates that approximate numerical tasks, such as less-than/greater-than judgments, activate cortical regions associated with spatial processes, while activities that rely exclusively upon recall, such as single-digit multiplication, do not (Dehaene, Piazza, Pinel, & Cohen, 2003). According to Dehaene (1997) our ability to map numerical values to spatial magnitudes is what is commonly referred to as "number sense," and grounds all mathematical reasoning.

Yet, the study of number sense extends beyond theoretical interest as recent evidence suggests a link between estimation and mathematical achievement. Along these lines Halberda, Mazzocco, and Feigenson (2008) discovered that 14-year-old's ability to discriminate between dot displays of varying cardinalities was highly correlated with achievement scores extending back to kindergarten. Likewise, Siegler and Booth (2004) found that individual differences on a number line estimation task are correlated with standardized test scores.

In the case of number line estimation individual differences may embody large, qualitative shifts in representation (Siegler & Opfer, 2003). Dehaene (1997) asserts that numerical symbols implicitly recruit a logarithmic representation that is more precise at smaller values. Siegler and Opfer (2003) found that young children, especially with larger numerical ranges, tend to apply this kind of logarithmic representation while estimating the position of a given value on a number line. Specifically, data of estimated magnitude over actual magnitude are best fit by a logarithmic function for these younger children. On the other hand, older children's data, in many cases, is best fit by a straight line.

The emergence of a linear representation has several possible causes and implications. In particular, Siegler and Opfer (2003) differentiate between two models of linear representation. In the *accumulator* model, adopted from Gibbon and Church (1981), noise in the mental representation for a numerical value increases in proportion to the mean. This representation implies increasing variability in the estimates as the magnitude increases. In the *linear-ruler* model – which was found to be a better fit for the data – variability in estimates has a constant relation to magnitude.

The authors suggest that the mature, linear representation is developed through cultural, particularly school-based, experience. Furthermore, evidence of less variability near landmark values along the number line (e.g. quartiles) demonstrates a specific means for implementing the linear-ruler model. One may even speculate that at the lowest-level number representation may be logarithmic or accumulator in nature, but at the level of conscious-level processing number concepts are modulated for specific tasks.

If the appeal of number line estimation tasks is due, in part, to its high ecological validity, one might then find it surprising that although number lines are a ubiquitous feature of elementary school classrooms, many students maintain immature, logarithmic representations. Yet, recent evidence suggests that the development of mature representations may be promoted through simple, economical interventions, such as playing linear board games (Siegler & Ramani, 2008) or providing corrective feedback (Opfer & Siegler, 2007) on values that maximize the logarithmic-linear difference. In the latter case many children demonstrated a logarithmic to linear shift within a few feedback trials.

Yet, the ease with which some children transition from a logarithmic to linear representation begs the question of whether these children already maintain a linear representation of whole numbers and simply learn to recruit it for the given task. From this perspective, "development" of a linear representation in these interventions may capture only the tail end of this learning process, only made possible through years of informal experience with numerical concepts (Ginsberg, 1983).

At the cost of ecological validity, an alternative approach to cognitive developmental research might imply the adoption of a task utilizing a novel, unique spatial representation of number. Within such a paradigm researchers would observe as children (or adults) struggle to construct meaning out of the task, although the task may be meaningless beyond the research setting. This research model would afford psychological researchers with a level of control that is unobtainable with common concepts.

As a compromise between ecological validity and control this study applies the numerical estimation paradigm to degree measure, which is an important element of mathematics, but is under-utilized in elementary school curriculum and therefore relatively unfamiliar to children (Clements & Battista, 1992). Considering that degree measure does not become a major component of curriculum, generally, until high school, research with degree measure provides an opportunity to study numerical development of older children.

Yet, degree measure is not a unitary concept, but is rather composed of two psychologically distinct spatial representations: degree as angle of intersection between lines and degree as rotation. While some tasks, such as LOGO programming, may confound the two concepts (Clements, Battista, & Sarama, 2001), other activities clearly demonstrate that children perceive physical models of each concept distinctly (Mitchelmore, 1998).

Given the unique spatial qualities of each representation, we should expect courses of development for rotation and angle concepts that may differ from whole number concepts. For example, Clements and Burns (2000) found that fourth grade students physically modeled angle values and curtailed the degree of embodiment with increasing expertise. Furthermore, both students and instructors focused on the representation of "benchmark" (or landmark) magnitudes, such as 90°. Although one might perform a degree estimation task by applying the same linear representation developed for the number line, albeit in a circular form, the emphasis on standard landmarks for degree measure suggests that performance with number lines and degree measure is likely to be quite different. Specifically, the mental representation for degree measure might rely upon the integration of continuous models of numbers and discrete abstractions of landmark values.

While the nature of a mental abstraction is a constant source of debate, the grounded or embodied cognition framework (Barsalou, 2008) asserts that all mental representations are composed of sensory-motor elements of experience. Specifically, perceptual symbols develop from frequent encounters with a meaningful type of object. In turn perceptual simulators develop to provide individuals a means of representing a concept in its perceptual absence (Barsalou, 1999).

In the case of angle and rotation, perceptual symbols are likely to embody landmark values. Given the perceptual salience of perpendicular lines – which can be discriminated from non-perpendicular lines by Amazonian tribesman (Dehaene & Izard, 2006) – we should expect that 90° angles are represented in this form. However, the perceptual symbol encoding 90° angles may only account for a limited range of valid right angles, such as right angles with sides oriented horizontally and vertically from the ground. Thus, perceptual symbols may develop in both robustness for particular symbols, and in number, overall.

In the case of rotation, the spatial mapping of language may play an important role in the embodiment of this numerical concept (Lakoff & Nunez, 2000). For example, the directives "turn around" or "turn to your right" may ground landmark values of 180° and 90°, respectively. Older students may develop other landmark values for common spatial transformations, such as a rotation of 45°.

Yet, all numerical tasks involving degree measure do not, explicitly, require landmark values. An angle measure of 117°, for example, is unlikely to have its own unique representation. However, individuals may shuttle between analog, continuous models and discrete, abstract models (Schwartz & Black, 1996). In this case one is likely to apply this process to numerical estimation by searching for relevant landmarks (e.g. 90°) and then applying an analog procedure, in a form of divide-and-conquer. As stated above, Siegler and Opfer (2003) suggest that this is the specific mechanism used by adults to implement a linear-ruler representation on the number line. However, since number lines may utilize an arbitrary range of values, linear representation may reflect the online development of landmarks, rather than the application of perceptual symbols from memory.

Although evidence for the application of landmark strategies is suggested by the pattern of variability in accuracy across numerical range, these accuracy measures reflect only the final judgment of the participant and may hide strategy-relevant features of the estimation process. On the other hand continuous, online measures of performance afford researchers a view of the specific process undertaken by a participant (Spivey, 2007). In this study we adapt a mouse-tracking paradigm in which mouse position and time is recorded at continuous, fine intervals. Specifically, as participants perform the estimation task the mouse's rotational orientation about the center of the screen is recorded to facilitate the analysis of inflections near landmark values.

In the following study we analyze estimation performance of relative experts (graduate students in education) to novices (middle-to-upper elementary school students). Given the adults likely experience with relevant geometric concepts, we expect that these participants are likely to use apply a process of shuttling between landmark and analog representations, which should result in a high proportion of mouse stops near landmark values and an overall linear representation.

The children, on the other hand, are less likely to be familiar with landmark values and may struggle to integrate them into the estimation task. Therefore we suspect that linear representations will be rare. Although these students, mostly in fourth grade, should clearly maintain a linear representation on the number line, we expect that, given the novelty of this task, they are likely to adopt a logarithmic representation. Furthermore, we expect there to be clear differences between adults and children in overall accuracy.

Method

Participants

Sixteen adults (mean age = 28.9, SD = 8.7) were recruited from an introductory cognitive psychology course as part of a research requirement. Sixteen children (mean age = 9.5, SD = .73) were recruited from an after-school program located in a low-SES neighborhood of a large city. The children consisted of two third-graders, one fifth-grader, and thirteen fourth graders.

Tasks

Both adults and children performed two distinct numerical estimation tasks on Apple MacBooks. Both tasks were programmed with Adobe Air 1.1 in the Adobe Flash CS3 development suite. The application covered the entire height of the screen (23 cms) and approximately 87% of the width (32.5 cms).

Both tasks required a single click (and release) on a circle within the display to initiate each trial. Upon completing each trial participants were required to click-and-hold the mouse button for a half second to "lock-in" their answer to reduce the frequency of accidental clicking. Although there was no time limit, if the participant made no motion with the mouse for more than ten seconds the trial was terminated.

In *angle construction* (Figure 1) the participant maneuvers the mouse to rotate one leg (9.2 cms long) of an isosceles triangle clockwise about a fixed vertex, while the other leg remained motionless – opening and closing the triangle. Participants were asked to manipulate a target angle, marked with a red arc, to reflect a target number of degrees. At 0° and 180° the figure becomes a straight line. Motion beyond 180° maintained the appearance at 180° and was recorded as 180°. Participants could move directly from 0° to 180° by moving the mouse counter-clockwise from the initial position.

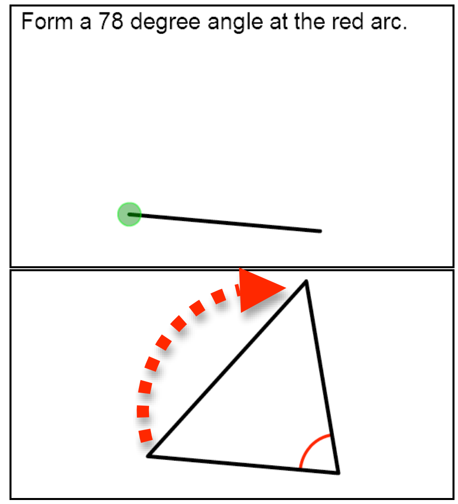

Figure 1: The top pane shows the initial display for an angle construction trial. Participants clicked within the circle to begin the trial. The lower pane shows a triangle that has been formed to match the target value. The arced arrow is superimposed here to demonstrate the vector of motion of the non-stationary vertex from its original position.

In *triangle rotation* (figure 2) the participants maneuvered the mouse to rotate an isosceles triangle about the center of the triangle. Participants were asked to rotate the triangle, clockwise, a target number of degrees from the triangle's initial orientation. A light gray triangle in the initial orientation of the triangle remained throughout the trial to provide a reference. The shape of the triangle was varied between trials by randomizing the angle measure at intersection of the triangles legs from 10° to 170°, although the length of the legs was constant (9.2 cms). Varying the shape was necessary to avoid the use of strategies involving static relationships between the moving triangle and the gray reference triangle. Participants could maneuver the triangle between 0° and 180°. Motion beyond 180° did not affect the appearance of the figure. Like angle construction, in some cases participants moved directly from 0° to 180° by moving the mouse counter-clockwise.

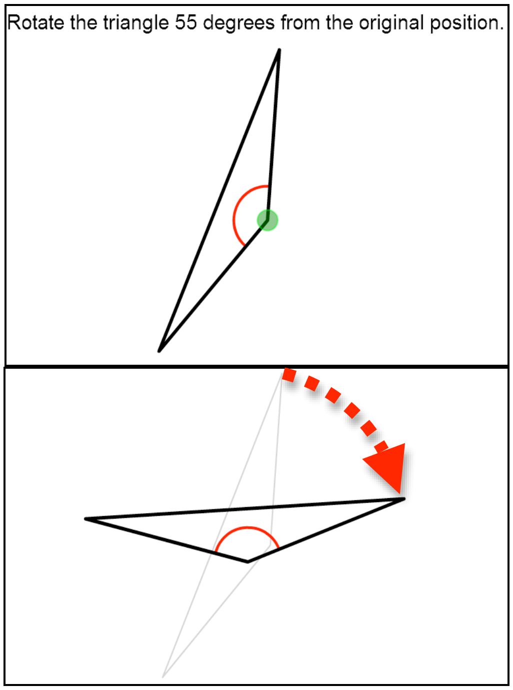

Figure 2: The top pane shows the initial display for a triangle rotation trial. Participants clicked within the circle at the vertex to begin. The lower pane shows how the triangle has been rotated to match the target value.

Procedure

Both children and adults were split into two groups of eight, varying task order. The children received a block of 20 non-feedback trials in both *angle construction* and *triangle rotation*. The adults received 120 trials, organized into six blocks, for each task. However, for the purpose of directly comparing adult and child performance, only the adults' first block for each task were analyzed here. The task was individually administered to adults in a private room. Children performed the task in a dedicated section of a classroom as their classmates completed homework.

Prior to the first block of each task participants received 5 practice trials. Each practice trial required the participant be within 15° of the target and displayed written, verbal feedback suggesting an increase or decrease. The practice values of 90°, 45°, 135°, 15°, and 180° were selected to represent a wide distribution of the range. However, in angle estimation the 180° trial was replaced with a 165° trial to maintain a triangular appearance of the display.

Each block was populated with target values in the range of 10° to 170°. Target values were selected randomly from 20 intervals of 8° over this range. In the interest of directly comparing the two tasks, angles greater than 180° were not used as they cannot form internal angles of a triangle.

Data Analysis

Prior to all analyses outlier trials were removed to eliminate cases of accidental clicks, which prematurely terminated trials. From observation of performance accidental clicks generally occurred near either extreme value (0° or 180°) or within a short time period (e.g. double-clicking). Therefore trials in which the participant moved the mouse less than 5° from the initial position, ended the trial within a degree of the 180° endpoint, or completed the trial in less than one second were removed from analysis. To reduce the likelihood that subjects intended degree measures in the outlier range we only used targets between 10° and 170°.

During each trial the current value of the manipulated degree measure was sampled at approximately every 40 msecs. Degree over time data was fit to a function and smoothed using the 'fda' package within R (Ramsay, Wickham, Graves, & Hooker, 2009). The first derivative of smoothed data, degree change over time, was then searched for values at or near zero for an extended period of time (500 msecs), indicating a stop point.

Stop points within 10° of specific landmark values were tallied and are referred to as *landmark stops*. Likewise stop points in 10° ranges just above and below the landmark ranges were tallied and are referred to as *near-landmark stops*. For example, 90° landmark stops included all stops between 80° and 100°, while 90° near-landmark stops occur in the ranges 70° to 80° and 100° to 110°. For 180° landmark stops were tallied between 170° and 180°, while near-landmark stops range between 160° and 170°.

Although stops in a landmark or near-landmark range could represent random behavior, subjects consistently applying a landmark strategy are more likely to stop within landmark than near-landmark range, while subjects stopping at random should be equally likely to stop within either range. Therefore we suggest that a high proportion of landmark to near-landmark stops indicates the explicit use of a landmark strategy. This landmark-to-near-landmark proportion was calculated as a statistic ranging from -1 (all near-landmark) to 1 (all landmark) by subtracting the count of near-landmark stops from the count of landmark stops and dividing by their total. For example, three landmark stops to one near landmark stop is a value of .50 [i.e. (3-1)/(3+1)]. On the other hand, one landmark stop to two near-landmark stops is a value of -.33 [i.e. (1-2)/(1+2)]. A value of zero indicates either an equal number of landmark and near-landmark stops or no stops.

Results

To determine the nature of a participant's representation of estimated magnitude vs. actual magnitude data was fit to a linear and logarithmic model. Participants were classified to "linear" or "log" representation to indicate the model that accounted for a larger proportion of their variance. In the case where neither model was a significant predictor of estimates participants were classified as "other." Also, the absolute deviations (residuals) from estimated to actual magnitudes of "linear" participants were fit to a linear model to determine the presence of scalar variability. Those participants with a positive slope, significantly different from zero (p < .05) were classified as "accumulator", while those with no trend of increasing deviations were classified as "linear-ruler." Table 1 indicates the distribution of models for adults and children in each task.

Table 1: Model frequencies across task and age.

Model:	Linear-ruler	Linear accum.	Log	Other
Adult–Angle	11	5	0	0
Child – Angle	4	2	5	5
Adult–Rotation	10	2	4	0
Child–Rotation	4	1	1	10

A chi-square test of this distribution indicates that adults and children vary significantly in both angle construction and triangle rotation [$\chi^2(3, N=32) = 14.55$ and 14.7, respectively, ps < .01]. Model frequencies between tasks, within age groups, did not differ significantly, ps > .05.

Furthermore, the means of all absolute deviation from estimated magnitude to target magnitude (error), indicating overall accuracy, showed a similar pattern (Table 2).

Table 2: Mean and standard deviation of absolute deviations from estimated and actual magnitudes.

	Mean Error	SD
Adult–Angle	9.78°	2.69
Child – Angle	35.7°	16.7
Adult–Rotation	16.5°	4.93
Child–Rotation	41.0°	13.9

A two-way ANOVA with task-type as a within subjects factor and age as a between subjects factor reveals a strong main effect for age [$F(1,31)=29.8$, $p<.000$] and a weaker, yet significant, effect for task type [$F(1,31)=4.6$, $p<.05$], indicating better performance for angle construction. Interaction between task and age was non-significant.

To analyze the use of landmarks each participant's trials were divided into 4 quadrants with target values in the ranges 0°-45°, 46°-90°, 91°-135°, and 136°-180°. For each participant the 180° landmark to near-landmark statistic was calculated for trials in quadrant four (136°-180°) and 90° in quadrant two and three (46°-135°). We then applied t-tests to determine whether these values differed significantly from zero, suggesting explicit use of the landmark. The means and p-values are shown in Table 3 below.

Table 3: Mean ratios of landmark to near-landmark stops and associated p-values.

Angle Construction	**Mean Ratio**		**Triangle Rotation**	**Mean Ratio**
Adult – 90° Q2-Q3	.37*		Adult – 90° Q2-Q3	.19
Child – 90° Q2-Q3	-.06		Child – 90° Q2-Q3	.25
Adult – 180° Q4	.25+		Adult – 180° Q4	.48**
Child – 180° Q4	-.06		Child – 180° Q4	.25*

+ p < .1 * p < .05 ** p < .01

For the rotation task both children and adults applied the 180° landmark strategy, while neither did so in the angle construction task. In angle construction only adults utilized a 90° landmark strategy. We applied two ANOVA models to compare age and task for each landmark within its associated region. With 90° landmarks there was a significant main effect of age [$F(1,31)=5.0$, $p < .05$], but not task-type ($p > .05$). With 180° landmarks there was a significant main effect for task-type [$F(1,31)=4.9$, $p < .05$], but not for age.

Discussion

Clearly, there is a large difference between adult and child performance in both tasks. This is certainly not surprising, given the difference in experience between the adults and children in the domain of geometry. Yet, considering that 10 of the 16 children's data for the rotation task could not be fit by either a linear or logarithmic function, the extent of this difference was surprising.

The graphs for individual subjects who were classified as "other" show either general randomness, crowding towards some arbitrary magnitude, or (in one case) a *negative* linear relationship. For these students, the mapping between numerical value and the chosen spatial representation was either meaningless or completely misconceived.

Another explanation is that these children simply refused to "play the game" correctly, and were simply applying their own, idiosyncratic rules. Yet, considering that only half as many children were classified as "other" for angle construction, it is unlikely that this behavior emerged from general disinterest. Rather, many students expressed their frustrations, especially during the rotation task, by telling test administrators that, "I don't know how to do this." Furthermore, in both tasks, several students were unsure as to which direction represented an increase in value.

Adults and children also differed greatly in the application of landmarks. While adults clearly used 90° landmarks for targets ranging from 45° to 135° in angle construction, children showed little evidence of this strategy. In fact, the children were slightly more likely to stop at near-landmark values (albeit at non-significant level). Both adults and children used the 180° landmark in the rotation task, but this may be an effect of the particular environment as 180° represents a clear physical boundary.

One may also reasonably claim that near-landmark stops were, in some cases, attempts at using specific landmarks. Yet, due to a general lack of precision, and difficulty with mouse control, children often stopped outside of the accepted landmark range suggesting that, with practice, landmark stops may replace near-landmark stops.

While, this data demonstrates relative extremes in numerical representation it cannot inform us about the path of development from novice to expert. Rather, we are left to ask whether adult performance result from a wealth of exposure to degree measures in particular or from a flexible understanding of the linear nature of numbers? From the latter perspective one might imagine that adults are able to imagine a curved number line with endpoints at 0° and 180° – which would enable linear performance with relatively little experience with degrees, specifically. Furthermore, given the important role that landmark values holds in adult performance, should child instruction focus on strategies incorporating landmark values or will mental representations for these landmarks emerge from exposure to the entire range of magnitudes?

Such questions suggest the potential of intervention studies to elucidate paths of development. Possible interventions to promote understanding of degree measure may include measuring angles, playing games aimed specifically at these numerical constructs, or situated activities such as LOGO programming. In particular our research team is currently investigating the latter two means of developing numerical understanding.

In a preliminary intervention study applying a LOGO-like environment and geometry curriculum with thirteen children (from this study), we have found a trend towards improvement in overall accuracy measure for angle construction [$t(12)= 2.1$, $p = .059$]. Furthermore, of these 13 students the number of students demonstrating a linear representation increased from four to nine.

Although the study of relatively novel numerical concepts is of theoretical interest, one might argue that if these concepts are so under-represented in curriculum then interventions at this level may be unnecessary or inappropriate. However, the National Council of Teachers of Mathematics (2000) stresses the important of geometric and spatial reasoning for children of all ages. We suspect that mastery of basic concepts, such as angle measure, serves as a grounding for higher-level conceptual skills, such as geometric constructions and proofs

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