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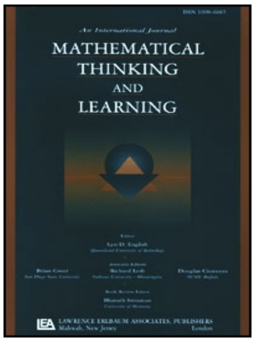

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Conceptualizing Perseverance in Problem Solving as Collective Enterprise

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ABSTRACT

Students are expected to learn mathematics such that when they encounter challenging problems they will persist. Creating opportunities for students to persist in problem solving is therefore argued as essential to effective teaching and to children developing positive dispositions in mathematical learning. This analysis takes a novel approach to perseverance by conceptualizing it as collective enterprise among learners in lieu of its more conventional treatment as an individual capacity. Drawing on video of elementary school children in two US classrooms ($n = 52$), this paper offers: (1) empirical examples that define perseverance as collective enterprise; (2) indicators of perseverance for teachers (and researchers) to support (and study) its emergence; and (3) evidence of how the task, peer dynamics, and student-teacher interactions afford or constrain its occurrence. The significance of perseverance as collective enterprise and as an object of design in developing effective learning communities, is discussed.

Introduction

In the United States, the idea of mathematical learning as more than disciplinary knowledge and practices appears in guiding texts like *Adding It Up* (2001), in which the National Research Council (NRC) argues that mathematically proficient children exhibit *productive dispositions* (Kilpatrick, Swafford, & Findell, 2001). The NRC defines productive dispositions as the "habitual inclination to see mathematics as sensible and useful, coupled with a belief in one's own efficacy" (p. 5). This analysis investigates one aspect of productive dispositions, which the NRC refers to as "diligence" or "steady effort" (p. 131). Appearing in mathematics frameworks in and beyond the United States, this notion of "diligence" is typically articulated as *perseverance* and conceptualized as an individual capacity. By focusing on problem solving in a group-based context, this analysis takes a novel approach to perseverance by conceptualizing it as a *collective enterprise* among mathematics learners. Doing so capitalizes on recent reforms in the United States that modestly gesture toward attending to individuals and collective outcomes (Common Core Standards Initiative [CCSI], 2010; National Council for Teachers of Mathematics [NCTM], 2014).

Problem solving is foundational to mathematical learning (English & Gainsburg, 2016; Pólya, 1981, 2014; Schoenfeld, 1985, 1992) such that the world over, children are expected to *persist* when they encounter challenging problems (CCSI, 2010; Department for Education and Employment [DfEE], 1999; Human Resources Development Working Group [HRDWG], 2009; Kilpatrick et al., 2001; NCTM, 2014). In Singapore, for example, mathematics frameworks center on problem solving where students develop an "attitude of perseverance" (HRDWG, 2009). The English National Curriculum similarly expects students to "overcome difficulties in problem solving" (DfEE, 1999, p. 5). In the United States, where this analysis takes place, creating opportunities for students to persist in problem solving is a tenet of effective teaching that is

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often described as creating the conditions for *productive struggle*[1] (Hiebert & Grouws, 2007; NCTM, 2014). Most recently, mathematics education reforms include practice standards that articulate perseverance as making sense of problems, modeling and reasoning about them, and monitoring or changing directions (CCSI, 2010). Given the ongoing global and national investment in perseverance,[2] we argue for an expansion of the notion from characterizing an individual, to what we term, perseverance as collective enterprise.

We define perseverance as collective enterprise, as peers engaging together in productive struggle with effort over time. Here, "productive struggle" refers to students grappling with important mathematical ideas that are just beyond reach (Hiebert & Grouws, 2007), while "effort" refers to children modeling or representing ideas to their peers, considering others' explanations, or monitoring their problem solving (CCSI, 2010; Schoenfeld, 1985). Shifting attention from perseverance as an individual trait to an attribute of group dynamics, is important in understanding how the context (e.g., peers, the teacher, the task) shape the emergence (or not) of perseverance in collaborative settings. Drawing on video of elementary school children working in groups to solve nonroutine algebraic tasks, this analysis makes three contributions. First, it offers empirical examples of what constitutes perseverance and what is meant by perseverance as collective enterprise. This defines the phenomenon more specifically, while rejecting the idea that whenever children solve a problem they have *de facto* persevered (i.e., finishing is persevering). Second, our methodological approach offers indicators of perseverance, which help identify *in situ* opportunities to support perseverance (i.e., Are they persisting?) rather than relying on retrospective accounts to assess its occurrence (i.e., Did they persist?). Third and finally, we offer an extended case analysis of perseverance as collective enterprise to illustrate how the task, peer dynamics, and student-teacher interactions afford or constrain its emergence.

Rationale

Just as perseverance is seen as important for an individual learning mathematics, we conjecture perseverance as collective enterprise is important to collaborative learning in mathematics. The NRC argues persevering is what makes a child believe she or he is a doer and learner of the discipline (National Research Council & Mathematics Learning Committee, 2001, p. 131), and prior research attests to its importance for individual learning (Bass & Ball, 2015; Duckworth, Peterson, Matthews, & Kelly, 2007). In complement to this perspective, we assert perseverance as collective enterprise offers a new vantage point from which to consider how children see themselves *and one another* as doers and learners of the discipline, which is essential for building equitable and effective learning communities (Boaler & Staples, 2008; Cohen & Lotan, 2014; Sherin, 2002; Silver & Smith, 1996). Our assertion derives from extensive research on peer collaboration, which argues a relationship between group dynamics and mathematical learning (Boaler, 2008; Cohen & Lotan, 2014; Esmonde & Langer-Osuna, 2013; Gresalfi, 2009; Sengupta-Irving, 2014; Sengupta-Irving & Enyedy, 2014), and problem solving in particular (DeBellis & Goldin, 2006; Goldin, 2000; Schoenfeld, 1992, 2014). Barron (2000, 2003), for example, argued that collaboration on interesting problems promises deep disciplinary engagement while also promoting individual agency. Others found that when structured and supported, students express and receive increased social and academic support in collaborative settings (Angier & Povey, 1999; Boaler & Staples, 2008; Cohen & Lotan, 2014; Leikin & Zaslavsky, 1997; Sengupta-Irving, 2014; Slavin, 2011), and exhibit greater responsibility for their learning (Boaler, 2015; Langer-Osuna, 2016; Yackel, Cobb, & Wood, 1991). Conceptualizing perseverance as collective enterprise introduces important problems of disciplinary practice, which this analysis addresses, including how teachers

[1] *Productive struggle* is the effort expended to make sense of ideas that are within reach but not immediately apparent (Hiebert & Grouws, 2007). Similarly, perseverance and persistence are said to occur as a result of opposition and struggle (Ryans, 1938a, p. 83). Thus, we conceptually parallel productive struggle with perseverance and persistence in analyzing children's collaborative problem solving.

[2] In 2015 the Spencer Foundation sponsored a collection of papers on perseverance in mathematics—see (Bass & Ball, 2015; Berry & Thunder, 2015; Middleton, Tallman, Hatfield, & Davis, 2015; Star, 2015; Taylor, 2015).

create opportunities for children to persevere, what teachers look and listen for to support children persevering, and understanding how the context (i.e., peers, teacher, task) might constrain its emergence.

Conceptual framework

Perseverance as individual and collective enterprise

The interpretive framework for this analysis builds on prior research regarding children's mathematics dispositions in relation to classroom practices. Drawing on Wenger (1998), Gresalfi and Cobb (2006, p. 50) used the term "dispositions" to include how students come to engage with the discipline in a particular context, and how they come to identify with the discipline more broadly. Gresalfi (2009) later operationalized dispositions in relation to two classroom components—students working with content and with peers. As Gresalfi explained, her analysis "did not consider more nuanced differences between students' engagement with content, such as giving up, trying multiple strategies or critiquing answers" (2009, p. 336) and later suggested "many other aspects of mathematical dispositions could quite usefully be pursued" (p. 363). This analysis investigates a more specific aspect of dispositions: children *not* giving up on each other in grappling with difficult mathematical ideas.

In framing her analysis of dispositions, Gresalfi (2009) delineated several perspectives on learning, two of which are relevant here. First, Gresalfi explained how an *individual* perspective focuses primarily on traits or characteristics. This describes how perseverance has been typically studied (e.g., DiCerbo, 2014; Feather, 1961, 1966; Ryans, 1938a, 1938b). Early on, for example, the notion of perseverance or persistence was described through popular aphorisms like "If at first you don't succeed, try, try, try again," (Ryans, 1938a, pp. 80–81). This literary trope conveys a view of persistence as an individual capacity ("*you* don't succeed") that is then measured as a function of time spent on task (Ryans, 1938b). This look to an individual's time on task or trials permeates much of the empirical history of this construct (e.g., Altshuler & Kassinove, 1975; Briggs & Johnson, 1942; Schofield, 1943; Sigman, Cohen, Beckwith, & Topinka, 1987). More recently, a study of video gaming measured persistence in relation to time and number of restarts at a given level (see DiCerbo, 2014).

The problem with an individual perspective on perseverance is that it presumes the child is the cause of success/failure (i.e., she *lacks* perseverance) without considering what role the context plays in supporting or constraining her efforts. Gresalfi (2009) referred to a more situated accounting of dispositions as taking an *individual-with-context* perspective. This recognizes how the task, physical materials, and peer or teacher interactions constrain or afford persistence. In collaborative contexts, mathematical practices are shared among collectives of students (e.g., pairs, groups). This joint engagement is seldom explicitly understood in relation to perseverance. Rather, it is traced back to the individual—how the task, peers, or the teacher advance an individual child's mathematical understanding or disposition over time. This latter tracing of outcomes back to the individual, while important, is not the goal of this analysis. This investigation conjectures that what transpires within the collective in relation to a particular task, and how the context (peers, teacher, task) shapes what transpires, traces back to collective outcomes. Warrant for this line of inquiry lies in the language of current US mathematics education reforms that advocate individual and collective learning opportunities. For example, in *Principles to Action* (2014), a guide to mathematics reforms, the NCTM asserted that effective teaching should consider perseverance an individual and collective enterprise:

> Effective teaching of mathematics consistently provides students, individually *and collectively*, with opportunities and supports to engage in productive struggle as they grapple with mathematical ideas and relationships. (NCTM, 2014, p. 48, emphasis added)

This passage invites the study of children grappling with important mathematical ideas and relationships collectively (i.e., perseverance as collective enterprise), where the unit of analysis is the group's problem solving efforts and not, for example, how one child is doing as a consequence of group work. In this analysis we therefore coordinate attention to the *in situ* opportunities for productive struggle, how such opportunities are taken up (or not) by the group, and how efforts at joint problem solving are sustained, advanced, or foreclosed on for the collective.

Learning and productive struggle

When Vygotsky (1978) formulated the idea of a Zone of Proximal Development (ZPD) he was wrestling with the paradoxical outcomes of seemingly more mature children being slowed by their experiences of schooling, while the opposite was occurring for seemingly less mature children (Del Rio & Alvarez, 2007). Vygotsky formulated ZPD as a way to anticipate development but also recognize its open, divergent, and idiosyncratic nature. Much of this openness hinged on what cultural and societal tools were at a child's disposal as loaned through shared discourse and activity with peers or adults. ZPD therefore describes the difference in what a child can achieve alone and what is possible when cooperating with an expert adult or more capable peer. As Vygotsky explained:

> We propose that an essential feature of learning is that it *creates* the zone of proximal development; that is, learning awakens a variety of internal developmental processes that are able to operate *only* when the child is interacting with people in his environment and in cooperation with his peers. (Del Rio & Alvarez, 2007, p. 279; 1978, pp. 89–90, emphasis added)

The idea that students can be awakened to possibilities unlike what happens when working alone, reverberates through numerous studies of collaborative learning (e.g., De Guerrero & Villamil, 2000; Doolittle, 1997; Nyikos & Hashimoto, 1997; Slavin, 2011). Studies of collaborative problem solving have even extended ZPD to show how peers negotiate and define task goals, and develop shared expectations (e.g., Forman, 1989, 1992; Miell & MacDonald, 2000; Newman, Griffin, & Cole, 1984). Forman (Forman, 1989; Forman & McPhail, 1993), for example, describes how peers create a "bi-directional ZPD" where mathematics learners construct and coordinate different views in order to problem solve together. Goos, Galbraith, and Renshaw (2002) leverage a similar understanding in what they refer to as "collaborative zones of proximal development" in their analyses of successful and unsuccessful cases of small group problem solving. The reach for language like "bidirectional" or "collaborative" distinguishes same status interactions from expert interactions. Indeed, which child is "more capable" in a group can vary from moment-to-moment or task-to-task and cannot be presumed a fixed ascription of a particular child (Engle, Langer-Osuna, & McKinney de Royston, 2014; Langer-Osuna, 2011; Sengupta-Irving, 2014). In studying metacognition and collaborative ZPD, Goos and colleagues (2002) explained how encountering a challenge stimulates collective mathematical activity (p. 218). This is important to the current analysis insofar as it suggests that productive struggle makes evident the creation of a collaborative ZPD.

Conceptually coordinating productive struggle and the collaborative ZPD during a group activity helps establish why perseverance as a collective enterprise matters to studying mathematical learning. Productive struggle refers to students' efforts at grappling with key ideas that are yet unformulated (Hiebert & Grouws, 2007). Productive struggle allows students to advance their conceptual understanding and apply what they learn to novel problems (NCTM, 2014; Kapur, 2010, 2014). Figure 1 depicts our conceptual coordination of productive struggle and the collaborative ZPD.

Since productive struggle marks the boundaries of what is yet mathematically unformulated but within reach, it too describes a zone of learning. In our view, this zone is bordered either by *no struggle* or *unnecessary struggle*. Depicted on the left in Figure 1, no struggle occurs when the solution is apparent: the task may be procedural and circumvents grappling with key ideas; the group may simply know what to do; or, maybe the group reaches consensus on an incorrect solution and that prevents them from grappling further. In any case, the difference between what students do alone and together is imperceptible and in effect, no collaborative zone of proximal development is created.

On the right in Figure 1 is the boundary between productive and unnecessary struggle, which also marks the terminal boundary of the collaborative ZPD. Unnecessary struggles are those that involve extreme levels of challenge created by overly difficult problems (Hiebert & Grouws, 2007) or that do little

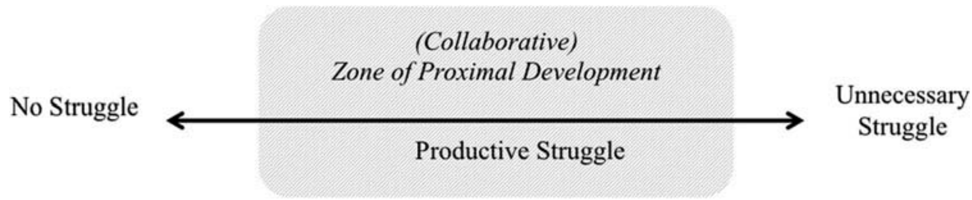

Figure 1. Conceptually coordinating productive struggle and the Zone of Proximal Development.

to advance sense making, explanations, or problem solving (Warshauer, 2014). We interpret unnecessary struggle to mean that despite an expert adult or more capable peer, the mathematical idea is beyond the group's grasp—beyond their collaborative ZPD.

The center of Figure 1 reflects productive struggle and corresponds to the idea of working within a collaborative ZPD. Here, students offer each other ideas for problem solving, model or represent their thinking to others, explain ideas and critique each other's reasoning, and so on. In short, students working within their collaborative ZPD are also engaged in productive struggle.

Overall, neither the ZPD nor productive struggle exists in advance of the group, task, or classroom norms and expectations. Where one group grapples another may solve without effort; then again, what creates unnecessary struggle for one group may not for another. Within a task, a group may grapple with one part of the task and subsequently breeze through the rest. Finally, the norms and expectations for collaboration may differ between classrooms and thus children appear to persevere in some settings but not others. To borrow and amend the language that Gresalfi (2009) offered: we study perseverance from an individual<u>s</u>-with-context perspective. Whether or not students are working within their ZPD and engaged in productive struggle is a function of the task, their past experiences and knowledge, their moment-to-moment interactions, and what support the teacher and classroom context offer.

Methods

Setting and participants

The data reported were collected at a progressive public charter elementary school in California where the teachers readily engage in research with University researchers. The school was ethnically and racially diverse, reporting a student population that was 36% Caucasian, 32% Latino/a, 14% Asian, 10% African American, and 8% "Other" at the time of the study.

The study occurred over 6 days in two fifth-grade classes. Each class period was 55 minutes in length. Fifty two of 54 students consented to all aspects of the study. Two groups of four students were selected at random in each class for video recording (n = 16). The teacher, Ms. Munroe, taught both classes. Ms. Munroe had 15 years of experience teaching math at the elementary and middle school levels. A typical lesson for Ms. Munroe had four parts: (1) a mini-lecture that directly instructs on the concept or topic; (2) small group discussion and problem solving time as she circulated; (3) whole-class discussion and student presentations; and (4) a teacher-led summary of the day's big ideas. As will be discussed next, we adhered to a similar four-part structure with some deliberate modifications.

Context of study

The first author and Ms. Munroe had collaborated previously in a teaching experiment exploring students' affect and productive disciplinary engagement in a 5-week data and statistics unit (see Sengupta-Irving & Enyedy, 2014; Sengupta-Irving, Redman, & Enyedy, 2013). In the teaching experiment, Ms. Munroe became convinced of inquiry as creating robust and rich learning opportunities. As a follow up, we sought a closer focus on students' collaborative problem solving. The first author designed six lessons in consultation with Ms. Munroe to advance conceptual learning of algebra through the use of group worthy tasks (Lotan, 2003). Notably, conceptual learning is a better goal for peer collaboration than procedural learning (Pai, Sears, & Maeda, 2015; Phelps & Damon, 1989) and group worthy tasks afforded students multiple entry points and pathways for problem solving together.

Lessons during the study were similar in structure to Ms. Munroe's typical lessons, as described previously. However, the opening minutes of class involved a Number Talk (Boaler, 2009; Parrish, 2010) rather than a mini-lesson that directly instructed on the relevant mathematical ideas or relationships. In Number Talks, students are presented with problems to solve mentally (e.g., 223 + 119). Students then describe their approaches to visualizing and solving the problem, as the teacher records each approach on the board and facilitates discussion between them. This visual and verbal affirmation of multiple problem solving methods was important because it communicated the plurality of how people see and think mathematically, and that there may be many viable approaches in a group. Additionally, Ms. Munroe leveraged what she learned from the previous study to encourage students in listening, offering ideas, clarifying expectations, and generally looking to the collective as a resource for problem solving (for a more general discussion of teacher support during collaboration see Dekker & Elshout-Mohr, 2004). The other components of the lesson were largely the same (i.e., 20–25 minutes of small group and 10–15 minutes of whole-class discussion), though whole-class discussions were more explicitly focused on students describing multiple problem solving approaches.

The lessons were designed around six algebraic tasks that had multiple entry points, solutions, and opportunities for interdependence and individual accountability. The tasks represented what Stein and Smith (2011) describe as having high cognitive demand, which means they required complex and nonalgorithmic thinking; an exploration and understanding of concepts, processes or relationships; self-monitoring of one's own cognitive engagement; and actively analyzing and examining constraints that limit strategies and solutions (p. 16). The tasks were selected because they afforded opportunities to develop and share various solutions, critique and evaluate those solutions, and engage in problem-solving activities like patterning, working with a smaller case, and so on. The tasks came from published curricular resources—for example, Interactive Mathematics Project, College Preparatory Mathematics, and Mathematics Education Collaborative patterning tasks (see Parker, 2009). Ms. Munroe requested that some tasks be done in pairs and some in four-person groups so she could later assess how participation may vary by group size. As will be discussed in in the data analysis section, we selected the two tasks (*Handshake* and *Cowpens*) that were implemented in four-person groups.

Handshake problem

This task required the group to represent their reasoning in words or images. The underlying concepts were permutations, graph theory, and discrete mathematics. The group must consider if the number of handshakes at a party is equal to half the number of people (i.e., 100 people, 50 handshakes). If true, how many handshakes for 86 people? If false, they must find a strategy to determine the number of handshakes for 86 people using pictures or words. We anticipated students would agree that the proposition is false and develop a strategy for 86 people based on smaller cases (e.g., 2 people, 3 people). The students were likely to draw or write a strategy that could be represented functionally as $H(N) = (N-1) + (N-2) + \ldots + 1$, which reflects the sum of handshakes (H) as a function of people (N). Figure 2 visually represents this solution beginning with N = 2 people.

In Figure 2, each person is a circle and each unique handshake is an arrow. The total handshakes is represented by $H(N) = \sum_{N-1}^{1} N or H(N) = \frac{1}{2}N(N-1)$. Note, students were not given these representations and were not expected to use algebraic notation or find a numeric answer to case 86.

Cowpens problem

Solving this task hinged on how students deconstructed the given visual representation in order to predict its growth (see Figure 3). In the figure that follows, grey squares are fencing units and white

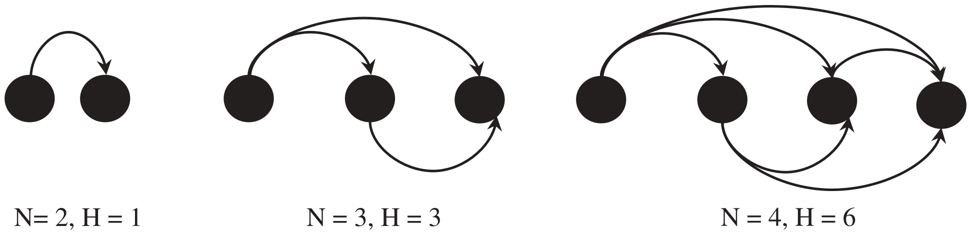

Figure 2. A deconstruction of the Handshake task and corresponding algebraic function.

squares are cows. The group must determine a relationship between the fencing units and cows for increasing numbers of cows (up to C cows). Students were expected to explain the growth pattern and to represent it algebraically. This afforded opportunities to develop multiplicative thinking, algebraic reasoning, and preliminary knowledge of variables and generalizations (English & Warren, 1998).

While there are multiple ways to deconstruct the images in Figure 3, Figure 4 represents the problem solving approach students used in this analysis.

Figure 4 connects the visual representations and terms in the resulting algebraic function $F(C) = 2C + 6$. In this form, $2C$ represents the horizontal top and bottom fencing (not inclusive of corners), while 6 represents the two three-unit long vertical sides of fencing. Other forms of the solution include, for example, $F(C) = 2(C + 2) + 2$.

Data sources

These data draw on video from two of the six lessons. With 55-minute classes and four focal groups, this resulted in approximately eight hours of video.

Data analysis

The goal of this analysis was to progress the idea of perseverance as collective enterprise through empirical illustrations that delineate the boundaries of the phenomenon, and show how it unfolds in

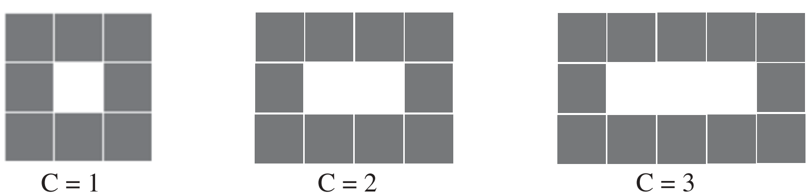

Figure 3. Visual representation given to students for Cowpens.

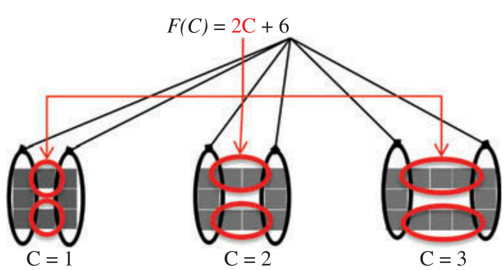

Figure 4. A deconstruction of Cowpens and corresponding algebraic function.

relation to peers, the task, or teacher support. Our analyses generally followed what Powell, Francisco, and Maher (2003) articulated as seven phases of video analysis.

The first phase of video analysis is *attentive viewing*. Both authors and two undergraduate research assistants watched videos of problem solving across the six tasks (~24 hours) to familiarize ourselves with the overall content. The second phase (*describing the video*) yielded 24 video content logs (Derry et al., 2010) that provided a time-stamped narrative of what transpired mathematically and interpersonally for each video. We then chose to focus on four-person group configurations—the maximum capacity of a "collective" in this particular study—as the unit of analysis (i.e., as the size of "collective" to focus on). This led to a purposive sample of eight of the 24 hours of video, which reflected all opportunities for group-based problem solving.

The third phase of video analysis involved identifying what are described as *critical events* (Maher, 2002; Maher & Martino, 1996). In reflecting on 14 years of studying a cohort of students working in groups, Maher (2002) described critical events as a means of tracing the development of ideas. As Maher described (2002, p. 7), the chronology of critical events defines a timeline of the present in relation to the past and future, which serves as the fodder for analysis. Here, we leverage this idea of critical events, where our critical events reflect interactions that evidence productive struggle. Reorienting our analyses from the chronology of peer interactions generally, to the chronology of critical events specifically, led us to create what we refer to as *struggle logs*.

Struggle logs involve systematically chronicling when a struggle emerged, notes on what led to the struggle, and a logging of the talk and activity that preceded and followed each episode. This novel approach to video logging allowed the authors to discuss and agree on the types of struggles encountered in the videos (i.e., unnecessary, productive), and to identify possible patterns in what talk and actions specifically indicated productive struggle. First working independently to categorize the kinds of indicators of productive struggle and then together, the authors agreed on five indicators of productive struggle in the logs:

(1) Conflict in declared solution or strategy
(2) Declaration of uncertainty about solution or strategy
(3) Declaration of inelegant or inefficient strategy
(4) Clarification of task expectations or features
(5) Seeking expert support (TA or teacher)

These indicators are somewhat related to the coding scheme Goos and colleagues (2002) used to describe the conversational moves occurring in their groups. Specifically, their coding scheme had three elements: *self-disclosure*, which was self-oriented talk to clarify one's own thinking; *feedback request*, which were self-oriented questions inviting critique; and *other-monitoring*, which was other-oriented talk to engage with peers' thinking (p. 199). Our first and fourth indicators may be described as *other-monitoring*, where students declare their solutions or seek clarification on the task with one another as a precursor to engaging with each other's thinking. Our second and third indicators align to feedback request, where declaring uncertainty or inefficiency invites others to modify one's thinking toward a more generative possibility. We reserve further discussion of our fifth and final indicator for later in this section.

We argue this preliminary list of *in situ* indicators is an analytic contribution of the study. We see this list as moving in the direction of assisting teachers or researchers in anticipating productive struggle, which in turn provides opportunities to support or advance students developing this capacity together. In Table 1, to further elaborate this analytic effort, we provide example struggle log entries for the four primary indicators.

Although at least one of these indicators preceded every episode of perseverance in problem solving identified in the videos, there were moments when the indicator was not followed by perseverance in problem solving. Consider for example, declared conflict in solutions. In some cases, students declared two solutions and then began explaining, modeling, justifying, and critiquing each other's reasoning. This would evidence productive struggle. At other times, students would agree to disagree and wait for the

Table 1. Example struggle log entries organized by indicators.

Group: Task	Time	Talk/Act	Description
1: Conflict in Declared Solutions			
Group 3: Cowpens	14:00	"We got 20." "…it's 26."	Eve and Layney (but mostly Eve) seem to have arrived at 26 by adding 14 to 12. … Xavier has drawn the case of 7 and literally counted the units of fencing to yield 20. [Eve and Xavier call out their (different) solutions and realizing the mismatch, begin convincing each other of their positions].
Group 1: Cowpens	20:00	"Neil, it's 116." "No it's not. How much do you want to bet?"	The build up to this moment includes Neil hearing that the other pair is working on 103 cows. He calls out the answer as 212 and at first Sage disagrees (even though she's not done). [Neil then challenges Sage asking, "How much do you want to bet?" after which they work on convincing each other.]
2: Declaration of Uncertainty About Solution or Strategy			
Group 4: Handshake	18:00	"So I'm confused, does it work now?"	The group's initial reaction was that the proposition works. But then the TA comes and essentially gives away that for three people there are three handshakes by doing it with them using his hands. Aisha concludes, "so half doesn't work!" This question comes up shortly thereafter when Destiny asks to clarify what just happened. … Aisha says it doesn't work if it's an odd number. … Destiny asks again, "So *does* it work now?"
Group 3: Cowpens	16:00	"Wait, how did we do the first one again?"	As the girls go on to 32 cows, Eve tries to apply what they know from the case of 7 cows to the case of 32 cows. But in so doing, the fragility of her understanding of case 7 emerges again. She starts by asking how many times 7 goes into 32 and Layney starts to answer noting that there will be remainder. Eve then asks, "Wait, how did we do the first one again?"
3: Declaration of Inelegant or Inefficient Strategy			
Group 1: Cowpens	16:00	"Girls, that's a lot of C's for me to count."	Sage and Paola are drawing out each of the 32 cows. When Teacher comes, Sage excitedly tells her that they found a pattern. Both girls explain the strategy to the teacher. The teacher sees their drawing and comments that's a lot of "Cs" to count. This prompts the girls to think of a strategy that applies to any given number of cows without having to draw out the Cowpens and number of cows.
Group 2: Handshake	19:40	"There's always an easier way."	Conrad and Nathan have completed the calculation of 85 x 86 accurately but do not know if that's the same as the sum of 85 to 1. Conrad's intuitive sense of it, however, is that that is far too many handshakes. Nathan agrees and they both laugh a bit about this. Then, Nathan says, "I think we're going to have to do it the long way [add 1 to 85 like the girls]." Conrad, reluctant to do it says, "There's always an easier way." "There's always an easier way?" Nathan repeats/asks.
4:: Clarifying Task Features Or Expectations			
Group 3: Handshake	14:40	"That's 6 in total, handshakes." "No it's not!"	The students are considering the case of three guests and Mason concludes it is 6 handshakes (he's counting each shake as two—two hand*s* shaking). Then Layney says it's 3 handshakes. This causes Eve and Layney to rise as the teacher says "try it, try it." In acting it out Mason agrees it's three shakes but he says the shake between Eve and Layney is "two in total, one for you and one for her" meaning he's counting it twice. This causes Layney to doubt herself and the teacher says to let Layney 'finish acting it out.' … Eve then gestures to the worksheet, reads it, and says, "It says, this counts as one handshake."
Group 2: Cowpens	17:46	"The cows are not blocks." "I know but the cows are inside the blocks."	Jillian has drawn the case of 7 and is counting the cows in her total. Heather sees this as she looks over her shoulder and says "you're not suppose to count the cows, the cows are not blocks." Jillian replies that that's true but they are inside of the blocks (this was clarified at the start of class when a student asked the same question).

teacher to intervene, so there was no productive struggle. Once again, the individual-with-context perspective anticipates this possibility. How one's peers react to a disagreement or whether or not the group has the (physical or conceptual) tools to resolve the disagreement, affords or constrains perseverance as collective enterprise. The fifth indicator, seeking expert support, was the most inconsistent in identifying productive struggle. As one can imagine, students seek out the teacher or TA for a variety of reasons. And, if it was in relation to productive struggle, it often co-occurred with another indicator (e.g., telling the teacher they cannot determine which solution is correct—indicator two, above).

At this point our analytic approach somewhat departed from what Powell and colleagues (2003) described as the remaining phases of analysis (i.e., transcription, coding, creating a storyline). Instead, we next selected particular cases for transcription that would demarcate the boundaries of perseverance as collective enterprise. There was no additional coding, although the extended case analysis that appears in the next section approximates what the Powell and colleagues called "creating a storyline" of perseverance as collective enterprise.

Results and analysis

Investigating an expanded notion of perseverance—perseverance as collective enterprise—led us to delineate three distinct phenomena in the data that are presented here in two sections. Recalling Figure 1, the first of these phenomena reflects problem solving in which productive struggle fails to emerge. We present this in Cases 1 and 2. The second of these phenomena, presented in Case 3, reflects critical events where problem solving *and* productive struggle occur, but no collaborative ZPD emerges. The third and final of these phenomena (Cases 4 and 5) reflect problem solving *and* productive struggle, *and* where a collaborative ZPD emerges (i.e., perseverance as collective enterprise). As would be expected, in presenting these varied empirical examples we pay greatest analytic attention to these latter cases. To summarize:

- Section I: What is Not Perseverance (as Collective Enterprise)?
  - Case 1: Problem solving but *not* persevering because of agreement on incorrect answer
  - Case 2: Problem solving but *not* persevering because solution is immediately obvious
  - Case 3: Perseverance in problem solving but *not* as collective enterprise
- Section II: What is Perseverance as Collective Enterprise?
  - Case 4: Perseverance in problem solving as collective enterprise (brief)
  - Case 5: Perseverance in problem solving as collective enterprise (extended)

Section I: What is not perseverance (as collective enterprise)?

Case 1 (<1 Min)

This is a case of problem solving but not perseverance because the group agrees on an *incorrect* relationship, which stops them from grappling further. Group 1 is discussing the first part of the handshake problem in which they must evaluate the given strategy: number of handshakes in a party is half the number of guests. Sage[3] proposes, "If every person shook everyone's hand it would be 99 shakes for that person. … So that means 99 times 100." Momentarily, Neil considers the possibility of it being 99 times 99 but Sage clarifies by drawing a line and gesturing to points on the line saying, "No, 100 people [the line]. Ninety nine shakes per each [the points]." This representation leads the group to think the solution is therefore the product of 100 and 99, and they quickly reach consensus on 9900 handshakes without further discussion. This case shows how agreeing to the incorrect solution forecloses on grappling with the mathematical relationship between handshakes and consecutive guests (i.e., one less handshake for each). Without reason to reconsider (i.e., no peer or teacher identifying their mistake), the group's problem solving ceases and no struggle emerges.

[3]All proper names are pseudonyms.

Case 2 (<1 Min)

This is a case of problem solving but not perseverance because the group quickly determines an accurate answer and fails to struggle. The group first agreed that the number of handshakes is not half the number of guests because the first of 100 people shakes 99 hands, which is more than half. This laid a conceptual foundation for the group to now grapple with the case of 86 guests. Jillian began by describing what the first of 86 guests does: "Okay, one person shakes (*gestures handshake*), 'Hi!' for 85 people. Then, the next person." Conrad then continues, "Yeah, 84 people," extending the logic to the next guest. Jillian continues, "Then the next person shakes hands with 83 people." Nathan listens and then says, "Subtract one each time!" and they smile triumphantly at one another. Conrad and Heather listen and agree such that the group quickly (within a minute) reaches consensus on (85 + 84+…+2 + 1) as the answer. Thus the group completes the task with no struggle and hence, no perseverance.

Case 3 (<1 Min)

This is a case of perseverance but not as collective enterprise. Here, Group 1 is deliberating the first part of the handshake problem. Conrad says the proposition is false, arguing, "If there's 100 people in a party, and then I have to shake 99 people, that's already more than 50." As Conrad offers his reasoning and explanation, Jillian looks to investigate with a smaller case. She settles on 26 guests so that each guest can be represented by a letter, and the handshakes would be represented by coupled letters (e.g., AB, AC … XZ, YZ). As Conrad beseeches Jillian to listen, Nathan agrees the proposition is false because "If you have 17 people … you can't half a handshake." Jillian and Heather, in contrast, continue to model and count the handshakes as coupled letters of the alphabet. As they do so, the students engage in the following exchanges:

10:37	1	Conrad	You don't have to do that. Jillian, listen to my simple method of figuring it out. You like to draw. It's a bad idea.
	2	Nathan	Oh my god! It's very simple.
	3	Conrad	So say there's 100 people. I have to shake 99 people already. So it's wrong.
	4	Nathan	[*To Jillian and Heather*:] Yeah, because 99 people is way more than 50.
	5	Jillian	[*Smiling at Nathan*] Okay, but now—*(Looks at her notebook again)*
	6	Nathan	—And even if there's 17 people, you can't half a handshake.
	7	Jillian	Can I see the paper? I want to see the paper. [*Holding task she reads*:] Evaluate the strategy—
	8	Conrad	[*To Nathan*:] I'll still have to shake 16 people, and another person has to shake more people.
	9	Nathan	[*To Conrad*:] I know. (*Nodding his head*)
11:05	10	Jillian	Okay, A, B. (*Continue forming coupled letters to represent handshakes*)

Conrad and Nathan directly challenge Jillian's pursuit of the alphabet strategy and judge her preference to couple letters a bad choice (Line 1). Nathan and Conrad's efforts to dissuade Jillian (and Heather) leads them to connect their mathematical justifications (Lines 4, 6, 8). Jillian and Heather continue as the others question and critique them (e.g., "You don't have to do everybody. How many letters are you doing, Jillian?") and demand they stop (e.g., "Don't you dare make more than three [letters]!"). Eventually, Conrad belittles, "This is why they're so slow Nathan, and we're so fast!"

This is a group (four-person) task in which we see Jillian and Heather engaged in productive struggle but not Nathan and Conrad. The group is constrained from persevering by peer dynamics. Nathan and Conrad mock Heather and Jillian rather than, for example, attempting the alphabet method and finding that more than 13 pairs of letters makes the proposition false. Similarly, Jillian and Heather seem unwilling to engage with the others' explanations. Thus, at the level of the group, the students fail to evidence perseverance as collective enterprise.

Summary of section I

In this section we presented three cases that disrupt the idea that whenever groups problem solve to resolution they have *de facto* persevered. Whether having reached consensus on the wrong or right solution, or because peer dynamics prevent a nonadversarial exchange of ideas, these cases all fail to evidence perseverance as collective enterprise for the group task.

Section II: What is perseverance as collective enterprise?

These two cases evidence perseverance as collective enterprise. Case 4 is relatively brief (approximately 3 minutes) while Case 5 transpires over approximately 10 minutes. Providing two cases of varying lengths reasserts the idea that perseverance is not defined by time on task but rather whether students are grappling with important mathematical ideas, and how their collective efforts are advanced or constrained in context.

Case 4 (Brief ~3 Minutes)

This is a case of perseverance as collective enterprise that begins with a narrative description of moment-to-moment interactions followed by analysis. Group 3 is grappling with how to prove that the number of handshakes is not half the number of guests. Ms. Munroe suggests the group stand up and investigate. Eve, Layney, and Mason act out the case of 3 guests (which yields 3 handshakes), by which they also clarify what a unique handshake means (see Table 1, Indicator 4, Group 3). Having resolved the proposition is false for odd values of guests, Eve initiates an effort to evaluate the proposition for even cases (i.e., 4, 6, 8).

15:48	1	Eve	No, like if you have six people: one, two, three, four, five, six [*Eve draws circles to represent people*] these ones shake hands, these ones shakes hands, these ones shakes hands, those ones shakes hands, these ones handshake, ooh, these ones shakes hands, these ones handshake, these ones shakes hands [*she is connecting the circles with lines. Looking up at the teacher*:] It's hard!
	2	Teacher	It is hard. But you have [unintelligible] here (*the teacher makes a circular gesture indicating Eve consider the group and then moves on*)
16:07	3	Mason	And these two, and these two. Oh, and these two also! (*Mason reaches across the table and gestures more connections for the circles*)
	4	Layney	Look, Eve, you forgot a lot of things!
	5	Mason	And then, wait, hold on (*Eve smiles and relinquishes the paper to Mason who picks up the pencil and adds more connections*)
	6	Layney	[*Laughing jovially*:] Let's just say she forgot a lot. [*Leaning toward Mason*:] And then you forgot that guy and, look this guy (*Layney then takes the pencil from Mason and adds more connections*)
	7	Xavier	[*As the three students work on a common paper, Xavier draws circles on his paper and is making similar connectors*]
	8	Mason	Oh, yeah! (*Adds more connections to Layney's work*)
16:25	9	Eve	Oh, wait! Let me see that. (*Eve stands and reaches for the pencil and paper from Layney*)

At this point, Eve realizes they need to change their random approach to tracking handshakes and when she takes the paper, draws six new circles and connects them systematically. Over the next 3 minutes or so the students try a variety of approaches. For example, Xavier continues developing an image on his paper with two rows of three circles, connected vertically and diagonally to indicate ten handshakes among six guests. Eve meanwhile explains to Layney and Mason that she can make a list of paired numbers to represent handshakes between numerals: 12, 13, 14, 15, 16, 23, 24, 25, 26, and so on (this is akin to Heather and Jillian pairing in Case 3). Using that approach, Eve concludes there are 15 handshakes, which prompts Xavier to look up and then return to his work. Xavier adds two more lines connecting the far corners of his representation. When Xavier finishes, he says, "Guys, I got 12. Eve, what did you get?" She answers 15 and comes around the table. As she does so, Layney looks on and Mason begins working on his paper to recreate the solution. The group reaches consensus on the solution by 18:40.

Case 4 analysis

The group evidences perseverance as collective enterprise. They are representing their mathematical ideas with images and bodies (as in the case of 3 guests). They are also monitoring their progress, as when Layney and Mason identify missing handshakes on Eve's representation and later, when Eve starts over to more systematically track handshakes. Although Xavier seems to be working apart from the others, his actions parallel the others. For example, he also uses circles to represent the six guests after watching Eve's initial attempts. He adds additional handshakes when he hears her call out a total. And, he solicits Eve's attention when he is done and together they clarify his understanding. We cannot explain why Xavier remains in his seat while the others do not, although he says to Layney at one point, "I don't like talking across the table." Nonetheless, we interpret Xavier's parallel and corresponding actions as joint problem solving with the others.

There are several contextual factors that afford perseverance as collective enterprise. The teacher encouraging them to act out the case of three guests allowed the group to visualize key features of the task (i.e., each handshake counts once; everyone shakes everyone else's hand but not themselves). This also revealed that the proposed strategy fails for odd cases, leaving them to grapple with even cases. Eve's choice of six guests offered the group a common object of attention that sustained their joint problem solving. After Eve declares, "It's hard!" (Line 1), the teacher's gesture to others returned Eve's attention to her peers. Taking up the opportunity for productive struggle, Layney and Mason monitor and contribute to Eve's work, which advances their problem solving efforts as they get closer to accounting for all of the handshakes. When Eve restarts their efforts with a more systematic (and trustworthy) approach, they reach the solution. Concomitantly, Xavier leverages Eve's approach, uses her utterances as impetus to revisit his work, and by comparing answers later, works with Eve to secure his understanding. Taken together, this case elucidates how constituent factors of the context afforded perseverance as collective enterprise. In fact, Case 4 shows how sometimes the collective can engage in parallel mathematical activity, even critiquing one another (i.e., Eve being told she forgot a lot of handshakes), but by virtue of repeatedly engaging one another, monitoring one another's progress, assisting one another, perseverance as collective enterprise was achieved.

Case 5 (Extended ~10 minutes)

This extended case of perseverance as collective enterprise involves Group 4 and the *Cowpens* task. The group engages in multiple productive struggles, often immediately following the pairs working in parallel before returning to the group to assess their reasoning and monitor their progress. Again, we first present a narrative description of what transpires moment-to-moment in the group, followed by analysis.

The underlying mathematical idea at the heart of this case (and the task) is finding a non-recursive relationship between the number of cows and fencing units to predict any case of cows. The pairs determine two relationships, neither of which proves sufficient (center column, Table 2).

In the opening moments, the students propose, evaluate, and justify an initial conjecture about the pattern's growth. As they speak, they point and count the fencing units in the visual representations and eventually align them to their conjecture.

12:17	1	Destiny	No, cause you add 2 [fences] from 2 cows [to get fences for case 3].
	2	Emilio	One two three—one two three for five six seven eight. One two three four five six seven eight nine ten.
	3	Aisha	One two three four five—you add two.
	4	Emilio	Yeah, you add two to every one.
	5	Aisha	So three is twelve. Right? Yeah, three cows is 12. Four cows will be 14.
	…		
	13	Aisha	[*To Destiny*:] Start from the bottom row, they [the top and bottom rows] each go up by one.
	14	Destiny	Yeah.
13:26	15	Emilio	Yeah that's true.

Table 2. What each student pair initially determines as the mathematical relationship versus the correct function.

Pair	The Pair's Function Relating Fences (F) to Cows (C)	Correct Function
Aisha and Destiny	$F(C) = F(C-1) + 2$	$F(C) = 2C + 6$
Emilio and Daniel	$F(C) = 2C$	

Destiny offers the first conjecture of how the pattern is growing (Line 1), which Aisha repeats (Line 3) and Emilio evaluates and endorses (Line 4). In Line 13, Aisha points to the images and links the agreed-upon verbal conjecture to rows of fencing in the visual representation and shortly thereafter Destiny and Emilio agree (Lines 13, 14, 15). Between their talk and gestures, these exchanges show the group problem solving though they have yet to engage in productive struggle. Moments after the last line, Ms. Munroe comments, “Good sharing, guys,” in response to their joint deliberations.

13:54	27	Aisha	[*Overhearing Emilio discuss his approach*] We’re supposed to be working in our group.
	28	Emilio	Okay but first can we, like, try it out and then we’ll tell you what we’re doing?
	29	Aisha	Yeah okay go.
	30	Destiny	20, 20!
	31	Aisha	Are you sure?
	32	Destiny	I’m sure.
	33	Emilio	[*Daniel whispers numbers consecutively 1 through 7 as Emilio responds*]: 2, 4, 6, 8, 10, 12, 14. 7 cows is 14.
	34	Daniel	Oh, I get it.
	35	Emilio	Right? Cause 7 times 2 is 14
	36	Aisha	[*To Destiny*]: It says here you have to draw [the case of 7 cows].
	37	Daniel	So it’s 64. 64, 206. Got it [*Raises hand to get the teacher*].
	38	Aisha	[*To Daniel*:] What?
	39	Emilio	[*Bangs the table*] We have to explain it to them.
14:41	40	Daniel	Fine. It’s just—you multiply—it says 7 cows.

Having established the fences will “grow by two,” Emilio suggests a slightly different approach to determining the number of fencing units for 7, 32, and 103 cows: He’ll count the case number aloud and Daniel should count up by two’s. In what follows Aisha reminds Emilio they are working as a group, which prompts him to ask permission to pursue his approach, and the two pairs diverge in their efforts.

When problem solving, Emilio forgets that the first case begins with eight units of fence (Line 33) and inaccurately concludes 7 cows yield 14 units of fence (Line 35). This does not remain a simple counting mistake—he then concludes that the fencing units are *double the number of cows* so that 32 and 103 cows yield 64 and 206 fencing units, respectively (Line 34). Meanwhile, Destiny and Aisha accurately determine 7 cows yield 20 fencing units (Line 30). The conflict in solutions was related to problem solving apart—ostensibly, had Emilio and Daniel written out each successive number of fences as Destiny had, or had they been reminded by Aisha to draw case 7, they would have reached

14:54	42	Aisha	So what’s your answer?
	43	Daniel	14.
	44	Aisha	[*Hand on forehead*] That is so wrong!
	45	Daniel	So what did you get?
	46	Destiny	Okay so it’s one two three four five six seven eight. Eight, you start at eight [fencing units for case 1].
	47	Aisha	If all of these go by two, [case] four would have fourteen.
	48	Daniel	I have no idea what you’re saying.
15:29	49	Destiny	This is twelve [*gestures total fencing for case 3*]. This is fourteen [*gestures total fencing*]. This is four [cows: *gestures to same image*].

the same conclusion. Rather than remain apart, however, the students know they are accountable to the collective throughout (Line 27, 28, 39) and following these exchanges, return to the group.

Together again, the group must grapple with two different solutions that represent two different relationships between cows and fences.

These exchanges demonstrate joint problem solving and productive struggle. Daniel, in declaring he does not understand what is being said, is grappling with a mathematical idea beyond his current thinking. Destiny and Aisha offer a methodic and reasoned explanation and justification (Lines 46, 47, 48). Emilio, moments after these exchanges, asks a TA, “Which one is right? Twenty or fourteen?” thereby evidencing he too is involved in the productive struggle. Before the TA responds, Destiny interjects with another effort to explain. Pointing to case 3 (the last given image) she says:

"Twenty. Oh my gosh! 'Cause it's 12 here. … It's right here. Four cows would *have to* be fourteen." At that point, Daniel concedes, "Dude, it's 20." His tone, gaze, and use of "Dude," strongly suggest Destiny convinced him and he is now joining her effort to convince Emilio.

Emilio remains unconvinced and instead, follows the TA's suggestion to draw case 7 as Daniel looks on. Destiny and Aisha pursue cases 32 and 103 using the "grow by two" rule. In what follows,

16:23	68	Aisha	Thirty-two cows. Okay, so at seven it's twenty.
	69	Destiny	Okay, can you count that by two?
	70	Aisha	Two, four. Okay let me tally. So twenty twenty-two. How about you tally. No, no, no, I'll tally, you count.
17:10	71	Destiny	22, 24, 26, 28, 30, 32, 34, 36, 38, 40, 42, 44, 46, 48, 50, 52, 54, 56, 58, 60, 62, 64, 66, 68. Okay, I think that's enough.
	…		
	73	Aisha	24, 25, 26, 27, 28, 29, 30, 31, 32. No, this is 32.
	74	Destiny	Then why did we put 7?
	75	Aisha	7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31. This is 68.
	76	Destiny	So 70. But somehow that seems wrong.
	…		
	79	Aisha	70. Let's just keep it for now. 70.
	80	Destiny	[*Looking ahead to 103 cows*] I don't want to count all the way.
17:57	81	Aisha	We're not gonna count 'til 100. We need a new strategy for that.

they persist in productive struggle, and indicate that by declaring uncertainty in their solution (Table 1, Indicator 2) and then identifying the limits of a recursive rule for large numbers of cows (Table 1, Indicator 3).

Because they are using a recursive function, Aisha and Destiny must move sequentially from the last case they solved up to case 32. Through tallying and counting they arrive at 70 fencing units but Destiny declares uncertainty about the solution ("But somehow that seems wrong") to which Aisha suggests they keep the answer as it is (Line 79). Soon thereafter they both admit their strategy is inefficient as Destiny declares she won't continue in this fashion for 103 cows and Aisha responds with "We need a new strategy for that" (Lines 80, 81). These turns of talk indicate a productive struggle to find a non-recursive relationship between cows and fences; to find a solution that remains beyond their immediate reach.

The next set of interactions begins when the teacher checks in. When the teacher comes to the table, Emilio explains, "We're still on number one," (i.e., 7 cows). Ms. Munroe suggests Aisha talk to Emilio and Daniel about case 32. Aisha explains to Emilio how she counted on from 7 cows and suggests, "Let's count together." Emilio, unconvinced and unable to track how Aisha builds from 7 cows, says repeatedly, "I'm sorry but I don't get that." Meanwhile, Destiny continues to express doubt that their solution of 70 fencing units for 32 cows is correct. Between Emilio being unconvinced and Aisha and Destiny's wavering certainty, the students are engaged in productive struggle: How can we be convinced of the mathematical relationship between cows and fences?

For a few moments the group appears stalled and unable to persist. This changes through continued discussion. It begins when Daniel leverages Aisha's explanation of 70 fencing units for 32 cows. At first, Daniel seeks only to confirm her answer by asking, "Wait, you got 70?" Aisha, thinking Daniel is merely writing down her answer accuses him of copying and says, "You have to figure out how we got it." In keeping with previous interactions, Aisha holds Daniel accountable to the norms of joint problem solving. When Daniel then claims he understands, Aisha rebukes, "You have to write it down!" What Aisha does not realize is that Daniel has found the correct function. What he has gleaned from his own drawings and now the group conversation is: (1) 7 cows yield 20 fencing units; (2) 32 cows yield 70 fencing units. Recalling the inaccurate conjecture of doubling cows to determine fences, Daniel determines the correct function by manipulating the below information on his paper, and focusing on the last two columns:

Cows	If Cows Were Doubled	Fencing
7	14	20
32	64	70

As he continues working on his paper, all three students are gazing intently at his writing and press him on his thinking. Emilio asks, "Where'd you get the six from?" "Yeah, what?" Destiny asks. "You can't just find a six," Aisha cautions. Looking at his paper and then the group, he explains, "You times 7 by 2 and you put 6, it's 20. Times 32 by 2 plus 6 [and it] is 70." Destiny then says, "Okay, then we did get the correct answer," to which Daniel says excitedly, "Yeah, yeah, you guys did!" These exchanges are important regarding collectivity. Destiny and Aisha determined a solution for 32 cows that they cannot confirm independently; Daniel uses their solution *as though* confirmed, which leads him to deduce the correct function; Destiny takes his deduction to be independent confirmation of her answer; and, Daniel confirms that she was right all along. By deducing the function F = 2C + 6 from what he overhears in the collective and what he and Emilio initially conjectured (F = 2C), Daniel effectively ends their problem solving efforts as a nonrecursive relationship between cows and fences emerges. Together the students then agree that the case of 103 yields 212 fencing units and the task is complete. However, not having grappled with the relationship visually, and Aisha not being fully convinced of why the function works, has two immediate consequences for the group moving forward.

20:41	122	Teacher	So what did you do, darling?
	123	Daniel	So first 7 times 2 plus 6.
	124	Teacher	What's the 6? Plus 6?
	125	Daniel	I don't know. We just got it.
	126	Teacher	You just got 6 out of the air? Shoop, I'm gonna add six? (*gestures to grabbing something out of the air*)
	…		
	129	Daniel	And then so we noticed that if we times 7 cows plus 2 equals 14 plus 6, it's 20. If you times 32 by 2, it equals 64 plus 6 is 70. So 103—
21:25	130	Teacher	But do you know what 6 is representing? I do, but I want to know if you do. The pattern works.

First, soon after Daniel declares his solution, he excitedly calls Ms. Munroe to show her his work, and a new productive struggle emerges: What is the relationship between the numeric pattern and the visual representation?

As this conversation reveals, Daniel deduced the correct mathematical relationship by having "noticed" (Line 129) a numeric pattern but cannot yet connect it to the visual representations. Without that connection, the task remains unaccomplished. When Ms. Munroe poignantly asks, "But do you know what six is representing?" she clarifies a new productive struggle for them to deliberate further. As she walked away, Emilio called out, "The blocks!" and then high fives Daniel shouting, "Yes!" A few minutes later and just before the whole-class discussion, Daniel whispers to Emilio, "I don't know how you found that out [what 6 represents]!" Emilio's response shows how he relied on the collective:

> Dude, because I saw three plus three [is] six. Because you know how Aisha said those three don't change? So I added them up. And when [Ms. Munroe] said that you grabbed six out of the air, I was like, they are the blocks on the side!

From the beginning, Aisha and Destiny have been problem solving through the representation—pointing and counting to images until the case of 32. During those deliberations, Emilio heard Aisha comment on the "six blocks" (referring to the constant three-block columns). Here, in grappling with how to connect the numeric relationship to the visual representation, he invokes what he learned in the collective and uses it to advance their understanding. This is to say, Emilio's recollection of Aisha connecting the visual representation to numeric values was instrumental to grappling with the relationship between the image and his function. As

Emilio explains where the six comes from to Daniel, it is unclear if Aisha and Destiny also understand. However, when the group embarks on the subsequent task as pairs, Aisha and Destiny easily leverage Emilio's idea toward a more challenging (and related) task.

Summary of section II

In this section we provided two illustrative cases of perseverance as collective enterprise. Both cases evidence students grappling with mathematical ideas or relationships that are within reach but as yet unformulated. Moreover, both cases evidence how interdependently perseverance was experienced by individuals (e.g., Xavier), pairs (e.g., Aisha and Destiny), triads (e.g., Eve, Layney, Mason) and, as was operationalized for "collectivity" in this analysis, the four-person group. In both cases students were working in their zone of proximal development, while simultaneously shaping a collaborative zone of proximal development. They were formulating, evaluating, and leveraging one another's mathematical reasoning when engaging in productive struggle over time. To note, despite problem solving apart from the collective at times, in both groups students reconvene, discuss their thinking and progress, and find new paths forward as a collective (see Figure 5). In this way, the students variedly positioned as more or less capable peers in relation to one another and required only modest support from the TA or teacher. Although modest, these more expert interventions helped sustain the groups in productive struggle—whether suggesting they draw out a case, embody the handshakes, or look to peers as resources. With discussion and the freedom to pursue their own paths of reasoning with concurrent support and accountability to the group, both cases demonstrate perseverance as collective enterprise.

Discussion

National mathematics frameworks expect children to meet problem solving challenges with exceptional resolve individually and collectively (NCTM, 2014), and assert this as one aspect of children developing positive dispositions toward the discipline (National Research Council, & Mathematics Learning Study Committee, 2001). Such frameworks put the onus on teachers to create opportunities for children to productively struggle (NCTM, 2014). Such policies, we argue, require greater description and detail to translate from page to practice. This analysis offers such description and detail by expanding current conceptualizations of perseverance as individual capacity to perseverance as collective enterprise. We presented five cases of children solving non-routine algebraic problems in order to empirically illustrate what we mean by perseverance as collective enterprise. In this discussion, we address the significance of those cases and how this conceptualization provides a new vantage point from which to think about the organization and design of supportive learning environments in mathematics.

Our expansion of perseverance from individual capacity to collective enterprise recruited ideas of productive struggle (Hiebert & Grouws, 2007) and coordinated them with the Vygotsky-inspired notion of collaborative ZPD (Goos et al., 2002). Our video analyses followed from this conceptual coordination (Figure 1) and centered on what Maher (2002) referred to as critical events—episodes of productive struggle within each group and over time. Thus a major contribution of this work are the conceptual and methodological approaches taken in creating struggle logs and subsequently, articulating *in situ* indicators of productive struggle (Table 1). Like teachers, researchers of collaborative learning need to define what counts as productive struggle and when those instances reflect perseverance as collective enterprise. The conceptual framework, analytic methods, and resulting indicators represent important elements in that effort.

The results of our analysis reveal an important relationship between productive struggle and perseverance as collective enterprise. In Cases 1 and 2 (Section I) we show how determining a solution (right or wrong) forecloses on productive struggle *and therefore* on perseverance as collective enterprise (i.e., no struggle, no perseverance). We also show the reverse is not true: productive struggle will not necessarily lead to perseverance as collective enterprise (Case 3, to be

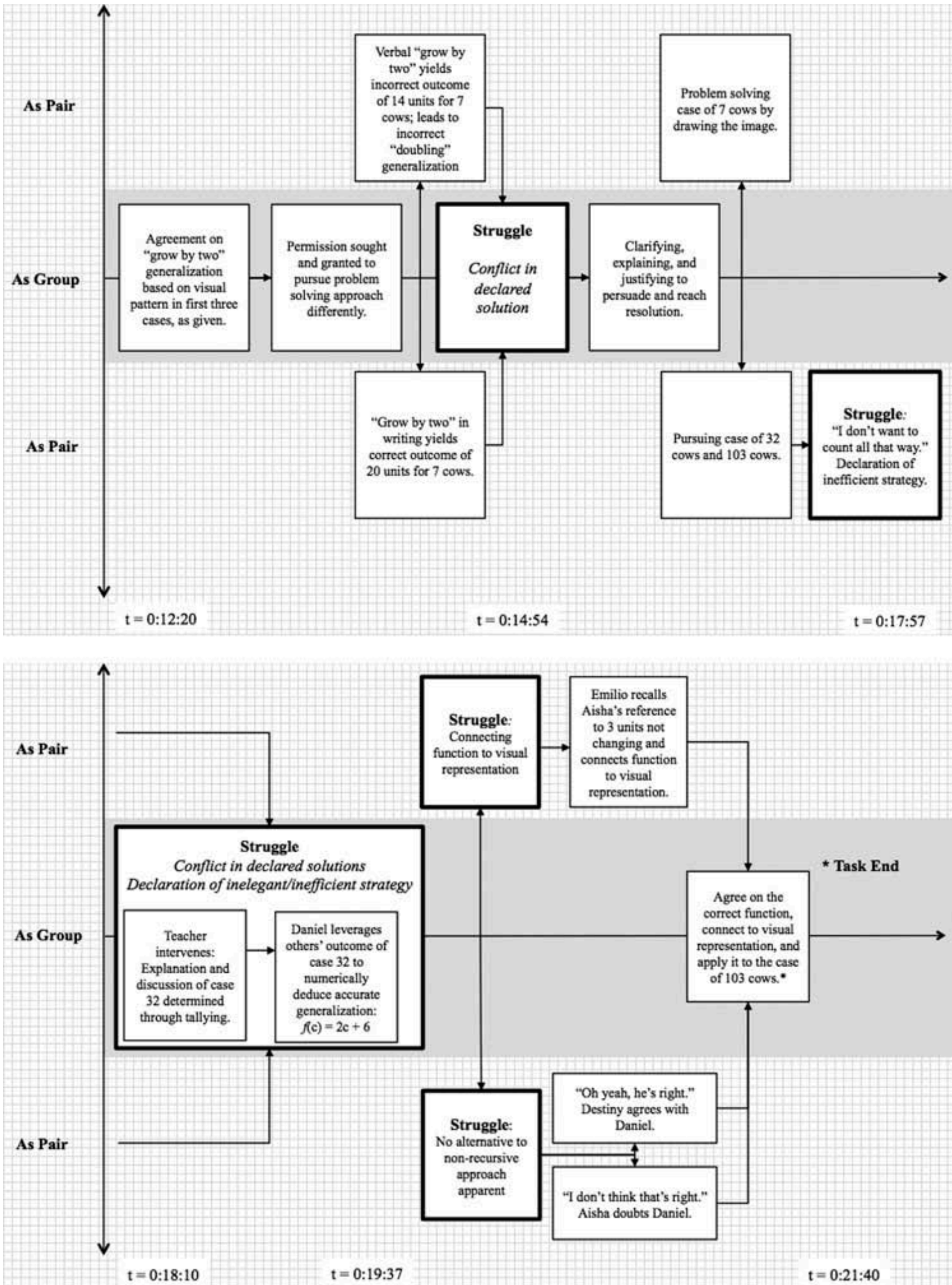

Figure 5. Case 5 timeline of productive struggle with *in situ* indicators.

discussed shortly). This opens the door to a new empirical question: Why wouldn't productive struggle lead to persevering collectively?

Among the reasons why perseverance as collective enterprise fails to emerge during children's collaborations, our data speak to the significance of peer dynamics on learning and collectivity (Barron, 2003; Cohen & Lotan, 2004, 2014; Kotsopoulos, 2010; Langer-Osuna, 2011). Using an individual-with-context perspective on learning (Gresalfi, 2009), the data show how peer dynamics both constrained and advanced children in persevering. For example, Conrad and Nathan's critique of Jillian and Heather's alphabet method (Case 3) foreclosed on persevering as a collective, which meant the students could not consider diverse perspectives in problem solving to gain a deeper understanding of the material. When Conrad refers to Heather and Jillian's legitimate (but perhaps inefficient) problem solving approach as why they are "so slow," its negative connotation (see Horn, 2007; Schoenfeld, 1992; Stevenson & Stigler, 1992) ruptured the good will necessary for persevering together. In contrast, Mason and Layney critiquing Eve's problem solving method (Case 4) led to the pencil and paper being passed about as they collectively attempted other approaches. Moreover, in this latter case, Ms. Munroe suggested Eve look to her peers for support, which reaffirms the criticality of teachers' roles in promoting perseverance as collective enterprise (to be discussed next). Thus, we identify how contextual factors necessarily complicate the relationship between productive struggle and perseverance as collective enterprise; a relationship that is perhaps more straightforward (or even irrelevant) when perseverance is conceived of as a trait or capacity of individuals.

Perseverance as collective enterprise represents a new vantage point for teachers creating norms and procedures that sustain equitable learning communities (Cohen, Brody, & Sapon-Shevin, 2004; Cohen & Lotan, 2014; Jansen, 2012; Megowan-Romanowicz, Middleton, Ganesh, & Joanou, 2013; Nasir, Cabana, Shreve, Woodbury, & Louie, 2014; Yackel & Cobb, 1996). For example, in Case 5, Aisha, Emilio, and Daniel all spoke or behaved in ways that indicated they were accountable to the group. The shared norm of accountability facilitated perseverance as collective enterprise, as the students intermittently posed their struggles to one another (see Figure 5). Ms. Munroe explicitly referenced this norm when praising the group for their collectivity ("Good sharing, guys"). Prior to the study, Ms. Munroe participated in professional development on Cognitively Guided Instruction (see Carpenter, Fennema, Franke, Levi, & Empson, 1999) and was accustomed to monitoring individual student thinking during small group discussions. In this study, she was taking on the idea of student-driven inquiry and learning to focus on the group as a unit of analysis (see Sengupta-Irving et al., 2013). In turn, her efforts to have children listen to one another's thinking at moments of productive struggle proved consequential to supporting perseverance as collective enterprise. Whether responding to Eve's declaration, "It's hard," by gesturing her peers as resources, or walking away from Emilio to deliberate on where the six in his equation came from, or asking Aisha and Destiny to engage their peers in a discussion of case 32, Ms. Munroe's active support of *collectively* persevering was significant. In contrast to Ms. Munroe's approach, there were multiple occasions when the TAs inserted themselves in ways that foreclosed on aspects of productive struggle. For example, on one occasion (see Table 1, Indicator 2, Group 4), the TA gave away that three guests yield three handshakes by using his own body to model the task. In contrast, when Ms. Munroe inserted herself into the same group for the same task, she encouraged the students to embody the task for themselves. When they finished she simply noted, "This is a strategy," thereby ratifying their joint activity as problem solving. Those cases in which Ms. Munroe inserted herself (i.e., Cases 4 and 5) we see evidence of perseverance as collective enterprise while those in which she is absent (Cases 1, 2, 3), we do not. This not only affirms the assertion of policy documents that teachers create opportunities for productive struggle (NCTM, 2014), but also suggests teachers are instrumental in supporting and sustaining perseverance as collective enterprise in the face of those struggles.

Conceptualizing perseverance as collective enterprise invites teachers and researchers to think of perseverance as an object of design, including tasks. The tasks used in this study (*Handshakes* and *Cowpens*) tilt toward opportunities for productive struggle because they were conceptual in nature, of high cognitive demand (Stein & Smith, 2011) and group worthy—each task had multiple entry points, solution pathways, and opportunities for interdependence and individual accountability (Lotan, 2003; also see Bass & Ball, 2015; for discussion of perseverance using the Trains & Ways task). We see this vividly in Cases 4 and 5. In Case 4, for example, the group moving fluidly from one representation to the next (e.g., random circles, aligned circles, lists of numbers) propels them forward in systematically solving the number of handshakes for six guests. In Case 5, the initial entry point of drawing out cases allowed Aisha and Destiny to make progress on the task while Emilio and

Daniel found traction by determining a function to describe the number pattern. When a productive struggle seized them all (i.e., the case of 103 cows), it was the interdependence of these approaches that led Emilio to find a resolution by blending Aisha's drawings and Daniel's algebraic function. Just as the shape of Figure 5 suggests (apart, together in struggle, apart), struggling collectively propelled the group forward in problem solving. Thus, the vantage point of perseverance as collective enterprise renews the importance of designing "interesting problems" (Barron, 2000) that necessarily engage children in mathematical practices—from making sense of problems, to modeling and reasoning about them, to monitoring or changing directions in their problem solving approach (Boaler & Sengupta-Irving, 2016; CCSI, 2010; Parker, 2009; Sengupta-Irving, 2016; Sengupta-Irving & Enyedy, 2014; Simon & Tzur, 2004).

Ultimately, like others, this analysis places a premium on the generative and significant work that children can accomplish together (Barron, 2003; Boaler, 2008; Cohen & Lotan, 2014; CCSI, 2010; Langer-Osuna, 2016; National Research Council, & Mathematics Learning Study Committee, 2001; Sengupta-Irving, 2014; Sfard & Kieran, 2001). Rather than ascribe meritorious traits like "persistent" (or "smart" or "fast") to individuals, we investigate how such an ascription could also describe a way of learning together. What resulted is an empirical dialogue over how to describe this phenomenon (and not individuals), how to anticipate its emergence *in situ* (i.e., indicators of productive struggle), and how to design for it (through tasks). Conceptualizing perseverance as collective enterprise adds an important dimension to the toolkit of ways we design for, describe, and assess supportive mathematics learning environments for all.

Conclusion

The nineteenth century short story, *The Seven Sticks* (Lindberg & McGuffey, 1879/1976), tells the tale of a man whose seven sons quarreled incessantly. Distressed by their discord, the man gives his sons a bundle of seven sticks, which he challenges them to break. After several attempts the sons give up. The man then unties the bundle and breaks the sticks individually. The moral of this story—that they are stronger together—persists today through popular aphorisms like "the sum is greater than its parts" or "two heads are better than one." Yet, this simple lesson is a difficult one to teach in an era of academic accountability that equates excellence with individual performance. Policies and practices that shift our national preoccupation with individual excellence toward also prizing what students can also accomplish together, represents an important opportunity to rebalance our educational priorities. Perseverance as collective enterprise capitalizes on this opportunity to build stronger and more supportive classroom communities—communities that will prove themselves unbreakable in the midst of great struggle. Indeed, conceptualizing perseverance as collective enterprise moves us closer to the ideal of equitable and effective mathematics learning community: communities where "good conversation" is the currency of the class, the type of conversation that feminist philosopher and research Jane Roland Martin (1985) described as: "Circular in form, co-operative in manner, and constructive in intent: [good conversation] is an interchange of ideas by those who see themselves not as adversaries but as human beings coming together to talk and listen and learn from one another" (p. 10). In conceptualizing perseverance as a collective capacity, we see great potential in designing classrooms where children come together, not as adversaries, but as allies in an effort to tame a common enemy: the unformulated mathematical idea that collective effort will bring within reach.

References

Altshuler, R., & Kassinove, H. (1975). The effects of skill and chance instructional sets, schedule of reinforcement, and sex on children's temporal persistence. *Child Development*, *46*, 258–262. doi:10.2307/1128861

Angier, C., & Povey, H. (1999). One teacher and a class of school students: Their perception of the culture of their mathematics classroom and its construction. *Educational Review*, *51*(2), 147–160. doi:10.1080/00131919997588

Barron, B. (2000). Achieving coordination in collaborative problem-solving groups. *The Journal of the Learning Sciences*, *9*(4), 403–436. doi:10.1207/S15327809JLS0904_2
Barron, B. (2003). When smart groups fail. *The Journal of the Learning Sciences*, *12*(3), 307–359. doi:10.1207/S15327809JLS1203_1
Bass, H., & Ball, D. L. (2015). *Beyond "You can do it!": Developing mathematical perseverance in elementary school*. Retrieved from www.spencer.org
Berry, R. Q., III, & Thunder, K. (2015). *Black learners' perseverance with mathematics: A qualitative metasynthesis*. Retrieved from www.spencer.org
Boaler, J. (2008). Promoting 'relational equity' and high mathematics achievement through an innovative mixed-ability approach. *British Educational Research Journal*, *34*(2), 167–194. doi:10.1080/01411920701532145
Boaler, J. (2009). *The elephant in the classroom: Helping children learn & love maths*. London, UK: Souvenir Press.
Boaler, J. (2015). *What's math got to do with it?: How teachers and parents can transform mathematics learning and inspire success*. New York, NY: Penguin.
Boaler, J., & Sengupta-Irving, T. (2016). The many colors of algebra: The impact of equity focused teaching upon student learning and engagement. *The Journal of Mathematical Behavior*, *41*, 179–190. doi:10.1016/j.jmathb.2015.10.007
Boaler, J., & Staples, M. (2008). Creating mathematical futures through an equitable teaching approach: The case of Railside School. *The Teachers College Record*, *110*(3), 608–645.
Briggs, A., & Johnson, D. M. (1942). A note on the relation between persistence and achievement on the final examination. *Journal of Educational Psychology*, *33*(8), 623. doi:10.1037/h0063193
Carpenter, T. P., Fennema, E., Franke, M. L., Levi, L., & Empson, S. B. (1999). *Children's mathematics: Cognitively guided instruction*. Portsmouth, NH: Heinemann.
Cohen, E. G., Brody, C. M., & Sapon-Shevin, M. (2004). *Teaching cooperative learning: The challenge for teacher education*. New York, NY: Suny Press.
Cohen, E. G., & Lotan, R. A. (2004). Equity in heterogeneous classrooms. In J. A. Banks & C. A. M. Banks (Eds.), *Handbook of research on multicultural education* (2nd ed., pp. 736–752). San Francisco, CA: Jossey-Bass.
Cohen, E. G., & Lotan, R. A. (2014). *Designing groupwork: Strategies for the heterogeneous classroom* (3rd ed.). New York, NY: Teachers College Press, Columbia University.
Common Core Standards Initiative. (2010). *Common core state standards for mathematics*. Retrieved from http://www.corestandards.org
DeBellis, V. A., & Goldin, G. A. (2006). Affect and meta-affect in mathematical problem solving: A representational perspective. *Educational Studies in Mathematics*, *63*(2), 131–147. doi:10.1007/s10649-006-9026-4
De Guerrero, M., & Villamil, O. S. (2000). Activating the ZPD: Mutual scaffolding in L2 peer revision. *The Modern Language Journal*, *84*(1), 51–68.
Dekker, R., & Elshout-Mohr, M. (2004). Teacher interventions aimed at mathematical level raising during collaborative learning. *Educational Studies in Mathematics*, *56*(1), 39–65. doi:10.1023/B:EDUC.0000028402.10122.ff
Del Rio, P., & Alvarez, A. (2007). Inside and outside the zone of proximal development: An ecofunctional reading of Vygotsky. In H. Daniels, M. Cole, & J.V. Wertsch (Eds.), *The Cambridge Companion to Vygotsky* (pp. 276–303). Cambridge, UK: Cambridge University Press.
Department for Education and Employment. (1999). *The national curriculum: Handbook for primary teachers in England*. London, UK: DfEE.
Derry, S. J., Pea, R. D., Barron, B., Engle, R. A., Erickson, F., Goldman, R., … Sherin, B. L. (2010). Conducting video research in the learning sciences: Guidance on selection, analysis, technology, and ethics. *The Journal of the Learning Sciences*, *19*(1), 3–53. doi:10.1080/10508400903452884
DiCerbo, K. E. (2014). Game-based assessment of persistence. *Educational Technology & Society*, *17*(1), 17–28.
Doolittle, P. E. (1997). Vygotsky's Zone of Proximal Development as a theoretical foundation for cooperative learning. *Journal on Excellence in College Teaching*, *8*(1), 83–103.
Duckworth, A. L., Peterson, C., Matthews, M. D., & Kelly, D. R. (2007). Grit: Perseverance and passion for long-term goals. *Journal of Personality and Social Psychology*, *92*(6), 1087. doi:10.1037/0022-3514.92.6.1087
Engle, R. A., Langer-Osuna, J. M., & McKinney de Royston, M. (2014). Toward a model of influence in persuasive discussions: Negotiating quality, authority, privilege, and access within a student-led argument. *Journal of the Learning Sciences*, *23*(2), 245–268. doi:10.1080/10508406.2014.883979
English, L. D., & Gainsburg, J. (2016). Problem solving in a 21st century mathematics curriculum. In L. D. English and D. Kirshner (Eds.), In *Handbook of International Research in Mathematics Education* (3rd ed.) (Vol. 3, pp. 313-335). New York, NY: Taylor and Francis.
English, L. D., & Warren, E. A. (1998). Introducing the variable through pattern exploration. *Mathematics Teacher*, *91*(2), 166–170.
Esmonde, I., & Langer-Osuna, J. M. (2013). Power in numbers: Student participation in mathematical discussions in heterogeneous spaces. *Journal for Research in Mathematics Education*, *44*(1), 288–315.
Feather, N. T. (1961). The relationship of persistence at a task to expectation of success and achievement related motives. *The Journal of Abnormal and Social Psychology*, *63*(3), 552. doi:10.1037/h0045671

Feather, N. T. (1966). Effects of prior success and failure on expectations of success and subsequent performance. *Journal of Personality and Social Psychology*, *3*(3), 287. doi:10.1037/h0022965

Forman, E. A. (1989). The role of peer interaction in the social construction of mathematical knowledge. *International Journal of Educational Research*, *13*, 55–70. doi:10.1016/0883-0355(89)90016-5

Forman, E. A. (1992). Discourse, intersubjectivity, and the development of peer collaboration: A Vygotskian approach. *Children's Development within Social Context*, *1*, 143–159.

Forman, E. A., & McPhail, J. (1993). Vygotskian perspective on children's collaborative problem-solving activities. In E. A. Forman, N. Minick, & C. A. Stone (Eds.), *57Contexts for learning: Sociocultural dynamics in children's development* (pp. 213–229). New York, NY: Oxford University Press.

Goldin, G. A. (2000). Affective pathways and representation in mathematical problem solving. *Mathematical Thinking and Learning*, *2*(3), 209–219. doi:10.1207/S15327833MTL0203_3

Goos, M., Galbraith, P., & Renshaw, P. (2002). Socially mediated metacognition: Creating collaborative zones of proximal development in small group problem solving. *Educational Studies in Mathematics*, *49*(2), 193–223. doi:10.1023/A:1016209010120

Gresalfi, M. S. (2009). Taking up opportunities to learn: Constructing dispositions in mathematics classrooms. *The Journal of the Learning Sciences*, *18*(3), 327–369. doi:10.1080/10508400903013470

Gresalfi, M. S., & Cobb, P. (2006). Cultivating students' discipline-specific dispositions as a critical goal for pedagogy and equity. *Pedagogies*, *1*(1), 49–57. doi:10.1207/s15544818ped0101_8

Hiebert, J., & Grouws, D. A. (2007). The effects of classroom mathematics teaching on students' learning. *Second Handbook of Research on Mathematics Teaching and Learning*, *1*, 371–404.

Horn, I. S. (2007). Fast kids, slow kids, lazy kids: Framing the mismatch problem in mathematics teachers' conversations. *The Journal of the Learning Sciences*, *16*(1), 37–79.

Human Resources Development Working Group. (2009). *Analysis of mathematics and science standards from the Asia-Pacific Economic Cooperation*. Retrieved from http://publications.apec.org/publication-detail.php?pub_id=163

Jansen, A. (2012). Developing productive dispositions during small-group work in two sixth-grade mathematics classrooms: Teachers' facilitation efforts and students' self-reported benefits. *Middle Grades Research Journal*, *7*(1), 37.

Kapur, M. (2010). Productive failure in mathematical problem solving. *Instructional Science*, *38*(6), 523–550. doi:10.1007/s11251-009-9093-x

Kapur, M. (2014). Productive failure in learning math. *Cognitive Science*, *38*(5), 1008–1022. doi:10.1111/cogs.2014.38.issue-5

Kilpatrick, J., Swafford, J., & Findell, B. (Eds.). (2001). *Adding it up: Helping children learn mathematics. Mathematics* Learning Study Committee, Center for Education, Division of Behavioral and Social Sciences and Education, National Research Council. Washington, DC: National Academy Press.

Kotsopoulos, D. (2010). When collaborative is not collaborative: Supporting student learning through self-surveillance. *International Journal of Educational Research*, *49*(4), 129–140. doi:10.1016/j.ijer.2010.11.002

Langer-Osuna, J. M. (2011). How Brianna became bossy and Kofi came out smart: Understanding the trajectories of identity and engagement for two group leaders in a project-based mathematics classroom. *Canadian Journal of Science, Mathematics and Technology Education*, *11*(3), 207–225. doi:10.1080/14926156.2011.595881

Langer-Osuna, J. M. (2016). The social construction of authority among peers and its implications for collaborative mathematics problem solving. *Mathematical Thinking and Learning*, *18*(2), 107–124. doi:10.1080/10986065.2016.1148529

Leikin, R., & Zaslavsky, O. (1997). Facilitating student interactions in mathematics in a cooperative learning setting. *Journal for Research in Mathematics Education*, *28*(3), 331–354. doi:10.2307/749784

Lindberg, S. W., & McGuffey, W. H. (1976). *The annotated McGuffey*. New York, NY: Van Nostrand Reinhold.

Lotan, R. A. (2003). Group-worthy tasks. *Educational Leadership*, *60*(6), 72–75.

Maher, C. A. (2002). How students structure their own investigations and educate us: What we've learned from a fourteen year study. In A. D. Cockburn, & E. Nardi (Eds.), *Proceedings of the 26th Annual Meeting for the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 31–46). Norwich, UK: University of East Anglia.

Maher, C. A., & Martino, A. M. (1996). The development of the idea of mathematical proof: A 5-year case study. *Journal for Research in Mathematics Education*, *27*, 194–214. doi:10.2307/749600

Martin, J. R. (1985). *Reclaiming a conversation*. New Haven, CT: Yale.

Megowan-Romanowicz, M. C., Middleton, J. A., Ganesh, T., & Joanou, J. (2013). Norms for participation in a middle school mathematics classroom and its effect on student motivation. *Middle Grades Research Journal*, *8*(1), 51.

Middleton, J. A., Tallman, M. A., Hatfield, N., & Davis, O. (2015). *Taking the severe out of perseverance: Strategies for building mathematical determination*. Retrieved from www.spencer.org

Miell, D., & MacDonald, R. (2000). Children's creative collaborations: The importance of friendship when working together on a musical composition. *Social Development*, *9*(3), 348–369. doi:10.1111/sode.2000.9.issue-3

Nasir, N. S., Cabana, C., Shreve, B., Woodbury, E., & Louie, N. (2014). *Mathematics for equity: A framework for successful practice*. New York, NY: Teachers College Press.

National Research Council, & Mathematics Learning Study Committee. (2001). *Adding it up: Helping children learn mathematics*. Washington, DC: National Academies Press.

National Council for Teachers of Mathematics. (2014). *Principles to actions: Ensuring mathematical success for all*. Reston, VA: Author.
Newman, D., Griffin, P., & Cole, M. (1984). Social constraints in laboratory and classroom tasks. In B. Rogoff & J. Lave (Eds.), *Everyday cognition: Its development in social context* (pp. 172–193). Cambridge, MA: Harvard University Press.
Nyikos, M., & Hashimoto, R. (1997). Constructivist theory applied to collaborative learning in teacher education: In search of ZPD. *The Modern Language Journal*, *81*(4), 506–517. doi:10.1111/modl.1997.81.issue-4
Pai, H. H., Sears, D. A., & Maeda, Y. (2015). Effects of small-group learning on transfer: A meta-analysis. *Educational Psychology Review*, *27*, 79–102.
Parker, R. (2009). Patterns: Foundations for algebraic reasoning. *The Mathematics Education Collaborative*. Retrieved from www.mec-math.org
Parrish, S. (2010). *Number talks: Helping children build mental math and computation strategies*. Sausalito, CA: Math Solutions.
Phelps, E., & Damon, W. (1989). Problem solving with equals: Peer collaboration as a context for learning mathematics and spatial concepts. *Journal of Educational Psychology*, *81*(4), 639. doi:10.1037/0022-0663.81.4.639
Pólya, G. (1981). *Mathematical discovery: On understanding, learning, and teaching problem solving* (Combined ed.). New York, NY: John Wiley & Sons.
Pólya, G. (2014). *How to solve it: A new aspect of mathematical method*. Princeton, NJ: Princeton University Press.
Powell, A. B., Francisco, J. M., & Maher, C. A. (2003). An analytical model for studying the development of learners' mathematical ideas and reasoning using videotape data. *The Journal of Mathematical Behavior*, *22*(4), 405–435. doi:10.1016/j.jmathb.2003.09.002
Ryans, D. G. (1938a). The meaning of persistence. *The Journal of General Psychology*, *19*(1), 79–96. doi:10.1080/00221309.1938.9711188
Ryans, D. G. (1938b). An experimental attempt to analyze persistent behavior: I. measuring traits presumed to involve "persistence". *The Journal of General Psychology*, *19*(2), 333–353. doi:10.1080/00221309.1938.9711208
Schoenfeld, A. H. (1985). *Mathematical problem-solving*. New York, NY: Academic Press.
Schoenfeld, A. H. (1992). Learning to think mathematically: Problem-solving, metacognition, and sense-making in mathematics. In D. Grouws (Ed.), *Handbook for research on mathematics teaching and learning* (pp. 334–370). New York: MacMillan.
Schofield, W. (1943). An attempt to measure 'persistence' in its relationship to scholastic achievement. *Journal of Experimental Psychology*, *33*(5), 440. doi:10.1037/h0060235
Sengupta-Irving, T. (2014). Affinity through mathematical activity: Cultivating democratic learning communities. *Journal of Urban Mathematics Education*, *7*(2), 31–54.
Sengupta-Irving, T. (2016). Doing things: Organizing for agency in mathematical learning. *The Journal of Mathematical Behavior*, *41*, 210–218. doi:10.1016/j.jmathb.2015.10.001
Sengupta-Irving, T., & Enyedy, N. (2015). Why engaging in mathematical practices may explain stronger outcomes in affect and engagement: Comparing student-driven with highly-guided inquiry. *Journal of the Learning Sciences*, *24*(4), 550–592.
Sengupta-Irving, T., Redman, E., & Enyedy, N. (2013). Re-storying practice: Using stories about students to advance mathematics education reform. *Teaching and Teacher Education*, *31*, 1–12. doi:10.1016/j.tate.2012.10.007
Sfard, A., & Kieran, C. (2001). Cognition as communication: Rethinking learning-by-talking through multi-faceted analysis of students' mathematical interactions. *Mind, Culture and Activity*, *8*(1), 42–76. doi:10.1207/S15327884MCA0801_04
Sherin, M. G. (2002). A balancing act: Developing a discourse community in a mathematics classroom. *Journal of Mathematics Teacher Education*, *5*(3), 205–233. doi:10.1023/A:1020134209073
Sigman, M., Cohen, S. E., Beckwith, L., & Topinka, C. (1987). Task persistence in 2-year-old preterm infants in relation to subsequent attentiveness and intelligence. *Infant Behavior and Development*, *10*(3), 295–305. doi:10.1016/0163-6383(87)90018-X
Silver, E., & Smith, M. (1996). Building discourse communities in mathematics classrooms: A worthwhile but challenging journey. In P. C. Elliott, & M. J. Kenney (Eds.), *Communication in mathematics: K–12 and beyond—1996 yearbook* (pp. 20–28). Reston, VA: National Council of Teachers of Mathematics.
Simon, M. A., & Tzur, R. (2004). Explicating the role of mathematical tasks in conceptual learning: An elaboration of the hypothetical learning trajectory. *Mathematical Thinking and Learning*, *6*(2), 91–104. doi:10.1207/s15327833mtl0602_2
Slavin, R. E. (2011). Instruction based on cooperative learning. In R. E. Mayer & P. A. Alexander (Eds.), *Handbook of research on learning and instruction* (pp. 344–360). New York, NY: Taylor & Francis.
Star, J. R. (2015). *When not to persevere: Nuances related to perseverance in mathematical problem solving*. Retrieved from www.spencer.org
Stein, M. K., & Smith, M. (2011). *5 practices for orchestrating productive mathematics discussions*. Reston, VA: National Council of Teachers of Mathematics.

Stevenson, H., & Stigler, J. (1992). *The learning gap: Why our schools are failing and what we can learn from Japanese and Chinese education*. New York: Summit Books.

Taylor, E. V. (2015). *Cultural considerations in support of mathematical perseverance: The role of context activation*. Retrieved from www.spencer.org

Vygotsky, L. S. (1978). *Mind in society: The psychology of higher mental functions*. Cambridge, MA: Harvard University Press.

Warshauer, H. (2014). Productive struggle in middle school mathematics classrooms. *Journal of Mathematics Teacher Education*, *17*(4). doi:10.1007/s10857-014-9286-3

Wenger, E. (1998). *Communities of practice: Learning, meaning, and identity*. Cambridge, UK: Cambridge University Press.

Yackel, E., & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education*, *27*, 458–477. doi:10.2307/749877

Yackel, E., Cobb, P., & Wood, T. (1991). Small-group interactions as a source of learning opportunities in second-grade mathematics. *Journal for Research in Mathematics Education*, *22*, 390–408. doi:10.2307/749187