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Entanglement, chaos, and causality in holography

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Entanglement, chaos, and causality in holography

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Entanglement, chaos, and causality in holography

To Xin Ying and Michael

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Curriculum Vitæ

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Publications

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[2] X. Dong, G. N. Remmen, D. Wang, W. W. Weng, and C.-H. Wu. *Holographic entanglement from the UV to the IR*,[arXiv:2308.07952]

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Abstract

Entanglement, chaos, and causality in holography

by

Diandian Wang

Holography remains one of the most powerful approaches to understanding quantum gravity. The currently best-understood realization of holography is AdS/CFT – a map between a quantum gravity theory in anti-de Sitter space (AdS) and a conformal field theory (CFT) on its boundary. In this thesis, several aspects of this duality are explored, with a focus on entanglement, chaos and causality. Entanglement plays an especially important role in holography because it is deeply connected to the emergence of spacetime; chaos is another key player in recent developments of holography because it is a characteristic feature of black holes that is expected to persist beyond the classical regime; finally, the idea of causality is a valued concept even in a quantum theory of gravity where the spacetime itself fluctuates, and the boundary CFT which has a well-defined notion of causality is a pragmatic starting point for understanding causality in a bulk theory of quantum gravity. One main theme of the thesis is to explore how these aspects depend on the theory. In particular, conclusions are drawn for whether certain statements that hold in General Relativity hold in general higher-derivative gravity and/or with higher-spin fields. Another theme is the connections between different concepts. In particular, boundary quantities that are apparently very different may be intricately related from the bulk perspective.

Contents

Curriculum Vitae **vii**

Abstract **ix**

1 Introduction **1**

2 Matching of butterfly velocities **8**
2.1 Operator size and the butterfly velocity 12
2.2 Butterfly velocities for f(Riemann) gravity 27
2.3 Equivalence of the two butterfly velocities 41
2.4 Discussion 53

3 Covariant pole skipping: bosonic fields **57**
3.1 General pole-skipping conditions 60
3.2 Matching of butterfly velocities 67
3.3 Discussion 71

4 Covariant pole skipping: gauge and fermionic fields **73**
4.1 Bosonic fields 75
4.2 Gauge fields 82
4.3 Fermionic fields 87
4.4 Bosonic and fermionic fields 95
4.5 Discussion 99

5 Causality in the gravitational path integral **103**
5.1 Specific examples 105
5.2 Operator dictionaries 106
5.3 Resolution 108
5.4 Discussion 114

A Exact linearity of the shockwave equation of motion **116**

B Proof of sum relations **119**

C Bosonic examples **121**
C.1 Minimally coupled scalar 121
C.2 Maxwell theory 123
C.3 Einstein gravity 126

D Fermionic examples **130**
D.1 Free Dirac spinor 130
D.2 Rarita-Schwinger field 133

Bibliography **137**

Chapter 1

Introduction

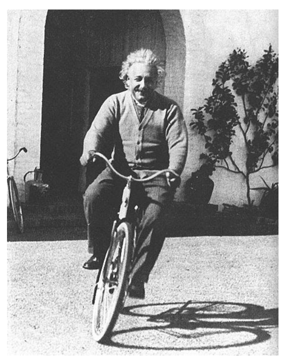

Figure 1.1: Einstein the biker.

Albert Einstein – a genius, a celebrity, a politician, a guy who once rode a bike in Santa Barbara (Fig. 1.1) – whatever you call him, he unarguably revolutionized our understanding of the universe, of spacetime – a word that doesn't carry much weight without his theories of Relativity.

In General Relativity, space and time are interwoven. Through this remarkable insight, it turns the description of the most prevalent force of Nature, the gravitational force, into a question of geometry. The universal attraction between objects, the bending of light, etc, all became natural in this language, because everything just goes in "straight lines", or more precisely *geodesics* – the most natural trajectory you can have on any curved geometry. As is often said, matter tells spacetime how to curve, and curved spacetime tells matter how to move.

With almost no free choice to make, Einstein wrote down the equation that governs it all:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G_N T_{\mu\nu}, \tag{1.1}$$

where G_N is Newton's constant, $g_{\mu\nu}$ is the metric, $R_{\mu\nu}$ is the Ricci curvature tensor, $T_{\mu\nu}$ is the stress tensor that incorporates the effects of matter, and Λ is the so-called cosmological constant. The inclusion of the cosmological constant was needed to keep the universe static for all past and future, which was in line with the thinking at the time. When it was later found that the universe was not static, Einstein called his inclusion of Λ his "greatest blunder".

As time went by, new evidence found that the universe is actually expanding at an accelerating rate, consistent with the inclusion of a positive Λ. So his "blunder" was in fact a surprising prediction. Nowadays, we call the maximally symmetric spacetime with $\Lambda > 0$ de Sitter spacetime, whereas $\Lambda = 0$ leads to Minkowski spacetime. Both $\Lambda > 0$ and $\Lambda = 0$ are being actively studied, and they contribute to our understanding of Nature at different scales.

Now, one may wonder, what happens if we choose $\Lambda < 0$. In this case, we get (asymptotically) anti-de Sitter (AdS) spacetimes. They have features not observed in our universe, so there does not seem to be much motivation for studying them, *a priori*. However, surprises are always found when one is not expecting them. These seemingly unmotivated spacetimes turned out to have very interesting mathematical structures and have been teaching us about the most important question in theoretical physics: how to quantize gravity?

To begin with, asymptotically AdS spacetimes have a timelike boundary. This makes it possible for the boundary itself to have a notion of causality. Secondly, AdS is like a box owing to its negative curvature, making gravity contained and avoiding pathology related to enclosing a gravitating region with a finite boundary. Helped by these features, along with an enormous amount of evidence, it is conjectured that gravity theories in an asymptotically AdS spacetime are dual to non-gravitational field theories living on its boundary. In the language of this duality, namely AdS/CFT, the AdS side is called the bulk theory, and the field theory side is called the boundary theory.

Such a duality is impressive for many reasons. To begin with, two theories live in different dimensions, which makes a local mapping between the two theories nonsensical. In fact, which region in the bulk maps to which region on the boundary is an important area of study. Moreover, gravity in the classical or semiclassical regime corresponds to the large N limit of the boundary field theory, where N is the number of degrees of freedom in the theory. Additionally, the strongly coupled region of the boundary theory is related to the low energy limit of the bulk, making it possible to obtain highly nontrivial results in the field theory from simple calculations in classical gravity. Finally, it is expected that this duality provides a possible way to investigate the existence of a UV-complete quantum gravity theory as progress is being made in the understanding of the map between the bulk and the boundary at finite N.

All these reasons motivate us to understand the duality better. In this dissertation, I will focus on a few aspects of this duality, including but not limited to: entanglement, chaos, and causality. In the rest of this introduction, I will give an overview of a few projects and explain how they facilitate the understanding of the aforementioned concepts.

The first project will be detailed in Chapter 2. It is centered around the topic of butterfly velocity, a concept in the study of quantum chaos which describes how fast localized information can spread in a quantum system. As mentioned earlier, strongly coupled and complicated boundary phenomena can have simple bulk descriptions, owing to the beautiful nature of the duality. Here, we will describe two simple ways of computing the boundary butterfly velocity in the bulk, one involving of the use of shockwaves, and another involving the so-called Ryu-Takayanagi surface, a geometrization of the boundary entanglement entropy. Computations are carried out for a large family of bulk theories, and the results all agree. Since shockwaves are connected to chaos, this is a precise investigation of the connection between chaos and entanglement. More precisely, we prove the equivalence of two holographic computations of the butterfly velocity in higher-derivative theories with Lagrangian built from arbitrary contractions of curvature tensors. The butterfly velocity characterizes the speed at which local perturbations grow in chaotic many-body systems and can be extracted from the out-of-time-order correlator. This leads to a holographic computation in which the butterfly velocity is determined from a localized shockwave on the horizon of a dual black hole. A second holographic computation uses entanglement wedge reconstruction to define a notion of operator size and determines the butterfly velocity from certain extremal surfaces. By direct computation, we show that these two butterfly velocities match precisely in the aforementioned class of gravitational theories. We also present evidence showing that this equivalence holds in all gravitational theories. Along the way, we prove a number of general results on shockwave spacetimes.

The second project will be presented in Chapters 3 and 4. It involves yet another computation of the butterfly velocity, through the idea of pole skipping. Pole skipping is a phenomenon exhibited by thermal Green's functions, which has a standard computation in the bulk. In this project, we show that for all higher-derivative gravity, the butterfly velocity computed using pole skipping agrees precisely with that defined using shockwaves. Moreover, a systematic analysis of the pole skipping phenomenon is presented, which works for all bulk theories with diffeomorphism-invariant Lagrangians. Furthermore, for theories containing higher-spin fields, a formula for the leading pole-skipping frequency is presented. which solely depends on the highest spin in the theory. More specifically, in Chapter 3, we study pole skipping in holographic CFTs dual to diffeomorphism invariant theories containing an arbitrary number of bosonic fields in the large N limit. Defining a weight to organize the bulk equations of motion, a set of general pole-skipping conditions are derived. In particular, the frequencies simply follow from general covariance and weight matching. In the presence of higher spin fields, we find that the imaginary frequency for the highest-weight pole-skipping point equals the higher-spin Lyapunov exponent which lies outside of the chaos bound. Without higher spin fields, we show that the energy density Green's function has its highest-weight pole skipping happening at a location related to the OTOC for arbitrary higher-derivative gravity, with a Lyapunov exponent saturating the chaos bound and a butterfly velocity matching that extracted from a shockwave calculation. We also suggest an explanation for this matching at the metric level by obtaining the on-shell shockwave solution from a regularized limit of the metric perturbation at the skipped pole. In Chapter 4, we revisit this formalism in theories with gauge symmetry and upgrade the pole-skipping condition so that it works without having to remove the gauge redundancy. We also extend it by incorporating fermions with general spins and interactions and show that their presence generally leads to a separate tower of pole-skipping points at frequencies $\mathrm{i}(l_f - s)2\pi T$, l_f being the highest half-integer spin in the theory and s taking all positive integer values. In addition to being useful for proving general statements, the covariant formalism is also convenient for practical computations, which we demonstrate using a selection of examples with spins $0, \frac{1}{2}, 1, \frac{3}{2}, 2$.

The third project will be explained in Chapter 5, and it touches on the bigger question of going beyond the semiclassical region in gravity. Since there is no definitive answer to the question of how to do this, we take the naive approach of including offshell metrics in the gravitational path integral as a starting point. This, however, immediately leads to an apparent puzzle due to the presence of metrics that violate boundary causality. Taken at face value, it would imply that the boundary theory is acausal, not a desired feature for a field theory. Interestingly, as we will see, the puzzle is resolved if the boundary conditions of the gravitational path integral are imposed carefully. Through this investigation, we learn something about potential subtleties in the endeavour of searching for a UV-complete theory of quantum gravity, along with lessons about the gravitational path integral. More specifically, the puzzle is as follows. Even for holographic theories that obey boundary causality, the full bulk Lorentzian path integral includes metrics that violate this condition. By causality of the boundary theory, the commutator of two field theory operators at spacelike-separated points on the boundary must vanish. However, if these points are causally related in a bulk metric, then the bulk calculation of the commutator will be nonzero. It would appear that the integral over all metrics of this commutator must vanish exactly for holography to hold. This is puzzling since it must also be true if the commutator is multiplied by any other operator. Upon careful treatment of boundary conditions in holography, we show how the bulk path integral leads to a natural resolution of this puzzle.

Throughout this dissertation, we will avoid fixating on a specific bulk theory because that may result in us accidentally drawing conclusions that are not generalizable. After all, even though we think of General Relativity, or Einstein gravity, as an excellent approximation at low energy, it is always corrected by higher-derivative operators. So ultimately it may not be helpful to find statements that only hold for Einstein gravity, especially given that our long-term goal is a quantum gravitational theory. By doing this, we draw conclusions that are robust against including small corrections to the Lagrangian; sometimes, this type of analysis can also put constraints on the low energy theory. The use of higher-derivative gravity, the inclusion of higher-spin fields, and the use of the gravitational path integral with an unspecified action, all contribute towards such a goal.

Finally, we conclude this introductory section by saying that the dissertation will be an adventure, one that attempts to demystify the beautiful realm of holography, and one that is only possible with all the work that has been done in the past, from that of Galilei, that of Newton, and that of Einstein, to that of Maldacena and that of everyone who worked in the field. The world of high energy physics is vast, but we will start somewhere; today, let us take a step in the adventure into entanglement, chaos, and causality.

Chapter 2

Matching of butterfly velocities

Chaos is prevalent in the physical world: small changes in initial conditions can lead to drastic variations in the outcome. This sensitive dependence on the initial state is known as the butterfly effect. In classical systems, chaotic dynamics is characterized by an exponential deviation between trajectories in phase space; the Lyapunov exponent parameterizes the degree of deviation.

In quantum mechanical systems, defining chaos is a more challenging task as the wavefunction is governed by a linear evolution [1]. Nevertheless, one can characterize quantum chaos by the strength of the commutator $[V(t), W(0)]$ between two generic operators with time separation t [2, 3].[1] One useful measure of the typical matrix elements of this commutator is the expectation value of $|[V(t), W(0)]|^2$ (where the square is defined as $|\mathcal{O}|^2 \equiv \mathcal{O}^\dagger\mathcal{O}$ for any operator $\mathcal{O}$). In a chaotic system, this quantity grows with t and becomes significant around the scrambling time t_* [5, 6], behaving like $e^{\lambda_L(t-t_*)}$. Here λ_L is the (quantum) Lyapunov exponent.

In this paper, we are interested in the spreading of the quantum butterfly effect in spatial directions. We therefore specialize W and V into local operators with spatial

[1] Random matrix theory provides a different way of characterizing quantum chaos from the spectral statistics [4].

separation x in $d-1$ spatial dimensions. The corresponding expectation value behaves like [7, 8, 9]

$$C_\beta(x,t) \equiv \langle |[V(x,t), W(0,0)]|^2 \rangle_\beta \sim e^{\lambda_L (t - t_* - |x|/v_B)}, \tag{2.1}$$

where $\langle \cdot \rangle_\beta$ denotes the expectation value in a thermal state with inverse temperature β. The quantity $C_\beta(x,t)$ characterizes the strength of the butterfly effect at (x,t) detected by a probe V following an earlier perturbation $W(0,0)$.

Here v_B is known as the *butterfly velocity*. It is the speed at which the region with $C_\beta(x,t) \gtrsim 1$ expands outward. Intuitively, the commutator probes how a small perturbation $W(0,0)$ spreads over the system. For two operators that are far separated, the commutator is zero at early times and becomes large at sufficiently late times. In particular, the region in which $C_\beta(x,t)$ is order unity gives a measure of the size of the operator $W(0,0)$ at time t, and the speed at which the size grows over time is precisely the butterfly velocity.

In recent years, there has been much interest in the role of chaotic dynamics in holographic quantum systems, particularly in the context of AdS/CFT [7, 10, 8, 11, 9, 12, 13]. In this context, a thermal state in the boundary CFT can be realized as a black hole in the bulk AdS spacetime [14]. The rapid thermalization of a local perturbation on the boundary can be understood from the fast scrambling dynamics of the black hole [5, 6, 15, 16]. In particular, the Lyapunov exponent is related to the exponential blueshift of early infalling quanta in the near-horizon region. For CFTs dual to Einstein gravity (possibly corrected by a finite number of higher-derivative terms), the Lyapunov exponent is universal and saturates [7, 10, 8, 11] the chaos bound $\lambda_L \leq 2\pi/\beta$ [17]. This can be understood from the universality of the near-horizon Rindler geometry. If one considers perturbations sent from sufficiently far in the past, the quanta are significantly blue-shifted by the Rindler geometry. At around a scrambling time, the backreaction on the background geometry can be described by a shockwave on the horizon [7, 10, 11]. The strength of the shockwave grows exponentially as we insert the perturbation at earlier and earlier times.

This observation leads to one way of obtaining the butterfly velocity. To see it concretely, consider a localized perturbation in the thermofield-double (TFD) state dual to a two-sided $(d+1)$-dimensional planar black hole. Inserting such a spatially localized perturbation on one boundary corresponds to injecting a small number of quanta which then proceed to fall towards the black hole in the bulk. The result of doing so is a localized shockwave [8]. The spatial region in which the shockwave has non-trivial support defines a size of the corresponding boundary operator. As we send the perturbation at earlier and earlier times, the size of this region grows, and this "speed of propagation" determines a butterfly velocity v_B.

An alternative way of calculating the operator size and the butterfly velocity in holography was proposed in [18]. It is based on entanglement wedge reconstruction, which states that a bulk operator within the entanglement wedge of any boundary subregion can be represented by some boundary operator on that subregion [19, 20, 21, 22]. Consider again a local operator inserted on the boundary, which in the bulk can be thought of as a particle falling into the black hole. By entanglement wedge reconstruction, a boundary region whose entanglement wedge contains the particle possesses full information about the boundary operator. The smallest spherical region that does so defines a notion of size for the boundary operator. At early times, the particle is near the boundary; the boundary region, and hence the operator size, is small. At very late times, say after a scrambling time, the particle is very close to the horizon and the entanglement wedge needs to extend deep into the bulk. The corresponding boundary region, and hence the operator size, is very large and in fact grows linearly with time. The corresponding speed, which we denote as $\widetilde{v}_B$, quantifies the growth of the operator size. It provides a second way of computing the butterfly velocity holographically.

These two holographic computations of the butterfly velocity appear to be very different and unrelated to each other. However, it was shown directly in [18] that the results of the two computations agree for Einstein gravity and for higher-derivative gravity with up to four derivatives on the metric.

The goal of this paper is to prove that the two computations of the butterfly velocity continue to agree in general higher-derivative theories of gravity. We will focus on the family of general f(Riemann) theories, namely those with Lagrangians built from arbitrary contractions of an arbitrary number of Riemann tensors:[2]

$$\mathcal{L} = \frac{1}{2}(R - 2\Lambda) + \lambda_1 R^2 + \lambda_2 R_{\mu\nu}R^{\mu\nu} + \lambda_3 R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + \lambda_4 R^3 + \cdots , \tag{2.2}$$

where the higher-derivative terms are viewed as perturbative corrections to the leading Einstein-Hilbert action. The main purposes of studying higher-derivative theories are twofold: (1) they arise generally as perturbative corrections to Einstein gravity in low energy effective theories of UV-complete models of quantum gravity such as string theory; (2) the agreement between two computations of the butterfly velocity for general higher-derivative theories would suggest an equivalence between the two methods themselves rather than a coincidence in certain theories. This equivalence then suggests a deeper connection between gravitational shockwaves and holographic entanglement, as well as providing further evidence for entanglement wedge reconstruction.

The rest of the paper is organized as follows. In Section 2.1, we begin with a detailed review of the two holographic calculations of the butterfly velocity. In Section 2.2, we derive general expressions for the two butterfly velocities in f(Riemann) theories. In Section 2.3, we prove $v_B = \widetilde{v}_B$ for this class of theories, which is our main result. In

[2]Nevertheless, in Section 2.4 we will discuss one example that is more general than f(Riemann) theories, where the two butterfly velocities continue to agree.

Section 2.4, we end with a discussion of this result and comment on potential future directions.

2.1 Operator size and the butterfly velocity

In chaotic many-body systems, the size of generic local operators — the spatial region on which the operator has large support — grows ballistically under Heisenberg time evolution. More specifically, consider a perturbation by such a local operator. Under chaotic evolution, information about the perturbation is *scrambled* amongst the local degrees of freedom and spreads throughout the system. The maximum speed at which this occurs is the butterfly velocity v_B. It can be regarded as a finite temperature analogue of the Lieb-Robinson bound [23, 9]. The existence of this bound on information scrambling is a general feature of chaotic systems.

One can define operator size more precisely using the square of the commutator for two generic local operators W and V:

$$\begin{aligned} C_\beta(x, t_W) &\equiv \langle [V(x,0), W(0,-t_W)]^\dagger [V(x,0), W(0,-t_W)] \rangle_\beta \\ &= 2 - 2\,\mathrm{Re}\,\langle V(x,0)^\dagger W(0,-t_W)^\dagger V(x,0) W(0,-t_W) \rangle_\beta , \end{aligned} \tag{2.3}$$

where $t_W > 0$ so that W is inserted at an earlier time than V and the expectation value is taken in a thermal state with inverse temperature β.[3] The second term in the second line is called an out-of-time-order correlator (OTOC) and carries all of the non-trivial information in (2.3). The exponential decay of the OTOC over time is commonly used as an indicator of chaos in quantum many-body systems.

Under chaotic time evolution, the commutator with $x = 0$ exhibits an exponential

[3] In going to the second line of (2.3), we have assumed the two operators W and V to be unitary. This is not a crucial assumption, but does simplify our discussion.

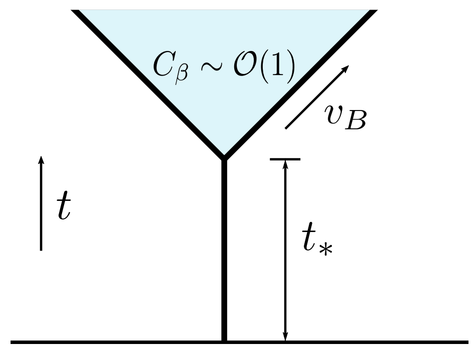

Figure 2.1: Illustration of the butterfly cone. A localized perturbation at some initial time scrambles everything at that location after a scrambling time t_*. This effect then spreads out spatially at the butterfly velocity v_B, defining the butterfly cone where C_β is of order one.

growth near the scrambling time t_*,[4] defined as the time at which the commutator becomes of order unity. Now considering non-zero x, information from a localized perturbation is scrambled among the local degrees of freedom and spreads throughout the system at the butterfly velocity v_B. In this scrambling regime, the behavior of the commutator has the following universal form [9],

$$C_\beta(x, t_W) \sim e^{\lambda_L (t_W - t_* - |x|/v_B)}. \tag{2.4}$$

The Lyapunov exponent λ_L gives a time scale for scrambling at a fixed spatial location, while v_B parametrizes the delay in scrambling due to spatial separation (see Figure 2.1). At time t_W after the insertion of the W perturbation, the commutator is order unity in the spatial region defined by $|x| \leq v_B(t_W - t_*)$, and is exponentially suppressed outside this region. This gives a precise notion of the size of an operator under time evolution.

A nice way to think about the OTOC, and hence the commutator, is as the overlap

[4]For example, in maximally chaotic many-body systems with $\mathcal{O}(N)$ degrees of freedom per site, t_* is approximately $\frac{\beta}{2\pi} \log N$.

$\langle\psi'|\psi\rangle$ between the two states

$$|\psi\rangle = V(x,0)W(0,-t_W)|\mathrm{TFD}\rangle_\beta \quad \text{and} \quad |\psi'\rangle = W(0,-t_W)V(x,0)|\mathrm{TFD}\rangle_\beta. \tag{2.5}$$

Here $|\mathrm{TFD}\rangle_\beta$ is the thermofield-double state:

$$|\mathrm{TFD}\rangle_\beta \equiv \frac{1}{Z(\beta)^{1/2}} \sum_n e^{-\beta E_n/2} |n\rangle_L |n\rangle_R, \tag{2.6}$$

where L denotes the original system and R denotes an identical copy, $|n\rangle$ is a complete set of energy eigenstates with energy E_n, and $Z(\beta)$ is the thermal partition function at inverse temperature β. The states (2.5) are obtained from $|\mathrm{TFD}\rangle_\beta$ by acting with operators in the L system. In particular, the state $|\psi\rangle$ corresponds to acting with W in the past at $t = -t_W$ and time evolving to $t = 0$ before inserting an operator V. The state $|\psi'\rangle$ corresponds to creating a perturbation V at $t = 0$, time evolving to the past, inserting W, and finally evolving back to $t = 0$. Under chaotic dynamics, the operator W is scrambled amongst degrees of freedom in its neighborhood. If W is inserted far enough in the past, it can interfere with the perturbation due to V and prevent it from reappearing at $t = 0$ in the state $|\psi'\rangle$. Consequently, the state $|\psi'\rangle$ would have a small overlap with $|\psi\rangle$ since the latter has V inserted at $t = 0$ by construction. This small overlap means that the commutator (2.3) is of order one, which defines the butterfly cone.

We now review two methods of calculating the butterfly velocity in holographic systems, which we call the shockwave method and the entanglement wedge method.

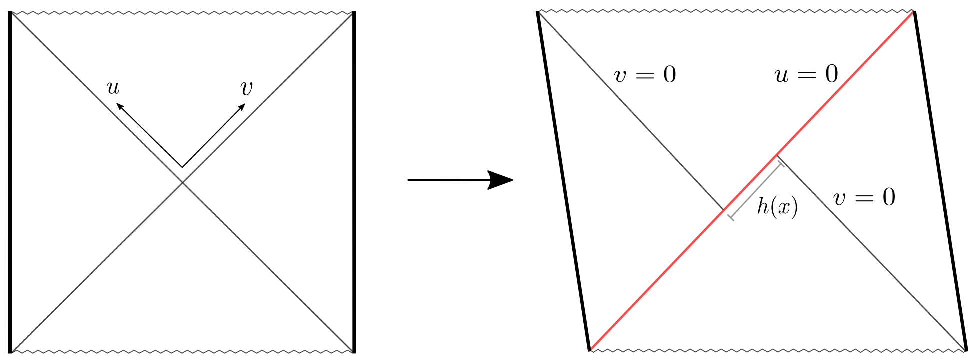

Figure 2.2: An illustration of the shockwave spacetime. By sending in a localized energy packet from the left boundary around a scrambling time in the past, the backreaction is described by a localized shockwave on the horizon at $u = 0$ with profile $h(x)$.

2.1.1 The shockwave method

One way of capturing the chaotic behavior in a holographic CFT is to introduce a perturbation at the asymptotic boundary and study the backreaction to the geometry as it propagates into the bulk. We will consider two copies of the CFT in a thermofield double state, which is dual to a two-sided eternal black hole in the bulk. Now let us act on the left boundary CFT with a local operator W at the origin $x = 0$ and boundary time $-t_W$:

$$W(0, -t_W)|\mathrm{TFD}\rangle = e^{-iH_L t_W} W(0,0) e^{iH_L t_W} |\mathrm{TFD}\rangle. \tag{2.7}$$

In the bulk, this perturbation corresponds to inserting an energy packet near the asymptotic boundary, which then falls towards the black hole. If we take t_W to be large, by the time it reaches $t = 0$, it will have gained considerable energy due to the exponentially large blueshift near the horizon, which then backreacts significantly on the spacetime. For large enough t_W,[5] the backreacted geometry is well-described by a shockwave along the horizon as shown in Figure 2.2.

To be more specific, let us consider a general $(d + 1)$-dimensional planar black hole.

[5]For the perturbation to be large (but not Planckian), the time at which we send in the particle needs to be around the scrambling time, $|t_W| \approx t_*$ [5, 6, 15, 16, 7].

The metric can be written in Kruskal coordinates as

$$ds^2 = 2A(uv)dudv + B(uv)dx^i dx^i, \tag{2.8}$$

where the functions $A(uv)$ and $B(uv)$ in general depend on the gravitational theory and the matter content, and $i \in \{1, \ldots, d-1\}$ labels the transverse directions. The two horizons live at $u = 0$ and $v = 0$. We will also rescale the transverse directions so that $B(0) = 1$.

A localized shockwave on the horizon at $u = 0$ is sourced by a change in the stress tensor due to a perturbation with initial asymptotic energy $\mathcal{E}$

$$\delta T_u^v = A^{-1} \mathcal{E} e^{\frac{2\pi}{\beta} t_W} \delta(u) \delta(x). \tag{2.9}$$

The prefactor $\mathcal{E} e^{\frac{2\pi}{\beta} t_W}$ can be thought of as the effective energy of the blueshifted perturbation. Note that t_W is not a spacetime coordinate but rather parameterizes the time at which the perturbation is inserted. Inserting the perturbation at earlier times increases this effective energy and results in a larger backreaction.

To solve for the backreaction, it is sufficient to perturb only the uu component of metric by an amount parameterized by some function $h(x)$,

$$ds^2 = 2A(uv)dudv + B(uv)dx^i dx^i - 2A(uv)h(x)\delta(u)du^2. \tag{2.10}$$

The function $h(x)$ — which we will refer to as the *shockwave profile* — is determined by the equations of motion. Assuming that the equations of motion for the unperturbed background solution (2.8),

$$E_{\mu\nu}\left(\equiv \frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}}\right) = T_{\mu\nu}, \tag{2.11}$$

are satisfied (where S is the gravitational part of the action), it suffices to consider the perturbed equations of motion

$$\delta E_{\mu}^{\nu} = \delta T_{\mu}^{\nu} \tag{2.12}$$

sourced by the shockwave stress tensor (2.9). For Einstein gravity, this shockwave equation of motion was first derived for a vacuum background by Dray and t'Hooft [24], and later generalized to allow a non-trivial background stress-tensor in [25].

Now we consider general theories of gravity. As we prove in Appendix A, the only non-trivial component of (2.12) is

$$\delta E_{u}^{v} = \delta T_{u}^{v} = A^{-1}\mathcal{E}e^{\frac{2\pi}{\beta}t_W}\delta(u)\delta(x). \tag{2.13}$$

Furthermore, in the same appendix we show that (2.13) truncates automatically at linear order in $h(x)$ (and its x^i-derivatives). Indeed, the equations of motion reduce to a single ODE for the shockwave profile $h(x)$, which we will refer to as the *shockwave equation.* As we will show, (2.13) can be solved for large $r = |x|$ with the following ansatz for the shockwave profile

$$h(x) \sim \frac{e^{\frac{2\pi}{\beta}t_W - \mu r}}{r^{\#}}, \tag{2.14}$$

where $\mu > 0$ and # is some integer that will not be important.

To see the connection to the OTOC, and hence the butterfly velocity, let us consider the overlap of the two states defined in (2.5) from the bulk perspective (see Figure 2.3). The states differ in the trajectory of the particle V, due to the shift $h(x)$ from the

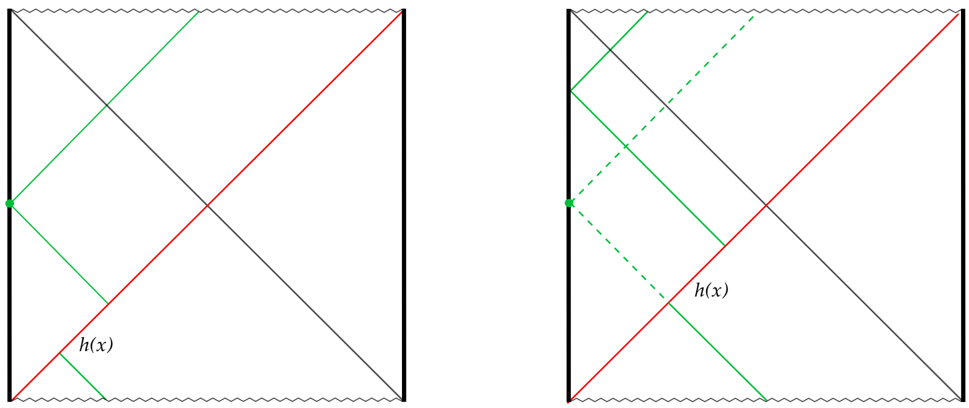

Figure 2.3: Bulk representations of $|\psi\rangle$ and $|\psi'\rangle$. For the state $|\psi\rangle$ (left), the shockwave (red) created by W is inserted first and the probe operator V is then inserted at $t = 0$, manifesting itself as a particle (green) propagating into the future and the past. On its way to the past, it encounters the already existent shockwave and is shifted by an amount $h(x)$. On the other hand, $|\psi'\rangle$ (right) is created by inserting V first and then placing the shockwave W. The shockwave alters the original trajectory of the V particle, such that it is shifted to the future of the shockwave.

shockwave W occurring either in the past or future evolution. Because the only difference between the two states is in the particle trajectory, the size of the overlap is controlled by the shift: it is close to unity when the h is small and close to zero when the h is large. Let us for concreteness define the boundary of the butterfly cone to be where $C_\beta = 1$, which translates to $\mathrm{Re}\,\langle\psi'|\psi\rangle = \frac{1}{2}$. This corresponds to some threshold value of the shift, say $h(x) = h_*$. According to (2.14), the size of the butterfly cone then grows as a function of t_W with the velocity

$$v_B = \frac{2\pi}{\beta\mu} \tag{2.15}$$

at large r. We identify this with the butterfly velocity. Solving the equation of motion (2.13) yields a value for μ and thus a value for v_B in terms of the functions $A(uv)$ and $B(uv)$ in the background metric (2.8).

Let us now look at some examples.

Einstein gravity Let us start with the simplest case of Einstein gravity, $\mathcal{L}_{\mathrm{EH}} = \frac{1}{2}(R - 2\Lambda)$. The calculations follow that in [7, 8], which we now review. Using (2.13), the shockwave equation is given by

$$\delta E_u^v = \frac{1}{B}\left(\partial_i\partial_i - \frac{d-1}{2}\frac{B'}{A}\right)h(x)\delta(u) = A^{-1}\mathcal{E}e^{\frac{2\pi}{\beta}t_W}\delta(x)\delta(u). \tag{2.16}$$

In the large-r limit, we can then substitute the ansatz (2.14), and obtain an algebraic equation for the parameter μ,

$$\mu^2 - \frac{d-1}{2}\frac{B_1}{A_0} = 0, \tag{2.17}$$

where $A_0 \equiv A(0)$ and $B_1 \equiv B'(0)$. Choosing the positive root for μ and using (2.15), we therefore find the butterfly velocity

$$v_B = \frac{2\pi}{\beta}\sqrt{\frac{2A_0}{(d-1)B_1}}. \tag{2.18}$$

Note that the equation for μ and thus the butterfly velocity depend only on the behavior of the metric near the $u = 0$ horizon. This is enforced by the overall factor of $\delta(u)$ in the equation of motion. As we will see, in the entanglement wedge method, this feature will be reproduced via a different mechanism — by taking a near-horizon limit of an extremal surface. This is one of the many distinctions between the two methods, making their agreement quite non-trivial.

Lovelock gravity As was shown in [8], for Gauss-Bonnet gravity whose Lagrangian is the Einstein-Hilbert term $\mathcal{L}_{\mathrm{EH}}$ plus

$$\mathcal{L}_{\mathrm{GB}} = \lambda_{\mathrm{GB}}(R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}), \tag{2.19}$$

the coupling constant $\lambda_{\rm GB}$ does not contribute to the shockwave equation (2.16).

We can further show that this is the case for Lovelock gravity, a general $2p$-derivative theory with second order equations of motion. The Lagrangian is $\mathcal{L}_{\rm EH}$ plus

$$\mathcal{L}_{2p} = \frac{\lambda_{2p}}{2^p}\delta^{\mu_1\nu_1\cdots\mu_p\nu_p}_{\rho_1\sigma_1\cdots\rho_p\sigma_p}R_{\mu_1\nu_1}{}^{\rho_1\sigma_1}\cdots R_{\mu_p\nu_p}{}^{\rho_p\sigma_p}, \tag{2.20}$$

where the generalized delta symbol is a totally antisymmetric product of Kronecker deltas, defined recursively as

$$\delta^{\mu_1\mu_2\cdots\mu_n}_{\nu_1\nu_2\cdots\nu_n} = \sum_{i=1}^{n}(-1)^{i+1}\delta^{\mu_1}_{\nu_i}\delta^{\mu_2\mu_3\cdots\mu_n}_{\nu_1\cdots\hat{\nu}_i\cdots\nu_n}. \tag{2.21}$$

Choosing $p = 1$ gives Einstein-Hilbert, while $p = 2$ reduces to Gauss-Bonnet. The metric equation of motion is given by

$$E^\alpha_\beta = -\frac{\lambda_{2p}}{2^p}\delta^{\alpha\mu_1\nu_1\cdots\mu_p\nu_p}_{\beta\rho_1\sigma_1\cdots\rho_p\sigma_p}R_{\mu_1\nu_1}{}^{\rho_1\sigma_1}\cdots R_{\mu_p\nu_p}{}^{\rho_p\sigma_p}. \tag{2.22}$$

We are interested in δE^v_u. It is not difficult to show that the only way to get a non-zero contraction is to make one of the Riemann tensors $R_{ui}{}^{vj}$ and the rest $R_{kl}{}^{mn}$. Using our metric ansatz (2.10), one can show that $\delta R_{kl}{}^{mn} = 0$, so any potential contribution can only come from $\delta R_{ui}{}^{vj} \propto \delta(u)$ multiplied by $p-1$ factors of $R_{kl}{}^{mn}$ (see Eq. (2.40) for the detailed expressions). The latter vanishes on the horizon of the planar black hole, due to the flatness of the transverse directions. Hence, we find that as long as $p > 1$ the Lovelock corrections do not contribute to the shockwave equation (2.16).

2.1.2 The entanglement wedge method

In holography, the butterfly effect manifests itself in another way as first observed in [18]. The intuition comes from entanglement wedge reconstruction [19, 20, 21, 22], which states that a given boundary region contains all of the information inside its entanglement wedge, which is a bulk region bounded by the boundary region and the corresponding extremal surface. As was argued in [18], this allows us to define a notion of operator size on the boundary.

Consider a thermal state in the boundary CFT dual to a planar black hole in the bulk. Let us perturb the boundary state by acting with a local operator. Under the chaotic time evolution, information from the perturbation is scrambled throughout the system. At late times, the boundary region over which this information is smeared propagates outwards at a constant velocity. In the bulk, the perturbation corresponds to a probe particle (or wavepacket) originating from the asymptotic boundary and falling towards the black hole, its trajectory determined from our choice of the bulk theory and the background geometry. According to entanglement wedge reconstruction, any boundary region whose entanglement wedge contains the particle should contain all the information of the corresponding boundary operator. In particular, we would like to consider the extremal surface which barely encloses the particle in its entanglement wedge. The corresponding boundary region then defines a size for the boundary operator.

To extract the butterfly velocity, we now study how this extremal surface changes as it follows the trajectory of the particle. Note that even though the location of the particle is time-dependent, the background spacetime is static[6] and at any given time we may use a Ryu-Takayanagi (RT) surface [26, 27] (instead of its dynamical generalization — the HRT surface [28]). At early times, the shape of the RT surface will depend sensitively on details of the background metric. However, at late times, the surface approaches the near-horizon region and exhibits a characteristic profile which propagates outwards at a constant velocity (see Figure 2.4). This velocity, which we can identify as the butterfly velocity $\widetilde{v}_B$, depends only on the bulk theory and the near-horizon geometry of the black hole (as long as the theory admits a black hole solution, which we can ensure by taking

[6]Here we do not need to analyze the backreaction of the particle, unlike in the shockwave method.

the coefficients of higher-derivative terms to be small so that a solution perturbatively close to the static planar black hole in Einstein gravity exists).

In general higher-derivative gravity, the location of the RT surface is determined by extremizing the holographic entanglement entropy functional S_{EE} among all bulk surfaces homologous to the corresponding boundary region [29, 30, 31, 32]. For the f(Riemann) theories that we focus on, the entropy functional S_{EE} can be found in [29]. The important terms for our purpose is

$$S_{\mathrm{EE}} = 2\pi \int d^{d-1}y\sqrt{\gamma}\Big\{ -\frac{\partial \mathcal{L}}{\partial R_{\mu\rho\nu\sigma}}\varepsilon_{\mu\rho}\varepsilon_{\nu\sigma} - \frac{\partial^2 \mathcal{L}}{\partial R_{\mu_1\rho_1\nu_1\sigma_1}\partial R_{\mu_2\rho_2\nu_2\sigma_2}} 2K_{\lambda_1\rho_1\sigma_1}K_{\lambda_2\rho_2\sigma_2}\times \\ \times \left[(n_{\mu_1\mu_2}n_{\nu_1\nu_2} + \varepsilon_{\mu_1\mu_2}\varepsilon_{\nu_1\nu_2})\, n^{\lambda_1\lambda_2} - (n_{\mu_1\mu_2}\varepsilon_{\nu_1\nu_2} + \varepsilon_{\mu_1\mu_2}n_{\nu_1\nu_2})\, \varepsilon^{\lambda_1\lambda_2}\right] + \cdots \Big\}, \quad (2.23)$$

where y denotes a set of coordinates on an appropriate codimension-2 surface, γ is the determinant of its induced metric, $K_{\lambda\rho\sigma}$ is its extrinsic curvature tensor, $n_{\mu\nu}$ is the induced metric (and $\varepsilon_{\mu\nu}$ is the Levi-Civita tensor) in the two orthogonal directions while vanishing in the remaining directions, and $\cdots$ denotes terms that are higher-order[7] in $K_{\lambda\rho\sigma}$ and its derivatives than the second-order term shown here. As we will explain in Section 2.2.3, these higher-order terms do not affect the calculation of the butterfly velocity. Note that Eq. (2.23) works in Lorentzian signature, which we obtain by analytically continuing the corresponding Euclidean expression via $\mathcal{L} \to -\mathcal{L}, n_{\mu\nu} \to n_{\mu\nu}, \varepsilon_{\mu\nu} \to -i\varepsilon_{\mu\nu}$.[8] Extremizing

[7]These higher-order terms are difficult to write down explicitly because of 'splitting' [29, 33, 34, 35], although they can in principle be determined by using appropriate equations of motion [29, 31, 32]. Fortunately, here we only need S_{EE} up to second order in K (and its derivatives), which can be obtained by setting $q_\alpha = 0$ in Eq. (3.30) of [29] and is free from the splitting difficulty — this follows roughly from the results of [29] but will also be proved carefully in a separate work [36]. Moreover, the same S_{EE} up to second order in K (and its derivatives) was derived using the second law of black hole thermodynamics [37].

[8]The extra factor of $-i$ can be traced back to the definition $\varepsilon_{\mu\nu} = \varepsilon_{ab}n^{(a)}_\mu n^{(b)}_\nu$ where $\varepsilon_{ab} = \sqrt{g}\,\widetilde{\varepsilon}_{ab}$, with $\widetilde{\varepsilon}_{ab}$ being the Levi-Civita symbol. Going to the Lorentzian version which comes with $\sqrt{-g}$ requires a factor of i. The minus sign is a result of going from the convention in [29] where $\widetilde{\varepsilon}_{\tau x} = -1$ to the standard Lorentzian convention $\widetilde{\varepsilon}_{tx} = 1$.

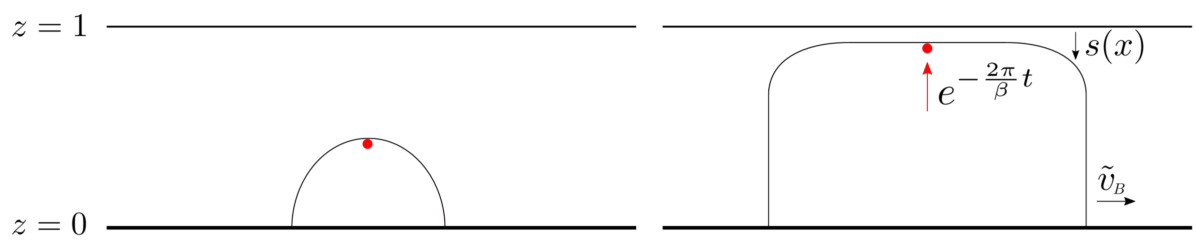

Figure 2.4: An illustration of the entanglement wedge method. A particle is released from the asymptotic boundary and we show the RT surface which barely encloses the particle. At early times (left), the profile of the RT surface depends sensitively on the metric away from the horizon; but at late times (right), the RT surface in the near-horizon region has a characteristic profile which propagates outwards at the butterfly velocity $\widetilde{v}_B$.

S_{EE} leads to a differential equation for the location of the RT surface, which we refer to as the *RT equation*.

For a general planar black hole, the metric is given in (2.8), which can be written in Poincaré-like coordinates as

$$ds^2 = \frac{1}{z^2}\left[-\frac{f(z)}{h(z)}dt^2 + \frac{dz^2}{f(z)} + dx^i dx^i\right], \tag{2.24}$$

where the horizon is located at $z = 1$. In the near-horizon region, we can Taylor expand the functions $f(z) = f_1(1-z)+f_2(1-z)^2+\cdots$ and $h(z) = h_0+h_1(1-z)+h_2(1-z)^2+\cdots$. As in [18], we will consider spherical boundary regions, for which the corresponding RT surfaces are also spherical. We use $r = |x|$ to denote the radial coordinate in the x^i directions. This reduces the RT equation to a simple ODE.

At late times, the probe particle is exponentially close to the horizon and follows a trajectory $1 - z(t) \sim e^{-\frac{4\pi}{\beta}t}$ given by the geodesic equation. As such, we will focus on the near-horizon region near $z = 1$. In particular, defining the RT surface by $z = Z(x)$, we can parametrize the near-horizon limit by writing

$$Z(x) = 1 - \epsilon s(x)^2, \tag{2.25}$$

where ϵ is a small positive number and we have defined a new function $s(x)$ which we call the *RT profile*. The profile is determined by solving the RT equation, which we Taylor expand around $\epsilon = 0$. The RT surface will stay close to the horizon for small r and start to depart the near-horizon region at some large radius $r = r_*$, where $\epsilon s(x)^2|_{r=r_*} \sim \mathcal{O}(1)$, after which it reaches the asymptotic boundary within an order one distance. Thus, the corresponding boundary region has approximately radius r_* and this defines a size of the boundary operator. To model this behavior, one can use an ansatz[9]

$$s(x) \sim \frac{e^{\widetilde{\mu} r}}{r^{\#}}, \tag{2.26}$$

where $\widetilde{\mu} > 0$ and $\#$ is some unimportant integer.

At each point in time, we demand the tip of the RT surface intersects the particle, which is enforced by setting $s(x = 0, t) \sim e^{-\frac{2\pi}{\beta} t}$. Therefore, the time-dependent RT profile is given by

$$s(x, t) \sim \frac{e^{\widetilde{\mu} r - \frac{2\pi}{\beta} t}}{r^{\#}}. \tag{2.27}$$

At any given time t, there is some radius $r = r_*(t)$ such that $\epsilon s(x,t)^2|_{r=r_*(t)} \sim \mathcal{O}(1)$. This in turn gives the radius of the boundary region (modulo an order one distance), which propagates outward with characteristic velocity

$$\widetilde{v}_B \equiv \frac{2\pi}{\beta \widetilde{\mu}}. \tag{2.28}$$

[9]This ansatz is valid when r is large enough (so that we may ignore higher-order corrections in $1/r$) but not too large (so that the RT surface has not exited the near-horizon region). This regime of validity is parametrically large for a large boundary region.

Solving the RT equation yields a value for $\widetilde{\mu}$ and thus a value for $\widetilde{v}_B$ in terms of the functions $f(z)$ and $h(z)$ in the background metric (2.24). This is the second way of calculating the butterfly velocity.

Let us now revisit the examples in Section 2.1.1 using the entanglement wedge method.

Einstein gravity For Einstein gravity with $\mathcal{L}_{\rm EH} = \frac{1}{2}(R - 2\Lambda)$, the entropy functional is simply given by the area:

$$S_{\rm EE} = 2\pi \int d^{d-1}y\sqrt{\gamma}. \tag{2.29}$$

The entire dependence of this functional on the choice of the surface is then in the determinant of the induced metric γ. Using (2.25) and expanding to linear order in ϵ, the induced metric of the surface is given by

$$\gamma_{ij}dx^i dx^j = \left[1 + 2\epsilon\left(s(r)^2 + \frac{2}{f_1}s'(r)^2\right)\right]dr^2 + r^2\Big(1 + 2\epsilon s(r)^2\Big)d\Omega_{d-2}^2 + \mathcal{O}(\epsilon^2), \tag{2.30}$$

where we have Taylor expanded $f(z) \equiv f_1(1-z) + f_2(1-z)^2 + \cdots$ near the horizon. This is a flat metric up to $\mathcal{O}(\epsilon)$ corrections, which is consistent with the fact that we are expanding near the horizon of a planar black hole. The determinant is then given by

$$\sqrt{\gamma} = r^{d-2} + \epsilon\, r^{d-2}\left((d-1)s(r)^2 + \frac{2}{f_1}s'(r)^2\right) + \mathcal{O}(\epsilon^2). \tag{2.31}$$

The RT equation is then determined by varying with respect to $s(r)$, which at leading order in ϵ gives

$$(d-1)s - \frac{2}{f_1}\left(s'' + (d-2)\frac{s'}{r}\right) = 0. \tag{2.32}$$

Substituting the ansatz (2.26) and dropping higher order terms in $1/r$ then gives

$$\widetilde{\mu}^2 = \frac{d-1}{2} f_1 = \frac{d-1}{2} \frac{B_1}{A_0}, \tag{2.33}$$

where the second equality follows from a coordinate transformation. Using (2.28), we thus find that the resulting butterfly velocity matches the shockwave result (2.18).

Lovelock gravity We will now show that the Lovelock corrections $\mathcal{L}_{2p}$ with $p > 1$ do not contribute to the RT equation, consistent with the shockwave calculation in Section 2.1.1.

It is well-known that the entropy functional for this theory is given by the Jacobson-Myers formula [38]

$$S_{\mathrm{EE},2p} = 2\pi\lambda_{2p} \int d^{d-1}y \sqrt{\gamma} \frac{p}{2^{p-2}} \delta^{i_1 j_1 \cdots i_{p-1} j_{p-1}}_{k_1 l_1 \cdots k_{p-1} l_{p-1}} \mathcal{R}_{i_1 j_1}{}^{k_1 l_1} \cdots \mathcal{R}_{i_{p-1} j_{p-1}}{}^{k_{p-1} l_{p-1}}, \tag{2.34}$$

where $\mathcal{R}_{ij}{}^{kl}$ is the intrinsic curvature of the codimension-2 surface. For an RT surface perturbatively close to the horizon, the induced metric is again given by (2.30). We can therefore immediately write $\mathcal{R}_{ij}{}^{kl} \sim \mathcal{O}(\epsilon)$, which implies

$$S_{\mathrm{EE},2p} \sim \mathcal{O}(\epsilon^{p-1}). \tag{2.35}$$

For $p > 2$, these terms are higher order in ϵ and therefore do not contribute. Thus these Lovelock corrections do not modify (2.33).

For the case of $p = 2$, namely Gauss-Bonnet gravity, a more refined argument is required. In this case, the entropy functional (2.34) depends on the intrinsic curvature scalar $\mathcal{R} = \mathcal{R}_{ij}{}^{ij}$ which, up to unimportant numerical factors, is given by

$$\mathcal{R} \sim \frac{\epsilon}{r}\frac{d}{dr}\Big(rs(r)s'(r)\Big) + \mathcal{O}(\epsilon^2), \tag{2.36}$$

where we have neglected terms that only contribute higher orders in $1/r$ to the RT equation compared to the Einstein-Hilbert term (2.31). The entropy is then given by

$$\begin{aligned} S_{\text{EE,GB}} &\sim \int d^{d-1}y\sqrt{\gamma}\,\mathcal{R} \sim \epsilon\int dr\, r^{d-3}\frac{d}{dr}\Big(rs(r)s'(r)\Big) + \mathcal{O}(\epsilon^2) \\ &\sim \epsilon\int dr\,\frac{d}{dr}\Big(r^{d-2}s(r)s'(r)\Big) - (d-3)\,\epsilon\int dr\, r^{d-3}s(r)s'(r) + \mathcal{O}(\epsilon^2). \end{aligned} \tag{2.37}$$

The first term is a total derivative and therefore does not contribute to the RT equation. The second term is down by a factor of $1/r$ compared to the contribution from the Einstein-Hilbert term (2.31). Since we are only interested in the leading large-r profile, this term will not contribute either. We therefore conclude that the Gauss-Bonnet correction does not modify (2.33), reproducing the result from [18].

2.2 Butterfly velocities for f(Riemann) gravity

In this section we derive general formulae for the butterfly velocities using the two holographic methods described in Section 2.1, valid for all f(Riemann) theories. Before diving into the derivations, we will lay out a few useful definitions and notations and calculate a few basic quantities related to the background metric.

2.2.1 Definitions, notations, and the spacetime

As reviewed in Section 2.1.1, the metric for the shockwave geometry is given by

$$ds^2 = 2A(uv)dudv + B(uv)dx^i dx^i - 2A(uv)h(x)\delta(u)du^2. \tag{2.38}$$

The background metric with $h = 0$ is simply the planar black hole (2.8). We will denote the functions A, B and their derivatives evaluated on the horizon at $u = 0$ by $A_n \equiv \partial^n A(uv)/\partial(uv)^n|_{u=0}$ and $B_n \equiv \partial^n B(uv)/\partial(uv)^n|_{u=0}$. Throughout our calculations, we set $B_0 = 1$ by rescaling the x^i coordinates.

The non-zero Christoffel symbols are

$$\begin{aligned}
&\Gamma^v{}_{uu} = -h\delta'(u) + vA^{-1}A'h\delta(u), \quad \Gamma^u{}_{uu} = vA^{-1}A', \quad \Gamma^v{}_{vv} = uA^{-1}A', \\
&\Gamma^i{}_{uu} = AB^{-1}h_j\delta^{ij}\delta(u), \quad \Gamma^v{}_{ui} = -h_i\delta(u), \quad \Gamma^v{}_{ij} = -\frac{1}{2}vA^{-1}B'\delta_{ij}, \\
&\Gamma^u{}_{ij} = -\frac{1}{2}uA^{-1}B'\delta_{ij}, \quad \Gamma^i{}_{ju} = \frac{1}{2}vB^{-1}B'\delta^i_j, \quad \Gamma^i{}_{jv} = \frac{1}{2}uB^{-1}B'\delta^i_j,
\end{aligned} \tag{2.39}$$

where $h_i \equiv \partial_i h(x)$, and we have dropped terms[10] containing $u\delta(u)$. It will become clear in our later calculations that the second term $vA^{-1}A'h\delta(u)$ in $\Gamma^v{}_{uu}$ is unimportant because it always enters the equations of motion together with at least one power of u.

The non-zero components of the Riemann tensor are

$$R_{uvuv} = A' + uvA'' - uvA^{-1}A'^2, \tag{2.40a}$$

$$R_{uivj} = -\frac{1}{2}\delta_{ij}\left(B' + uvB'' - \frac{1}{2}uvB^{-1}B'^2\right), \tag{2.40b}$$

$$R_{uiuj} = \frac{\delta_{ij}}{2}v^2\left(A^{-1}A'B' + \frac{1}{2}B^{-1}B'^2 - B''\right) + \left(Ah_{ij} + \frac{\delta_{ij}}{2}B'h\right)\delta(u), \tag{2.40c}$$

[10] We can drop these terms at this stage because, as we will see, they will not be multiplied by any negative powers of u in the equations of motion. This allows us to use the identities $u\delta(u) = 0$ and $u\delta'(u) = -\delta(u)$, viewed as equality of distributions.

$$R_{vivj} = \frac{\delta_{ij}}{2} u^2 \left(A^{-1} A' B' + \frac{1}{2} B^{-1} B'^2 - B'' \right), \tag{2.40d}$$

$$R_{ijkl} = \frac{1}{2} uv A^{-1} B'^2 (\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}), \tag{2.40e}$$

where we have again discarded terms which vanish as a distribution. We see that the only correction due to h is in R_{uiuj}, and the correction is linear in h. The same statement applies for ${R_{ui}}^{vj}$. The fact that, given the specific ansatz (2.38), these are the only possible non-zero components of the Riemann tensor is crucial in establishing our proof.

For notational convenience, we define the following tensors and use them to denote the corresponding values evaluated in the background solution and on the $u = 0$ horizon:

$$\begin{gathered} C_{\rho\sigma} \equiv \frac{\partial \mathcal{L}}{\partial g^{\rho\sigma}}, \quad D^{\mu\nu\rho\sigma} \equiv \frac{\partial \mathcal{L}}{\partial R_{\mu\nu\rho\sigma}}, \quad {F^{\mu\nu\rho\sigma}}_{\lambda\tau} \equiv \frac{\partial^2 \mathcal{L}}{\partial R_{\mu\nu\rho\sigma} \partial g^{\lambda\tau}}, \\ G_{\mu\nu\rho\sigma} \equiv \frac{\partial^2 \mathcal{L}}{\partial g^{\mu\nu} \partial g^{\rho\sigma}}, \quad H^{\mu\nu\rho\sigma\mu'\nu'\rho'\sigma'} \equiv \frac{\partial^2 \mathcal{L}}{\partial R_{\mu\nu\rho\sigma} \partial R_{\mu'\nu'\rho'\sigma'}}. \end{gathered} \tag{2.41}$$

For example, $C_{\rho\sigma}$ denotes the value of $\partial \mathcal{L} / \partial g^{\rho\sigma}$ in the background $h = 0$ solution and on the $u = 0$ horizon. Here, we have viewed the Lagrangian $\mathcal{L}$ as a function of $R_{\mu\nu\rho\sigma}$ and $g^{\mu\nu}$, i.e., we lower all indices on the Riemann tensor and raise all indices on the metric and treat these two types of tensors as independent variables when taking derivatives. Throughout this paper, we define derivatives with respect to tensors such as $R_{\mu\nu\rho\sigma}$ and $g^{\mu\nu}$ in the standard way; for example, $\partial \mathcal{L} / \partial R_{\mu\nu\rho\sigma}$ has the Riemann symmetry and satisfies the identity $\delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial R_{\mu\nu\rho\sigma}} \delta R_{\mu\nu\rho\sigma}$.

Since the i, j indices can only appear in the combination δ_{ij} for any background quantity, we can define the following notation where the transverse components are stripped away:

$$D^{aibj} \equiv D^{ab} \delta^{ij}, \quad {F^{aibj}}_{cd} \equiv {F^{ab}}_{cd} \delta^{ij}, \quad {F^{abcd}}_{ij} \equiv F^{(2)abcd} \delta_{ij},$$

$$H^{aibjckdl} \equiv H^{(1)abcd}\delta^{ij}\delta^{kl} + \frac{H^{(2)abcd}}{2}(\delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk}), \quad H^{abcd} \equiv H^{(1)abcd} + \frac{H^{(2)abcd}}{d-1},$$
$$H^{abcdeifj} \equiv H^{(3)abcdef}\delta^{ij}, \quad H^{abcdijkl} \equiv H^{(4)abcd}(\delta^{ik}\delta^{jl} - \delta^{il}\delta^{jk}),$$

where $a, b, c, \dots \in \{u, v\}$ and $i, j, k, \dots$ are the transverse directions.

2.2.2 The shockwave method

For general f(Riemann) theories, we start by writing down the equations of motion:

$$E_{\mu\nu} \equiv \frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} = E^{(1)}_{\mu\nu} + E^{(2)}_{\mu\nu} + E^{(3)}_{\mu\nu} + E^{(4)}_{\mu\nu}, \tag{2.42}$$

with

$$\begin{aligned} E^{(1)}_{\mu\nu} &= -g_{\mu\nu}\mathcal{L}, \quad E^{(2)}_{\mu\nu} = 2\frac{\partial \mathcal{L}}{\partial g^{\mu\nu}}, \\ E^{(3)\mu\nu} &= -2\frac{\partial \mathcal{L}}{\partial R_{\mu\lambda\rho\sigma}} R^{\nu}{}_{\lambda\rho\sigma}\Big|_{\mathrm{sym}(\mu\nu)}, \quad E^{(4)\mu\nu} = 4\left(\frac{\partial \mathcal{L}}{\partial R_{\mu\rho\nu\sigma}}\right)_{;\sigma;\rho}\Bigg|_{\mathrm{sym}(\mu\nu)}, \end{aligned} \tag{2.43}$$

where the Lagrangian $\mathcal{L}$ is viewed as a function of $R_{\mu\nu\rho\sigma}$ and $g^{\mu\nu}$. Here the parenthesized numbers $(1), \dots, (4)$ merely label the various terms and do not have any physical meaning.

We view the shockwave spacetime (2.38) as a perturbation from the background geometry (2.8) with

$$\delta g_{uu} = -2Ah\delta(u). \tag{2.44}$$

The only non-zero component of the perturbation of the inverse metric is

$$\delta g^{vv} = 2A^{-1}h\delta(u). \tag{2.45}$$

As we show in Appendix A, the only component of the equations of motion E^{ν}_{μ} that receives a non-vanishing perturbation from the shockwave is E_u^v, with

$$\delta E_u^v = g_{uv}\delta E^{vv} + E^{uv}\delta g_{uu}. \tag{2.46}$$

We now calculate these two terms separately.

For δE^{vv}, we have

$$\delta E^{vv} = \delta E^{(1)vv} + \delta E^{(2)vv} + \delta E^{(3)vv} + \delta E^{(4)vv}, \tag{2.47a}$$

where

$$\delta E^{(1)vv} = -\,\delta g^{vv}\mathcal{L} = -\frac{2\mathcal{L}}{A}h\delta(u), \tag{2.47b}$$

$$\begin{aligned}
\delta E^{(2)vv} =& 4\delta g^{vv}g^{uv}C_{uv} + 2(g^{uv})^2 G_{uuvv}\delta g^{vv} + 8(g^{uv})^2 F^{uu}{}_{uu}\delta^{ij}\delta R_{uiuj} \\
=& \left[\frac{8}{A^2}C_{uv} + \frac{4}{A^3}G_{uuvv} + \frac{8F^{uu}{}_{uu}}{A^2}\left(A\partial_i\partial_i + \frac{d-1}{2}B'\right)\right] h\delta(u),
\end{aligned} \tag{2.47c}$$

$$\begin{aligned}
\delta E^{(3)vv} =& -\left(4D^{vuvu}R_{vuvu} + 4D^{viuj}R_{viuj}\right)\delta g^{vv} - 4D^{viuj}g^{uv}\delta R_{uiuj} \\
& - 4F^{viaj}{}_{vv}g^{uv}R_{uiaj}\delta g^{vv} - 16H^{uiujvkvl}g^{uv}R_{ukvl}\delta R_{uiuj} \\
=& \Bigg[-\frac{8A'}{A}D^{uvuv} + 4(d-1)\frac{B'}{A}\left(D^{uv} + F^{vv}{}_{vv}\right) \\
& - \left(4D^{uv} - 8(d-1)H^{uuvv}B'\right)\left(\partial_i\partial_i + \frac{d-1}{2}\frac{B'}{A}\right)\Bigg] h\delta(u),
\end{aligned} \tag{2.47d}$$

$$\begin{aligned}
\delta E^{(4)vv} =& 4\delta\left(\nabla_\rho\nabla_\sigma\frac{\partial\mathcal{L}}{\partial R_{v\rho v\sigma}}\right)\\
=& 4F^{vivj}{}_{vv}\delta g^{vv}{}_{,j,i} + 16H^{ukulvivj}\delta R_{ukul,j,i}\\
&+ 4\delta\Gamma^{v}{}_{uj,i}D^{uivj} + 4\delta\Gamma^{i}{}_{uu,i}D^{vuvu} + 4\delta\Gamma^{v}_{uu}D^{vuu\sigma}{}_{;\sigma}\\
=& \Bigg[\frac{8}{A}F^{vv}{}_{vv}\partial_i\partial_i + 8H^{(1)uuvv}\left(2A\partial_i\partial_i\partial_j\partial_j + (d-1)B'\partial_i\partial_i\right)\\
&+ 8H^{(2)uuvv}\left(2A\partial_i\partial_i\partial_j\partial_j + B'\partial_i\partial_i\right) - 4D^{uv}\partial_i\partial_i + 4D^{uvuv}\frac{A}{B}\partial_i\partial_i\\
&- 2(d-1)\frac{B'}{A}D^{uv} - \left(8\frac{A'}{A} + 2(d-1)\frac{B'}{B}\right)D^{uvuv} + 8\frac{A'}{A^2}F^{uvuv}{}_{uv}\\
&+ 4(d-1)\frac{B'}{B^2}F^{(2)uvuv} - 16\left(2A'' - \frac{A'^2}{A}\right)H^{uvuvuvuv}\\
&- 32(d-1)\left(\frac{B'^2}{4B} - B''\right)H^{(3)uvuvuv} + 4(d-1)(d-2)\frac{B'^2}{A}H^{(4)uvuv}\Bigg]h\delta(u).
\end{aligned} \tag{2.47e}$$

In arriving at this, it is important that we use the distributional identity $u\delta'(u) = -\delta(u)$ (see Footnote 10). The δ-function sets $u = 0$ so the quantities are all evaluated on the horizon. In deriving (2.47), it is useful to note the following simplifying properties. First, in the background solution, every extra v-index downstairs (beyond those paired with a u-index downstairs or a v-index upstairs) costs a factor of u. Similarly, a single i-type index cannot contribute in the background solution since it must come paired with another such index to form a Kronecker delta.

To work out the second term on the right-hand side of (2.46), we find an expression for E^{uv} in the background solution on the horizon:

$$E^{uv} = E^{(1)uv} + E^{(2)uv} + E^{(3)uv} + E^{(4)uv}, \tag{2.48a}$$

where

$$E^{(1)uv} = -\frac{\mathcal{L}}{A}, \tag{2.48b}$$

$$E^{(2)uv} = \frac{2C_{uv}}{A^2}, \tag{2.48c}$$

$$E^{(3)uv} = 2(d-1)\frac{B'}{A}D^{uv} - \frac{4A'}{A}D^{uvuv}, \tag{2.48d}$$

$$\begin{aligned} E^{(4)uv} = &\frac{8A'}{A^2}F^{uvuv}{}_{uv} + 4(d-1)B'F^{(2)uvuv} - 16\left(2A'' - \frac{A'^2}{A}\right)H^{uvuvuvuv} \\ &+ 16(d-1)\left(2B'' - \frac{B'^2}{2}\right)H^{(3)uvuvuv} + 4(d-1)(d-2)\frac{B'^2}{A}H^{(4)uvuv} \\ &+ 2\left(\frac{4A'}{A} + (d-1)B'\right)D^{uvuv} - 2(d-1)\frac{B'}{A}D^{uv}. \end{aligned} \tag{2.48e}$$

Finally, collecting all the terms into (2.46) and plugging in the ansatz for $h(x)$ yields

$$\lim_{r\to\infty}\frac{1}{4h(x)}\delta E_u^v\Big|_{h(x)\sim\frac{e^{-\mu r}}{r^{\#}}} \equiv \delta(u)f_{\rm SW}(\mu) = 0, \tag{2.49}$$

where we have taken the large-r limit and neglected higher-order terms in $1/r$, and

$$\begin{aligned} f_{\rm SW}(\mu) = &\frac{C_{uv}}{A_0^2} + \left(\frac{2A_1}{A_0} + \frac{d-1}{2}B_1\right)D^{uvuv} - (d-1)B_1F^{(2)uvuv} \\ &- \frac{2A_1}{A_0^2}F^{uvuv}{}_{uv} + (d-1)\frac{B_1}{A_0^2}\left(F^{uu}{}_{uu} + F^{vv}{}_{vv}\right) + \frac{G_{uuvv}}{A_0^3} \\ &+ (d-1)^2\frac{B_1^2}{A_0}H^{uuvv} + 2(d-1)\left(B_1^2 - 4B_2\right)H^{(3)uvuvuv} \\ &- (d-1)(d-2)\frac{B_1^2}{A_0}H^{(4)uvuv} + 4\left(2A_2 - \frac{A_1^2}{A_0}\right)H^{uvuvuvuv} \\ &+ \left(-2D^{uv} + A_0D^{uvuv} + \frac{2}{A_0}\left(F^{uu}{}_{uu} + F^{vv}{}_{vv}\right) + 4(d-1)B_1H^{uuvv}\right)\mu^2 \\ &+ 4A_0\left(H^{(1)uuvv} + H^{(2)uuvv}\right)\mu^4 \end{aligned} \tag{2.50}$$

is a function of quantities on the horizon through imposing $\delta(u)$. This is a quartic[11] equation for μ and a quadratic equation for μ^2. The correct root is the one that is positive and continuously connected to the unperturbed Einstein gravity result given in Eq. (2.17). We then extract the butterfly velocity v_B from (2.15).

2.2.3 The entanglement wedge method

Let us now move on to the entanglement wedge method. As reviewed in Section 2.1.2, our objective is to find the size of the smallest spherical boundary region whose entanglement wedge encloses a probe particle falling into the black hole.

We will work in the same coordinate system as used for the shockwave method. This will make it easier to see the matching with the shockwave result, but at the expense of making the time translation symmetry slightly less explicit. The metric is the planar black hole given by

$$ds^2 = 2A(uv)dudv + B(uv)dx^i dx^i. \tag{2.51}$$

We would now like to derive the RT equation for a spherical boundary region in this background using (2.23). Before we proceed, let us make the following simplifying observations:

1. Entanglement surfaces anchored to a single boundary can never penetrate the horizon, so we can choose to work in one of the exterior patches and exploit the time translation symmetry. Because of this symmetry, we only need to look for the RT surface rather than the HRT surface. This means we can restrict to the $u = -v$ hypersurface in order to get the spatial profile of the entanglement surface. This is the $t = 0$ surface in the original (t, z, x^i) coordinates.

[11] It is quartic because we are considering f(Riemann) theories, which have only up to four derivatives acting on a single factor of the metric in the equations of motion (2.43).

2. Since we are interested in the near-horizon limit, the butterfly velocity can be calculated by extremizing the entropy functional S_{EE} with respect to a candidate RT surface defined by $uv = -\epsilon s(x)^2$ to linear order in ϵ. Since $u = -v$, we can consider each factor of u or v as contributing a factor of $\sqrt{\epsilon}$.

3. The entropy functional S_{EE} given in Eq. (2.23) is only accurate at second order in the extrinsic curvature K and its derivatives, but this is sufficient to determine the butterfly velocity. In particular, higher-order terms in K and its derivatives are suppressed by additional powers of ϵ, and can thus be neglected in the near-horizon limit.

4. The $\varepsilon^{\lambda_1\lambda_2}$ term in Eq. (2.23) vanishes when restricting to RT surfaces. To see this, go to a coordinate system where the time translation symmetry is manifest (so ∂_t is the timelike Killing vector field), and note that $K_{\lambda\rho\sigma}$ vanishes if $\lambda = t$ but $\varepsilon^{\lambda_1\lambda_2}$ vanishes unless one of λ_1 and λ_2 is t.

5. We can write $K_{\lambda_1\rho_1\sigma_1}K_{\lambda_2\rho_2\sigma_2}n^{\lambda_1\lambda_2} = K_{2\rho_1\sigma_1}K_{2\rho_2\sigma_2}$, where '2' denotes the direction of the second normal vector (which is orthogonal to the t direction), i.e., $K_{2\rho\sigma} = h^\mu_\rho h^\nu_\sigma \nabla_\mu n^{(2)}_\nu$, where h^μ_ν is the projector onto the codimension-2 surface. This is a simplification due to the first observation above: the extrinsic curvature $K_{1\rho\sigma} \equiv K_{t\rho\sigma} = 0$ because of the time reflection symmetry at $t = 0$.

Implementing these simplifications and writing $S_{\text{EE}} = 2\pi \int d^{d-1}y \sqrt{\gamma}\mathcal{L}_{\text{EE}}$, we have

$$\begin{aligned} \mathcal{L}_{\text{EE}} = &- \frac{\partial \mathcal{L}}{\partial R_{\mu\rho\nu\sigma}} \varepsilon_{\mu\rho}\varepsilon_{\nu\sigma} \\ &- \frac{\partial^2 \mathcal{L}}{\partial R_{\mu_1\rho_1\nu_1\sigma_1} \partial R_{\mu_2\rho_2\nu_2\sigma_2}} 2K_{2\rho_1\sigma_1}K_{2\rho_2\sigma_2} \left(n_{\mu_1\mu_2}n_{\nu_1\nu_2} + \varepsilon_{\mu_1\mu_2}\varepsilon_{\nu_1\nu_2}\right). \end{aligned} \tag{2.52}$$

We call the second term the extrinsic curvature term.

The non-zero components of the Riemann tensor in the background solution are again given by (2.40) but with h set to zero, i.e., without the shockwave. With our candidate entanglement surface defined on $uv = -\epsilon s(x)^2$, the components of the two normals are then given by

$$\begin{aligned}
n_u^{(1)} &= \sqrt{\frac{vA(uv)}{2u}}, \quad n_u^{(2)} = \frac{-v}{\sqrt{2uv/A(uv) + 4\epsilon^2 s^2 s_j s_j / B(uv)}}, \\
n_v^{(1)} &= \sqrt{\frac{uA(uv)}{2v}}, \quad n_v^{(2)} = \frac{-u}{\sqrt{2uv/A(uv) + 4\epsilon^2 s^2 s_j s_j / B(uv)}}, \\
n_i^{(1)} &= 0, \quad n_i^{(2)} = \frac{-2\epsilon s s_i}{\sqrt{2uv/A(uv) + 4\epsilon^2 s^2 s_j s_j / B(uv)}},
\end{aligned} \tag{2.53}$$

where s_i stands for $\partial_i s(x)$. In deriving this, we used the fact that t is a function of u/v, $n^{(1)} \sim dt$, and $n^{(2)} \sim df$ where $f = uv + \epsilon s^2$.

Next, we need the following tensors, defined by

$$n_{\mu\nu} = -n_\mu^{(1)} n_\nu^{(1)} + n_\mu^{(2)} n_\nu^{(2)}, \tag{2.54}$$

$$\varepsilon_{\mu\nu} = n_\mu^{(1)} n_\nu^{(2)} - n_\mu^{(2)} n_\nu^{(1)}. \tag{2.55}$$

To linear order in ϵ, the non-zero components of $\varepsilon_{\mu\nu}$ are given by

$$\begin{aligned}
\varepsilon_{uv} &= A_0 - \epsilon \left(s^2 A_1 - A_0^2 s_j s_j \right), \\
\varepsilon_{ui} &= \sqrt{\frac{-\epsilon\, v}{u}} A_0 s_i + \mathcal{O}(\epsilon^{3/2}), \\
\varepsilon_{vi} &= \sqrt{\frac{-\epsilon\, u}{v}} A_0 s_i + \mathcal{O}(\epsilon^{3/2}).
\end{aligned} \tag{2.56}$$

It turns out that we will only need the $\mathcal{O}(1)$ term in $n_{\mu\nu}$, and the only non-zero component at this order is

$$n_{uv} = A_0 + \mathcal{O}(\epsilon). \tag{2.57}$$

We will also need the extrinsic curvatures. To leading order, the only non-zero component is

$$K_{2ij} = \sqrt{\frac{-\epsilon}{2A_0}}\left(-B_1\delta_{ij}s + A_0 s_{ij}\right). \tag{2.58}$$

We can now derive the contributions to $S_{\rm EE}$ at linear order in ϵ. For any quantity X, we denote the linear order coefficient in a Taylor expansion in ϵ by ΔX. Then $\Delta S_{\rm EE} = 2\pi \int d^{d-1}y\, \Delta(\sqrt{\gamma}\mathcal{L}_{\rm EE})$ is given by

$$\Delta(\sqrt{\gamma}\mathcal{L}_{\rm EE}) \equiv (\Delta\sqrt{\gamma})\bar{\mathcal{L}}_{\rm EE} + \sqrt{\bar{\gamma}}\left(\Delta\mathcal{L}^{(1)}_{\rm EE} + \Delta\mathcal{L}^{(2)}_{\rm EE} + \Delta\mathcal{L}^{(3)}_{\rm EE}\right), \tag{2.59}$$

$$\bar{\mathcal{L}}_{\rm EE} = -D^{\mu\rho\nu\sigma}\bar{\varepsilon}_{\mu\rho}\bar{\varepsilon}_{\nu\sigma}, \tag{2.60}$$

$$\Delta\mathcal{L}^{(1)}_{\rm EE} = -\bar{\varepsilon}_{\mu\rho}\bar{\varepsilon}_{\nu\sigma}\Delta\left(\frac{\partial\mathcal{L}}{\partial R_{\mu\rho\nu\sigma}}\right), \tag{2.61}$$

$$\Delta\mathcal{L}^{(2)}_{\rm EE} = -\Delta(\varepsilon_{\mu\rho}\varepsilon_{\nu\sigma})D^{\mu\rho\nu\sigma}, \tag{2.62}$$

$$\Delta\mathcal{L}^{(3)}_{\rm EE} = -\Delta\left(2K_{2\rho_1\sigma_1}K_{2\rho_2\sigma_2}\right)\left(\bar{n}_{\mu_1\mu_2}\bar{n}_{\nu_1\nu_2} + \bar{\varepsilon}_{\mu_1\mu_2}\bar{\varepsilon}_{\nu_1\nu_2}\right)H^{\mu_1\rho_1\nu_1\sigma_1\mu_2\rho_2\nu_2\sigma_2}, \tag{2.63}$$

where the barred quantities are evaluated on the horizon at $\epsilon = 0$. Note that quantities such as $D^{\mu\rho\nu\sigma}$ and $H^{\mu_1\rho_1\nu_1\sigma_1\mu_2\rho_2\nu_2\sigma_2}$ do not need to have bars because they are already defined to be evaluated on the horizon. The last piece is the only contribution from the extrinsic curvature term of $S_{\rm EE}$ since $K_{2ij} = \mathcal{O}(\sqrt{\epsilon})$.

The determinant of the induced metric is given by

$$\sqrt{\gamma} = 1 - \epsilon\left(A_0 s_j s_j + \frac{d-1}{2}B_1 s^2\right) + \mathcal{O}(\epsilon^2), \tag{2.64}$$

which can be derived by substituting $uv = -\epsilon s(x)^2$ into the metric and expanding the identity $\det\exp M = \exp\operatorname{Tr} M$ to linear order. Then

$$\frac{1}{4A_0^2}(\Delta\sqrt{\gamma})\bar{\mathcal{L}}_{\rm EE} = \left(A_0 s_j s_j + \frac{d-1}{2}B_1 s^2\right)D^{uvuv}, \tag{2.65}$$

where we have used the fact that only $\varepsilon_{uv} \neq 0$ at zeroth order.

For the next term, we need

$$\Delta\left(\frac{\partial \mathcal{L}}{\partial R_{\mu\rho\nu\sigma}}\right) = H^{\mu\rho\nu\sigma\mu'\rho'\nu'\sigma'}\Delta R_{\mu'\rho'\nu'\sigma'} + 2F^{\mu\rho\nu\sigma}{}_{uv}\Delta g^{uv} + F^{\mu\rho\nu\sigma}{}_{ij}\Delta g^{ij}, \tag{2.66}$$

$$g^{uv} = \frac{1}{A(-\epsilon s^2)} = \frac{1}{A_0} + \epsilon\frac{A_1}{A_0^2}s^2 + \mathcal{O}(\epsilon^2), \tag{2.67}$$

$$g^{ij} = \frac{\delta^{ij}}{B(-\epsilon s^2)} = \delta^{ij}\left(1 + \epsilon B_1 s^2 + \mathcal{O}(\epsilon^2)\right). \tag{2.68}$$

We also notice that H vanishes if the numbers of lower u and v indices do not match (each upper u is considered one lower v and vice versa). Then we have

$$\begin{aligned}\frac{1}{4A_0^2}\sqrt{\bar{\gamma}}\Delta\mathcal{L}_{\mathrm{EE}}^{(1)} = \Bigg[&4\left(2A_2 - \frac{A_1^2}{A_0}\right)H^{uvuvuvuv} - 4\left(2B_2 - \frac{1}{2}B_1^2\right)\delta_{ij}H^{uvuvuivj} \\ &+ \frac{B_1^2}{2A_0}\left(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}\right)H^{uvuvijkl} - \frac{2A_1}{A_0^2}F^{uvuv}{}_{uv} - B_1\delta^{ij}F^{uvuv}{}_{ij}\Bigg]s^2. \end{aligned}\tag{2.69}$$

Using the expressions for $\varepsilon_{\mu\nu}$ above, the third term is simply given by

$$\frac{1}{4A_0^2}\sqrt{\bar{\gamma}}\Delta\mathcal{L}_{\mathrm{EE}}^{(2)} = 2D^{uvuv}\left(\frac{A_1}{A_0}s^2 - A_0 s_j s_j\right) - 2D^{uivj}s_i s_j. \tag{2.70}$$

In the last term $\Delta\mathcal{L}_{\mathrm{EE}}^{(3)}$, we notice that only K_{2ij} has low enough order in ϵ to contribute, so we have

$$\frac{1}{4A_0^2}\sqrt{\bar{\gamma}}\Delta\mathcal{L}_{\mathrm{EE}}^{(3)} = H^{uiujvkvl}\left[\frac{B_1^2}{A_0}\delta_{ij}\delta_{kl}s^2 - 2B_1 s(s_{ij}\delta_{kl} + s_{kl}\delta_{ij}) + 4A_0 s_{ij}s_{kl}\right]. \tag{2.71}$$

Finally, putting everything together, the total contribution to the entropy functional at linear order in ϵ is given by

$$\begin{aligned}
&\frac{1}{4A_0^2}\Delta(\sqrt{\gamma}\mathcal{L}_{\text{EE}})\\
=&\left[\left(\frac{2A_1}{A_0}+\frac{d-1}{2}B_1\right)D^{uvuv}-B_1\delta^{ij}F^{uvuv}{}_{ij}-\frac{2A_1}{A_0^2}F^{uvuv}{}_{uv}\right.\\
&\quad+\frac{B_1^2}{A_0}\delta_{ij}\delta_{kl}H^{uiujvkvl}+2\left(B_1^2-4B_2\right)\delta_{ij}H^{uvuvuivj}\\
&\quad\left.+\frac{B_1^2}{2A_0}\left(\delta_{il}\delta_{jk}-\delta_{ik}\delta_{jl}\right)H^{uvuvijkl}+4\left(2A_2-\frac{A_1^2}{A_0}\right)H^{uvuvuvuv}\right]s^2\\
&-\left(A_0D^{uvuv}\delta_{ij}+2D^{uivj}\right)s_is_j-2H^{uiujvkvl}\Big[B_1s\left(s_{ij}\delta_{kl}+s_{kl}\delta_{ij}\right)-2A_0s_{ij}s_{kl}\Big].
\end{aligned}\tag{2.72}$$

To obtain the butterfly velocity, we vary $\Delta S_{\text{EE}}=2\pi\int d^{d-1}y\,\Delta(\sqrt{\gamma}\mathcal{L}_{\text{EE}})$ with respect to $s(x)$ and then substitute our ansatz $s(x)\sim\frac{e^{\widetilde{\mu}r}}{r^{\#}}$ from (2.26), keeping only leading terms in $1/r$. It is not hard to see that the number of x^i-derivatives on s will be the number of factors of $\widetilde{\mu}$. From this, we obtain a polynomial equation for $\widetilde{\mu}$:

$$\lim_{r\to\infty}\frac{1}{16\pi A_0^2 s(x)}\frac{\delta\left(\Delta S_{\text{EE}}\right)}{\delta s(x)}\bigg|_{s(x)\sim\frac{e^{\widetilde{\mu}r}}{r^{\#}}}\equiv f_{\text{EE}}(\widetilde{\mu})=0,\tag{2.73}$$

where

$$\begin{aligned}
f_{\text{EE}}(\widetilde{\mu})=&\left(\frac{2A_1}{A_0}+\frac{d-1}{2}B_1\right)D^{uvuv}-(d-1)B_1F^{(2)uvuv}-\frac{2A_1}{A_0^2}F^{uvuv}{}_{uv}\\
&\quad+(d-1)^2\frac{B_1^2}{A_0}H^{uuvv}+2(d-1)\Big(B_1^2-4B_2\Big)H^{(3)uvuvuv}\\
&\quad-(d-1)(d-2)\frac{B_1^2}{A_0}H^{(4)uvuv}+4\left(2A_2-\frac{A_1^2}{A_0}\right)H^{uvuvuvuv}\\
&+\Big(2D^{uv}+A_0D^{uvuv}-4B_1(d-1)H^{uuvv}\Big)\widetilde{\mu}^2+4A_0\Big(H^{(1)uuvv}+H^{(2)uuvv}\Big)\widetilde{\mu}^4.
\end{aligned}\tag{2.74}$$

Notice that all coefficients only involve quantities evaluated on the horizon; this is true as well in the shockwave calculation, where it is enforced by the presence of $\delta(u)$. Similar to the shockwave result (2.50), this is a quartic equation for $\widetilde{\mu}$ and a quadratic equation for $\widetilde{\mu}^2$, from which we choose the positive root continuously connected to the result for Einstein gravity (2.17). The butterfly velocity $\widetilde{v}_B$ is then obtained from $\widetilde{\mu}$ using (2.28).

Before proceeding to show that the two butterfly velocities we have derived agree, let us pause for a moment and use the results from this and the previous subsection in two explicit examples.

First, consider the following four-derivative correction to Einstein gravity:

$$\mathcal{L} \supset R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma}R_{\mu'\nu'\rho'\sigma'}g^{\mu\mu'}g^{\nu\nu'}g^{\rho\rho'}g^{\sigma\sigma'}. \tag{2.75}$$

The non-vanishing tensor components are given by

$$\begin{aligned}
C_{uv} &= 2(R_{uvuv})^2(g^{uv})^3 + 4R_{uivj}R_{vkul}g^{uv}\delta^{ik}\delta^{jl}\big|_{u=0} = \frac{8A_1^2}{A_0^3} + 2(d-1)\frac{B_1^2}{A_0},\\
G_{uuvv} &= 8R_{uivj}R_{ukvl}\delta^{ik}\delta^{jl} + 8R_{uvuv}R_{uvvu}(g^{uv})^2\big|_{u=0} = 2(d-1)B_1^2 - \frac{8A_1^2}{A_0^2},\\
D^{uvuv} &= 2R^{uvuv}|_{u=0} = \frac{2A_1}{A_0^4}, \quad D^{uivj} = 2R^{uivj}|_{u=0} = -\frac{B_1}{A_0^2}\delta^{ij},\\
F^{uvuv}{}_{uv} &= \frac{4}{A_0^3}R_{uvuv}\bigg|_{u=0} = \frac{4A_1}{A_0^3}, \quad F^{uivj}{}_{uv} = 2R^{uivj}g_{uv}\big|_{u=0} = -\frac{B_1}{A_0}\delta^{ij},\\
F^{uiuj}{}_{uu} &= F^{vivj}{}_{vv} = 4R^{uivj}g_{uv}|_{u=0} = -\frac{2B_1}{A_0}\delta^{ij},\\
H^{uvuvuvuv} &= \frac{1}{2}(g^{uv})^4|_{u=0} = \frac{1}{2A_0^4}, \quad H^{uiujvkvl} = \frac{1}{4A_0^2}(\delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk}).
\end{aligned} \tag{2.76}$$

Substituting these expressions into either (2.50) or (2.74) reproduces the result in [18].

The second example is a top-down theory in AdS$_5$ obtained by dimensionally reducing 10-dimensional type IIB supergravity on S^5. Its bulk Lagrangian contains the leading-order α' correction

$$\mathcal{L} \supset \frac{\gamma}{2}\left(C^{\alpha\beta\gamma\delta}C_{\mu\beta\gamma\nu}C_{\alpha}{}^{\rho\sigma\mu}C^{\nu}{}_{\rho\sigma\delta} + \frac{1}{2}C^{\alpha\delta\beta\gamma}C_{\mu\nu\beta\gamma}C_{\alpha}{}^{\rho\sigma\mu}C^{\nu}{}_{\rho\sigma\delta}\right), \tag{2.77}$$

where $\gamma \sim \alpha'^3$ is the higher-derivative coupling constant and C is the Weyl tensor. Applying either (2.50) or (2.74) reproduces the butterfly velocity calculated in [39] for this theory.[12]

2.3 Equivalence of the two butterfly velocities

In Section 2.2, we derived general expressions for the butterfly velocities from two distinct holographic calculations — the shockwave method and the entanglement wedge method. More specifically, we have obtained polynomial equations for the parameters μ and $\widetilde{\mu}$ given by $f_{\mathrm{SW}}(\mu) = 0$ and $f_{\mathrm{EE}}(\widetilde{\mu}) = 0$, respectively.

In both cases, to solve for the value of μ (or $\widetilde{\mu}$) we treat the higher-derivative couplings perturbatively and choose the positive root that is continuously connected to the value in Einstein gravity. Recalling that the butterfly velocities are related to these parameters via (2.15) and (2.28), it then suffices to prove that f_{SW} and f_{EE} are the same function.

With f_{SW} given in (2.50) and f_{EE} given in (2.74), the two functions are the same if the following two equations hold in the background solution and on the $u = 0$ horizon:

$$C_{uv} + g^{uv}G_{uuvv} - 2\delta^{ij}R_{uivj}\left(F^{uu}{}_{uu} + F^{vv}{}_{vv}\right) = 0, \tag{2.78a}$$

$$\left(F^{uu}{}_{uu} + F^{vv}{}_{vv}\right) - 2g_{uv}D^{uv} - 8g_{uv}\delta^{ij}R_{uivj}H^{uuvv} = 0. \tag{2.78b}$$

In the rest of this section, we use 'on the background' to mean 'in the background solution and on the $u = 0$ horizon'. In writing the above two equations, we have used the fact

[12] We thank Sašo Grozdanov for bringing this result to our attention.

that $g^{uv} = A_0^{-1}$ and $\delta^{ij} R_{uivj} = -\frac{1}{2}(d-1)B_1$ on the background. We will refer to (2.78a) and (2.78b) as the first and second relations, respectively.

We will now prove the above relations for any given choice of Lagrangian involving contractions of Riemann tensors. To that end, it is useful to think of the terms in (2.78a) and (2.78b) as differential operators acting on the Lagrangian $\mathcal{L}$. For example, C_{uv} can be thought of as the operator

$$\widehat{C}_{uv} = \frac{\partial}{\partial g^{uv}} \tag{2.79}$$

acting on the Lagrangian $\mathcal{L}$ with the result evaluated on the background. A useful quantity to define is

$$\widetilde{R}_{ab} = \frac{1}{d-1}\delta^{ij} R_{aibj}, \tag{2.80}$$

which we use to rewrite

$$\delta_{kl}\frac{\partial}{\partial R_{akbl}} = \frac{1}{4}\frac{\partial}{\partial \widetilde{R}_{ab}}, \tag{2.81}$$

where the factor 1/4 is a symmetry factor from the Riemann tensor. For example, when we act $\partial/\partial \widetilde{R}_{uu}$ on a function of $R_{\mu\rho\nu\sigma}$ such as $\mathcal{L}$, we take the derivative with respect to $\widetilde{R}_{uu}$ while holding the traceless part of R_{uiuj} fixed. Now we can rewrite (2.78a) and (2.78b) by defining two operators

$$\widehat{O}_1 = \frac{\partial}{\partial g^{uv}} + g^{uv}\frac{\partial^2}{\partial g^{uu}\partial g^{vv}} - \frac{1}{2}\widetilde{R}_{uv}\left(\frac{\partial^2}{\partial \widetilde{R}_{uu}\partial g^{uu}} + (u \to v)\right), \tag{2.82a}$$

$$\widehat{O}_2 = \widetilde{R}_{uv}\frac{\partial}{\partial \widetilde{R}_{uv}} + \widetilde{R}_{uv}^2\frac{\partial^2}{\partial \widetilde{R}_{uu}\partial \widetilde{R}_{vv}} - \frac{1}{2}g^{uv}\widetilde{R}_{uv}\left(\frac{\partial^2}{\partial \widetilde{R}_{uu}\partial g^{uu}} + (u \to v)\right). \tag{2.82b}$$

Our goal is then to prove that

$$\widehat{O}_i\mathcal{L} = 0, \qquad i = 1, 2 \tag{2.83}$$

on the background. This will be the goal of the remainder of this section.

For any Lagrangian composed of a covariant combination of an arbitrary number of the Riemann tensor and inverse metric, we expand it by decomposing the sum over any dummy index into two sums, one over $\{u, v\}$ and the other over the x^i directions. This can be written in the following schematic form

$$\mathcal{L} = \sum L, \tag{2.84}$$

where L is an object of the form

$$L = g^{AA} \cdots g^{AA} R_{AAAA} \cdots R_{AAAA} R_{AIAI} \cdots R_{AIAI} \mathcal{X}^{I \cdots I}_{A \cdots A}. \tag{2.85}$$

Here, A denotes any a-type index labelling either u or v, I denotes any i-type index, and the tensor $\mathcal{X}$ is a product of any number of inverse metric and Riemann tensor components not explicitly shown in (2.85), i.e., g^{II}, g^{AI}, R_{IIII}, R_{AIII}, R_{AAII}, and R_{AAAI}. Different A (or I) indices may specialize to different a-type (or i-type) indices. As a concrete example, (2.84) for $\mathcal{L} = g^{\mu\nu} g^{\rho\sigma} R_{\mu\rho\nu\sigma}$ can be written as $\mathcal{L} = g^{ab} g^{cd} R_{acbd} + g^{ab} g^{ij} R_{aibj} + \cdots$ where the first term $g^{ab} g^{cd} R_{acbd}$ is of the form $g^{AA} g^{AA} R_{AAAA}$ and the second term $g^{ab} g^{ij} R_{aibj}$ is of the form $g^{AA} R_{AIAI} g^{II}$, with $a, b, c, \cdots \in \{u, v\}$ and $i, j, k, \ldots$ labelling transverse coordinates. Notice that all a-type and i-type indices are contracted.

As we are only interested in $\widehat{O}_i \mathcal{L}$ on the background, we may simplify (2.85) significantly by dropping those terms that vanish eventually. In particular, g^{AI}, R_{IIII}, R_{AIII}, R_{AAII}, and R_{AAAI} all vanish on the background.[13] As the derivatives in $\widehat{O}_i$ do not involve these components, if L in (2.85) contains any of these components, it would vanish after acting with $\widehat{O}_i$ and evaluating on the background. Therefore, we can restrict L to those that do not contain any of these components. Similarly, the traceless part $R_{aibj} - \widetilde{R}_{ab} \delta_{ij}$

[13] This can be verified by setting $h = 0$ and $u = 0$ in (2.40).

of R_{aibj} vanishes on the background, and as the derivatives in $\widehat{O}_i$ do not involve this traceless part, we can restrict L to those that do not contain the traceless part, and thus we may replace all instances of R_{AIAI} in (2.85) with $\widetilde{R}_{AA}$. Therefore, we replace (2.85) with

$$L = g^{AA} \cdots g^{AA} R_{AAAA} \cdots R_{AAAA} \widetilde{R}_{AA} \cdots \widetilde{R}_{AA}, \tag{2.86}$$

up to a multiplicative constant that we do not need to keep track of.

Now define *index loops* by connecting the two indices of g^{ab}, the two indices of $\widetilde{R}_{ab}$, the first two indices of R_{abcd}, and its last two indices. For example, the term $g^{ab}\widetilde{R}_{ab}$ has a single (index) loop. In general, L contains one or more loops, and the two antisymmetric pairs of indices in any $R_{abcd} = R_{[ab][cd]}$ need not be part of the same loop. In order for a loop not to vanish on the background, it must consist of alternating u and v: either $(uvuv \cdots uv)$ or $(vuvu \cdots vu)$. For example, $g^{ab}g^{cd}g^{ef}R_{afbc}\widetilde{R}_{de}$ consists of a single loop $(abcdef)$, with non-vanishing contributions on the background

$$g^{uv}g^{uv}g^{uv}R_{uvvu}\widetilde{R}_{vu} + (u \leftrightarrow v), \tag{2.87}$$

while $g^{ab}g^{cd}g^{ef}R_{adbc}\widetilde{R}_{ef}$ has the two loops $(abcd)$ and (ef), with non-vanishing contributions on the background

$$g^{uv}g^{uv}g^{uv}R_{uvvu}\widetilde{R}_{uv} + g^{uv}g^{uv}g^{vu}R_{uvvu}\widetilde{R}_{vu} + (u \leftrightarrow v). \tag{2.88}$$

It turns out that it is sufficient to prove $\widehat{O}_i L = 0$ for L made of a single loop, because even for L made of multiple loops, $\widehat{O}_i$ must act entirely on a single loop to have a chance to be non-trivial: in particular, if we act the two derivatives in any second-derivative term of (2.82) — such as $\partial^2/\partial\widetilde{R}_{uu}\partial g^{uu}$ — on two different loops, one of the two loops would have to contain an extra factor of g^{uu}, $\widetilde{R}_{vv}$, R_{vvab}, or R_{abvv}, thus vanishing on the background.[14]

Therefore, from now on we consider a general L made of a single loop. It may be written as

$$L = g^{a_1 b_1} g^{a_2 b_2} \cdots g^{a_n b_n} \mathcal{T}_{a_1 b_1 a_2 b_2 \cdots a_n b_n} \tag{2.89}$$

where n is the number of g^{ab} factors and the tensor $\mathcal{T}$ is a product of a suitable number of R_{abcd} and $\widetilde{R}_{ab}$.[15] Our goal is thus to prove

$$\widehat{O}_i L = 0, \qquad i = 1, 2, \tag{2.90}$$

on the background for any L of the form (2.89). The general statement (2.83) then follows.

Before proceeding, let us introduce some useful terminology. For simplicity, we rename the g^{ab} factors appearing in (2.89) so that the loop is precisely $(a_1 b_1 a_2 b_2 \cdots a_n b_n)$. For any neighboring pair of inverse metrics $g^{a_k b_k}$, $g^{a_{k+1} b_{k+1}}$ (where $k = 1, 2, \cdots, n$ and $a_{n+1} \equiv a_1$, $b_{n+1} \equiv b_1$), the b_k, a_{k+1} indices are either (1) contracted with some $R_{b_k a_{k+1} cd}$ or $R_{cd b_k a_{k+1}}$, or (2) contracted with $\widetilde{R}_{b_k a_{k+1}}$. In the first case, we say that there is an R-contraction between $g^{a_k b_k}$ and $g^{a_{k+1} b_{k+1}}$, while in the second case, we say that there is an $\widetilde{R}$-contraction between $g^{a_k b_k}$ and $g^{a_{k+1} b_{k+1}}$. More generally, for any $k \leq l$ we say that there is an R-contraction between $g^{a_k b_k}$ and $g^{a_l b_l}$ if there is an R-contraction between any neighboring pair among $g^{a_k b_k}, g^{a_{k+1} b_{k+1}}, \cdots, g^{a_l b_l}$, and we say that there is an R-contraction *not* between $g^{a_k b_k}$ and $g^{a_l b_l}$ if there is an R-contraction between any of the other neighboring pairs (i.e., among $g^{a_1 b_1}, \cdots, g^{a_k b_k}$ or $g^{a_{l+1} b_{l+1}}, \cdots, g^{a_n b_n}$). Similar statements apply for $\widetilde{R}$-contractions. Note that the number of R-contractions and $\widetilde{R}$-

[14] For $\widetilde{R}_{vv}$ this is because it is proportional to u^2, and thus vanishes on the horizon.

[15] Although the two antisymmetric pairs of indices in R_{abcd} need not be part of the same loop, this does not affect our analysis because $\widehat{O}_i$ does not involve R_{abcd} at all; there is no harm in including the other antisymmetric pair of indices in L even if they are not in the same loop.

contractions add up to n. As an example, in the loop $g^{a_1b_1}g^{a_2b_2}g^{a_3b_3}R_{a_1b_3b_1a_2}\widetilde{R}_{b_2a_3}$, there is an R-contraction between $g^{a_3b_3}$ and $g^{a_1b_1}$, as well as between $g^{a_1b_1}$ and $g^{a_2b_2}$; and there is an $\widetilde{R}$-contraction between $g^{a_2b_2}$ and $g^{a_3b_3}$.

First relation We now prove the first relation

$$\widehat{O}_1 L = 0 \tag{2.91}$$

on the background for any L of the form (2.89).

First, consider $\widehat{O}_1^{(1)}L$ where

$$\widehat{O}_1^{(1)} \equiv \frac{\partial}{\partial g^{uv}}. \tag{2.92}$$

On the background, we have

$$\widehat{O}_1^{(1)}L = \widehat{O}_1^{(1)}L_1^{(1)} \tag{2.93}$$

where $L_1^{(1)}$ is the sum of the two terms in (2.89) where the loop $(a_1b_1a_2b_2\cdots a_nb_n)$ consists of alternating u and v: either $(uvuv\cdots uv)$ or $(vuvu\cdots vu)$. These two terms differ by a factor of $(-1)^m$ where m is the total number of R-contractions,[16] because $\widetilde{R}_{uv} = \widetilde{R}_{vu}$ but exchanging u and v in each R-contraction (i.e., each antisymmetric pair of indices in R_{abcd}) costs a minus sign. In other words,

$$\begin{aligned} L_1^{(1)} &= g^{uv}\cdots g^{uv}\mathcal{T}_{uv\cdots uv} + g^{vu}\cdots g^{vu}\mathcal{T}_{vu\cdots vu} \\ &= \left[1+(-1)^m\right](g^{uv})^n\mathcal{T}_{uv\cdots uv} \end{aligned} \tag{2.94}$$

and

$$\widehat{O}_1^{(1)}L = \frac{1}{2}n g_{uv} L_1^{(1)}, \tag{2.95}$$

[16]Note that m need not be an even integer because the two antisymmetric pairs of indices in a Riemann tensor can be in different loops.

where the factor of $1/2$ comes from the symmetry of g^{ab}.

Second, consider $\widehat{O}_1^{(2)}L$ where

$$\widehat{O}_1^{(2)} \equiv g^{uv} \frac{\partial^2}{\partial g^{uu} \partial g^{vv}}. \tag{2.96}$$

On the background, we have

$$\widehat{O}_1^{(2)} L = \widehat{O}_1^{(2)} L_1^{(2)} \tag{2.97}$$

where $L_1^{(2)}$ is the sum of all terms in (2.89) where the loop $(a_1 b_1 a_2 b_2 \cdots a_n b_n)$ is alternating except for two 'defects' at $g^{a_k b_k} = g^{uu}$ and $g^{a_l b_l} = g^{vv}$, for any $k \neq l$. The two derivatives in $\widehat{O}_1^{(2)}$ act precisely on these two defects. If $k < l$, compared to the alternating loop $(uvuv \cdots uv)$ we are exchanging u and v in all R- and $\widetilde{R}$-contractions between $g^{a_k b_k}$ and $g^{a_l b_l}$. Since each R-contraction costs a minus sign and each $\widetilde{R}$-contraction gives a plus sign, such a loop contributes

$$(-1)^{s_{kl}} g^{uu} g^{vv} (g^{uv})^{n-2} \mathcal{T}_{uv \cdots uv} \tag{2.98}$$

to $L_1^{(2)}$, where s_{kl} (sometimes also written as $s_{k,l}$) is defined to be the number of R-contractions between $g^{a_k b_k}$ and $g^{a_l b_l}$. By definition we have $s_{kl} = s_{lk}$ and $s_{kk} = 0$, with no summation implied.

If $k > l$, compared to the alternating loop $(uvuv \cdots uv)$ we are exchanging u and v in all R- and $\widetilde{R}$-contractions *not* between $g^{a_k b_k}$ and $g^{a_l b_l}$. Since there is a total of m R-contractions and thus the number of R-contractions *not* between $g^{a_k b_k}$ and $g^{a_l b_l}$ is $m - s_{kl}$, such a loop contributes

$$(-1)^{m - s_{kl}} g^{uu} g^{vv} (g^{uv})^{n-2} \mathcal{T}_{uv \cdots uv} \tag{2.99}$$

to $L_1^{(2)}$.

Combining the above two cases, we find

$$\begin{aligned} L_1^{(2)} &= \sum_{k<l} (-1)^{s_{kl}} g^{uu} g^{vv} (g^{uv})^{n-2} \mathcal{T}_{uv\cdots uv} + \sum_{k>l} (-1)^{m-s_{kl}} g^{uu} g^{vv} (g^{uv})^{n-2} \mathcal{T}_{uv\cdots uv} \\ &= [1+(-1)^m] \sum_{k<l} (-1)^{s_{kl}} g^{uu} g^{vv} (g^{uv})^{n-2} \mathcal{T}_{uv\cdots uv} \end{aligned} \tag{2.100}$$

and

$$\widehat{O}_1^{(2)} L = [1+(-1)^m] \sum_{k<l} (-1)^{s_{kl}} (g^{uv})^{n-1} \mathcal{T}_{uv\cdots uv} = \sum_{k<l} (-1)^{s_{kl}} g_{uv} L_1^{(1)}. \tag{2.101}$$

Third, consider $\widehat{O}_1^{(3)} L$ where

$$\widehat{O}_1^{(3)} \equiv -\frac{1}{2} \widetilde{R}_{uv} \frac{\partial^2}{\partial \widetilde{R}_{uu} \partial g^{uu}}. \tag{2.102}$$

On the background, we have

$$\widehat{O}_1^{(3)} L = \widehat{O}_1^{(3)} L_1^{(3)} \tag{2.103}$$

where $L_1^{(3)}$ is the sum of all terms in (2.89) where the loop $(a_1 b_1 a_2 b_2 \cdots a_n b_n)$ is alternating except for two 'defects' at $g^{a_k b_k} = g^{uu}$ and $\widetilde{R}_{b_l a_{l+1}} = \widetilde{R}_{uu}$, for any k, l, whether or not they are equal. If $k \leq l$, compared to the alternating loop $(uvuv \cdots uv)$ we are exchanging u and v in all R- and $\widetilde{R}$-contractions between $g^{a_k b_k}$ and $g^{a_l b_l}$. Such a loop contributes

$$\frac{(-1)^{s_{kl}} + (-1)^{s_{k,l+1}}}{2} g^{uu} (g^{uv})^{n-1} \frac{\widetilde{R}_{uu}}{\widetilde{R}_{uv}} \mathcal{T}_{uv\cdots uv} \tag{2.104}$$

to $L_1^{(3)}$. This expression is nice because it applies to any l satisfying $k \leq l$, whether or not there is actually an $\widetilde{R}$-contraction between $g^{a_l b_l}$ and $g^{a_{l+1} b_{l+1}}$. If there is, we have $s_{k,l+1} = s_{kl}$ and (2.104) gives the correct contribution. If not, there must be an R-

contraction instead between $g^{a_l b_l}$ and $g^{a_{l+1} b_{l+1}}$, so we find $s_{k,l+1} = s_{kl} + 1$ and (2.104) vanishes.

If $k > l$, compared to the alternating loop $(uvuv \cdots uv)$ we are exchanging u and v in all R- and $\widetilde{R}$-contractions *not* between $g^{a_k b_k}$ and $g^{a_{l+1} b_{l+1}}$. Such a loop contributes

$$\frac{(-1)^{m-s_{kl}} + (-1)^{m-s_{k,l+1}}}{2} g^{uu} (g^{uv})^{n-1} \frac{\widetilde{R}_{uu}}{\widetilde{R}_{uv}} \mathcal{T}_{uv\cdots uv} \tag{2.105}$$

to $L_1^{(3)}$. Again, this expression vanishes if there is actually an R-contraction between $g^{a_l b_l}$ and $g^{a_{l+1} b_{l+1}}$.

Combining the above two cases, we find

$$\begin{aligned} L_1^{(3)} &= \left[\sum_{k \le l} \frac{(-1)^{s_{kl}} + (-1)^{s_{k,l+1}}}{2} + \sum_{k > l} \frac{(-1)^{m-s_{kl}} + (-1)^{m-s_{k,l+1}}}{2} \right] g^{uu} (g^{uv})^{n-1} \frac{\widetilde{R}_{uu}}{\widetilde{R}_{uv}} \mathcal{T}_{uv\cdots uv} \\ &= [1 + (-1)^m] \left[\frac{n}{2} + \sum_{k<l} (-1)^{s_{kl}} \right] g^{uu} (g^{uv})^{n-1} \frac{\widetilde{R}_{uu}}{\widetilde{R}_{uv}} \mathcal{T}_{uv\cdots uv} \end{aligned} \tag{2.106}$$

and

$$\begin{aligned} \widehat{O}_1^{(3)} L &= -\frac{1}{2} [1 + (-1)^m] \left[\frac{n}{2} + \sum_{k<l} (-1)^{s_{kl}} \right] (g^{uv})^{n-1} \mathcal{T}_{uv\cdots uv} \\ &= -\frac{1}{2} \left[\frac{n}{2} + \sum_{k<l} (-1)^{s_{kl}} \right] g_{uv} L_1^{(1)}. \end{aligned} \tag{2.107}$$

Finally, consider $\widehat{O}_1^{(4)} L$ where

$$\widehat{O}_1^{(4)} = -\frac{1}{2} \widetilde{R}_{uv} \frac{\partial^2}{\partial \widetilde{R}_{vv} \partial g^{vv}}. \tag{2.108}$$

This can be obtained from $\widehat{O}_1^{(3)} L$ by exchanging u with v. This leads to

$$\widehat{O}_1^{(4)} L = (-1)^m \widehat{O}_1^{(3)} L = \widehat{O}_1^{(3)} L. \tag{2.109}$$

Combining all four pieces of $\widehat{O}_1$, we find

$$\begin{aligned}\widehat{O}_1 L &= \left(\widehat{O}_1^{(1)} + \widehat{O}_1^{(2)} + \widehat{O}_1^{(3)} + \widehat{O}_1^{(4)}\right) L \\ &= \left(\frac{n}{2} + \sum_{k<l}(-1)^{s_{kl}} - 2\frac{1}{2}\left[\frac{n}{2} + \sum_{k<l}(-1)^{s_{kl}}\right]\right) g_{uv} L_1^{(1)} = 0,\end{aligned} \tag{2.110}$$

thus establishing the first relation.

Second relation We now prove the second relation

$$\widehat{O}_2 L = 0 \tag{2.111}$$

on the background. The calculation is similar to that of the first relation.

First, consider $\widehat{O}_2^{(1)} L$ where

$$\widehat{O}_2^{(1)} = \widetilde{R}_{uv} \frac{\partial}{\partial \widetilde{R}_{uv}}. \tag{2.112}$$

On the background, we have

$$\widehat{O}_2^{(1)} L = \widehat{O}_2^{(1)} L_2^{(1)} \tag{2.113}$$

where $L_2^{(1)}$ is equal to $L_1^{(1)}$ in (2.94). This gives

$$\widehat{O}_2^{(1)} L = \frac{1}{2}(n-m) L_2^{(1)}, \tag{2.114}$$

where $n-m$ is the number of $\widetilde{R}$-contractions in the loop and the factor of 1/2 comes from the symmetry of $\widetilde{R}_{ab}$.

Second, consider $\widehat{O}_2^{(2)} L$ where

$$\widehat{O}_2^{(2)} = \widetilde{R}_{uv}^2 \frac{\partial^2}{\partial \widetilde{R}_{uu} \partial \widetilde{R}_{vv}}. \tag{2.115}$$

On the background, we have

$$\widehat{O}_2^{(2)} L = \widehat{O}_2^{(2)} L_2^{(2)} \tag{2.116}$$

where $L_2^{(2)}$ is the sum of all terms in (2.89) where the loop $(a_1 b_1 a_2 b_2 \cdots a_n b_n)$ is alternating except for two defects at $\widetilde{R}_{b_k a_{k+1}} = \widetilde{R}_{uu}$ and $\widetilde{R}_{b_l a_{l+1}} = \widetilde{R}_{vv}$, for any $k \neq l$. If $k < l$, compared to the alternating loop $(uvuv \cdots uv)$ we are exchanging u and v in all R- and $\widetilde{R}$-contractions between $g^{a_{k+1} b_{k+1}}$ and $g^{a_l b_l}$. Such a loop contributes

$$\frac{(-1)^{s_{k+1,l}} + (-1)^{s_{kl}}}{2} \frac{(-1)^{s_{k+1,l}} + (-1)^{s_{k+1,l+1}}}{2} (-1)^{s_{k+1,l}} (g^{uv})^n \frac{\widetilde{R}_{uu}}{\widetilde{R}_{uv}} \frac{\widetilde{R}_{vv}}{\widetilde{R}_{uv}} \mathcal{T}_{uv \cdots uv} \tag{2.117}$$

to $L_2^{(2)}$. As with (2.104), this expression applies to any l satisfying $k < l$, regardless of whether the b_k, a_{k+1} and b_l, a_{l+1} indices are contracted to some $\widetilde{R}_{b_k a_{k+1}}$ and $\widetilde{R}_{b_l a_{l+1}}$.

If $k > l$, compared to the alternating loop $(uvuv \cdots uv)$ we are exchanging u and v in all R- and $\widetilde{R}$-contractions *not* between $g^{a_k b_k}$ and $g^{a_{l+1} b_{l+1}}$. Such a loop contributes

$$\frac{(-1)^{m-s_{l+1,k}} + (-1)^{m-s_{lk}}}{2} \frac{(-1)^{m-s_{l+1,k}} + (-1)^{m-s_{l+1,k+1}}}{2} (-1)^{m-s_{l+1,k}} (g^{uv})^n \frac{\widetilde{R}_{uu}}{\widetilde{R}_{uv}} \frac{\widetilde{R}_{vv}}{\widetilde{R}_{uv}} \mathcal{T}_{uv \cdots uv} \tag{2.118}$$

to $L_2^{(3)}$.

Using the sum relations that we show in Appendix B, the prefactors in (2.117) and (2.118) after summing over k, l simplify to

$$\sum_{k<l} \frac{(-1)^{s_{k+1,l}} + (-1)^{s_{kl}}}{2} \frac{(-1)^{s_{k+1,l}} + (-1)^{s_{k+1,l+1}}}{2} (-1)^{s_{k+1,l}} = \frac{m}{2} + \sum_{k<l} (-1)^{s_{kl}} \tag{2.119a}$$

and

$$\sum_{k>l} \frac{(-1)^{m-s_{l+1,k}} + (-1)^{m-s_{lk}}}{2} \frac{(-1)^{m-s_{l+1,k}} + (-1)^{m-s_{l+1,k+1}}}{2} (-1)^{m-s_{l+1,k}}$$

$$= (-1)^m \left[\frac{m}{2} + \sum_{k<l} (-1)^{s_{kl}} \right]. \quad (2.119b)$$

Combining the two cases, we find

$$\widehat{O}_2^{(2)} L = [1 + (-1)^m] \left[\frac{m}{2} + \sum_{k<l} (-1)^{s_{kl}} \right] (g^{uv})^n \mathcal{T}_{uv\cdots uv} = \left[\frac{m}{2} + \sum_{k<l} (-1)^{s_{kl}} \right] L_2^{(1)}. \quad (2.120)$$

Finally, consider $O_2^{(3)} L$ and $O_2^{(4)} L$ where

$$\widehat{O}_2^{(3)} \equiv -\frac{1}{2} g^{uv} \widetilde{R}_{uv} \frac{\partial^2}{\partial \widetilde{R}_{uu} \partial g^{uu}}, \qquad \widehat{O}_2^{(4)} \equiv -\frac{1}{2} g^{uv} \widetilde{R}_{uv} \frac{\partial^2}{\partial \widetilde{R}_{vv} \partial g^{vv}}. \quad (2.121)$$

They were worked out in (2.107) and (2.109), respectively. We therefore simply quote the results here:

$$\widehat{O}_2^{(3)} L = \widehat{O}_2^{(4)} L = -\frac{1}{2} \left[\frac{n}{2} + \sum_{k<l} (-1)^{s_{kl}} \right] L_2^{(1)}. \quad (2.122)$$

Combining all four pieces of $\widehat{O}_2$, we find

$$\begin{aligned} \widehat{O}_2 L &= \left(\widehat{O}_2^{(1)} + \widehat{O}_2^{(2)} + \widehat{O}_2^{(3)} + \widehat{O}_2^{(4)} \right) L \\ &= \left(\frac{1}{2}(n-m) + \left[\sum_{k<l} (-1)^{s_{kl}} + \frac{m}{2} \right] - \left[\frac{n}{2} + \sum_{k<l} (-1)^{s_{kl}} \right] \right) L_2^{(1)} = 0. \end{aligned} \quad (2.123)$$

We have therefore proven that the two functions f_{SW} and f_{EE} are the same for any f(Riemann) theory, as claimed. This immediately implies our main result $v_B = \widetilde{v}_B$ via (2.15) and (2.28).

2.4 Discussion

In this paper, we have shown that the butterfly velocity can be calculated using two distinct methods in holography: the shockwave method or the entanglement wedge method. We proved that the two methods give the same result for any f(Riemann) theory by direct computation. To find the butterfly velocity, we have solved the metric perturbation in the shockwave calculation and the near-horizon shape of extremal surfaces in the entanglement wedge calculation. In both methods, we have also taken a large-radius expansion in the transverse directions. After finding general expressions using both methods, their matching was not immediate. Nevertheless, exploiting the symmetry of the background solution on the horizon, we have shown that the difference indeed vanishes.

While our calculations show explicitly that the two methods are equivalent for a large class of theories, a deeper and more intuitive understanding of the equivalence remains an interesting open question. In particular, the holographic entanglement entropy formula was derived by evaluating the gravitational action on a Euclidean conical geometry and varying it with respect to the conical angle [40, 29, 30, 31, 32], whereas the shockwave equation is derived in a Lorentzian spacetime with no conical defects. Furthermore, the shockwave profile (2.14) is exponentially decreasing in r, but the RT profile (2.26) is exponentially increasing in r. All these distinctions make the two methods appear very different, and finding a more direct way to connect them will likely shed light on the relationship between holographic entanglement and gravitational dynamics in general.

We now describe some potential future directions:

More general gravitational theories: It would be interesting to see if the equivalence holds beyond f(Riemann) theories. To that end, we have worked out an example whose Lagrangian depends explicitly on the covariant derivative and found that the two methods continue to agree. More precisely, the Lagrangian contains

$$\mathcal{L} \supset \nabla_\mu R \nabla^\mu R, \tag{2.124}$$

and we find that its contributions to $f_{\mathrm{SW}}(\mu)$ and $f_{\mathrm{EE}}(\mu)$ are equal (in $d = 3$) and given by

$$\begin{aligned} &\frac{72B_2A_1}{A_0^4} + \frac{8B_1A_2}{A_0^4} - \frac{16B_1A_1^2}{A_0^5} - \frac{26B_1^3}{A_0^3} - \frac{24A_3}{A_0^4} - \frac{108A_1^3}{A_0^6} + \frac{64B_2B_1}{A_0^3} - \frac{12B_1^2A_1}{A_0^4} \\ &\quad + \frac{120A_2A_1}{A_0^5} - \frac{48B_3}{A_0^3} - \left(\frac{8B_1^2}{A_0^2} + \frac{4A_1^2}{A_0^4} + \frac{12B_1A_1}{A_0^3}\right)\mu^2 + \left(\frac{4B_1}{A_0} + \frac{2A_1}{A_0^2}\right)\mu^4. \end{aligned} \tag{2.125}$$

The holographic entanglement entropy functional for this theory can be found in [33]. This example suggests that the two methods continue to agree in higher-derivative theories beyond f(Riemann). It would be interesting to prove this generally, including cases where the gravitational theory is coupled to matter fields with general interactions. It would also be interesting to understand this better in the context of string theory, perhaps building on the results of [11, 41].

Beyond the butterfly velocity: It would also be interesting to see if other properties of the OTOC related to shockwave quantities besides the butterfly velocity can be connected to properties of the entanglement wedge, further strengthening the link between gravity and entanglement.

Connections to the Wald entropy: An interesting connection between gravitational shockwaves and the Wald entropy was found in [42]. It was shown that the shockwave and microscopic deformations of the Wald entropy were related by a thermodynamic relation on the horizon. Since our main result establishes a connection between shockwaves and the generalized entropy (2.23), it would be worth investigating to what extent their result can be related to ours.

Constraints on higher-derivative couplings: As a potential application of our results, one could try to understand the constraints on higher-derivative couplings from the perspective of quantum chaos. To avoid issues related to unitarity and causality at finite couplings, we have treated the higher-derivative interactions perturbatively. The signs of these couplings appear constrained by the butterfly velocity. For example, in $d = 2$ the butterfly velocity equals the speed of light in Einstein gravity. Therefore, requiring it be subluminal with higher-derivative corrections imposes constraints on the signs of the couplings. Given our expressions for a large class of higher-derivative theories, it would be interesting to see if requiring the butterfly velocity be subluminal can provide further constraints.

Relation to pole-skipping: Throughout the paper we have focused on two methods of calculating the butterfly velocity — the shockwave method and the entanglement wedge method. However, it has been suggested that the butterfly velocity (and more generally the OTOC) is also related to the phenomenon of pole-skipping [43, 44, 45]. In the gravitational context, this is related to the appearance of special points in Fourier space of the Einstein equations near the horizon, from which the Lyapunov exponent and butterfly velocity can be extracted. Although both the pole-skipping calculation and the shockwave method involve finding solutions to certain metric perturbations, the exact details are different. Nevertheless, explicit calculations show that this third way of calculating the butterfly velocity indeed matches with the first two both in Gauss-Bonnet gravity and in the presence of the leading α' correction (2.77) [39]. It would be interesting to explore their connections further.

Asymptotically flat spacetimes: Finally, both methods we discussed rely only on the near-horizon geometry and are therefore potentially generalizable beyond AdS spacetimes, such as asymptotically flat spacetimes, perhaps along the lines of [46].

Chapter 3

Covariant pole skipping: bosonic fields

The out-of-time-order correlator (OTOC), an important quantity containing characteristics of chaos, can be calculated holographically in a shockwave spacetime [7, 8, 9, 10, 11]. For a localized perturbation to a chaotic system at temperature T, the OTOC between a perturbation W at $x = t = 0$ and a probe operator V at a later time t behaves as

$$\langle V(x,t)WV(x,t)W\rangle \sim 1 - e^{\lambda_L(t-t_*-|x|/v_B)}, \tag{3.1}$$

where t_* is called the scrambling time. This defines the Lyapunov exponent, λ_L, and the butterfly velocity, v_B. For classical bulk gravitational theories, λ_L saturates the chaos bound $\lambda_L \leq 2\pi T$ [17], so they are said to be maximally chaotic. The butterfly velocity, however, depends on the theory [8, 18, 47, 48, 49, 50].

More recently, it was discovered that the quantities λ_L and v_B may already show up in features of the energy density retarded Green's function through a phenomenon called *pole skipping* [43, 44, 45]. It was first found numerically for pure Einstein gravity [43] and later studied analytically for Einstein gravity with matter [45]. See also [39, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66] for holographic and [67, 68, 69, 70, 71, 72] for boundary studies.

The retarded Green's function is the relation between a source and its response. Holographically, the Green's function of an operator dual to a bulk dynamic field X (suppressing indices) is given by [73, 74]

$$G_R(\omega, k) = \left(\lim_{r\to\infty} \frac{\Pi(r;\omega,k)|_{X_R}}{X_R(r;\omega,k)} \right) \Bigg|_{X_0=0}, \tag{3.2}$$

where X_R is the bulk solution satisfying Dirichlet boundary condition $X_R \to X_0$ at infinity and ingoing wave boundary condition at the horizon, and Π is its conjugate variable in a radial foliation. In terms of an asymptotic expansion, it is proportional to the ratio between the coefficient of the normalizable falloff and that of the non-normalizable falloff. A quasinormal mode, by definition, does not have a non-normalizable divergence, so the poles of the Green's function are identified with the quasinormal spectrum.

Generically, X_R is uniquely determined from X_0, and G_R is therefore well-defined. However, a would-be pole can sometimes get multiplied by a zero, resulting in an ill-defined limit. This happens at a special frequency and momentum,

$$\omega = i\lambda_L, \qquad k = \frac{i\lambda_L}{v_B}, \tag{3.3}$$

where λ_L and v_B are the Lyapunov exponent and the butterfly velocity extracted from a holographic OTOC calculation (3.1) in Einstein gravity minimally coupled to a large class of matter fields [43, 45].

To explain this universality, [45] discovered a feature of Einstein's equation at the horizon. Expanding metric perturbations around a stationary planar black hole in terms of Fourier modes, a particular component of Einstein's equation evaluated at the horizon was found to be trivial at (3.3) so that there exists one fewer constraint. This implies an extra degree of freedom of the ingoing modes and consequently an ambiguity in the bulk solution X_R and in turn the Green's function G_R.

Later it was discovered that pole skipping happens more generally at other locations and for other types of Green's functions [52, 53, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 65]. See also [85, 86] for higher-derivative corrections and [87] for a zero-temperature example. However, unlike the one at (3.3), the other skipped poles are unrelated to chaos.

We put these in the same framework by considering general diffeomorphism invariant bulk theories with matter fields that are not necessarily minimally coupled. For simplicity, we only consider bosonic fields and leave a discussion of fermionic fields to the last section. By defining a weight, we can separate the equations of motion into different groups and evaluate them in a given order. This allows us to find the frequencies of the skipped poles and the corresponding momenta in general. This is done in Section 3.1. Furthermore, we observe a relation between higher-weight pole-skipping frequencies and higher-spin Lyapunov exponents and use it to justify the removal of a bounded tower of higher spin fields from consideration in the remaining sections.

In Section 3.2, it is shown that, for general higher-derivative gravitational theories, the butterfly velocity can be obtained from the highest-weight equation of motion, and it agrees with the butterfly velocity obtained via a shockwave calculation. This generalizes the matching for Gauss-Bonnet gravity and Einstein gravity with a string theory correction at $O(\alpha'^3)$ [39]. We also try to explain this matching between pole skipping and chaos in the same section. By regularizing the metric perturbation at the chaotic skipped pole with a Gaussian distribution in the frequency Fourier space, we obtain a metric that is regular at the horizon. Extending it to a Kruskal–Szekeres coordinate patch and taking the regulator away, we show that this metric perturbation localizes to the past horizon in a distributional sense, like the shockwave metric. We end with a summary and a discussion of potential future directions in Section 3.3.

3.1 General pole-skipping conditions

The metric for a general stationary planar black hole can be written in ingoing Eddington-Finkelstein coordinates as

$$ds^2 = -f(r)dv^2 + 2dvdr + h(r)dx^i dx^i, \tag{3.4}$$

where $f(r_0) = 0$ at the horizon $r = r_0$ and $i = 1, ..., d$. The non-vanishing Christoffel components are given by

$$\begin{aligned} \Gamma^v_{vv} &= \frac{1}{2}f', \quad \Gamma^v_{ij} = -\frac{1}{2}h'\delta_{ij}, \quad \Gamma^r_{vr} = -\frac{1}{2}f', \\ \Gamma^i_{rj} &= \frac{h'}{2h}\delta^i_j, \quad \Gamma^r_{vv} = \frac{1}{2}ff', \quad \Gamma^r_{ij} = -\frac{1}{2}fh'\delta_{ij}. \end{aligned} \tag{3.5}$$

For simplicity, we assume that background matter fields are stationary, isotropic and homogeneous in x^i, and regular at both past and future horizons, like the metric.

Now, if we define a *pseudo-weight* for any tensor component as the number of lower v-indices minus that of lower r-indices, where an upper v is considered a lower r and vice versa, then any background tensor component (ones constructed from the stationary background metric and matter fields) with positive weight needs to vanish at the horizon. We prove this next.

In Kruskal–Szekeres coordinates, defined via

$$U = -e^{-f'(r_0)(v-2r_*)/2}, \quad V = e^{f'(r_0)v/2}, \tag{3.6}$$

where $dr_*/dr = 1/f(r)$, one can similarly define a *boost weight* as the number of lower V-indices minus that of lower U-indices [37]. Then, the boost symmetry ($V \mapsto aV$, $U \mapsto U/a$) requires that a background quantity with boost weight $n > 0$ must scale like U^n times a function of the product UV, and regularity at the bifurcate horizon requires this function to be non-singular as $UV \to 0$. Therefore, at the future horizon ($U = 0$), this vanishes. Relating this to quantities in ingoing Eddington-Finkelstein coordinates, using

$$dv = \frac{2}{f'(r_0)}\frac{dV}{V}, \quad dr = \frac{f(r)}{f'(r_0)}\left(\frac{dV}{V} + \frac{dU}{U}\right) \tag{3.7}$$

for each lower index V or U of a tensor T, we have (suppressing other indices)

$$T_V = \frac{\partial v}{\partial V}T_v + \frac{\partial r}{\partial V}T_r = \frac{2}{f'(r_0)V}\left(T_v + \frac{1}{2}f(r)T_r\right) \tag{3.8}$$

and

$$T_U = \frac{\partial r}{\partial U}T_r = \frac{1}{U}\frac{f(r)}{f'(r_0)}T_r. \tag{3.9}$$

We see that each V-index maps to a v-index and each U-index maps to an r-index (all lower indices here but upper ones work similarly) up to terms that are of higher order in f. Given that background quantities with positive boost weight and f vanish at the horizon, we arrive at the conclusion that the same is true if we replace boost weight with pseudo-weight. From now on, we no longer need to mention boost weight and will refer to pseudo-weight simply as *weight*.[1]

[1] The name pseudo-weight emphasizes the fact that it does not correspond to any symmetry transformation, unlike boost weight which characterizes how a tensor component transforms under the boost symmetry. In fact, the boost transformation is just a translation in v in ingoing Eddington-Finkelstein coordinates, and tensor components in this coordinate system do not transform non-trivially under it. The property we need for positive-pseudo-weight quantities is inherited from a more fundamental feature

To describe ingoing quasinormal modes at the horizon, for any dynamic field X, we expand its perturbation around the stationary background in the Fourier space as

$$\delta X(r, v, x) = \delta X(r)\, e^{-i\omega v + ikx}. \tag{3.10}$$

For Einstein gravity, writing Einstein's equation as $E_{\mu\nu} = T_{\mu\nu}$, a particular component under perturbation, δE^r_v, is proportional to

$$\left(k^2 - i\frac{d}{2}\omega h'\right)\delta g_{vv} + (\omega - i2\pi T)\left[\omega\delta g_{ii} + 2k_i\delta g_{vi}\right]. \tag{3.11}$$

On the horizon, for matter perturbations that are regular enough, the stress tensor component $\delta T^r_v = 0$ [45], and prefactors in δE^r_v can be tuned to zero by choosing (3.3). As a consequence, Einstein's equation provides one fewer constraint, which serves as an explanation for the universal behaviour of the energy density Green's function with low-spin matter fields coupled to Einstein gravity [45].

Now consider an arbitrary diffeomorphism invariant theory defined with a local action $S = S_g + S_M$ where the gravitational part S_g is given by

$$S_g = \int d^{d+2}x\, \sqrt{-g}\, \mathcal{L}\left(g, R, \nabla, \Phi\right), \tag{3.12}$$

and S_M is part of the action with only minimally coupled matter fields, artificially separated from the rest for later convenience. Here $\mathcal{L}$ can be an arbitrary function of the metric, g, and an arbitrary number of bosonic matter fields collectively denoted as Φ. More specifically, $\mathcal{L}$ can be written as a sum of contractions between an arbitrary number of the metric, curvature tensors, matter fields, and an arbitrary number of covariant

about boost weight via a coordinate transformation and is only true because we can drop terms that vanish at the horizon.

derivatives of them.

The metric equation of motion is defined as

$$E_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S_g}{\delta g^{\mu\nu}} = -\frac{2}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}} = T_{\mu\nu}. \tag{3.13}$$

The remaining equations of motion are given by $\delta S/\delta \Phi = 0$, indices suppressed. Now, to obtain (3.2), the idea is to perturb the dynamical fields and apply the equations of motion everywhere. However, it turns out sufficient to consider the near-horizon expansion of all perturbations in order to study pole skipping. For readability, we introduce the following compact notation: we use $\delta\mathcal{E} = 0$ to denote collectively all the perturbed equations of motions and their radial derivatives (∇_r) evaluated on the horizon. These are essentially the coefficients of a near-horizon Taylor expansion. We further define $\delta\mathcal{E}_p$ as the subset of $\delta\mathcal{E}$ with weight p, organized into a vector, and denote its number of components as $|\delta\mathcal{E}_p|$.

Similarly, we collect perturbations of all dynamics fields (including both the metric and matter) and their radial derivatives with weight q into $\delta\mathcal{X}_q$ (all evaluated on the horizon). For example,

$$\delta\mathcal{X}_2 = (\delta g_{vv}, \nabla_r \delta B_{vvv}, ...) \tag{3.14}$$

and

$$\delta\mathcal{X}_0 = (\delta g_{ij}, \nabla_r \delta g_{vi}, \nabla_r \nabla_r \delta g_{vv}, \delta A_i, \nabla_r \delta A_v, ...). \tag{3.15}$$

With these definitions, we can now write

$$\delta\mathcal{E}_p = \sum_q \mathcal{M}_{p,q}(\omega, k)\, \delta\mathcal{X}_q, \tag{3.16}$$

where each $\mathcal{M}_{p,q}(\omega, k)$ is a matrix of size $|\delta\mathcal{E}_p| \times |\delta\mathcal{X}_q|$. To arrive at this form, first commute all ∇_r's through ∇_i's and ∇_v's to the rightmost location before substituting the Fourier expansion and evaluating the ∇_v's and ∇_i's. By definition, the radial derivatives are then absorbed into $\delta\mathcal{X}_q$. For later convenience, we also commute all ∇_i to the right of ∇_v.

We now prove a useful property that, for $p > q$,

$$\mathcal{M}_{p,q}(\omega, k) \propto [\omega - (p-1)\omega_0] \ldots [\omega - q\omega_0]\,, \tag{3.17}$$

where $\omega_0 = i2\pi T = if'(r_0)/2$.

Begin by noticing that, for a given p and q',

$$\delta\mathcal{E}_p \sim F(g, R, \nabla, \Phi)(\nabla_v)^k(\nabla_i)^l(\nabla_r)^m \delta X_{q'+m} \tag{3.18}$$

before substituting the Fourier expansion, where F is some c-number tensor component constructed out of g, R, ∇ and Φ such as $R_{virj}A^\mu\nabla_\mu\phi$ evaluated on the horizon of the background configuration, and $\delta X_{q'+m}$ is the perturbation to some component of a dynamic field X with weight $q' + m$ – not evaluated on the horizon until acted upon by all the derivative operators in front. Next, notice that the only way to raise weight is with ∇_v because any background tensor with positive weight vanishes on the horizon. Therefore, to raise the weight of $(\nabla_r)^m\delta X_{q'+m}$ to that of $\delta\mathcal{E}_p$, one needs $k \geq p - q'$. From (3.5), it is straightforward to show that, on the horizon,

$$\nabla_v T \propto \left(\partial_v - \frac{n}{2}f'(r_0)\right) T \tag{3.19}$$

for a general tensor component T with weight n; therefore, evaluating $(\nabla_v)^k$ and substituting (3.10) gives at least a factor of $[\omega - (p-1)\omega_0]\ldots[\omega - q'\omega_0]$. Finally, the remaining part $(\nabla_i)^l(\nabla_r)^m\delta X_{q'+m}$ evaluates to a number of terms, each proportional to $\delta\mathcal{X}_q$ for some $q \geq q'$. This follows from (3.5), where any Christoffel symbol appearing in $\nabla_i T$ vanishes if multiplying an object with lower weight than T. This concludes our proof of (3.17).

We now discuss the general conditions for pole skipping. We take as an assumption that pole skipping happens whenever an equation of motion becomes trivial.[2] Suppose the highest weight of $\delta\mathcal{X}$ is q_0, then the highest weight of $\delta\mathcal{E}$ is also q_0 (since the action, being a scalar, has weight zero). Consequently, for any positive integer s, once we set

$$\omega = (q_0 - s)\,\omega_0, \tag{3.20}$$

all $\mathcal{M}_{p,q}(\omega, k)$ with $p > q_0 - s \geq q$ are then set to zero (assuming they are not all automatically zero). Now consider the square matrix

$$M_s(k) \equiv \begin{pmatrix} \mathcal{M}_{q_0,q_0} & \dots & \mathcal{M}_{q_0,q_0-s+1} \\ \dots & \dots & \dots \\ \mathcal{M}_{q_0-s+1,q_0} & \dots & \mathcal{M}_{q_0-s+1,q_0-s+1} \end{pmatrix} \tag{3.21}$$

where (3.20) has been substituted. The full set of equations of motion $\delta\mathcal{E}_p, \forall p$, does not determine $\delta\mathcal{X}_q, \forall q$, when

$$\det\, M_s(k) = 0. \tag{3.22}$$

The equations (3.20) and (3.22) are therefore the generalized pole-skipping conditions (for any given $s \geq 1$), assuming the second one has solutions. If the theory has a highest spin field with bounded spin l, then $q_0 = l$ and the pole-skipping frequencies are $(l-s)\,\omega_0$, consistent with observations made in [53, 67, 84] and in particular reproducing the posi-

[2]It was pointed out in [53] that there are so-called anomalous points at which triviality of equations of motion at the horizon does not imply dependence on $\delta\omega/\delta k$ for small deviations from the point, but these points were identified as a different class of skipped poles where the limit does depend on higher order quantities such as $(\delta k)^2$ [78]. This justifies our assumption here.

tions of pole skipping at Matsubara frequencies first found in [53]. The second condition is a polynomial equation for k, and the roots are then the pole-skipping momenta, which could be more than one. The order of the polynomial increases with the size of the matrix, and therefore there will be generically more pole-skipping points at larger s (lower ω).

The first pole skipping happens at $s = 1$ at frequency $\omega = (q_0 - 1)\,\omega_0 = i(q_0 - 1)2\pi T$. Suppose there exists an equation of motion with e.g. three lower v-indices. In that case, there will be a skipped pole at $2\omega_0 = i4\pi T$, and the field perturbation (3.10) will grow like $\exp(4\pi T t)$. On this ground, we expect (finitely many) higher spin fields to violate the chaos bound. This is supported by an independent calculation of the spin-l Lyapunov exponent, $\lambda_L^l = (l - 1)2\pi T$ [12] and is consistent with the findings of [67, 88]. Bounded higher spin fields also suffer from causality violation [89], which is another reason to exclude them from consideration in the next section. Notice, however, that equations of motion for fields with no dynamics automatically have $\mathcal{M}_{p,q}(\omega, k) = 0$ for $p > q$ due to the nonappearance of ∇_v, so they do not become trivial from non-trivial; therefore, they do not violate the chaos bound, in agreement with [12] where pure AdS_3 higher spin gravity was exempt from their argument for bound violation.

If $q_0 = 2$, which is the case for an arbitrary metric theory coupled to matter fields of spin no larger than two, then the bound is satisfied and in fact saturated. We will discuss this further in the next section.

For $q_0 < 2$, such as a scalar or vector field without gravitational backreaction, there is no growing mode and therefore no relation to chaos, but an infinite number of skipped poles still exist and constrain the structure of Green's functions [52, 53].

3.2 Matching of butterfly velocities

For arbitrary higher-derivative gravity coupled to scalar, vector or form fields, $q_0 = 2$ (from the metric) and the highest-weight skipped pole has $\omega = i2\pi T$. We now show that the corresponding butterfly velocity matches that obtained from the OTOC.

In this case, the only dynamic field with weight 2 is $\delta\mathcal{X}_2 = \delta g_{vv}$, and the corresponding equation of motion is $\delta\mathcal{E}_2 = \delta E_{vv} - \delta T_{vv} = 0$. The perturbation to the stress tensor component δT_{vv} does not necessarily vanish, but δT_v^r $(= \delta T_{vv} - T_{rv}\delta g_{vv})$ does vanish for matter fields regular on the horizon [45]. We will make this restriction in order to compare results with OTOC: the metric shockwave also has vanishing δT_v^r. Therefore, the pole-skipping conditions with $s = 1$ are given by

$$\omega = \omega_0, \quad \det\, M_1 = \frac{\delta E_v^r}{\delta g_{vv}} = 0. \tag{3.23}$$

This gives a polynomial equation for k with only even powers (by symmetry). In cases where the polynomial is of quartic order or higher, one can take the view that all corrections to Einstein gravity should be treated perturbatively so only the roots continuously connected to Einstein gravity are physical. But as we will see, the matching is evident without a perturbative treatment.

For the class of theories we consider,

$$\delta E_v^r = \sum_{k,l} H_{k,l}(f, h, \partial_r, \Phi)(\partial_v)^k(\partial_i)^l \delta g_{vv} \tag{3.24}$$

for some non-covariant c-number coefficients $H_{k,l}$. The non-trivial statement that no ∂_r acts on δg_{vv} and none of the other components such as δg_{vi} can appear follow directly from the weight argument. As an example, consider the Einstein gravity equation of motion (3.11) studied in [45]. Since δg_{ij} has weight zero, it has to pick up a factor of

ω to get to weight one and then a factor of $(\omega - \omega_0)$ to get to weight two, similarly for δg_{vi} which only needs to raise its weight by one. Another simplification in Einstein gravity is due to the fact of it being two-derivative. It is not possible for (3.11) to contain a term like for example $\partial_r \delta g_{vi}$: this quantity has weight zero and therefore needs two v-derivatives to go to two, but it already has one derivative itself.

To compare this with the shockwave calculation, we move to Kruskal–Szekeres coordinates defined in (3.6). Then $UV = -e^{f'(r_0)r_*}$, and the metric is given by

$$\mathrm{d}s^2 = 2A(UV)\mathrm{d}U\mathrm{d}V + B(UV)dx^i dx^i, \tag{3.25}$$

$$A(UV) = \frac{2}{f'(r_0)^2}\frac{f(r)}{UV}, \quad B(UV) = h(r). \tag{3.26}$$

In general higher-derivative gravity and for a shockwave along $V = 0$, the only non-trivial component of δE^μ_ν perturbed by a local source is δE^U_V [50]. For a general perturbation, δg_{vv}, translating to ingoing Eddington-Finkelstein coordinates, this component is given by

$$\delta E^U_V = \frac{U}{V}\left(\frac{2}{f(r)}\delta E^r_v + \delta E^r_r - \delta E^v_v - \frac{f(r)}{2}\delta E^v_r\right). \tag{3.27}$$

Compared to the first term, others are suppressed with extra factors of $f(r)$, so they vanish when evaluated on the horizon. Similarly, $\delta T^U_V \propto \delta T^r_v$, but recall that this vanishes for regular matter configurations. Therefore,

$$\begin{aligned} 0 = \delta E^U_V &= \frac{2UV}{f(r)}\frac{1}{V^2}\sum_{k,l} H_{k,l}(\partial_v)^k(\partial_i)^l \delta g_{vv} \\ &= \frac{2UV}{f(r)}\frac{1}{V^2}\sum_{k,l} H_{k,l}(\partial_i)^l\left(\frac{2}{f'(r_0)}V\partial_V\right)^k \delta g_{vv} \\ &= \frac{2UV}{f(r)}\sum_{k,l} H_{k,l}(\partial_i)^l\left(\frac{2}{f'(r_0)}(V\partial_V + 2)\right)^k \frac{\delta g_{vv}}{V^2} \end{aligned}$$

$$= \frac{2UV}{f(r)} \sum_{k,l} \tilde{H}_{k,l} (\partial_i)^l \, (V\partial_V)^k \, \frac{\delta g_{vv}}{V^2}$$
$$= \frac{1}{A} \sum_{k,l} \tilde{H}_{k,l} (\partial_i)^l \, (V\partial_V)^k \, \delta g_{VV}, \tag{3.28}$$

where we used the transformation $\partial_v = \frac{2}{f'(r_0)} V\partial_V$ in going to the second line, and a trick

$$\frac{1}{V^2} V\partial_V = (V\partial_V + 2) \, \frac{1}{V^2} \tag{3.29}$$

in going to the third line. The fourth line follows from a reorganization of the sum with new coefficients $\tilde{H}_{k,l}$ and the last line follows from

$$\delta g_{VV}(V, x) = \frac{4\delta g_{vv}(v, x)}{f'(r_0)^2 V^2}. \tag{3.30}$$

The special thing about $\omega = \omega_0$ is that,

$$\delta g_{vv} \sim e^{-i\omega_0 v} = e^{-\frac{i2\omega_0}{f'(r_0)} \log V} = V, \tag{3.31}$$

and therefore

$$\delta g_{VV}(V, x) \sim \frac{1}{V} \, e^{-ikx}. \tag{3.32}$$

Compare this with a linearized shockwave perturbation

$$\delta g_{VV} \sim \delta(V) \, e^{-\mu x}, \tag{3.33}$$

where $\mu = 2\pi T / v_B$ upon using $\delta E^U_V = 0$ (outside of a localized source in x). Noticing that $\delta(V)$ has the same distributional behavior as $1/V$ under $V\partial_V$ [43], e.g., $V\delta'(V) = -\delta(V)$ and $V d(1/V)/dV = -1/V$, it follows that $k = i2\pi T / v_B$ upon using (3.28) for the perturbation (3.32), thereby extending (3.3) to general higher-derivative gravity and hence some of the results of [45, 39].

Given the similarity between $1/V$ and $\delta(V)$ and the role this similarity plays in establishing the equivalence of these two calculations of the butterfly velocity, it is natural to wonder whether there is a more direct connection between them. An immediate obstacle is the divergence of the function $1/V$ at the past horizon $V = 0$. We mitigate this problem with an unnormalized[3] regularization of the Fourier space delta function along the real frequency line:

$$\int d\xi\, \delta(\xi) \to \int_{-\infty}^{\infty} d\xi\, e^{-\xi^2/a}, \tag{3.34}$$

giving rise to a mode

$$\delta g_{vv} = \sqrt{\pi a}\, e^{-\frac{a}{4}(\lambda_L v)^2} e^{\lambda_L v}. \tag{3.35}$$

To compare with the shockwave metric (3.33), we convert this to Kruskal-Szekeres coordinates. Using (3.30),

$$\delta g_{VV} = \begin{cases} 0, & V < 0 \\ \frac{\sqrt{\pi a}}{\lambda_L^2} \frac{1}{V}\, e^{-a(\log V)^2/4}, & V \geq 0 \end{cases} \tag{3.36}$$

where we have used the fact that the perturbation vanishes exactly behind the past horizon. This function is finite and integrates to a constant for finite a, and it vanishes everywhere off the horizon as $a \to 0$. It therefore behaves as a regularized $\delta(V)$. Taking the regulator away, this becomes a shockwave localized at $V = 0$.[4]

[3]We should note that the need for the unnormalized regulator arises from the need to remove the divergence. Alternatively, one can use a normalized delta function regulator and remove the divergence at the end using a subtraction not unlike the minimal subtraction in dimensional regularization.

[4]Physically, this suggests that the shockwave solution encodes part of the physical content of the quasinormal mode at the highest-weight pole-skipping frequency. The renormalization procedure throws away some information irrelevant for computing the butterfly velocity.

3.3 Discussion

We have defined a quantity called weight to organize bulk equations of motion and exploited its convenience to show that pole skipping happens in holographic CFTs dual to quite general diffeomorphism invariant bulk theories. As a result, the pole-skipping frequencies show up at $(q_0 - s)\,\omega_0$ for all $s \in \mathbb{Z}^+$, where $\omega_0 = i2\pi T$, and q_0 is defined as the weight of the highest-weight object. In particular, a theory that has a bounded highest spin larger than two in general gives rise to $q_0 > 2$, which leads to very fast scrambling that violates the chaos bound. It is therefore reasonable to disallow a finite tower of higher spin fields, in addition to causality reasons [89]. This brings down q_0 to two, and, with this restriction, the metric is the field that can have the highest weight. This is the main reason behind the universality of the special pole-skipping point at $\omega = i\lambda_L$ and $k = i\lambda_L/v_B$, where $\lambda_L = 2\pi T$, and v_B is defined via a OTOC calculation.

In other words, for maximally chaotic holographic theories, instead of needing to compute a four-point function, the retarded Green's function already knows about the butterfly velocity, and its dependence on the bulk theory is exactly the same as an OTOC would predict. It would be interesting to test this statement for non-holographic maximally chaotic theories.[5] Furthermore, there are now three ways of computing the butterfly velocity: (i) using entanglement wedge, (ii) using shockwave and (iii) using pole skipping. We proved the equivalence between the second and third *prescriptions* themselves.[6]

The restriction of the discussion to bosons is for simplicity, and the generalization to include fermions should be completely analogous. For minimally coupled spinors on

[5]For non-maximally chaotic theories, the predictions from pole skipping could differ from OTOC results [90, 72, 91].

[6]Evidence for the general equivalence between the first two was presented in [18, 50]; evidence for the general equivalence between the last two was presented in [45, 39]. Here our emphasis is on the equivalence of the methods and not the equality of the results.

a fixed background, pole skipping has been shown to happen at $\omega = (q_0 - s)\,\omega_0$ for a half integer $q_0 = 1/2$ and positive integers s [76]; with a spin-3/2 Rarita-Schwinger field, q_0 becomes 3/2 [82]. Both of the examples fit the pattern that the leading pole skipping happens at $(q_0 - 1)\,\omega_0$, and if one allows both bosonic and fermionic fields with arbitrary couplings between them, one might expect that both q_0 and s can be half integers. It might be of use to analyze this with the weight argument, perhaps beginning by rephrasing the current discussion in a spin connection language.

We should summarize three assumptions that were used: (i) the existence of a finite q_0; (ii) the non-triviality of equation (3.20), i.e., the entries set to zero by this equations are not already all zero; and (iii) equation (3.22) has solutions. We expect that assumption i can be lifted with more careful analysis, but assumptions ii and iii are essential. Given any theory, one needs to check whether these are satisfied. For example, Vasiliev gravity violates assumption i as it contains an infinite tower of higher spin fields; this is consistent with it being dual to a sector of a free theory [92], which does not exhibit chaos.

Another condition implicit in our discussion is the restriction to finite temperatures. Extremal black holes do not have a bifurcate surface, so the property derived from regularity at the bifurcate surface no longer applies. Furthermore, poles in the Green's function get replaced by branch cuts [93, 87]. Accordingly, a generalization of our argument to zero temperature will be non-trivial.

We also showed that the shockwave metric could be obtained from a regularized mode of the metric perturbation. This serves as an explanation for the similarities between the two calculations and the equivalence regardless of the theory. One might try different regulators or use different subtraction schemes to find a more regulator-independent relation.

Chapter 4

Covariant pole skipping: gauge and fermionic fields

Pole skipping refers to the multivalued nature of Green's functions at special points in the momentum space where lines of zeros intersect lines of poles [43, 44, 45]. In holographic theories at large N and finite temperature T, this phenomenon can be understood through the existence of extra ingoing modes at the black hole horizon, resulting in non-uniqueness of the bulk solution that determines the holographic retarded Green's function [45]. The first example of pole skipping was found in Einstein gravity, where it happens at frequency $\omega = \mathrm{i}\lambda_L$ and momentum $k = \mathrm{i}\lambda_L/v_B$ [43, 45], λ_L being the Lyapunov exponent [7, 10, 11, 17] and v_B being the butterfly velocity [8, 9] for this theory. One can therefore say that the analytic structure of the Green's function contains some information about the chaotic properties of the quantum system, especially true when the stress tensor dominates chaos [90, 72, 91].

For bulk theories containing bosonic fields, it was noticed through numerous examples that pole skipping happens for more general theories and at many more positions in the momentum space [55, 57, 81, 94, 62, 54, 77, 78, 95, 96, 53, 64, 39, 52, 58, 97, 59, 61, 66, 98, 60, 63, 84, 56, 65, 51, 75, 86, 79, 80, 83, 99, 100, 85]. Moreover, even though the number of pole-skipping points at each frequency and the corresponding momenta depend on the details of the theory, there is a universal pattern: pole skipping happens at frequencies $\omega = \mathrm{i}(l_b - s)2\pi T$ for all positive s, where l_b is the highest spin in the theory. For $l_b \geq 2$, we can write the leading frequency as $\omega = \mathrm{i}\lambda_L$ because $\lambda_L = (l_b - 1)2\pi T$ is the Lyapunov exponent for a theory with a spin l_b [12, 67, 88]. To derive this general statement, [101] used what we will refer to as the *covariant expansion formalism.* Expanding bulk fields around the black hole horizon, not using partial derivatives, ∂_r, but using covariant derivatives, ∇_r, one finds that certain properties of the equations of motion become manifest. The covariant expansion formalism also provides an algorithm for locating skipped poles, making it automatable on a computer program.

Interestingly, this pattern of frequencies for bosonic fields was found to hold analogously in some examples with fermionic fields as well. For the theory of a minimally coupled Dirac fermion, the leading frequency was found to be $-\mathrm{i}\pi T$ [76], and, for the theory of a minimally coupled Rarita-Schwinger field, it was found to be $\mathrm{i}\pi T$ [82]. We will show that this pattern holds for fermionic theories in general. More explicitly, for a theory of fermions (with no gravitational backreaction) with the highest spin being l_f, pole skipping happens universally at frequencies $\mathrm{i}(l_f - s)2\pi T$ for all $s \in \mathbb{Z}^+$.

More generally, a theory will contain both bosonic and fermionic fields that are dynamic. For example, supergravity theories have both. We will argue that there would be two towers of pole-skipping frequencies starting at $\mathrm{i}(l_s - 1)2\pi T$ and $\mathrm{i}(l_f - 1)2\pi T$ respectively. This is regardless of how the bosons and fermions are coupled.

The covariant expansion method relies on a classification of linearized perturbations of bulk fields and an analysis of equations of motion on this basis. In a theory with gauge symmetry, not all equations of motion are independent. This redundancy is commonly dealt with by restricting to only gauge-invariant quantities [102], which is a theorydependent procedure. Since part of the motivation of [101] was to provide an algorithm, it is thus helpful if we have a systematic procedure for finding the pole-skipping points given a Lagrangian while sidestepping the gauge analysis. In this paper, we present a pole-skipping condition that works without having to remove the gauge redundancy, and we will refer to it as the *gauge-covariant condition*.

We will begin by reviewing the covariant expansion formalism in the bosonic case in Section 4.1, followed by a discussion of gauge redundancy which leads to the gauge-covariant pole-skipping condition in Section 4.2. We then present the analogous formalism for theories with only fermions in Section 4.3. In Section 4.4, we present an argument for the general pattern of pole-skipping frequencies when both bosonic and fermionic fields are present. We then discuss some consequences and future directions in Section 4.5. Some examples with bosonic fields and fermionic fields are presented in Appendices C and D respectively, where gauge symmetry is present in two of the bosonic and one of the fermionic examples.

4.1 Bosonic fields

We now review the covariant expansion formalism of [101] which was in the context of general holographic theories with bosonic fields. Consider a local diffeomorphism-invariant action of the form

$$S = \int \mathrm{d}^{d+2}x \sqrt{-g}\, \mathcal{L}\left(g, R, \nabla, \Phi\right), \tag{4.1}$$

where $\mathcal{L}$ is constructed from contractions of the metric, g, the Riemann tensor, $R_{\mu\nu\rho\sigma}$, bosonic matter fields which are collectively denoted as Φ, and their covariant derivatives such as $\nabla_\lambda R_{\mu\nu\rho\sigma}$ and $\nabla_\mu \nabla_\nu \Phi$. An example of a term that can appear in $\mathcal{L}$ is

$\phi R_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma}$ for some scalar field ϕ and some field strength $F_{\mu\nu}$.

At large N and finite temperature T, a boundary state in thermal equilibrium is described by a stationary black hole in the bulk [14]. Let us write its metric in ingoing Eddington-Finkelstein coordinates as

$$\mathrm{d}s^2 = -f(r)\mathrm{d}v^2 + 2\mathrm{d}v\mathrm{d}r + h(r)\mathrm{d}x^i\mathrm{d}x^i, \tag{4.2}$$

where $f(r_0) = 0$ at the horizon $r = r_0$ and $i \in \{1, \ldots, d\}$ labels the transverse directions. We will refer to it as the background metric. We can also have stationary background matter fields, which, as a simplifying assumption, are isotropic and homogeneous in x^i. Furthermore, all background fields should be regular at both past and future horizons.

For future reference, the non-vanishing Christoffel symbols for our background metric are

$$\begin{aligned} \Gamma^v_{vv} &= \frac{1}{2}f', \qquad \Gamma^v_{ij} = -\frac{1}{2}h'\delta_{ij}, \qquad \Gamma^r_{vr} = -\frac{1}{2}f', \\ \Gamma^i_{rj} &= \frac{h'}{2h}\delta^i_j, \qquad \Gamma^r_{vv} = \frac{1}{2}ff', \qquad \Gamma^r_{ij} = -\frac{1}{2}fh'\delta_{ij}, \end{aligned} \tag{4.3}$$

and the non-vanishing components of the Riemann tensor are

$$\begin{aligned} R_{vrvr} &= \frac{1}{2}f'', \quad R_{virj} = -\frac{1}{4}f'h'\delta_{ij}, \quad R_{rirj} = \frac{h'^2 - 2hh''}{4h}\delta_{ij}, \\ R_{vivj} &= \frac{1}{4}ff'h'\delta_{ij}, \quad R_{ijkl} = -\frac{1}{4}fh'^2\left(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}\right). \end{aligned} \tag{4.4}$$

Quasinormal modes are perturbations of the dynamical fields on the black hole background that satisfy the linearized equations of motion [103, 104]. They are identified with poles of the retarded Green's function [105, 73, 74]. Pole skipping happens when the equations of motion (along with boundary conditions at the horizon and the asymptotic infinity) do not uniquely determine the modes, leading to an ambiguity in the Green's function where the value depends on how the limit is taken in the frequency-momentum space (ω, k) [45, 53]. Pole skipping can therefore be studied by examining the equations of motion. It turns out sufficient to expand the equations of motion perturbatively in the radial direction away from the future horizon $r = r_0$. This is a manifestation of the expectation that the near-horizon geometry is responsible for many universal aspects of the dual thermal system.

A single Fourier mode of linear perturbation of a dynamical field X takes the form

$$\delta X(r, v, x) = \delta X(r)\, e^{-\mathrm{i}\omega v + \mathrm{i}kx}, \tag{4.5}$$

where kx is a shorthand for $k_i x^i$. The Fourier coefficient $\delta X(r)$ still depends on the radial coordinate r because the radial direction is not a coordinate of the boundary theory; since we are computing the boundary Green's function, albeit in the bulk, the radial direction should not be Fourier transformed. Often, the next step is to expand this function as a Taylor series around the horizon $r = r_0$:

$$\delta X(r) = \sum_{n=0}^{\infty} \frac{((\partial_r)^n \delta X)|_{r=r_0}}{n!} (r - r_0)^n. \tag{4.6}$$

However, it was noticed in [101] that

$$((\nabla_r)^n \delta X)|_{r=r_0}\,, \quad n \geq 0, \tag{4.7}$$

form a more convenient basis for the near-horizon degrees of freedom. As we will see, working with covariant expressions like these turns out to be a key idea that helps reveal various hidden symmetries of the equations of motion. We shall denote this set by $\delta\mathcal{X}$; for example, $\delta\mathcal{X} = [\delta g_{vv}, \delta g_{vi}, \nabla_r \delta g_{vv}, \dots]|_{r=r_0}$. Similarly, let us use $\delta\mathcal{E} = 0$ to denote perturbed equations of motions and their covariant radial derivatives evaluated on the horizon.

In order to organize the equations of motion, it is useful to define the *weight* [37, 101] for a general tensor component as

$$\#(\text{lower } v) - \#(\text{lower } r) - \#(\text{upper } v) + \#(\text{upper } r), \tag{4.8}$$

i.e., each v-index carries a weight of $+1$ if downstairs and -1 if upstairs, and the opposite is true for r-indices; transverse indices $i, j, \ldots$ do not carry weights. With this, define $\delta\mathcal{X}_p$ as the subset of $\delta\mathcal{X}$ that has weight p and similarly for $\delta\mathcal{E}_p$. We will think of and refer to them as vectors to compactify some of our equations, but it is nothing more than a notational convenience. For example, in Einstein gravity,

$$\begin{aligned}
\delta\mathcal{X}_2 &= [\delta g_{vv}]|_{r=r_0}, \\
\delta\mathcal{X}_1 &= [\nabla_r \delta g_{vv}, \delta g_{vi}]|_{r=r_0}, \\
\delta\mathcal{X}_0 &= [\nabla_r \nabla_r \delta g_{vv}, \nabla_r \delta g_{vi}, \delta g_{ij}, \delta g_{vr}]|_{r=r_0}.
\end{aligned} \tag{4.9}$$

More generally, with matter fields, say $B_{\mu\nu\rho}$ and A_μ, it could contain additional terms like

$$\delta\mathcal{X}_0 = [\cdots, \nabla_r^3 \delta B_{vvv}, \nabla_r^2 \delta B_{vvi}, \delta B_{ijk}, \delta A_i, \nabla_r \delta A_v, \cdots]|_{r=r_0}. \tag{4.10}$$

The near-horizon expansions of the perturbed equations of motion can now be compactly written as

$$\delta\mathcal{E}_p = \sum_q \mathcal{M}_{p,q}(\omega, k)\, \delta\mathcal{X}_q, \tag{4.11}$$

where a Fourier mode of the form (4.5) has been substituted. It is worth mentioning that $\mathcal{M}_{p,q}(\omega, k)$ for each p and q is a $|\delta\mathcal{E}_p|$-by-$|\delta\mathcal{X}_q|$ matrix, the modulus sign denoting the length of the vectors. The definition of the weight has allowed us to divide the infinite matrix into these finite ones, each labeled by p and q. This division leads to an important property that, for $p > q$,

$$\mathcal{M}_{p,q}(\omega, k) \propto [\omega - (p-1)\omega_0] \ldots [\omega - q\omega_0] \,, \tag{4.12}$$

where

$$\omega_0 \equiv \mathrm{i}2\pi T = \mathrm{i}f'(r_0)/2. \tag{4.13}$$

The proof of this statement uses the covariance of the equation of motion and the symmetry of the background fields including in particular the metric (4.2). To see the origin of these factors, notice that a component of a linearized equation of motion is a sum of terms with the following form:

$$\delta\mathcal{E}_p = \sum F(g, R, \nabla, \Phi)(\nabla_v)^k(\nabla_i)^l(\nabla_r)^m \delta X_{q'+m}, \tag{4.14}$$

where F is a tensor component constructed from the background fields. To get into this form, one needs to use the Riemann tensor to commute the covariant derivatives, and it is important that ∇_v derivatives are pushed through to the very left for our purpose.

A consequence of the background fields being stationary is that any tensor component constructed from background fields with a positive weight must vanish, so F must have a non-positive weight. When $p > q'$, we must then have $k \geq p - q'$ to balance the weights on both sides. The action of $(\nabla_v)^k$ along with (4.5) then immediately leads to the factors

$$[\omega - (p-1)\omega_0] \ldots [\omega - q'\omega_0] \tag{4.15}$$

because they have a very simple action on a general tensor component T with weight w when evaluated on the horizon:

$$\nabla_v T = \left(\partial_v - \frac{\mathrm{w}}{2} f'(r_0)\right) T. \tag{4.16}$$

It is a straightforward exercise to show this using the definition of the covariant derivative and the associated Christoffel symbols explicitly given in (4.3).

To go from (4.14) to (4.11), besides substituting the Fourier mode and unpacking $(\nabla_v)^k$, we still need to unpack $(\nabla_i)^l (\nabla_r)^m \delta X_{q'+m}$, which can lead to additional terms, each proportional to $\delta \mathcal{X}_q$ for some q. From (4.3), one should notice that the only non-vanishing Christoffel symbols that can appear in ∇_i are Γ^v_{ij} and Γ^j_{ri}, i.e.,

$$\begin{aligned} \nabla_i T_{\dots\mu\dots} &= \partial_i T + \cdots - \Gamma^\rho_{\mu i} T_{\dots\rho\dots} + \cdots \\ &= \partial_i T + \cdots - \delta^j_\mu \Gamma^v_{ji} T_{\dots v\dots} - \delta^r_\mu \Gamma^j_{ri} T_{\dots j\dots} + \cdots , \end{aligned} \tag{4.17}$$

so it can at most turn a lower i-index into a lower v-index or a lower r into a lower j, thereby increasing the weight by one in either case. The same is true for upper indices. In other words, we must have $q \geq q'$. This concludes the derivation of (4.12) because (4.15) contains at least as many factors as needed, with the extra factors playing no obvious role.

Pole skipping happens whenever an equation of motion becomes trivial. This is our starting point and can be thought of as our definition. A more detailed explanation for why this implies the existence of an extra ingoing mode can be found in [53]. This extra ingoing solution would lead to an ambiguity in the holographic retarded Green's function, which is computed via finding the solution to the linearized equations of motion with certain asymptotic boundary conditions and ingoing boundary conditions at the future horizon. This ambiguity was known to be related to the ambiguity of the retarded Green's function at special values $\omega = \omega_*$ and $k = k_*$, where it takes the form $G \sim 0/0$.

To see why the Green's function is ambiguous or multi-valued at pole-skipping points, expand the numerator and the denominator in terms of small $\delta\omega$ and δk around such a point:

$$G(\omega_* + \delta\omega, k_* + \delta k) \sim \frac{0 + a(\delta\omega) + b(\delta k)}{0 + c(\delta\omega) + d(\delta k)} \sim \frac{a(\delta\omega/\delta k) + b}{c(\delta\omega/\delta k) + d}. \tag{4.18}$$

This illustrates the dependence of the value of $G(\omega_*, k_*)$ on the direction in which it is approached in the (ω, k) plane. In the special case $a = c = 0$, called anomalous in [53], the Green's function does not depend on $\delta\omega/\delta k$ to leading order, but the limit would now depend on higher order quantities such as $(\delta k)^2$ [78]. In either case, we have $0/0$, so the pole is skipped.

We now turn to the condition under which an equation of motion trivializes. From property (4.12), it follows that, for any integer $s > 0$, matrices $\mathcal{M}_{p,q}(\omega, k)$ with $p > q_0 - s \geq q$ vanish identically when

$$\omega = (q_0 - s)\,\omega_0. \tag{4.19}$$

Then the infinite dimension matrix

$$\mathcal{M}_\infty(\omega, k) \equiv \begin{bmatrix} \mathcal{M}_{q_0,q_0} & \mathcal{M}_{q_0,q_0-1} & \ldots \\ \mathcal{M}_{q_0-1,q_0} & \mathcal{M}_{q_0-1,q_0-1} & \ldots \\ \ldots & \ldots & \ldots \end{bmatrix} \tag{4.20}$$

takes the form

$$\mathcal{M}_\infty((q_0-s)\omega_0,k)=\left[\begin{array}{c|c} M_s(k) & 0 \\ \hline \cdots & \cdots \end{array}\right]. \tag{4.21}$$

As soon as we make the finite submatrix

$$M_s(k)\equiv\left.\begin{bmatrix} \mathcal{M}_{q_0,q_0} & \ldots & \mathcal{M}_{q_0,q_0-s+1} \\ \ldots & \ldots & \ldots \\ \mathcal{M}_{q_0-s+1,q_0} & \ldots & \mathcal{M}_{q_0-s+1,q_0-s+1} \end{bmatrix}\right|_{\omega=(q_0-s)\omega_0} \tag{4.22}$$

degenerate, a certain linear combination of the equations of motion $\delta\mathcal{E}_p$ with $p>q_0-s$ would become trivial, leaving a certain linear combination of the degrees of freedom $\delta\mathcal{X}_q$ with $q>q_0-s$ free. This generally happens at discrete values of $|k|\equiv\sqrt{k_ik_i}$.

To summarize, the general pole-skipping condition is given by

$$\boxed{\omega=(q_0-s)\,\omega_0 \quad \text{and} \quad \det\, M_s(k)=0} \tag{4.23}$$

for any $s\in\mathbb{Z}^+$.

4.2 Gauge fields

We have just reviewed the covariant expansion formalism for general holographic theories. It works well for bosonic fields without gauge symmetry. For gauge theories, the formalism works only after removing the gauge redundancy. To understand why there might be subtleties if we do not do so, realize that a pure gauge perturbation should automatically satisfy all equations of motion by their very nature.

To see how this is reflected in the covariant expansion formalism, write a general gauge parameter as

$$\delta\xi(v,r,x) = \delta\xi(r)e^{-\mathrm{i}\omega v+\mathrm{i}kx}, \tag{4.24}$$

where ξ could carry Lorentz indices. For example, $\delta\xi(v,r,x) = \Lambda(v,r,x)$ in Maxwell theory where the gauge transformation is given by $A_\mu \to A_\mu + \nabla_\mu\Lambda$. As another example, in General Relativity, $\delta\xi = \zeta_\mu$, where the gauge transformation is given by $g_{\mu\nu} \to g_{\mu\nu} + \nabla_\mu\zeta_\nu + \nabla_\nu\zeta_\mu$. With this, just like how we defined $\delta\mathcal{X}_q$ around (4.7), we define $\delta\Xi_u$ to be the subset of $\nabla_r^n\delta\xi|_{r=r_0}$ with weight u.

With this, we can write a general pure-gauge perturbation as

$$\delta\mathcal{X}_q = \sum_u \mathcal{T}_{q,u}(\omega,k)\,\delta\Xi_u. \tag{4.25}$$

Continuing with the Maxwell example, $\delta A_\mu = \nabla_\mu\delta\xi = \nabla_\mu\Lambda$. Writing out the components of (4.25), the first few orders are given by[1]

$$\begin{bmatrix} \delta A_v \\ \delta A_i \\ \nabla_r \delta A_v \\ \delta A_r \\ \nabla_r \delta A_i \\ \nabla_r^2 \delta A_v \\ \cdots \end{bmatrix} = \begin{bmatrix} -\mathrm{i}\omega \\ \mathrm{i}k_i \\ 0 \\ 0 \\ -\mathrm{i}k_i h'/2h \\ 0 \\ \cdots \end{bmatrix} \Lambda + \begin{bmatrix} 0 \\ 0 \\ -\mathrm{i}\omega + f'/2 \\ 1 \\ \mathrm{i}k_i \\ f''/2 \\ \cdots \end{bmatrix} \nabla_r \Lambda + \cdots . \tag{4.26}$$

The gravitational case is very similar and will be presented in Appendix C.3.

Once again, $\mathcal{T}_{q,u}$ satisfies the property of being proportional to $[\omega - (q-1)\omega_0] \ldots [\omega - u\omega_0]$ for $q > u$ for exactly the same reason, namely that we need sufficiently many ∇_v to raise the weight from u to q, as reviewed in Section 4.1.

A pure-gauge perturbation has the property that it automatically satisfies the equations of motion

$$\delta \mathcal{E}_p = \sum_{q,u} \mathcal{M}_{p,q} \, \mathcal{T}_{q,u} \, \delta \Xi_u = 0 \quad \forall p. \tag{4.27}$$

This can help us understand some features of the matrices $\mathcal{M}_{p,q}$. At $\omega = (q_0 - s)\omega_0$ where $s > 0$, as explained earlier, the entries of $\mathcal{M}_{p,q}$ with $p > q_0 - s \geq q$ will be zero. At the same time, for the same reason, the entries of $\mathcal{T}_{q,u}$ with $q > q_0 - s \geq u$ will be zero. Therefore, the infinite-dimensional statement (4.27) reduces to a finite one:

$$\sum_{q=q_0-s+1}^{q_0} \sum_{u=q_0-s+1}^{u_0} \mathcal{M}_{p,q} \, \mathcal{T}_{q,u} \, \delta \Xi_u = 0$$

[1]Here, everything is evaluated on the horizon, but from now on we will frequently avoid writing $(\cdot)|_{r=r_0}$ when it is clear that the quantity should be evaluated on the horizon.

$$\Rightarrow \sum_{q=q_0-s+1}^{q_0} \mathcal{M}_{p,q} \left(\mathcal{T}_{q,u_0} \delta\Xi_{u_0} + ... + \mathcal{T}_{q,q_0-s+1} \delta\Xi_{q_0-s+1} \right) = 0, \quad q_0 - s + 1 \leq p \leq q_0. \quad (4.28)$$

Since $\delta\Xi_u$ is the gauge parameter, we can choose its value for different u independently. Furthermore, for a given u, we can even choose its $|\delta\Xi_u|$ entries independently. From (4.28), it is now clear that, for $\delta\Xi_u$ with a given $u > q_0 - s$, each $\mathcal{T}_{q,u}\delta\Xi_u$ (with the range of q restricted to $q_0 - s + 1 \leq q \leq q_0$) belongs to the kernel of $M_s(k)$.

Before moving on, let us comment on the visibility of gauge redundancy at different orders. Notice that $u_0 < q_0$ usually. For example, in Maxwell theory, the highest weight for $\delta\mathcal{X}$ is $q_0 = 1$ owing to A_v, whereas the highest weight for $\delta\xi = \Lambda$ is $u_0 = 0$ as it is a scalar; similarly, in Einstein or higher-derivative gravity, $q_0 = 2$ owing to δg_{vv}, but $u_0 = 1$ because $\delta\xi = \zeta_\mu$ is a vector. This means that, in both cases, the matrix $M_s(k)$ has no kernel at leading order ($s = 1$). This is due to the fact that the range of u-index in (4.28) is empty if $u_0 < q_0 - s + 1$. As soon as s is large enough, however, a kernel will exist for all larger values of s.

In summary, the determinant of $M_s(k)$ will always vanish automatically, i.e., without having to fine-tune k, for all $s \geq q_0 - u_0 + 1$. This invalidates our earlier proposal for the pole-skipping condition (4.23) as we expect to turn $\det M_s(k)$ to zero only at special k.

Here is the moral of the story. This automatic degeneracy, as we have just seen, is a manifestation of gauge redundancy, i.e., gauge symmetry makes some equations of motion redundant. Pole skipping, however, is a statement about the physical bulk solution having an extra degree of freedom. Therefore, if we want to stick to (4.23) as our pole-skipping condition, we would have to remove the redundancy at the onset. This can be done by e.g. restricting to the physical (gauge-invariant) degrees of freedom and their equations of motion as in [102].

In practice, we find it convenient to skip the step of figuring out the gauge symmetry of the theory and finding the physical degrees of freedom. This is particularly advantageous if whether a theory has gauge symmetry depends on the values of certain coupling constants of the theory. For example, for a theory of Rarita-Schwinger field on curved space with mass m, there is gauge symmetry if $m = d/2$ and the curved background satisfies vacuum Einstein's equation with a negative cosmological constant, and no gauge symmetry otherwise. We study this example in Appendix D.2.

With the understanding above, we now present what we call the gauge-covariant version of (4.23):

$$\boxed{\omega = (q_0 - s)\,\omega_0 \quad \text{and} \quad \dim(\ker M_s(k)) \nearrow} \tag{4.29}$$

for each $s \geq 1$. To state it in words, for a given pole-skipping frequency, pole skipping happens for values of k that increase the dimension of the kernel (from its dimension at generic values of k).[2]

As examples, we use this new prescription to study Maxwell theory in Appendix C.2, Einstein gravity in Appendix C.3, and Rarita-Schwinger theory in Appendix D.2. For other examples where there is no gauge symmetry, the condition (4.29) reduces to (4.23).

[2]Incidentally, increasing the dimension of the kernel is different from setting non-zero diagonal entries of the Jordan normal form to zero. As an example, the matrix

$$\begin{bmatrix} 0 & 1 \\ 0 & x \end{bmatrix} \tag{4.30}$$

has eigenvalues 0 and x, but setting $x = 0$ will not expand its kernel from one dimension to two. Terminology-wise, increasing the algebraic multiplicity does not necessarily increase the geometric multiplicity. The geometric multiplicity of an eigenvalue is the dimension of its eigenspace. The algebraic multiplicity of an eigenvalue is its multiplicity as a root of the characteristic polynomial $\det(\lambda \mathbb{1} - M)$ for the matrix M. A given eigenvalue's algebraic multiplicity is equal to or greater than its geometric multiplicity. The kernel's dimension is the geometric multiplicity of the eigenvalue zero.

4.3 Fermionic fields

Let us now turn to bulk theories with only fermionic degrees of freedom. In this case, bosonic fields (including the metric) exist only as fixed backgrounds. This assumption is justified for example in theories with minimally coupled fermionic matter where the matter action carries an extra factor of G_N relative to the gravitational action since the gravitational backreaction can be neglected at leading order in G_N. Alternatively, if the fermionic background is trivial, the stress tensor remains zero at linear order, so we can neglect the backreaction in this case, too.

Another motivation for this section is to demonstrate how the techniques used in the bosonic case generalize, as many tools from this section will be useful when we consider general theories with both types of dynamical fields in the next section.

To begin with, the metric is again given by (4.2), repeated here for readability:

$$\mathrm{d}s^2 = -f(r)\mathrm{d}v^2 + 2\mathrm{d}v\mathrm{d}r + h(r)\mathrm{d}x^i\mathrm{d}x^i. \tag{4.31}$$

The transverse direction x^1 is now x for notational simplicity. To couple to fermions on a curved background, we need to introduce the tetrad, or frame fields. Just like how a useful choice of coordinates made various properties of the metric manifest, a specific choice for the tetrad will be similarly useful. Taking $(\bar{v}, \bar{r}, \bar{x}^i)$ as coordinates for the $(d+2)$-dimensional Minkowski spacetime with

$$\eta_{\bar{v}\bar{v}} = -1, \quad \eta_{\bar{r}\bar{r}} = 1, \quad \eta_{\bar{\imath}\bar{\jmath}} = \delta_{\bar{\imath}\bar{\jmath}}, \tag{4.32}$$

we choose the frame fields to have components

$$e_v^{\bar{v}} = \frac{1}{2}\left(1+f\right), \quad e_r^{\bar{v}} = -1, \quad e_v^{\bar{r}} = \frac{1}{2}\left(1-f\right), \quad e_r^{\bar{r}} = 1, \quad e_i^{\bar{\jmath}} = \sqrt{h}\,\delta_i^{\bar{\jmath}}, \tag{4.33}$$

which satisfy $g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu$. Bars over the flat space indices are used to distinguish them from the curved spacetime indices, and we use the Latin alphabet $a, b, \ldots$ to denote them abstractly.

The associated non-vanishing spin connections are then given by

$$(\omega_{\bar{v}\bar{r}})_v = -\frac{1}{2}f', \quad (\omega_{\bar{v}\bar{i}})_j = \frac{1}{4}\frac{h'}{\sqrt{h}}(1-f)\,\delta_{\bar{i}j}, \quad (\omega_{\bar{r}\bar{i}})_j = -\frac{1}{4}\frac{h'}{\sqrt{h}}(1+f)\,\delta_{\bar{i}j}. \tag{4.34}$$

Note that the barred indices are anti-symmetric. We also use ∇_μ to denote the full covariant derivative, whose action depends on the object it acts on. For example, for a spinor ψ,

$$\nabla_\mu \psi = \partial_\mu \psi + \frac{1}{4}(\omega_{ab})_\mu \Gamma^{ab}\psi, \tag{4.35}$$

and for a vector-spinor ψ_μ,

$$\nabla_\mu \psi_\nu = \partial_\mu \psi_\nu + \frac{1}{4}(\omega_{ab})_\mu \Gamma^{ab}\psi_\nu - \Gamma^\rho_{\mu\nu}\psi_\rho. \tag{4.36}$$

To avoid unnecessary complications caused by the dimension-dependent nature of gamma matrices, we will from now on work in $2+1$ bulk dimensions; the generalization to higher dimensions is straightforward and will be discussed briefly at the end of the section. With $d = 1$, we have the following three gamma matrices:

$$\Gamma^{\bar{v}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \Gamma^{\bar{r}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \Gamma^{\bar{x}} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{4.37}$$

In curved spacetime coordinates,

$$\Gamma^{v} = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}, \quad \Gamma^{r} = \begin{bmatrix} 0 & f \\ 1 & 0 \end{bmatrix}, \quad \Gamma^{x} = \frac{1}{\sqrt{h}}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{4.38}$$

Define projectors

$$P_{\pm} = \frac{1 \mp \sqrt{h}\Gamma^x}{2} = \frac{1 \mp \Gamma^{\bar{x}}}{2}, \tag{4.39}$$

which, for a general fermionic quantity X (potentially carrying Lorentz indices), has the following effect:

$$X = \begin{bmatrix} X_+ \\ X_- \end{bmatrix}, \quad P_+ X = \begin{bmatrix} X_+ \\ 0 \end{bmatrix}, \quad P_- X = \begin{bmatrix} 0 \\ X_- \end{bmatrix}. \tag{4.40}$$

Just like how we decomposed Lorentzian tensors into its components when considering bosons, we do the analogous thing here of decomposing all fermionic quantities into $\pm$ components. For example, they could be $\delta\psi_+, \nabla_r \delta\psi_-, \nabla_r \nabla_r \delta\Psi_{v,+}$, etc. An operator acting on a spinor carries two spinor indices. For such objects, we decompose in the same way and write

$$O = \begin{bmatrix} O_+{}^+ & O_+{}^- \\ O_-{}^+ & O_-{}^- \end{bmatrix}. \tag{4.41}$$

This allows us to easily generalize the definition of the weight. We define a lower $\pm$ to carry $\pm 1/2$ weight and an upper $\pm$ to carry $\mp 1/2$ in addition to the contributions from Lorentz indices, i.e., the total weight is equal to

$$\begin{aligned} &\#(\text{lower } v) && - \#(\text{lower } r) && - \#(\text{upper } v) && + \#(\text{upper } r) \\ &+ \frac{1}{2}\#(\text{lower } +) && - \frac{1}{2}\#(\text{lower } -) && - \frac{1}{2}\#(\text{upper } +) && + \frac{1}{2}\#(\text{upper } -). \end{aligned} \tag{4.42}$$

The analogue of (4.11) can be written as

$$\delta\mathcal{E}_p = \sum_q \mathcal{M}_{p,q}(\omega,k)\,\delta\mathcal{X}_q, \tag{4.43}$$

which looks exactly the same, but p and q are now half-integers. Each entry in $\delta\mathcal{E}_p$ or $\delta\mathcal{X}_q$ is not just a tensor component, but a tensor-spinor projected onto one of the two eigenspaces of Γ^x. As before, we absorb ∇_r derivatives into the definition of $\delta\mathcal{X}$, i.e., ∇_r should not be unpacked. In the bosonic case, we have explained that $(\nabla_r)^n\delta X|_{r=r_0}$ form a better basis than partial derivatives. Here, we are going one step further by saying that

$$(\nabla_r)^n \begin{bmatrix} \delta X_+ \\ \delta X_- \end{bmatrix}\Bigg|_{r=r_0} = \begin{bmatrix} ((\nabla_r)^n\delta X)_+|_{r=r_0} \\ ((\nabla_r)^n\delta X)_-|_{r=r_0} \end{bmatrix} \tag{4.44}$$

is a good basis for packaging things.

Recall that the most important property of the matrix $\mathcal{M}_{p,q}(\omega,k)$ is (4.12). We now proceed to show that it holds for fermions as well. A general component of the equation of motion for a fermion can be written as

$$(\nabla_r)^n\delta \begin{bmatrix} E_+ \\ E_- \end{bmatrix} = \sum F(g,R,\nabla,\Phi,\Psi)\,(\nabla_v)^k\,(\nabla_i)^l\,(\nabla_r)^m\delta \begin{bmatrix} X_+ \\ X_- \end{bmatrix}, \tag{4.45}$$

where F is a component of a spacetime tensor and at the same time a spinor operator, and the sum is over different terms of this form. Because the three gamma matrices together with the identity matrix form a complete basis for all 2-by-2 matrices and because F is itself a component of a covariant tensor, we can decompose it into

$$F(g,R,\nabla,\Phi,\Psi) = V_\mu(g,R,\nabla,\Phi,\Psi)\Gamma^\mu + F_0(g,R,\nabla,\Phi,\Psi)\mathbb{1}, \tag{4.46}$$

where V_μ is a vector and F_0 is a scalar, neither of which carries spinor indices.

The fermionic analogue of (4.14) can be written as

$$\delta\mathcal{E}_p = \sum \mathcal{F}(g, R, \nabla, \Phi, \Psi) \left(\nabla_v\right)^k \left(\nabla_i\right)^l \left(\nabla_r\right)^m \delta X_{q'+m}. \tag{4.47}$$

What is different from (4.14) is that $\delta\mathcal{E}$ and δX both carry a $\pm$ index, and $\mathcal{F}$, being a projected component of an operator in the spinor space, carries two of them. We emphasize again that ∇_μ here contains spin connections, so they are also operators on the spinors.

Like in the bosonic case, we first need to show that $\mathcal{F}$ vanishes on the horizon if it has a positive weight. In addition to bosonic constituents, fermionic fields are also potential ingredients. The trick here is to realize that $\mathcal{F}$ is a component of the matrix $V_\mu \Gamma^\mu + F_0 \mathbb{1}$. Since V_μ and F_0 do not carry spinor indices, the boost symmetry argument in the bosonic case applies still. Following our definition of the weight in (4.42), it is straightforward to check that Γ^μ and $\mathbb{1}$ all have this property. For example, the only entry in the identity matrix that carries a positive weight is the $(\mathbb{1})_+{}^-$ component, which is zero indeed; the components of

$$\Gamma^r = \begin{bmatrix} (\Gamma^r)_+{}^+ & (\Gamma^r)_+{}^- \\ (\Gamma^r)_-{}^+ & (\Gamma^r)_-{}^- \end{bmatrix} = \begin{bmatrix} 0 & f \\ 1 & 0 \end{bmatrix} \tag{4.48}$$

have weights $1, 2, 0$ and 1, so the only one that does not have to vanish on the horizon is $(\Gamma^r)_-{}^+$; for

$$\Gamma^v = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}, \tag{4.49}$$

the components all have non-positive weights, so there is no requirement for any component to be zero even though some of them are. Since $\mathcal{F}$ is built from V_μ, F_0, Γ and $\mathbb{1}$, the fact that $\mathcal{F} = 0$ in (4.47) if it has a positive weight is now guaranteed.

To show (4.12), we just need to look at ∇_v, which is the only way to increase the weight, knowing that $\mathcal{F}$ cannot. Consider a spinor ψ first:

$$\nabla_v \begin{bmatrix} \psi_+ \\ \psi_- \end{bmatrix} = \begin{bmatrix} (\partial_v - f'/4)\psi_+ \\ (\partial_v + f'/4)\psi_- \end{bmatrix}. \tag{4.50}$$

This clearly satisfies

$$\nabla_v T \propto \left(\partial_v - \frac{\mathrm{w}}{2} f'(r_0)\right) T \tag{4.51}$$

since $\psi_\pm$ have weights $\pm 1/2$. For a more general tensor-spinor, we just need to take into account Christoffel symbols in ∇_v, but those were exactly the same as for bosons! This concludes the proof that $\mathcal{F}$ is proportional to (4.15).

To show that $\mathcal{M}_{p,q}$ is proportional to (4.12), which is what we really need, we must show $q \geq q'$, where q' is the weight of $(\nabla_r)^m \delta X_{q'+m}$ appearing in (4.47) and q is what appears in (4.43). They differ because ∇'s will need to operate on the quantities to their right before evaluating everything on the horizon. In addition to Christoffel symbols, spin connections also appear in this process. Notice that ∇_i acts on a spinor as

$$(\nabla_i \delta\psi)_+ = \partial_x \delta\psi_+ - \frac{h' f}{4\sqrt{h}} \delta\psi_-, \tag{4.52}$$

$$(\nabla_i \delta\psi)_- = \partial_x \delta\psi_- + \frac{h'}{4\sqrt{h}} \delta\psi_+. \tag{4.53}$$

Combined with (4.17), we see that ∇_i will only turn a tensor-spinor component into another tensor-spinor component with a higher weight (when evaluated at the horizon). This ensures that $q \geq q'$. Finally, substitution of the Fourier expansion gives the desired factors (4.12).

In fact, this concludes the discussion of bulk theories with only dynamical fermions (in three dimensions), because the rest follows in exactly the same way as in the bosonic case. The only difference is that p, q are half-integers. Even the conclusion reads the same as before: pole skipping happens at frequencies $(q_0 - s)\,\omega_0$ for $s = 1, 2, \ldots$, where q_0 is the highest weight present in the theory, now a half-integer.

So far, we have not said anything about gauge symmetry. Fortunately, this goes through just like in the bosonic case. Without gauge symmetry, the pole-skipping condition is given in (4.23); with gauge symmetry, we could either restrict to gauge-invariant quantities and use the same condition, or we can use the gauge-covariant condition (4.29). We note, however, that the gauge parameter $\delta\Xi_u$ appearing in (4.25) would have to become a fermionic one, i.e., u should take half-integer values. Having a bosonic gauge parameter in a theory of only dynamical fermions is neither common nor within the scope of the current section, but it does belong to the more general class we study in Section 4.4.

Higher dimensions

To define a spinor on curved spacetime, we need to define the gamma matrices in Minkowski space first. The gamma matrices in Minkowski space $\mathbb{R}^{1,D-1}$ satisfy the Clifford algebra $C\ell(1, D-1)$: $\{\gamma^a, \gamma^b\} = 2\eta^{ab}$. We can define these gamma matrices recursively, starting from two dimensions, where we can choose

$$\gamma_2^0 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \gamma_2^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{4.54}$$

The $(2n+1)$-dimensional gamma matrices are then defined using the $(2n)$-dimensional gamma matrices by

$$\gamma_{2n+1}^a = \gamma_{2n}^a, \qquad a = 0, \ldots, 2n-1,$$

$$\gamma_{2n+1}^{2n} = \mathrm{i}^{n+1}\gamma_{2n}^{0}...\gamma_{2n}^{2n-1}. \tag{4.55}$$

Similarly, the $(2n)$-dimensional gamma matrices are defined from $(2n-1)$-dimensional gamma matrices by

$$\begin{aligned} \gamma_{2n}^{a} &= \gamma_{2n-1}^{a} \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad a = 0, \ldots, 2n-2, \\ \gamma_{2n}^{2n-1} &= \mathbb{1} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \end{aligned} \tag{4.56}$$

It is then straightforward to check that these matrices satisfy the Clifford algebra. In this construction, gamma matrices are $2^{\lfloor \frac{D}{2} \rfloor} \times 2^{\lfloor \frac{D}{2} \rfloor}$ matrices.

Now we can define the Γ-matrices in the $(\bar{v}, \bar{r}, \bar{x}^i)$ coordinates by

$$\begin{aligned} \Gamma_D^{\bar{v}} &= \gamma_D^0, \qquad \Gamma_D^{\bar{r}} = \gamma_D^1, \\ \Gamma_D^{\bar{\imath}} &= \gamma_D^i \delta_{\bar{\imath},i-1}, \qquad \bar{\imath} = 1, \ldots, d. \end{aligned} \tag{4.57}$$

For $D = 3$ $(d = 1)$, this reproduces (4.37).

The projectors to the subspaces are defined by

$$P_+ = \frac{1 + \Gamma^{\bar{v}}\Gamma^{\bar{r}}}{2} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \mathbb{1}_{2^{\lfloor D/2-1 \rfloor}}, \quad P_- = \frac{1 - \Gamma^{\bar{v}}\Gamma^{\bar{r}}}{2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \mathbb{1}_{2^{\lfloor D/2-1 \rfloor}}. \tag{4.58}$$

Here, even though each subspace would have more degrees of freedom than in $D = 3$ (e.g., $\psi_\pm$ each has $2^{\lfloor D/2-1 \rfloor}$ components), in the basis we have defined, there is no need to distinguish them further as in [76, 82]. We attribute this difference to the choice of projectors.

Using these Γ-matrices, most of the argument for $D = 3$ goes through in the same way, but some aspects must be generalized. For example, instead of having Γ^μ and $\mathbb{1}$ in (4.46) as a complete basis for the general operator F, a complete basis in higher dimensions is formed using $\mathbb{1}$ and anticommutators of all the gamma matrices [106]. Explicitly, in even dimensions ($D = 2k$), a complete basis for $2^k \times 2^k$ matrices is given by

$$\mathbb{1} \cup \left\{ \Gamma^{[\mu_1}\Gamma^{\mu_2}...\Gamma^{\mu_m]} \mid m = 1, ..., 2k \right\}, \tag{4.59}$$

and in odd dimensions ($D = 2k + 1$), a complete basis for $2^k \times 2^k$ matrices is given by

$$\mathbb{1} \cup \left\{ \Gamma^{[\mu_1}\Gamma^{\mu_2}...\Gamma^{\mu_m]} \mid m = 1, ..., k \right\}. \tag{4.60}$$

4.4 Bosonic and fermionic fields

We discussed general theories with either dynamical bosonic fields or fermionic fields in earlier sections. It is then natural to ask whether the argument generalizes to the case when both are present. A naive expectation might be that, if we have already worked out the pole-skipping points for a theory with e.g. only bosonic fields, adding fermions will not change them even though it may lead to more. This is not always correct because fermions can appear even in the bosonic equations of motion, adding extra terms proportional to fermionic perturbations, so the special frequencies that could turn the original bosonic equations of motion trivial would no longer necessarily do so. In other words, the linearized bosonic equations of motion $\delta\mathcal{E}_B$ takes the form

$$\delta\mathcal{E}_B = \mathcal{M}_{BB}\delta\mathcal{X}_B + \mathcal{M}_{BF}\delta\mathcal{X}_F, \tag{4.61}$$

where $\delta\mathcal{X}_B$ and $\delta\mathcal{X}_F$ are bosonic and fermionic perturbations respectively, so even if special frequencies and momenta set $\mathcal{M}_{BB}$ to zero, $\mathcal{M}_{BF}$ can still be non-zero, preventing a linear combination of $\delta\mathcal{E}_B$ from necessarily becoming trivial.

In this section, we will consider this general case. To begin with, write covariant expansion coefficients of the equations of motion evaluated on the horizon as

$$\delta\mathcal{E}_p = \sum_q \mathcal{M}_{p,q}(\omega,k)\,\delta\mathcal{X}_q. \tag{4.62}$$

This looks exactly like (4.11) and (4.43), but p,q can now both be integers or half-integers. When p is an integer, this is a bosonic equation of motion, receiving contributions from both bosonic field perturbations which have integer q's and fermionic field perturbations which have half-integer q's; when p is a half-integer, this is a fermionic equation of motion, again receiving contributions from both integer and half-integer q's.

Section 4.2 presented a gauge-covariant formalism. It would be natural if we now proceed with the current section gauge-covariantly. However, as we will see, when both bosons and fermions are present, it is not obvious whether there exists any systematic and practical way of locating the pole-skipping momenta k. Nevertheless, we can still derive a general pattern of pole-skipping frequencies ω. Since the main motivation for the gauge-covariant formalism was to compute the pole-skipping momenta with less effort, its advantage is lost if we are uncommitted to that goal. As a result, we find it easier to derive our statements after removing gauge redundancy. We will comment more on this issue at the end of the section.

In this approach, we still write (4.62), now with the understanding that only physical (gauge-invariant) degrees of freedom and their corresponding equations of motion are included. Let us also organize the basis so that the matrix $\mathcal{M}_\infty$ defined in (4.20) divides into four blocks of infinite size:

$$\mathcal{M}_\infty = \left[\begin{array}{ccc|ccc} \mathcal{M}_{l_b,l_b} & \mathcal{M}_{l_b,l_b-1} & \cdots & \mathcal{M}_{l_b,l_f} & \mathcal{M}_{l_b,l_f-1} & \cdots \\ \mathcal{M}_{l_b-1,l_b} & \mathcal{M}_{l_b-1,l_b-1} & \cdots & \mathcal{M}_{l_b-1,l_f} & \mathcal{M}_{l_b-1,l_f-1} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \hline \mathcal{M}_{l_f,l_b} & \mathcal{M}_{l_f,l_b-1} & \cdots & \mathcal{M}_{l_f,l_f} & \mathcal{M}_{l_f,l_f-1} & \cdots \\ \mathcal{M}_{l_f-1,l_b} & \mathcal{M}_{l_f-1,l_b-1} & \cdots & \mathcal{M}_{l_f-1,l_f} & \mathcal{M}_{l_f-1,l_f-1} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{array}\right], \tag{4.63}$$

where l_b is the highest integer weight and l_f is the highest half-integer weight. With this separation, we can give each of the blocks a name so that $\delta\mathcal{E} = \mathcal{M}_\infty \delta\mathcal{X}$ can be written as

$$\left[\begin{array}{c} \delta\mathcal{E}_B \\ \hline \delta\mathcal{E}_F \end{array}\right] = \left[\begin{array}{c|c} \mathcal{M}_{BB} & \mathcal{M}_{BF} \\ \hline \mathcal{M}_{FB} & \mathcal{M}_{FF} \end{array}\right] \left[\begin{array}{c} \delta\mathcal{X}_B \\ \hline \delta\mathcal{X}_F \end{array}\right]. \tag{4.64}$$

Recall that pole skipping happens when $\mathcal{M}_\infty$ is degenerate. When we only have bosonic fields, we only have $\mathcal{M}_{BB}$; when we only have fermionic fields, we only have $\mathcal{M}_{FF}$. The problem with $\mathcal{M}_{BF}$ and $\mathcal{M}_{BB}$ is that they interpolate between a half-integer weight and an integer weight, so we do not have a relation like (4.12). (The argument does not generalize to this case.)

As is by now clear, a main theme of the covariant expansion formalism is to reorganize things to manifest hidden features, so it is what we will now do once again. Acting on both sides of (4.62) with the invertible matrix

$$\mathcal{U} \equiv \begin{bmatrix} \mathbb{1} & -\mathcal{M}_{BF}\mathcal{M}_{FF}^{-1} \\ \\ 0 & \mathbb{1} \end{bmatrix} \tag{4.65}$$

defines a new basis for $\delta\mathcal{E}$ while keeping the same basis for $\delta\mathcal{X}$. This physically means that we are taking linear combinations of the original equations of motion. In this basis, (4.62) becomes

$$\begin{gathered} \mathcal{U}\delta\mathcal{E} = \mathcal{U}\mathcal{M}_\infty\delta\mathcal{X} \\ \begin{bmatrix} \delta\mathcal{E}_B - \mathcal{M}_{BF}\mathcal{M}_{FF}^{-1}\delta\mathcal{E}_F \\ \\ \delta\mathcal{E}_F \end{bmatrix} = \begin{bmatrix} \mathcal{M}_{BB} - \mathcal{M}_{BF}\mathcal{M}_{FF}^{-1}\mathcal{M}_{FB} & 0 \\ \\ \mathcal{M}_{FB} & \mathcal{M}_{FF} \end{bmatrix} \begin{bmatrix} \delta\mathcal{X}_B \\ \\ \delta\mathcal{X}_F \end{bmatrix}. \end{gathered} \tag{4.66}$$

With the top-right block set to zero, we have now reduced the problem of finding the conditions under which an equation of motion in the upper-left block becomes trivial. But this is the same problem as in the bosonic case! We conclude that pole skipping happens at frequencies $\mathrm{i}(l_b - s)2\pi T$ for $s \in \mathbb{Z}^+$. Similarly, we can use another invertible matrix to turn the lower-left block to zero, which immediately leads to the conclusion that pole skipping *also* happens at frequencies $\mathrm{i}(l_f - s)2\pi T$ for $s \in \mathbb{Z}^+$. This concludes the proof.

We should note that this proof uses the inverse of an infinite-dimensional matrix $\mathcal{M}_{FF}$ to eliminate the whole upper-right block of the equation of motion matrix. The inverse should exist after removing gauge redundancy because all equations of motion are linearly independent. It would be good to prove this rigorously. The existence of the inverse is certainly sufficient for the next steps, but it may not be necessary. In particular, when we look for pole skipping by studying (4.66), we only need finitely many rows of the upper-right block of $\mathcal{U}\mathcal{M}_\infty$ to vanish at each order in s. This suggests that it might be possible to remove the need to invert an infinite matrix from the argument.

To go from the current approach to a gauge-covariant approach is in principle easy. The inverse of $\mathcal{M}_{FF}$ might not exist with gauge symmetry.[3] The physical reason why we need to invert $\mathcal{M}_{FF}$ is because we want to use the fermionic equations of motion to turn fermionic perturbations into bosonic ones. This procedure can be of course performed in the gauge-covariant approach, but instead of inverting the whole matrix, we should only invert the "physical part" of the matrix. More precisely, we should project out the kernel before performing the inverse. We have avoided doing that to keep the equations simple.

4.5 Discussion

In this paper, we studied pole skipping in the presence of gauge and fermionic fields.

In the presence of gauge fields, we presented a pole-skipping condition that automatically deals with gauge symmetry. This upgrade is in fact quite practical as it allows one to compute pole-skipping points for any theory with a given Lagrangian without having to worry about removing the redundancy which usually involves determining the gauge-invariant combinations of field components. This condition reduces to the one in [101] in the absence of gauge symmetry.

For theories with only fermionic fields, we provided a formalism that is parallel to the bosonic case. This formalism is again practically useful and allows one to locate the pole-skipping points systematically. With this extension, we found that pole skipping

[3] It certainly would not exist if the gauge parameter is fermionic and only acts on the fermions. Similarly the inverse of $\mathcal{M}_{BB}$ would not exist if the gauge parameter is bosonic and only acts on the bosons. This follows from the discussion in Section 4.2.

generally happens at frequencies $\mathrm{i}(l_f - s)2\pi T$ for positive integer s, where l_f is the highest spin in the fermionic theory.

We then applied the formalism to theories with both dynamic bosonic and fermionic fields. In this case, we provided an argument for the pole-skipping frequencies, namely that there is one tower of pole skipping at frequencies $\mathrm{i}(l_b - s)2\pi T$ and another tower with $\mathrm{i}(l_f - s)2\pi T$. This statement is nontrivial as the presence of fermionic fields highly influences the pole-skipping momenta of the bosonic tower and vice versa. Unlike the purely bosonic or fermionic cases, the argument here is rather abstract – it might be difficult to actually do the computations for a specific theory due to the need to invert an infinite matrix. It would be interesting to see if this requirement can be lifted. There is one situation, however, where we do not need this inverse even when both dynamical bosons and fermions are present: when the background fermions are all zero, the bosonic and fermionic perturbations then decouple ($\mathcal{M}_{BF} = \mathcal{M}_{FB} = 0$ in (4.64)), so the problem becomes equivalent to a purely bosonic theory plus a purely fermionic theory. This is quite common e.g. for supersymmetric black holes.

Let us now comment on the connection to chaos. Since the leading pole-skipping frequency is given by $\mathrm{i}(l-1)2\pi T$, where l is the highest spin in the theory (either integer or half-integer), for $l \geq 3/2$, it seems that this frequency is positive in the imaginary direction, meaning that the Fourier mode (which is proportional to $e^{-\mathrm{i}\omega v}$) will grow exponentially in the retarded time. This already suggests a connection to chaos. More quantitatively, this connection was explained in [45] by comparing the form of the OTOC and the leading pole-skipping mode in Einstein gravity with matter. This connection was further explained in [101] at the level of the metric: the shockwave solution for general higher-derivative gravity [50] is found to be a limit of the quasinormal mode responsible for the leading pole-skipping point.

In Einstein gravity or higher-derivative gravity, $l = 2$ and the leading frequency is

$i\lambda_{L,\max}$, where $\lambda_{L,\max} = 2\pi T$ is the Lyapunov exponent for maximal chaos [17]. For higher spins, the leading pole-skipping point still happens at $i\lambda_L$ but now $\lambda_L > \lambda_{L,\max}$, consistent with the well-known result that (finitely many) higher-spin fields violate the chaos bound [89, 17, 107, 108]. It would be interesting to extend the connection between the leading pole-skipping point and the OTOC to $l > 2$. To achieve this, one might first generalize the shockwave solution to general higher-spin theories, perhaps in the same way that shockwave solutions were constructed for a general higher-derivative theory (Appendix A of [50]). The analysis in [101] then suggests that the higher-spin shockwave can be obtained as a limit of the leading pole-skipping mode. Since shockwaves tell us about the OTOC, this would be enough to connect the dots.

Moreover, it would be interesting, though arguably more challenging, to do the same for half-integer l. In other words, can we find shockwave solutions in gravitational theories where the highest-spin field is fermionic and use that to build the connection between the leading pole-skipping frequency and the OTOC? A rigid way to compute the Lyapunov exponent in the presence of fermionic fields might require a scattering perspective [11], and it is *a priori* not clear whether there is a classical limit where the OTOC is described by a "fermionic" shockwave. The quasinormal mode at the leading pole-skipping point, however, suggests that there might be.

So far, this formalism has restricted to planar black holes with planar symmetry. Evidence has found that the connection between pole skipping and the OTOC holds even for rotating black holes [56, 87, 64, 109, 98]. It would be interesting to generalize the covariant expansion formalism in this direction and use it to prove this connection for general higher-derivative gravity.

In maximally chaotic systems, the form of pole skipping and the OTOC are constrained by symmetry [44, 67, 110, 69, 70]. In the bulk, we have seen how they are constrained by the boost symmetry of the black hole. Beyond maximal chaos, the connection between pole skipping and the OTOC is more subtle [90, 72, 71]. It would be interesting to understand the more general relation between them better, perhaps in effective models of non-maximal chaos [111, 112, 113].

Recent work has found pole skipping on non-black hole backgrounds [114, 115]. It is not clear whether there exists a similar covariant expansion formalism beyond black holes as the formalism relies heavily on the boost symmetry. However, since the example in [115] was obtained via a double Wick rotation from a black hole spacetime, it is plausible that pole skipping happens only for those spacetimes that are related to black holes via analytical continuation, in which case the analytically continued symmetry generator might again play an important role.

Finally, we should mention that this formalism works well in practice for any number of fields with very general interactions even though we did not present such examples. For this reason, this formalism might be helpful for interesting computations that were previously considered too complicated.

Chapter 5

Causality in the gravitational path integral

A bulk description of nonperturbative quantum gravity is not yet available. A popular approach is to consider a path integral over metrics. We investigate some aspects of this approach. The metric could turn out to be only a low-energy approximation of other fundamental degrees of freedom. Our results would plausibly still apply, since the issues we address arise already in metrics with curvature well below the Planck (and string) scale. We begin with a Lorentzian formulation of holography, in which one integrates over asymptotically anti-de Sitter (AdS) spacetimes and matter fields (with certain boundary conditions) to compute correlation functions in a dual quantum field theory. One often works in a large N or semiclassical limit and only includes classical supergravity solutions and small perturbations of them. Classically, spacetimes satisfy the null energy condition so the Gao-Wald theorem [116] ensures that the bulk preserves boundary causality. In other words, the fastest way to send a signal between two observers on the boundary is via a path that stays on the boundary. No trajectory that enters the bulk can arrive sooner. Semiclassically, the achronal averaged null energy condition ensures boundary causality [117]. (A necessary and sufficient geometric condition is given in [118].)

However, in full quantum gravity (finite N), the bulk path integral includes metrics that violate boundary causality. That is, two boundary points that are spacelike-separated on the boundary can nevertheless be timelike-separated with respect to some bulk metrics. We will see that these boundary causality violating metrics are not "rare" – they include open sets in the space of metrics. These metrics describe causally well-behaved bulk geometries – there are no closed timelike curves or any causal pathology. So there is no reason to exclude them from the bulk path integral. Hence one might worry that they could contribute to causality violations in the dual field theory, which would be a problem.

Consider, for example, the commutator of an operator $\mathcal{O}$ at two spacelike-separated boundary points: $[\mathcal{O}(x), \mathcal{O}(y)]$. This must vanish identically, but holography says that it should be given by a bulk path integral over metrics and a matter field ϕ dual to $\mathcal{O}$. If one computes the commutator $[\phi(p), \phi(q)]$ in each metric g, and then follows the standard limiting procedure of taking $p \to x$ and $q \to y$, one generically expects a nonzero answer whenever x and y are causally related with respect to g. This is not yet a contradiction: the integral of this over metrics g weighted by $e^{\mathrm{i}S_g}$ could still vanish exactly.

However, it is not just one quantity that must integrate to zero. In principle, $\mathcal{O}$ can be any operator in the dual field theory, and the commutator could be multiplied by any other operator. Their corresponding bulk expressions would all be nonzero in some metrics g, but their integrals over g must still vanish identically. Since the gravitational weighting $e^{\mathrm{i}S_g}$ and measure are independent of the operator insertions, this seems unlikely.

By studying the boundary conditions for the bulk path integral computation of commutators, we identify some freedom whenever the boundary theory is causal and unitary. We show that there is a way to use this freedom to make boundary causality manifest.

5.1 Specific examples

For $b \in \mathbb{R}$, consider the family

$$g_{(b)} = -f_b(r)\,\mathrm{d}t^2 + f_b(r)^{-1}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega_2^2. \tag{5.1}$$

One requires $f_b(r) \approx r^2/\ell^2$ asymptotically, $f_b(0) = 1$ and $f_b'(0) = 0$ for differentiability at $r = 0$. We connect this family at $b = 0$ to AdS_4 by setting $f_0(r) = r^2/\ell^2 + 1$. In pure AdS, two boundary points are null-separated on the boundary if and only if they are null-separated in the bulk. Hence we explore boundary causality in (5.1) by comparing to AdS. Restricting to radial null geodesics, the boundary-to-boundary crossing time is

$$t_\infty = 2\int_0^\infty \frac{\mathrm{d}r}{f_b(r)}. \tag{5.2}$$

Hence (5.1) violates boundary causality if $t_\infty < t_\infty^{\mathrm{AdS}} = \pi\ell$; e.g., $f_b(r) > f_0(r)$ everywhere would suffice. For instance, setting $\ell = 1$, (5.1) with $f_b(r) = r^2 + 1 + b\ r^2(1 + r^4)^{-1}$ violates boundary causality for $b > 0$.

These static configurations only contribute to a path integral where the initial and final surfaces have induced metric $\mathrm{d}r^2/f_b(r) + r^2\mathrm{d}\Omega_2^2$. One might ask if there exist classical solutions interpolating between such surfaces. The answer is yes: global AdS gives the desired induced metric with $f_b(r) \geq f_0(r)$ on spacelike surfaces with $t = b\,h(r)$ where h must just fall off like $1/r^2$ or faster.

Alternatively, to match a static AdS slice, one can just make b time-dependent near the surface, going to zero on it. More generally, one can match any given induced metric by making $g_{(b)}$ appropriately time-dependent without affecting the causality violating region.

Although we have focused on four-dimensional examples, it is clear that similar examples exist in all dimensions greater than two. In particular, if x is timelike-related to y in one metric, it remains timelike-related in any nearby metric. Hence boundary causality is violated in open sets in the space of metrics.

5.2 Operator dictionaries

Holographic duality is generally formulated as an equivalence between bulk and boundary partition functions [119, 120]:

$$\mathcal{Z}_{\text{bulk}}[\phi_0] = \mathcal{Z}_{\text{bdy}}[\phi_0]. \tag{5.3}$$

Here, $\mathcal{Z}_{\text{bdy}}$ is defined on some fixed Lorentzian manifold $(\mathcal{B}, \gamma)$, whereas $\mathcal{Z}_{\text{bulk}}$ path-integrates over spacetimes $(\mathcal{M}, g)$ having $(\mathcal{B}, \gamma)$ as their conformal boundary. The symbol ϕ_0 is a placeholder for all boundary conditions for dynamical fields on the bulk side, and for sources of operator deformations of the action on the boundary side. To make progress, most work in the literature studies (5.3) perturbatively in $1/N$, where $\mathcal{Z}_{\text{bulk}}$ is amenable to a saddlepoint approximation (plus corrections). Here we test one aspect of (5.3) as a statement of holography in full quantum gravity, i.e., nonperturbatively in $1/N$.

To study real-time correlation functions, we assume (5.3) applies in Lorentzian signature. From the standpoint of the boundary theory, $\mathcal{Z}_{\text{bdy}}[\phi_0]$ is a standard field-theoretic generating functional. Explicitly [120],

$$\mathcal{Z}_{\text{bdy}}[\phi_0] = \left\langle \exp\left\{ \mathrm{i} \int_{\mathcal{B}} \sum_{\alpha} \phi_0^{\alpha}\, \mathcal{O}_{\alpha} \right\} \right\rangle_{\text{bdy}}, \tag{5.4}$$

where $\langle \cdot \rangle_{\text{bdy}}$ denotes the boundary path integral over quantum fields, and each $\mathcal{O}_\alpha$ is a gauge-invariant operator sourced by ϕ_0^α. In light of (5.3), the bulk object $\mathcal{Z}_{\text{bulk}}[\phi_0]$ may also be regarded as a generating functional of boundary correlators of the form

$$G \equiv \langle \mathcal{O}_n(x_n) \cdots \mathcal{O}_1(x_1) \rangle_{\mathrm{bdy}} \,. \tag{5.5}$$

If G is time-ordered with respect to boundary time, then

$$G = \frac{1}{\mathrm{i}^n} \frac{\delta}{\delta \phi_0^n(x_n)} \cdots \frac{\delta}{\delta \phi_0^1(x_1)} \left. \frac{\mathcal{Z}_{\mathrm{bulk}}[\phi_0]}{\mathcal{Z}_{\mathrm{bulk}}[0]} \right|_{\phi_0 = 0} . \tag{5.6}$$

The rationale so far is that of [119, 120], later coined the "differentiate dictionary" in [121].

In practice though, one usually takes a leap and follows the logic of [122, 123] to argue that a boundary correlator should be nothing but an appropriate limit of bulk ones. More specifically, if ϕ_α is the bulk field dual to $\mathcal{O}_\alpha$ and the latter has conformal dimension Δ_α, one may propose [1]

$$G \overset{?}{\propto} \int \mathcal{D}g \, e^{\mathrm{i} S_g} \lim_{z \to 0} z^{\Delta_\Sigma} \left\langle \phi_n(x_n, z) \cdots \phi_1(x_1, z) \right\rangle_g , \tag{5.7}$$

where $\Delta_\Sigma = \sum_\alpha \Delta_\alpha$, $\langle \cdot \rangle_g$ denotes the bulk path integral over quantum fields in a fixed metric g, and (x, z) are Fefferman-Graham coordinates [124]. (These provide an unambiguous asymptotic location for the bulk operators.) Alternatively, one could consider (5.7) with $z \to 0$ taken outside the integral. While physically both seem like reasonable proxies for boundary correlators, it is unclear whether they are equal or mathematically consistent with (5.6).

When gravity is treated semiclassically, these two options do agree. In this case, they were referred to as the "extrapolate dictionary" in [121], where their consistency with

[1] Time-ordering has been intentionally omitted below since the notion of time order in the bulk is subtle – see the next section.

(5.6) was carefully studied. We deviate from [121] by studying Lorentzian physics and at a nonperturbative level.

For any causal field theory, local operators at spacelike boundary separation commute. In the bulk, there appears to be a problem with the extrapolate dictionary stemming from the presence of off-shell contributions to the gravitational path integral that violate boundary causality. Similar problems arise regardless of when one takes the $z \to 0$ limit, so consider (5.7) for definiteness. For a commutator of operators at spacelike separation, $\langle [\mathcal{O}(x), \mathcal{O}(y)] \rangle_{\mathrm{bdy}} = 0$, so (5.7) becomes

$$\int \mathcal{D}g \, e^{\mathrm{i}S_g} \lim_{z \to 0} z^{2\Delta} \left\langle [\phi(x,z), \phi(y,z)] \right\rangle_g \stackrel{?}{=} 0. \tag{5.8}$$

As explained above, there are open sets of bulk metrics for which x and y are causally related and the commutator is generically nonzero. So why should the integral vanish exactly?

5.3 Resolution

We now explain an important subtlety omitted from the above discussion. Consider the commutator

$$\left\langle [\phi(p), \phi(q)] \right\rangle_g \equiv \left\langle \phi(p)\phi(q) \right\rangle_g - \left\langle \phi(q)\phi(p) \right\rangle_g , \tag{5.9}$$

with each term computed via a path integral. Although we have used the bulk notation $\langle \cdot \rangle_g$, this also applies to the boundary by replacing $(\mathcal{M}, g)$ with $(\mathcal{B}, \gamma)$. Now, the path integral on the original manifold always computes time-ordered correlators. To compute an out-of-time-order correlator, one needs to manufacture from the original manifold a

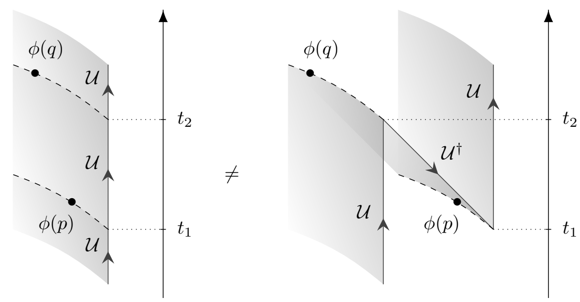

Figure 5.1: Inequivalent timefolds for computing a two-point function for a field ϕ inserted with different time orderings. Point q is to the future of p, each respectively lying at times t_1 and $t_2 > t_1$. Left: time-ordered correlator $\langle \phi(q)\phi(p)\rangle$, where time-evolution is implemented by $\mathcal{U}$ everywhere. Right: nontrivial timefold for the out-of-time-order correlator $\langle \phi(p)\phi(q)\rangle$, with backwards time-evolution via $\mathcal{U}^\dagger$.

timefold as follows. Consider an n-point function for a field ϕ,

$$\langle \phi_+ | \phi(p_n) \cdots \phi(p_2)\phi(p_1) | \phi_- \rangle_g \,, \tag{5.10}$$

where $\phi_\pm$ are initial and final states on Cauchy surfaces $\Sigma_\pm$. Starting at Σ_-, one should use the evolution operator $\mathcal{U}$ up to a Cauchy surface containing p_1 and insert $\phi(p_1)$. One should then evolve to p_2 for the next insertion. If p_2 is to the future of p_1, one could just use $\mathcal{U}$ again. However, if p_2 is to the past of p_1, one should evolve backwards with $\mathcal{U}^\dagger$ for the insertion of $\phi(p_2)$. This procedure must be repeated for all n insertions before finally evolving to Σ_+. The whole process produces a timefold, a new zigzagged spacetime implementing the correct ordering of operators (see Fig. 5.1). In the path integral, backward components receive weighting $e^{-\mathrm{i}S_g}$, instead of the usual $e^{\mathrm{i}S_g}$ on forward ones.

If two points are timelike-related, the commutator is a difference between two correlators computed on distinct timefolds, so will be generically nonzero. In the case of spacelike-separated points, however, the two points can be inserted on the same Cauchy surface. Hence the same timefold qualifies for a computation of either ordering, yielding a vanishing commutator. Again, the discussion here applies to both boundary commutators $\langle[\mathcal{O}(x), \mathcal{O}(y)]\rangle_{\text{bdy}}$ and bulk commutators $\langle[\phi(x,z), \phi(y,z)]\rangle_g$ on a specific metric, and therefore one has both boundary and bulk timefolds.

In general, the field theory $\mathcal{Z}_{\text{bdy}}$ is naturally defined on some manifold $\mathcal{B}$, and $\mathcal{Z}_{\text{bdy}}[\mathcal{B}, \phi_0]$ is the generating functional of time-ordered correlators on $\mathcal{B}$. To compute an out-of-time-order correlator, one needs to extend $\mathcal{Z}_{\text{bdy}}$ to an appropriate timefold spacetime $\mathcal{F}$ constructed out of $\mathcal{B}$ as described above. Then $\mathcal{Z}_{\text{bdy}}[\mathcal{F}, \phi_0]$ allows one to compute the desired correlator. The statement of holography in terms of partition functions must account for this. Hence one has to more explicitly rewrite (5.3) as

$$\mathcal{Z}_{\text{bulk}}[\mathcal{F}, \phi_0] = \mathcal{Z}_{\text{bdy}}[\mathcal{F}, \phi_0], \tag{5.11}$$

where $\mathcal{Z}_{\text{bulk}}[\mathcal{F}, \phi_0]$ path-integrates over all metrics with the timefold $\mathcal{F}$ as conformal boundary, and boundary conditions ϕ_0 appropriately distributed on $\mathcal{F}$.

We will call the regions of spacetime separated by the creases the "sheets" of the timefold. For a causal and unitary boundary theory, if x and y are spacelike-separated, $\langle\mathcal{O}(y)\mathcal{O}(x)\rangle$ can be computed on either a trivial timefold (i.e. the original spacetime with no foldings), or a nontrivial timefold with $\mathcal{O}(x)$ on some sheet and $\mathcal{O}(y)$ on some other sheet with the same result. For example, evolution forward and back – without operator insertions – is the identity. It is not obvious that the corresponding bulk partition functions will agree since we do not know (independent of holography) that the gravitational path integral describes unitary evolution. This leads to a potential ambiguity in the definition of $\mathcal{Z}_{\text{bulk}}$. Since we are trying to understand how a basic property of the boundary theory follows from the bulk path integral, we have to resolve this ambiguity. If the boundary theory is causal and unitary, we adopt a *minimal-timefold* approach, introducing a fold only when needed to represent operators at causally-related points where the later operator appears first in the correlator. When folding minimally, backward evolution just needs to sweep causal diamonds between insertions at non-time-ordered points (including small neighborhoods for spacelike creases). Minimal timefolds are introduced to disambiguate inequivalent computations which should nonetheless give equivalent results – if the boundary theory is acausal or non-unitary, however, results will generically disagree and the minimal-timefold prescription does not apply.

We now show that with the minimal timefold, both the differentiate and extrapolate dictionaries predict no violation of boundary causality. Consider first the differentiate dictionary. Equation (5.6) tells us how the time-ordered correlators on the boundary are computed from the bulk, but we have so far not included out-of-time-order correlators in this equality. Extending the dictionary to general n-point functions, one has

$$G = \frac{1}{\mathrm{i}^n} \frac{\delta}{\delta\phi_0^n(x_n)} \cdots \frac{\delta}{\delta\phi_0^1(x_1)} \left. \frac{\mathcal{Z}_{\text{bulk}}[\mathcal{F}, \phi_0]}{\mathcal{Z}_{\text{bulk}}[\mathcal{F}, 0]} \right|_{\phi_0 = 0}, \tag{5.12}$$

where $\mathcal{F}$ is a minimal *boundary* timefold constructed so as to order the operator insertions as given in G (cf. (5.6) when the left-hand side is time-ordered, in which case $\mathcal{F}$ is just $\mathcal{B}$). If x and y are spacelike-separated boundary points, the minimal timefold for a two-point correlator is trivial. Since this is the same for both terms in the commutator, the only difference between them is the order in which one takes the derivatives with respect to ϕ_0. Since these derivatives commute, the commutator obviously vanishes. Note that this is completely independent of whether x and y are causally related with respect to some bulk metrics. All we use is that the integral over all metrics and matter fields is some functional of the boundary conditions ϕ_0 on a trivial timefold. If x and y are causally related on the boundary, the situation is different. In that case, each term in the commutator requires a different minimal timefold: a trivial one for the time-ordered correlator, and a nontrivial one for the out-of-time-order correlator. In the latter, the source for the earlier operator is moved to the second sheet, so the commutator can be nonzero.

We now turn to the extrapolate dictionary as in (5.7). For a bulk metric relating x and y causally, one would expect $\langle\phi(x,z)\phi(y,z)\rangle_g \neq \langle\phi(y,z)\phi(x,z)\rangle_g$ since they are computed on different bulk timefolds. This nonzero commutator on g is the origin of the causality puzzle. However, there is a subtlety: The boundary condition on g must be the same as for the differentiate dictionary. Namely, one must have a trivial boundary timefold when x and y are spacelike-related on the boundary. If one computes the commutator by a path integral over fields on the bulk metric g, the causality puzzle is resolved as follows. Recall that a field path integral on a trivial timefold always gives the time-ordered correlator. So when the two operators are in the asymptotic region, inside the bulk path integral each of the two terms in the commutator (5.9) should come with a time-ordering with respect to g, $\langle\mathcal{T}\phi(x,z)\phi(y,z)\rangle_g = \langle\mathcal{T}\phi(y,z)\phi(x,z)\rangle_g$, yielding a vanishing commutator. Indeed, if the two points are timelike-related in the bulk, in the limit $z\to 0$ a non-time-ordered correlator would require a bulk metric with a nontrivial timefold asymptotically, violating our boundary condition. In this case, the naive bulk commutator $\langle[\phi(x,z),\phi(y,z)]\rangle_g$ plays no role in computing the boundary commutator $\langle[\mathcal{O}(x),\mathcal{O}(y)]\rangle_{\text{bdy}}$.

Even with no timefold on the boundary, the bulk path integral includes timefolds that trivialize asymptotically [2]. Intuitively, the amount of time that one evolves backward goes to zero as $z\to 0$. This has no effect on the differentiate dictionary, since derivatives only act on the boundary conditions. However, for the extrapolate dictionary, it introduces an ambiguity in the location of the bulk operators ϕ. If there are different sheets of the bulk timefold, one has to specify which sheet the operator is on, in addition to giving its location on the sheet. If we use (5.7) one expects this ambiguity to have no effect since

[2]This is related to the fact that the bulk path integral should impose gravitational constraints, necessitating integration over both positive and negative lapse [125].

we take the limit $z \to 0$ for each metric where the timefold becomes trivial.

In contrast, there is a potential problem with taking $z \to 0$ after integration. For any nonzero z, there are bulk metrics where for timelike-separated points (x, z) and (y, z) there is a timefold that is trivial asymptotically, but both the time-ordered and out-of-time-order correlators can be obtained by placing the operators on different sheets [3]. In other words, the bulk contains a timefold that allows a nonzero commutator, but in the region outside the location of the operators, the timefold decays and completely disappears at the boundary. Under these conditions, it would appear that the integrand can be nonzero, and one again is left with the puzzle of why the integral over metrics vanishes exactly.

However, the above occurs only if we allow bulk operators to change sheets depending on their ordering in a correlator. Since the choice of sheet is not fixed by the boundary correlator that we are trying to reproduce, one can adopt the rule that one fixes the ambiguity in the extrapolate dictionary by picking one sheet of a timefold for each operator independent of the location of the operator in the correlator. With this understanding, the integrand remains zero for each g (even before taking $z \to 0$), consistent with boundary causality.

Finally, we extend the ordering prescription for the bulk path integrand to general n-point functions, allowing for timefolds inside the bulk. This upgrades (5.7) to

$$G \propto \int \mathcal{D}g \, e^{\mathrm{i}S_g} \lim_{z \to 0} z^{\Delta_\Sigma} \left\langle \mathcal{P} \phi_n(x_n, z) \cdots \phi_1(x_1, z) \right\rangle_g \tag{5.13}$$

and analogously if $z \to 0$ is taken after integration. Here g is restricted to metrics that asymptote to the minimal boundary timefold required to order the field theory correlator

[3]Since the bulk timefolds do not have to be minimal, a nonzero commutator can be obtained from a single timefold, by moving the earlier operator from one sheet to another. For example, on the right of Fig. 5.1, one can compute the commutator by moving $\phi(p)$ to the first sheet.

correctly and $\mathcal{P}$ is the ordering operator enforced by the field path integral on g, which reduces to the time-ordering operator $\mathcal{T}$ when g is a trivial timefold (cf. the order followed by the arrows in Fig. 5.1).

5.4 Discussion

In this paper, we have explained how the bulk theory manages to respect causality of a unitary boundary theory despite the bulk path integral involving boundary causality violating metrics. To do so, we formulated the bulk computation of correlators using minimal timefolds, where it becomes manifest that boundary microcausality prevails, i.e., local operators still commute at spacelike separation. With non-minimal timefolds we would still expect microcausality to hold so long as the boundary theory is causal and unitary. This way, however, one would have to rely on unexpected cancellations in the integral over metrics – in this sense, minimal timefolds are nothing but a simpler representation of the problem (cf. charge conservation via symmetry arguments vs checking all orders in perturbation theory).

Our results apply to both the differentiate dictionary as given in (5.12) and the extrapolate dictionary in the form (5.7). It also applies to the latter if one integrates first, with a suitable rule for how to place operators on bulk timefolds that are trivial at the boundary. The basic reason for this is that a nonzero commutator for two asymptotic bulk operators requires a bulk timefold that is nontrivial at infinity, but the minimal timefold for two spacelike-separated points on the boundary is trivial. So the bulk dual of the commutator of two field theory operators at spacelike-separated points vanishes.

The bulk dual of the commutator of two stress energy tensors on the boundary does not involve bulk matter fields. Nevertheless our argument using the differentiate dictionary still applies. Since the boundary metric is the source for the stress tensor, one should compute the path integral over bulk geometries with a general boundary metric. One then functionally varies the boundary metric at the location of the stress tensors. If they are spacelike-related, there is no timefold on the boundary and hence the commutator vanishes. (There is an issue with even defining the extrapolate dictionary nonperturbatively in this case arising from the apparent need to split the metric degrees of freedom into background and fluctuations.) Our argument also applies to fermionic operators which anticommute at spacelike separations. For the differentiate dictionary, the sign difference comes from the fermionic sources for these operators anticommuting.

Although off-shell boundary causality violating metrics do not pose a problem to microcausality of the field theory, such metrics do contribute to general correlators. It would be interesting to understand what, if any, are the implications of this quantum gravity effect. We do not believe there are examples of holography in which a bulk classical solution violates boundary causality, but if there are, the boundary theory would have to be acausal or non-unitary. In this case, computations are timefold-dependent and our resolution does not apply.

Besides microcausality, there are other properties the field theory is expected to have, which in general impose further constraints on the bulk state classically and semiclassically. For example, the invariance of von Neumann entropy under unitary transformations requires that the causal wedge be inside the entanglement wedge [20, 21]. (This is stronger than and implies boundary causality [117].) Another requirement states that, for two causally-related bulk points, boundary regions encoding each of them cannot be totally spacelike [126]. (This extends our discussion from local operators to operators supported in subregions.) In a nonperturbative gravitational path integral there will be configurations violating this. It would be interesting to understand how the bulk path integral preserves these properties.

Appendix A

Exact linearity of the shockwave equation of motion

In this appendix, we show that there are no non-linear contributions, i.e., $\mathcal{O}(h^2)$, to the equation of motion from the shockwave perturbation in any higher-derivative theory of gravity including, but not limited to, f(Riemann). As an aside, we will also show that the only component of the equations of motion perturbed by the shockwave is E_u^v.

To achieve this, it will be useful to define a notion of *chirality*. Consider a (not necessarily covariant) tensor of the form $X_{b_1\cdots b_n}^{a_1\cdots a_\ell}$ built out of $g_{\mu\nu}$, $g^{\mu\nu}$ and ∂_μ, where indices $a_1,\cdots,a_\ell$, $b_1,\cdots,b_n$ can be either u or v, and we have suppressed i-type indices on X. We define the chirality of any of its components as

$$\chi = \#(v \text{ superscripts}) - \#(v \text{ subscripts}) - \#(u \text{ superscripts}) + \#(u \text{ subscripts}). \quad \text{(A.1)}$$

We refer to any tensor component with $\chi = 0$ as being non-chiral, and otherwise as being chiral. For example, the components g_{uu} and E_u^v are chiral since both have $\chi = 2$, while R_{uivj} is non-chiral since it has $\chi = 0$.

For all higher-derivative gravity theories, the equations of motion involve the metric, the Riemann tensor, and covariant derivatives. We can rewrite them using only the metric, the inverse metric, and partial derivatives. The only metric component that contains $h\delta(u)$ is $g_{uu} = -2Ah\delta(u)$; similarly, the only inverse metric component having $h\delta(u)$ is $g^{vv} = 2A^{-1}h\delta(u)$. A general term in E_u^v therefore takes the form

$$E_u^v \supset (\partial_v)^N X, \qquad X = X_0(\partial_u^{n_1} g_{uu})(\partial_u^{n_2} g_{uu})\cdots(\partial_u^{n_k} g_{uu})(g^{vv})^m, \quad \text{(A.2)}$$

where we have collected all v-derivatives into the beginning of the expression (so that they are understood to act on particular parts of X but not necessarily on X as a whole), and collected everything that does not involve g_{uu}, its u-derivatives, or g^{vv} into X_0. As $g^{uu} = g_{vv} = 0$, X_0 is a product of g_{uv}, $\partial_u^{\#} g_{uv}$, and g^{uv}. Let χ_0 be the chirality of X_0; it is equal to the total number of u-derivatives, and thus always non-negative. As g_{uv} is a function of uv only, each ∂_u acting on g_{uv} produces a factor of v, and we find

$$X_0 = v^{\chi_0} f_0(uv) \tag{A.3}$$

where $f_0(uv)$ is some function of uv. Since E_u^v has chirality 2, we need

$$N = 2m + 2k - 2 + \sum_{i=1}^{k} n_i + \chi_0 \tag{A.4}$$

for the chirality of the term in (A.2) to agree.

Since v appears only in the combination uv in all metric functions, each ∂_v in (A.2) produces a factor of u unless it acts on an explicit factor of v produced by ∂_u. In general, the ∂_u acting on $g_{uu} = -2A(uv)h\delta(u)$ in (A.2) can act either on $A(uv)$ or the δ-function.

Let us first consider the simplest case where all ∂_u shown in (A.2) act on the δ-function. In this case, using (A.3) we find

$$X = v^{\chi_0} f_1(uv)\delta^{(n_1)}(u)\cdots\delta^{(n_k)}(u)\,(\delta(u))^m\,, \tag{A.5}$$

where $f_1(uv)$ is some function of uv. Therefore, the term (A.2) in E_u^v behave at most as

$$\widetilde{X} \equiv (\partial_v)^N X \sim u^{N-\chi_0}\delta^{(n_1)}(u)\cdots\delta^{(n_k)}(u)\,(\delta(u))^m\,, \tag{A.6}$$

keeping only the leading dependence on u. Here we have acted as many ∂_v as possible on v^{χ_0}; if not, we would get subleading contributions that are suppressed by additional powers of u. We will show momentarily that the leading contribution (A.6), understood as a distribution, vanishes under the condition (A.4) unless it is actually $\delta(u)$ or $u^n\delta^{(n)}(u)$ for some n. Thus any subleading contribution suppressed by additional powers of u would always vanish as a distribution.

Now consider the more general case where not all ∂_u shown in (A.2) act on the δ-function. Every ∂_u that does not act on the δ-function must act on $A(uv)$ and produce an additional factor of v (for one more ∂_v to act on) — thus the net effect on the term $\widetilde{X}$ in (A.6) is to decrease one of the n_i by 1 and effectively increase χ_0 by 1. This preserves the condition (A.4), so it does not change our argument below.

We now show that the distribution (A.6) vanishes under the condition (A.4) unless it is actually $\delta(u)$ or $u^n\delta^{(n)}(u)$ for some n. To see this, we regularize the δ-functions in (A.6) as narrow Gaussian functions:

$$\delta(u) \to \frac{\#}{\epsilon} e^{-u^2/\epsilon^2}, \tag{A.7}$$

and integrate it against a test function $f(u)$:

$$I \equiv \int_{-\infty}^{\infty} du\, \widetilde{X}(u) f(u). \tag{A.8}$$

We find

$$\begin{aligned} I &\sim \int_{-\infty}^{\infty} du\, f(u) u^{N-\chi_0} \left(\frac{u}{\epsilon^2}\right)^{\sum_{i=1}^k n_i} \frac{1}{\epsilon^{k+m}} e^{-\frac{(k+m)u^2}{\epsilon^2}} \\ &\sim \epsilon f(0) \epsilon^{2m+2k-2+\sum_{i=1}^k n_i} \epsilon^{-\sum_{i=1}^k n_i} \frac{1}{\epsilon^{k+m}} \\ &= f(0)\epsilon^{k+m-1}, \end{aligned} \tag{A.9}$$

where we have used (A.4) in going to the second line. In the first line, we have written down a contribution to the regularized $\widetilde{X}(u)$ where all u-derivatives act on the exponent of e^{-u^2/ϵ^2}; every u-derivative that does not act on the exponent would remove a factor of u^2/ϵ^2 from the first line, but would not change the final result.

We now take the $\epsilon \to 0$ limit. By construction, $k + m \geq 1$ since we are interested in corrections to the equations of motion due to the shockwave which have at least one factor of $\delta(u)$ or its derivative. If $k + m > 1$, then the integral I vanishes as we send ϵ to zero. We are left with only two cases: either $k = 0, m = 1$ where $\widetilde{X} \sim \delta(u)$, or $k = 1, m = 0$ where $\widetilde{X} \sim u^n \delta^{(n)}(u)$ for some n. In either case, the term is a well-defined distribution and linear in h, concluding our proof for E_u^v.

Finally, consider other components of the equations of motion, e.g., E_i^v, E_v^v, etc. They have $\chi \leq 1$, so we must have more powers of ∂_v compared to (A.4), and the corresponding distribution must have more powers of u compared to (A.6). Thus the integral I would go like at least $\mathcal{O}(\epsilon^{k+m})$, which vanishes in the $\epsilon \to 0$ limit as long as $k+m \geq 1$. Therefore, other components of the equations of motion are not perturbed by the shockwave.

Appendix B

Proof of sum relations

In this appendix, we prove the sum relations (2.119a) and (2.119b) used in the main proof. We will need the following identities

$$\sum_{k=1}^{n}(-1)^{s_{kk}} = \sum_{k=1}^{n} 1 = n, \tag{B.1a}$$

$$\sum_{k=1}^{n}(-1)^{s_{k,k+1}} = \sum_{k=1}^{n} 1 + \sum_{k=1}^{n}\left[(-1)^{s_{k,k+1}} - 1\right] = n - 2\sum_{k=1}^{n}\delta_{s_{k,k+1},1} = n - 2m. \tag{B.1b}$$

In the sums below, the summation variables k, l are always within the range $[1, n]$.

Beginning with (2.119a), we prove it by writing

$$\begin{aligned}
&\sum_{k<l}\left[(-1)^{s_{k+1,l}} + (-1)^{s_{kl}}\right]\left[(-1)^{s_{k+1,l}} + (-1)^{s_{k+1,l+1}}\right](-1)^{s_{k+1,l}} \\
&= \sum_{k<l}\left[1 + (-1)^{s_{k,k+1}}\right]\left[1 + (-1)^{s_{l,l+1}}\right](-1)^{s_{k+1,l}} \\
&= \sum_{k<l}(-1)^{s_{k+1,l}} + \sum_{k<l}(-1)^{s_{kl}} + \sum_{k<l}(-1)^{s_{k+1,l+1}} + \sum_{k<l}(-1)^{s_{k,l+1}} \\
&= \left(\sum_{k\le l}(-1)^{s_{kl}} - \sum_{l=1}^{n}(-1)^{s_{1,l}}\right) \\
&\qquad + \sum_{k<l}(-1)^{s_{kl}} + \sum_{k<l}(-1)^{s_{kl}} + \left(\sum_{k\le l}(-1)^{s_{k,l+1}} - \sum_{k=1}^{n}(-1)^{s_{k,k+1}}\right) \\
&= \left(\sum_{k<l}(-1)^{s_{kl}} + \sum_{k=1}^{n}(-1)^{s_{kk}} - \sum_{l=1}^{n}(-1)^{s_{1,l}}\right) + \sum_{k<l}(-1)^{s_{kl}} + \sum_{k<l}(-1)^{s_{kl}} \\
&\qquad + \left(\sum_{k<l}(-1)^{s_{kl}} + \sum_{k=1}^{n}(-1)^{s_{k,n+1}} - \sum_{k=1}^{n}(-1)^{s_{k,k+1}}\right)
\end{aligned}$$

$$
\begin{aligned}
&= 4\sum_{k<l}(-1)^{s_{kl}} + \sum_{k=1}^{n}\left[(-1)^{s_{kk}} - (-1)^{s_{k,k+1}}\right] \\
&= 4\sum_{k<l}(-1)^{s_{kl}} + 2m, \qquad \text{(B.2)}
\end{aligned}
$$

where we have used $s_{1,k} = s_{k,1} = s_{k,n+1}$ in going to the second-to-last line, and used (B.1a) and (B.1b) in going to the last line.

Similarly, we prove (2.119b) by writing

$$
\begin{aligned}
&\sum_{k>l}\left[(-1)^{m-s_{l+1,k}} + (-1)^{m-s_{lk}}\right]\left[(-1)^{m-s_{l+1,k}} + (-1)^{m-s_{l+1,k+1}}\right](-1)^{m-s_{l+1,k}} \\
&= (-1)^m \sum_{k>l}\left[(-1)^{s_{l+1,k}} + (-1)^{s_{lk}}\right]\left[(-1)^{s_{l+1,k}} + (-1)^{s_{l+1,k+1}}\right](-1)^{s_{l+1,k}} \\
&= (-1)^m \left[4\sum_{k<l}(-1)^{s_{kl}} + 2m\right] \qquad \text{(B.3)}
\end{aligned}
$$

where in going to the last line we have used the fact that the sum is the same as (B.2) with $k \leftrightarrow l$.

Appendix C

Bosonic examples

C.1 Minimally coupled scalar

Consider the following theory for a massive scalar field on the black hole background:

$$\mathcal{L} = -\frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}m^2\phi^2. \tag{C.1}$$

This has been considered in [75] at leading weight and in [53] at lower weights. In this section, we perform the calculation in the covariant expansion formalism.

Its equation of motion is given by

$$E \equiv \left(\nabla^2 - m^2\right)\phi = 0. \tag{C.2}$$

Taking covariant derivatives in the radial direction,

$$\begin{aligned}(\nabla_r)^n \delta E &= (\nabla_r)^n \delta\left[\left(\nabla^2 - m^2\right)\phi\right] \\ &= (\nabla_r)^n \left(\nabla^2 - m^2\right)\delta\phi,\end{aligned} \tag{C.3}$$

where everything is evaluated at $r = r_0$ *after* derivatives are taken. Notice that we have passed the variation operator, δ, through the background differential operator $(\nabla^2 - m^2)$ because the background is fixed. In the notation reviewed in Section 4.1,

$$\delta\mathcal{E} = [\delta E, \nabla_r\delta E, \nabla_r\nabla_r\delta E, ...]|_{r=r_0}. \tag{C.4}$$

The components have weights $0, -1, -2, ...$ In other words,

$$\delta\mathcal{E}_0 = [\delta E|_{r=r_0}], \quad \delta\mathcal{E}_{-1} = [(\nabla_r\delta E)|_{r=r_0}], \quad \text{etc.} \tag{C.5}$$

Since there is only one component for each weight in this example, $|\delta\mathcal{E}_p| = 1$ for all $p \in \mathbb{Z}_{\leq 0}$. We will omit the brackets for one-by-one matrices from now on.

Since this theory has only one dynamical field which has spin zero, the highest-weight equation of motion has $p = q_0 = 0$:

$$\begin{aligned}
\delta\mathcal{E}_0 &= g^{\mu\nu}\nabla_\mu\nabla_\nu\delta\phi - m^2\delta\phi \\
&= 2g^{vr}\nabla_v\nabla_r\delta\phi + g^{rr}\nabla_r\nabla_r\delta\phi + g^{ij}\nabla_i\nabla_j\delta\phi - m^2\delta\phi \\
&= \left(g^{ij}\nabla_i\nabla_j - m^2\right)\underbrace{\delta\phi}_{[0]} + 2g^{vr}\nabla_v\underbrace{\nabla_r\delta\phi}_{[-1]} + g^{rr}\underbrace{\nabla_r\nabla_r\delta\phi}_{[-2]} \\
&= \begin{bmatrix}(g^{ij}\nabla_i\nabla_j - m^2) & 2g^{vr}\nabla_v & g^{rr}\end{bmatrix}\begin{bmatrix}\delta\phi \\ \nabla_r\delta\phi \\ \nabla_r^2\delta\phi\end{bmatrix} \\
&= \begin{bmatrix}\left(g^{ij}(\partial_i\nabla_j - \Gamma^\mu_{ij}\nabla_\mu) - m^2\right) & 2g^{vr}\nabla_v & g^{rr}\end{bmatrix}\begin{bmatrix}\delta\phi \\ \nabla_r\delta\phi \\ \nabla_r^2\delta\phi\end{bmatrix}.
\end{aligned} \tag{C.6}$$

From (4.3), Γ^μ_{ij} is only non-zero for $\mu = v$ (on the horizon), so

$$\begin{aligned}
&\delta\mathcal{E}_0 \\
&= \begin{bmatrix}\left(h^{-1}\delta^{ij}(\partial_i\partial_j + \tfrac{1}{2}h'\delta_{ij}\partial_v) - m^2\right) & 2\nabla_v & f\end{bmatrix}\begin{bmatrix}\delta\phi \\ \nabla_r\delta\phi \\ \nabla_r^2\delta\phi\end{bmatrix} \\
&= \begin{bmatrix}\left(-h^{-1}k_ik_i + \tfrac{1}{2}h^{-1}h'(-\mathrm{i}\omega)d - m^2\right) & 2(-\mathrm{i}\omega + \tfrac{1}{2}f') & 0\end{bmatrix}\begin{bmatrix}\delta\phi \\ \nabla_r\delta\phi \\ \nabla_r^2\delta\phi\end{bmatrix}.
\end{aligned} \tag{C.7}$$

Notice that we have terms with various weights q on the right hand side, even though we are only considering the equation of motion with weight $p = 0$. Also, in the process, we have canonicalized the expression by moving ∇_r to the right of ∇_v according to the prescription. It is easy at this order because $[\nabla_\mu, \nabla_\nu] = 0$ when acting on scalars.

Also, recall that we have defined

$$\delta\mathcal{X} = \begin{bmatrix}\delta\mathcal{X}_0 \\ \delta\mathcal{X}_{-1} \\ \delta\mathcal{X}_{-2} \\ \cdots\end{bmatrix} = \begin{bmatrix}\delta\phi \\ \nabla_r\delta\phi \\ \nabla_r^2\delta\phi \\ \cdots\end{bmatrix}, \tag{C.8}$$

so what we have written down was $\delta\mathcal{E}_p = \sum_q \mathcal{M}_{p,q}\delta\mathcal{X}_q$ for $p = 0$.

Now let $\omega = -\omega_0$, then this equation reduces to

$$-h^{-1}k^2 - \frac{d}{2}h^{-1}h'2\pi T - m^2 = 0, \tag{C.9}$$

$$\implies \quad k^2 = -d\pi T h' - m^2 h, \tag{C.10}$$

which is exactly (2.16) of [53]. In our language, this comes from the pole-skipping condition (4.23) with $s = 1$, i.e., $\det M_1(k) = 0$, which in this case involves only $\mathcal{M}_{0,0}$.

Let us now include the next value of p, which is -1. For simplicity, take $d = 1$. Then

$$\begin{aligned}
\delta\mathcal{E}_{-1} &= \nabla_r(g^{\mu\nu}\nabla_\mu\nabla_\nu - m^2)\delta\phi \\
&= \left(\frac{2hh'' - h'^2}{4h^2}(-2\mathrm{i}\omega + f'/2) + \frac{k^2 h'}{h^2}\right)\delta\phi \\
&+ \left(\frac{f'h'}{4h} - \frac{k^2}{h} + \frac{h'}{2h}(-\mathrm{i}\omega + f'/2) + 2f'' - m^2\right)\nabla_r\delta\phi \\
&+ 2(-\mathrm{i}\omega + f')\nabla_r^2\delta\phi.
\end{aligned} \tag{C.11}$$

According to (4.23), the second set of poke-skipping points appears at the frequency $\omega = -2\omega_0$ and when the determinant of $M_2(k)$ vanishes, where

$$M_2(k) = \begin{bmatrix} -h^{-1}k^2 - \frac{1}{2}f'h^{-1}h' - m^2 & -f' \\ -\frac{3}{8}h^{-2}(2hh'' - h'^2)f' + k^2h'h^{-2} & -k^2h^{-1} + 2f'' - m^2 \end{bmatrix}. \tag{C.12}$$

On the BTZ background, where

$$f(r) = r^2\left(1 - \frac{r_0^2}{r^2}\right), \quad h(r) = r^2, \tag{C.13}$$

and for $m = 0$, the pole-skipping points at this order are given by

$$k^2 = \frac{r_0^2}{2(r_0 - 2)}\left[4 \pm r_0\left(\mp 4 + r_0\left(\mp 3 + r_0^{-2}\sqrt{(4 + 3r_0^2)(4 - 8r_0 + 7r_0^2)}\right)\right)\right]. \tag{C.14}$$

It is straightforward to continue to higher orders. We will go to higher orders in some more complicated examples.

C.2 Maxwell theory

Consider Maxwell theory whose Lagrangian is given by

$$\mathcal{L} = \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{C.15}$$

where

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu. \tag{C.16}$$

This is our first example with a gauge symmetry: $A_\mu \to A_\mu + \nabla_\mu\Lambda$. This example will demonstrate how gauge redundancy is reflected in the formalism and how the gauge-covariant pole-skipping condition works.

The equation of motion is given by

$$E_\nu = \nabla^\mu F_{\mu\nu} = \nabla^\mu \nabla_\mu A_\nu - \nabla^\mu \nabla_\nu A_\mu. \tag{C.17}$$

We simplify the calculation by setting $d = 1$ in this example, so

$$\begin{aligned}\delta E_v &= \nabla^\mu \nabla_\mu \delta A_v - \nabla^\mu \nabla_v \delta A_\mu \\ &= \nabla_v \nabla_r \delta A_v + \frac{1}{h}\nabla_x^2 \delta A_v - \nabla_v^2 \delta A_r - \frac{1}{h}\nabla_x \nabla_v \delta A_x.\end{aligned} \tag{C.18}$$

Unpacking the ∇'s and evaluating it on the horizon, we have:

$$\delta\mathcal{E}_1 = (\delta E_v)|_{r=r_0} = (-\mathrm{i}\omega)\nabla_r \delta A_v - \frac{1}{h}k^2 \delta A_v + \mathrm{i}\omega\left(-\mathrm{i}\omega + \frac{f'}{2}\right)\delta A_r - \frac{k\omega}{h}\delta A_x. \tag{C.19}$$

It is easy to see that the first pole-skipping point appears at

$$(\omega, k) = (0, 0) \tag{C.20}$$

because $M_1(k)$ is just the coefficient in front of δA_v. We note that gauge symmetry is not yet visible at this order because $s = 1$ is smaller than $q_0 - u_0 + 1 = 2$, as explained in Section 4.2.

At the next order $s = 2$, we have both $\nabla_r \delta E_v$ and δE_x to consider. By following the prescription, we compute

$$M_2(k) = \begin{bmatrix} -\frac{k^2}{h} & \frac{\mathrm{i}kf'}{2h} & -\frac{f'}{2} \\ \frac{\mathrm{i}kh'}{2h} & \frac{f'h'}{4h} & -\mathrm{i}k \\ \frac{k^2h'}{h^2} & -\frac{\mathrm{i}kf'h'}{2h^2} & \frac{f'h'-4k^2}{4h} \end{bmatrix}, \tag{C.21}$$

where we have chosen the basis elements in the following order:

$$\delta\mathcal{X}_1 \oplus \delta\mathcal{X}_0 = [\delta A_v, \delta A_i, \nabla_r \delta A_v]|_{r=r_0}, \tag{C.22}$$

$$\delta\mathcal{E}_1 \oplus \delta\mathcal{E}_0 = [\delta E_v, \delta E_i, \nabla_r \delta E_v]|_{r=r_0}. \tag{C.23}$$

From now on, we will only state the basis for $\delta\mathcal{X}$, and it should be understood that the basis for $\delta\mathcal{E}$ is chosen analogously, as above. The determinant of this matrix is zero for any value k. More precisely, this matrix has a one-dimensional kernel. This is due to the fact that the pure-gauge perturbation $\delta A_\mu = \nabla_\mu \Lambda$ automatically satisfies the equations of motion. Using the language of (4.28), the kernel of $M_2(k)$ is spanned by

$$\begin{bmatrix} \delta A_v \\ \delta A_x \\ \nabla_r \delta A_v \end{bmatrix} = \begin{bmatrix} \mathrm{i}f'/2 \\ k \\ 0 \end{bmatrix} \Lambda. \tag{C.24}$$

Here $\delta\xi|_{r_0} = \Lambda|_{r_0}$ is the only degree of freedom of the gauge parameter at this order, so the kernel is one-dimensional. See later for cases with larger kernels. The pole-skipping points at this order can now be found using (4.29). At

$$k^2 = -\frac{1}{4}f'h', \tag{C.25}$$

the dimension of the kernel increases from 1 to 2.

At the next order, $s = 3$ and $\omega = (q_0 - s)\omega_0 = -2\omega_0$, choosing

$$\delta\mathcal{X}_1 \oplus \delta\mathcal{X}_0 \oplus \delta\mathcal{X}_{-1} = [\delta A_v, \delta A_i, \nabla_r\delta A_v, \delta A_r, \nabla_r\delta A_i, \nabla_r^2\delta A_v]|_{r=r_0} \tag{C.26}$$

as the basis, the relevant matrix is given by

$$M_3(k) = \tag{C.27}$$

$$\begin{bmatrix} -\frac{k^2}{h} & \frac{\mathrm{i}kf'}{h} & -f' & -\frac{1}{2}f'^2 & 0 & 0 \\ \frac{\mathrm{i}kh'}{2h} & 0 & -\mathrm{i}k & \frac{1}{2}\mathrm{i}kf' & -f' & 0 \\ \frac{k^2h'}{h^2} & -\frac{3\mathrm{i}kf'h'}{4h^2} & \frac{f'h'-4k^2}{4h} & \frac{f'^2h'+2hf'f''}{8h} & \frac{\mathrm{i}kf'}{2h} & -\frac{f'}{2} \\ 0 & -\frac{\mathrm{i}kh'}{2h^2} & -\frac{h'}{2h} & \frac{2hf''-f'h'-4k^2}{4h} & -\frac{\mathrm{i}k}{h} & -1 \\ \frac{\mathrm{i}k(2hh''-3h'^2)}{4h^2} & \frac{hf''h'+hf'h''-f'h'^2}{2h^2} & \frac{\mathrm{i}kh'}{h} & -\frac{1}{2}\mathrm{i}kf'' & f''+\frac{f'h'}{2h} & -\mathrm{i}k \\ \frac{k^2(hh''-2h'^2)}{h^3} & \frac{k(-hf''h'-4hf'h''+6f'h'^2)}{4\mathrm{i}h^3} & A & B & \frac{hkf''+2kf'h'}{2\mathrm{i}h^2} & \frac{hf''+f'h'-2k^2}{2h} \end{bmatrix}$$

where

$$A = \frac{hf''h' + 2hf'h'' - 2f'h'^2 + 8k^2h'}{4h^2},$$
$$B = \frac{-2h^2f''^2 + 2hf'^2h'' - 2f'^2h'^2 - hf'f''h'}{8h^2},$$

which already has a two-dimensional kernel spanned by

$$\begin{bmatrix} \delta A_v \\ \delta A_x \\ \nabla_r\delta A_v \\ \delta A_r \\ \nabla_r\delta A_x \\ \nabla_r^2\delta A_v \end{bmatrix} = \begin{bmatrix} -f' \\ \mathrm{i}k \\ 0 \\ 0 \\ -\frac{\mathrm{i}kh'}{2h} \\ 0 \end{bmatrix}\Lambda + \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{2}f' \\ 1 \\ \mathrm{i}k \\ \frac{1}{2}f'' \end{bmatrix}\nabla_r\Lambda, \tag{C.28}$$

where $\delta\xi|_{r_0} = \Lambda|_{r_0} = \delta\Xi_0$ and $\nabla_r\delta\xi|_{r_0} = \nabla_r\Lambda|_{r_0} = \delta\Xi_{-1}$ are the two gauge parameters appearing in the sum (4.28). The pole-skipping momenta are given by the solutions to

$$\left(k^2+\frac{1}{2}f'h'\right)\left(k^2+f'h'-hf''\right)-hf'^2h''=0. \tag{C.29}$$

They will increase the dimension of the kernel from 2 to 3. On the BTZ black ground (C.13), they simplify to

$$k^2=r_0^2\left(-2\pm 2\sqrt{2}\right). \tag{C.30}$$

C.3 Einstein gravity

Consider the Einstein-Hilbert action with a negative cosmological constant:

$$S=\frac{1}{16\pi G_N}\int \mathrm{d}^{d+2}x\sqrt{-g}(R-2\Lambda),\quad \Lambda=-\frac{d(d+1)}{2\ell^2},\quad \ell=1. \tag{C.31}$$

Compared to the other examples we study, this one is computationally the hardest. This will hopefully illustrate the advantage of using the covariant expansion method: it is fully automatable. This also serves as another illustration of the nature of gauge symmetry, which in this case is given by: $g_{\mu\nu}\to g_{\mu\nu}+\nabla_\mu\zeta_\nu+\nabla_\nu\zeta_\mu$.

Einstein's equation is given by

$$E_{\mu\nu}=R_{\mu\nu}-\frac{2\Lambda}{d}g_{\mu\nu}. \tag{C.32}$$

The linearized Einstein's equation is given by

$$\begin{aligned}\delta E_{\mu\nu}=&\frac{1}{2}\left(-\nabla_\alpha\nabla^\alpha\delta g_{\mu\nu}-\nabla_\mu\nabla_\nu\delta g^\alpha{}_\alpha+\nabla_\mu\nabla^\alpha\delta g_{\nu\alpha}+\nabla_\nu\nabla^\alpha\delta g_{\mu\alpha}\right)\\&+\frac{1}{2}g^{\alpha\beta}R_{\mu\alpha}\delta g_{\nu\beta}+\frac{1}{2}g^{\alpha\beta}R_{\nu\alpha}\delta g_{\mu\beta}-g^{\alpha\rho}g^{\beta\sigma}R_{\mu\alpha\nu\beta}\delta g_{\rho\sigma}-\frac{2\Lambda}{d}\delta g_{\mu\nu}.\end{aligned} \tag{C.33}$$

We will first keep d general but turn to $d=2$ later for concreteness; if one wishes, one can keep d general throughout the whole calculation. To avoid repetition, we now state the order in which the basis elements are presented throughout this example:

$$\begin{aligned}&[\delta g_{vv},\delta g_{v1},\delta g_{v2},\nabla_r\delta g_{vv},\delta g_{vr},\delta g_{11},\delta g_{12},\delta g_{22},\nabla_r\delta g_{v1},\nabla_r\delta g_{v2},\nabla_r^2\delta g_{vv},\delta g_{r1},\\&\delta g_{r2},\nabla_r\delta g_{vr},\nabla_r\delta g_{11},\nabla_r\delta g_{12},\nabla_r\delta g_{22},\nabla_r^2\delta g_{v1},\nabla_r^2\delta g_{v2},\nabla_r^3\delta g_{vv},\cdots]|_{r=r_0},\end{aligned} \tag{C.34}$$

where the subscripts 1 and 2 abbreviate x^1 and x^2. To begin with, consider the highest-weight equation of motion

$$\delta\mathcal{E}_2=\delta E_{vv}|_{r_0}=\left[-\frac{1}{2h}\partial_i\partial_i+\frac{dh'}{4h}\left(\partial_v-f'\right)-\frac{1}{2}f''-\frac{2\Lambda}{d}\right]\delta g_{vv}$$

$$+\frac{1}{h}\left(\partial_v-\frac{1}{2}f'\right)\partial_i\delta g_{vi}-\frac{1}{2h}\partial_v\left(\partial_v-\frac{1}{2}f'\right)\delta g_{ii}. \tag{C.35}$$

Again, the gauge symmetry is not visible at this order, as $s=1$ is smaller than $q_0-u_0+1=2-1+1=2$. We can therefore easily read off the location of the first skipped pole:

$$\omega=\omega_0,\quad k^2=-\frac{d}{4}f'h', \tag{C.36}$$

where we have used the fact that the background metric satisfies Einstein's equation.

At the next order ($s=2$, $\omega=0$), we have (for general d)

$$\begin{aligned}\delta E_{vi}|_{r_0}&=(d-2)\frac{h'}{4h}\partial_i\delta g_{vv}+\left[-\frac{1}{2h}\partial_j\partial_j+\frac{h'}{4h}\left(\partial_v-2f'\right)-\frac{2\Lambda}{d}\right]\delta g_{vi}+\frac{1}{2h}\partial_i\partial_j\delta g_{vj}\\&\quad+\frac{1}{2}\partial_i\nabla_r\delta g_{vv}+\frac{1}{2}\partial_v\partial_i\delta g_{vr}-\frac{1}{2}\partial_v\nabla_r\delta g_{vi}+\frac{1}{2}\partial_v\left(\partial_v+\frac{1}{2}f'\right)\delta g_{ri},\end{aligned} \tag{C.37}$$

$$\begin{aligned}\nabla_r\delta E_{vv}|_{r_0}&=\left[-\frac{1}{2h}\partial_i\partial_i+\frac{dh'}{4h}(\partial_v-f')-\frac{1}{2}f''\right]'\delta g_{vv}-\left[\frac{h'}{h^2}\partial_v+\frac{1}{2}\left(\frac{f'}{h}\right)'\right]\partial_i\delta g_{vi}\\&\quad+\left[-\frac{1}{2h}\partial_i\partial_i+\frac{dh'}{4h}(\partial_v-f')-\frac{1}{2}f''-\frac{2\Lambda}{d}\right]\nabla_r\delta g_{vv}+\frac{dh'}{4h}\partial_v\delta g_{vr}\\&\quad+\frac{1}{4}\left(\frac{f'}{h}\right)'\partial_v\delta g_{ii}+\frac{1}{h}\partial_v\partial_i\nabla_r\delta g_{vi}-\frac{1}{2h}\partial_v\left(\partial_v+\frac{1}{2}f'\right)\nabla_r\delta g_{ii}.\end{aligned} \tag{C.38}$$

We can easily write down the matrix $M_2(k)$ using the expressions above. The determinant of this matrix is automatically zero. Its kernel is spanned by the pure-gauge perturbations $\delta g_{\mu\nu}=\nabla_\mu\zeta_\nu+\nabla_\nu\zeta_\mu$ with weights $q\geq 1$. In the near-horizon covariant expansion, this is

$$\begin{bmatrix}\delta g_{vv}\\ \delta g_{vi}\\ \nabla_r\delta g_{vv}\end{bmatrix}=\begin{bmatrix}-f'\\ \mathrm{i}k_i\\ -f''\end{bmatrix}\zeta_v, \tag{C.39}$$

where $\zeta_v|_{r_0}=\delta\Xi_1$ is the only gauge parameter appearing in (4.28) at this order. For simplicity, we now specialize to a specific background, the Schwarzschild-AdS$_4$ black hole, which has

$$f(r)=r^2\left(1-\frac{r_0^3}{r^3}\right),\quad h(r)=r^2. \tag{C.40}$$

Now our matrix simplifies to

$$M_2(k) = \begin{bmatrix} \frac{k_1^2+k_2^2}{2r_0^2} & -\frac{3\mathrm{i}k_1}{2r_0} & -\frac{3\mathrm{i}k_2}{2r_0} & 0 \\ 0 & \frac{k_2^2}{2r_0^2} & -\frac{k_1 k_2}{2r_0^2} & \frac{\mathrm{i}k_1}{2} \\ 0 & -\frac{k_1 k_2}{2r_0^2} & \frac{k_1^2}{2r_0^2} & \frac{\mathrm{i}k_2}{2} \\ -\frac{k_1^2+k_2^2}{r_0^3} & \frac{3\mathrm{i}k_1}{r_0^2} & \frac{3\mathrm{i}k_2}{r_0^2} & \frac{k_1^2+k_2^2}{2r_0^2} \end{bmatrix}. \tag{C.41}$$

As we described above for general background, there is a one-dimensional kernel of this matrix (C.39). On our chosen background, it reduces to

$$\begin{bmatrix} \delta g_{vv} \\ \delta g_{v1} \\ \delta g_{v2} \\ \nabla_r \delta g_{vv} \end{bmatrix} = \begin{bmatrix} -3 \\ \mathrm{i}k_1 \\ \mathrm{i}k_2 \\ 0 \end{bmatrix} \zeta_v. \tag{C.42}$$

The product of nonzero diagonal entries of its Jordan normal form is

$$k^4(k^2 + 6r_0^2). \tag{C.43}$$

Setting $k = 0$ will increase the dimension of the kernel from 1 to 4, while $k^2 = -6r_0^2$ will *not*. According to (4.29) and with the caveat mentioned in Footnote 2, pole skipping only happens at $k = 0$.

We can continue to find the skipped poles at the next order ($s = 3$), where $\omega = -\omega_0$. We now need all equations of motion with weights greater or equal to 0. The relevant matrix is worked out to be

$$M_3(k) = \tag{C.44}$$

$$\begin{bmatrix} A & -\frac{3\mathrm{i}k_1}{r_0} & -\frac{3\mathrm{i}k_2}{r_0} & 0 & 0 & -\frac{9}{4} & 0 & -\frac{9}{4} & 0 & 0 & 0 \\ 0 & \frac{k_2^2}{2r_0^2} - \frac{3}{4} & -\frac{k_1 k_2}{2r_0^2} & \frac{\mathrm{i}k_1}{2} & \frac{3}{4}\mathrm{i}k_1 r_0 & 0 & -\frac{3\mathrm{i}k_2}{4r_0} & \frac{3\mathrm{i}k_1}{4r_0} & \frac{3r_0}{4} & 0 & 0 \\ 0 & -\frac{k_1 k_2}{2r_0^2} & \frac{k_1^2}{2r_0^2} - \frac{3}{4} & \frac{\mathrm{i}k_2}{2} & \frac{3}{4}\mathrm{i}k_2 r_0 & \frac{3\mathrm{i}k_2}{4r_0} & -\frac{3\mathrm{i}k_1}{4r_0} & 0 & 0 & \frac{3r_0}{4} & 0 \\ -\frac{k^2}{r_0^3} + \frac{3}{2} & \frac{9\mathrm{i}k_1}{2r_0^2} & \frac{9\mathrm{i}k_2}{2r_0^2} & A & -\frac{9r_0}{2} & \frac{9}{4r_0} & 0 & \frac{9}{4r_0} & -\frac{3\mathrm{i}k_1}{2r_0} & -\frac{3\mathrm{i}k_2}{2r_0} & 0 \\ 0 & \frac{\mathrm{i}k_1}{2r_0^3} & \frac{\mathrm{i}k_2}{2r_0^3} & \frac{1}{r_0} & \frac{k^2}{2r_0^2} + 3 & \frac{3}{4r_0^2} & 0 & \frac{3}{4r_0^2} & \frac{\mathrm{i}k_1}{2r_0^2} & \frac{\mathrm{i}k_2}{2r_0^2} & \frac{1}{2} \\ 1 & \frac{2\mathrm{i}k_1}{r_0} & \frac{\mathrm{i}k_2}{r_0} & r_0 & k_1^2 + 3r_0^2 & \frac{k_2^2}{2r_0^2} + \frac{9}{4} & -\frac{k_1 k_2}{r_0^2} & \frac{k_1^2}{2r_0^2} + \frac{3}{4} & \mathrm{i}k_1 & 0 & 0 \\ 0 & \frac{\mathrm{i}k_2}{2r_0} & \frac{\mathrm{i}k_1}{2r_0} & 0 & k_1 k_2 & 0 & \frac{3}{2} & 0 & \frac{\mathrm{i}k_2}{2} & \frac{\mathrm{i}k_1}{2} & 0 \\ 1 & \frac{\mathrm{i}k_1}{r_0} & \frac{2\mathrm{i}k_2}{r_0} & r_0 & k_2^2 + 3r_0^2 & \frac{k_2^2}{2r_0^2} + \frac{3}{4} & -\frac{k_1 k_2}{r_0^2} & \frac{k_1^2}{2r_0^2} + \frac{9}{4} & 0 & \mathrm{i}k_2 & 0 \\ 0 & \frac{9}{4r_0} - \frac{k_2^2}{r_0^3} & \frac{k_1 k_2}{r_0^3} & -\frac{\mathrm{i}k_1}{2r_0} & \frac{3\mathrm{i}k_1}{4} & 0 & \frac{3\mathrm{i}k_2}{4r_0^2} & -\frac{3\mathrm{i}k_1}{4r_0^2} & \frac{k_2^2}{2r_0^2} - \frac{9}{4} & -\frac{k_1 k_2}{2r_0^2} & \frac{\mathrm{i}k_1}{2} \\ 0 & \frac{k_1 k_2}{r_0^3} & \frac{9}{4r_0} - \frac{k_1^2}{r_0^3} & -\frac{\mathrm{i}k_2}{2r_0} & \frac{3\mathrm{i}k_2}{4} & -\frac{3\mathrm{i}k_2}{4r_0^2} & \frac{3\mathrm{i}k_1}{4r_0^2} & 0 & -\frac{k_1 k_2}{2r_0^2} & \frac{k_1^2}{2r_0^2} - \frac{9}{4} & \frac{\mathrm{i}k_2}{2} \\ \frac{3\left(k^2 - r_0^2\right)}{r_0^4} & -\frac{15\mathrm{i}k_1}{r_0^3} & -\frac{15\mathrm{i}k_2}{r_0^3} & \frac{3}{r_0} - \frac{2\left(k^2\right)}{r_0^3} & 9 & -\frac{27}{4r_0^2} & 0 & -\frac{27}{4r_0^2} & \frac{6\mathrm{i}k_1}{r_0^2} & \frac{6\mathrm{i}k_2}{r_0^2} & A \end{bmatrix}$$

where

$$A = \frac{1}{2}\left(\frac{k^2}{r_0^2} - 3\right), \quad k^2 = k_1^2 + k_2^2.$$

Here, the dimension of the kernel is already 4, an indication of the size of the gauge group (diffeomorphism). The kernel is spanned by

$$\begin{bmatrix} \delta g_{vv} \\ \delta g_{v1} \\ \delta g_{v2} \\ \nabla_r \delta g_{vv} \\ \delta g_{vr} \\ \delta g_{11} \\ \delta g_{12} \\ \delta g_{22} \\ \nabla_r \delta g_{v1} \\ \nabla_r \delta g_{v2} \\ \nabla_r^2 \delta g_{vv} \end{bmatrix} = \begin{bmatrix} -6 \\ \mathrm{i}k_1 \\ \mathrm{i}k_2 \\ 0 \\ 0 \\ 2 \\ 0 \\ 2 \\ -ik_1 \\ -\mathrm{i}k_2 \\ -6 \end{bmatrix} \zeta_v + \begin{bmatrix} 0 \\ -\frac{3}{2} \\ 0 \\ 0 \\ 0 \\ 2\mathrm{i}k_1 \\ \mathrm{i}k_2 \\ 0 \\ -\frac{3}{2} \\ 0 \\ 0 \end{bmatrix} \zeta_1 + \begin{bmatrix} 0 \\ 0 \\ -\frac{3}{2} \\ 0 \\ 0 \\ 0 \\ \mathrm{i}k_1 \\ 2\mathrm{i}k_2 \\ 0 \\ -\frac{3}{2} \\ 0 \end{bmatrix} \zeta_2 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ \mathrm{i}k_1 \\ \mathrm{i}k_2 \\ 0 \end{bmatrix} \nabla_r \zeta_v, \tag{C.45}$$

where $\delta\Xi_1 = [\zeta_v]|_{r=r_0}$ and $\delta\Xi_0 = [\zeta_1, \zeta_2, \nabla_r\zeta_v]|_{r=r_0}$ parameterize the four dimensions of the kernel. The product of nonzero diagonal entries of its Jordan normal form is given by

$$(k^2 - 3r_0^2)(k^2 + 6r_0^2)(k^2 + 9r_0^2)(k^4 + 9r_0^4)(k^4 + 15k^2r_0^2 + 18r_0^4). \tag{C.46}$$

Among the roots of this expression, it can be checked that $k^2 = 3k_0^2$, $k^4 = -9k_0^4$ increases the dimension of the kernel.

We can continue doing this, but the size of the matrix is getting unmanageable. We will just state the results for the next two orders. At $s = 4$ ($\omega = -2\omega_0$), pole-skipping momenta are given by

$$k^4 = 18r_0^4, \quad k^4 = -18r_0^4. \tag{C.47}$$

At $s = 5$ ($\omega = -3\omega_0$), they are given by

$$k^4 = 27r_0^4, \quad k^4 = -27r_0^4, \quad k^2 = -15r_0^2. \tag{C.48}$$

All the results that overlap with (5.17) and (E.8) of [53] agree.

Appendix D

Fermionic examples

D.1 Free Dirac spinor

Consider a theory of a minimally coupled free Dirac field on curved spacetime:

$$\mathcal{L} = \mathrm{i}\bar{\psi}\left(\Gamma^\mu \nabla_\mu - m\right)\psi. \tag{D.1}$$

Pole skipping for this theory has been studied in [76]. In this section, we perform the analysis using the formalism of Section 4.3. For simplicity, we work in three bulk dimensions ($d = 1$). To begin with, the linear order perturbation to the equation of motion

$$E = (\Gamma^\mu \nabla_\mu - m)\psi \tag{D.2}$$

is given by

$$\begin{aligned}
\delta E &= \Gamma^\mu \nabla_\mu \delta\psi - m\delta\psi \\
&= \Gamma^v \nabla_v \delta\psi + \Gamma^r \nabla_r \delta\psi + \Gamma^x \nabla_x \delta\psi - m\delta\psi \\
&= \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} (\partial_v - f'/4)\delta\psi_+ \\ (\partial_v + f'/4)\delta\psi_- \end{bmatrix} + \begin{bmatrix} 0 & f \\ 1 & 0 \end{bmatrix} \begin{bmatrix} (\nabla_r \delta\psi)_+ \\ (\nabla_r \delta\psi)_- \end{bmatrix} \\
&\quad + \begin{bmatrix} -1/\sqrt{h} & 0 \\ 0 & 1/\sqrt{h} \end{bmatrix} \begin{bmatrix} \partial_x & -h' f/(4\sqrt{h}) \\ h'/(4\sqrt{h}) & \partial_x \end{bmatrix} \begin{bmatrix} \delta\psi_+ \\ \delta\psi_- \end{bmatrix} - m \begin{bmatrix} \delta\psi_+ \\ \delta\psi_- \end{bmatrix}.
\end{aligned} \tag{D.3}$$

When evaluated at the horizon,

$$\delta\mathcal{E}_{1/2} = \delta E_+|_{r_0} = -\left(\frac{1}{\sqrt{h}}\partial_x + m\right)\delta\psi_+ + 2\left(\partial_v + \frac{f'}{4}\right)\delta\psi_-. \tag{D.4}$$

According to (4.23), the first pole skipping point happens at frequency $\omega = -\frac{1}{2}\omega_0$, and the corresponding momentum can be easily found by setting $\det M_1(k)$, which in this case is the prefactor in front of $\delta\psi_+$, to zero, after substituting the Fourier expansion (4.5). Solving for k immediately leads to $k = \mathrm{i}m\sqrt{h}$.

To find the next pole-skipping point, we need

$$\delta\mathcal{E}_{-1/2} = \left.\begin{bmatrix} \delta E_- \\ (\nabla_r \delta E)_+ \end{bmatrix}\right|_{r_0}, \tag{D.5}$$

where

$$\begin{aligned} \delta E_-|_{r_0} &= \frac{h'}{4h}\delta\psi_+ + \left(\frac{1}{\sqrt{h}}\partial_x - m\right)\delta\psi_- + (\nabla_r\delta\psi)_+, \qquad &(\text{D.6})\\ (\nabla_r\delta E)_+|_{r_0} &= \frac{h'}{2h^{3/2}}\partial_x\delta\psi_+ + \left(\frac{1}{2}f'' + \frac{f'h'}{4h}\right)\delta\psi_- \\ &\quad - \left(\frac{1}{\sqrt{h}}\partial_x + m\right)(\nabla_r\delta\psi)_+ + 2\left(\partial_v + \frac{3}{4}f'\right)(\nabla_r\delta\psi)_-. &(\text{D.7})\end{aligned}$$

From the general argument, we know that setting $\omega = -\frac{3}{2}\omega_0$ would kill all terms involving $\delta\mathcal{X}_q$ with $q \leq -3/2$. One can check explicitly that the prefactor in front of $(\nabla_r\delta\psi)_-$ vanishes. As always, we are left with a square matrix to analyze:

$$\begin{bmatrix} \delta E_+ \\ \delta E_- \\ (\nabla_r\delta E)_+ \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{h}}\mathrm{i}k - m & -f' & 0 \\ \frac{h'}{4h} & \frac{1}{\sqrt{h}}\mathrm{i}k - m & 1 \\ \frac{h'}{2h^{3/2}}\mathrm{i}k & \frac{f''}{2} + \frac{f'h'}{4h} & -\frac{1}{\sqrt{h}}\mathrm{i}k - m \end{bmatrix} \begin{bmatrix} \delta\psi_+ \\ \delta\psi_- \\ (\nabla_r\delta\psi)_+ \end{bmatrix}. \tag{D.8}$$

The determinant of the square matrix is evaluated to be

$$\det M_2(k) = \frac{\mathrm{i}kf''}{2\sqrt{h}} + \frac{1}{2}mf'' - \frac{\mathrm{i}kf'h'}{2h^{3/2}} - \frac{\mathrm{i}k^3}{h^{3/2}} - \frac{k^2 m}{h} - \frac{\mathrm{i}km^2}{\sqrt{h}} - m^3. \tag{D.9}$$

On the BTZ background (C.13), the pole-skipping momenta can be found by solving

$$0 = \det M_2(k) = -\frac{\mathrm{i}}{r_0^3}(k + \mathrm{i}mr_0)(k - \mathrm{i}(m-1)r_0)(k - \mathrm{i}(m+1)r_0), \tag{D.10}$$

with solutions

$$k = -\mathrm{i}mr_0, \quad \mathrm{i}(m-1)r_0, \quad \mathrm{i}(m+1)r_0. \tag{D.11}$$

At the next order ($s = 3$ and $\omega_3 = -\frac{5}{2}\omega_0$), taking

$$\delta\mathcal{X}_{1/2} \oplus \delta\mathcal{X}_{-1/2} \oplus \delta\mathcal{X}_{-3/2} = [\delta\psi_+, \delta\psi_-, \nabla_r\delta\psi_+, \nabla_r\delta\psi_-, \nabla_r^2\delta\psi_+]|_{r_0} \tag{D.12}$$

as our basis (in the order presented),

$$M_3(k) =$$

$$\begin{bmatrix} -m-\frac{\mathrm{i}k}{\sqrt{h}} & -2f' & 0 & 0 & 0 \\ \frac{h'}{4h} & -m+\frac{\mathrm{i}k}{\sqrt{h}} & 1 & 0 & 0 \\ \frac{\mathrm{i}kh'}{2h^{3/2}} & \frac{1}{4}\left(2f''+\frac{f'h'}{h}\right) & -m-\frac{\mathrm{i}k}{\sqrt{h}} & -f' & 0 \\ -\frac{h'^2-hh''}{4h^2} & -\frac{\mathrm{i}kh'}{2h^{3/2}} & \frac{h'}{4h} & -m+\frac{\mathrm{i}k}{\sqrt{h}} & 1 \\ \frac{\mathrm{i}k\left(2hh''-3h'^2\right)}{4h^{5/2}} & \frac{h\left(2f^{(3)}h+f''h'\right)-2f'\left(h'^2-hh''\right)}{4h^2} & \frac{\mathrm{i}kh'}{h^{3/2}} & 2f''+\frac{f'h'}{2h} & -m-\frac{\mathrm{i}k}{\sqrt{h}} \end{bmatrix}. \tag{D.13}$$

On the BTZ background (C.13), the pole-skipping momenta are found by solving

$$\begin{aligned} 0 &= \det M_3(k) \qquad\qquad (\text{D.14}) \\ &= \frac{-\mathrm{i}}{r_0^5}(k-\mathrm{i}mr_0)(k+\mathrm{i}(m-1)r_0)(k+\mathrm{i}(m+1)r_0)(k-i(m-2)r_0)(k-\mathrm{i}(m+2)r_0). \end{aligned}$$

At the next order ($s=4$ and $\omega_4=-\frac{7}{2}\omega_0$), taking our basis as

$$[\delta\psi_+,\delta\psi_-,\nabla_r\delta\psi_+,\nabla_r\delta\psi_-,\nabla_r^2\delta\psi_+,\nabla_r^2\delta\psi_-,\nabla_r^3\delta\psi_+]|_{r_0}, \tag{D.15}$$

the relevant matrix is given by

$$M_4(k) = \tag{D.16}$$
$$\begin{bmatrix} -m-\frac{\mathrm{i}k}{\sqrt{h}} & -3f' & 0 & 0 & 0 & 0 & 0 \\ \frac{h'}{4h} & -m+\frac{\mathrm{i}k}{\sqrt{h}} & 1 & 0 & 0 & 0 & 0 \\ \frac{\mathrm{i}kh'}{2h^{3/2}} & \frac{1}{4}\left(2f''+\frac{f'h'}{h}\right) & -m-\frac{\mathrm{i}k}{\sqrt{h}} & -2f' & 0 & 0 & 0 \\ -\frac{h'^2-hh''}{4h^2} & -\frac{\mathrm{i}kh'}{2h^{3/2}} & \frac{h'}{4h} & -m+\frac{\mathrm{i}k}{\sqrt{h}} & 1 & 0 & 0 \\ \frac{\mathrm{i}k\left(2hh''-3h'^2\right)}{4h^{5/2}} & A & \frac{\mathrm{i}kh'}{h^{3/2}} & 2f''+\frac{f'h'}{2h} & -m-\frac{\mathrm{i}k}{\sqrt{h}} & -f' & 0 \\ \frac{2h'^3+h^2h^{(3)}-3hh'h''}{4h^3} & \frac{\mathrm{i}k\left(3h'^2-2hh''\right)}{4h^{5/2}} & -\frac{h'^2-hh''}{2h^2} & -\frac{\mathrm{i}kh'}{h^{3/2}} & \frac{h'}{4h} & -m+\frac{\mathrm{i}k}{\sqrt{h}} & 1 \\ \frac{\mathrm{i}k\left(15h'^3+4h^2h^{(3)}-18hh'h''\right)}{8h^{7/2}} & B & \frac{3\mathrm{i}k\left(2hh''-3h'^2\right)}{4h^{5/2}} & C & \frac{3\mathrm{i}kh'}{2h^{3/2}} & \frac{3}{4}\left(6f''+\frac{f'h'}{h}\right) & -m-\frac{\mathrm{i}k}{\sqrt{h}} \end{bmatrix}$$

where

$$\begin{aligned} A &= \frac{h\left(2f^{(3)}h+f''h'\right)-2f'\left(h'^2-hh''\right)}{4h^2}, \\ B &= \frac{3f'\left(2h'^3+h^2h^{(3)}-3hh'h''\right)+h\left(f^{(3)}hh'-3f''h'^2+h\left(2hf^{(4)}(r)+3f''h''\right)\right)}{4h^3}, \\ C &= \frac{5f^{(3)}}{2}+\frac{3hf''h'-6f'\left(h'^2-hh''\right)}{4h^2}. \end{aligned}$$

On the BTZ background (C.13), the determinant simplifies to

$$\begin{aligned} \det M_4(k) = &-\frac{\mathrm{i}}{r_0^7}(k+\mathrm{i}mr_0)(k-\mathrm{i}(m-1)r_0)(k-\mathrm{i}(m+1)r_0) \\ &\times(k+\mathrm{i}(m-2)r_0)(k+\mathrm{i}(m+2)r_0)(k-\mathrm{i}(m-3)r_0)(k-\mathrm{i}(m+3)r_0). \end{aligned} \tag{D.17}$$

Again, setting this to zero gives the corresponding pole-skipping momenta at this frequency, which one can easily read off from the expression.

This procedure can be continued to higher orders systematically, but we will stop here to save space. To all the orders we have presented, the locations exactly match those found in [76].

D.2 Rarita-Schwinger field

Consider the following action for the spin-$\frac{3}{2}$ Rarita-Schwinger field, ψ_μ, on a curved background:

$$S = -\frac{1}{16\pi G_N}\int \mathrm{d}^{d+2}x\,\sqrt{-g}\left(\bar{\psi}_\mu\Gamma^{\mu\nu\rho}\nabla_\nu\psi_\rho + m\bar{\psi}_\mu\Gamma^{\mu\nu}\psi_\nu\right). \tag{D.18}$$

This theory has been considered in [82]. This is an interesting example not only because this is the only example of ours where the dynamic field carries both Lorentz and spinor indices but also because it has a gauge symmetry only when we tune the mass m to a special value.

The equation of motion for ψ_μ is given by

$$E^\mu(\psi_\nu) = \Gamma^{\mu\nu\rho}\nabla_\nu\psi_\rho + m\Gamma^{\mu\nu}\psi_\nu = 0. \tag{D.19}$$

This action has a gauge symmetry when $m = m_c \equiv d/2$ if the background satisfies vacuum Einstein's equation with a negative cosmological constant. To see this, consider the transformation

$$\delta\psi_\mu = \left(\nabla_\mu - \frac{1}{2}\Gamma_\mu\right)\chi, \tag{D.20}$$

under which the equation of motion changes by

$$\begin{aligned}
\delta E^\mu|_{m_c} &= \Gamma^{\mu\nu\rho}\nabla_\nu\left(\nabla_\rho - \frac{1}{2}\Gamma_\rho\right)\chi + \frac{d}{2}\Gamma^{\mu\nu}\left(\nabla_\nu - \frac{1}{2}\Gamma_\nu\right)\chi \\
&= g^{\mu\nu}\left(R_{\nu\rho} - \frac{1}{2}g_{\nu\rho}R\right)\Gamma^\rho\chi - \frac{1}{2}\Gamma^{\mu\nu\rho}\Gamma_\rho\nabla_\nu\chi + \frac{d}{2}\Gamma^{\mu\nu}\nabla_\nu\chi - \frac{d}{4}\Gamma^{\mu\nu}\Gamma_\nu\chi \\
&= \frac{1}{2}\left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R - \frac{d(d+1)}{2}g^{\mu\nu}\right)\Gamma_\nu\chi = 0,
\end{aligned} \tag{D.21}$$

where $\Gamma^{\mu\nu}\Gamma_\nu = (d+1)\Gamma^\mu$, $\Gamma^{\mu\nu\rho}\Gamma_\rho = d\Gamma^{\mu\nu}$, and the AdS length has been set to 1.

We again work in three bulk dimensions ($d = 1$). The highest-weight equation of motion is given by

$$\delta\mathcal{E}_{3/2} = \delta E_{v+}|_{r_0} = -\left(\frac{1}{\sqrt{h}}\partial_x + m\right)\delta\psi_{v+} - \frac{1}{\sqrt{h}}\left(\partial_v - \frac{1}{4}f'\right)\delta\psi_{x+}. \tag{D.22}$$

The first skipped pole is then located at $\omega = \frac{1}{2}\omega_0$, $k = \mathrm{i}m\sqrt{h}$. To find the next few points, we need

$$\begin{aligned}
\delta E_{v-}|_{r_0} &= -\frac{h'}{4h}\delta\psi_{v+} - \left(\frac{1}{\sqrt{h}}\partial_x - m\right)\delta\psi_{v-} - \frac{m}{\sqrt{h}}\delta\psi_{x+} + \frac{1}{\sqrt{h}}\left(\partial_v + \frac{1}{4}f'\right)\delta\psi_{x-}, \\
\delta E_{x+}|_{r_0} &= -2m\sqrt{h}\delta\psi_{v-} + \sqrt{h}(\nabla_r\psi)_{v+} - \sqrt{h}\left(\partial_v + \frac{1}{4}f'\right)\delta\psi_{r+}, \\
\nabla_r\delta E_{v+}|_{r_0} &= \frac{h'}{2h^{3/2}}\partial_x\delta\psi_{v+} + \frac{f'h'}{4h}\delta\psi_{v-} + \frac{f'h' - hf''}{4h^{3/2}}\delta\psi_{x+} - \left(\frac{1}{\sqrt{h}}\partial_x + m\right)(\nabla_r\psi)_{v+}.
\end{aligned} \tag{D.23}$$

At the next order ($s = 2$, $\omega = -\frac{1}{2}\omega_0$), in the basis

$$\delta\mathcal{X}_{3/2} \oplus \delta\mathcal{X}_{1/2} = [\delta\psi_{v+}, \delta\psi_{v-}, \delta\psi_{x+}, \nabla_r\delta\psi_{v+}]|_{r_0}, \tag{D.24}$$

the square matrix we are interested in is given by

$$M_2(k) = \begin{bmatrix} -m - \frac{\mathrm{i}k}{\sqrt{h}} & 0 & -\frac{f'}{2\sqrt{h}} & 0 \\ -\frac{h'}{4h} & m - \frac{\mathrm{i}k}{\sqrt{h}} & -\frac{m}{\sqrt{h}} & 0 \\ 0 & -2m\sqrt{h} & 0 & \sqrt{h} \\ \frac{\mathrm{i}kh'}{2h^{3/2}} & \frac{f'h'}{4h} & \frac{f'h'-hf''}{4h^{3/2}} & -m - \frac{\mathrm{i}k}{\sqrt{h}} \end{bmatrix}, \tag{D.25}$$

which has determinant

$$\begin{aligned}
&\det M_2(k) \\
&= -\frac{k^2 f''}{4h} - \frac{1}{4}m^2 f'' + \frac{\mathrm{i}kmf'h'}{4h^{3/2}} + \frac{3m^2 f'h'}{4h} - \frac{f'^2 h'^2}{32h^2} + \frac{2k^2m^2}{h} - \frac{4\mathrm{i}km^3}{\sqrt{h}} - 2m^4.
\end{aligned} \tag{D.26}$$

As mentioned at the beginning of the section, there is a gauge symmetry for the Rarita-Schwinger field when the background Einstein's equation is satisfied. In our case, they constrain the metric as follows:

$$(h'(r))^2 - 2h(r)h''(r) = 0, \quad f'(r)h'(r) - 4h(r) = 0, \quad f''(r) = 2. \tag{D.27}$$

Using the last two equations, the determinant becomes

$$\frac{1}{2h}(4m^2 - 1)(-2\mathrm{i}km\sqrt{h} - m^2h + h + k^2). \tag{D.28}$$

For the generic case $m \neq m_c$, the determinant is non-zero and pole skipping happens at those k's that make the determinant zero. For $m = m_c$, the determinant vanishes automatically, consistent with what we showed in Section 4.2. In this special case, we need to use the gauge-covariant version of pole-skipping conditions (4.29). This gives

$$k = -\frac{\mathrm{i}}{2}\sqrt{h}. \tag{D.29}$$

This momentum increases the dimension of the kernel from 1 to 2. We should emphasize that one cannot locate the pole-skipping points in this case by taking $m \to m_c$ after finding pole-skipping points for $m \neq m_c$, i.e., the procedures do not commute.

To understand why the matrix has a kernel of dimension 1, consider the pure-gauge perturbation

$$\delta\psi_\mu = \left(\nabla_\mu - \frac{1}{2}\Gamma_\mu\right)\delta\xi = \left(\nabla_\mu - \frac{1}{2}\Gamma_\mu\right)\chi. \tag{D.30}$$

The relevant part of (4.25) at $\omega = -\frac{1}{2}\omega_0$ is

$$\begin{bmatrix} \delta\psi_{v+} \\ \delta\psi_{v-} \\ \delta\psi_{x+} \\ (\nabla_r\delta\psi)_{v+} \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} f' \\ 1 \\ -(\sqrt{h}+2\mathrm{i}k) \\ \frac{1}{2}f'' \end{bmatrix}\chi_+. \tag{D.31}$$

This is indeed an eigenvector of the 4×4 matrix (D.23) with eigenvalue 0, assuming background Einstein's equation and $m = m_c$.

At $s = 3$ ($\omega = -\frac{3}{2}\omega_0$), using the basis

$$\begin{aligned} &\delta\mathcal{X}_{3/2} \oplus \delta\mathcal{X}_{1/2} \oplus \delta\mathcal{X}_{-1/2} \\ &= [\delta\psi_{v+}, \delta\psi_{v-}, \delta\psi_{x+}, \nabla_r\delta\psi_{v+}, \delta\psi_{x-}, \delta\psi_{r+}, \nabla_r\delta\psi_{v-}, \nabla_r\delta\psi_{x+}, \nabla_r^2\delta\psi_{v+}]|_{r_0}, \end{aligned} \tag{D.32}$$

the matrix we are interested in is

$$M_3(k) = \tag{D.33}$$
$$\begin{bmatrix}
-m-\frac{\mathrm{i}k}{\sqrt{h}} & 0 & -\frac{f'}{\sqrt{h}} & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{h'}{4h} & m-\frac{\mathrm{i}k}{\sqrt{h}} & -\frac{m}{\sqrt{h}} & 0 & -\frac{f'}{2\sqrt{h}} & 0 & 0 & 0 & 0 \\
0 & -2\sqrt{h}m & 0 & \sqrt{h} & 0 & \frac{\sqrt{h}f'}{2} & 0 & 0 & 0 \\
\frac{\mathrm{i}kh'}{2h^{3/2}} & \frac{f'h'}{4h} & \frac{f'h'-hf''}{4h^{3/2}} & -m-\frac{\mathrm{i}k}{\sqrt{h}} & 0 & 0 & 0 & -\frac{f'}{2\sqrt{h}} & 0 \\
0 & 0 & 0 & 0 & 0 & \sqrt{h}m & \sqrt{h} & 0 & 0 \\
0 & 0 & -\frac{h'}{2h^{3/2}} & 0 & \frac{2m}{\sqrt{h}} & m+\frac{\mathrm{i}k}{\sqrt{h}} & 0 & -\frac{1}{\sqrt{h}} & 0 \\
\frac{h'^2-hh''}{4h^2} & \frac{\mathrm{i}kh'}{2h^{3/2}} & 0 & -\frac{h'}{4h} & \frac{hf''+f'h'}{4h^{3/2}} & 0 & m-\frac{\mathrm{i}k}{\sqrt{h}} & -\frac{m}{\sqrt{h}} & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{1}{4}\sqrt{h}f'' & -2\sqrt{h}m & 0 & \sqrt{h} \\
\frac{\mathrm{i}k\left(2hh''-3h'^2\right)}{4h^{5/2}} & A & B & \frac{\mathrm{i}kh'}{h^{3/2}} & 0 & 0 & \frac{f'h'}{2h} & \frac{f'h'}{2h^{3/2}} & -m-\frac{\mathrm{i}k}{\sqrt{h}}
\end{bmatrix}$$

where

$$A = \frac{hf''h' - 2f'\left(h'^2 - hh''\right)}{4h^2},$$
$$B = \frac{h\left(f''h' - f^{(3)}h\right) - 2f'\left(h'^2 - hh''\right)}{4h^{5/2}}.$$

Again, the determinant is automatically zero when the background Einstein equation is satisfied and $m = m_c$. In this case, the kernel of this matrix is spanned by the pure-gauge perturbations

$$\begin{bmatrix} \delta\psi_{v+} \\ \delta\psi_{v-} \\ \delta\psi_{x+} \\ (\nabla_r\delta\psi)_{v+} \\ \delta\psi_{x-} \\ \delta\psi_{r+} \\ (\nabla_r\delta\psi)_{v-} \\ (\nabla_r\delta\psi)_{x+} \\ (\nabla_r^2\delta\psi)_{v+} \end{bmatrix} = \begin{bmatrix} -f' \\ -\frac{1}{2} \\ \frac{1}{2}\sqrt{h} + \mathrm{i}k \\ -\frac{1}{4}f'' \\ \frac{h'}{4\sqrt{h}} \\ 0 \\ 0 \\ -\frac{\mathrm{i}kh'}{2h} \\ -\frac{1}{4}f^{(3)} \end{bmatrix} \chi_+ + \begin{bmatrix} 0 \\ -\frac{f'}{2} \\ 0 \\ 0 \\ -\frac{1}{2}\sqrt{h} + \mathrm{i}k \\ -1 \\ \frac{f''}{4} \\ -\frac{f'h'}{4\sqrt{h}} \\ 0 \end{bmatrix} \chi_- + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\frac{f'}{2} \\ 0 \\ 1 \\ -\frac{1}{2} \\ \frac{1}{2}(\sqrt{h} + 2\mathrm{i}k) \\ 0 \end{bmatrix} (\nabla_r\chi)_+. \tag{D.34}$$

According to our gauge-covariant pole-skipping condition (4.29), we need to look for values of k that increase the dimension of $\ker M_3(k)$. Curiously, there turns out to be none at this order. Incidentally, the values

$$k = \frac{\mathrm{i}}{2}\sqrt{h}, \quad -\frac{3\mathrm{i}}{2}\sqrt{h}, \quad \frac{5\mathrm{i}}{2}\sqrt{h} \tag{D.35}$$

would increase the number of zeros in the characteristic polynomial but not the dimension of the kernel. See Footnote 2 for this distinction.

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