Tropical Linear Spaces and Applications

by

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A dissertation submitted in partial satisfaction of the
requirements for the degree of
Doctor of Philosophy

in

Mathematics

in the

Graduate Division

of the

University of California, BERKELEY

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Spring 2012

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Abstract

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Tropical geometry is an area of mathematics that has enjoyed a quick development in the last 15 years. It can be seen as a tool for translating problems in algebraic geometry to combinatorial problems in convex polyhedral geometry. In this way, tropical geometry has proved to be very succesful in different areas of mathematics like enumerative algebraic geometry, phylogenetics, real algebraic geometry, mirror symmetry, and computational algebra.

One of the most basic tropical varieties are tropical linear spaces, which are obtained as tropicalizations of classical linear subspaces of projective space. They are polyhedral complexes with a very rich combinatorial structure related to matroid polytopes and polytopal subdivisions.

In Chapter 1 we give a basic introduction to tropical geometry and tropical linear spaces, and review the basic theory of tropical linear spaces that was developed by Speyer in [Spe08].

In Chapter 2 we study a family of functions on the class of matroids, which are "well behaved" under matroid polytope subdivisions. In particular, we prove that the ranks of the subsets and the activities of the bases of a matroid define valuations for the subdivisions of a matroid polytope into smaller matroid polytopes.

The pure spinor space is an algebraic set cut out by the quadratic Wick relations among the 2^n principal subPfaffians of an $n \times n$ skew-symmetric matrix. Its points correspond to n-dimensional isotropic subspaces of a $2n$-dimensional vector space. In Chapter 3 we tropicalize this picture, and we develop a combinatorial theory of tropical Wick vectors and tropical linear spaces that are tropically isotropic. We characterize tropical Wick vectors in terms of subdivisions of Δ-matroid polytopes. We also examine to what extent the Wick relations form a tropical basis. Our theory generalizes several results for tropical linear spaces and valuated matroids to the class of Coxeter matroids of type D.

In Chapter 4 we study tropical linear spaces locally: For any basis B of the matroid underlying a tropical linear space L, we define the local tropical linear space L_B to be the set of all vectors $v \in L$ that make B a basis of maximal v-weight. The tropical linear space L is the union of all its local tropical linear spaces, which we prove are homeomorphic to

Euclidean space. We also study the combinatorics of local tropical linear spaces, and we prove that they are combinatorially dual to mixed subdivisions of a Minkowski sum of simplices. We use this duality to produce tight upper bounds on their f-vectors. We introduce a certain class of tropical linear spaces called conical tropical linear spaces, and we give a simple proof that they satisfy the f-vector conjecture. Along the way, we give an independent proof of a conjecture of Herrmann and Joswig posed in a first version of [HJS11].

In Chapter 5 we introduce the cyclic Bergman fan of a matroid M. This is a simplicial polyhedral fan supported on the tropical linear space $\mathcal{T}(M)$ of M, which is amenable to computational purposes. It slightly refines the nested set structure on $\mathcal{T}(M)$, and its rays are in bijection with flats of M which are either cyclic flats or singletons. We give a fast algorithm for calculating it, making some computational applications of tropical geometry now viable. We develop a C++ implementation, called TropLi, which is available online. Based on it, we also give an implementation of a ray shooting algorithm for computing vertices of Newton polytopes of A-discriminants.

Contents

Acknowledgments

During my time at Berkeley I have benefited enormously from the help and support of amazing mentors and friends. I am deeply indebted to Bernd Sturmfels for his constant guidance and support. His extraordinary enthusiasm and insights have taught me a lot. I am incredibly grateful to Federico Ardila for his continuous help and encouragement throughout my mathematical career. My thanks also go to Olga Holtz for her invaluable support during my studies.

I have been very fortunate to have had many wonderful people around me during these years. I would like to thank all my friends and academic siblings, who have made my life in Berkeley very stimulating and enjoyable. I am greatly indebted to Laura, who gave me her unconditional love and support whenever I needed it. Finally, I would like to thank my family for all their love and encouragement. Without them, I would not have made it this far.

Chapter 1

Introduction

In this chapter we discuss the motivation, background and main objects of study of this thesis. For a much more detailed introduction to tropical geometry and tropical linear spaces, the reader can consult [Jos12, MS12, Spe08].

1.1 Tropical Geometry

We start we a brief introduction to some of the basic notions of tropical geometry that we will use in the rest of this work. The field of **Puiseux series** in the variable t over the complex numbers is the algebraically closed field $\mathbb{C}\{\{t\}\} := \bigcup_{n=1}^{\infty} \mathbb{C}((t^{\frac{1}{n}}))$. Its elements are formal power series of the form $f = \sum_{k=k_0}^{+\infty} c_k \cdot t^{\frac{k}{N}}$, where N is a positive integer, k_0 is any integer, and the coefficients c_k are complex numbers. The field $\mathbb{C}\{\{t\}\}$ comes equipped with a valuation val : $\mathbb{C}\{\{t\}\} \to \mathbb{Q} \cup \infty$ that makes it a valuated field: take val(f) to be the least exponent r such that the coefficient of t^r in f is nonzero (so val$(0) = \infty$). If $Y \subseteq \mathbb{C}\{\{t\}\}^n$, we define its **valuation** to be the set

$$\operatorname{val}(Y) := \{(\operatorname{val}(y_1), \operatorname{val}(y_2), \ldots, \operatorname{val}(y_n)) \in (\mathbb{Q} \cup \infty)^n : (y_1, y_2, \ldots, y_n) \in Y\}.$$

Denote by $\mathbb{T} := (\mathbb{R} \cup \infty, \oplus, \odot)$ the **tropical semiring** of real numbers with ∞ together with the binary operations **tropical addition** $\oplus$ and **tropical multiplication** $\odot$, defined as $x \oplus y = \min(x, y)$ and $x \odot y = x + y$. Given a multivariate polynomial

$$P = \sum_{a_1, a_2, \ldots, a_n} f_{a_1, a_2, \ldots, a_n} \cdot X_1^{a_1} \cdot X_2^{a_2} \cdot \cdots \cdot X_n^{a_n} \quad \in \quad \mathbb{C}\{\{t\}\}[X_1, X_2, \ldots, X_n],$$

we define its **tropicalization** to be the tropical polynomial obtained by substituting the operations in P by their tropical counterparts and the coefficients by their corresponding valuations. The tropicalization of P is then equal to

$$\operatorname{trop}(P) := \bigoplus_{a_1, a_2, \ldots, a_n} \operatorname{val}(f_{a_1, a_2, \ldots, a_n}) \odot x_1^{a_1} \odot x_2^{a_2} \odot \cdots \odot x_n^{a_n},$$

where exponentiation should be understood as repeated application of tropical multiplication. Given any subset $I \subseteq \mathbb{C}\{\{t\}\}[X_1, X_2, \dots, X_n]$, we define its tropicalization to be the set of tropical polynomials

$$\mathrm{trop}(I) := \{\mathrm{trop}(P) : P \in I\}.$$

The notion of "zero set" for a tropical polynomial is defined as follows. A tropical polynomial p in n variables is the tropical sum (or minimum) of tropical monomials

$$p = \bigoplus_{a_1, a_2, \dots, a_n} v_{a_1, a_2, \dots, a_n} \odot x_1^{a_1} \odot x_2^{a_2} \odot \cdots \odot x_n^{a_n},$$

where the coefficients $v_{a_1, a_2, \dots, a_n}$ are elements of $\mathbb{T}$, and only finitely many of them are not equal to ∞. The **tropical hypersurface** $\mathcal{T}(p) \subseteq \mathbb{T}^n$ is then defined as the set of points $(x_1, x_2, \dots, x_n) \in \mathbb{T}^n$ such that this minimum is attained in at least two different terms of p (or it is equal to ∞). For example, the tropical hypersurface defined by the tropical polynomial $p = 1 \odot x \oplus 0 \odot y^2 \oplus (-2)$ is the set of points in $\mathbb{T}^2$ where the minimum $\min(1 + x, 2y, -2)$ is achieved at least twice (see Figure 1.1).

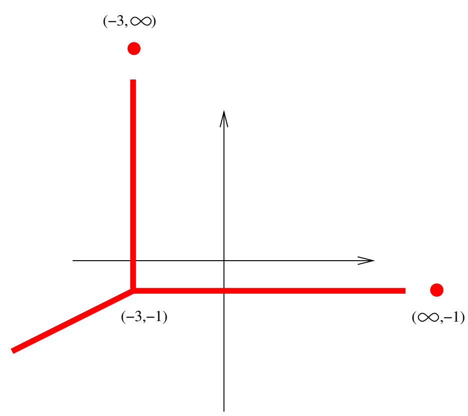

Figure 1.1: A tropical hypersurface

If T is a set of tropical polynomials in n variables, the **tropical prevariety** described by them is $\mathcal{T}(T) := \bigcap_{p \in T} \mathcal{T}(p)$. If $I \subseteq \mathbb{C}\{\{t\}\}[X_1, X_2, \dots, X_n]$ is an ideal then the tropical prevariety $\mathcal{T}(\mathrm{trop}(I))$ is called a **tropical variety**. If the ideal I is generated by some set of polynomials $S \subseteq I$, it is *not* necessarily true that the tropical variety defined by I is equal to the tropical prevariety defined by S, not even if we impose the condition that S be a universal Gröbner basis for I. When it does happen that $\mathcal{T}(\mathrm{trop}(I)) = \mathcal{T}(\mathrm{trop}(S))$ we say that S is a **tropical basis** for I. The notion of tropical basis is very subtle, and it is in general very hard (both theoretically and computationally) to determine if a given set of generators forms such a basis. For an excellent example illustrating these difficulties, the reader is invited to see [CJR11].

The Fundamental Theorem of Tropical Geometry establishes the connection between the "algebraic tropicalization" of an ideal and the "geometric tropicalization" of its corresponding variety. A proof of it can be found in [MS12].

Theorem 1.1.1 (Fundamental Theorem of Tropical Algebraic Geometry)**.** *Let I be an ideal of $\mathbb{C}\{\{t\}\}[X_1, X_2, \ldots, X_n]$ and $X := V(I) \subseteq \mathbb{C}\{\{t\}\}^n$ its associated algebraic set. Then*

$$\mathcal{T}(\mathrm{trop}(I)) \cap (\mathbb{Q} \cup \infty)^n = \mathrm{val}(X).$$

Moreover, if I is a prime ideal then $\mathcal{T}(\mathrm{trop}(I)) \cap \mathbb{R}^n$ is a pure connected polyhedral complex of the same dimension as the irreducible variety X.

In this way, tropical geometry allows us to get information about the variety X just by studying the combinatorially defined polyhedral complex $\mathcal{T}(\mathrm{trop}(I))$. This approach has been very fruitful in many cases, and has led to many beautiful results. See [Jos12, MS12] for a detailed introduction to tropical geometry.

1.2 Tropical Linear Spaces

Tropical linear spaces are one of the most basic objects in tropical geometry. They are obtained as tropicalizations of classical linear subspaces of projective space, and they play a prominent role in several contexts like the study of tropicalizations of varieties obtained as the image of a linear subspace under a monomial map [DFS07], or the study of realizability questions and intersection theory in tropical geometry (see [Sha10], [FR10], [KP09]).

Let $K = \mathbb{C}\{\{t\}\}$, and let $m \leq n$ be nonnegative integers. Denote $[n] := \{1, 2, \ldots, n\}$, and let $\binom{[n]}{m}$ be the set of subsets of size m of the set $[n]$. The set of m-dimensional linear subspaces of K^n is parametrized by the **Grassmannian** $\mathrm{Gr}_{m,n} \subseteq \mathbb{P}_K{}^{\binom{[n]}{m}-1}$ in the following way. Any m-dimensional subspace V of K^n can be written as the rowspace of an $m \times n$ matrix $A \in K^{m \times n}$. To the subspace V we associate a vector $P \in \mathbb{P}_K{}^{\binom{[n]}{m}-1}$ of **Plücker coordinates**, defined as $P_I := \det A_I$ for $I \in \binom{[n]}{m}$, where A_I denotes the maximal minor of A whose columns are indexed by I. The Grassmannian $\mathrm{Gr}_{m,n}$ is then defined to be the projective variety consisting of the Plücker coordinates of all m-dimensional subspaces of K^n. It is a parameter space for the set of these subspaces, that is, the function

$$\begin{array}{ccc} \{m\text{-dimensional subspaces}\} & \longrightarrow & \mathrm{Gr}_{m,n} \\ \mathrm{rowspace}\, A & \longmapsto & (\det A_I)_{I \in \binom{[n]}{m}}. \end{array}$$

is a bijection.

The ideal defining the Grassmannian is called the **Plücker ideal**, and it is generated by special quadratic relations called the **Plücker relations**. The shortest ones are called **3-term Plücker relations**, and they have the form

$$P_{Sab} \cdot P_{Scd} - P_{Sac} \cdot P_{Sbd} + P_{Sad} \cdot P_{Sbc} = 0,$$

where $S \subseteq [n]$ has size $m-2$, and $a, b, c, d \in [n] - S$ are distinct.

The **tropical Grassmannian** $\mathrm{TGr}_{m,n} \subseteq \mathbb{T}^{\binom{[n]}{m}}$ and the **Dressian** $\mathrm{Dr}_{m,n} \subseteq \mathbb{T}^{\binom{[n]}{m}}$ are the tropical variety and the tropical prevariety defined by the Plücker relations, that is,

$$\mathrm{TGr}_{m,n} := \bigcap_{\substack{f \in \text{Plücker} \\ \text{ideal}}} \mathcal{T}(f) \qquad \subseteq \qquad \mathrm{Dr}_{m,n} := \bigcap_{\substack{f \text{ Plücker} \\ \text{relation}}} \mathcal{T}(f).$$

Note that we are working over the tropical semiring $\mathbb{T}$ of real numbers *together with* ∞. Even after intersecting them with $\mathbb{R}^{\binom{[n]}{m}}$, the inclusion between the tropical Grassmannian and the Dressian is in general strict, except in the cases where $m \leq 2$, $m \geq n-2$, or $m = 3$ and $n = 6$ (see [SS04]).

A vector $p \in \mathrm{Dr}_{m,n} \subseteq \mathbb{T}^{\binom{[n]}{m}}$ is called a **tropical Plücker vector** of rank m. By definition, a vector p is a tropical Plücker vector if p satisfies the tropical Plücker relations, that is, for any $S, T \in 2^{[n]}$ satisfying $|S| = m-1$ and $|T| = m+1$, the minimum

$$\min_{i \in T \setminus S} (p_{Si} + p_{T-i}) \tag{1.2.1}$$

is achieved at least twice (i.e., for at least two different values of i) or it is equal to ∞. In the literature, tropical Plücker vectors have also been studied under the name of **valuated matroids**, but using the opposite sign convention to ours [DW92].

In general, it is not enough that a vector $p \in \mathbb{T}^{\binom{[n]}{m}}$ satisfies the tropical 3-term Plücker relations for it to be a tropical Plücker vector. However, if we impose some conditions on the coordinates of p that are distinct from ∞, then the 3-term Plucker relations are enough (see Corollary 3.5.2).

The classical correspondence between linear subspaces and points in the Grassmannian also holds tropically. More specifically, there is a bijection L that makes the following diagram commute:

$$\begin{array}{ccc} \mathrm{Gr}_{m,n} & \longleftrightarrow & \{m\text{-dimensional subspaces of } K^n\} \\ \Big\downarrow \scriptstyle{\mathrm{val}} & & \Big\downarrow \scriptstyle{\mathrm{val}} \\ \mathrm{TGr}_{m,n} & \overset{L}{\longleftrightarrow} & \{\text{tropicalizations of } m\text{-dimensional linear subspaces}\}. \end{array}$$

In fact, the function L can be defined for any tropical Plücker vector (not only the ones in $\mathrm{TGr}_{m,n}$), as we describe below.

Definition 1.2.1. Let $p \in \mathbb{T}^{\binom{[n]}{m}}$ be a tropical Plücker vector. Suppose that $S \subseteq [n]$ has size $m+1$ and the vector $c_S \in \mathbb{T}^n$ defined as

$$(c_S)_i := \begin{cases} p_{S-i} & \text{if } i \in S, \\ \infty & \text{otherwise;} \end{cases} \tag{1.2.2}$$

is not equal to $\vec{\infty} := (\infty, \infty, \ldots, \infty) \in \mathbb{T}^n$. In this case, any vector of the form $c_S + \lambda \cdot \mathbf{1}$ with $\lambda \in \mathbb{R}$ is called a (valuated) **circuit** of p, where $\mathbf{1} := (1, 1, \ldots, 1) \in \mathbb{R}^n$.

Two vectors $x, y \in \mathbb{T}^n$ are said to be **tropically orthogonal**, denoted by $x \top y$, if the minimum $\min(x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$ is achieved at least twice (or it is equal to ∞). If $X \subseteq \mathbb{T}^n$ then its **tropically orthogonal set** is $X^\top := \{y \in \mathbb{T}^n \mid y \top x \text{ for all } x \in X\}$.

Definition 1.2.2. Let $p \in \mathbb{T}^{\binom{[n]}{m}}$ be a tropical Plücker vector, and denote by $\mathcal{C}(p) \subseteq \mathbb{T}^n$ the set of all circuits of p. The space $L(p) := \mathcal{C}(p)^\top \subseteq \mathbb{T}^n$ is called the **tropical linear space** associated to p.

In [Spe08], Speyer studied in detail combinatorial properties of tropical linear spaces associated to tropical Plücker vectors whose coordinates are never equal to ∞, i.e., tropical Plücker vectors in $\mathbb{R}^{\binom{[n]}{m}}$. He proved that the polyhedral complex $L(p) \cap \mathbb{R}^n$ is a pure polyhedral complex of dimension m, and formulated the following conjecture.

Conjecture 1.2.3 (The f-vector conjecture)**.** *If $p \in \mathbb{R}^{\binom{[n]}{m}}$ is a tropical Plücker vector then the polyhedral complex $L(p) \cap \mathbb{R}^n$ has at most $\binom{n-i-1}{i-1}\binom{n-2i}{m-i}$ faces of dimension i that become bounded after modding out by the lineality space generated by the vector $(1, 1, \ldots, 1) \in \mathbb{R}^n$.*

In [Spe08] he proved this conjecture in a few special cases, and in [Spe09] he proved it for tropical linear spaces which arise as the tropicalization of a classical linear subspace of a vector space in characteristic zero (i.e., tropical linear spaces that correspond to tropical Plücker vectors in the tropical Grassmannian). In the general case, the conjecture remains open.

1.3 A Polytopal Perspective

A matroid is a combinatorial object which unifies several notions of independence. We start with some basic definitions. For more information on matroid theory we refer the reader to [Oxl92].

Definition 1.3.1. A **matroid** M is a pair $(E, \mathcal{B})$ consisting of a finite set E and a collection of subsets $\mathcal{B}$ of E, called the **bases** of M, which satisfies the basis exchange axiom: If $B_1, B_2 \in \mathcal{B}$ and $b_1 \in B_1 - B_2$, then there exists $b_2 \in B_2 - B_1$ such that $B_1 - b_1 \cup b_2 \in \mathcal{B}$. The matroid M is said to be **loopless** if for any $e \in E$ there is a basis $B \in \mathcal{B}$ such that $e \in B$.

Example 1.3.2. If E is a finite set of vectors in a vector space, then the maximal linearly independent subsets of E are the bases of a matroid. The matroids arising in this way are called **representable**.

Example 1.3.3. If $k \leq n$ are positive integers, then the subsets of size k of $[n] = \{1, \ldots, n\}$ are the bases of a matroid, called the **uniform matroid** $U_{k,n}$.

In [Spe08], Speyer presented a very useful polytopal perspective on tropical linear spaces that we now describe. To any collection $\mathcal{S}$ of subsets of $[n]$ we can associate a 0/1 polytope $\Gamma(\mathcal{S}) := \text{convex}\{e_S \mid S \in \mathcal{S}\} \subseteq \mathbb{R}^n$, where $e_S := \sum_{i \in S} e_i$. Matroids can be easily characterized from this point of view (see [GGMS87]): A collection $\mathcal{S} \subseteq 2^{[n]}$ is the collection of bases of a matroid M over the ground set $[n]$ if and only if its associated polytope $\Gamma(\mathcal{S})$ has only edges of the form $e_i - e_j$ for $i, j \in [n]$ distinct. In this case, the polytope $\Gamma(M) := \Gamma(S)$ is called a **matroid polytope**.

A **subdivision** of a polytope P is a set of polytopes $S = \{P_1, \ldots, P_m\}$, whose vertices are vertices of P, such that $P_1 \cup \cdots \cup P_m = P$, and for all $1 \leq i < j \leq m$, if the intersection $P_i \cap P_j$ is nonempty then it is a proper face of both P_i and P_j.

The hypersimplex $\Delta_{m,n} \subseteq \mathbb{R}^n$ is the polytope defined as $\Delta_{m,n} := \text{convex}\{e_S \mid S \in \binom{[n]}{m}\}$. Any vector $p \in \mathbb{R}^{\binom{[n]}{m}}$ induces a polytopal subdivision of $\Delta_{m,n}$, as follows. The vector $p \in \mathbb{R}^{\binom{[n]}{m}}$ can be thought of as a height function on the vertices of $\Delta_{m,n}$, giving rise to the "lifted polytope" $\Gamma(p) := \text{convex}\{(e_S, p_S) \in \mathbb{R}^{n+1} \mid S \in \binom{[n]}{m}\}$. Projecting the lower facets of $\Gamma(p)$ (i.e., its facets whose outward normal vector has a negative $(n+1)$st coordinate) back to $\mathbb{R}^n$, we get a polytopal subdivision $\mathcal{D}_p$ of the polytope $\Delta_{m,n}$, called the **regular subdivision** induced by p.

Tropical Plücker vectors admit a beautiful characterization in this language (see [Spe08]): a vector $p \in \mathbb{R}^{\binom{[n]}{m}}$ is a tropical Plücker vector if and only if the regular subdivision $\mathcal{D}_p$ is a matroid polytope subdivision, i.e., it is a subdivision of $\Delta_{m,n}$ into matroid polytopes.

Now, if $p \in \mathbb{R}^{\binom{[n]}{m}}$ is a tropical Plücker vector, it was also proved in [Spe08] that a vector $v \in \mathbb{R}^n$ is in the tropical linear space $L(p)$ associated to p if and only if the projection back to $\mathbb{R}^n$ of the face of the polytope $\Gamma(p)$ that maximizes the dot product with the vector $(v, -1) \in \mathbb{R}^{n+1}$ is the matroid polytope associated to a loopless matroid. In particular, this implies that $L_p \cap \mathbb{R}^n$ is a polyhedral complex dual to the faces of the subdivision $\mathcal{D}_p$ that correspond to loopless matroids.

Example 1.3.4. Let $n = 4$, $m = 2$, and consider the vector $p \in \mathbb{R}^{\binom{[n]}{m}}$ defined as

$$p_S := \begin{cases} 1 & \text{if } S = 12 \text{ or } S = 34, \\ 0 & \text{if } S = 13 \text{ or } S = 14 \text{ or } S = 23 \text{ or } S = 24. \end{cases}$$

The hypersimplex $\Delta_{2,4}$ is the convex hull of all 0/1 vectors in $\mathbb{R}^4$ having exactly two coordinates equal to 1. This polytope lives in the 3-dimensional hyperplane defined by

$x_1 + x_2 + x_3 + x_4 = 2$, and is in fact a regular octahedron. The regular subdivision $\mathcal{D}_p$ induced by p is consists of two square pyramids meeting at their base (and all their faces), as depicted in the left of Figure 1.3.4. Since all faces of this subdivision are matroid polytopes, this ensures that p is a tropical Plücker vector. The tropical linear space $L(p)$ is then dual to all the faces of this subdivision corresponding to loopless matroids, as drawn in red on the right side of Figure 1.3.4.

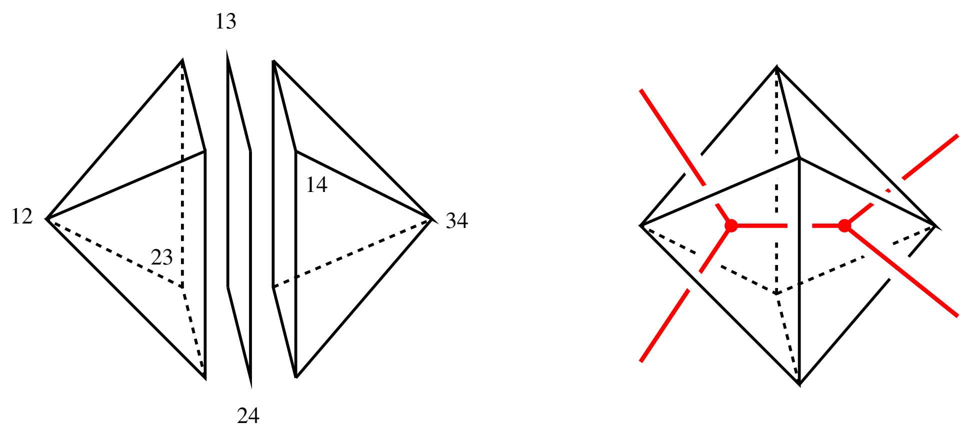

Figure 1.2: A regular matroid subdivision and its associated tropical linear space.

Chapter 2

Matroid Valuations

The material of this chapter is joint work with Federico Ardila and Alex Fink. It was published in the *Canadian Journal of Mathematics* under the title “Valuations for matroid polytope subdivisions” [AFR10].

2.1 Introduction

Aside from its wide applicability in many areas of mathematics, one of the pleasant features of matroid theory is the availability of a vast number of equivalent points of view. Among many others, one can think of a matroid as a notion of independence, a closure relation, or a lattice. One point of view has gained prominence due to its applications in algebraic geometry, combinatorial optimization, and Coxeter group theory: that of a matroid as a polytope. This chapter is devoted to the study of functions of a matroid which are amenable to this point of view.

To each matroid M one can associate a (basis) *matroid polytope* $\Gamma(M)$, which is the convex hull of the indicator vectors of the bases of M. One can recover M from $\Gamma(M)$, and in certain instances $\Gamma(M)$ is the fundamental object that one would like to work with. For instance, matroid polytopes play a crucial role in the matroid stratification of the Grassmannian [GGMS87]. They allow us to invoke the machinery of linear programming to study matroid optimization questions [Sch03]. They are also the key to understanding that matroids are just the type A objects in the family of Coxeter matroids [BGW03].

The subdivisions of a matroid polytope into smaller matroid polytopes have appeared prominently in different contexts: in compactifying the moduli space of hyperplane arrangements (Hacking, Keel and Tevelev [HKT06] and Kapranov [Kap93]), in compactifying fine Schubert cells in the Grassmannian (Lafforgue [Laf99, Laf03]), and in the study of tropical linear spaces (Speyer [Spe08]).

Billera, Jia and Reiner [BJR09] and Speyer [Spe08, Spe09] have shown that some important functions of a matroid, such as its quasisymmetric function and its Tutte polynomial, can be thought of as nice functions of their matroid polytopes: they act as valuations on the subdivisions of a matroid polytope into smaller matroid polytopes.

The purpose of this chapter is to show that two much stronger functions are also valuations. Consider the matroid functions

$$f_1(M) = \sum_{A \subseteq [n]} (A, r_M(A)) \quad \text{and} \quad f_2(M) = \sum_{B \text{ basis of } M} (B, E(B), I(B)),$$

regarded as formal sums. Here r_M denotes matroid rank, and $E(B)$ and $I(B)$ denote the sets of externally and internally active elements of B.

Theorems 2.5.1 and 2.5.4. *The functions f_1 and f_2 are valuations for matroid polytope subdivisions: for any subdivision of a matroid polytope $\Gamma(M)$ into smaller matroid polytopes $\Gamma(M_1), \ldots, \Gamma(M_m)$, these functions satisfy*

$$f(M) = \sum_i f(M_i) - \sum_{i<j} f(M_{ij}) + \sum_{i<j<k} f(M_{ijk}) - \cdots ,$$

where $M_{ab\ldots c}$ is the matroid whose polytope is $\Gamma(M_a) \cap \Gamma(M_b) \cap \cdots \cap \Gamma(M_c)$.

The chapter is organized as follows. In Section 2.2 we present some background information on matroids and matroid polytope subdivisions. In Section 2.3 we define valuations under matroid subdivisions, and prove an alternative characterization of them. In Section 2.4 we present a useful family of valuations, which we use to prove Theorems 2.5.1 and 2.5.4 in Section 2.5. Finally in Section 2.6 we discuss related work.

2.2 Matroid Subdivisions

We start by recalling some of the definitions given in Chapter 1.

Definition 2.2.1. A **matroid** M is a pair $(E, \mathcal{B})$ consisting of a finite set E and a collection of subsets $\mathcal{B}$ of E, called the **bases** of M, which satisfies the *basis exchange axiom:* If $B_1, B_2 \in \mathcal{B}$ and $b_1 \in B_1 - B_2$, then there exists $b_2 \in B_2 - B_1$ such that $B_1 - b_1 \cup b_2 \in \mathcal{B}$.

We will find it convenient to allow $(E, \emptyset)$ to be a matroid; this is not customary.

A subset $A \subseteq E$ is **independent** if it is a subset of a basis. All the maximal independent sets contained in a given set $A \subseteq E$ have the same size, which is called the **rank** $r_M(A)$ of A. In particular, all the bases have the same size, which is called the rank $r(M)$ of M.

Example 2.2.2. If E is a finite set of vectors in a vector space, then the maximal linearly independent subsets of E are the bases of a matroid. The matroids arising in this way are called **representable**, and motivate much of the theory of matroids.

Example 2.2.3. If $k \leq n$ are positive integers, then the subsets of size k of $[n] = \{1, \ldots, n\}$ are the bases of a matroid, called the **uniform matroid** $U_{k,n}$.

Example 2.2.4. Given positive integers $1 \leq s_1 < \ldots < s_r \leq n$, the sets $\{a_1, \ldots, a_r\}$ such that $a_1 \leq s_1, \ldots, a_r \leq s_r$ are the bases of a matroid, called the **Schubert matroid** $SM_n(s_1, \ldots, s_r)$. These matroids were discovered by Crapo [Cra65] and rediscovered in various contexts; they have been called Catalan matroids [Ard03], PI-matroids [BJR09], generalized Catalan matroids [BdM06], and freedom matroids [CS05], among others. We prefer the name Schubert matroid, which highlights their relationship with the stratification of the Grassmannian into Schubert cells [BLVS$^+$99, Section 2.4].

The following geometric representation of a matroid is central to our study.

Definition 2.2.5. Given a matroid $M = ([n], \mathcal{B})$, the (basis) **matroid polytope** $\Gamma(M)$ of M is the convex hull of the indicator vectors of the bases of M:

$$\Gamma(M) = \text{convex}\{e_B : B \in \mathcal{B}\}.$$

For any $B = \{b_1, \ldots, b_r\} \subseteq [n]$, by e_B we mean $e_{b_1} + \cdots + e_{b_r}$, where $\{e_1, \ldots, e_n\}$ is the standard basis of $\mathbb{R}^n$.

When we speak of "a matroid polytope", we will refer to the polytope of a specific matroid, in its specific position in $\mathbb{R}^n$. The following elegant characterization is due to Gelfand and Serganova:

Theorem 2.2.6. *[GGMS87] Let $\mathcal{B}$ be a collection of r-subsets of $[n]$ and let $\Gamma(\mathcal{B}) = \text{convex}\{e_B : B \in \mathcal{B}\}$. The following are equivalent:*
1. $\mathcal{B}$ is the collection of bases of a matroid.
2. Every edge of $\Gamma(\mathcal{B})$ is a parallel translate of $e_i - e_j$ for some $i, j \in [n]$.

When the statements of Theorem 2.2.6 are satisfied, the edges of $\Gamma(\mathcal{B})$ correspond exactly to the pairs of different bases B, B' such that $B' = B \setminus i \cup j$ for some $i, j \in [n]$. Two such bases are called **adjacent bases**.

A **subdivision** of a polytope P is a set of polytopes $S = \{P_1, \ldots, P_m\}$, whose vertices are vertices of P, such that

- $P_1 \cup \cdots \cup P_m = P$, and
- for all $1 \leq i < j \leq m$, if the intersection $P_i \cap P_j$ is nonempty, then it is a proper face of both P_i and P_j.

The **faces** of the subdivision S are the faces of the P_i; it is easy to see that the interior faces of S (i.e. faces not contained in the boundary of P) are exactly the non-empty intersections between some of the P_i.

Definition 2.2.7. A **matroid polytope subdivision** is a subdivision of a matroid polytope $\Gamma = \Gamma(M)$ into matroid polytopes $\Gamma_1 = \Gamma(M_1), \ldots, \Gamma_m = \Gamma(M_m)$. We will also refer to this as a **matroid subdivision** of the matroid M into $M_1, \ldots, M_m$.

The lower-dimensional faces of the subdivision, which are intersections of subcollections of the Γ_i, are also of interest. Given a set of indices $A = \{a_1, \ldots, a_s\} \subseteq [m]$, we will write $\Gamma_A = \Gamma_{a_1 \cdots a_s} := \bigcap_{a \in A} \Gamma_a$. By convention, $\Gamma_\emptyset = \Gamma$. Since any face of a matroid polytope is itself a matroid polytope, it follows that any nonempty Γ_A is the matroid polytope of a matroid, which we denote M_A.

Because of the small number of matroid polytopes in low dimensions, there is a general lack of small examples of matroid subdivisions. In two dimensions the only matroid polytopes are the equilateral triangle and the square, which have no nontrivial matroid subdivisions. In three dimensions, the only nontrivial example is the subdivision of a regular octahedron (with bases $\{12, 13, 14, 23, 24, 34\}$) into two square pyramids (with bases $\{12, 13, 14, 23, 24\}$ and $\{13, 14, 23, 24, 34\}$, respectively); this subdivision is shown in Figure 2.1.

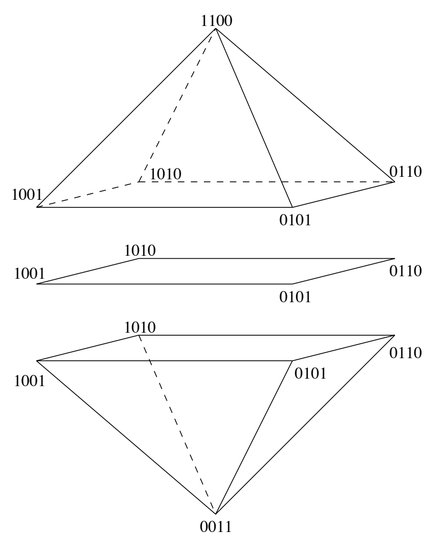

Figure 2.1: The matroid subdivision of a regular octahedron into two square pyramids.

Example 2.2.8. A more interesting example is the following subdivision [BJR09, Example 7.13]: Let $M_1 = SM_6(2, 4, 6)$ be the Schubert matroid whose bases are the sets $\{a, b, c\} \subseteq [6]$ such that $a \leq 2, b \leq 4$, and $c \leq 6$. The permutation $\sigma = 345612$ acts on the ground set [6] of M_1, thus defining the matroids $M_2 = \sigma M_1$ and $M_3 = \sigma^2 M_1$. (Note that σ^3 is the identity.) Then $\{M_1, M_2, M_3\}$ is a subdivision of $M = U_{3,6}$. One can easily generalize this construction to obtain a subdivision of $U_{a,ab}$ into a isomorphic matroids.

2.3 Valuations under Matroid Subdivisions

We now turn to the study of matroid functions which are valuations under the subdivisions of a matroid polytope into smaller matroid polytopes. Throughout this section, $\mathsf{Mat} = \mathsf{Mat}_n$ will denote the set of matroids with ground set $[n]$, and G will denote an arbitrary abelian group. As before, given a subdivision $\{M_1, \ldots, M_m\}$ of a matroid M and a subset $A \subseteq [m]$, M_A is the matroid whose polytope is $\bigcap_{a \in A} \Gamma(M_a)$.

Definition 2.3.1. A function $f : \mathsf{Mat} \to G$ is a **valuation under matroid subdivision**, or simply a **valuation**, if for any subdivision $\{M_1, \ldots, M_m\}$ of a matroid $M \in \mathsf{Mat}$, we have

$$\sum_{A \subseteq [m]} (-1)^{|A|} f(M_A) = 0 \tag{2.3.1}$$

or, equivalently,

$$f(M) = \sum_i f(M_i) - \sum_{i<j} f(M_{ij}) + \sum_{i<j<k} f(M_{ijk}) - \cdots \tag{2.3.2}$$

Recall that, contrary to the usual convention, we have allowed $\emptyset = ([n], \emptyset)$ to be a matroid. We will also adopt the convention that $f(\emptyset) = 0$ for all the matroid functions considered in this chapter.

Remark 2.3.2. This use of the term *valuation* is standard in convex geometry [McM93]. It should not be confused with the unrelated notion of a matroid valuation found in the theory of valuated matroids [DW92], nor with the notion of field valuation discussed in Chapter 1.

Many important matroid functions are well-behaved under subdivision. Let us start with some easy examples.

Example 2.3.3. The function Vol, which assigns to each matroid $M \in \mathsf{Mat}$ the n-dimensional volume of its polytope $\Gamma(M)$, is a valuation. This is clear since the lower-dimensional faces of a matroid subdivision have volume 0.

Example 2.3.4. The **Ehrhart polynomial** $E_P(x)$ of a lattice polytope P in $\mathbb{R}^d$ is the polynomial such that, for a positive integer n, $E_P(n) = |nP \cap \mathbb{Z}^d|$ is the number of lattice points contained in the n-th dilate nP of P [Sta97, Section 4.6]. By the inclusion-exclusion formula, the function $E : \mathsf{Mat} \to \mathbb{R}[x]$ defined by $E(M) = E_{\Gamma(M)}(x)$ is a valuation.

Example 2.3.5. The function $b(M) =$ (number of bases of M) is a valuation. This follows from the fact that the only lattice points in $\Gamma(M)$ are its vertices, which are the indicator vectors of the bases of M; so $b(M)$ is the evaluation of $E(M)$ at $x = 1$.

Before encountering other important valuations, let us present an alternative way of characterizing them.

Theorem 2.3.6. *A function $f : \mathsf{Mat} \to G$ is a valuation if and only if, for any matroid subdivision S of $\Gamma = \Gamma(M)$,*

$$f(M) = \sum_{F \in \mathrm{int}(S)} (-1)^{\dim(\Gamma) - \dim(F)} f(M(F)), \tag{2.3.3}$$

where the sum is over the interior faces of the subdivision S, and $M(F)$ denotes the matroid whose matroid polytope is F.

To prove Theorem 2.3.6 we first need to recall some facts from topological combinatorics. These can be found, for instance, in [Sta97, Section 3.8].

Definition 2.3.7. A **regular cell complex** is a finite set $C = \{\sigma_1, \sigma_2, \ldots, \sigma_s\}$ of pairwise disjoint and nonempty **cells** $\sigma_i \subseteq \mathbb{R}^d$ such that for any $i \in [s]$:

1. $\overline{\sigma_i} \approx \mathbb{B}^{m_i}$ and $\overline{\sigma_i} \setminus \sigma_i \approx \mathbb{S}^{m_i - 1}$ for some nonnegative integer m_i, called the **dimension** of σ_i.

2. $\overline{\sigma_i} \setminus \sigma_i$ is the union of some other σ_js.

Here $\overline{\sigma_i}$ denotes the topological closure of σ_i and $\approx$ denotes homeomorphism. Also $\mathbb{B}^l$ and $\mathbb{S}^l$ are the l-dimensional closed unit ball and unit sphere, respectively. The **underlying space** $|C|$ of C is the topological space $\sigma_1 \cup \cdots \cup \sigma_s$.

Definition 2.3.8. Let C be a regular cell complex, and let c_i be the number of i-dimensional cells of C. The *Euler characteristic* of C is:

$$\chi(C) = \sum_{\sigma \in C} (-1)^{\dim(\sigma)} = \sum_{i \in \mathbb{N}} (-1)^i c_i = c_0 - c_1 + c_2 - c_3 \cdots .$$

The **reduced Euler characteristic** of C is $\tilde{\chi}(C) = \chi(C) - 1$. A fundamental fact from algebraic topology is that the Euler characteristic of C depends solely on the homotopy type of the underlying space $|C|$.

Definition 2.3.9. For a regular cell complex C, let $P(C)$ be the poset of cells of C, ordered by $\sigma_i \leq \sigma_j$ if $\overline{\sigma_i} \subseteq \overline{\sigma_j}$. Let $\widehat{P}(C) = P(C) \cup \{\hat{0}, \hat{1}\}$ be obtained from $P(C)$ by adding a minimum and a maximum element.

Definition 2.3.10. The **Möbius function** $\mu : \mathrm{Int}(P) \to \mathbb{Z}$ of a poset P assigns an integer to each closed interval of P, defined recursively by

$$\mu_P(x, x) = 1, \qquad \sum_{x \leq a \leq y} \mu(x, a) = 0 \text{ for all } x < y.$$

It can equivalently be defined in the following dual way:

$$\mu_P(x, x) = 1, \qquad \sum_{x \leq a \leq y} \mu(a, y) = 0 \text{ for all } x < y.$$

The following special case of Rota's Crosscut Theorem is a powerful tool for computing the Möbius function of a lattice.

Theorem 2.3.11. *[Rot64] Let L be any finite lattice. Then for all $x \in L$,*

$$\mu(\hat{0}, x) = \sum_B (-1)^{|B|},$$

where the sum is over all sets B of atoms of L such that $\bigvee B = x$.

Finally, we recall an important theorem which relates the topology and combinatorics of a regular cell complex.

Theorem 2.3.12. *[Sta97, Proposition 3.8.9] Let C be a regular cell complex such that $|C|$ is a manifold, with or without boundary. Let $P = \widehat{P}(C)$. Then*

$$\mu_P(x,y) = \begin{cases} \tilde{\chi}(|C|) & \text{if } x = \hat{0} \text{ and } y = \hat{1}, \\ 0 & \text{if } x \neq \hat{0}, y = \hat{1}, \text{ and } x \text{ is on the boundary of } |C|, \\ (-1)^{l(x,y)} & \text{otherwise,} \end{cases}$$

where $l(x,y)$ is the number of elements in a maximal chain from x to y.

We are now in a position to prove Theorem 2.3.6.

Proof of Theorem 2.3.6. Let $S = \{M_1, \ldots, M_m\}$ be a matroid subdivision of the matroid M. Let $\{\Gamma_1, \ldots, \Gamma_m\}$ and Γ be the corresponding polytopes. Notice that the (relative interiors of the) faces of the subdivision S form a regular cell complex whose underlying space is Γ. Additionally, the poset $\widehat{P}(S)$ is a lattice, since it has a meet operation $\sigma_i \wedge \sigma_j = \operatorname{int}(\overline{\sigma}_i \cap \overline{\sigma}_j)$ and a maximum element.

We will show that

$$\sum_{F \in \operatorname{int}(S)} (-1)^{\dim(\Gamma) - \dim(F)} f(M(F)) = \sum_i f(M_i) - \sum_{i<j} f(M_{ij}) + \sum_{i<j<k} f(M_{ijk}) - \cdots \quad (2.3.4)$$

which will establish the desired result in view of (2.3.2). In the right hand side, each term is of the form $f(M(F))$ for an interior face F of the subdivision S and moreover, all interior faces F appear. The term $f(M(F))$ appears with coefficient

$$\sum_{A \subseteq [m] \,:\, M_A = M(F)} (-1)^{|A|+1}.$$

This is equivalent to summing over the sets of coatoms of the lattice $\widehat{P}(S)$ whose meet is F. By Rota's Crosscut Theorem 2.3.11, when applied to the poset $\widehat{P}(S)$ turned upside down, this sum equals $-\mu_{\widehat{P}(S)}(F, \hat{1})$. Theorem 2.3.12 tells us that this is equal to $(-1)^{l(F,\hat{1})-1} = (-1)^{\dim(\Gamma)-\dim(F)}$, as desired. □

2.4 A Powerful Family of Valuations

Definition 2.4.1. Given $X \subseteq \mathbb{R}^n$, let $i_X : \mathsf{Mat} \to \mathbb{Z}$ be defined by

$$i_X(M) = \begin{cases} 1 & \text{if } \Gamma(M) \cap X \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

Our interest in these functions is that, under certain hypotheses, they are valuations under matroid subdivisions. They are a powerful family for our purposes because many valuations of interest, in particular those of Section 2.5, can be obtained as linear combinations of evaluations of these valuations.

Theorem 2.4.2. *If $X \subseteq \mathbb{R}^n$ is convex and open, then i_X is a valuation.*

Proof. Let $M \in \mathsf{Mat}$ be a matroid and S be a subdivision of $\Gamma = \Gamma(M)$. We can assume that $\Gamma \cap X \neq \emptyset$, or else the result is trivial. We can also assume that X is bounded since $i_X = i_{X\cap[0,1]^n}$.

We will first reduce the proof to the case when X is an open polytope in $\mathbb{R}^n$. By the Hahn-Banach separation theorem [Rud73, Theorem 3.4], for each face F of S such that $F \cap X = \emptyset$ there exists an open halfspace H_F containing X and disjoint from F. Let

$$X' = \bigcap_{F\cap X=\emptyset} H_F$$

be the intersection of these halfspaces. Then $X' \supseteq X$ and $X' \cap F = \emptyset$ for each face F not intersecting X, so $i_{X'}$ and i_X agree on all the matroids of this subdivision. If we define X'' as the intersection of X' with some open cube containing Γ, $i_{X''}$ and i_X agree on this subdivision and X'' is an open polytope.

We can therefore assume that X is an open polytope in $\mathbb{R}^n$; in particular it is full-dimensional. Note that $X \cap \operatorname{int}(\Gamma)$ is the interior $\operatorname{int}(R)$ of some polytope $R \subseteq \Gamma$. Since R and Γ have the same dimension, $R \approx \mathbb{B}^{\dim(\Gamma)}$ and $\partial R \approx \mathbb{S}^{\dim(\Gamma)-1}$. If F is a face of the subdivision S and σ is a face of the polytope R, let $c_{F,\sigma} = \operatorname{int}(F) \cap \operatorname{int}(\sigma)$. Since $c_{F,\sigma}$ is the interior of a polytope, it is homeomorphic to a closed ball and its boundary to the corresponding sphere. Define

$$C = \{c_{F,\sigma} : c_{F,\sigma} \neq \emptyset\}$$
$$\partial C = \{c_{F,\sigma} : c_{F,\sigma} \neq \emptyset \text{ and } \sigma \neq R\}.$$

The elements of C form a partition of R and in this way C is a regular cell complex whose underlying space is R. Similarly, ∂C is a regular subcomplex whose underlying space is ∂R. Note that if F is an interior face of S, $c_{F,R} = \operatorname{int}(F) \cap \operatorname{int}(R) \neq \emptyset$ if and only if $F \cap X \neq \emptyset$, and in this case $\dim(c_{F,R}) = \dim(F)$.

We then have

$$\begin{aligned}
\sum_{F\in \mathrm{int}(S)} (-1)^{\dim(F)} i_X(M(F)) &= \sum_{\substack{F\in \mathrm{int}(S)\\ F\cap X\neq\emptyset}} (-1)^{\dim(F)} \\
&= \sum_{c_{F,R}\neq\emptyset} (-1)^{\dim(c_{F,R})} \\
&= \sum_{c\in C}(-1)^{\dim(c)} - \sum_{c\in\partial C}(-1)^{\dim(c)} \\
&= \chi(R)-\chi(\partial R) \\
&= 1-\left(1+(-1)^{\dim(\Gamma)-1}\right) \\
&= (-1)^{\dim(\Gamma)} i_X(M),
\end{aligned}$$

which finishes the proof in view of Theorem 2.3.6. □

Corollary 2.4.3. *If $X \subseteq \mathbb{R}^n$ is convex and closed, then i_X is a valuation.*

Proof. As before, we can assume that X is bounded since $i_X = i_{X\cap[0,1]^n}$. Now let S be a subdivision of $\Gamma = \Gamma(M)$ into m parts. For all $A \subseteq [m]$ such that $X \cap \Gamma_A = \emptyset$, the distance $d(X, \Gamma_A)$ is positive since X is compact and Γ_A is closed. Let $\epsilon > 0$ be smaller than all those distances, and define the convex open set

$$U = \{x \in \mathbb{R}^n : d(x, X) < \epsilon\}.$$

For all $A \subseteq [m]$ we have that $X \cap \Gamma_A \neq \emptyset$ if and only if $U \cap \Gamma_A \neq \emptyset$. By Theorem 2.4.2,

$$\sum_{A\subseteq[m]} (-1)^{|A|} i_X(M_A) = \sum_{A\subseteq[m]} (-1)^{|A|} i_U(M_A) = 0$$

as desired. □

In particular, i_P is a valuation for any polytope $P \subseteq \mathbb{R}^n$.

Proposition 2.4.4. *The constant function $c(M) = 1$ for $M \in \mathsf{Mat}$ is a valuation.*

Proof. This follows from $c(M) = i_{[0,1]^n}$. □

Proposition 2.4.5. *If $X \subseteq \mathbb{R}^n$ is convex, and is either open or closed, then the function $\overline{i_X} : \mathsf{Mat} \to \mathbb{Z}$ defined by*

$$\overline{i_X}(M) = \begin{cases} 0 & \text{if } \Gamma(M)\cap X \neq \emptyset, \\ 1 & \text{otherwise,} \end{cases}$$

is a valuation.

Proof. Notice that $\overline{i_X} = 1 - i_X$, which is the sum of two valuations. □

2.5 Subset Ranks and Basis Activities are Valuations

We now show that there are two surprisingly fine valuations of a matroid: the ranks of the subsets and the activities of the bases.

2.5.1 Rank Functions

Theorem 2.5.1. *Let G be the free abelian group on symbols of the form (A, s), $A \subseteq [n]$, $s \in \mathbb{Z}_{\geq 0}$. The function $F : \mathsf{Mat} \to G$ defined by*

$$F(M) = \sum_{A \subseteq [n]} (A, r_M(A))$$

is a valuation.

Proof. It is equivalent to show that the function $f_{A,s} : \mathsf{Mat} \to \mathbb{Z}$ defined by

$$f_{A,s}(M) = \begin{cases} 1 & \text{if } r_M(A) = s, \\ 0 & \text{otherwise,} \end{cases}$$

is a valuation. Define the polytope

$$P_{A,s} = \left\{ x \in [0,1]^n : \sum_{i \in A} x_i \geq s \right\}.$$

A matroid M satisfies that $r_M(A) = s$ if and only if it has a basis B with $|A \cap B| \geq s$, and it has no basis B such that $|A \cap B| \geq s + 1$. This is equivalent to $\Gamma(M) \cap P_{A,s} \neq \emptyset$ and $\Gamma(M) \cap P_{A,s+1} = \emptyset$. It follows that $f_{A,s} = i_{P_{A,s}} - i_{P_{A,s+1}}$, which is the sum of two valuations. □

2.5.2 Basis Activities

One of the most important invariants of a matroid is its **Tutte polynomial**:

$$T_M(x, y) = \sum_{A \subseteq [n]} (x-1)^{r(M)-r(A)} (y-1)^{|A|-r(A)}.$$

Its importance stems from the fact that many interesting invariants of a matroid satisfy the **deletion-contraction recursion**, and every such invariant is an evaluation of the Tutte polynomial [BO92].

Definition 2.5.2. Let B be a basis of the matroid $M = ([n], \mathcal{B})$. An element $i \in B$ is said to be **internally active** with respect to B if $i < j$ for all $j \notin B$ such that $B - i \cup j \in \mathcal{B}$. Similarly, an element $i \notin B$ is said to be **externally active** with respect to B if $i < j$ for all $j \in B$ such that $B - j \cup i \in \mathcal{B}$. Let $I(B)$ and $E(B)$ be the sets of internally and externally active elements with respect to B.

Theorem 2.5.3. *(Tutte, Crapo [BO92]) The Tutte polynomial of a matroid is*

$$T_M(x, y) = \sum_{B \text{ basis of } M} x^{|I(B)|} y^{|E(B)|}.$$

Theorem 2.5.4. *Let G be the free abelian group generated by the triples (B, E, I), where $B \subseteq [n]$, $E \subseteq [n] \setminus B$ and $I \subseteq B$. The function $F : \mathsf{Mat} \to G$ defined by*

$$F(M) = \sum_{B \text{ basis of } M} (B, E(B), I(B)) \tag{2.5.1}$$

is a valuation.

Before proving this result, let us illustrate its strength with an example. Consider the subdivision of $M = U_{3,6}$ into three matroids M_1, M_2, and M_3 described in Example 2.2.8. Table 2.1 shows the external and internal activity with respect to each basis in each one of the eight matroids M_A arising in the subdivision. The combinatorics prescribed by Theorem 2.5.4 is extremely restrictive: in any row, any choice of (E, I) must appear the same number of times in the M_As with $|A|$ even and in the M_As with $|A|$ odd.

	M		M_1		M_2		$M_{1,2}$		M_3		$M_{1,3}$		$M_{2,3}$		$M_{1,2,3}$	
B	$E(B)$	$I(B)$	$E(B)$	$I(B)$	$E(B)$	$I(B)$	$E(B)$	$I(B)$	$E(B)$	$I(B)$	$E(B)$	$I(B)$	$E(B)$	$I(B)$	$E(B)$	$I(B)$
123	$\emptyset$	123	$\emptyset$	123												
124	$\emptyset$	12	$\emptyset$	12												
125	$\emptyset$	12	$\emptyset$	12					$\emptyset$	125	$\emptyset$	125				
126	$\emptyset$	12	5	12					$\emptyset$	12	5	12				
134	$\emptyset$	1	$\emptyset$	1	$\emptyset$	134	$\emptyset$	134								
135	$\emptyset$	1	$\emptyset$	1	$\emptyset$	13	$\emptyset$	13	$\emptyset$	15	$\emptyset$	15	$\emptyset$	135	$\emptyset$	135
136	$\emptyset$	1	5	1	$\emptyset$	13	5	13	$\emptyset$	1	5	1	$\emptyset$	13	5	13
145	$\emptyset$	1	3	1	$\emptyset$	1	3	1	3	15	3	15	3	15	3	15
146	$\emptyset$	1	35	1	$\emptyset$	1	35	1	3	1	35	1	3	1	35	1
156	$\emptyset$	1							$\emptyset$	1						
234	1	$\emptyset$	1	$\emptyset$	1	34	1	34								
235	1	$\emptyset$	1	$\emptyset$	1	3	1	3	1	5	1	5	1	35	1	35
236	1	$\emptyset$	15	$\emptyset$	1	3	15	3	1	$\emptyset$	15	$\emptyset$	1	3	15	3
245	1	$\emptyset$	13	$\emptyset$	1	$\emptyset$	13	$\emptyset$	13	5	13	5	13	5	13	5
246	1	$\emptyset$	135	$\emptyset$	1	$\emptyset$	135	$\emptyset$	13	$\emptyset$	135	$\emptyset$	13	$\emptyset$	135	$\emptyset$
256	1	$\emptyset$							1	$\emptyset$						
345	12	$\emptyset$			12	$\emptyset$										
346	12	$\emptyset$			12	$\emptyset$										
356	12	$\emptyset$			12	3			12	$\emptyset$			12	3		
456	123	$\emptyset$			123	$\emptyset$			123	$\emptyset$			123	$\emptyset$		

Table 2.1: External and internal activities for a subdivision of $U_{3,6}$

We will divide the proof of Theorem 2.5.4 into two lemmas.

Lemma 2.5.5. *Let $B \subseteq [n]$, $E \subseteq [n] \setminus B$ and $I \subseteq B$. Let*

$$V(B,E,I) = \{A \subseteq [n] : e_A - e_B = e_a - e_b \text{ with } a \in E \text{ and } a > b, \\ \text{or with } b \in I \text{ and } a < b\}$$

and

$$P(B,E,I) = \text{convex}\left\{\frac{e_A + e_B}{2} : A \in V(B,E,I)\right\}.$$

Then for any matroid $M \in \mathsf{Mat}$, we have that $\Gamma(M) \cap P(B,E,I) = \emptyset$ if and only if

- *B is not a basis of M, or*
- *B is a basis of M with $E \subseteq E(B)$ and $I \subseteq I(B)$.*

To illustrate this lemma with an example, consider the case $n = 4$, $B = \{1,3\}$, $E = \{2\}$ and $I = \{3\}$. Then $V(B,E,I) = \{\{1,2\},\{2,3\}\}$. Figure 2.2 shows the polytope $P = P(B,E,I)$ inside the hypersimplex, whose vertices are the characteristic vectors of the 2-subsets of [4]. The polytope of the matroid M_1 with bases $\mathcal{B}_1 = \{\{1,2\},\{1,4\},\{2,3\},\{3,4\}\}$ does not intersect P because B is not a basis of M_1. The polytope of the matroid M_2 with bases $\mathcal{B}_2 = \{\{1,3\},\{1,4\},\{3,4\}\}$ does not intersect P either, because B is a basis of M_2, but 2 is externally active with respect to B and 3 is internally active with respect to B. Finally, the polytope of the matroid M_3 with bases $\mathcal{B}_3 = \{\{1,3\},\{2,3\},\{3,4\}\}$ does intersect P, since B is a basis of M_3 and 2 is not externally active with respect to B; the intersection point $\frac{1}{2}(0110 + 1010)$ "certifies" this.

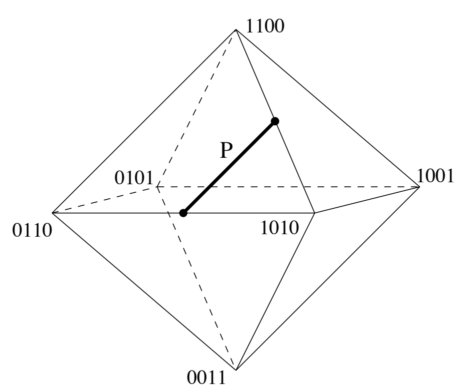

Figure 2.2: The polytope $P = P(B,E,I)$ inside $\Gamma(U_{2,4})$

Proof. Assume B is a basis of M. For $a \notin B$, a is externally active with respect to B if and only if there are no edges in $\Gamma(M)$ of the form $e_a - e_b$ with $a > b$ which are incident to e_B. In the same way, for $b \in B$, b is internally active with respect to B if and only if there are no edges in $\Gamma(M)$ of the form $e_a - e_b$ with $a < b$ which are incident to e_B. Since the vertices of $P(B, E, I)$ are precisely the midpoints of these edges when $a \in E$ and $b \in I$, if $\Gamma(M) \cap P(B, E, I) = \emptyset$ then $E \subseteq E(B)$ and $I \subseteq I(B)$.

To prove the other direction, suppose that $\Gamma(M) \cap P(B, E, I) \neq \emptyset$. First notice that, since $P(B, E, I)$ is on the hyperplane $x_1 + x_2 + \cdots + x_n = |B|$ and $\Gamma(M)$ is on the hyperplane $x_1 + x_2 + \cdots + x_n = r(M)$, we must have $|B| = r(M)$. Moreover, since the vertices v of $P(B, E, I)$ satisfy $e_B \cdot v = r(M) - 1/2$ then B must be a basis of M, or else the vertices w of $\Gamma(M)$ would all satisfy $e_B \cdot w \leq r(M) - 1$.

Now let $q \in \Gamma(M) \cap P(B, E, I)$. Since $q \in \Gamma(M)$, we know that q is in the cone with vertex e_B generated by the edges of $\Gamma(M)$ incident to e_B. In other words, if $A_1, A_2, \ldots, A_m$ are the bases adjacent to B,

$$q = e_B + \sum_{i=1}^{m} \lambda_i (e_{A_i} - e_B),$$

where the λ_i are all nonnegative. If we let $e_{c_i} - e_{d_i} = e_{A_i} - e_B$, then

$$q = e_B + \sum_{i=1}^{m} \lambda_i (e_{c_i} - e_{d_i}).$$

On the other hand, since $q \in P(B, E, I)$,

$$q = \sum_{A \in V(B,E,I)} \gamma_A \frac{e_A + e_B}{2},$$

where the γ_A are nonnegative and add up to 1. Setting these two expressions equal to each other we obtain

$$q = e_B + \sum_{i=1}^{m} \lambda_i (e_{c_i} - e_{d_i}) = \sum_{A \in V(B,E,I)} \gamma_A \frac{e_A + e_B}{2}$$

and therefore

$$r = q - e_B = \sum_{i=1}^{m} \lambda_i (e_{c_i} - e_{d_i}) = \sum_{A \in V(B,E,I)} \gamma_A \frac{e_A - e_B}{2}.$$

For $A \in V(B, E, I)$ we will let $e_{a_A} - e_{b_A} = e_A - e_B$. We have

$$r = \sum_{i=1}^{m} \lambda_i (e_{c_i} - e_{d_i}) = \sum_{A \in V(B,E,I)} \gamma_A \frac{e_{a_A} - e_{b_A}}{2}. \tag{2.5.2}$$

Notice that there is no cancellation of terms in either side of (2.5.2), since the d_is and the b_As are elements of B, while the c_is and the a_As are not. Let $r = (r_1, r_2, \ldots, r_n)$ and let k be the largest integer for which r_k is nonzero.

Assume that $k \notin B$. From the right hand side of (2.5.2) and taking into account the definition of $V(B,E,I)$, we have that $k \in E$. From the left hand side we know there is an i such that $c_i = k$. But then $e_{c_i} - e_{d_i}$ is an edge of $\Gamma(M)$ incident to e_B, and $d_i < k = c_i$ by our choice of k. It follows that k is not externally active with respect to B. In the case that $k \in B$, we obtain similarly that $k \in I$, and that $d_j = k$ for some j. Thus $e_{c_j} - e_{d_j}$ is an edge of $\Gamma(M)$ incident to e_B and $c_j < k = d_j$, so k is not internally active with respect to B. In either case we conclude that $E \nsubseteq E(B)$ or $I \nsubseteq I(B)$, which finishes the proof. □

Lemma 2.5.6. *Let B be a subset of $[n]$, and let $E \subseteq [n] \setminus B$ and $I \subseteq B$. The function $G_{B,E,I} : \mathsf{Mat} \to \mathbb{Z}$ defined by*

$$G_{B,E,I}(M) = \begin{cases} 1 & \text{if } B \text{ is a basis of } M, E = E(B) \text{ and } I = I(B), \\ 0 & \text{otherwise,} \end{cases}$$

is a valuation.

Proof. To simplify the notation, we will write $\overline{i_B}$ instead of $\overline{i_{\{e_B\}}}$. We will prove that $G(B,E,I) = G'(B,E,I)$ where

$$G'_{B,E,I}(M) = (-1)^{|E|+|I|} \cdot \sum_{\substack{E \subseteq X \subseteq [n] \\ I \subseteq Y \subseteq [n]}} (-1)^{|X|+|Y|} \left(\overline{i_{P(B,X,Y)}}(M) - \overline{i_B}(M) \right), \tag{2.5.3}$$

which is a sum of valuations.

Let $M \in \mathsf{Mat}$. If B is not a basis of M then $\overline{i_B}(M) = 1$, and by Lemma 2.5.5 we have $\overline{i_{P(B,X,Y)}}(M) = 1$ for all X and Y. Therefore $G'_{B,E,I}(M) = 0 = G_{B,E,I}(M)$ as desired. If B is a basis of M then $\overline{i_B}(M) = 0$; and we use Lemma 2.5.5 to rewrite (2.5.3) as

$$\begin{aligned} G'_{B,E,I}(M) &= (-1)^{|E|+|I|} \cdot \sum_{\substack{E \subseteq X \subseteq E(B) \\ I \subseteq Y \subseteq I(B)}} (-1)^{|X|+|Y|} \\ &= (-1)^{|E|+|I|} \cdot \sum_{E \subseteq X \subseteq E(B)} (-1)^{|X|} \cdot \sum_{I \subseteq Y \subseteq I(B)} (-1)^{|Y|} \\ &= \begin{cases} 1 & \text{if } E = E(B) \text{ and } I = I(B), \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

as desired. □

Proof of Theorem 2.5.4. The coefficient of (B,E,I) in the definition of (2.5.1) is $G_{B,E,I}(M)$, so the result follows from Lemma 2.5.6. □

Theorem 2.5.4 is significantly stronger than the following result of Speyer which motivated it:

Corollary 2.5.7. *(Speyer, [Spe08]) The Tutte polynomial (and therefore any of its evaluations) is a valuation under matroid subdivisions.*

Proof. By Theorem 2.5.3, $T_M(x, y)$ is the composition of the function $h : G \to \mathbb{Z}[x, y]$ defined by $h(B, E, I) = x^{|I|}\, y^{|E|}$ with the function F of Theorem 2.5.4. □

2.6 Related Work

Previous to our work, Billera, Jia and Reiner [BJR09] and Speyer [Spe08, Spe09] had studied various valuations of matroid polytopes. A few months after the completion of our work, we learned about Derksen's results on this topic [Der09], which were obtained independently and roughly simultaneously. Their approaches differ from ours in the basic fact that they are concerned with matroid invariants which are valuations, whereas our matroid functions are not necessarily constant under matroid isomorphism; however there are similarities. We outline their main invariants here.

In his work on tropical linear spaces [Spe08], Speyer shows that the Tutte polynomial is a valuative invariant. He also defines in [Spe09] a polynomial invariant $g_M(t)$ of a matroid M which arises in the K-theory of the Grassmannian. It is not known how to describe $g_M(t)$ combinatorially in terms of M.

Given a matroid $M = (E, \mathcal{B})$, a function $f : E \to \mathbb{Z}_{>0}$ is said to be M**-generic** if the minimum value of $\sum_{b \in B} f(b)$ over all bases $B \in \mathcal{B}$ is attained just once. Billera, Jia, and Reiner study the valuation

$$QS(M) = \sum_{f\ M\text{-generic}} \prod_{b \in E} x_{f(b)},$$

which takes values in the ring of **quasi-symmetric functions** in the variables $x_1, x_2, \ldots$; *i.e.*, the ring generated by

$$\sum_{i_1 < \ldots < i_r} x_{i_1}^{\alpha_1} \cdots x_{i_r}^{\alpha_r}$$

for all tuples $(\alpha_1, \ldots, \alpha_r)$ of positive integers.

Derksen's valuation is given by

$$G(M) := \sum_{\mathbf{A}} U(r_M(A_1) - r_M(A_0), \ldots, r_M(A_n) - r_M(A_{n-1}))$$

where $\mathbf{A} = (A_0, \ldots, A_n)$ ranges over all maximal flags of M, and

$$\{U(\mathbf{r}) : \mathbf{r} \text{ a finite sequence of nonnegative integers}\}$$

is a particular basis for the ring of quasi-symmetric functions. Derksen's invariant can be defined more generally on polymatroids. He shows that the Tutte polynomial and the quasisymmetric function of Billera, Jia and Reiner are specializations of $G(M)$, and asks whether $G(M)$ is universal for valuative invariants in this setting.

For the remainder of this section, $F(M)$ will denote the function of our Theorem 2.5.1. Since $F(M)$ is not a matroid invariant, it cannot be a specialization of $g_M(t), QS(M)$, or $G(M)$. In the other direction, we suspect that, like the Tutte polynomial, Speyer's polynomial $g_M(t)$ is a specialization of $F(M)$. As one would expect, $G(M)$ and $QS(M)$ are not specializations of $F(M)$. One linear combination that certifies this is set out in Table 2.2, in which, to facilitate carrying out the relevant checks for $F(M)$, the relevant matroids are specified via their rank functions.

However, one can give a valuation which is similar in spirit to our $F(M)$ and specializes to Derksen's valuation $G(M)$.

Proposition 2.6.1. *The function* $H : \mathsf{Mat} \to G^n$ *defined by*

$$H(M) = \sum_{\mathbf{A}} \big((A_1, r(A_1)), \ldots, (A_n, r(A_n))\big),$$

where $\mathbf{A} = (A_1, \ldots, A_n)$ *ranges over all maximal flags of* M*, is a valuation.*

Proof. The proof is a straightforward extension of our argument for Theorem 2.5.1. With the notation of that proof, checking whether a matroid M satisfies $r_M(A_i) = r_i$ for some fixed vector $\mathbf{r} = (r_i)$, *i.e.* whether the term $((A_1, r_1), \ldots, (A_n, r_n))$ is present in $H(M)$, is equivalent to checking that $\Gamma(M)$ intersects P_{A_i,r_i} and not P_{A_i,r_i+1} for each i.

Observe that if $\Gamma(M)$ intersects P_{A_i,s_i} for all i then $r(A_i) \geq s_i$ and, since $\mathbf{A}$ is a flag, we can choose a single basis of M whose intersection with A_i has at least s_i elements for each i. Therefore $\Gamma(M)$ intersects $P_{A_1,s_1} \cap \cdots \cap P_{A_n,s_n}$.

Consider the sum

$$\sum (-1)^{e_1 + \ldots + e_n} i_{P_{\mathbf{A},\mathbf{r}+\mathbf{e}}}(M) \tag{2.6.1}$$

where the sum is over all $\mathbf{e} = (e_1, ..., e_n) \in \{0,1\}^n$, and where $P_{\mathbf{A},\mathbf{r}+\mathbf{e}}$ is the intersection $P_{A_1,r_1+e_1} \cap \cdots \cap P_{A_n,r_n+e_n}$. By our previous observation this sum equals

$$\left(\sum_{e_1} (-1)^{e_1} i_{P_{A_1,r_1+e_1}}(M)\right) \cdots \left(\sum_{e_n} (-1)^{e_n} i_{P_{A_n,r_n+e_n}}(M)\right),$$

which is 1 if the term $((A_1, r_1), \ldots, (A_n, r_n))$ is present in $H(M)$, and is 0 otherwise. All the terms in (2.6.1) are valuations, hence H is a valuation. □

S	$\emptyset$	1	2	12	3	13	23	123
$r_{M_1}(S)$	0	1	1	1	0	1	1	1
$r_{M_2}(S)$	0	1	0	1	1	1	1	1
$r_{M_3}(S)$	0	0	1	1	1	1	1	1
$r_{M_4}(S)$	0	1	1	1	1	1	1	1
$r_{M_5}(S)$	0	1	1	2	0	1	1	2
$r_{M_6}(S)$	0	1	0	1	1	2	1	2
$r_{M_7}(S)$	0	0	1	1	1	1	2	2
$r_{M_8}(S)$	0	1	1	2	1	2	2	2
$r_{M_9}(S)$	0	0	1	1	1	1	2	2
$r_{M_{10}}(S)$	0	1	1	2	1	2	2	3
$r_{M_{11}}(S)$	0	1	1	2	1	1	2	2
$r_{M_{12}}(S)$	0	1	1	2	1	2	2	3

S	4	14	24	124	34	134	234	1234
$r_{M_1}(S)$	0	1	1	1	0	1	1	1
$r_{M_2}(S)$	1	1	1	1	1	1	1	1
$r_{M_3}(S)$	1	1	1	1	1	1	1	1
$r_{M_4}(S)$	1	1	1	1	1	1	1	1
$r_{M_5}(S)$	0	1	1	2	0	1	1	2
$r_{M_6}(S)$	1	2	1	2	1	2	1	2
$r_{M_7}(S)$	1	1	2	2	1	1	2	2
$r_{M_8}(S)$	1	2	2	2	1	2	2	2
$r_{M_9}(S)$	1	1	2	2	2	2	3	3
$r_{M_{10}}(S)$	1	1	2	2	2	2	3	3
$r_{M_{11}}(S)$	1	2	2	3	2	2	3	3
$r_{M_{12}}(S)$	1	2	2	3	2	2	3	3

i	1	2	3	4	5	6	7	8	9	10	11	12
c_i	-1	1	-1	1	1	-1	-1	1	2	-2	-2	2

Table 2.2: The top table contains the rank functions of twelve matroids M_i on $[4]$, $i = 1, \dots, 12$. The bottom table shows coefficients c_i such that $\sum c_i F(M_i) = 0$ but $\sum c_i G(M_i) \neq 0$ and $\sum c_i QS(M_i) \neq 0$.

Chapter 3

Isotropical Linear Spaces

This chapter presents single-authored material that was published in the *Journal of Combinatorial Theory, Series A* under the title "Isotropical Linear Spaces and Valuated Delta-Matroids" [Rin12].

3.1 Introduction

Let n be a positive integer, and let V be a $2n$-dimensional vector space over an algebraically closed field K of characteristic 0. Fix a basis $e_1, e_2, \ldots, e_n, e_{1^*}, e_{2^*}, \ldots, e_{n^*}$ for V, and consider the symmetric bilinear form on V defined as

$$Q(x, y) = \sum_{i=1}^{n} x_i \, y_{i^*} + \sum_{i=1}^{n} x_{i^*} \, y_i,$$

for any two $x, y \in V$ with coordinates

$$x = (x_1, \ldots, x_n, x_{1^*}, \ldots, x_{n^*}) \text{ and } y = (y_1, \ldots, y_n, y_{1^*}, \ldots, y_{n^*}).$$

An n-dimensional subspace $U \subseteq V$ is called (totally) **isotropic** if for all $u, v \in U$ we have $Q(u, v) = 0$, or equivalently, for all $u \in U$ we have $Q(u, u) = 0$. Denote by $2^{[n]}$ the collection of subsets of the set $[n] := \{1, 2, \ldots, n\}$. The space of pure spinors $\text{Spin}^{\pm}(n)$ is an algebraic set in projective space $\mathbb{P}^{2^{[n]}-1}$ that parametrizes totally isotropic subspaces of V. Its defining ideal is generated by very special quadratic equations, known as Wick relations. We will discuss these relations in Section 3.2. Since any linear subspace $W \subseteq K^n$ defines an isotropic subspace $U := W \times W^{\perp} \subseteq K^{2n}$, all Grassmannians $G(k, n)$ can be embedded naturally into the space of pure spinors, and in fact, Wick relations can be seen as a natural generalization of Plücker relations.

In [Spe08], Speyer studied tropical Plücker relations, tropical Plücker vectors (or valuated matroids [DW92]), and their relation with tropical linear spaces. In his study he showed that these objects have a beautiful combinatorial structure, which is closely related to matroid polytope decompositions. In this chapter we will study the tropical variety and prevariety defined by all Wick relations, the combinatorics satisfied by the vectors in these spaces (valuated Δ-matroids [DW91]), and their connection with tropical linear spaces that are tropically isotropic (which we will call isotropical linear spaces). Much of our work can be seen as a generalization to type D of some of the results obtained by Speyer, or as a generalization of the theory of Δ-matroids to the "valuated" setup.

In Section 3.4 we will also be interested in determining for what values of n the Wick relations form a tropical basis. We will provide an answer for all $n \neq 6$:

Theorem 3.4.5. *If $n \leq 5$ then the Wick relations are a tropical basis; if $n \geq 7$ then they are not.*

We conjecture that, in fact, for all $n \leq 6$ the Wick relations are a tropical basis.

We will say that a vector $p \in \mathbb{T}^{2^{[n]}}$ with coordinates in the tropical semiring $\mathbb{T} := \mathbb{R} \cup \{\infty\}$ is a tropical Wick vector if it satisfies the tropical Wick relations. A central object for our study of tropical Wick vectors will be that of an even Δ-matroid [Bou87]. Even Δ-matroids are a natural generalization of classical matroids, and much of the theory of matroids can be extended to them. In particular, their associated polytopes are precisely those $0/1$ polytopes whose edges have the form $\pm e_i \pm e_j$, with $i \neq j$. In this sense, even Δ-matroids can be seen as Coxeter matroids of type D, while classical matroids correspond to Coxeter matroids of type A. We will present all the necessary background on even Δ-matroids in Section 3.3. Tropical Wick vectors will be valuated Δ-matroids: real functions on the set of bases of an even Δ-matroid satisfying certain "valuated exchange property" which is amenable to the greedy algorithm (see [DW91]). We will prove in Section 3.5 that in fact tropical Wick vectors can be characterized in terms of even Δ-matroid polytope subdivisions:

Theorem 3.5.4. *The vector $p \in \mathbb{T}^{2^{[n]}}$ is a tropical Wick vector if and only if the regular subdivision induced by p is a subdivision of an even Δ-matroid polytope into even Δ-matroid polytopes.*

We give a complete list of all even Δ-matroids up to isomorphism on a ground set of at most 5 elements, together with their corresponding spaces of valuations, in the website

`http://math.berkeley.edu/~felipe/delta/` .

In Section 3.6 we will extend some of the theory of even Δ-matroids to the valuated setup. We say that a vector $x = (x_1, x_2, \ldots, x_n, x_{1^*}, x_{2^*}, \ldots, x_{n^*}) \in \mathbb{T}^{\mathbf{2n}}$ with coordinates in the tropical semiring $\mathbb{T} := \mathbb{R} \cup \{\infty\}$ is admissible if for all i we have that at most one of x_i and x_{i^*} is not equal to ∞. Based on this notion of admissibility we will define duality, circuits, and cycles for a tropical Wick vector p, generalizing the corresponding definitions for even Δ-matroids. We will be mostly interested in studying the cocycle space of a tropical Wick vector, which can be seen as an analog in type D to the tropical linear space associated to a tropical Plücker vector. We will study some of its properties, and in particular, we will give a parametric description of it in terms of cocircuits:

Theorem 3.6.9. *The cocycle space $\mathcal{Q}(p) \subseteq \mathbb{T}^{\mathbf{2n}}$ of a tropical Wick vector $p \in \mathbb{T}^{2^{[n]}}$ is equal to the set of admissible vectors in the tropical convex hull of the cocircuits of p.*

We will then specialize our results to tropical Plücker vectors, unifying in this way several results for tropical linear spaces given by Murota and Tamura [MT01], Speyer [Spe08], and Ardila and Klivans [AK06].

In Section 3.7 we will define isotropical linear spaces and study their relation with tropical Wick vectors. We will give an effective characterization in Theorem 3.7.3 for determining when a tropical linear space is isotropical, in terms of its associated Plücker vector. We will also show that the correspondence between isotropic linear spaces and points in the pure spinor space is lost after tropicalizing; nonetheless, we will prove that this correspondence still holds when we restrict our attention only to admissible vectors:

Theorem 3.7.5. *Let $K = \mathbb{C}\{\{t\}\}$ be the field of Puiseux series. Let $U \subseteq K^{\mathbf{2n}}$ be an isotropic subspace, and let w be its corresponding point in the space of pure spinors $\mathrm{Spin}^{\pm}(n)$. Suppose $p \in \mathbb{T}^{2^{[n]}}$ is the tropical Wick vector obtained as the valuation of w. Then the set of admissible vectors in the tropicalization of U is the cocycle space $\mathcal{Q}(p) \subseteq \mathbb{T}^{\mathbf{2n}}$ of p.*

3.2 Isotropic Linear Spaces and Spinor Varieties

Let n be a positive integer, and let V be a $2n$-dimensional vector space over an algebraically closed field K of characteristic 0, with a fixed basis $e_1, e_2, \ldots, e_n, e_{1^*}, e_{2^*}, \ldots, e_{n^*}$. Denote by $2^{[n]}$ the collection of subsets of the set $[n] := \{1, 2, \ldots, n\}$. In order to simplify the notation, if $S \in 2^{[n]}$ and $a \in [n]$ we will write Sa, $S - a$, and $S\Delta a$ instead of $S \cup \{a\}$, $S \setminus \{a\}$, and $S\Delta\{a\}$, respectively. Given an n-dimensional isotropic subspace $U \subseteq V$, one can associate to it a vector $w \in \mathbb{P}^{2^{[n]}-1}$ of Wick coordinates as follows. Write U as the rowspace of some $n \times 2n$ matrix M with entries in K. If the first n columns of M are linearly independent, we can row-reduce the matrix M and assume that it has the form $M = [I|A]$, where I is the identity matrix of size n and A is an $n \times n$ matrix. The fact that U is isotropic is equivalent to the property that the matrix A is skew-symmetric. The vector $w \in \mathbb{P}^{2^{[n]}-1}$ is then defined as

$$w_{[n]\setminus S} := \begin{cases} \mathrm{Pf}(A_S) & \text{if } |S| \text{ is even,} \\ 0 & \text{if } |S| \text{ is odd;} \end{cases}$$

where $S \in 2^{[n]}$ and $\mathrm{Pf}(A_S)$ denotes the Pfaffian of the principal submatrix A_S of A whose rows and columns are indexed by the elements of S. If the first n columns of M are linearly dependent then we proceed in a similar way but working over a different affine chart of $\mathbb{P}^{2^{[n]}-1}$. In this case, we can first reorder the elements of our basis (and thus the columns of M) using a permutation of $\mathbf{2n} := \{1, 2, \ldots, n, 1^*, 2^*, \ldots, n^*\}$ consisting of transpositions of the form (j, j^*) for all j in some index set $J \subseteq [n]$, so that we get a new matrix that can be row-reduced to a matrix of the form $M' = [I|A]$ (with A skew-symmetric). We then compute the Wick coordinates as

$$w_{[n]\setminus S} := \begin{cases} (-1)^{\mathrm{sg}(S,J)} \cdot \mathrm{Pf}(A_{S\Delta J}) & \text{if } |S\Delta J| \text{ is even,} \\ 0 & \text{if } |S\Delta J| \text{ is odd;} \end{cases}$$

where $(-1)^{\mathrm{sg}(S,J)}$ is some sign depending on S and J that will not be important for us. The vector $w \in \mathbb{P}^{2^{[n]}-1}$ of Wick coordinates depends only on the subspace U, and the subspace U can be recovered from its vector w of Wick coordinates as

$$U = \bigcap_{T\subseteq[n]} \left\{ x \in V : \sum_{i\in T}(-1)^{\mathrm{sg}(i,T)}\, w_{T-i} \cdot x_i \;+ \sum_{j\notin T}(-1)^{\mathrm{sg}(j,T)}\, w_{Tj} \cdot x_{j^*} \right\}, \qquad (3.2.1)$$

where again the signs $(-1)^{\mathrm{sg}(i,T)}$ and $(-1)^{\mathrm{sg}(j,T)}$ will not matter for us. The following example might help make things clear.

Example 3.2.1. Take $n = 4$, and let U be the isotropic subspace of $\mathbb{C}^{\mathbf{8}}$ given as the rowspace of the matrix

$$M = \begin{array}{c} \begin{matrix} \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} & \mathbf{1}^* & \mathbf{2}^* & \mathbf{3}^* & \mathbf{4}^* \end{matrix} \\ \begin{pmatrix} 1 & 0 & -1 & 0 & 0 & 1 & 0 & 2 \\ 0 & 1 & 3 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -3 & 1 & -5 \\ 0 & 0 & 5 & 1 & -2 & 0 & 0 & 0 \end{pmatrix} \end{array}.$$

Since the first four columns of M are linearly dependent, in order to find the Wick coordinates of U we first swap columns 3 and 3^* (so J will be equal to $\{3\}$), getting the matrix

$$M' = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & -1 & 2 \\ 0 & 1 & 0 & 0 & -1 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 & 1 & -3 & 0 & -5 \\ 0 & 0 & 0 & 1 & -2 & 0 & 5 & 0 \end{pmatrix}.$$

Some Wick coordinates of U are then

$$\begin{aligned} |w_{134}| &= \mathrm{Pf}(A_{2\Delta 3}) = \mathrm{Pf}(A_{23}) = 3, \\ |w_{3}| &= \mathrm{Pf}(A_{124\Delta 3}) = \mathrm{Pf}(A_{1234}) = 1 \cdot (-5) - (-1) \cdot 0 + 2 \cdot 3 = 1, \\ |w_{24}| &= 0. \end{aligned}$$

The **space of pure spinors** is the set $\mathrm{Spin}^{\pm}(n) \subseteq \mathbb{P}^{2^{[n]}-1}$ of Wick coordinates of all n-dimensional isotropic subspaces of V, and thus it is a parameter space for these subspaces. It is an algebraic set, and it decomposes into two isomorphic irreducible varieties as $\mathrm{Spin}^{\pm}(n) =$

$\mathrm{Spin}^+(n) \sqcup \mathrm{Spin}^-(n)$, where $\mathrm{Spin}^+(n)$ consists of all Wick coordinates w whose **support** $\mathrm{supp}(w) := \{S \in 2^{[n]} : w_S \neq 0\}$ is made of even-sized subsets, and $\mathrm{Spin}^-(n)$ consists of all Wick coordinates whose support is made of odd-sized subsets. The irreducible variety $\mathrm{Spin}^+(n)$ is called the **spinor variety**; it is the projective closure of the image of the map sending an $n \times n$ skew-symmetric matrix to its vector of Pfaffians. Its defining ideal consists of all polynomial relations among the Pfaffians of a skew-symmetric matrix, and it is generated by the following quadratic relations:

$$\sum_{i=1}^{s}(-1)^i\, w_{\tau_i\sigma_1\sigma_2\cdots\sigma_r} \cdot w_{\tau_1\tau_2\cdots\hat{\tau_i}\cdots\tau_s} + \sum_{j=1}^{r}(-1)^j\, w_{\sigma_1\sigma_2\cdots\hat{\sigma_j}\cdots\sigma_r} \cdot w_{\sigma_j\tau_1\tau_2\cdots\tau_s}, \tag{3.2.2}$$

where $\sigma, \tau \in 2^{[n]}$ have odd cardinalities r, s, respectively, and the variables w_σ are understood to be alternating with respect to a reordering of the indices, e.g. $w_{2134} = -w_{1234}$ and $w_{1135} = 0$. The ideal defining the space of pure spinors is generated by all quadratic relations having the form (3.2.2), but now with $\sigma, \tau \in 2^{[n]}$ having any cardinality. These relations are known as **Wick relations**. The shortest nontrivial Wick relations are obtained when $|\sigma \Delta \tau| = 4$, in which case they have the form

$$w_{Sabcd} \cdot w_S - w_{Sab} \cdot w_{Scd} + w_{Sac} \cdot w_{Sbd} - w_{Sad} \cdot w_{Sbc}$$

and

$$w_{Sabc} \cdot w_{Sd} - w_{Sabd} \cdot w_{Sc} + w_{Sacd} \cdot w_{Sb} - w_{Sbcd} \cdot w_{Sa},$$

where $S \subseteq [n]$ and $a, b, c, d \in [n] \backslash S$ are distinct. These relations will be of special importance for us; they will be called the **4-term Wick relations**. For more information about spinor varieties and isotropic linear spaces we refer the reader to [Man09, Pro07, SV10].

It is not hard to check that if W is any linear subspace of K^n then $U := W \times W^\perp$ is an n-dimensional isotropic subspace of $K^{\mathbf{2n}}$ whose Wick coordinates are the Plücker coordinates of W, so Wick vectors and Wick relations can be thought as a generalization of Plücker vectors and Plücker relations. When studying linear subspaces of a vector space and their corresponding Plücker coordinates, it is natural to investigate what are all possible supports of such Plücker vectors. This leads immediately to the notion of (realizable) matroids. In our case, the study of all possible supports of Wick vectors leads us directly to the notion of Delta-matroids (or Δ-matroids), which generalize classical matroids.

3.3 Delta-Matroids

In this section we review some of the basic theory of Δ-matroids and even Δ-matroids. These generalize matroids in a very natural way, and have also the useful feature of being characterized by many different sets of axioms. For a much more extensive exposition of matroids and Δ-matroids, the reader can consult [Oxl92, Bou87, BGW03, Bou97, Bou98].

3.3.1 Bases

Our first description of Δ-matroids is the following.

Definition 3.3.1. A **Δ-matroid** (or **Delta-matroid**) is a pair $M = (E, \mathcal{B})$, where E is a finite set and $\mathcal{B}$ is a nonempty collection of subsets of E satisfying the following **symmetric exchange axiom**:

- For all $A, B \in \mathcal{B}$ and for all $a \in A\Delta B$, there exists $b \in A\Delta B$ such that $A\Delta\{a, b\} \in \mathcal{B}$.

Here Δ denotes symmetric difference: $X\Delta Y = (X \setminus Y) \cup (Y \setminus X)$. The set E is called the **ground set** of M, and $\mathcal{B}$ is called the collection of **bases** of M. We also say that M is a Δ-matroid over the set E. In this chapter we will usually work with Δ-matroids over the set $[n]$.

Delta-matroids are a natural generalization of classical matroids; in fact, it is easy to see that matroids are precisely those Δ-matroids whose bases have all the same cardinality (the reader not familiar with matroids can take this as a definition).

Delta-matroids are a special class of Coxeter matroids, and they have appeared in the literature under many other names like: Lagrangian matroids, symmetric matroids, 2-matroids or metroids (see [BGW03] for more information). The name Δ-matroid is meant to emphasize the analogy in Definition 3.3.1 with the exchange axiom for classical matroids.

It is important to note that the exchange axiom for Δ-matroids does not require the elements b and a to be distinct. Doing so leads us to the more specific notion of even Δ-matroid, which will play a central role in the rest of this chapter.

Definition 3.3.2. An **even Δ-matroid** (or **even Delta-matroid**) is a Δ-matroid $M = (E, \mathcal{B})$ satisfying the following stronger exchange axiom:

- For all $A, B \in \mathcal{B}$ and for all $a \in A\Delta B$, there exists $b \in A\Delta B$ such that $b \neq a$ and $A\Delta\{a, b\} \in \mathcal{B}$.

An even Δ-matroid is called a Lagrangian orthogonal matroid in [BGW03].

The following proposition follows easily from the definitions, and it motivates the terminology we use.

Proposition 3.3.3. *Let M be a Δ-matroid. Then M is an even Δ-matroid if and only if all the bases of M have the same parity.*

It should be mentioned that the bases of an even Δ-matroid can all have odd cardinality; unfortunately, the name used for even Δ-matroids might be a little misleading.

The notion of duality for matroids generalizes naturally to Δ-matroids.

Definition 3.3.4. Let $M = (E, \mathcal{B})$ be an (even) Δ-matroid. Directly from the definition it follows that the collection

$$\mathcal{B}^* := \{E \setminus B : B \in \mathcal{B}\}$$

is also the collection of bases of an (even) Δ-matroid M^* over E. We will refer to M^* as the **dual** (even) Δ-matroid to M.

The somewhat simple exchange axiom defining even Δ-matroids implies the following much stronger exchange axiom (see [BGW03]).

Proposition 3.3.5. *Let M be an even Δ-matroid. Then M satisfies the following* ***strong exchange axiom****:*

- *For all $A, B \in \mathcal{B}$ and for any $a \in A\Delta B$, there exists $b \in A\Delta B$ such that $b \neq a$ and both $A\Delta\{a, b\}$ and $B\Delta\{a, b\}$ are in $\mathcal{B}$.*

General Δ-matroids do not satisfy an analogous strong exchange axiom, as the reader can easily verify.

3.3.2 Representability

As we mentioned before, our interest in even Δ-matroids comes from the study of the possible supports that a Wick vector can have. The following proposition establishes the desired connection.

Proposition 3.3.6. *Let V be a $2n$-dimensional vector space over the field K. If $U \subseteq V$ is an n-dimensional isotropic subspace with Wick coordinates w, then the subsets in* $\mathrm{supp}(w) := \{S \in 2^{[n]} : w_S \neq 0\}$ *form the collection of bases of an even Δ-matroid $M(U)$ over $[n]$. An even Δ-matroid arising in this way is said to be a* ***representable*** *even Δ-matroid (over the field K).*

If M is a matroid over the ground set $[n]$ then we have two different notions of representability: representability as a classical matroid by a linear subspace of K^n and representability as an even Δ-matroid by an n-dimensional isotropic subspace of $K^{\mathbf{2n}}$. It was shown by Bouchet in [Bou88] that these two notions agree, so representability for even Δ-matroids in fact generalizes representability for matroids.

Representability is a very subtle property of matroids. Some work has succeeded in studying this property over fields of very small characteristic, but there is no simple and useful characterization of representable matroids over a field of characteristic zero. The study of representability for even Δ-matroids shares the same difficulties, and there seems to be almost no research done in this direction so far.

3.3.3 Matroid Polytopes

A very important and useful way of working with matroids is via their associated polytopes. These polytopes and their subdivisions will play a very important role in the rest of the chapter.

Given any collection $\mathcal{B}$ of subsets of $[n]$ one can associate to it the polytope

$$\Gamma_{\mathcal{B}} := \text{convex}\{e_S : S \in \mathcal{B}\},$$

where $e_S := \sum_{i \in S} e_i$ is the indicator vector of the subset S. The following theorem characterizes the polytopes associated to classical matroids, and thus it gives us a geometrical way of thinking about matroids.

Theorem 3.3.7 (Gelfand, Goresky, MacPherson and Serganova [GGMS87])**.** *If $\mathcal{B} \subseteq 2^{[n]}$ is nonempty then $\mathcal{B}$ is the collection of bases of a matroid if and only if all the edges of the polytope $\Gamma_{\mathcal{B}}$ have the form $e_i - e_j$, where $i, j \in [n]$ are distinct.*

Theorem 3.3.7 is just a special case of a very general and fundamental theorem characterizing the associated polytopes of a much larger class of matroids, called Coxeter matroids (see [BGW03]). In the case of even Δ-matroids it takes the following form.

Theorem 3.3.8. *If $\mathcal{B} \subseteq 2^{[n]}$ is nonempty then $\mathcal{B}$ is the collection of bases of an even Δ-matroid if and only if all the edges of the polytope $\Gamma_{\mathcal{B}}$ have the form $\pm e_i \pm e_j$, where $i, j \in [n]$ are distinct.*

These results allow us think of matroids and even Δ-matroids in terms of irreducible root systems: classical matroids should be thought of as the class of matroids of type A, and even Δ-matroids as the class of matroids of type D.

3.3.4 Circuits and Symmetric Matroids

We will now describe a notion of circuits for even Δ-matroids that generalizes the notion of circuits for matroids. For this purpose we will introduce symmetric matroids, an concept equivalent to Δ-matroids. We will present here only the basic properties needed for the rest of the chapter; a much more detailed description can be found in [BGW03].

Consider the sets

$$[n] := \{1, 2, \ldots, n\} \text{ and } [n]^* := \{1^*, 2^*, \ldots, n^*\}.$$

Define the map $* : [n] \to [n]^*$ by $i \mapsto i^*$ and the map $* : [n]^* \to [n]$ by $i^* \mapsto i$. We can think of $*$ as an involution of the set $\mathbf{2n} := [n] \cup [n]^*$, where for any $j \in \mathbf{2n}$ we have $j^{**} = j$. If $J \subseteq \mathbf{2n}$ we define $J^* := \{j^* : j \in J\}$. We say that the set J is **admissible** if $J \cap J^* = \emptyset$, and that it is a **transversal** if it is an admissible subset of size n. For any $S \subseteq [n]$, we define its **extension** $\bar{S} \subseteq \mathbf{2n}$ to be the transversal given by $\bar{S} := S \cup ([n] \setminus S)^*$, and for any transversal J we will define its **restriction** to be the set $J \cap [n]$. Extending and restricting are clearly bijections (inverse to each other) between the set $2^{[n]}$ and the set of transversals $\mathcal{V}(n)$ of $\mathbf{2n}$.

Definition 3.3.9. Given a Δ-matroid $M = ([n], \mathcal{B})$, the **symmetric matroid** associated to M is the collection $\bar{\mathcal{B}}$ of transversals defined as $\bar{\mathcal{B}} := \{\bar{B} : B \in \mathcal{B}\}$.

There is of course no substantial difference between a Δ-matroid and its associated symmetric matroid; however, working with symmetric matroids will allow us to simplify the forthcoming definitions.

Definition 3.3.10. Let $M = ([n], \mathcal{B})$ be an even Δ-matroid over $[n]$. A subset $S \subseteq \mathbf{2n}$ is called **independent** in M if it is contained in some transversal $\bar{B} \in \bar{\mathcal{B}}$, and it is called **dependent** in M if it is not independent. A subset $C \subseteq \mathbf{2n}$ is called a **circuit** of M if C is a minimal dependent subset which is admissible. A **cocircuit** of M is a circuit of the dual even Δ-matroid M^*. The set of circuits of M will be denoted by $\mathcal{C}(M)$, and the set of cocircuits by $\mathcal{C}^*(M)$. An admissible union of circuits of M is called a **cycle** of M. A **cocycle** of M is a cycle of the dual even Δ-matroid M^*.

The next observation shows that our definition of circuits for even Δ-matroids indeed generalizes the concept of circuits for matroids.

Proposition 3.3.11. *Let $M = ([n], \mathcal{B})$ be a matroid. Denote by $\mathcal{C}$ its collection of (classical) circuits and by $\mathcal{K}$ its collection of (classical) cocircuits. Then the collection of circuits of M, when considered as an even Δ-matroid, is*

$$\{C : C \in \mathcal{C}\} \cup \{K^* : K \in \mathcal{K}\}.$$

Many of the results about circuits in matroids generalize to this extended setup. We will just state here some of the facts that we will use later; their proofs can be found in [BGW03].

Proposition 3.3.12. *Let $M = ([n], \mathcal{B})$ be an even Δ-matroid. Suppose $\bar{B} \in \bar{\mathcal{B}}$ and $j \in \mathbf{2n} \backslash \bar{B}$. Then $\bar{B} \cup j$ contains a unique circuit $C(\bar{B}, j)$, called the **fundamental circuit** of j over $\bar{B}$. It is given by*

$$C(\bar{B}, j) = \left\{ i \in \bar{B} : \bar{B} \Delta \{j, j^*, i, i^*\} \in \bar{\mathcal{B}} \right\} \cup j.$$

Proposition 3.3.13. *Let M be an even Δ-matroid. If C is a circuit of M and K is a cocircuit of M then $|C \cap K| \neq 1$.*

Example 3.3.14. Take $n = 3$, and let U be the isotropic subspace of $\mathbb{C}^{\mathbf{6}}$ defined as the rowspace of the matrix

$$M = \begin{array}{c} \begin{matrix} \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{1}^* & \mathbf{2}^* & \mathbf{3}^* \end{matrix} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 & 0 & 2 \\ 0 & 0 & 1 & 1 & -2 & 0 \end{pmatrix} \end{array}.$$

The even Δ-matroid M represented by U has bases $\mathcal{B} = \{123, 1, 2, 3\}$, corresponding to the support of its vector of Wick coordinates. Its associated polytope is the tetrahedron with vertices $(1,1,1),(1,0,0),(0,1,0),(0,0,1)$, whose edges are indeed of the form $\pm e_i \pm e_j$. The circuits of M are the admissible subsets $1^*23, 12^*3, 123^*, 1^*2^*3^*$. The dual even Δ-matroid M^* has bases $\mathcal{B}^* = \{\emptyset, 12, 13, 23\}$. The cocircuits of M are the admissible subsets $123, 12^*3^*, 1^*23^*, 1^*2^*3$.

3.3.5 Minors and Rank

We end up this section with a very brief discussion of minors and rank for even Δ-matroids. More information can be found in [Bou97, Bou98].

Definition 3.3.15. Let $M = ([n], \mathcal{B})$ be an even Δ-matroid. Given $S \in 2^{[n]}$, consider the subcollections of bases $\mathcal{B}^+ := \{B \in \mathcal{B} : |B \cap S| \text{ is maximal}\}$ and $\mathcal{B}^- := \{B \in \mathcal{B} : |B \cap S| \text{ is minimal}\}$. It is not hard to check that the collections $\mathcal{B}^+ \setminus S := \{B \setminus S : B \in \mathcal{B}^+\}$ and $\mathcal{B}^- \setminus S := \{B \setminus S : B \in \mathcal{B}^-\}$ are collections of bases of even Δ-matroids over the ground set $[n] \setminus S$. These even Δ-matroids are called the **contraction** of S from M and the **deletion** of S from M, respectively, and are denoted by M/S and $M \setminus S$. The deletion of S from M is sometimes called the **restriction** of M to the ground set $[n] \setminus S$. A **minor** of M is an even Δ-matroid that can be obtained by a sequence of contractions and deletions from the matroid M.

Note that deletion and contraction are operations dual to each other: for any $S \in 2^{[n]}$ we have $(M/S)^* = M^* \setminus S$.

We now define the rank function of an even Δ-matroid by means of its associated symmetric matroid.

Definition 3.3.16. Let $M = ([n], \mathcal{B})$ be an even Δ-matroid. The **rank** in M of an admissible subset $J \subseteq \mathbf{2n}$ is defined as

$$r_M(J) := \max_{\bar{B} \in \bar{\mathcal{B}}} |\bar{B} \cap J|.$$

The following proposition can be checked without too much difficulty.

Proposition 3.3.17. *Let $M = ([n], \mathcal{B})$ be an even Δ-matroid, and let $S \in 2^{[n]}$. Then the rank functions of the contraction M/S and the deletion $M \setminus S$ are given by*

$$r_{M/S}(T) = r_M(T \cup S) - r_M(S),$$
$$r_{M \setminus S}(T) = r_M(T \cup S^*) - r_M(S^*);$$

where $T \subseteq \mathbf{2n} \setminus (S \cup S^*)$ is any admissible subset.

3.4 Tropical Wick Relations

We now turn to the study of the tropical prevariety and tropical variety defined by the Wick relations. Denote by $2^{[n]}$ the collection of subsets of the set $[n] := \{1, 2, \ldots, n\}$. If $S \in 2^{[n]}$ and $a \in [n]$, we write Sa, $S - a$, and $S\Delta a$ instead of $S \cup \{a\}$, $S \setminus \{a\}$, and $S\Delta\{a\}$, respectively.

Definition 3.4.1. A vector $p = (p_S) \in \mathbb{T}^{2^{[n]}}$ is called a **tropical Wick vector** if it satisfies the tropical Wick relations, that is, for all $S, T \in 2^{[n]}$ the minimum

$$\min_{i \in S\Delta T}(p_{S\Delta i} + p_{T\Delta i}) \tag{3.4.1}$$

is achieved at least twice (or it is equal to ∞). The **Δ-Dressian** $\Delta\mathrm{Dr}(n) \subseteq \mathbb{T}^{2^{[n]}}$ is the space of all tropical Wick vectors in $\mathbb{T}^{2^{[n]}}$, i.e., the tropical prevariety defined by the Wick relations.

Tropical Wick vectors have also been studied in the literature under the name of **valuated Δ-matroids** (see [DW91, Mur96]), and in a more general setup under the name of M-convex functions on jump systems (see [Mur06]).

The following definition will be central to our study, and it is the reason why working over $\mathbb{R} \cup \infty$ and not just $\mathbb{R}$ is fundamental for us.

Definition 3.4.2. The **support** of a vector $p = (p_S) \in \mathbb{T}^{2^{[n]}}$ is the collection

$$\mathrm{supp}(p) := \{S \subseteq [n] : p_S \neq \infty\}.$$

We will later see (Theorem 3.5.1) that the support of any tropical Wick vector consists of subsets whose cardinalities have all the same parity, so the Δ-Dressian decomposes as the disjoint union of two tropical prevarieties: the **even Δ-Dressian** $\Delta\mathrm{Dr}^+(n) \subseteq \mathbb{T}^{2^{[n]}}$ (consisting of all tropical Wick vectors whose support has only subsets of even cardinality) and the **odd Δ-Dressian** $\Delta\mathrm{Dr}^-(n) \subseteq \mathbb{T}^{2^{[n]}}$ (defined analogously).

One of the main advantages of allowing our vectors to have ∞ entries is that tropical Wick vectors can be seen as a generalization of tropical Plücker vectors (or valuated matroids), as explained below.

Definition 3.4.3. A tropical Wick vector $p = (p_S) \in \mathbb{T}^{2^{[n]}}$ is called a **tropical Plücker vector** (or a **valuated matroid**) if all the subsets in $\mathrm{supp}(p)$ have the same cardinality r_p, called the **rank** of p. The name is justified by noting that in this case, the tropical Wick relations become just the tropical Plücker relations: For all $S, T \in 2^{[n]}$ such that $|S| = r_p - 1$ and $|T| = r_p + 1$, the minimum

$$\min_{i \in T \setminus S}(p_{Si} + p_{T-i}) \tag{3.4.2}$$

is achieved at least twice (or it is equal to ∞). The space of tropical Plücker vectors of rank k is called the **Dressian** $\mathrm{Dr}(k, n)$; it is the tropical prevariety defined by the Plücker relations of rank k.

Tropical Plücker vectors play a central role in the combinatorial study of tropical linear spaces done by Speyer (see [Spe08]). In his paper he only deals with tropical Plücker vectors whose support is the collection of all subsets of $[n]$ of some fixed size k; we will later see that our definition is the "correct" generalization to more general supports.

Definition 3.4.4. The **tropical pure spinor space** $\mathrm{TSpin}^{\pm}(n) \subseteq \mathbb{T}^{2^{[n]}}$ is the tropicalization of the space of pure spinors, i.e., it is the tropical variety defined by the ideal generated by all Wick relations. A tropical Wick vector in the tropical pure spinor space is said to be **realizable**. The decomposition of the Δ-Dressian into its even an odd parts induces a decomposition of the tropical pure spinor space as the disjoint union of two "isomorphic" tropical varieties $\mathrm{TSpin}^{+}(n)$ and $\mathrm{TSpin}^{-}(n)$, namely, the tropicalization of the spinor varieties $\mathrm{Spin}^{+}(n)$ and $\mathrm{Spin}^{-}(n)$ described in Section 3.2. The tropicalization $\mathrm{TSpin}^{+}(n) \subseteq \mathbb{T}^{2^{[n]}}$ of the even part $\mathrm{Spin}^{+}(n)$ will be called the **tropical spinor variety**.

By definition, we have that the tropical pure spinor space $\mathrm{TSpin}^{\pm}(n)$ is contained in the Δ-Dressian $\Delta\mathrm{Dr}(n)$. A first step in studying representability of tropical Wick vectors (i.e. valuated Δ-matroids) is to determine when these two spaces are the same, or equivalently, when the Wick relations form a tropical basis. Our main result in this section answers this question for almost all values of n.

Theorem 3.4.5. *If $n \leq 5$ then the tropical pure spinor space $\mathrm{TSpin}^{\pm}(n)$ is equal to the Δ-Dressian $\Delta\mathrm{Dr}(n)$, i.e., the Wick relations form a tropical basis for the ideal they generate. If $n \geq 7$ then $\mathrm{TSpin}^{\pm}(n)$ is strictly smaller than $\Delta\mathrm{Dr}(n)$; in fact, there is a vector in the even Δ-Dressian $\Delta\mathrm{Dr}^{+}(n)$ whose support consists of all even-sized subsets of $[n]$ which is not in the tropical spinor variety $\mathrm{TSpin}^{+}(n)$.*

As a corollary, we get the following result about representability of even Δ-matroids.

Corollary 3.4.6. *Let M be an even Δ-matroid on a ground set of at most 5 elements. Then M is a representable even Δ-matroid over any algebraically closed field of characteristic 0.*

We will postpone the proof of Theorem 3.4.5 and Corollary 3.4.6 until Section 3.5, after we have studied some of the combinatorial properties of tropical Wick vectors. To show that the tropical pure spinor space and the Δ-Dressian agree when $n \leq 5$ we will make use of Anders Jensen's software Gfan [Jen]. It is still unclear what happens when $n = 6$. In this case, the spinor variety is described by 76 nontrivial Wick relations (60 of which are 4-term Wick relations) on 32 variables, and a Gfan computation requires a long time to finish. We state the following conjecture.

Conjecture 3.4.7. *The tropical pure spinor space $\mathrm{TSpin}^{\pm}(6)$ is equal to the Δ-Dressian $\Delta\mathrm{Dr}(6)$.*

If Conjecture 3.4.7 is true, or even if $\mathrm{TSpin}^{\pm}(6)$ and $\Delta\mathrm{Dr}(6)$ agree just on all vectors having as support all even-sized subsets of $[n]$, we could extend the proof of Corollary 3.4.6 to show that all even Δ-matroids over a ground set of at most 6 elements are representable over any algebraically closed field of characteristic 0.

3.5 Tropical Wick Vectors and Delta-Matroid Subdivisions

In this section we provide a description of tropical Wick vectors in terms of polytopal subdivisions. It allows us to deal with tropical Wick vectors in a purely geometric way. We start with a useful local characterization, which was basically proved by Murota in [Mur06].

Theorem 3.5.1. *Suppose $p = (p_S) \in \mathbb{T}^{2^{[n]}}$ has nonempty support. Then p is a tropical Wick vector if and only if the following two conditions are satisfied:*

(a) The support supp(p) of p is the collection of bases of an even Δ-matroid over $[n]$.

(b) The vector p satisfies the 4-term tropical Wick relations: For all $S \in 2^{[n]}$ and all $a, b, c, d \in [n] \setminus S$ distinct, the minima

$$\begin{aligned} &\min(p_{Sabcd} + p_S, p_{Sab} + p_{Scd}, p_{Sac} + p_{Sbd}, p_{Sad} + p_{Sbc}) \\ &\min(p_{Sabc} + p_{Sd}, p_{Sabd} + p_{Sc}, p_{Sacd} + p_{Sb}, p_{Sbcd} + p_{Sa}) \end{aligned} \tag{3.5.1}$$

are achieved at least twice (or are equal to ∞).

Proof. If p is a tropical Wick vector then, by definition, p satisfies the 4-term tropical Wick relations. To show that supp(p) is an even Δ-matroid, suppose $A, B \in$ supp(p) and $a \in A\Delta B$. Take $S = A\Delta a$ and $T = B\Delta a$. The minimum in equation (3.4.1) is then a finite number, since $p_{S\Delta a} + p_{T\Delta a} = p_A + p_B$ is finite. Therefore, this minimum is achieved at least twice, so there exists $b \in S\Delta T = A\Delta B$ such that $b \neq a$ and $p_{S\Delta b} + p_{T\Delta b} = p_{A\Delta\{a,b\}} + p_{B\Delta\{a,b\}} < \infty$. This implies that $A\Delta\{a, b\}$ and $B\Delta\{a, b\}$ are both in supp(p), which shows that supp(p) satisfies the strong exchange axiom for even Δ-matroids.

The reverse implication is basically a reformulation of the following characterization given by Murota (done in greater generality for M-convex functions on jump systems; for details see [Mur06]): If supp(p) is the collection of bases of an even Δ-matroid over $[n]$ then p is a tropical Wick vector if and only if for all $A, B \in$ supp(p) such that $|A\Delta B| = 4$, there exist $a, b \in A\Delta B$ distinct such that $p_A + p_B \geq p_{A\Delta\{a,b\}} + p_{B\Delta\{a,b\}}$. ☐

As a corollary, we get the following local description of tropical Plücker vectors.

Corollary 3.5.2. *Suppose $p = (p_S) \in \mathbb{T}^{2^{[n]}}$ has nonempty support. Then p is a tropical Plücker vector if and only if the following two conditions are satisfied:*

(a) The support supp(p) of p is the collection of bases of matroid over $[n]$ (of rank r_p).

(b) The vector p satisfies the 3-term tropical Plücker relations: For all $S \in 2^{[n]}$ such that $|S| = r_p - 2$ and all $a, b, c, d \in [n] \setminus S$ distinct, the minimum

$$\min(p_{Sab} + p_{Scd}, p_{Sac} + p_{Sbd}, p_{Sad} + p_{Sbc})$$

is achieved at least twice (or it is equal to ∞).

Proof. The 3-term tropical Plücker relations are just the 4-term tropical Wick relations in the case where all the subsets in supp(p) have the same cardinality. □

Corollary 3.5.2 shows that our notion of tropical Plücker vector is indeed a generalization of the one given by Speyer in [Spe08] to the case where supp(p) is not necessarily the collection of bases of a uniform matroid.

It is worth mentioning that the assumptions on the support of p are essential in the local descriptions given above. As an example of this, consider the vector $p \in \mathbb{T}^{\mathcal{P}(6)}$ defined as

$$p_I := \begin{cases} 0 & \text{if } I = 123 \text{ or } I = 456, \\ \infty & \text{otherwise.} \end{cases}$$

The vector p satisfies the 3-term tropical Plücker relations, but its support is not the collection of bases of a matroid and thus p is not a tropical Plücker vector.

Definition 3.5.3. Given a vector $p = (p_S) \in \mathbb{T}^{2^{[n]}}$, denote by $\Gamma_p \subseteq \mathbb{R}^n$ its **associated polytope**

$$\Gamma_p := \text{convex}\{e_S : S \in \text{supp}(p)\}.$$

The vector p induces naturally a regular subdivision $\mathcal{D}_p$ of Γ_p in the following way. Consider the vector p as a height function on the vertices of Γ_p, so "lift" vertex e_S of Γ_p to height p_S to obtain the **lifted polytope** $\Gamma'_p = \text{convex}\{(e_S, p_S) : S \in \text{supp}(p)\} \subseteq \mathbb{R}^{n+1}$. The **lower faces** of Γ'_p are the faces of Γ'_p minimizing a linear form $(v, 1) \in \mathbb{R}^{n+1}$; their projection back to $\mathbb{R}^n$ form the polytopal subdivision $\mathcal{D}_p$ of Γ_p, called the **regular subdivision induced by** p.

We now come to the main result of this section. It describes tropical Wick vectors as the height vectors that induce "nice" polytopal subdivisions. After finishing this work, it was pointed out to the author that an equivalent formulation of this result had already been proved by Murota in [Mur97], under the language of maximizers of an even Δ-matroid.

Theorem 3.5.4. *Let $p = (p_S) \in \mathbb{T}^{2^{[n]}}$. Then p is a tropical Wick vector if and only if the regular subdivision $\mathcal{D}_p$ induced by p is an even Δ-matroid subdivision, i.e., it is a subdivision of an even Δ-matroid polytope into even Δ-matroid polytopes.*

Proof. Assume p is a tropical Wick vector. By condition (a) in Theorem 3.5.1, we know that Γ_p is an even Δ-matroid polytope. Let $Q \subseteq \mathbb{R}^n$ be one of the polytopes in $\mathcal{D}_p$. By definition, Q is the projection back to $\mathbb{R}^n$ of the face of the lifted polytope $\Gamma'_p \subseteq \mathbb{R}^{n+1}$ minimizing some linear form $(v, 1) \in \mathbb{R}^{n+1}$, and thus

$$\text{vertices}(Q) = \left\{ e_R \in \{0,1\}^n : p_R + \sum_{j \in R} v_j \text{ is minimal} \right\}.$$

To show that Q is an even Δ-matroid polytope, suppose e_A and e_B are vertices of Q, and assume $a \in A\Delta B$. Let $S = A\Delta a$ and $T = B\Delta a$. Since p is a tropical Wick vector, the minimum $\min_{i\in S\Delta T}(p_{S\Delta i} + p_{T\Delta i})$ is achieved at least twice or it is equal to ∞. Adding $\sum_{j\in S} v_j + \sum_{j\in T} v_j$, we get that the minimum

$$\min_{i\in S\Delta T}\left(\left(p_{S\Delta i} + \sum_{j\in S\Delta i} v_j\right) + \left(p_{T\Delta i} + \sum_{j\in T\Delta i} v_j\right)\right) \tag{3.5.2}$$

is achieved at least twice or it is equal to ∞. Since the minimum over all $R \in 2^{[n]}$ of $p_R + \sum_{j\in R} v_j$ is achieved when $R = A$ and $R = B$, it follows that the minimum (3.5.2) is achieved when $i = a$ and it is finite. Therefore, there exists $b \in S\Delta T = A\Delta B$ such that $b \neq a$ and

$$\left(p_{S\Delta b} + \sum_{j\in S\Delta b} v_j\right) + \left(p_{T\Delta b} + \sum_{j\in T\Delta b} v_j\right) = \left(p_A + \sum_{j\in A} v_j\right) + \left(p_B + \sum_{j\in B} v_j\right),$$

so $e_{S\Delta b}$ and $e_{T\Delta b}$ are also vertices of Q. This shows that the subsets corresponding to the vertices of Q satisfy the strong exchange axiom (see Proposition 3.3.5), and thus Q is an even Δ-matroid polytope.

Now, suppose $\mathcal{D}_p$ is an even Δ-matroid subdivision. We have that $\operatorname{supp}(p)$ is the collection of bases of an even Δ-matroid, so by Theorem 3.5.1 it is enough to prove that p satisfies the 4-term tropical Wick relations. If this is not the case then for some $S \in 2^{[n]}$ and $a, b, c, d \in [n] \setminus S$ distinct, one of the two minima in (3.5.1) is achieved only once (and it is not equal to infinity). It is easy to check that the corresponding sets

$$\begin{aligned} &\{e_{Sabcd}, e_S, e_{Sab}, e_{Scd}, e_{Sac}, e_{Sbd}, e_{Sad}, e_{Sbc}\} \cap \{e_S : S \in \operatorname{supp}(p)\} \\ &\{e_{Sabc}, e_{Sd}, e_{Sabd}, e_{Sc}, e_{Sacd}, e_{Sb}, e_{Sbcd}, e_{Sa}\} \cap \{e_S : S \in \operatorname{supp}(p)\} \end{aligned} \tag{3.5.3}$$

are the set of vertices of faces of Γ_p. This implies that $\mathcal{D}_p$ contains an edge joining the two vertices that correspond to the term where this minimum is achieved, which is not an edge of the form $\pm e_i \pm e_j$, so by Theorem 3.3.8 the subdivision $\mathcal{D}_p$ is not an even Δ-matroid subdivision. □

Note that Theorem 3.5.1 can now be seen as a local criterion for even Δ-matroid subdivisions: the regular subdivision induced by p is an even Δ-matroid subdivision if and only if the subdivisions it induces on the polytopes whose vertices are described by the sets of the form (3.5.3) are even Δ-matroid subdivisions. These polytopes are all isometric (when p has maximal support), and they are known as the 4-demicube. This is a regular 4-dimensional polytope with 8 vertices and 16 facets; a picture of its Schlegel diagram, created using Robert Webb's Great Stella software [Web], is shown in Figure 3.1. The 4-demicube plays the same role for even Δ-matroid subdivisions as the hypersimplex $\Delta(2, 4)$ (an octahedron) for classical matroid subdivisions.

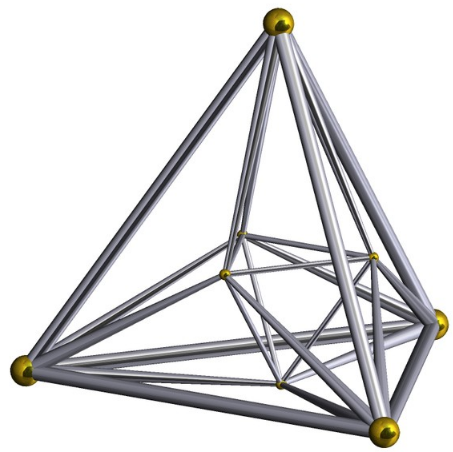

Figure 3.1: Schlegel diagram of the 4-demicube

If we restrict Theorem 3.5.4 to the case where all subsets in $\mathrm{supp}(p)$ have the same cardinality, we get the following corollary. It generalizes the results of Speyer in [Spe08] for subdivisions of a hypersimplex.

Corollary 3.5.5. *Let $p \in \mathbb{T}^{2^{[n]}}$. Then p is a tropical Plücker vector if and only if the regular subdivision $\mathcal{D}_p$ induced by p is a matroid subdivision, i.e., it is a subdivision of a matroid polytope into matroid polytopes.*

We are now in position to prove Theorem 3.4.5 and its corollary.

Proof of Theorem 3.4.5. For $n \leq 5$, we used Anders Jensen's software Gfan [Jen] to compute both the tropical spinor variety $\mathrm{TSpin}^+(n)$ and the even Δ-Dressian $\Delta\mathrm{Dr}^+(n)$, and we then checked that they were equal. At the moment, Gfan does not support computations with vectors having coordinates equal to ∞, so we split our computation into several parts. We first computed all possible even Δ-matroids on a ground set of at most 5 elements, getting a list of 35 even Δ-matroids up to isomorphism. We then used Gfan to compute for each of these even Δ-matroids M, the set of vectors in the tropical spinor variety and in the even Δ-Dressian whose support is the collection of bases of M. We finally checked that for all M these two sets were the same. A complete list of the 35 even Δ-matroids up to isomorphism and their corresponding spaces can be found in the website

`http://math.berkeley.edu/~felipe/delta/` .

The most important of these spaces is obtained when M is the even Δ-matroid whose bases are all even-sized subsets of the set [5]. It is the finite part of the even Δ-Dressian $\Delta\mathrm{Dr}^+(5)$ (and the tropical spinor variety $\mathrm{TSpin}^+(5)$), and it is described by 10 nontrivial Wick relations on 16 variables. Using Gfan we computed this space to be a pure simplicial 11-dimensional polyhedral fan with a 6-dimensional lineality space. After modding out by this lineality space we get a 5 dimensional polyhedral fan whose f-vector is $(1, 36, 280, 960, 1540, 912)$. By Theorem 3.5.4, all vectors in this fan induce an even Δ-matroid subdivision of the polytope Γ_M associated to M, which is known as the 5-demicube. As an example of this, the 36 rays in the fan correspond to the coarsest nontrivial even Δ-matroid subdivisions of Γ_M, which come in two different isomorphism classes: 16 isomorphic hyperplane splits of Γ_M into 2 polytopes, and 20 isomorphic subdivisions of Γ_M into 6 polytopes. The 912 maximal cones in the fan correspond to the finest even Δ-matroid subdivisions of Γ_M, which come in four different isomorphism classes: 192 isomorphic subdivisions into 11 pieces, and 720 subdivisions into 12 pieces, divided into 3 distinct isomorphism classes of sizes 120, 120, and 480, respectively. A complete description of all these subdivisions can also be found in the website

`http://math.berkeley.edu/~felipe/delta/` ;

they were computed with the aid of the software polymake [GJ00].

We now move to the case $n \geq 7$. Recall the notion of rank for even Δ-matroids discussed in Section 3.3.5. We will prove that for any even Δ-matroid M with rank function r_M, the vector $p = (p_T) \in \mathbb{R}^{2^{[n]}}$ defined as

$$p_T := \begin{cases} -r_M(\bar{T}) & \text{if } |T| \text{ is even,} \\ \infty & \text{otherwise;} \end{cases}$$

is a tropical Wick vector (where $\bar{T} := T \cup ([n] \setminus T)^*$). By Theorem 3.5.1, it is enough to prove that for any $S \in 2^{[n]}$ and any $a, b, c, d \in [n] \setminus S$ distinct, p satisfies the 4-term tropical Wick relations given in (3.5.1). Since the rank function of M satisfies $r_M(S \cup I) = r_{M/S}(I) + r_M(S)$ (see Proposition 3.3.17), we can assume that $S = \emptyset$. In a similar way, by restricting our matroid to the ground set $\{a, b, c, d\}$ we see that it is enough to prove our claim for even Δ-matroids over a ground set of at most 4 elements. There are 11 even Δ-matroids up to isomorphism in this case (see `http://math.berkeley.edu/~felipe/delta/`), and it is not hard to check that for all of them the assertion holds.

Now, take M to be an even Δ-matroid which is not representable over $\mathbb{C}$ (for example, let M be any matroid having the Fano matroid as a direct summand). In this case, the linear form $(0, 0, \ldots, 0, 1) \in \mathbb{R}^{n+1}$ attains its minimum on the lifted polytope Γ'_p at the vertices corresponding to the bases of M, so the corresponding even Δ-matroid subdivision $\mathcal{D}_p$ has as one of its faces the even Δ-matroid polytope of M. Since M is not representable over $\mathbb{C}$, the tropical Wick vector p is not in the tropical pure spinor space, by Lemma 3.5.6 below. □

Lemma 3.5.6. *If $p \in \mathbb{T}^{2^{[n]}}$ is a representable tropical Wick vector then all the faces in the regular subdivision $\mathcal{D}_p$ induced by p are polytopes associated to even Δ-matroids which are representable over $\mathbb{C}$.*

Proof. Suppose p is a representable tropical Wick vector. Without loss of generality, we can assume that all the entries of p are in $\mathbb{Q} \cup \infty$, so by the Fundamental Theorem of Tropical Geometry, p can be obtained as the valuation of the vector of Wick coordinates corresponding to some n-dimensional isotropic subspace $U \subseteq \mathbb{C}\{\{t\}\}^{\mathbf{2n}}$. Applying a suitable change of coordinates, we might assume as well that U is the rowspace of some $n \times 2n$ matrix of the form $[I|A]$, where I is the identity matrix of size n and A is an $n \times n$ skew-symmetric matrix. Let $v \in \mathbb{R}^n$, and suppose the face of the lifted polytope Γ'_p minimizing the linear form $(v, 1) \in \mathbb{R}^{n+1}$ projects back to $\mathbb{R}^n$ to the polytope of an even Δ-matroid M. The bases of M are then the subsets $S \in 2^{[n]}$ at which $p'_S := p_S + \sum_{i \in S} v_i$ is minimal. Multiplying the rows and columns of the matrix A by appropriate powers of t (namely, multiplying row i and column i by t^{-v_i}), we see that the vector $(p'_S) \in \mathbb{T}^{2^{[n]}}$ is also a representable tropical Wick vector, so we might assume that $v = \overrightarrow{0} \in \mathbb{R}^n$. We can also add a scalar to all entries of p and assume that $\min_{S \in 2^{[n]}} p_S = 0$. Now, if $w = w(t)$ is a Wick vector in $\mathbb{C}\{\{t\}\}^{2^{[n]}}$ whose valuation is p then the vector $w(0)$ obtained by substituting in w the variable t by 0 is a Wick vector with entries in $\mathbb{C}$ whose support is precisely the collection of bases of M, thus M is representable over $\mathbb{C}$. ☐

Proof of Corollary 3.4.6. The proof of Theorem 3.4.5 shows that the existence of an even Δ-matroid over the ground set $[n]$ which is not representable over $\mathbb{C}$ implies that the tropical pure spinor $\mathrm{TSpin}^{\pm}(n)$ space is strictly smaller than the Δ-Dressian $\Delta\mathrm{Dr}(n)$, so all even Δ-matroids on a ground set of at most 5 elements are representable over $\mathbb{C}$. Moreover, since the representability of an even Δ-matroid M over a field K is a first order property of the field K, any even Δ-matroid which is representable over $\mathbb{C}$ is also representable over any algebraically closed field of characteristic 0. ☐

3.6 The Cocycle Space

In this section we define the notion of circuits, cocircuits and duality for tropical Wick vectors (i.e. valuated Δ-matroids), and study the space of vectors which are "tropically orthogonal" to all circuits. The admissible part of this space will be called the cocycle space, for which we give a parametric representation.

Most of our results can be seen as a generalization of results for matroids and even Δ-matroids to the "valuated" setup. For this purpose it is useful to keep in mind that for any even Δ-matroid $M = ([n], \mathcal{B})$, by Theorem 3.5.4 there is a natural tropical Wick vector associated to it, namely, the vector $p_M \in \mathbb{T}^{2^{[n]}}$ defined as

$$(p_M)_I := \begin{cases} 0 & \text{if } I \in \mathcal{B}, \\ \infty & \text{otherwise.} \end{cases}$$

In fact, as we will see below, this perspective on even Δ-matroids makes tropical geometry an excellent language for working with them.

Definition 3.6.1. Suppose $p = (p_S) \in \mathbb{T}^{2^{[n]}}$ is a tropical Wick vector. It follows easily from the definition that the vector $p^* = (p^*_S) \in \mathbb{T}^{2^{[n]}}$ defined as $p^*_S := p_{[n]\setminus S}$ is also a tropical Wick vector, called the **dual tropical Wick vector** to p. Note that the even Δ-matroid associated to p^* is the dual even Δ-matroid to the one associated to p.

Definition 3.6.2. Recall that a subset $J \subseteq \mathbf{2n}$ is said to be admissible if $J \cap J^* = \emptyset$. An admissible subset of $\mathbf{2n}$ of size n is called a transversal; the set of all transversals of $\mathbf{2n}$ is denoted by $\mathcal{V}(n)$. For any subset $S \in 2^{[n]}$ we defined its extension to be the transversal $\bar{S} := S \cup ([n] \setminus S)^* \subseteq \mathbf{2n}$. There is of course a bijection $S \mapsto \bar{S}$ between $2^{[n]}$ and $\mathcal{V}(n)$.

Now, let $p = (p_S) \in \mathbb{T}^{2^{[n]}}$ be a tropical Wick vector. It will be convenient for us to work with the natural **extension** $\bar{p} \in \mathbb{T}^{\mathcal{V}(n)}$ of p defined as $\bar{p}_{\bar{S}} := p_S$. For any $T \in 2^{[n]}$ we define the vector $c_T \in \mathbb{T}^{\mathbf{2n}}$ (also denoted $c_{\bar{T}}$) as

$$(c_T)_i = (c_{\bar{T}})_i := \begin{cases} \bar{p}_{\bar{T}\Delta\{i,i^*\}} & \text{if } i \in \bar{T}, \\ \infty & \text{otherwise.} \end{cases}$$

It can be easily checked that if $\operatorname{supp}(c_T) \neq \emptyset$ then $\operatorname{supp}(c_T)$ is one of the fundamental circuits of the even Δ-matroid M_p whose collection of bases is $\operatorname{supp}(p)$ (see Proposition 3.3.12). We will say that the vector $c \in \mathbb{T}^{\mathbf{2n}}$ is a **circuit** of the tropical Wick vector p if $\operatorname{supp}(c) \neq \emptyset$ and there is some $T \in 2^{[n]}$ and some $\lambda \in \mathbb{R}$ such that $c = \lambda \odot c_T$ (or in classical notation, $c = c_T + \lambda \cdot \mathbf{1}$, where $\mathbf{1}$ denotes the vector in $\mathbb{T}^{\mathbf{2n}}$ whose coordinates are all equal to 1). Since every circuit of M_p is a fundamental circuit, we have

$$\mathcal{C}(M_p) = \{\operatorname{supp}(c) : c \text{ is a circuit of } p\},$$

so we see that this notion of circuits indeed generalizes the notion of circuits for even Δ-matroids to the "valuated" setup. The collection of circuits of p will be denoted by $\mathcal{C}(p) \subseteq \mathbb{T}^{\mathbf{2n}}$. A **cocircuit** of the tropical Wick vector p is just a circuit of the dual vector p^*, i.e., a vector of the form $\lambda \odot c^*_T$, where $c^*_T \in \mathbb{T}^{\mathbf{2n}}$ (also denoted $c^*_{\bar{T}}$) is the vector

$$(c^*_T)_i = (c^*_{\bar{T}})_i := \begin{cases} \bar{p}_{\bar{T}\Delta\{i,i^*\}} & \text{if } i \notin \bar{T}, \\ \infty & \text{otherwise.} \end{cases}$$

The collection of cocircuits of p will be denoted by $\mathcal{C}^*(p) \subseteq \mathbb{T}^{\mathbf{2n}}$.

We now define the concept of "tropical orthogonality", which is just the tropicalization of the usual notion of orthogonality in terms of the dot product.

Definition 3.6.3. Two vectors $x, y \in \mathbb{T}^N$ are said to be **tropically orthogonal**, denoted by $x \top y$, if the minimum

$$\min(x_1 + y_1, x_2 + y_2, \ldots, x_N + y_N)$$

is achieved at least twice (or it is equal to ∞). If $X \subseteq \mathbb{T}^N$ then its **tropically orthogonal set** is

$$X^\top := \{y \in \mathbb{T}^N : y \top x \text{ for all } x \in X\}.$$

Under these definitions, tropical Wick relations can be stated in a very simple form.

Proposition 3.6.4. *Let $p \in \mathbb{T}^{2^{[n]}}$ be a tropical Wick vector. Then any circuit of p is tropically orthogonal to any cocircuit of p.*

Proof. It suffices to prove that for all $S, T \in 2^{[n]}$, the vectors c_S and c^*_T are tropically orthogonal, which is exactly the content of the tropical Wick relations. □

Proposition 3.6.4 can be seen as a generalization of the fact that a circuit and a cocircuit of an even Δ-matroid cannot intersect in exactly one element.

We now turn to the study of the space of vectors which are tropically orthogonal to all circuits. Our motivation for this will be clear later, when we deal with tropical linear spaces.

Definition 3.6.5. A vector $x \in \mathbb{T}^{\mathbf{2n}}$ is said to be **admissible** if $\text{supp}(x)$ is an admissible subset of $\mathbf{2n}$. Let $p \in \mathbb{T}^{2^{[n]}}$ be a tropical Wick vector. If $x \in \mathcal{C}(p)^\top$ is admissible then x will be called a **cocycle** of p. The set of all cocycles of p will be called the **cocycle space** of p, and will be denoted by $\mathcal{Q}(p) \subseteq \mathbb{T}^{\mathbf{2n}}$.

Proposition 3.6.6. *Suppose $p \in \mathbb{T}^{2^{[n]}}$ is a tropical Wick vector, and let M be the even Δ-matroid whose collection of bases is* $\text{supp}(p)$*. Then*

- *If $x \in \mathcal{C}(p)^\top$ has nonempty support then* $\text{supp}(x)$ *is a dependent subset in M^*.*
- *The cocycles of p having minimal nonempty support (with respect to inclusion) are precisely the cocircuits of p.*
- *For any two cocircuits c^*_1 and c^*_2 of p with the same support there is a $\lambda \in \mathbb{R}$ such that $c^*_1 = \lambda \odot c^*_2$.*

Proof. Assume that $x \in \mathcal{C}(p)^\top$ has nonempty independent support in M^*, so there exists a basis $B \in \mathcal{B}(M)$ such that $\text{supp}(x) \cap \bar{B} = \emptyset$. Take $j \in \text{supp}(x)$, and consider the admissible subset $J := \bar{B}\Delta\{j, j^*\}$. The circuit c_J of p satisfies $j \in \text{supp}(c_J) \subseteq \bar{B} \cup j$, so $\text{supp}(x) \cap \text{supp}(c_J) = \{j\}$ and thus x cannot be tropically orthogonal to c_J.

Now, Proposition 3.6.4 tells us that all cocircuits of p are cocycles of p. Suppose x is a cocycle with minimal nonempty support, and fix $j \in \text{supp}(x)$. Since $\text{supp}(x)$ is an admissible dependent subset in M^* and $\mathcal{Q}(p)$ contains all cocircuits of p, we have that $\text{supp}(x)$ must be a cocircuit of M. Therefore, there is a basis $B \in \mathcal{B}(M)$ such that $(\text{supp}(x) - j) \cap \bar{B} = \emptyset$. For any $k \in \text{supp}(x) - j$, consider the admissible subset $J_k := \bar{B}\Delta\{k, k^*\}$. We have that $k \in \text{supp}(x) \cap \text{supp}(c_{J_k})$, $\text{supp}(x) \subseteq (\mathbf{2n} \setminus \bar{B}) \cup j$ and $\text{supp}(c_{J_k}) \subseteq \bar{B} \cup k$, so we must have $\text{supp}(x) \cap \text{supp}(c_{J_k}) = \{j, k\}$, since $x \top c_{J_k}$. We thus have

$$x_j + (c_{J_k})_j = x_k + (c_{J_k})_k,$$

so

$$x_k - x_j = (c_{J_k})_j - (c_{J_k})_k = p_{\bar{B}\Delta\{k,k^*\}\Delta\{j,j^*\}} - p_{\bar{B}}. \tag{3.6.1}$$

Since equation (3.6.1) is true for any $k \in \operatorname{supp}(x) - j$ (and also for $k = j$), it follows that

$$x = c^*_{\bar{B}\Delta\{j,j^*\}} + (x_j - p_{\bar{B}}) \cdot \mathbf{1}, \tag{3.6.2}$$

so x is a cocircuit of p as required. Finally, the above discussion shows that if c_1^* and c_2^* are cocircuits of p with the same support then both of them can be written in the form given in equation (3.6.2) (using the same B and j), so there is a $\lambda \in \mathbb{R}$ such that $c_1^* = c_2^* + \lambda \cdot \mathbf{1}$. ☐

We will now give a parametric description for the cocycle space $\mathcal{Q}(p) \subseteq \mathbb{T}^{\mathbf{2n}}$ of a tropical Wick vector $p \in \mathbb{T}^{2^{[n]}}$. For this purpose we first introduce the concept of tropical convexity. More information about this topic can be found in [DS04].

Definition 3.6.7. A set $X \subseteq \mathbb{T}^N$ is called **tropically convex** if it is closed under tropical linear combinations, i.e., for any $x_1, x_2, \ldots, x_r \in X$ and any $\lambda_1, \lambda_2, \ldots, \lambda_r \in \mathbb{T}$ we have that $\lambda_1 \odot x_1 \oplus \lambda_2 \odot x_2 \oplus \cdots \oplus \lambda_r \odot x_r \in X$. For any $a_1, a_2, \ldots, a_r \in \mathbb{T}^N$, their **tropical convex hull** is defined to be

$$\operatorname{tconvex}(a_1, a_2, \ldots, a_r) := \{\lambda_1 \odot a_1 \oplus \lambda_2 \odot a_2 \oplus \cdots \oplus \lambda_r \odot a_r : \lambda_1, \lambda_2, \ldots, \lambda_r \in \mathbb{T}\};$$

it is the smallest tropically convex set containing the vectors $a_1, a_2, \ldots, a_r$. A set of the form $\operatorname{tconvex}(a_1, a_2, \ldots, a_r)$ is usually called a **tropical polytope**.

Lemma 3.6.8. *Let $p = (p_S) \in \mathbb{T}^{2^{[n]}}$ be a tropical Wick vector. If $x \in \mathbb{T}^{\mathbf{2n}}$ is in the cocycle space $\mathcal{Q}(p)$ of p then x is in the tropical convex hull of the cocircuits of p.*

Proof. Let M denote the even Δ-matroid whose collection of bases is $\operatorname{supp}(p)$. Let $x \in \mathcal{Q}(p)$, and suppose $j \in \operatorname{supp}(x)$.

Assume first that $\{j\}$ is an independent set in M^*, and take a basis $B \in \mathcal{B}(M^*)$ such that $j \in \bar{B}$, the number of elements in $\bar{B} \cap \operatorname{supp}(x)$ is as large as possible, and

$$p'_{\bar{B}} := p^*_{\bar{B}} + \sum_{l \in \operatorname{supp}(x) \cap \bar{B}} x_l \tag{3.6.3}$$

is as small as possible (using that order of precedence). Now, consider the admissible subset $J := (\mathbf{2n} \setminus \bar{B})\Delta\{j, j^*\}$, and denote $c_j := c_J$. Since x is a cocycle of p, we have that $x \top c_j$, so there is a $k \in \mathbf{2n} - j$ such that the minimum

$$\min_{l \in \mathbf{2n}} (x_l + (c_j)_l)$$

is attained when $l = k$. It follows that

$$x_k + (c_j)_k \leq x_j + (c_j)_j < \infty, \tag{3.6.4}$$

so in particular $k \in \operatorname{supp}(x) \cap \operatorname{supp}(c_j)$. Note that, since $\operatorname{supp}(c_j) \subseteq J$, we have that $k \in \mathbf{2n} \setminus \bar{B}$. Let $J' := (\mathbf{2n} \setminus \bar{B})\Delta\{k, k^*\}$, and consider the cocircuit $c_j^* := c^*_{J'}$ of p. The support of c_j^* is the fundamental circuit in M^* of k over $\bar{B}$, so our choice of B and the fact that x is admissible imply that $\operatorname{supp}(c_j^*) \subseteq \operatorname{supp}(x)$ (see Proposition 3.3.12). Moreover, since $k \in \operatorname{supp}(c_j) \cap \operatorname{supp}(c_j^*)$, $\operatorname{supp}(c_j^*) \subseteq \bar{B} \cup k$, and $\operatorname{supp}(c_j) \subseteq (\mathbf{2n} \setminus \bar{B}) \cup j$, by Proposition 3.6.4 we must have $\operatorname{supp}(c_j) \cap \operatorname{supp}(c_j^*) = \{j, k\}$ and

$$(c_j)_j + (c_j^*)_j = (c_j)_k + (c_j^*)_k. \tag{3.6.5}$$

Now, note that for any $l \in \operatorname{supp} c_j^* - j$, our choice of B minimizing (3.6.3) implies that $p'_{\bar{B}} \leq p'_{\bar{B}\Delta\{k,k^*\}\Delta\{l,l^*\}}$. Since x is admissible, this means that

$$(c_j^*)_k - (c_j^*)_l = p^*_{\bar{B}} - p^*_{\bar{B}\Delta\{k,k^*\}\Delta\{l,l^*\}} \leq x_k - x_l. \tag{3.6.6}$$

Moreover, (3.6.4) and (3.6.5) tell us that

$$(c_j^*)_j - (c_j^*)_k = (c_j)_k - (c_j)_j \leq x_j - x_k, \tag{3.6.7}$$

and adding (3.6.6) and (3.6.7) we get

$$(c_j^*)_j - (c_j^*)_l \leq x_j - x_l. \tag{3.6.8}$$

Now, consider the cocircuit $d_j^* := c_j^* - ((c_j^*)_j - x_j) \cdot \mathbf{1}$ of p. We have $(d_j^*)_j = x_j$, and if $l \in \operatorname{supp} d_j^* - j = \operatorname{supp} c_j^* - j$ then (3.6.8) implies that $(d_j^*)_l \geq x_l$.

In the case $\{j\}$ is a cocircuit of M, take d_j^* to be the cocircuit of p given by

$$(d_j^*)_l := \begin{cases} x_j & \text{if } l = j, \\ \infty & \text{otherwise.} \end{cases}$$

By the above discussion, we have that $x = \min_{j\in\operatorname{supp}(x)} d_j^*$, so x is in the tropical convex hull of the cocircuits of p as desired. □

We will now state the main theorem of this section.

Theorem 3.6.9. *Let $p \in \mathbb{T}^{2^{[n]}}$ be a tropical Wick vector. Then the cocycle space $\mathcal{Q}(p) \subseteq \mathbb{T}^{\mathbf{2n}}$ of p is the set of admissible vectors in the tropical convex hull of the cocircuits of p.*

Proof. One implication is given by Lemma 3.6.8. For the reverse implication, it is not hard to see that if $y \in \mathbb{T}^{\mathbf{2n}}$ then the set $\{y\}^\top$ is tropically convex, and since any intersection of tropically convex sets is tropically convex, any set of the form $Y^\top$ with $Y \subseteq \mathbb{T}^{\mathbf{2n}}$ is tropically convex. Therefore, since the space $\mathcal{C}(p)^\top$ contains all the cocircuits of p, it contains their tropical convex hull, so the result follows. □

Theorem 3.6.9 implies that if p is a tropical Wick vector and M is its associated even Δ-matroid then the set of supports of all cocycles of p is precisely the set of cocycles of M (see Definition 3.3.10). This shows that our definition of cocycles for tropical Wick vectors extends the usual definition of cocycles for even Δ-matroids to the valuated setup.

Corollary 3.6.10. *Let $p \in \mathbb{T}^{2^{[n]}}$ be a tropical Wick vector. Then $\mathcal{Q}(p^*) \subseteq \mathbb{T}^{2n}$ is the set of admissible vectors in $\mathcal{Q}(p)^\top$.*

Proof. Since $\mathcal{Q}(p)$ contains all cocircuits of p, taking orthogonal sets we get that all admissible vectors in $\mathcal{Q}(p)^\top$ are also in $\mathcal{Q}(p^*)$. On the other hand, by definition, we have that $\mathcal{Q}(p)^\top$ contains all the circuits of p, and since $\mathcal{Q}(p)^\top$ is tropically convex, $\mathcal{Q}(p)^\top$ contains their tropical convex hull. Applying Theorem 3.6.9 to p^* we get that $\mathcal{Q}(p^*)$ is contained in the set of admissible vectors of $\mathcal{Q}(p)^\top$. □

3.6.1 Tropical Linear Spaces

We will now specialize some of the results presented above to tropical Plücker vectors (i.e. valuated matroids). In this way we will unify several results for tropical linear spaces given by Murota and Tamura in [MT01], Speyer in [Spe08], and Ardila and Klivans in [AK06]. Unless otherwise stated, all matroidal terminology in this section will refer to the classical matroidal notions and not to the Δ-matroidal notions discussed above.

Definition 3.6.11. Let $p = (p_S) \in \mathbb{T}^{2^{[n]}}$ be a tropical Plücker vector of rank r_p. For $T \in 2^{[n]}$ of size $r_p + 1$, we define the vector $d_T \in \mathbb{T}^n$ as

$$(d_T)_i := \begin{cases} p_{T-i} & \text{if } i \in T, \\ \infty & \text{otherwise.} \end{cases}$$

If $\text{supp}(d_T) \neq \emptyset$ then $\text{supp}(d_T)$ is one of the fundamental circuits of the matroid M_p whose collection of bases is $\text{supp}(p)$. We will say that the vector $d \in \mathbb{T}^n$ is a **Plücker circuit** of p if $\text{supp}(d) \neq \emptyset$ and there is some $T \in 2^{[n]}$ of size $r_p + 1$ and some $\lambda \in \mathbb{R}$ such that $d = \lambda \odot d_T$ (or in classical notation, $d = d_T + \lambda \cdot \mathbf{1}$, where $\mathbf{1}$ denotes the vector in $\mathbb{T}^n$ whose coordinates are all equal to 1). Since every circuit of M_p is a fundamental circuit, we have

$$\mathcal{C}(M_p) = \{\text{supp}(d) : d \text{ is a Plücker circuit of } p\},$$

so this notion of Plücker circuits generalizes the notion of circuits for matroids to the "valuated" setup. The collection of Plücker circuits of p will be denoted by $\mathcal{PC}(p)$. A **Plücker cocircuit** of p is just a Plücker circuit of the dual vector p^*, i.e., a vector of the form $\lambda \odot d_T^*$ where $T \in 2^{[n]}$ has size $n - r_p - 1$ and $d_T^* \in \mathbb{T}^n$ denotes the vector

$$(d_T^*)_i := \begin{cases} p_{T \cup i} & \text{if } i \notin T, \\ \infty & \text{otherwise.} \end{cases}$$

The collection of Plücker cocircuits of p will be denoted by $\mathcal{PC}^*(p)$.

The reason we are using the name "Plücker circuits" is just so that they are not confused with the circuits of p in the Δ-matroidal sense; a more appropriate name (but not very practical for our purposes) would be "circuits in type A" (while the Δ-matroidal circuits are "circuits in type D").

The following definition was introduced by Speyer in [Spe08].

Definition 3.6.12. Let $p \in \mathbb{T}^{2^{[n]}}$ be a tropical Plücker vector. The space $L_p := \mathcal{PC}(p)^\top \subseteq \mathbb{T}^n$ is called the **tropical linear space** associated to p.

The tropical linear space L_p should be thought of as the space of cocyles of p "in type A" (while $\mathcal{Q}(p)$ is the space of cocycles of p "in type D").

Tropical linear spaces have a very special geometric importance that we now describe. We will only mention some of the basic facts, the reader can consult [Spe08] for much more information and proofs. Consider the n-dimensional vector space $V := \mathbb{C}\{\{t\}\}^n$ over the field $K := \mathbb{C}\{\{t\}\}$, and suppose W is a k-dimensional linear subspace of V with Plücker coordinates $P \in K^{\binom{n}{k}}$. Let $p \in \mathbb{T}^{\binom{n}{k}} \subseteq \mathbb{T}^{2^{[n]}}$ be the valuation of the vector P. Since P satisfies the Plücker relations, the vector p is a tropical Plücker vector. Under this setup, Speyer proved that the tropicalization of the linear space W (i.e. the tropical variety associated to its defining ideal) is precisely the tropical linear space L_p. Also, if $W^\perp$ is the corresponding orthogonal linear subspace then the tropicalization of $W^\perp$ is the tropical linear space L_{p^*}. It is also shown in [Spe08] that if p is any tropical Plücker vector (not necessarily realizable by a subspace W of V) of rank r_p then the polyhedral complex $L_p \cap \mathbb{R}^n$ is a pure polyhedral complex of dimension r_p.

The following proposition will allow us to apply the "type D" results that we got in previous sections to the study of tropical linear spaces.

Proposition 3.6.13. *Let $p \in \mathbb{T}^{2^{[n]}}$ be a tropical Plücker vector, and let $L_p \subseteq \mathbb{T}^n$ be its associated tropical linear space. Then, under the natural identification $\mathbb{T}^{\mathbf{2n}} \cong \mathbb{T}^n \times \mathbb{T}^n$, we have $\mathcal{C}(p)^\top = L_p \times L_{p^*}$.*

Proof. It is not hard to check that the circuits of p are precisely the vectors of the form $(d, \overrightarrow{\infty}) \in \mathbb{T}^{\mathbf{2n}}$ with $d \in \mathbb{T}^n$ a Plücker circuit of p (where $\overrightarrow{\infty}$ denotes the vector in $\mathbb{T}^n$ with all coordinates equal to ∞), and of the form $(\overrightarrow{\infty}, d^*) \in \mathbb{T}^{\mathbf{2n}}$ with $d^* \in \mathbb{T}^n$ a Plücker cocircuit of p; so the result follows directly from the definitions. □

The following theorem provides a parametric description of any tropical linear space. It was first proved by Murota and Tamura in [MT01]. In the case of realizable tropical linear spaces it also appears in work of Yu and Yuster [YY07].

Theorem 3.6.14. *Suppose $p \in \mathbb{T}^{2^{[n]}}$ is a tropical Plücker vector. Then the tropical linear space $L_p \subseteq \mathbb{T}^n$ is the tropical convex hull of the Plücker cocircuits of p.*

Proof. The cocircuits of p are the vectors of the form $(d^*, \vec{\infty}) \in \mathbb{T}^{\mathbf{2n}}$ with $d^* \in \mathbb{T}^n$ a Plücker cocircuit of p, and of the form $(\vec{\infty}, d) \in \mathbb{T}^{\mathbf{2n}}$ with $d \in \mathbb{T}^n$ a Plücker circuit of p; so the result follows from Proposition 3.6.13 and Theorem 3.6.9. □

It is instructive to see what Theorem 3.6.14 is saying when applied to tropical Plücker vectors with only zero and infinity entries (what is sometimes called the "constant coefficient case" in tropical geometry). In this case, since the complements of unions of cocircuits of the associated matroid M are exactly the flats of M, we get precisely the description of the tropical linear space in terms of the flats of M that was given by Ardila and Klivans in [AK06].

Another useful application to the study of tropical linear spaces is the following. It was also proved by Murota and Tamura in [MT01].

Theorem 3.6.15. *If $p \in \mathbb{T}^{2^{[n]}}$ is a tropical Plücker vector then $L_{p^*} = L_p^\top$. In particular, for any tropical linear space L, we have $(L^\top)^\top = L$.*

Proof. By Proposition 3.6.13 we have that $L_{p^*} = \mathcal{C}(p^*)^\top \cap (\mathbb{T}^n \times \{\vec{\infty}\}) = \mathcal{Q}(p^*) \cap (\mathbb{T}^n \times \{\vec{\infty}\})$, so the result follows from Corollary 3.6.10. □

One can also apply these ideas to prove the following result of Speyer in [Spe08].

Proposition 3.6.16. *There is a bijective correspondence between tropical linear spaces and tropical Plücker vectors (up to tropical scalar multiplication).*

Proof. Propositions 3.6.13 and 3.6.15 show that one can recover $\mathcal{C}(p)^\top$ from the tropical linear space L_p. Proposition 3.6.6 shows that one can recover the cocircuits of p (and thus p, up to a scalar multiple of $\mathbf{1}$) from $\mathcal{C}(p)^\top$. □

3.7 Isotropical Linear Spaces

Definition 3.7.1. Let $L \subseteq \mathbb{T}^{\mathbf{2n}}$ be an n-dimensional tropical linear space. We say that L is (totally) **isotropic** if for any two $x, y \in L$ we have that the minimum

$$\min(\, x_1 + y_{1^*} \,,\, \ldots \,,\, x_n + y_{n^*} \,,\, x_{1^*} + y_1 \,,\, \ldots \,,\, x_{n^*} + y_n \,)$$

is achieved at least twice (or it is equal to ∞). In this case, we also say that L is an **isotropical linear space**. Note that if $K = \mathbb{C}\{\{t\}\}$ and $V = K^{\mathbf{2n}}$, the tropicalization of any n-dimensional isotropic subspace U of V (see Section 3.2) is an isotropical linear space $L \subseteq \mathbb{T}^{\mathbf{2n}}$. In this case we say that L is **isotropically realizable** by U.

Not all isotropical linear spaces that are realizable are isotropically realizable. As an example of this, take $n = 2$ and let $L \subseteq \mathbb{T}^{\mathbf{2n}}$ be the tropicalization of the rowspace of the matrix

$$\begin{array}{c} \begin{matrix} \mathbf{1} & \mathbf{2} & \mathbf{1}^* & \mathbf{2}^* \end{matrix} \\ \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \end{pmatrix}. \end{array}$$

The tropical linear space L is a realizable isotropical linear space (as can be seen from Theorem 3.7.3), but it is easy to check that it cannot be isotropically realizable.

We mentioned in Section 3.2 that if U is an isotropic linear subspace then its vector of Wick coordinates w carries all the information of U. One might expect something similar to hold tropically, that is, that the valuation of the Wick vector w still carries all the information of the tropicalization of U. This is not true, as the next example shows.

Example 3.7.2. We present two n-dimensional isotropic linear subspaces of $\mathbb{C}\{\{t\}\}^{\mathbf{2n}}$ whose corresponding tropicalizations are distinct tropical linear spaces, but whose Wick coordinates have the same valuation. Take $n = 4$. Let U_1 be the 4-dimensional isotropic linear subspace of $\mathbb{C}\{\{t\}\}^{\mathbf{8}}$ defined as the rowspace of the matrix

$$M_1 = \begin{array}{c} \begin{matrix} \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} & \mathbf{1}^* & \mathbf{2}^* & \mathbf{3}^* & \mathbf{4}^* \end{matrix} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 2 & 2 \\ 0 & 1 & 0 & 0 & -1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & -2 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -2 & -2 & -1 & 0 \end{pmatrix}, \end{array}$$

and U_2 be the 4-dimensional isotropic linear subspace of $\mathbb{C}\{\{t\}\}^{\mathbf{8}}$ defined as the rowspace of

$$M_2 = \begin{array}{c} \begin{matrix} \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} & \mathbf{1}^* & \mathbf{2}^* & \mathbf{3}^* & \mathbf{4}^* \end{matrix} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 2 & 4 \\ 0 & 1 & 0 & 0 & -1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & -2 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -4 & -2 & -1 & 0 \end{pmatrix}. \end{array}$$

Their corresponding tropical linear spaces L_1 and L_2 are distinct since, for example, the Plücker coordinate indexed by the subset 343^*4^* is nonzero for U_1 but zero for U_2. However, the Wick coordinates of U_1 and U_2 are all nonzero scalars (the ones indexed by even subsets), and thus their valuations give rise to the same tropical Wick vector.

It is important to have an effective way for deciding if a tropical linear space is isotropical or not. For this purpose, if $v \in \mathbb{T}^{\mathbf{2n}}$, we call its **reflection** to be the vector $v^r \in \mathbb{T}^{\mathbf{2n}}$ defined as $v_i^r := v_{i^*}$. If $X \subseteq \mathbb{T}^{\mathbf{2n}}$ then its reflection is the set $X^r := \{x^r : x \in X\}$. The following theorem gives us a simple criterion for identifying isotropical linear spaces.

Theorem 3.7.3. *Let $L \subseteq \mathbb{T}^{\boldsymbol{2n}}$ be a tropical linear space with associated tropical Plücker vector p (whose coordinates are indexed by subsets of $\boldsymbol{2n}$). Then the following are equivalent:*

1. *L is an n-dimensional isotropical linear space.*
2. *$L^{\top} = L^{r}$.*
3. *$p_{\boldsymbol{2n}\setminus T} = p_{T^*}$ for all $T \subseteq \boldsymbol{2n}$.*

Proof. By Proposition 3.6.16 we know that two tropical linear spaces are equal if and only if their corresponding tropical Plücker vectors are equal, so (2) $\leftrightarrow$ (3) follows from Theorem 3.6.15. To see that (1) $\leftrightarrow$ (2), note that L is an isotropical linear space if and only if L is tropically orthogonal to the reflected tropical linear space L^r, that is, if and only if $L^r \subseteq L^{\top}$. Since $\dim(L^{\top}) = 2n - \dim(L) = 2n - \dim(L^r)$, the result follows from Lemma 3.7.4 below. □

Lemma 3.7.4. *If $L_1 \subseteq L_2$ are two tropical linear spaces of the same dimension then $L_1 = L_2$.*

Proof. Let p_1 and p_2 be the corresponding tropical Plücker vectors, and let M_1 and M_2 be their associated matroids. By Theorem 3.6.14 we have that every cocircuit of p_1 is in the tropical convex hull of the cocircuits of p_2, so in particular, any cocircuit of M_1 is a union of cocircuits of M_2. This is saying that M_2^* is a quotient of M_1^* (see [Oxl92], Proposition 7.3.6), and since M_1^* and M_2^* have the same rank, we have $M_1^* = M_2^*$ ([Oxl92], Corollary 7.3.4). But then, in view of Proposition 3.6.6 and Proposition 3.6.13, the cocircuits of p_1 and p_2 are the same, so in fact $L_1 = L_2$. □

Note that Theorem 3.7.3 describes the set of isotropical linear spaces (or more precisely, their associated tropical Plücker vectors) as an intersection of the Dressian $\mathrm{Dr}(n, 2n)$ with a linear subspace.

If L is an isotropical linear space which is isotropically realizable by U then we have seen that the valuation p of the Wick vector w associated to U does not determine L. Nonetheless, the following theorem shows that p does determine the admissible part of L.

Theorem 3.7.5. *Let $L \subseteq \mathbb{T}^{\boldsymbol{2n}}$ be an n-dimensional isotropical linear space which is isotropically realizable by the subspace $U \subseteq \mathbb{C}\{\{t\}\}^{\boldsymbol{2n}}$. Let $p \in \mathbb{T}^{2^{[n]}}$ be the tropical Wick vector obtained as the valuation of the Wick vector w associated to U. Then the set of admissible vectors in L is the cocycle space $\mathcal{Q}(p) \subseteq \mathbb{T}^{\boldsymbol{2n}}$.*

Proof. Equation (3.2.1) in Section 3.2 implies that the circuits of p are tropically orthogonal to all the elements of L, so $L \subseteq \mathcal{C}(p)^{\top}$ and thus the admissible vectors of L are in $\mathcal{Q}(p)$. On the other hand, it can be easily checked that the valuation of the Wick vector associated to the isotropic subspace $U^{\perp}$ is precisely the dual tropical Wick vector p^*, so repeating the same argument we have that $L^{\top} \subseteq \mathcal{C}(p^*)^{\top}$. Taking orthogonal sets we get that $L \supseteq (\mathcal{C}(p^*)^{\top})^{\top} \supseteq \mathcal{C}(p^*)$, and since L is tropically convex, Theorem 3.6.9 implies that the set of admissible vectors in L contains $\mathcal{Q}(p)$. □

Chapter 4

Local Tropical Linear Spaces

The material presented in this chapter is work in progress. It will be expanded and published in the future.

4.1 Introduction

Tropical linear spaces are one of the most basic objects in tropical geometry. They are obtained as tropicalizations of classical linear subspaces, and they play a prominent role in several contexts like the study of tropicalizations of varieties obtained as the image of a linear subspace under a monomial map [DFS07], or the study of realizability questions and intersection theory in tropical geometry (see [FR10], [KP09], [Sha10]).

In [Spe08] Speyer studied the combinatorial structure of tropical linear spaces, and in particular, he showed that tropical linear spaces can be described as polyhedral complexes dual to subdivisions of matroid polytopes. He also formulated a conjecture on the upper bound for the f-vector of a tropical linear space:

Conjecture 4.1.1 (The f-vector conjecture [Spe08])**.** *If L is an m-dimensional tropical linear space in $\mathbb{R}^n$ then L has at most $\binom{n-i-1}{i-1}\binom{n-2i}{m-i}$ faces of dimension i that become bounded after modding out by the lineality space generated by the vector $(1, 1, \ldots, 1) \in \mathbb{R}^n$.*

This conjecture implies that the total number of i-dimensional faces of an m-dimensional tropical linear space in $\mathbb{R}^n$ is at most $\binom{n-i-1}{m-i}\binom{2n-m-1}{i-1}$.

In [Spe08], Speyer proved the f-vector conjecture in a few special cases like $i = 1$ and $m = n/2$. Later in [Spe09], he proved it for tropical linear spaces which are realizable over a field of characteristic zero. The conjecture is still open in the general case.

Tropical linear spaces have also been studied in connection to Dressians and tropical Grassmannians. In [HJJS09], several combinatorial results on tropical planes were developed to study the Dressians $\mathrm{Dr}(3, n)$ and the tropical Grassmannians $\mathrm{TGr}(3, n)$, with an emphasis in the case $n = 7$. In [HJS11] some of these results were extended and applied to study the case $n = 8$, achieving a combinatorial characterization of all rays in the Dressian $\mathrm{Dr}(3, 8)$.

In this chapter we study tropical linear spaces locally: For any basis B of the matroid underlying a tropical linear space L, we define the local tropical linear space L_B to be the subcomplex of L consisting of all vectors $v \in L$ that make B a basis of maximal v-weight. As discussed in Section 4.2, the space L_B consists then of all cells of L that can be "seen" from the vertex e_B of the underlying matroid polytope $\Gamma(M)$. The tropical linear space L is the union of all its local tropical linear spaces, which we prove are homeomorphic to Euclidean space.

We study the combinatorics of local tropical linear spaces, and we prove that they are combinatorially dual to mixed subdivisions of a Minkowski sum of simplices. We use this duality to produce tight upper bounds on their f-vectors. We also introduce a certain class of tropical linear spaces called conical tropical linear spaces, and we give a simple proof that they satisfy the f-vector conjecture. Along the way, we give an independent proof of a conjecture of Herrmann and Joswig posed in a first version of [HJS11].

4.2 Definition and Basic Notions

Let $m \leq n$ be nonnegative integers, and denote by $\mathbb{T} := \mathbb{R} \cup \{\infty\}$ the tropical semiring of real numbers *including infinity*. A vector $p \in \mathbb{T}^{\binom{[n]}{m}}$ is called a **tropical Plücker vector** of rank m if it satisfies the tropical Plücker relations, that is, for any $S, T \in 2^{[n]}$ satisfying $|S| = m-1$ and $|T| = m+1$, the minimum

$$\min_{i \in T \setminus S} (p_{Si} + p_{T-i}) \tag{4.2.1}$$

is achieved at least twice (i.e., for at least two different values of i) or it is equal to ∞. It follows that the **support** $\operatorname{supp}(p) := \{B \in \binom{[n]}{m} \mid p_B \neq \infty\}$ of p is the collection of bases of matroid over $[n]$, called the **underlying matroid** of p (see Corollary 3.5.2). In the literature, tropical Plücker vectors have also been studied under the name of **valuated matroids**, but using the opposite sign convention to ours [DW92]. The space of all tropical Plücker vectors is called the **Dressian**, and it is denoted as $\mathrm{Dr}_{m,n} \subseteq \mathbb{T}^{\binom{[n]}{m}}$.

Tropical Plücker vectors can be described in term of polyhedral subdivisions in the following way. To any collection $\mathcal{S}$ of subsets of $[n]$ we can associate a 0/1 polytope $\Gamma(\mathcal{S}) := \operatorname{convex}\{e_S \mid S \in \mathcal{S}\} \subseteq \mathbb{R}^n$, where $e_S := \sum_{i \in S} e_i$. Matroids can be easily characterized from this point of view (see [GGMS87]): A collection $\mathcal{S} \subseteq 2^{[n]}$ is the collection of bases of a matroid M over the ground set $[n]$ if and only if its associated polytope $\Gamma(\mathcal{S})$ has only edges of the form $e_i - e_j$ for $i, j \in [n]$ distinct. In this case, the polytope $\Gamma(M) := \Gamma(S)$ is called a **matroid polytope**.

A **subdivision** of a polytope P is a set of polytopes $S = \{P_1, \ldots, P_m\}$, whose vertices are vertices of P, such that $P_1 \cup \cdots \cup P_m = P$, and for all $1 \leq i < j \leq m$, if the intersection $P_i \cap P_j$ is nonempty then it is a proper face of both P_i and P_j. Any vector $p \in \mathbb{T}^{\binom{[n]}{m}}$ induces a polytopal subdivision of $\Gamma := \Gamma(\operatorname{supp}(p))$ as follows. The vector $p \in \mathbb{T}^{\binom{[n]}{m}}$ can be thought of as a height function on the vertices of Γ, giving rise to the "lifted polytope" $\Gamma(p) := \mathrm{convex}\{(e_S, p_S) \in \mathbb{R}^{n+1} \mid S \in \mathrm{supp}(p)\}$. Projecting the lower facets of $\Gamma(p)$ (i.e., its facets whose outward normal vector has a negative $(n+1)$st coordinate) back to $\mathbb{R}^n$, we get a polytopal subdivision $\mathcal{D}_p$ of Γ, called the **regular subdivision** induced by p.

Tropical Plücker vectors admit a beautiful characterization in this language (see Corollary 3.5.5): A vector $p \in \mathbb{T}^{\binom{[n]}{m}}$ is a tropical Plücker vector if and only if the regular subdivision $\mathcal{D}_p$ is a **matroid polytope subdivision**, i.e., it is a subdivision of a matroid polytope into matroid polytopes.

Let $p \in \mathbb{T}^{\binom{[n]}{m}}$ be a tropical Plücker vector. Suppose $S \subseteq [n]$ is such that $|S| = m+1$ and the vector $c_S \in \mathbb{T}^n$ defined by

$$(c_S)_i := \begin{cases} p_{S-i} & \text{if } i \in S, \\ \infty & \text{otherwise;} \end{cases} \tag{4.2.2}$$

is not equal to $\vec{\infty} := (\infty, \infty, \dots, \infty) \in \mathbb{T}^n$. In this case, any vector of the form $c_S + \lambda \cdot \mathbf{1}$ with $\lambda \in \mathbb{R}$ is called a (valuated) **circuit** of p, where $\mathbf{1} := (1, 1, \dots, 1) \in \mathbb{R}^n$. Note that its support $\mathrm{supp}(c_S + \lambda \cdot \mathbf{1}) = \mathrm{supp}(c_S) := \{i \in [n] \mid (c_S)_i \neq \infty\}$ is a circuit of the underlying matroid M of p. For any basis $B \subseteq [n]$ of M and any $e \in [n]$ there is a unique circuit of M contained in $B \cup e$ (containing the element e), which is called the **fundamental circuit** $C(e, B)$ of e over B. If the support of c_S is a fundamental circuit of M over some basis B then we say that c_S is a **fundamental circuit** of p over the basis B. It follows from Proposition 3.6.6 (see also [MT01]) that if two circuits of p have the same support $D \subseteq [n]$ then they differ by a scalar multiple of the vector $\mathbf{1}$, that is, the two circuits are the same in tropical projective space $\mathbb{TP}^{n-1} := (\mathbb{T}^n - \vec{\infty})/\mathbb{R} \cdot \mathbf{1}$.

Valuated circuits satisfy the following valuated elimination property, which generalizes the classical elimination axiom for circuits of a matroid.

Proposition 4.2.1 ([MT01])**.** *Let $p \in \mathbb{T}^{\binom{[n]}{m}}$ be a tropical Plücker vector. If $d, e \in \mathbb{T}^n$ are two circuits of p and $a, b \in [n]$ are such that $d_a < e_a$ and $d_b = e_b \neq \infty$, then there exists a circuit $f \in \mathbb{T}^n$ of p satisfying $f_b = \infty$, $f_a = d_a$, and $f \geq \min(d, e)$.*

Two vectors $x, y \in \mathbb{T}^n$ are said to be **tropically orthogonal**, denoted by $x \top y$, if the minimum $\min(x_1 + y_1, x_2 + y_2, \dots, x_n + y_n)$ is achieved at least twice (or it is equal to ∞). If $X \subseteq \mathbb{T}^n$ then its **tropically orthogonal set** is $X^\top := \{y \in \mathbb{T}^n \mid y \top x \text{ for all } x \in X\}$.

Let $p \in \mathbb{T}^{\binom{[n]}{m}}$ be a tropical Plücker vector, and denote by $\mathcal{C}(p) \subseteq \mathbb{T}^n$ the set of all circuits of p. The space $L(p) := \mathcal{C}(p)^\top \subseteq \mathbb{T}^n$ is called the **tropical linear space** associated to p. Tropical linear spaces were introduced and studied by Speyer in [Spe08].

As discussed in Chapter 1, tropical linear spaces play a very important role in tropical geometry: Consider the n-dimensional vector space $V := \mathbb{C}\{\{t\}\}^n$ over the field of Puiseux series $K := \mathbb{C}\{\{t\}\}$, and suppose W is an m-dimensional linear subspace of V with Plücker coordinates $P \in K^{\binom{[n]}{m}}$. Let $p \in \mathbb{T}^{\binom{[n]}{m}}$ be the valuation of the vector P. Since P satisfies the

Plücker relations, the vector p is a tropical Plücker vector. Under this setup, the tropical linear space $L(p)$ is precisely the tropicalization of the linear space W.

Definition 4.2.2. Let $p \in \mathbb{T}^{\binom{[n]}{m}}$ be a tropical Plücker vector with underlying matroid M. If B is a basis of M and $v \in \mathbb{R}^n$, we define the v**-weight** of B (with handicap p) to be $w_p(v, B) := -p_B + \sum_{i \in B} v_i$. For any $v \in \mathbb{R}^n$, the collection of bases of M with maximal v-weight is the collection of bases of a matroid M_v on the ground set $[n]$. Note that M_v is the matroid corresponding to the face of $\mathcal{D}_p$ obtained as the projection of the face of $\Gamma(p)$ that maximizes the dot product with the vector $(v, -1) \in \mathbb{R}^{n+1}$.

Now, for any basis B of M, denote by Σ_B the set of vectors $v \in \mathbb{R}^n$ such that M_v contains the basis B. The **local tropical linear space** $L(p)_B$ is defined as $L(p)_B := L(p) \cap \Sigma_B$.

Suppose $p \in \mathbb{R}^{\binom{[n]}{m}}$ is a tropical Plücker vector (with no coordinates equal to ∞). The vector p induces a regular matroid subdivision $\mathcal{D}_p$ of the hypersimplex $\Delta_{m,n}$. As we mentioned in Section 1.3, it was shown in [Spe08] that the tropical linear space $L(p) \cap \mathbb{R}^n$ consists of all vectors $v \in \mathbb{R}^n$ such that M_v is a loopless matroid. In particular, $L(p) \cap \mathbb{R}^n$ is a polyhedral complex dual to the faces of $\mathcal{D}_p$ that correspond to loopless matroids. If B is a basis of the uniform matroid $U_{m,n}$, then the local tropical linear space $L(p)_B$ consists of the cells of $L(p) \cap \mathbb{R}^n$ which are dual to faces of $\mathcal{D}_p$ that correspond to loopless matroids and *contain the vertex* e_B. All these results hold more generally for any tropical Plücker vector in $\mathbb{T}^{\binom{[n]}{m}}$, as we will show in Proposition 4.2.5.

Example 4.2.3. Let $n = 4$, $m = 2$, and consider the vector $p \in \mathbb{R}^{\binom{[n]}{m}}$ defined as

$$p_S := \begin{cases} 1 & \text{if } S = 12 \text{ or } S = 34, \\ 0 & \text{if } S = 13 \text{ or } S = 14 \text{ or } S = 23 \text{ or } S = 24. \end{cases}$$

The hypersimplex $\Delta_{2,4}$ is the convex hull of all 0/1 vectors in $\mathbb{R}^4$ having exactly two coordinates equal to 1. This polytope lives in the 3-dimensional hyperplane defined by $x_1 + x_2 + x_3 + x_4 = 2$, and is in fact a regular octahedron. The regular subdivision $\mathcal{D}_p$ induced by p is consists of two square pyramids meeting at their base, as depicted in the left of Figure 4.1. Since all faces of this subdivision are matroid polytopes then this ensures that p is a tropical Plücker vector. The tropical linear space $L(p) \cap \mathbb{R}^n$ is then dual to all the faces of this subdivision which correspond to loopless matroids, as drawn in green and red on the right side of Figure 4.1. The local tropical linear space around the basis $B = \{1, 4\}$ consists of the cells of $L(p) \cap \mathbb{R}^n$ that are dual to faces of the subdivision containing the vertex e_{14} and corresponding to loopless matroids, which are precisely the green cells in the picture.

Any tropical linear space $L(p) \cap \mathbb{R}$ is the union of all its tropical linear spaces $L(p)_B$, for B a basis of its underlying matroid M, so we can attempt to understand tropical linear spaces by studying them locally.

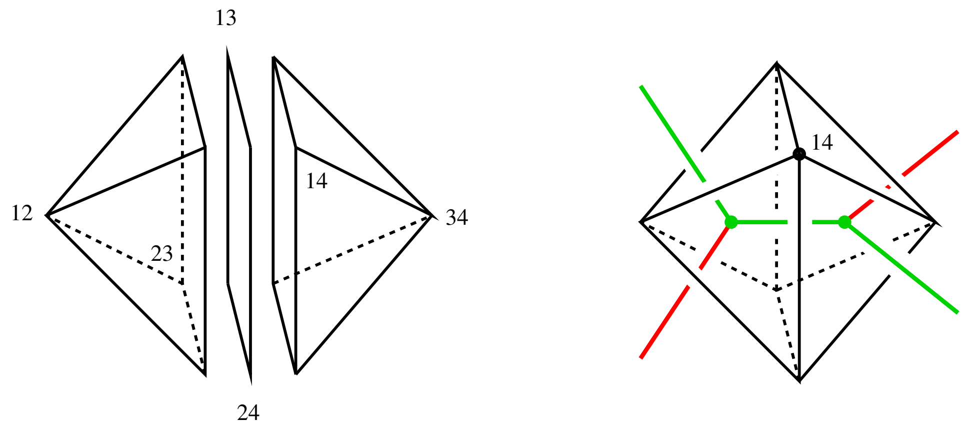

Figure 4.1: A regular subdivision induced by a tropical Plücker vector, and its associated tropical linear space.

Lemma 4.2.4. *Let $p \in \mathbb{T}^{\binom{[n]}{m}}$ be a tropical Plücker vector, and let B be a basis of its underlying matroid M. For any $v \in \Sigma_B$, v is in the local tropical linear space $L(p)_B$ if and only if v is tropically orthogonal to all* fundamental *circuits of p over the basis B.*

Proof. Assume by contradiction that $v \in \Sigma_B$ is tropically orthogonal to all fundamental circuits of p over the basis B, but v is not in L. Let d be a circuit of p which is not tropically orthogonal to v, and take d such that $D := \operatorname{supp}(d)$ contains as few elements outside of B as possible. Since the circuit D is not a fundamental circuit over B, D contains at least two elements not in B. Let $a \in D$ be the unique element such that $d_a+v_a = \min\{d_i+v_i \mid i \in [n]\}$, and let $b \in D - B$ be different from a. After adding a suitable scalar multiple of the vector $\mathbf{1}$, we can assume that $d_b = p_B$. Let $S := B \cup b$, and consider the circuit $e := c_S$ of p, as defined in Equation (4.2.2). Its support $\operatorname{supp}(e)$ is the fundamental circuit $E := C(b, B)$ of b over B. Since $v \in \Sigma_B$, it follows that

$$e_b + v_b = p_B + v_b \leq p_{S-i} + v_i = e_i + v_i \tag{4.2.3}$$

for any $i \in E$. Note that in fact this inequality holds for any $i \in [n]$. We thus have $d_a + v_a < d_b + v_b = p_B + v_b \leq e_a + v_a$, so $d_a < e_a$. Applying Proposition 4.2.1, we get that there is a circuit f of p such that $f_b = \infty$, $f_a = d_a$, and $f \geq \min(d, e)$. We have

$$f_a + v_a = d_a + v_a < d_i + v_i$$

for any $i \in [n]$ different from a, and therefore

$$f_a + v_a < d_b + v_b = e_b + v_b \leq e_i + v_i$$

for any $i \in [n]$ (the last inequality in the previous line comes from (4.2.3)). Since $f \geq \min(d, e)$, these last two inequalities imply that $f_a + v_a < f_i + v_i$ for any $i \in [n]$ different from

a, so $\min_{i\in[n]}(f_i + v_i)$ is achieved only once at $i = a$. But this means that f is a circuit of p which is not tropically orthogonal to v, and whose support $F := \text{supp}(f)$ has fewer elements outside of B than the circuit D (since $F \subseteq D \cup E - b \subseteq B \cup D - b$), which contradicts our choice of d. □

Lemma 4.2.4 can be stated in polyhedral terms in the following way. It generalizes to arbitrary tropical Plücker vectors in $\mathbb{T}^{\binom{[n]}{m}}$ the description of tropical linear spaces given in Section 1.3.

Proposition 4.2.5. *Let $p \in \mathbb{T}^{\binom{[n]}{m}}$ be a tropical Plücker vector with underlying matroid M. A vector $v \in \mathbb{R}^n$ is in the tropical linear space $L(p)$ if and only if M_v is a loopless matroid. In particular, $L(p) \cap \mathbb{R}^n$ is a polyhedral complex dual to the faces of $\mathcal{D}_p$ that correspond to loopless matroids.*

Proof. Suppose B is a basis of M and $a \in [n] \setminus B$. The (valuated) circuit $c := c_{B\cup a} \in \mathbb{T}^n$ of p is the fundamental circuit of a over the basis B. A vector $v \in \mathbb{R}^n$ is tropically orthogonal to c if and only if $\min_{b\in B\cup a} p_{B\cup a-b} + v_b$ is achieved at least twice. Equivalently, v is tropically orthogonal to c if and only if $\max_{b\in C(a,B)} w_p(v, B \cup a - b)$ is achieved at least twice, where $C(a, B)$ denotes the fundamental circuit in M of a over the basis B.

Now, let $v \in \mathbb{R}^n$ and take B a basis of maximal v-weight. According to Lemma 4.2.4, $v \in L(p)$ if and only if v is tropically orthogonal to all fundamental circuits of p over the basis B. Our discussion above implies that this is the case if and only if for any $a \in [n] \setminus B$ there exists $b \in B$ such that $B \cup a - b$ is also a basis of maximal v-weight. It follows that $v \in L(p)$ if and only if the matroid M_v has no loops, as desired. □

Lemma 4.2.4 implies that local tropical linear spaces are homeomorphic to Euclidean space, as stated in next theorem. It generalizes Theorem 4.2 in [FS05].

Theorem 4.2.6. *Any m-dimensional local tropical linear space $L(p)_B$ is homeomorphic to $\mathbb{R}^m$. More specifically, if $B = \{b_1, b_2, \ldots, b_m\} \subseteq [n]$ then the function $f_B : \mathbb{R}^m \to \mathbb{R}^n$ sending a vector $x = (x_1, x_2, \ldots, x_m) \in \mathbb{R}^m$ to the vector $f_B(x) \in \mathbb{R}^n$ defined by*

$$(f_B(x))_i := \begin{cases} x_j & \text{if } i = b_j \text{ for some } j, \\ \min\limits_{b_j\in C(i,B)-i} x_j + p_{B\cup i-b_j} - p_B & \text{if } i \in [n] \setminus B; \end{cases}$$

is a piecewise linear homeomorphism between $\mathbb{R}^m$ and $L(p)_B$.

Proof. We first prove that the image of f_B lies in Σ_B. Let $x \in \mathbb{R}^m$, and denote $v := f_B(x) \in \mathbb{R}^n$. Assume by contradiction that there is a basis A of M such that $w_p(v, A) > w_p(v, B)$, and take A such that $|A \setminus B|$ is minimal. Let $a \in A \setminus B$, and define $S := A - a$, $T := B \cup a$. Since p is a tropical Plücker vector, the minimum in (4.2.1) is attained at least twice, so there exists a $b \in B$ such that $p_A + p_B \geq p_{A-a\cup b} + p_{B-b\cup a}$. Subtracting $\sum_{i\in A} v_i + \sum_{j\in B} v_j$ on both sides we get $w_p(v,A)+w_p(v,B) \leq w_p(v,A-a\cup b)+w_p(v,B-b\cup a)$. But the definition of f_B implies that $w_p(v,B-b\cup a) \leq w_p(v,B)$, so it follows that $w_p(v,A) \leq w_p(v,A-a\cup b)$, contradicting our choice of A.

Now, it follows directly from the definition that any vector in the image of f_B is tropically orthogonal to all fundamental circuits of p over the basis B, so Lemma 4.2.4 ensures that the image of f_B lies in $L(p)_B$. Also, f_B is clearly an injective function. Moreover, if v is any vector in $L(p)_B$ then for any $i \in [n] \setminus B$,

$$\min_{b\in C(i,B)} p_{B\cup i-b} + v_b = p_B + v_i,$$

so it follows that f_B is surjective onto $L(p)_B$. □

4.3 Mixed Subdivisions and Face Vectors

In this section we apply the Cayley trick for subdivisions of a product of simplices to study the combinatorial properties of local tropical linear spaces.

Suppose that $p \in \mathbb{T}^{\binom{[n]}{m}}$ is a tropical Plücker vector without any coordinates equal to ∞, so that its underlying matroid M is the uniform matroid $U_{m,n}$. The regular subdivision $\mathcal{D} := \mathcal{D}_p$ induced by p is then a matroid subdivision of the hypersimplex $\Gamma := \Delta_{m,n}$. Let B be any basis of M. As discussed above, the local tropical linear space $L(p)_B$ is a polyhedral complex dual to the faces of $\mathcal{D}$ that contain the vertex e_B and correspond to loopless matroids, so in order to study this local tropical linear space it is enough to study how the subdivision $\mathcal{D}$ looks "around" the vertex e_B.

Denote by Γ_B the subpolytope of Γ obtained as the convex hull of all vertices adjacent to e_B, i.e., $\Gamma_B := \text{convex}\{e_A \mid A \in \mathcal{B}(M) \text{ and } |A \setminus B| = 1\}$. Note that Γ_B is the intersection of Γ with the affine hyperplane $h_B := \{\mathbf{x} \in \mathbb{R}^n \mid \sum_{i\in B} x_i = m-1\}$. The polytope Γ_B is called the **vertex figure** of Γ around the vertex e_B (or around the basis B). The subdivision of Γ_B obtained by intersecting the subdivision $\mathcal{D}$ with the polytope Γ_B is called the **local subdivision** induced by p (or induced by $\mathcal{D}$) around the basis B, and it is denoted by $\mathcal{D}_B$. Since the polytope Γ_B is equal to $\text{convex}\{e_B+e_i-e_j \mid i \notin B \text{ and } j \in B\} \cong \Delta_{n-m-1} \times \Delta_{m-1}$, we can use the Cayley trick to relate the subdivision $\mathcal{D}_B$ to a mixed subdivision of $(n-m)\cdot \Delta_{m-1}$, as described below.

The Cayley trick is a procedure that allows us to encode a subdivision of a product of simplices using a mixed subdivision of a Minkowski sum of simplices. More specifically, suppose $\mathcal{S}$ is a subdivision of the product $\Delta_{n-m-1} \times \Delta_{m-1}$. Denote by $o \in \mathbb{R}^{n-m}$ the centroid of the simplex Δ_{n-m-1}, that is, $o = \frac{1}{n-m}(e_1+e_2+\ldots+e_{n-m})$. Intersecting the subdivision $\mathcal{S}$ with the polytope $\{o\} \times \Delta_{n-m}$ we get a mixed subdivision of $\frac{1}{n-m} \cdot \sum_{i=1}^{n-m} \Delta_{m-1}$, which after scaling can be thought of as a mixed subdivision $\mathcal{M}$ of $(n-m)\cdot\Delta_{m-1}$. Note that this procedure defines a bijection between the faces of $\mathcal{M}$ of dimension d and the faces of $\mathcal{S}$ of dimension $d+(n-m-1)$ which are not contained in $\text{bd}(\Delta_{n-m-1}) \times \Delta_{m-1}$, where $\text{bd}(\Delta_{n-m-1})$

denotes the boundary of the simplex Δ_{n-m-1}. Moreover, this bijection preserves inclusion, so it is an isomorphism between the face poset of the mixed subdivision $\mathcal{M}$ and the subposet of the face poset of the subdivision $\mathcal{D}$ consisting of all faces which are not contained in $\mathrm{bd}(\Delta_{n-m-1}) \times \Delta_{m-1}$.

If we apply the Cayley trick to the subdivision $\mathcal{D}_B$ of the polytope $\Gamma_B = \mathrm{convex}\{e_B + e_i - e_j \mid i \notin B \text{ and } j \in B\} \cong \Delta_{n-m-1} \times \Delta_{m-1}$, we obtain a mixed subdivision $\mathcal{M}_B$ of $(n-m) \cdot \Delta_{m-1}$ whose face poset is isomorphic to the subposet of the face poset of $\mathcal{D}_B$ consisting of all faces which are not contained in $\mathrm{bd}(\Delta_{n-m-1}) \times \Delta_{m-1}$, i.e., all faces of $\mathcal{D}_B$ that correspond to loopless matroids. We thus have the following result.

Proposition 4.3.1. *The local tropical linear space $L(p)_B$ is combinatorially dual to the mixed subdivision $\mathcal{M}_B$ of $(n-m) \cdot \Delta_{m-1}$.*

It can be proved that in fact any regular subdivision of Γ_B is induced by a regular matroid subdivision of Γ.

Proposition 4.3.2 ([Kap93, Corollary 1.4.14], [HJS11, Corollary 6])**.** *For any regular subdivision $\mathcal{S}$ of the polytope Γ_B, there is a tropical Plücker vector p such that its associated regular matroid subdivision $\mathcal{D}$ of Γ restricts to the subdivision $\mathcal{S}$ when intersected with the vertex figure Γ_B.*

It follows from the two previous propositions that studying the combinatorics of local tropical linear spaces is equivalent to studying the combinatorics of mixed subdivisions of a Minkowski sum of simplices, as stated by the following theorem.

Theorem 4.3.3. *A poset P is isomorphic to the face poset of an m-dimensional local tropical linear space in $\mathbb{R}^n$ if and only if P is isomorphic to the dual face poset of a coherent mixed subdivision of $(n-m) \cdot \Delta_{m-1}$.*

This duality between local tropical linear spaces and mixed subdivisions of a Minkowski sum of simplices can be used to get a bound on the f-vector of local tropical linear spaces.

Proposition 4.3.4. *The number of i-dimensional faces of an m-dimensional local tropical linear space in $\mathbb{R}^n$ which become bounded after modding out by the lineality space generated by the vector $(1, 1, \ldots, 1)$ is at most*

$$\binom{n-i-1}{n-m-i,\, i-1,\, m-i} = \binom{n-i-1}{i-1}\binom{n-2i}{m-i},$$

and the number of i-dimensional faces without any boundedness constraint is at most

$$\frac{n-m}{n-i} \cdot \binom{n-1}{n-m,\, i-1,\, m-i} = \binom{n-i-1}{m-i}\binom{n-1}{i-1}.$$

Furthermore, for any m and n there is a local tropical linear space that achieves all these bounds.

Proof. By Proposition 4.3.1, the i-dimensional (bounded) faces of a local tropical linear space $L(p)_B$ are in correspondence with the (interior) faces of codimension $i-1$ in the associated mixed subdivision $\mathcal{M}_B$ of $(n-m)\cdot\Delta_{m-1}$. The maximum number of faces is attained when the mixed subdivision $\mathcal{M}_B$ is a fine mixed subdivision, so the result follows by substituting $s=n-m$, $r=m$, and $k=m-i$ in the following lemma. The existence of a tropical linear space satisfying these bounds follows from Proposition 4.3.2. □

Lemma 4.3.5. *The number of k-dimensional interior faces in any fine mixed subdivision of $s\cdot\Delta_{r-1}$ is equal to*

$$\binom{s-1+k}{s-r+k,\ r-1-k,\ k}, \tag{4.3.1}$$

and the total number of k-dimensional faces is equal to

$$\frac{s}{s+k}\cdot\binom{r+s-1}{s,\ r-1-k,\ k}.$$

Proof. Interior faces of dimension k in a fine mixed subdivision of $s\cdot\Delta_{r-1}$ are in correspondence with interior faces of dimension $k+s-1$ in an associated triangulation of the product $\Delta_{r-1}\times\Delta_{s-1}$. Products of simplices are equidecomposable polytopes, that is, all its trangulations have the same f-vector. Moreover, since faces of a product of simplices are also product of simplices, it follows that all triangulations of a product of simplices have the same number of interior faces in each dimension. The number of interior faces in a triangulation of a product of simplices was studied in [DS04, Corollary 25] in connection to tropical polytopes, from which 4.3.1 follows.

The total number of faces can be computed by adding interior faces over all faces of $s\cdot\Delta_{r-1}$. For any $1\le l\le r$, there are $\binom{r}{l}$ faces of $s\cdot\Delta_{r-1}$ isomorphic to $s\cdot\Delta_{l-1}$, so the total number of k-dimensional faces in a fine mixed subdivision of $s\cdot\Delta_{r-1}$ is equal to

$$\begin{aligned}\sum_{l=1}^{r}\binom{r}{l}\binom{s-1+k}{s-l+k,\ l-1-k,\ k} &= \sum_{l=1}^{r}\binom{r}{l}\binom{s-1+k}{k}\binom{s-1}{s-l+k}\\ &= \binom{s-1+k}{k}\cdot\sum_{l=1}^{r}\binom{r}{l}\binom{s-1}{s+k-l}\\ &= \binom{s-1+k}{k}\cdot\binom{r+s-1}{s+k}\\ &= \frac{s}{s+k}\cdot\binom{r+s-1}{s,\ r-1-k,\ k},\end{aligned}$$

as desired. □

4.4 Conical Tropical Linear Spaces

In this section we introduce a certain class of tropical linear spaces called conical tropical linear spaces, and we give a simple proof that they satisfy the f-vector conjecture. We start by describing more in depth the relation between regular subdivisions of $\Delta_{n-m-1} \times \Delta_{m-1}$ and regular matroid subdivisions of the hypersimplex $\Delta_{m,n}$ stated in Proposition 4.3.2.

Let B be a basis of the uniform matroid $U_{m,n}$. A regular subdivision $\mathcal{S}$ of the polytope $\Gamma_B \cong \Delta_{n-m-1} \times \Delta_{m-1}$ is obtained by lifting its vertices to some heights and then projecting back the lower faces of the resulting polytope. The set of heights on the vertices of this product of simplices can be encoded as a matrix $V \in \mathbb{R}^{m\times(n-m)}$. The **augmented matrix** of V is the $m \times n$ matrix $\bar{V}$ whose maximal submatrix consisting of its columns indexed by B is equal to the tropical identity matrix of size m (i.e., the $m \times m$ matrix with zeroes on the diagonal and ∞ in the rest of its entries), and whose maximal submatrix consisting of its columns indexed by $[n] \setminus B$ is equal to V. Define the vector $\tau_V \in \mathbb{R}^{\binom{[n]}{m}}$ as $(\tau_V)_A := \operatorname{tdet}(\bar{V}_A)$, where $\bar{V}_A$ denotes the $m \times m$ submatrix of $\bar{V}$ whose columns are indexed by the elements of A, and tdet denotes the tropical determinant. More explicitly, if $A = \{a_1, a_2, \ldots, a_m\}$ then

$$(\tau_V)_A = \min_{\sigma \in S_m} \left(\bar{V}_{a_1,\sigma(a_1)} + \bar{V}_{a_2,\sigma(a_2)} + \cdots + \bar{V}_{a_m,\sigma(a_m)} \right).$$

It follows from this construction that the vector τ_V is a tropical Plücker vector. The regular subdivision induced by τ_V on the hypersimplex $\Delta_{m,n}$ is then a regular matroid subdivision $\mathcal{D}$. Moreover, it is not hard to see that (see [Kap93, Corollary 1.4.14] and [HJS11, Corollary 6]) the subdivision $\mathcal{D}_B$ induced by $\mathcal{D}$ on the vertex figure Γ_B is equal to the original subdivision $\mathcal{S}$.

Given a polyhedral subdivision Σ of a polytope P, let us denote by $\mathcal{I}(\Sigma)$ the graded poset of interior faces of Σ ordered by reverse inclusion.

The following proposition was proved for $m \leq 3$ and conjectured for general m in a first version of [HJS11]. It was later proved in a second version of their paper. Our proof was obtained independently, and presents different ideas to the ones used in their approach.

Proposition 4.4.1. *Let $\mathcal{S}$ be a regular subdivision of $\Gamma_B \cong \Delta_{n-m-1} \times \Delta_{m-1}$ induced by the matrix $V \in \mathbb{R}^{m\times(n-m)}$, and let $\mathcal{D}$ be the matroid subdivision of $\Gamma = \Delta_{m,n}$ induced by $\tau_V \in \mathbb{R}^{\binom{[n]}{m}}$. Then the posets $\mathcal{I}(\mathcal{S})$ and $\mathcal{I}(\mathcal{D})$ are isomorphic.*

Proof. We first prove that every facet of $\mathcal{D}$ contains the vertex e_B, and thus every facet of $\mathcal{D}$ intersects Γ_B in a facet of the subdivision $\mathcal{S} = \mathcal{D}_B$. Note that a sufficiently small perturbation on the matrix V produces a refinement on both subdivisions $\mathcal{S}$ and $\mathcal{D}$, so without loss of generality we can assume that $\mathcal{S}$ is a triangulation of Γ_B.

If $\mathcal{S}$ is a triangulation of Γ_B then, as discussed in Proposition 4.3.4 and Lemma 4.3.5, the number of facets of $\mathcal{S} = \mathcal{D}_B$ is exactly

$$\binom{n-2}{n-m-1,\, 0,\, m-1} = \binom{n-2}{m-1}.$$

Each of these facets arises as the intersection of a facet of $\mathcal{D}$ with the vertex figure Γ_B. It was proved by Speyer in [Spe08, Theorem 6.1] that any matroid subdivision of the hypersimplex $\Delta_{m,n}$ has at most $\binom{n-2}{m-1}$ facets, so this implies that the subdivision $\mathcal{D}$ has exactly $\binom{n-2}{m-1}$ facets, all of which contain the vertex e_B.

Now, every interior face of $\mathcal{D}$ is equal to an intersection of facets of $\mathcal{D}$, so all interior faces contain the vertex e_B. It follows that the map from $\mathcal{I}(\mathcal{D})$ to $\mathcal{I}(\mathcal{S})$ sending the face F to the face $F \cap \Gamma_B$ is an isomorphism, as desired. □

Proposition 4.4.1 implies that the regular matroid subdivision $\mathcal{D}$ induced by τ_V is simply the "cone" from vertex e_B over the subdivision $\mathcal{S}$, in such a way that all the facets of $\mathcal{D}$ contain the vertex e_B. Any matroid subdivision arising in this way is called a **conical matroid subdivision**. A tropical linear space dual to a conical matroid subdivision is called a **conical tropical linear space**. Note that conical tropical linear spaces are precisely those tropical linear spaces such that all its bounded faces lie in a single local tropical linear space.

Example 4.4.2. Suppose $m = 2$. In this case, the space of tropical Plücker vectors with finite coordinates agrees with the space of phylogenetic trees on n leaves (see [SS04]). In fact, after modding out by its lineality space, any tropical linear space dual to a matroid subdivision of the hypersimplex $\Delta_{2,n}$ is homeomorphic to a tree with n (unbounded) leaves. Each of its unbounded rays is dual to a face of $\Delta_{2,n}$ of the form $x_i = 1$, so they are naturally labeled by the numbers $1, 2, \ldots, n$. The local tropical linear space around a basis $B = \{i, j\}$ is simply the unique path in the tree between leaves i and j. It follows that a tropical linear space is conical if and only if there is a path between two of its leaves containing all internal vertices, that is, if and only if it is a caterpillar tree.

It was proved in [HJS11] that the map $\tau : \mathbb{R}^{m\times(n-m)} \to \mathrm{Dr}_{m,n}$ defined by $V \mapsto \tau_V$ is a combinatorial embedding of the secondary fan of $\Delta_{n-m-1} \times \Delta_{m-1}$ into the Dressian $\mathrm{Dr}_{m,n} \cap \mathbb{R}^{\binom{[n]}{m}}$, that is, it is an injective map between polyhedral fans preserving dimension and the inclusion relation between the cones. The image of τ is precisely the set of tropical Plücker vectors corresponding to conical tropical linear spaces.

Theorem 4.4.3. *Any conical tropical linear space satisfies the f-vector conjecture.*

Proof. As discussed above, $L(p)$ is a conical tropical linear space if and only if there exists some local tropical linear space $L(p)_B$ containing all bounded faces of $L(p)$. The result follows from Proposition 4.3.4. □

Since any subdivision of Γ_B can be refined to a triangulation, any conical matroid subdvision of $\Delta_{m,n}$ can be refined to a conical matroid subdivision with exactly $\binom{n-i-1}{i-1}\binom{n-2i}{m-i}$ bounded faces of codimension i (see Propositions 4.3.4 and 4.3.5). This implies that any conical tropical linear space can be subdivided into a conical tropical linear space whose f-vector attains the upper bound predicted by the f-vector conjecture (Conjecture 4.1.1). However, there are tropical linear spaces attaining this upper bound which are not conical.

If $m = 2$, for example, any tropical linear space homeomorphic to a trivalent tree which is not a caterpillar tree also attains this upper bound, but it is not a conical tropical linear space. It would be very interesting to clarify exactly what part of the tropical Grassmannian $\mathrm{TGr}_{m,n}$ corresponds to conical tropical linear spaces, in the case where $m \geq 3$.

Chapter 5

Computing Tropical Linear Spaces

The content of this chapter will be published in the *Journal of Symbolic Computation* in a paper with the same title [Rin11]. The present version has only minor changes for consistency with previous chapters.

5.1 Introduction

Let A be an $m \times n$ complex matrix of rank m, with columns $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n \in \mathbb{C}^m$. We denote by $M(A)$ its associated matroid, i.e., the matroid on the ground set $[n] := \{1, 2, \ldots, n\}$ encoding the linear dependences in $\mathbb{C}^m$ among the columns of A. The circuits of $M(A)$ are then the sets $C \subseteq [n]$ such that there is a minimal dependence among the columns of A of the form $\sum_{i \in C} \lambda_i \mathbf{a}_i = 0$. See [Oxl92] for an introductory reference to matroid theory.

The **tropical linear space** $\mathcal{T}(M)$ of any matroid M over the ground set $[n]$ is the set of vectors $v \in \mathbb{R}^n$ such that for any circuit C of M, the minimum $\min\{v_i \mid i \in C\}$ is attained at least twice (i.e., there exist $j, k \in C$ distinct such that $v_j = v_k = \min\{v_i \mid i \in C\}$). In the case where M is the matroid associated to a complex matrix $A \in \mathbb{C}^{m \times n}$, $\mathcal{T}(M)$ agrees with the tropicalization of the linear subspace rowspace$(A) \subseteq \mathbb{C}^n$ (using the trivial valuation on $\mathbb{C}$). We will not consider in this chapter tropical linear spaces obtained by tropicalizing using non-trivial valuations; for a discussion about these more general tropical linear spaces and their beautiful combinatorics the reader is invited to see [Spe08].

Tropical linear spaces are one of the most basic objects in tropical geometry, and interest in them has increased substantially in the last few years. They are the local building blocks for abstract smooth tropical varieties, they play a key role in defining a well-behaved tropical intersection product, and they are central objects for studying realizability questions in tropical geometry (see [FR10], [KP09], [Sha10]). They are also fundamental for the study of tropicalizations of varieties obtained as the image of a linear subspace under a monomial map [DFS07]. It is thus desirable in many situations to have an explicit description of them as polyhedral fans, i.e., as a list of polyhedral cones in $\mathbb{R}^n$ on which it is possible to perform different computations.

There are several natural polyhedral fan structures that can be given to the tropical linear space of a matroid M. In [FS05], Feichtner and Sturmfels described a whole family $\mathcal{N}$ of polyhedral fans, all of them supported on the tropical linear space $\mathcal{T}(M)$. They compared these fans to the coarsest polyhedral structure on $\mathcal{T}(M)$, called the **Bergman fan** $\mathcal{B}(M)$ of M, which is induced by the normal fan of the matroid polytope associated to M. The finest fan structure in the family $\mathcal{N}$ is called the **fine subdivision** of $\mathcal{T}(M)$. It was studied by Ardila and Klivans in [AK06], where they used it to show that the intersection of the tropical linear space $\mathcal{T}(M)$ with the $(n-1)$-dimensional unit sphere is homeomorphic to the order complex of the lattice of flats of M, and thus to a wedge of spheres. The coarsest fan structure in the family $\mathcal{N}$ is called the (coarsest) **nested set fan** of M, and was studied in depth in [FS05]. In particular, Feichtner and Sturmfels proposed an algorithm for computing the nested set fan of M by gluing together "local" tropical linear spaces. In general, their algorithm has the inconveniences of having to go over all rank(M)! possible total orders on the elements of each basis of M, and of performing the computation of each maximal cone in the nested set fan a multiple number of times.

In Section 5.2 we introduce the **cyclic Bergman fan** $\Phi(M)$ of M, which is a simplicial polyhedral fan also supported on the tropical linear space $\mathcal{T}(M)$. The maximal cones of $\Phi(M)$ are described using some interesting combinatorial objects that we call "compatible pairs". We prove that the rays of $\Phi(M)$ are in correspondence with flats of the matroid M that are either cyclic flats or singletons, showing that $\Phi(M)$ is in general a little finer than the nested set fan of M. In Section 5.3 we present an effective algorithm for computing the cyclic Bergman fan of any matroid M that overcomes the difficulties present in [FS05]. We carry out a C++ implementation of our algorithm in the case M is the matroid associated to an integer matrix A. The resulting software, called `TropLi`, computes tropical linear spaces with great speed. It can also be used to compute basic matroidal information about the matrix A, like its collection of bases, circuits, or its Tutte polynomial. `TropLi` can be obtained at the website

`http://math.berkeley.edu/~felipe/tropli/` .

In Section 5.4 we give examples of a few computations done with it and report on its performance. Finally, in Section 5.5 we describe how our computation of tropical linear spaces can be used to compute vertices of Newton polytopes of A-discriminants. A C++ implementation of this procedure is also available online.

5.2 The Cyclic Bergman Fan

In this section we introduce the cyclic Bergman fan $\Phi(M)$ of a matroid M. It is a simplicial polyhedral fan supported on the tropical linear space $\mathcal{T}(M)$ of M amenable to computational purposes.

Let M be any rank m matroid on the ground set $[n]$ having no loops and no coloops. Suppose $I \subseteq [n]$ is an independent set of the matroid M and $e \in [n]$ is an element not in I such that $I \cup \{e\}$ is dependent. There is a unique circuit of M contained in $I \cup \{e\}$ (containing the element e), which is called the **fundamental circuit** $C(e, I)$ of e over I. It can be described as

$$C(e, I) = \{e\} \cup \{i \in I \mid I - \{i\} \cup \{e\} \text{ is independent}\}. \tag{5.2.1}$$

Now, let $B \subseteq [n]$ be a basis of the matroid M. Let $\Sigma_B \subseteq \mathbb{R}^n$ be the polyhedral cone consisting of all vectors v that make B a basis of maximal v-weight, i.e., such that $\sum_{i \in B} v_i$ is maximal among all bases of M. The set $\mathcal{T}(M)_B := \mathcal{T}(M) \cap \Sigma_B$ is called the **local tropical linear space** of M around the basis B. Note that this terminology agrees with the notion of local tropical linear space discussed in Chapter 4.

Example 5.2.1. Consider the 3×6 matrix

$$A := \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & -1 \end{pmatrix}.$$

The matroid $M := M(A)$ is in this case a graphical matroid, namely, the cycle matroid of the graph G presented in Figure 5.1. The circuits of M correspond to minimal cycles of G and the bases of M correspond to spanning trees of G.

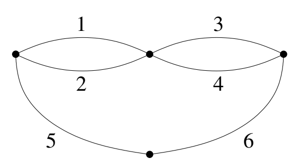

Figure 5.1: A graph G

The tropical linear space $\mathcal{T}(M)$ is then the set of vectors $v \in \mathbb{R}^6$ such that $v_1 = v_2$, $v_3 = v_4$, and $\min(v_1, v_3, v_5, v_6)$ is attained twice. It is naturally a polyhedral fan with six maximal cones, corresponding to the six posibilities for the two positions where $\min(v_1, v_3, v_5, v_6)$ is attained. For the basis $B := \{1, 5, 6\}$, the corresponding local tropical linear space $\mathcal{T}(M)_B$ is the set of vectors $v \in \mathcal{T}(M)$ that satisfy $v_3 = \min(v_1, v_3, v_5, v_6)$, which consists of only three of the six maximal cones described above. Note that each of these maximal cones is in several local tropical linear spaces. For example, the cone described by $v_5 \geq v_1 = v_2 = v_3 = v_4 \leq v_6$ is in the local tropical linear space corresponding to the bases $\{1, 5, 6\}$, $\{2, 5, 6\}$, $\{3, 5, 6\}$, and $\{4, 5, 6\}$.

The following theorem and its corollary appear in the work of Feichtner and Sturmfels [FS05], and they are a special case of our work in Chapter 4. They show that, although the tropical linear space $\mathcal{T}(M)$ might have a complicated combinatorial structure, all local tropical linear spaces are much faster to compute. In order to make our presentation self contained, we give here a completely combinatorial proof of their result.

Theorem 5.2.2 ([FS05]). *Let B be a basis of the matroid M. For any $v \in \Sigma_B$, v is in the local tropical linear space $\mathcal{T}(M)_B$ if and only if the minimum $\min\{v_i \mid i \in C\}$ is attained at least twice for any* fundamental *circuit C over the basis B.*

Proof. Assume by contradiction that $v \in \Sigma_B$ is such that the minimum $\min\{v_i \mid i \in C\}$ is attained at least twice for all fundamental circuits C over B, but v is not in $\mathcal{T}(M)$. Let D be a circuit of M such that $\min\{v_i \mid i \in D\}$ is attained only once, and take D containing as few elements outside of B as possible. Since D is not a fundamental circuit over B, the circuit D contains at least two elements not in B. Let $a \in D$ be the element such that $v_a = \min\{v_i \mid i \in D\}$, and let $b \in D - B$ be different from a. Consider the fundamental circuit $C := C(b, B)$ of b over B. Since B is a basis of maximal v-weight, Equation (5.2.1) implies that the minimum $\min\{v_i \mid i \in C\}$ is attained at b, that is, $v_b \leq v_c$ for any $c \in C$. In particular, we have that $a \notin C$. Applying the strong circuit elimination axiom (see [Oxl92, Proposition 1.4.11]) to the circuits C and D, with the elements $b \in C \cap D$ and $a \in D - C$, we get that there is a circuit $E \subseteq C \cup D - \{b\}$ containing the element a. But then E is a circuit such that $\min\{v_i \mid i \in E\}$ is attained only once (at $i = a$), and E has fewer elements outside of B than the circuit D, which is a contradiction. □

Corollary 5.2.3. *Let $B = \{b_1, b_2, \ldots, b_m\} \subseteq [n]$ be a basis of the matroid M. The function $f_B : \mathbb{R}^m \to \mathbb{R}^n$ sending a vector $x = (x_1, x_2, \ldots, x_m) \in \mathbb{R}^m$ to the vector $f_B(x) \in \mathbb{R}^n$ defined by*

$$(f_B(x))_i := \begin{cases} x_j & \text{if } i = b_j \text{ for some } j, \\ \min\limits_{b_j \in C(i,B)-\{i\}} x_j & \text{if } i \in [n] - B; \end{cases}$$

is a piecewise linear homeomorphism between $\mathbb{R}^m$ and the local tropical linear space $\mathcal{T}(M)_B$.

We now define the combinatorial objects that we will use to study the cyclic Bergman fan. Fix a basis $B \subseteq [n]$ of M. For any $k \in [n] - B$, denote $F_k := C(k, B) - \{k\}$ (note that $F_k \neq \emptyset$ since M has no loops). Let v be any vector in the local tropical linear space $\mathcal{T}(M)_B$, and suppose $\mathcal{J}$ is a total order on B such that for any $a, b \in B$ we have $v_a < v_b \implies a <_{\mathcal{J}} b$ (note that for generic v this condition determines $\mathcal{J}$ uniquely). This total order $\mathcal{J}$ induces a function $p : [n] - B \to B$ defined by $p(k) :=$ "$\mathcal{J}$-smallest element in F_k". We say that p is the **preference function** induced by the total order $\mathcal{J}$. According to Corollary 5.2.3, this preference function p is encoding which coordinates attain the minima described in Theorem 5.2.2, that is, $\min\{v_i \mid i \in C(k, B)\} = v_k = v_{p(k)}$ for all $k \in [n] - B$. Let $\mathcal{L}$ denote the restriction of the total order $\mathcal{J}$ to the image $\mathrm{Im}(p)$ of p. We call the pair $(p, \mathcal{L})$ a **compatible pair** (with respect to the basis B) induced by the vector v. Note that a non-generic vector $v \in \mathcal{T}(M)_B$ might induce several different compatible pairs with respect to B, corresponding to different choices of the total order $\mathcal{J}$.

Example 5.2.4. Let A be the 4×7 matrix

$$A := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 3 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 \end{pmatrix}.$$

Consider the basis $B := \{1, 2, 3, 4\}$ of the matroid $M := M(A)$. The fundamental circuits over B are $C(5, B) = \{2, 4, 5\}$, $C(6, B) = \{1, 2, 4, 6\}$, and $C(7, B) = \{1, 2, 3, 7\}$. Let $v = (0, 5, 2, 3, 3, 0, 0) \in \mathbb{R}^7$. It is not hard to see that the basis B is a basis of maximal v-weight, so $v \in \Sigma_B$. Since the minimum $\min\{v_i \mid i \in C\}$ is attained at least twice for each fundamental circuit C over the basis B, Theorem 5.2.2 implies that $v \in \mathcal{T}(M)_B$. There is a unique total order $\mathcal{J}$ on the elements of B satisfying $v_a < v_b \implies a <_{\mathcal{J}} b$, namely $1 <_{\mathcal{J}} 3 <_{\mathcal{J}} 4 <_{\mathcal{J}} 2$. The preference function p induced by $\mathcal{J}$ is then given by $p(5) = 4$, $p(6) = 1$, and $p(7) = 1$. The compatible pair $(p, \mathcal{L})$ induced by v in this way consists of the preference function p together with the linear order $1 <_{\mathcal{L}} 4$. Note that a different order $\mathcal{L}$ on the image of p would not be compatible with the preference function p. In fact, if the order $\mathcal{J}$ satisfied $4 <_{\mathcal{J}} 1$ then it would not be possible that $p(6) = 1$.

Proposition 5.2.5. *Let $(p, \mathcal{L})$ be a compatible pair (with respect to the basis B). The set of vectors v in the local tropical linear space $\mathcal{T}(M)_B$ that induce the pair $(p, \mathcal{L})$ is an m-dimensional polyhedral cone $\Gamma(p, \mathcal{L}) \subseteq \mathbb{R}^n$. Its lineality space is generated by the vector $(1, 1, \ldots, 1) \in \mathbb{R}^n$. After modding out by this lineality space, the cone $\Gamma(p, \mathcal{L})$ is a simplicial polyhedral cone whose extremal rays can all be taken to be $0/1$ vectors.*

Proof. Let $\mathcal{Q} = \mathcal{Q}(p)$ be the partition of the set $[n]$ with m blocks $Q_b := \{b\} \cup p^{-1}(\{b\})$, for $b \in B$. Note that if a vector $v \in \mathcal{T}(M)_B$ induces the pair $(p, \mathcal{L})$ then v has to be constant on each of the blocks of $\mathcal{Q}$, that is, for any i, j in the same block of $\mathcal{Q}$ we must have $v_i = v_j$. We will construct a directed caterpillar tree $T = T(p, \mathcal{L})$ with set of vertices $\mathcal{Q}$ encoding all further restrictions on the coordinates of such a vector v: If there is a directed path in T from Q_b to $Q_{b'}$ then v must satisfy $v_i \leq v_j$ for $i \in Q_b$ and $j \in Q_{b'}$.

Since $\mathcal{L}$ is a total order on the elements in the image of p, it naturally induces a total order on the non-singleton blocks of $\mathcal{Q}$. We start the construction of T as a directed path whose vertices are all the non-singleton blocks of $\mathcal{Q}$, with their position in the path matching the order prescribed by $\mathcal{L}$ (i.e., it is possible to walk from Q_b to $Q_{b'}$ if $b <_{\mathcal{L}} b'$). Now, for every $c \in B - \mathrm{Im}(p)$, add a directed edge from the non-singleton block Q_b to the block $Q_c = \{c\}$, where b is the $\mathcal{L}$-largest element in the image of p for which there is a $k \in [n] - B$ such that $k \in Q_b$ (i.e. $p(k) = b$) and $c \in F_k$. Note that such a b is guaranteed to exist since the matroid M has no coloops. Corollary 5.2.3 ensures that the directed tree T constructed in this way encodes precisely all the conditions on the coordinates of a vector v for it to be a vector in the local tropical linear space $\mathcal{T}(M)_B$ inducing the pair $(p, \mathcal{L})$. More specifically, $v \in \mathcal{T}(M)_B$ and v induces $(p, \mathcal{L})$ if and only if v is constant on the blocks of $\mathcal{Q}$ and for any directed edge $Q_b \to Q_{b'}$ in T we have $v_b \leq v_{b'}$.

Now, it is easy to see that the set $\Gamma(p, \mathcal{L})$ of vectors v satisfying the conditions imposed by T is a polyhedral cone with lineality space generated by the vector $(1, 1, \ldots, 1) \in \mathbb{R}^n$. Moreover, after modding out by its lineality space, the cone $\Gamma(p, \mathcal{L})$ can be described as the positive span of $m-1$ linearly independent 0/1 vectors, as follows. Think of T as a partial order on the blocks of $\mathcal{Q}$, and for any $b \in B$ define w_b as the sum of all coordinate vectors e_i such that i is in the union of all blocks in $\mathcal{Q}$ greater than or equal to Q_b (according to T). The cone $\Gamma(p, \mathcal{L})$ is then equal to the positive span of the vectors $\{w_b \mid b \in B \text{ and } w_b \neq (1, 1, \ldots, 1)\}$. □

Example 5.2.6. Let M and B be defined as in Example 5.2.4. We showed that the pair $(p, \mathcal{L})$ is a compatible pair, where p is given by $p(5) = 4$, $p(6) = 1$, $p(7) = 1$, and $\mathcal{L}$ is the total order $1 <_{\mathcal{L}} 4$. Following the proof of Proposition 5.2.5, the partition $\mathcal{Q}(p)$ for this preference function is $\{\{1, 6, 7\}, \{2\}, \{3\}, \{4, 5\}\}$. The directed caterpillar tree T associated to the pair $(p, \mathcal{L})$ is depicted in Figure 5.2. It encodes the conditions for a vector $v \in \mathbb{R}^7$ for it to induce the compatible pair $(p, \mathcal{L})$: v induces $(p, \mathcal{L})$ if and only if $v_3 \geq v_1 = v_6 = v_7 \leq v_4 = v_5 \leq v_2$. These equalities and inequalities define the simplicial polyhedral cone $\Gamma(p, \mathcal{L})$ in $\mathbb{R}^7$. After modding out by the lineality space $\mathbb{R} \cdot (1, 1, \ldots, 1)$, the cone $\Gamma(p, \mathcal{L})$ is generated by the rays e_2, e_{245}, e_3.

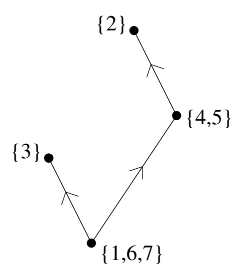

Figure 5.2: A directed caterpillar tree T

We will later prove in Theorem 5.2.11 that the extremal rays of the cones $\Gamma(p, \mathcal{L})$ are precisely the indicator vectors of all flats of the matroid M that are either cyclic flats or singletons.

It was pointed out to the author that the construction given in the proof of Proposition 5.2.5 of the polyhedral cones $\Gamma(p, \mathcal{L})$ from the directed trees $T(p, \mathcal{L})$ agrees with a more general construction of Postnikov, Reiner, and Williams described in [PRW08]. In their paper, a "braid" polyhedral cone $\sigma_Q \subseteq \mathbb{R}^n$ is associated to every preposet Q on the set $[n]$. In the case the Hasse diagram of the preposet Q is a (directed) tree T, their construction of the cone σ_Q agrees exactly with our construction of the cone $\Gamma(p, \mathcal{L})$.

It follows from our discussion that for any basis B of M, the local tropical linear space $\mathcal{T}(M)_B$ is the union over all compatible pairs $(p, \mathcal{L})$ with respect to B of the simplicial cones $\Gamma(p, \mathcal{L})$. However, since local tropical linear spaces corresponding to different bases might intersect nontrivially, compatible pairs with respect to different bases might give rise to the same cone. In order to find a canonical pair representing each cone, we say that a preference function $p : [n]-B \to B$ is **regressive** if $p(k) < k$ for all $k \in [n]-B$. If $(p, \mathcal{L})$ is a compatible pair and p is a regressive preference function, we say that $(p, \mathcal{L})$ is a **regressive compatible pair**.

Theorem 5.2.7. *The tropical linear space $\mathcal{T}(M)$ is the union over all bases B and all regressive compatible pairs $(p, \mathcal{L})$ with respect to B of the simplicial cones $\Gamma(p, \mathcal{L})$. Moreover, if $(p_1, \mathcal{L}_1)$ and $(p_2, \mathcal{L}_2)$ are different regressive compatible pairs (possibly with respect to different bases B_1 and B_2), then the intersection of $\Gamma(p_1, \mathcal{L}_1)$ and $\Gamma(p_2, \mathcal{L}_2)$ is a proper common face of both cones.*

Proof. In order to show that $\mathcal{T}(M)$ is the union of all cones $\Gamma(p, \mathcal{L})$ with p regressive, let v be any vector in $\mathcal{T}(M)$. Let B be the first basis with respect to lexicographic order which has maximal v-weight (i.e., such that $\sum_{i \in B} v_i$ is maximal). The vector v is then in the local tropical linear space $\mathcal{T}(M)_B$, so it induces some compatible pair $(p, \mathcal{L})$ with respect to B. Assume by contradiction that p is not a regressive preference function, so there exists a $k \in [n] - B$ such that $l := p(k) > k$. Since $l \in C(k, B)$, Equation (5.2.1) implies that $B' := B - \{l\} \cup \{k\}$ is also a basis of M. However, we have $v_k = v_l$, and thus B' is also a basis of maximal v-weight, contradicting our choice of B.

Now, suppose that $(p_1, \mathcal{L}_1)$ and $(p_2, \mathcal{L}_2)$ are any two compatible pairs. It follows from the description in terms of directed trees given in the proof of Proposition 5.2.5 that the cones $\Gamma(p_1, \mathcal{L}_1)$ and $\Gamma(p_2, \mathcal{L}_2)$ intersect in a common face. Moreover, if $(p_1, \mathcal{L}_1)$ and $(p_2, \mathcal{L}_2)$ are distinct regressive compatible pairs then the corresponding directed trees $T(p_1, \mathcal{L}_1)$ and $T(p_2, \mathcal{L}_2)$ are different. It follows that this intersection has to be a proper face of both cones. □

Definition 5.2.8. The **cyclic Bergman fan** $\Phi(M)$ of M is the simplicial fan in $\mathbb{R}^n$ whose maximal cones are the cones $\Gamma(p, \mathcal{L})$ with $(p, \mathcal{L})$ a (regressive) compatible pair. The support of $\Phi(M)$ is the tropical linear space $\mathcal{T}(M)$ of the matroid M.

As we will see later, the cyclic Bergman fan structure $\Phi(M)$ on $\mathcal{T}(M)$ is a little finer than the (coarsest) nested set structure on $\mathcal{T}(M)$ that was described in [FS05]. However, working with $\Phi(M)$ seems to be better for computational purposes, since its maximal cones are in one-to-one correspondence with effectively computable regressive compatible pairs. An explicit example of how the different fan structures on $\mathcal{T}(M)$ might look like is given in Example 5.2.12.

We now study the rays of the cyclic Bergman fan $\Phi(M)$.

Definition 5.2.9. A flat $F \subseteq [n]$ of M is called a **cyclic flat** if it is equal to a union of circuits of M. Equivalently, F is a cyclic flat if and only if F is a flat of M and $[n] - F$ is a flat of the dual matroid M^*.

Lemma 5.2.10. *Let F be a cyclic flat, and suppose $I \subseteq F$ is an independent set spanning F. Then F is a union of fundamental circuits over I.*

Proof. Denote by U the union of all fundamental circuits over I (which are contained in F), and assume by contradiction that $U \subsetneq F$. Since $F - I \subseteq U$, there exists some $i \in I$ such that $i \notin U$. Let $C \subseteq F$ be some circuit containing i such that $|C - I|$ is as small as possible. Let a be some element in $C - I$, and denote by D the fundamental circuit of a over I. Applying the strong circuit elimination axiom (see [Oxl92, Proposition 1.4.11]) to the circuits C and D, with the elements $a \in C \cap D$ and $i \in C - D$, we get that there is a circuit $C' \subseteq F$ containing i and contained in $C \cup D - \{a\}$, contradicting our choice of C. □

Theorem 5.2.11. *The rays of the cyclic Bergman fan $\Phi(M)$ (after modding out by the lineality space generated by the vector $(1, 1, \ldots, 1) \in \mathbb{R}^n$) are precisely the rays generated by the vectors $e_F := \sum_{i \in F} e_i$, where $F \subsetneq [n]$ is a flat of M which is either a cyclic flat or a singleton.*

Proof. Recall the description of the maximal cones of $\Phi(M)$ and their extremal rays in terms of directed trees given in the proof of Proposition 5.2.5. If $(p, \mathcal{L})$ is a compatible pair with respect to the basis B then from this description we see that all the extremal rays of the cone $\Gamma(p, \mathcal{L})$ have the form $\mathbb{R}_{\geq 0} \cdot e_F$, where $F = \{b\}$ for some $b \in B$ or $F \subsetneq [n]$ is a union of fundamental circuits over B. Moreover, since e_F is in the tropical linear space $\mathcal{T}(M)$, F must be a flat of M.

Now, suppose $F = \{b\}$ is a flat of M. If B is a basis of M containing b and v is a generic vector in the local tropical linear space $\mathcal{T}(M)_B$ such that $v_b = \max_{a \in B} v_a$, then the singleton $\{b\}$ appears as one of the leaves in the directed tree corresponding to the compatible pair induced by v, so e_b is an extremal ray of the corresponding maximal cone.

In the case F is a cyclic flat, let B be a basis of M intersecting F in as many elements as possible. The vector $v := e_F$ is then in the local tropical linear space $\mathcal{T}(M)_B$. Let $\mathcal{J}$ be any total order on B satisfying $v_a < v_b \implies a <_{\mathcal{J}} b$, and let $(p, \mathcal{L})$ be the compatible pair induced by $\mathcal{J}$. By Lemma 5.2.10, the directed tree associated to the pair $(p, \mathcal{L})$ has a node Q such that the set of elements that appear in nodes greater than or equal to Q is precisely F. It follows that e_F is an extremal ray of the cone $\Gamma(p, \mathcal{L})$. □

We now discuss how the different fan structures on $\mathcal{T}(M)$ that have been studied in the literature compare to the cyclic Bergman fan $\Phi(M)$. Let us assume that the matroid M is a connected matroid. The coarsest subdivision of $\mathcal{T}(M)$ is called the **Bergman fan** $\mathcal{B}(M)$ of M, and it was studied in [FS05]. It is the fan structure on $\mathcal{T}(M)$ inherited from the normal fan to the matroid polytope of M. The rays in this fan (after modding out by the lineality space) are all the vectors of the form e_F with F a “flacet” of M ($F \subseteq [n]$ is a “flacet” of M if the matroid $M|F$ obtained by restricting to F and the matroid M/F obtained by contracting F are both connected matroids). The Bergman fan is refined by the **nested set fan** (also studied in [FS05]), whose rays are the vectors e_F with F a connected flat (i.e., a flat F such that $M|F$ is connected), and whose maximal cones correspond to maximal nested sets of connected flats of M. This fan is in turn refined by the cyclic Bergman fan $\Phi(M)$, whose rays are the vectors e_F with F a flat which is either cyclic or a singleton, and whose maximal cones correspond to regressive compatible pairs. Finally, the cyclic Bergman fan $\Phi(M)$ is subdivided by the **fine subdivision** of $\mathcal{T}(M)$, which was studied in [AK06]. In this fine subdivision the rays are the vectors e_F with F any flat, and the maximal cones correspond to maximal chains of flats. Since the last three of these fans are simplicial fans, one way of measuring how different these fan structures on $\mathcal{T}(M)$ are is to measure how different the following sets of flats of M are:

$$\{F \text{ connected flat}\} \subseteq \{F \text{ flat, either cyclic or singleton}\} \subseteq \{F \text{ flat}\}.$$

A criterion for when the Bergman fan is equal to the nested set fan can be found in Theorem 5.3 of [FS05].

Example 5.2.12. Let M be the graphical matroid defined in Example 5.2.1. The Bergman fan $\mathcal{B}(M)$ is the coarsest fan structure on the tropical linear space $\mathcal{T}(M)$, and it consists of the six maximal cones discussed in Example 5.2.1. This coarsest fan structure is also equal to the nested set fan of M. After modding out by the lineality space generated by the vector $(1, 1, \ldots, 1) \in \mathbb{R}^6$, the fan $\mathcal{B}(M)$ has four rays $e_{12}, e_{34}, e_5, e_6 \in \mathbb{R}^6$, corresponding to the four nontrivial connected flats of the matroid M (which are also "flacets" of M). There is one more cyclic flat of M which is not connected: the flat $\{1, 2, 3, 4\}$. This implies that the cyclic Bergman fan $\Phi(M)$ strictly refines the fan $\mathcal{B}(M)$. In fact, the maximal cone of $\mathcal{B}(M)$ described by $v_1 = v_2 \geq v_5 = v_6 \leq v_3 = v_4$ gets subdivided into two smaller cones by the new ray e_{1234} of the fan $\Phi(M)$. The fine subdivision of $\mathcal{T}(M)$ is a fan with ten rays, corresponding to the ten nontrivial flats of M. In the fine subdivision, each of the six maximal cones of $\mathcal{B}(M)$ gets subdivided by a new ray into two cones, to get a total of twelve maximal cones.

5.3 Computing Compatible Pairs

Let M be a rank m matroid on the ground set $[n]$ having no loops and no coloops. The cyclic Bergman fan $\Phi(M)$ described in Section 5.2 allows us to develop an algorithm for computing the tropical linear space $\mathcal{T}(M)$ of M in an effective way. As it was discussed above, the maximal cones of $\Phi(M)$ are in bijection with regressive compatible pairs, so the key idea for a fast calculation of $\Phi(M)$ lies in coming up with a good way of computing all possible regressive compatible pairs with respect to a given basis B. The way compatible pairs were defined made use of a total order $\mathcal{J}$ on the elements of B to construct the pair, but it would not be a very good idea to go over all possible such total orders if m is not very small. Instead, what we do is to construct recursively each compatible pair $(p, \mathcal{L})$ by building up p and $\mathcal{L}$ *at the same time*. Algorithm 1 describes a general procedure that achieves this goal. As we mentioned before, our algorithm has two important features that

Algorithm 1: Computing the cyclic Bergman fan $\Phi(M)$

Input: A rank m matroid M having no loops and no coloops.
Output: A list of all maximal cones in the cyclic Bergman fan $\Phi(M)$.

for each *$B \subseteq [n]$ basis of M* **do**
  • Compute fundamental circuits:
  for each $k \in [n] - B$ **do**
    Compute $F_k := C(k, B) - \{k\}$.
  • Compute recursively all regressive compatible pairs $(p, \mathcal{L})$ with respect to B:
  Initialize $p = \emptyset$ and $\mathcal{L} = \emptyset$, and let k be the first element in $[n] - B$.
  Apply the recursive procedure `Pref(`$k, p, \mathcal{L}$`)` described below.
  Procedure `Pref(`$k, p, \mathcal{L}$`)`**:**
    if $k = end$ **then**
      Output the constructed pair $(p, \mathcal{L})$.
    else
      if $\mathrm{Im}(p) \cap F_k \neq \emptyset$ **then**
        Define $p(k) :=$ "$\mathcal{L}$-smallest element in $\mathrm{Im}(p) \cap F_k$".
        Let k' by the first element in $[n] - B$ greater than k (or $k' = end$ if k is the last element in $[n] - B$).
        Apply `Pref(`$k', p, \mathcal{L}$`)`.
      for each *$b \in F_k - \mathrm{Im}(p)$ such that $b < k$* **do**
        for each *total order $\mathcal{L}'$ on the set $\mathrm{Im}(p) \cup \{b\}$ that extends the total order $\mathcal{L}$* **do**
          if *there is no $l < k$ in $[n] - B$ satisfying both $b \in F_l$ and $b <_{\mathcal{L}'} p(l)$* **then**
            Define $p(k) = b$.
            Let k' by the first element in $[n] - B$ greater than k (or $k' = end$ if k is the last element in $[n] - B$).
            Apply `Pref(`$k', p, \mathcal{L}'$`)`.
  • Output the corresponding cones:
  for each *pair $(p, \mathcal{L})$ output in the previous step* **do**
    Compute the corresponding directed tree $T(p, \mathcal{L})$, as described in the proof of Proposition 5.2.5.
    Output the cone $\Gamma(p, \mathcal{L})$.

make it very fast compared to other existing algorithms: it does not have to go over all $m!$ total orders on the elements of each basis B, and moreover, each cone in the fan is computed exactly once, so there is no need to store them in memory or compare them with previously computed cones.

The pseudocode for Algorithm 1 deserves some explanation. Its main part consists of the recursive procedure $\mathrm{Pref}(k, p, \mathcal{L})$, which computes for $k \in [n] - B$ all possible ways of defining $p(k)$ given that we have already computed $p(j)$ for all $j < k$ (with $j \in [n] - B$), and that we already have a total order $\mathcal{L}$ on $\mathrm{Im}(p) := \{p(j)\}_{j<k}$. The block starting on Line 12 deals with the case where $p(k)$ is defined to be an element already in $\mathrm{Im}(p)$, in which case $p(k)$ can only be defined as the $\mathcal{L}$-smallest element in $\mathrm{Im}(p) \cap F_k$. The block starting on Line 16 deals with the case where $p(k)$ is defined to be a new element not in $\mathrm{Im}(p)$. In this case, the condition on Line 18 makes sure that the definition of $p(k)$ will not affect the compatibility of the pair $(p, \mathcal{L})$.

5.4 TropLi: A C++ Implementation

We developed a C++ implementation of the pseudocode described in Algorithm 1 for the case when the matroid M is given as the matroid associated to an $m \times n$ integer matrix A of rank m (having no loops and no coloops). In this case, if B is a basis of M and $k \in [n] - B$, we compute the set $F_k := C(k, B) - \{k\}$ by first row-reducing the matrix A in such a way that the submatrix of A consisting of the columns indexed by B is the identity, and then looking at the nonzero entries in the column indexed by k. A few minor changes were made to the pseudocode in Algorithm 1 in order to improve the efficiency of our implementation. For example, the order of the loops described by Line 16 and Line 17 was reversed, so that the amount of times the condition in Line 18 has to be checked is reduced significantly.

In order to run through all bases B of the matroid M, our code simply lists each subset of $[n]$ of size m and tests directly if the corresponding columns are a basis of $\mathbb{C}^m$. It makes use of the C++ library LEDA [Gmb] for carrying out all row operations on the matrix A with exact integer arithmetic. A much more effective way of listing all bases of the matroid M would be to make use of Avis and Fukuda's reverse search algorithm [AF96], which will be implemented in future versions of our code.

The result is a fast software tool for computing the cyclic Bergman fan $\Phi(A) := \Phi(M(A))$ of an integer matrix A, called `TropLi`. This software, together with documentation on how to use it, are available online at the website

`http://math.berkeley.edu/~felipe/tropli/`.

`TropLi` can also be used to compute some basic information about the matroid $M(A)$, like a list of all its bases, all its circuits, or its Tutte polynomial.

We now present a few computations done using `TropLi` and report on its performance. All of the computations were performed on a laptop computer with a 2.0 GHz Intel Core 2 processor and 2 GB RAM.

Example 5.4.1. Let A be the 4×8 matrix whose columns correspond to the affine coordinates of the 8 vertices of the three-dimensional unit cube. Running `TropLi` with this matrix A as input takes just a few milliseconds, and produces lists of all rays and all maximal cones in the cyclic Bergman fan $\Phi(A)$. The first list shows that there are 20 rays in the fan $\Phi(A)$, each of them specified as a 0/1 vector in $\mathbb{R}^8$. The second list tells us that $\Phi(A)$ contains 80 maximal cones, where each maximal cone is specified by its set of extremal rays. If instead we take A to be the 5×16 matrix whose columns are the affine coordinates of the vertices of the four-dimensional unit cube, `TropLi` still takes a fraction of a second and computes $\Phi(A)$ to be a fan with 176 rays and 2720 maximal cones.

Running `TropLi` with the flag "`-compare`" produces a comparison between the cyclic Bergman fan $\Phi(A)$ and the Bergman fan $\mathcal{B}(A)$. For the three-dimensional cube these two fans are the same, and thus equal to the nested set fan. For the four-dimensional cube the Bergman fan has 2600 maximal cones and it is thus a strict coarsening of the cyclic Bergman fan, even though the cyclic Bergman fan and the nested set fan are still equal (see Example 5.9 in [FS05]). Our program outputs a list showing which maximal cones of $\Phi(A)$ are part of the same maximal cone in $\mathcal{B}(A)$.

Example 5.4.2. Consider the 4×13 matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & -1 & 0 & -2 & -1 & 0 & -3 & -2 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & -1 & -2 & 0 & -1 & -2 & -3 \end{pmatrix}.$$

The orthogonal complement of the rowspace of A is the rowspace of the 9×13 matrix

$$A^{\perp} = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & -1 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 2 & -2 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 & -1 & 1 & 0 & -1 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 4 & -6 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & -1 \end{pmatrix}.$$

The matroid $M(A^{\perp})$ is the dual matroid M^* to the matroid $M := M(A)$. As we will discuss in Section 5.5, computing the tropical linear space of this dual matroid can be used as the key ingredient for computing the tropicalization of the A-discriminantal variety. For this matrix A, this variety is the hypersurface defined by the condition on the coefficients of a general affine linear form $l(x, y)$ and a general cubic polynomial $g(x, y)$ so that the curves $l(x, y) = 0$ and $g(x^{-1}, y^{-1}) = 0$ are tangent.

A Maple implementation of the algorithm described in [FS05] for computing tropical linear spaces locally takes many hours to compute $\mathcal{T}(M^*)$. As mentioned above, for each basis of the matroid M^* (there are 430 of them) it has to go through all $9! = 362\,880$ possible orderings of the rows of $A^{\perp}$. It also computes each maximal cone several times, so it has to compare each cone produced with the list of previously computed cones to see if it is a new cone or not. Running `TropLi` on the matrix $A^{\perp}$ computes the 9 dimensional fan $\Phi(M^*)$ (which has 29 rays and 2466 maximal cones) in less than a second.

Example 5.4.3. Let A be the 5×20 matrix

$$A = \begin{pmatrix} 1&1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1 \\ 0&2&0&0&1&0&0&1&1&0&0&2&0&0&1&0&0&1&1&0 \\ 0&0&2&0&0&1&0&1&0&1&0&0&2&0&0&1&0&1&0&1 \\ 0&0&0&2&0&0&1&0&1&1&0&0&0&2&0&0&1&0&1&1 \end{pmatrix}.$$

The orthogonal complement of the rowspace of A can be described as the rowspace of the 15×20 matrix

$$A^{\perp} = \begin{pmatrix} -1&0&0&0&1&0&1&0&-1&0&0&0&0&0&0&0&0&0&0&0 \\ 1&-1&0&0&0&0&0&0&0&0&-1&1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&2&0&-2&0&-1&1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0&-1&0&-1&0&0&0&1&0&0&0&0&0 \\ 0&-1&0&0&0&1&0&0&0&0&0&1&0&0&0&-1&0&0&0&0 \\ 1&-1&0&0&0&0&-1&0&0&1&0&1&0&0&0&-1&0&0&0&0 \\ -1&0&0&0&0&0&5&0&-4&0&0&2&1&0&0&-2&-1&0&0&0 \\ 1&0&0&0&0&0&-1&0&0&0&-1&0&0&0&0&0&1&0&0&0 \\ 2&-1&0&0&0&1&-5&0&3&0&0&0&0&1&0&0&0&-1&0&0 \\ 0&0&0&1&0&0&0&-1&0&0&0&0&0&-1&0&0&0&1&0&0 \\ 1&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&-1&0&1&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&-1&-1&1&0 \\ 0&-1&0&0&0&0&0&1&0&0&0&1&0&0&-1&0&1&0&0&-1 \\ 0&0&0&0&0&0&1&0&-1&0&-1&0&0&0&0&0&1&1&0&-1 \\ 0&0&1&0&0&0&3&0&-4&0&0&2&0&0&0&-3&0&0&0&1 \end{pmatrix}.$$

Again, computing the tropicalization of the rowspace of $A^{\perp}$ can be used for studying the tropicalization of the A-discriminantal variety, which now corresponds to the condition on the coefficients of two general quadratic polynomials in three variables for their corresponding surfaces to be tangent.

Running `TropLi` with the matrix $A^{\perp}$ as input computes the 15-dimensional fan $\Phi(A^{\perp}) \subseteq \mathbb{R}^{20}$, which has 172 rays and 475 722 maximal cones. All the computation takes just 60 seconds. A lot of this time is actually spent testing which subsets of the set $\{1, 2, \ldots, 20\}$ of size 15 are bases of the matroid $M(A^{\perp})$ by row-reducing all 15×15 submatrices of $A^{\perp}$. However, since $M(A^{\perp})$ is the dual matroid to $M(A)$, we can instead use A to compute the bases of M (by reducing the 5×5 submatrices of A) and then take their complement to get the bases of $M(A^{\perp})$. Also, the fundamental circuits $C_{M^*}(k, B)$ in $M^* = M(A^{\perp})$ can be computed from the fundamental circuits in M, since $j \in C_{M^*}(k, B)$ if and only if $k \in C_M(j, [n] - B)$. This method of computing $\Phi(A^{\perp})$ has been implemented in `TropLi`, and can be accessed by running it on the matrix A using the flag "`-dual`". In this way, the computation of $\Phi(A^{\perp})$ is even shorter: 30 seconds.

Example 5.4.4. Let A be the 6×30 matrix

$$A = \begin{pmatrix} 1&1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1 \\ 0&2&0&0&1&0&0&1&1&0&0&2&0&0&1&0&0&1&1&0&0&2&0&0&1&0&0&1&1&0 \\ 0&0&2&0&0&1&0&1&0&1&0&0&2&0&0&1&0&1&0&1&0&0&2&0&0&1&0&1&0&1 \\ 0&0&0&2&0&0&1&0&1&1&0&0&0&2&0&0&1&0&1&1&0&0&0&2&0&0&1&0&1&1 \end{pmatrix}.$$

The orthogonal complement of the rowspace of A is then the rowspace of a 24×30 matrix $A^{\perp}$. The A-discriminant is in this case the condition on the Cayley octad obtained as the intersection of three general quadratic surfaces in 3-space (see [PSV11]) to acquire a double point. Running `TropLi` on the matrix $A^{\perp}$ we see that $\Phi(A^{\perp})$ is a 24-dimensional fan in $\mathbb{R}^{30}$

having 929 rays and 154 495 683 maximal cones. The computation of all these 150 million cones takes a little more than 5 hours. If instead we run `TropLi` with the matrix A as input and using the flag "`-dual`", the computation of $\Phi(A^\perp)$ takes less than 4 hours.

5.5 An Application: Computing A-Discriminants

Let A be an $m \times n$ integer matrix of rank m with columns $\mathbf{a}_1, \dots, \mathbf{a}_n \in \mathbb{Z}^m$, and suppose that the vector $(1, 1, \dots, 1)$ is in the rowspace of A. The columns of A determine a collection of Laurent monomials $\mathbf{x}^{\mathbf{a}_1}, \dots, \mathbf{x}^{\mathbf{a}_n}$ in the ring $\mathbb{C}[x_1^{\pm 1}, \dots, x_m^{\pm 1}]$ in a natural way. Consider the space $\mathbb{C}^A$ of all Laurent polynomials whose support is contained in this set of monomials, i.e., polynomials of the form $f(\mathbf{x}) = \sum_{i=1}^n c_i \cdot \mathbf{x}^{\mathbf{a}_i}$, where the c_is are complex coefficients. Let $\nabla_A \subseteq \mathbb{C}^A$ be the Zariski closure of the set of all f in $\mathbb{C}^A$ that define a singular hypersurface in the torus $(\mathbb{C}^*)^m$, that is, for which there exists $\mathbf{z} \in (\mathbb{C}^*)^m$ such that

$$f(\mathbf{z}) = \frac{\partial f}{\partial x_i}(\mathbf{z}) = 0 \quad \text{for all } i = 1, 2, \dots, m.$$

The variety ∇_A is an irreducible variety defined over $\mathbb{Q}$, called the **A-discriminantal variety**. When $\nabla_A \subseteq \mathbb{C}^A$ is a subvariety of codimension 1, the irreducible integral polynomial Δ_A in the coefficients of f that defines ∇_A is called the **A-discriminant** (Δ_A is defined up to sign).

Of special interest is the case when A is a general matrix whose first s rows $r_1, \dots, r_s \in \mathbb{Z}^n$ are given by $r_j = \sum_{i \in I_j} e_i$ for some partition $\{I_1, \dots, I_s\}$ of $[n]$ (see examples 5.4.2, 5.4.3, and 5.4.4). In this case, the space $\mathbb{C}^A$ consists of polynomials of the form

$$f = x_1 \cdot f_1(x_{s+1}, \dots, x_m) + \cdots + x_s \cdot f_s(x_{s+1}, \dots, x_m),$$

where f_j is a polynomial on the variables $\mathbf{x}' = \{x_{s+1}, \dots, x_m\}$ whose support is contained in the set of monomials determined by the submatrix of A with rows indexed by $\{s+1, \dots, m\}$ and columns indexed by I_j. The A-discriminantal variety is then the Zariski closure of the set of such polynomials $f_1(\mathbf{x}'), \dots, f_s(\mathbf{x}')$ that have a common root in the torus where their gradient vectors $(\partial f_j / \partial x)_{x \in \mathbf{x}'}$ are linearly dependent. In the case where $s = 2$ this corresponds to the condition on the polynomials f_1, f_2 for their corresponding hypersurfaces to be tangent. If $s = |\mathbf{x}'|$ then this is the condition on the polynomials $f_1, \dots, f_s$ for the (finite) variety that they define to have a double point. In this case, the A-discriminant is also called their **mixed discriminant** (see [CCD$^+$]). If $s \geq |\mathbf{x}'| + 1$ and the matrix A is *essential* (see [DFS07]), then this is simply the condition on the polynomials $f_1, \dots, f_s$ for them to have a common root, so the A-discriminant is the same as their resultant. An extensive geometric treatment of all these notions can be found in [GKZ08].

Computing A-discriminants is in general a very hard computational task. Even for very small matrices A, the degree of Δ_A and its number of monomials can be quite large. From the definition, A-discriminants can in principle be computed by solving an elimination problem in the ring $\mathbb{C}[x_1^{\pm 1}, \ldots, x_m^{\pm 1}]$, but due to the huge size of these polynomials a Gröbner bases approach does not go too far.

In [DFS07], Dickenstein, Feichtner, and Sturmfels proposed a way of getting a handle on A-discriminants via tropical geometry. They proved that if $A^\perp$ denotes a Gale dual of the matrix A, i.e., an $(n-m)\times n$ matrix whose rowspace is equal to the orthogonal complement of the rowspace of A, then the tropicalization $\mathcal{T}(\nabla_A)$ of the variety ∇_A can be computed as the Minkowski sum of the tropical linear space $\mathcal{T}(M(A^\perp))$ and the rowspace of A. In the case where ∇_A has codimension 1, this tropicalization $\mathcal{T}(\nabla_A)$ is equal to the $(n-1)$-dimensional skeleton of the normal fan of the Newton polytope $NP(\Delta_A)$ of Δ_A. They used this to describe a "ray shooting" algorithm to recover vertices of $NP(\Delta_A)$ from $\mathcal{T}(\nabla_A)$, which goes as follows (see Theorem 1.2 in [DFS07]). Assume the columns of A span the integer lattice $\mathbb{Z}^m$. Suppose w is a generic vector in $\mathbb{R}^n$, and let $u \in \mathbb{Z}^n$ be the vertex of $NP(\Delta_A)$ minimizing the dot product $u \cdot w$. Then u can be computed as

$$u_i = \sum_{\sigma \in \mathcal{C}_{i,w}} \left|\det(A^t, \sigma_1, \ldots, \sigma_{n-m-1}, e_i)\right|, \qquad (5.5.1)$$

where $\mathcal{C}_{i,w}$ denotes the set of all maximal cones σ in the nested set fan of $M(A^\perp)$ satisfying $(w + \mathbb{R}_{>0} \cdot e_i) \cap (\sigma + \operatorname{rowspace} A) \neq \emptyset$, and $\sigma_1, \ldots, \sigma_{n-m-1}$ are the 0/1 extremal rays of the cone σ after modding out by its lineality space. An essential component in this procedure for computing vertices of $NP(\Delta_A)$ is to compute the tropical linear space $\mathcal{T}(M(A^\perp))$. It is possible, however, to replace in this formula the nested set fan of $M(A^\perp)$ by the cyclic Bergman fan $\Phi(A^\perp)$, which can be computed more easily.

Based on our implementation `TropLi` for computing cyclic Bergman fans, we developed a C++ code that computes vertices of Newton polytopes of A-discriminants in the way described above. Given an integer matrix A and a vector $w \in \mathbb{Z}^n$, it computes a vertex u of $NP(\Delta_A)$ minimizing the dot product $u \cdot w$. In the case where w is not generic and this minimum is attained at several vertices of $NP(\Delta_A)$, the code uses a symbolic perturbation approach to compute one of these vertices at random. This software tool can also be obtained at the website

`http://math.berkeley.edu/~felipe/tropli/` .

Example 5.5.1. Consider the 4×16 matrix

$$A = \begin{pmatrix} 1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1 \\ 0&0&1&0&1&2&0&-1&0&-2&-1&0&-3&-2&-1&0 \\ 0&1&0&2&1&0&0&0&-1&0&-1&-2&0&-1&-2&-3 \end{pmatrix}$$

Running our program with the matrix A as input and the flag "`-random 100`" computes 100 random vertices of the Newton polytope of the A-discriminant Δ_A. In this case, Δ_A is the condition on a general quadratic polynomial $f(x, y)$ and a general cubic polynomial $g(x, y)$ for the curves $f(x, y) = 0$ and $g(x^{-1}, y^{-1}) = 0$ to be tangent. Our code computes the fan $\Phi(A^\perp)$ in the same way `TropLi` does (with the matrix A as input and using the flag "`-dual`"), and for each maximal cone σ computed it checks if $\sigma + \text{rowspace}(A) \subseteq \mathbb{R}^{16}$ has codimension 1. If this is not the case then the cone σ will not contribute to the sum in Equation 5.5.1. The cyclic Bergman fan $\Phi(A)$ has 18 045 maximal cones, 6 675 of which have codimension 1 after adding the rowspace of A. This initial computation takes 22 seconds. The code then performs the ray shooting algorithm for 100 random values of the vector w using the 6 675 cones computed before, and outputs the corresponding 100 vertices u of the Newton polytope of Δ_A. It also prints the A-degree of the A-discriminant Δ_A, i.e., the vector $A \cdot u$ for u any point of $NP(\Delta_A)$ (this does not depend on the choice of u). In this case, the A-degree is equal to $(24, 22, -6, -6)$, so in particular we see that Δ_A is a homogeneous polynomial of degree 46. This second part of the computation takes 7 minutes.

Of course one would like to compute all the vertices of the Newton polytope $NP(\Delta_A)$. In general, however, due to the very large number of vertices of $NP(\Delta_A)$ and number of maximal cones in $\mathcal{T}(\nabla_A)$, one has to be quite clever about the way the vectors w are chosen. Choosing them at random and waiting until all vertices of $NP(\Delta_A)$ have been computed is in general not viable. A great example illustrating all these difficulties and a few ways to overcome them is given in [CTY10].

Very recently, an effective algorithm for recovering the normal fan of the Newton polytope of a polynomial from the support of its tropical hypersurface was proposed in [JY11]. This algorithm has already been implemented in the software package Gfan [Jen]. It can be used to take the description of $\mathcal{T}(\nabla_A)$ as a sum of a tropical linear space and a classical linear space and compute from it the normal fan of the Newton polytope of the A-discriminant Δ_A. From this normal fan it is possible to recover the exact coordinates of the vertices of the Newton polytope of Δ_A by keeping track of the multiplicities of the codimension 1 cones.

Now, even if we have a list of all the vertices of the Newton polytope $NP(\Delta_A)$, recovering the polynomial Δ_A is no easy task. One way to do it is to consider a generic polynomial whose monomials correspond to the lattice points in $NP(\Delta_A)$, and then imposing the condition that it vanishes on the image of the rational parametrization of ∇_A given in Proposition 4.1 of [DFS07]. This translates into a linear system of equations on the coefficients of this generic polynomial whose solution space corresponds to the coefficients of the A-discriminant Δ_A. Note that, however, this procedure requires first computing all lattice points in the polytope $NP(\Delta_A)$ and then solving a very large system of linear equations, so a more effective approach would be desirable.

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