Cognitive Development and Infinity in the Small: Paradoxes and Consensus

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Abstract

Throughout history the concept of infinity has played an important role in almost every branch of human knowledge. Paradoxically, very little effort has been made by the various theoretical schools in Cognitive Science to study this fascinating aspect of human mental activity. The study of subdivision offers an interesting subject matter to address the question of how the idea of infinity in the small emerge in our minds. 32 students, aged 8, 10, 12 and 14 (high and low intellectual–academic performers), participated in this study, in which a version of one of Zeno's paradoxes was analyzed by means of individual interviews. Results suggest that between ages 10 and 12, a certain intuition of the entailments of subdivision emerges, remaining very labile afterwards and being very influenced by the context. 66% of the 12- and 14-year-old children said that the process involved in the paradox comes to an end. Less than 25% considered (with deep hesitations) the possibility that the process might continue endlessly. This suggests that the classic piagetian view that the indefinite subdivision is mastered at the period of formal operations must be reassessed. Some epistemological consequences based on an embodied-cognition oriented perspective are discussed.

Introduction

Since the dawn of civilizations, the idea of infinity has played an important role in almost every branch of human knowledge, fascinating and thrilling philosophers, theologians, scientists and mathematicians. Full of counter-intuitive features, infinity has always been a controversial and elusive concept and its study has presented countless difficulties and disputes. Such an important and peculiar concept of human mental activity provides a very interesting subject matter for cognitive science for at least two reasons: firstly, because of the important role played in the different disciplines of human knowledge, and secondly, because it is a rich and representative concept of a dimension of mental activity not based on direct experience. Paradoxically, very little effort has been made by the various theoretical schools to study such a fascinating aspect of human mental activity.

History of mathematics, and studies in mathematical thinking and developmental psychology show that infinity in the small -involved in iterated subdivision- has been much more controversial and elusive than infinity in the large (Núñez, 1993a), and that children begin to understand the idea of infinity in the small much later than infinity in the large (Langford, 1974; Núñez, 1993b; Piaget & Inhelder, 1948; Taback, 1975). Thus, while 8-year-old children easily recognize the endless nature of the counting process, it is not until about the age of 11-12 that they start to consider indefinite subdivision as a legitimate problem (Núñez, 1993a). The present study intends to contribute to the understanding of the development of the idea of infinity in the small from a cognitive[2] viewpoint.

The Problem

About 2500 years ago in Elea, now southern Italy, a disciple of the philosopher Parmenides, known as Zeno the Eleatic, conceived some paradoxes presumably in order to prove the inconsistency in the Pythagorean ideas of multiplicity and change and to argue in favor of the unity and the permanence of being, fundamental principles of his school of thought. Different versions of these paradoxes have come down to us, but amazingly, their paradoxical features have always puzzled philosophers and mathematicians, and even today still intrigue us. A shortened version of one of these paradoxes could be stated as follows. Imagine that we are asked to go from a point A to a point B, but we are told to do so by following a rule which says: first go half of the way, then half of what remains, then half of what remains, and so on. Do we ever reach point B? Since each step only covers half of the remaining distance at the previous step, there will always be a short distance to be covered. Therefore we never reach point B. Of course, this is not what our experience tells us when we go from one place to another, from whence the paradox arises.

From a cognitive point of view, the study of paradoxes such as Zeno's offers an interesting subject matter to

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[2]"Cognitive" as subject matter, not as a view from the "cognitivist" school.

approach the question of how the mind builds the idea of infinity in the small, contributing thus to the poorly studied domain of infinity. It has been suggested elsewhere (Núñez, 1993a), that the paradoxical aspects of the arguments propounded by Zeno, as opposed to other situations in which infinity is involved but that don't lead to paradoxical considerations, could be partially explained in terms of a particularly complex coordination of attributes being iterated indefinitely. This, based on observations made by Langford (1974), Piaget & Inhelder (1948) and Taback (1975) regarding differences between infinity in the small and in the large, as well as those by Tall (1980) concerning differences between cardinal and measuring features of infinity. In the example given above, if one follows the rule of the problem, there are two attributes that one has to iterate simultaneously, namely, the number of steps one has to make and the distance that each of these steps covers. On the one hand, observe that the **type** of iteration is different for each attribute, because the number of steps increases whereas the distance covered by the steps decreases. Let's call these two **types** of iteration, *divergent* and *convergent*, respectively. On the other hand, the **nature** of the content being iterated is also different for both attributes: the number of steps refers to *cardinality* (number of steps) whereas the distance covered by the steps refers to *space*. Unlike situations in which two attributes of either same **type** or same **nature** are indefinitely iterated in a coordinated situation, the coordination that takes place in Zeno's paradox must incorporate heterogeneous components. That is, different **types** of iterations (*divergent* and *convergent*) and different **natures** of the content (*cardinality* and *space*), which makes it qualitatively more complex.

In general, any situation involving subdivision, that might lead to the conception of the idea of infinity in the small[3], presents this structure: a simultaneous coordination of an increasing number of steps (number of iterations) and a decreasing partial result (in absolute value) to be operated in the next iteration. From a cognitive viewpoint it is interesting then, to study how this complex coordination emerges in mental activity and what forms it takes across different conceptual contexts.

The following presents some qualitative aspects of an investigation that was inserted in a broader project conceived to study developmental and psycho-cognitive aspects underlying the idea of infinity in a much larger sense.

Method

Procedure

Two variables related to intellectual and academic performance were studied. The former was measured by the Raven test and the latter measured by the marks obtained both in mathematics and French (official language at school). The Raven test was administrated in the classroom during school time. The study was performed by means of individual interviews (50 minutes) and was introduced to the students as a study of psychology (mathematics was not mentioned at all). The interviews took place in a room located in the school and were videotaped. The question about Zeno's paradox was included.

Subjects

Thirty-two students from two schools in the city of Fribourg (Switzerland) , aged 8, 10, 12 and 14 years old, were interviewed. Each age group (8 subjects) was formed of half low and half high intellectual-academic performers (percentiles 0-33 and 66-100 in each age group respectively, taken from a larger population; N=172). Boys and girls were equally represented.

Material

The problem was presented orally as an open-ended one, and the question was: "Imagine that we want to go from this side of the table to the other side. First we are told to go half of the way, then we continue with half of what remains, then half of what remains, and so on. Do we ever reach the other side of the table?" Sometimes the subject performing the action was a small insect. The students were allowed to use a variety of materials such as paper, pencils, rulers, etc., which were on the table.

Results

In general, the children's answers could be classified in three main categories, namely, those saying that one reaches the destination, those saying that one does not arrive (either because the process gets stuck, or because it only approaches the destination), and those coming from a hesitating subject defending two different answers. Subjects of all four age groups gave answers of the first two categories. The last category was only observed in the older groups. The following presents the rationale behind these views. No gender differences were found.

8-year-old Group

In this age group, there was a difference between the arguments given by the high and low intellectual–academic performance subgroups, the former showing a more analytical attitude towards the problem. Low performers tended to consider the question as a very trivial one and easily responded with an obvious "we arrive", as if the iteration were to be performed just a couple of times in order to cover the distance.

Based on the idea that in the end one gets stuck due to the smallness of the steps, two children considered the possibility that one might not arrive, and one said that one never arrives. All three were high performers. They vaguely perceived that one might potentially continue for a while making small steps, but in the end they solved this

[3]Of course, this is not the only way that might lead to conceive infinity in the small. In non-standard analysis, for example, infinitesimals are seen as the reciprocals of infinitely big numbers. Nevertheless, from a developmental viewpoint, the idea of subdivision seems to be a more basic one.

unstable situation by altering the conditions of the problem, either by altering the conditions of the setting (new destination), of the action (i.e., stopping the iteration), or both. An example of the former is the following[4]:

Ban +(7;9): *If we always go half way we arrive very close, but we are forced to always do halves, always the half.* So? ... (thinks) ... *there is always a small half.* We get closer but we can not arrive? *We get closer ... but if we go half of the way there is just a step left, and if we go the* (other) *half, there is just one half left, ... and we go that half and we arrive* (not so convinced). You doubt? ... (thinks and reanalyzes with his fingers near the edge of the table), ... *or maybe there is a small piece left before the arrival and we decide that that's the place where we have to stop.* How is that? *We put the arrival before the place where we want to go, so then we are sure that we arrive.*

10-year-old Group

Like in the previous group, there was a difference between the high and low performers, similar to that described above. 10-year-old children presented more hesitations than the 8-year-old ones and some of them had two different positions, but like the youngest group, sometimes the conflict was solved by altering the conditions of the setting for the last iterations (i.e., new destination).

As mentioned in the *Procedure*, other contents were also covered in the interview. Among others, these included a problem described elsewhere (Núñez, 1993b), in which plane geometrical figures were transformed following an iterative process with similar construction (different **type** of iteration and **nature** of the content). It is worth noticing that some 10-year-old high performers presented an infinitist position (i.e., acceptance of an endless iteration) in Zeno's paradox but a finitist position in the geometrical problem, and vice-versa. Thus, Euo +(10:1) who was clearly convinced by his arguments in favor of the endless iteration involved in the geometrical problem, did not even hesitate to say that the iteration in the paradox comes to an end:

Euo +(10;1): So do we arrive to the other side? *Yes, certainly, ... if we make the half, the half, the half, the half, the half, the half, the half, the half, ... afterwards, we arrive, for sure.* I have a friend that says that we can not arrive because there is always a half that you have to make (the Exp. shows the steps on the table and the Subject responds immediately with the following step, as if he wanted to complete the "other half"). ... *and once again, the half, once again, the half, ... we will arrive.* And if we only make it in our imagination? *In our imagination and in physics also* (we arrive).

Ode, +(9,9), on the contrary, maintained a finitist position in the geometrical problem but suggested the endless nature of the iteration in the paradox:

Ode +(9;9): Does it arrive? ... (examines the "last" steps) ... *always the half?* (the Subject asks). *It's really, ... if it is there it makes the half* (follow the steps on the table) *it will take a long time because afterwards it makes just one step, ... and then the half, and then always the half, ... no, it doesn't arrive. There, for example* (shows the edge of the table) *it is there, and after makes the half, arrives there, and then there, ... no I don't think that I will arrive by always going a half.*

12- and 14-year-old Groups

Unlike the 8-and 10-year-old groups, we didn't find relevant differences between the arguments of high and low intellectual–academic performers in the older groups. The patterns of answers in the 12- and 14-year-old groups were quite similar to each other, so we are going to analyze them as one.

Nearly 2/3 of the 12- and 14-year-old children (high and low performers together) gave answers of the forms "It will take a long time, but we will arrive" or "One will arrive because in the end one will not be able to make the half, it will be too close" (altering the conditions of action). Among these subjects some showed deep hesitations holding antagonic positions, often opposing ideas based on immediate physical considerations (although not necessarily concrete) and ideas based on the fact that the iteration leads to approach the destination (although not necessarily abstract):

Mal -(14;8): Does it arrive? (thinks, whispers) ... *I'm not going to measure the table, but I think we don't arrive exactly, maybe about a millimeter away. I think that we will arrive, but that we are not going to arrive immediately, ... if we have every time, ... well, I don't know, I think that it will arrive exactly at the point, and I think that it will not arrive exactly. I have two opinions. I don't know whether it will be exactly or at about a millimeter away.*

Others changed their mind when the scale of the problem was amplified:

Frc -(14;7): Do we arrive? *Oh yes, you're almost there, ...* (thinks), *but it is always smaller and smaller, ... I think that it must arrive.* And if the distance is the one between Switzerland and Sweden? *Oh no, then I don't think so, ... It always remains a short way, ... it is shorter and shorter, afterwards there is a half* (check with the fingers at the edge of the table), *one must make it, but you almost don't advance anymore, it is too tight. ... Uf! It is hard.* So do we arrive? *I don't know. I think that for short distances we arrive, but for long distances I don't think so, ... I'm not so sure, but I don't think so.* And what about distances in between? ... (thinks), ... *Yes, we arrive I think, because afterwards there will be short distances.*

Augmenting the distance into a qualitatively much bigger one, seems to put the "last" steps of the iteration further apart, leaving room for further iterations, such that the arrival is not attainable anymore. Qualitatively, this view sees middle distances as belonging to a similar family as the short ones, that is, sharing the same features at the end of the process.

Finally, less than 25% of the 12- and 14-year-old children presented arguments in favor of the idea that one will only approach the destination, but never reach it, and only one subject was truly sure about it. In general, these arguments

[4]Text in italics corresponds to subjects' utterances. The symbols + and indicate high or low intellectual-academic performance respectively. The age is also indicated (years; months).

were characterized: by being loaded with doubts and hesitations; by respecting the conditions of setting and action; and sometimes by the emergence of the idea of infinity, explicitly:

Joé +(14;4): ... (thinks) ..., *at infinity we will never arrive, it goes beyond imagination, but if we see* (show the table) *we will arrive but in fact we don't arrive, ... I don't know, it surpasses the imagination. If we always make a half of the way that we have to do, it will be difficult to arrive, ... then it will be microscopic, infinitely small.*

The only subject who was sure about the impossibility of reaching the destination, despite of the fact that the iteration continues endlessly, showed a pragmatic approach throughout the whole interview. He never engaged himself in extremely abstract and speculative reasoning, although he conceived the objects of the problem as theoretical ones in which the physical constraints were not relevant for the analysis:

Stn -(14;1): *No, we will never arrive.* Even if we always advance? *Even, because we make only the half. We will get close, we will be very close to the edge* (table) *but we will never arrive there. ... there will always be a short way to be done.* Even if we get always closer? *Even.*

He follows the conditions of the problem avoiding any speculative consideration, focusing on the fact that there is always something left no matter how many steps one makes. The position obtained by any iteration is never the final destination and therefore the iteration continues endlessly.

Discussion

Among the variables studied, only age evinced clear–cut differences in the arguments (except for ages 12 and 14). No gender differences were found. The role of intellectual-academic performance seems to have only been significant for the 8-and 10-year-old groups. At these ages the arguments given by the high performers were more analytical and tended to consider the problem as such, whereas those given by low performers tended to consider the problem as a trivial one (reaching the destination after a couple of steps was evident, so that, no further considerations were necessary). The observation that no major intellectual-academic differences were found in the older groups could be partially explained by the fact that, on the one hand, the validity of the Raven test (Intelligence) decreases as age increases due to a ceiling effect, and on the other hand, in higher grades academic performance becomes a more complex phenomenon such that factors other than cognitive ones play an important role (e.g., self-esteem or motivational changes during early adolescence).

In mathematical terms, the version of Zeno's paradox presented here deals with a series of the form $d/2 + d/4 + d/8 + \ldots$ (d=distance). Using this language to describe children's views (which, of course, is not theirs), we could say that we have observed approaches of the following forms:

a) Altering the conditions of action very early. $d/2 + d/4 + \ldots + d/2^k + d/2^k = d$. In general k has a value between 3 and 5. The last two steps are unnoticedly considered as equal, leading to an exact arrival. This approach was observed among low performers aged 8 and 10. The problem then becomes trivial.

b) Altering the conditions of the setting. $d/2 + d/4 + d/8 + \ldots + d/2^k = e$. Suddenly a new destination which defines a new distance $e = \sum_{k=1}^{p} d/2^k$ (where p is a large finite integer) is considered, allowing the arrival at the new destination (in general $e<d$). This approach was observed among some high performers aged 8 and 10, and seems to emerge as a solution to reconcile early intuitions about long lasting *convergent* iterations (subdivision).

c) Altering the conditions of action. $d/2 + d/4 + d/8 + \ldots + d/2^k < d$, where k is a large finite integer. The process gets stuck due to the smallness of the steps and cannot continue. This approach was observed among all four age groups, but k tended to be larger at ages 12 and 14 than at age 10.

d) Respecting the conditions of setting and action. $d/2 + d/4 + d/8 + \ldots < d$. The process only approaches the destination and continues endlessly. It was observed with only one subject (14 years old). Nevertheless, nearly 25% of the 12- and 14-year–old subjects also referred to it, although with deep hesitations involving also approach c).

An approach of the form $d/2 + d/4 + d/8 + \ldots = d$, which is "mathematically correct" was never observed.

In order to conceive the paradoxical situation raised by Zeno's problem, one must be able to distinguish the elements involved and respect the conditions of the problem (i.e., setting: keeping the original distance to be covered; action: continuing the endless iteration). Approaches a), b), and c) refer to alterations of the conditions of the problem such that they don't give rise to paradoxical situations. Our paradox then, is not children's paradox.

The traditional views in philosophy of mathematics (platonism, constructivism, formalism) consider the existence of mathematical objects (or formulas in the case of formalism), in various degrees and forms, to be independent of human beings. Mathematics is seen as objectively structured, independent of any understanding. Here, the analysis we present is based in a totally different view, in which cognition is seen as an embodied phenomenon, being close to theoretical positions such as those defended by the biologist H. Maturana (1987) or the linguist G. Lakoff (1987). Mathematics thus is conceived as being totally dependent on human beings; conceived as emerging from the interaction of biological beings that evolve in their medium, and therefore depending on the very nature of the interwoven process of embodied concepts, social interactions, and language. In other words: no human beings, hence no mathematics; different biological beings, different mathematics. From this point of view, Mathematics emerges through a language shaped historically and based on a consensus grounded on bodily experiences. In Zeno's paradox, respecting the conditions of the problem takes place within this consensus. The consensus in turn, depends, among others, on the biological structure of the subjects (e.g., developing neurobiological structure) participating in the ongoing process of the definition of what the consensus is. In cognitive developmental psychology this issue is essential because the conceptual world that emerges from the cognitive activity of young children is based on a consensus which is different from ours (because their neurobiology, their language, their embodied cognition is different). Thus, what we call rigor in nowadays' mathematics[5] gives rise to different meanings at these ages, as does the respect for the conditions of Zeno's paradox.

From this point of view, the unnoticed alterations of the conditions of Zeno's paradox by children (setting and action) takes place in a different domain of consensus which we happen to not share. This becomes evident when the youngest subjects, as determined in their developing neurobiological structure, enact a consensual world clearly different from ours when the meaning of the *convergent* **type** of iteration is concerned. In fact, before the age of 12 there is a profound and striking ontological difference between infinity in the large (*divergent* **type**) and infinity in the small (*convergent* **type**). As a 10–year–old clearly stated:

Yak +(10;1): If that is the "bigger" infinity, as you say, when you say "smaller" infinity, is it the same thing but in the other sense? *Yes, and there is only one difference. At a certain moment it becomes so small that we can not even know where it is.* So what is the difference then, between the "bigger" infinity and the "smaller" infinity? *At a certain moment, when we are in the smaller infinity, it stops, whereas in the bigger infinity it could continue until, ... infinity.*

Thus, there are endless infinities and stopped up infinities, both being infinities! Notice that the latter contradicts our very notion of "In–finity". At 8 years–old, although the idea of endless (related to *divergent* **type** and infinity in the large) is already present in their consensual world, the distinction "infinity in the small" (related to *convergent* **type** iterations) simply does not exist. It seems that between the ages of 10 and 12, a certain intuition of the iterations of *convergent* **type** and their entailments emerges, remaining very labile afterwards such that it is very much influenced by the figural and conceptual context. Thus, 10-year–old high performers, and 12- and 14-year–old students gave different answers with different arguments to isomorphic situations in which the context had been changed (e.g., distance to be covered). Contrary to what Piaget has said (1948) and according to our observations, subdivision is not mastered at the age of formal operations.

Finally, it is worth mentioning that these observations are based on a specific adaptation of Zeno's paradox and the coordination of heterogeneous components (different **type** and **nature**). It would be interesting to study similar situations controlling certain contextual factors that might play a role in the cognitive activity due to the labile intuitions of the subjects regarding infinity in the small: (a) Considering qualitatively different distances d (e.g., between the sides of the table, between two places in a building, in a city, in a country, etc.); (b) Considering values other than 1 and 2 for p and q, respectively, in the series $\sum_{k=1}^{\infty} [p(q-p)^{k-1}/q^k]d$, such that the iteration is not only performed by halving (e.g., also 1/10 or 3/4); (c) Presenting the problem in a regressive manner as well (like Zeno's paradox of movement), and not only as a progressive one as studied here; (d) Focusing the question not only on the series $\sum_{k=1}^{\infty} d/2^k = d$ ("Do we arrive?") but also on $\lim_{n\to\infty}(d/2^k) = 0$ (the elements of the sequence: "What happens with the steps?") The study of the role played by these factors might reveal more about how our minds build the idea of subdivision and infinity in the small.

[5] Rigor is consensual. History of Calculus shows that rigor had different meaning for the Greeks (Eudoxe, Archimedes), the mathematicians of the 17th century (Leibniz, Newton), 19th century (Weierstrass) and 20th century (Robinson).

References

Lakoff, G. (1987). *Women, Fire, and Dangerous Things: What Categories Reveal About the Mind.* Chicago: University Press.

Langford, P.E. (1974). Development of Concepts of Infinity and Limit in Mathematics. *Archives de Psychologie, 42* (167-168), 311-322.

Maturana, H. and Varela, F. (1987). *The Tree of Knowledge: The Biological Roots of Human Understanding.* Boston: New Science Library.

Núñez, R. (1993a). *En Deçà du Transfini: Aspects Psychocognitifs sous-jacents au Concept d'Infini en Mathémathiques.* Fribourg: University Press.

Núñez, R. (1993b). Big and Small Infinities: Psycho-cognitive Aspects. In *Proceedings of the 17th International Conference Psychology of Mathematics Education, (Tsukuba, Japan),* Vol. II, pp. 121-128.

Piaget, J. & Inhelder, B. (1948). *La Représentation de l'Espace chez l'Enfant.* Paris: P.U.F.

Taback, S. (1975). The child's concept of limit. In Rosskopf, M. F. (Ed.), Children's Mathematical Concepts: *Six Piagetian Studies in Mathematics Education* (pp. 111-144). New York: Teachers College Press.

Tall, D. O. (1980). The notion of infinite measuring number and its relevance in the intuition of infinity. *Educational Studies in Mathematics, 11,* 271-284.