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DIFFERENTIAL CONSTITUTIVE EQUATIONS FOR AGING VISCOELASTIC MATERIALS

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ABSTRACT

Constitutive equations for aging viscoelastic materials are formulated as differential equations with variable coefficients. Particular attention is devoted to the second order equation; the creep function is derived and asymptotic behavior is studied. Comparison is made with commonly accepted expressions for creep functions of concrete. Extension to nonlinear behavior is discussed.

1. Introduction

The first description of linear viscoelastic stress-strain relations was apparently given by Boltzmann (1874), and the procedure he utilized was to represent the constitutive law in terms of a superposition integral. In later theoretical developments, Volterra (1930) also used such a representation. The so-called "weighting" or "memory" functions which occur as kernels in these integral formulations of the stress-strain relations are the creep or relaxation functions (or their derivatives), and represent the response of the material to unit step (or impulse) function inputs.

Most of the research in linear viscoelasticity which came after the work of Volterra dealt with nonaging, or time invariable, materials (called materials of the "closed cycle" type by Volterra). For such materials it has been found convenient to give alternative representations of the constitutive law in terms of: (1) differential equations; and (2) the frequency response of the material to sinusoidal input (i.e., the complex compliance or modulus). In connection with the differential equation approach, rheological (spring-and-dashpot) models have been extensively used as a heuristic device.

These three different representations are, of course, mathematically related to each other (see Gross (1953)), but, in practice, each has its own peculiar advantages and disadvantages. It would seem that from the point of view of the stress analyst, concerned with the practical solution of boundary value problems, the most convenient representation would be the one requiring the least amount of computational effort. From this standpoint, the most efficient formulation appears to be the differential equation form of lowest order which represents the behavior of the real material with sufficient accuracy.

Since the researches of Volterra, relatively little theoretical work has been done in the field of aging, or time-variable, viscoelastic media. Most of the investigations in that area have dealt with concrete, the first general representation of concrete as an aging, linear viscoelastic material apparently being that given by Maslov (1940), who represented the constitutive law by means of a superposition integral, in much the same way as did Volterra for general linear hereditary phenomena. Most of the later work with the creep of concrete also utilized the integral representation of the stress-strain relation (McHenry 1943, Arutiunian 1952, Ross 1958). The possible utility of differential equations with time-variable coefficients to represent the constitutive laws of aging viscoelastic media appears to be an area only slightly investigated, and one worthy of more intensive research. For materials such as concrete, which appear to have a limited number of creep mechanisms (i.e., a limited relaxation spectrum) in comparison to high polymers, relatively low order differential equations ought to be sufficient to satisfactorily approximate the viscoelastic behavior.

Recent investigations concerned with temperature effects in nonaging viscoelastic media have led naturally to time-variable stress-strain relations, the time-variability being induced by the unsteady temperature field (Morland and Lee 1960). In this case, it is also possible to utilize either the integral, differential equation, or complex impedence representation of the induced time-variable stress-strain relation; but the large majority of studies in this area do not utilize the complex impedence representation. On the basis of their investigations in this field, Hilton and Clements (1964) state that the differential equation representation appears to be the more convenient one.

Studies in other fields dealing with time-variable hereditary phenomena--notably in general systems theory--also indicate the desirability of employing differential equations to represent the basic character of the system under consideration. Stubberud (1964) gives a brief account of the merits and drawbacks of the three different representations, and develops the differential equation formulation relatively extensively. The fact that a considerable body of theory in the field of differential equations already exists is also an important feature in favor of such a representation.

Furthermore, the extension of linear constitutive relations of aging viscoelastic materials into the nonlinear region of behavior is probably easier to accomplish by use of the differential equation, rather than the integral, representation. In studies of the rate-dependent flow of metals, the "mechanical equation of state" (a first-order nonlinear differential equation) has been used quite successfully (e.g. Lubahn and Felgar 1961).

Integral representations of constitutive relations for nonlinear viscoelastic behavior are of course possible, and Leaderman (1943) and Rabotnov (1948) have given special forms. Extensive tests have not yet been carried out to verify the range of validity of these representations. However, it appears that even if valid, such forms are quite complex to use in the solution of boundary value problems. Arutiunian, who used the integral representation in the linear range, attempted to use a particular integral representation (based on the Leaderman form), in the nonlinear range. But he found it convenient, in the solution of problems, to transform the integral representation into differential-equation form.

The more general integral representation in terms of a Volterra-Fréchet functional expansion can be used to represent nonlinear viscoelastic behavior to a sufficient degree of accuracy, but such a relationship is even more difficult to work with than the special forms of Leaderman and Rabotnov. Therefore it seems worthwhile to investigate the use of nonlinear differential equations to represent aging, nonlinear viscoelastic behavior. If low order differential equations suffice, the prospect of obtaining a greater amount of quantitative results by their use appears more likely than by the use of nonlinear integral representations.

2. Aging Materials Described by Second-Order Differential Equations

(i) Notation.

In the development to follow, stress will be denoted by σ, strain by ε, time by t, and time differentiation by a superscript dot. While the notation will be that for one-dimensional stress and strain, it will be equally valid for m-dimensional states (m=2,3) if σ and ε are interpreted as vectors and the coefficients as square matrices in n-space, where $n = m(m+1)/2$, provided the algebraic operations are carried out in the matrix sense.

(ii) Development.

In the case of non-aging materials, it has often been stated (Reiner 1943, Lee 1950) that the simplest differential constitutive equation capable of describing complex material behavior is the equation of the Burgers body,

$$a_0 \ddot{\sigma} + a_1 \dot{\sigma} + a_2 \sigma = b_0 \ddot{\varepsilon} + b_1 \dot{\varepsilon} \tag{1}$$

the coefficients a_i, b_i being constant. The coefficient a_2 represents steady (unbounded) creep; in the case of bounded creep, a_2 vanishes, and we have

$$a_0 \ddot{\sigma} + a_1 \dot{\sigma} = b_0 \ddot{\varepsilon} + b_1 \dot{\varepsilon} \tag{2}$$

which may be integrated once to yield the equation of the "standard solid",

$$a_0 \dot{\sigma} + a_1 \sigma = b_0 \dot{\varepsilon} + b_1 \varepsilon \tag{3}$$

Equations (2) and (3) are equivalent if they are supplemented with equivalent initial conditions. There exists, however, an important difference between them, analogous to the difference between two possible descriptions of an elastic body,

$$\sigma = E\varepsilon \quad , \quad \dot{\sigma} = E\dot{\varepsilon} \tag{4}$$

the second of which is a hypoelastic description with no reference to a natural state. It is well known that for finite deformations, elasticity and hypoelasticity are not equivalent (Truesdell 1955). It is readily shown that in the case of an aging material, the difference is non-trivial even for infinitesimal deformations, since the descriptions

$$\sigma = E(t)\,\varepsilon \quad , \quad \dot{\sigma} = E(t)\,\dot{\varepsilon} \tag{5}$$

stand for fundamentally different kinds of behavior: the first represents, essentially, an aging viscoelastic material with infinitely short memory, while the second represents a conservative material (exhibiting no creep or relaxation). More generally, an aging viscoelastic material described by

$$p_0(t)\,\sigma^{(n)} + \cdot\cdot\cdot + p_n(t)\,\sigma = q_0(t)\,\varepsilon^{(n)} + \cdot\cdot\cdot + q_n(t)\,\varepsilon \tag{6}$$

where $q_n(t)$ does not vanish, has the property that in a creep test under a constant stress σ, the ultimate strain is

$$\varepsilon(\infty) = \lim_{t \to \infty} \left\{ \left[q_n(t) \right]^{-1} p_n(t) \right\} \tag{7}$$

independent of the time at which loading occurs. Since known aging materials (such as concrete) do not have this property, they cannot be described by differential constitutive equations of the form (6), but rather, of a form in which $q_n(t)$ vanishes identically:

$$p_0(t)\,\sigma^{(n)} + \cdot\cdot\cdot + p_n(t)\,\sigma = q_0(t)\,\varepsilon^{(n)} + \cdot\cdot\cdot + q_{n-1}(t)\,\dot{\varepsilon} \tag{8}$$

It will be assumed, by analogy with the case of non-aging materials, that the simplest realistic case of (8) is the second-order equation,

$$p_0(t)\ddot{\sigma} + p_1(t)\dot{\sigma} + p_2(t)\sigma = q_0(t)\ddot{\varepsilon} + q_1(t)\dot{\varepsilon} \tag{9}$$

which will be rewritten as

$$\frac{d}{dt}\left[\dot{\sigma}/E(t)\right] + \alpha(t)\dot{\sigma} + \beta(t)\sigma = \ddot{\varepsilon} + \gamma(t)\dot{\varepsilon} \tag{10}$$

The creep function, defined as the strain at time t due to a unit stress applied at time τ, corresponding to (10) is

$$\varphi(t,\tau) = 1/E(\tau) + \int_{\tau}^{t} e^{-\theta(\lambda)}\left[\chi(\lambda) - \chi(\tau) + \psi(\tau)\right] d\lambda \tag{11}$$

where

$$\theta(t) = \int_{t_0}^{t} \gamma(\lambda)\, d\lambda \qquad (t_0 \text{ arbitrary})$$

$$\dot{\chi}(t) = e^{\theta(t)}\beta(t)$$

$$\psi(t) = e^{\theta(t)}\left[\alpha(t) - \gamma(t)/E(t)\right]$$

The following relations between the coefficients appearing in (10) and the creep function (11) can be easily verified:

$$1/E(t) = \varphi(t,t) \tag{12}$$

$$\gamma(t) = -\frac{\partial}{\partial t}\ln \varphi_{t\tau}(t,\tau) \tag{13}$$

$$\alpha(t) = \varphi_t(t,t) + \gamma(t)/E(t) \tag{14}$$

$$\beta(t) = \varphi_{tt}(t,\tau) + \chi(t)\,\varphi_t(t,\tau) \tag{15}$$

where the subscripts t, τ denote partial differentiation. Note that the right-hand sides of equations (13) and (15) are in general functions of t and τ, but the description of material behavior by equation (10) requires that they be independent of τ.

Equation (13) breaks down if

$$\varphi_{tt}(t,\tau) = 0$$

This can be shown to be the case if and only if

$$\psi(t) - \chi(t) = \text{constant}$$

In this case equation (10) can be shown to be equivalent to the first-order equation.

$$\dot{\varepsilon} = \dot{\sigma}/E(t) + \eta(t)\,\sigma \tag{16}$$

where

$$\eta(t) = \varphi_t(t,\tau) \tag{17}$$

(iii) Asymptotic Behavior.

The special case of (10) which is the aging-material analogue of (2) corresponds to $\beta(t) = 0$. However, this is neither a necessary nor a sufficient condition for bounded creep. Necessary and sufficient conditions for the boundedness of solutions of differential equations have been listed by Cesari (1963). In the case of equation (10), the condition can be derived from the solution (11), namely, for any stress history beginning at

$t = t_o$ and bounded in (t_o, ∞), the strain history is bounded in (t_o, ∞) if and only if

$$e^{\theta(t)} \int_{t_0}^{\infty} e^{-\theta(t)}\, dt < \infty \qquad \text{(a)}$$

(18)

and

$$\int_{t_0}^{\infty} \left[\int_{t_0}^{t} e^{-\theta(t)+\theta(\tau)} |\beta(t)|\, d\tau \right] dt < \infty \qquad \text{(b)}$$

A sufficient condition for (a) is $\theta(t) = O(t^a)$, i.e. $\gamma(t) = O(t^{a-1})$ as $t \to \infty$, with $a > 0$. If $\lim_{t\to\infty} |\beta(t)| < \infty$, then the integral

$$\int_{t_0}^{t} e^{\theta(\tau)} \beta(\tau)\, d\tau \sim e^{\theta(t)} \beta(t)/\gamma(t) \quad \text{as } t \to \infty$$

so that the bracketed integrand in (b) behaves as

$$\beta(t)\, t^{1-a}$$

as $t \to \infty$. A sufficient condition for bounded creep is therefore

$$\gamma(t) = O(t^{a-1}), \quad \beta(t) = O(t^{b-2}) \qquad \text{as } t \to \infty$$

with $a > 0$, $a > b$.

If $\gamma(t) = O(t^{-1})$ as $t \to \infty$, then (a) is satisfied if and only if $\lim_{t \to \infty} t\,\gamma(t) = k > 1$, since then the integrand of (a) tends to t^{-k}. In this case a sufficient condition for (b), and hence for bounded creep, is

$$\beta(t) = O(t^{-2-c}) \qquad \text{as } t \to \infty$$

with $c > 0$.

If, finally, $\gamma(t) = O(t^{-1-d})$, $d > 0$, then (a) cannot be satisfied; hence creep will be unbounded, except in the special case in which

$$\psi(t) = \chi(t)$$

corresponding to equation (16). In this case, the necessary and sufficient condition for bounded creep is

$$\int_{t_0}^{\infty} |\eta(t)|\, dt < \infty \tag{19}$$

(iv) Special cases.

The special case corresponding to equation (16) was used by Dischinger (1937) to represent creep of concrete.

A more interesting special case of (10) corresponds to $\varkappa(t) = C$ (constant); then we may put

$$\Theta(t) = Ct$$

and the creep function becomes

$$\varphi(t,\tau) = 1/E(\tau) + f(\tau)\left[1 - e^{-C(t-\tau)}\right] + g(t) - g(\tau) \tag{20}$$

where

$$f(\tau) = \frac{1}{C}\left[\psi(\tau) - \varkappa(\tau)\right] e^{-C\tau}, \quad g(t) = \int_{t_0}^{t} e^{-C\lambda} \varkappa(\lambda)\, d\lambda$$

Equation (20) corresponds essentially to the creep function proposed for concrete by Yashin (1959), on the basis of experimental evidence. Earlier, a special case of (20), namely that corresponding to g(t) = const (i.e. $\varkappa(t) = 0$), had been used by Arutiunian (1952). More recently a creep function of the form (20), with g(t) = K log t (K constant), was formulated on the basis of the microstructure of concrete by Hansen (1960).

If the creep function (20) is applied to the problem of creep recovery, i.e. the case of a stress applied at time t_o and removed at time t_1, the resulting strain at time $t > t_1$ has the form

$$A + Be^{-Ct}, \tag{21}$$

where A and B are functions of t_0 and t_1. Creep recovery governed by (18) has been observed experimentally in concrete by L'Hermite (1962).

3. Remarks

It is interesting to note that almost all of the currently used expressions used to represent creep functions of concrete are contained, as special cases, within the solution to the general second-order differential equation constitutive relation proposed herein. This indicates the surprising generality of this rather low-order representation. Probably, by a suitable choice of the time-variable coefficients in the second-order differential equation, such a representation would be sufficient to describe the constitutive relation for a variety of aging materials.

Arutiunian (1952) suggested the possibility of a creep function for aging concrete more general than that given by equation (20), containing a sum of negative exponential terms. Such creep functions can be obtained from a differential-equation representation of the form considered here, but of order higher than two.

In certain special cases, it is possible to derive differential constitutive laws for aging materials from rheological models with time-variable elements. For example, equation (16) may be regarded as representing a time-varying Maxwell model. It must be borne in mind, however, that there is no unambiguous definition of elastic response (see equation (5)). Furthermore, even if a particular definition of elastic response is adopted, the governing equation of the time-varying rheological model is, in general, not a differential, but an integro-differential, equation. For these reasons rheological models appear to be of little value for aging materials.

An obviously useful area of application of the differential-equation representation of the type proposed here is that of creep buckling of structures made of aging viscoelastic materials. A fairly extensive body of theory on the stability and asymptotic behavior of differential equations with variable coefficients is in existence (Bellman 1953, Cesari 1962), so that this knowledge can be applied to such stability problems. Since the theory extends to certain classes of nonlinear differential equations as well, deeper insight into the problem of the stability of aging nonlinear viscoelastic structures may be gained by the use of differential equations to describe the constitutive law of the material.

References

Arutiunian, N. Kh.	1952	Some Questions in the Theory of Creep (in Russian), (Moscow: Gos. Tekh. Teor. Izdat.)
Bellman, R.	1953	Stability Theory of Differential Equations (New York: McGraw-Hill)
Boltzmann, L.	1874	Sitzber. Ak. Wiss. Wien, 70, 275; Wiss Abh. 1, 616
Cesari, L.	1963	Asymptotic Behavior and Stability Problems in Ordinary Differential Equations 2nd Ed. (Berlin: Springer)
Dischinger, F.	1937	Bauing. 18, 539
Gross, B.	1953	Mathematical Structure of the Theories of Viscoelasticity (Paris: Hermann et Cie.)
Hansen, T. C.	1960	Creep and Stress Relaxation of Concrete (Stockholm: Sw. Cem. & Concr. Res. Inst.)
Hilton, H. H. & Clements, J. R.	1964	Proc. Conf. Thermal Loading & Creep, p. 6-17, (London: Inst. Mech. Eng.)
Leaderman, H.	1943	Elastic and Creep Properties of Filamentous Materials (Washington: Textile Foundation)
Lee, E. H.	1960	Structural Mechanics, Proc. 1st Symp. on Naval Struct. Mech., p. 456 (New York: Pergamon)
L'Hermite, R.	1962	Chemistry of Cement, Proc. 4th Int. Symp. 2, 659 (Washington: U. S. Bureau of Standards)
Lubahn, J. D. & Felgar, R. P.	1961	Plasticity and Creep of Metals, Ch. 7 (New York; Wiley)
Maslov, G. N.	1940	Izv. Nauchno-Issl. Inst. Gidrot. 28, 175.
McHenry, D.	1943	Proc. A.S.T.M. 43, 1069

Morland, L. W. & Lee, E. H.	1960	Trans. Soc. Rheol. 4, 233
Rabotnov, Yu. N.	1948	Vest. Mosk. Gos. Univ., 3, No. 10
Reiner, M.	1954	Building Materials, Ch. 1 (Amsterdam: North-Holland)
Ross, A. D.	1958	J. Am. Concr. Inst., 29, 739
Stubberud, A. R.	1964	Analysis and Synthesis of Time-Varying Linear Systems (Berkeley: U.C. Press)
Truesdell, C.	1955	J. Rat. Mech. Anal., 4, 83
Volterra, V.	1930	Theory of Functionals, Ch. 6 (London: Blackie & Co.)
Yashin, A. V.	1959	Research on Properties of Concrete and Reinforced Concrete Structures (in Russian), p. 18 (Moscow: Gos. Stroi, Izdat.)