

UC San Diego

Scripps Institution of Oceanography Technical Report

Title

Recapitulation of the Research Leading to the Treatise on "Physical Hydrodynamics with Applications to Dynamical Meteorology" as recorded by V. Bjerknes, 1933

Permalink

<https://escholarship.org/uc/item/0kf7q43b>

Author

Berger, Wolfgang H

Publication Date

2007-01-24

Recapitulation of the Research Leading to the Treatise on “Physical Hydrodynamics with Applications to Dynamical Meteorology” as recorded by V. Bjerknes, 1933

Wolfgang H. Berger

Vilhelm Frimann Koren Bjerknes, Norwegian physicist (1862-1951), was a central figure in the pioneering stage of modern meteorology and oceanography. His meteorological theories (especially with regard to cyclogenesis along the polar front) helped establish the basis for weather forecasting in temperate and high latitudes. His mathematical treatment of water motions in the sea helped establish the foundations for dynamic oceanography.

V. Bjerknes was the leading figure of the “Bergen School” of the physics of atmosphere and ocean, early in the 20th century. The central goal of the “school” was to bring the principles of hydrodynamics and thermodynamics into the daily business of meteorology, which was focused on weather prediction. The most important result was the elucidation of the processes associated with frontal development and the genesis and evolution of storm systems along the polar front. As V. Bjerknes points out in the essay that is the subject of this entry, the circumstances in Bergen, downwind from a major region of cyclogenesis, were uniquely suited to this purpose.

The Bergen methodology soon moved into oceanography, where it provided the foundations for dynamical treatment of currents and eddies. A central figure in this process of conceptual “technology transfer” from meteorology to oceanography (and also from northern Europe to the U.S.) was Harald U. Sverdrup (1888-1957), erstwhile assistant of V. Bjerknes, and subsequently renowned polar explorer, world-leader in oceanography, and Director of Scripps Institution of Oceanography (1936-1948). It seems appropriate to reflect on the contributions of the “Bergen School” both in the context of the opening of the new Bjerknes Centre for Climate Research in Bergen (Dec 2002) and the centennial birthday of Scripps Institution of Oceanography (2003). A recent article by Sigbjørn Grønås (2005) celebrates the contributions of Vilhelm Bjerknes and provides much additional background on his life and work (In: H. Drange et al., eds., 2005, *The Nordic Seas*, AGU Geophys. Monogr. 158, 357-366.)

What follows is an essay by Vilhelm Bjerknes, translated from the German text by Wolfgang H. Berger, Scripps Institution of Oceanography, UCSD, La Jolla, California, and presented with the approval of Springer-Verlag, the successor to the original publisher of the book by V. Bjerknes and associates, Julius Springer, Berlin.

In V. Bjerknes’ original essay, references are given in the text and in footnotes. Here I have moved references to the reference section, with the exception of the introductory listing of the papers of C.A. Bjerknes, and one other, cited in passing. Notes to “Sections” and “Chapters” are in the original format and refer to the main four-author treatise to which Bjerknes’ essay is appended. My notes within the text are in brackets. The equations embodying the “circulation theorem” of V. Bjerknes are given after the reference section. They have been taken from the treatise; V. Bjerknes considered them central to his work.

Bibliography with Historical Explanations

by V. Bjerknes

Professor at the University of Oslo and Research Associate of the Carnegie Institute in Washington, D.C.

Citation: V. Bjerknes, 1933, Bibliographie mit historischen Erläuterungen. In: V. Bjerknes, J. Bjerknes, H. Solberg and T. Bergeron, Physikalische Hydrodynamik mit Anwendung auf die Dynamische Meteorologie. Julius Springer, Berlin, pp. 777-790.

This book carries forward from a string of ideas, which has been quietly spun for close to a century with but little interaction with the contemporary sciences. Almost nothing will be found on the subject in textbooks or symposia. Therefore, a bibliography is given here with the aim to illuminate developments with short historical annotations, without attempting comprehensiveness.

As reported in my biography of C. A. Bjerknes (1), the chain of ideas begins in the forties of the last century, when C. A. Bjerknes, still a pupil or beginning student, read Euler's letters to a German princess and as a result acquired a predilection in opposition to the prevailing theory of action at a distance. [C. A. Bjerknes was the father of V. Bjerknes.] However, the exterior circumstances of his life only allowed him much later to begin his planned attack on the problem of action-at-a-distance, and too late to permit him to finish his work. Only preliminary communications and the beginnings of longer treatises are extant from his hand, the historically most important of which are listed below as follows:

(1) C. A. Bjerknes: Om de indre Tilstande i et inkompressibelt ubegrænset Fluidum, hvori en Kugle bevæger sig, idet den forandrer Volum. Christiania (Oslo) Videnskabsselskabs Forhandlinger 1863.

(2) C. A. Bjerknes: Om den samtidige Bevægelse av kugleformige Legemer i et inkompressibelt Fluidum. De skandinaviske Naturforskeres tiende Møde, Christiania 1868.

(3) C. A. Bjerknes: Sur le mouvement simultané de corps sphériques variables dans un fluide infini et incompressible. Premier Mémoire. Christiania Videnskabsselsk. Forhand. 1871.

(4) C. A. Bjerknes: Foreløbige Meddelelser om de Kræfter der opstaa, naar kugleformige Legemer, idet de udføre Dilatations- og Kontraktionssvingninger bevæge sig i et inkompressibelt Fluidum. Ibid. 1875.

(5) C. A. Bjerknes: Hydrodynamiske Analogier til de statisk elektriske og de magnetiske Kræfter. "Naturen", Christiania 1880.

(These works are all accessible in “Ostwalds Klassiker der exakten Naturwissenschaften” Nr. 195, in German translation: “Hydrodynamische Fernkräfte von C. A. Bjerknes”, Leipzig 1915.)

(6) C. A. Bjerknes: Über die Druckkräfte, die durch gleichzeitige, mit Contractionen und Dilationen verbundene Bewegungen von mehreren kugelförmigen in einer inkompressiblen Flüssigkeit befindlichen Körpern entstehen. Erster Aufsatz. Göttinger Nachrichten 1876.

A systematic account of the results of C. A. Bjerknes first appeared in a series of lectures published in the years 1900 to 1902 (2). However, in that book only results concerning spherical bodies are treated. What was available on cylindrical bodies, in addition, was disregarded for reasons that are given in the biography (1).

The method of C. A. Bjerknes consisted of an explicit solution of the problems concerning the motion of spherical, later also cylindrical, bodies in a homogenous and incompressible fluid, as well as a subsequent discussion of the solutions found, from his special point of view. However, the direct-geometric and inverse-dynamical analogy of the hydrodynamic phenomena with the electric and magnetic ones, as found by him, was only proven in this manner for bodies of this certain form. While working on the book of lectures on the theory of C. A. Bjerknes (in the years 1893-1901), I looked for more general methods to carry out this type of investigation. I suspected that it would be profitable to this end to replace the extraneous bodies in the shapes of spheres or cylinders with *fluid bodies*. To preserve the analogy [for the cases illustrated in the main treatise, concerning rotating fluid bodies within a fluid of different density] (Figs. 15 and 16, p. 142), I concluded that for the vector of specific motion magnitude in the transition layer between ambient fluid and fluid body there had to be a *tendency for eddy formation*. Further analytical investigation led me immediately to the equation describing eddy formation (Section 40, Eq. 5B, p. 138) [see Appendix to this translation, (a)]. Later there emerged automatically the parallel and much simpler equation for eddies concerning velocity only (Section 40, Eq. 5A or 7A) [Appendix, (b, c)]. These rules concerning eddy formation pertain to perfect fluids of the most general type. Thus, the field of *physical hydrodynamics* had been entered. I formulated the equations in March 1897 and presented them in my lectures at the University of Stockholm on the 9th and 10th April of the same year. Following subsequent investigation of the twofold lamellar vectors, which appear in all applications of the equations, I published two linked accounts on the matter (3,4).

From the beginning, I was greatly surprised that even the extremely simple equation describing velocity eddies was not already known for a long time. That equation is obviously as readily accessible by intuition as it is difficult to avoid analytically when deriving the Helmholtz conservation rule, either in the original form or in the later form provided by Kelvin. For the direct path to the conservation equations proceeds via the general eddy formation equation. Whether one takes notice of it or not depends only on whether one introduces the limiting barotropic condition somewhat later or somewhat

earlier. Nevertheless, as far as I could ascertain, only one author has earlier drawn attention to this formulation (5). Remarkably, he did so without making any separate application of the equation, apart from an obvious return to the conservation formula. Even more surprising, it seems, is the fact that even now, after more than thirty years, one unfailingly finds the conservation equations in the hydrodynamic textbooks and only rarely the eddy formation equations. There seems to have been a general reluctance to go beyond the limits of classical hydrodynamics.

Generalized theory of the hydrodynamic force field phenomena

The circulation theorem for the specific motion magnitude permitted important inferences, but did not yield a complete theory of the hydrodynamic force field phenomena. In three articles published between 1904 and 1906 (6, 7, 8) it was possible to give the transformation of the hydrodynamic equations of a fluid system of greatest generality, and which leads to the *first* series of hydroelectric or hydromagnetic phenomena (Chapter 5). (An alternative method for deriving very general results about hydrodynamic forces acting at a distance is given by Mr. Almanzi: “Sulle attrazione Newtoniana di origine idrodinamica.” Rendiconti dei Lincei, Roma 1914.)

When beginning the series of lectures at the University of Columbia (8) I still had the erroneous notion that there is only *one* series of hydroelectric or hydromagnetic phenomena. And I intended to apply the very transformation developed for the first series also to the special case of permanent motion. But to my surprise I realized that this led into contradiction, and that there should exist an independent *second* series of hydroelectric and hydromagnetic phenomena with a very different assignment of hydrodynamic variables to the electric or magnetic ones. One of the former attendants of my lectures, Prof. A. P. Wills, took on the task to search for the new transformation (given now in Chapter VI), which leads to the second series of these phenomena (9). Subsequently I gave a review of the two analogies in a book published in 1909 (10). The possibility to search for new electromagnetic phenomena on the basis of the hydrodynamic analogy is discussed in a separate report (11).

Technical applications of hydrodynamics were never within my sphere of interests, and neither in that of my father. As a consequence the technical-hydrodynamic literature blooming since the beginnings of flying was entirely unknown to me. The conference of hydrodynamicists in Innsbruck in 1922, in which I participated, had a great surprise in store for me: I recognized in the forces fundamental to wing lift dynamics familiar patterns from the theory of hydroelectric and hydromagnetic phenomena. This resulted in an article published for the conference (12) as well as other explanations [summarized in the treatise] (Section 73, pp. 263-275).

The direction of the geophysical and cosmic-physical tasks ahead

The wide-ranging applicability of the circulation equations is largely due to their general validity, which is independent of the manner in which the solenoid systems are formed that determine the circulation accelerations. In the atmosphere, temperature and humidity

can be the cause, in the sea temperature and salinity. Everything points to a discussion of the great atmospheric and oceanic circulation motions. The beginning was made in these articles published between 1898 and 1902 (13, 14, 15).

In those years, I had many contacts with Fridtjof Nansen, who was then working on the scientific results of his North-Polar Expedition. During the discussion of his very puzzling experiences with “dead water” I expressed the conjecture that internal gravity waves might be involved, and at Nansen’s suggestion I asked V. Walfrid Ekman, then my student at the University of Stockholm, to investigate the phenomenon theoretically and experimentally. The results aroused in me a great interest for discontinuity surfaces within the atmosphere and within the ocean, and their possible wave motions, a fact that had a positive effect on our subsequent studies. — During a visit of Nansen in Stockholm, in 1900, the wind-driven currents observed during his expedition were the subject of discussion (16). I suggested to have Ekman investigate these theoretically, likewise, in some detail. He was called, received Nansen’s exposition of the problem, and already by the same evening had found the current distribution according to the *Ekman Spiral* (p. 496), which was to prove fundamental in both oceanography and meteorology.

I would hardly have occupied myself subsequently with the sciences of oceanography and meteorology, unfamiliar to me, if it had not been for J. W. Sandström, then another of my students at the University of Stockholm. He was ready to pursue practical studies in this direction and to work out the numeric and graphical auxiliary methods for the purpose. I mention from this time especially four articles from his pen, in which he made in fact the first diagnostic applications of the circulation theorem (17-20).

Following these studies, we planned to produce a book on the application of the circulation theorem in meteorology and hydrography. For this book Sandström calculated extensive tables for the quantitative determination of the solenoid systems in the sea according to hydrographic observations, and for the atmosphere according to the aerologic observations then emerging in an organized fashion.

But before the work was completed there arose the need for an extraordinary expansion of our task. I was becoming convinced that dynamical meteorology in the end only offers a single problem worth solving, but which at the same time includes all others: the task to predict the future conditions of the atmosphere. In this I did not think of the rushed efforts of the practical weather prediction, but of a discussion of atmospheric processes based on the equations of hydrodynamics and thermodynamics. The problem of weather prediction was to be seen as a mathematically determined problem: At a given point in time the initial condition of the atmosphere was to be documented as completely and transparently as possible, based on a sufficient number of observations – the problem of *diagnosis* – and then the *main problem of prognosis* was to be solved through integration of the equations of hydrodynamics and thermodynamics, if necessary by numerical methods [“mechanische Quadratur”] if analytical methods proved not feasible. The numerical procedure can be visualized as following a simple program whereby the task is split into two partial tasks, one concerning dynamics and the other thermodynamics, and which are carried out for appropriately short time steps in sequence:

(1) The dynamical partial task: *to calculate the change in position of the moving air masses for a short time span.*

(2) The thermodynamic partial task: *to calculate the conditions in which the moving air masses arrive at their new positions.*

This plan was developed in an article published in 1904 (21). It is true I had little hope in solving the task in a definitively satisfactory manner, but the program yielded a research plan that showed a secure path to follow. Mr. Sandström found the task both attractive and demanding; the work was begun, first with the introductory and absolutely fundamental problem of *diagnosis*, for which the system of tables produced by Sandström immediately proved of great value. Above all, we applied ourselves to the task to produce diagnoses of real-life atmospheric conditions, as complete as possible, and based on the international aerologic ascents then organized by Hergesell. First results were published in 1906 (22).

When the Carnegie Institution of Washington [D.C.] promised support for our further work, we planned for a substantial treatise (23), in which both for atmosphere and ocean the problems of diagnosis and prognosis would be treated in as complete a fashion as possible. Two introductory volumes, both dedicated to the diagnosis problem, were soon completed (24, 25). The third volume, “Dynamics”, which is to conclude the series treating both the diagnosis and the prognosis problems is still in preparation.

The circumstances for working on the problems of this third volume appeared very favorable indeed with my taking on a professorship at the University of Leipzig, early in 1913, with the assignment to establish a geophysical institute for precisely this research area.

The institute was readily organized with the active help from the first Assistant Dr. R. Wenger and immediately began producing publications in this field of research (26, 27, 28). By the means of seminars, we attempted to gain an overview of the earlier works on the great meteorological problems and to stay current with the new literature. We then considered, as was common practice, that the theories of Doves were out of date and done for, especially owing to the synoptic charts. These gave a false appearance of continuity, as we now know, because of the insufficient resolution of the network of observations. The name of Blasius was unknown to us and remained so. Our attention was drawn at that time to the features now known as “polar fronts”, drawn by v. Ficker in 1910 (29). In view of the plethora of the daily weather maps, all of which appeared to indicate continuity, we could only see interesting exceptions to the rule even in these signs of discontinuity, and we made no attempt to pursue this important lead. In contrast, important consequences arose from the fact that the meteorological work of Helmholtz from the year 1888 (30) was presented in the seminars, not once but several times. Using a theoretical approach, Helmholtz here arrives at the concept of a “polar front” running parallel to latitude, and concludes the essay with vague statements that could be interpreted, when read now, as though he had envisioned the wave theory of cyclones. Considering that elsewhere (and also specifically in the article under discussion) he always refers to cyclones and anticyclones as independent formations, without tying them

to wave motion and polar front, and that he only provides for application of wave motions to wavy clouds and small-scale meteorology in his subsequent important works on wave motions, but does not seek application to the large atmospheric disturbances, I reserve judgment on whether such a reading is warranted. But whatever the case may be, there is no question that these statements [of Helmholtz] had great importance for our further work, as will be presently elucidated.

The Series II of the publications of the Institute contains excursions into various directions whose aim was to clarify which paths might lead forward. Contributors were Th. Hesselberg and H. U. Sverdrup, then my assistants at Carnegie, R. Wenger, the First Assistant at the Institute in Leipzig (and my successor there; he died in 1922), and A. Friedmann (who died in 1925, while Director of the Geophysical Central Institute in Leningrad). I mention especially the work of Hesselberg and Sverdrup on friction, which even then was expanded into applications by H. Solberg, but which led to preliminary publication only much later (31). Only in the present treatise are these results published completely (Sections 134 and 135). Furthermore [I mention] the study by Sverdrup on the North Atlantic trade winds, and the various works of Hesselberg, which led me to produce the article on a prognostic principle of dynamical meteorology (32), the main results of which are given in Section 136. Contemporaneous studies by Sandström (33), which had been the subject of harsh but unjustified criticism in Sweden, caused me to write the article on thermodynamic machines subject to gravity (34).

In some ways the most important technical paper of that time was the article on wave motions in compressible heavy fluids (35), not because of the results per se, but because of the new direction it led into. The above-mentioned statement of Helmholtz led me to consider that atmospheric disturbances, as all disturbances, necessarily start as small disturbances and thus can be studied using *linearized equations*. However, linearized equations appear to yield, without exception, solutions that describe *wave motions*, both stable and unstable. *Consequently* [it seemed to me that] *also atmospheric disturbances, the cyclones, must begin as waves*. I did not yet have the slightest idea how to visualize the transition from the not yet observed wave to the well-known cyclonic eddy. However, it should be possible to find the connection if one were to develop, from the theoretical side, the theory of waves in the most general manner possible, and at the same time, from the empirical side, investigate the structure and history of development of the cyclones in as much detail as possible. The treatment in (35) initiated this theoretical work. However, the difficult circumstances arising from the war [WW I] interrupted the work; a second communication, which already incorporated various results here presented in Chap. VIII, remained unpublished, as did several initial results on the effects of the Earth's rotation on the wave motion.

The main task of the Institute, in any case, was to work on the synoptic representation of atmospheric conditions as published in the Series I of the Reports of the Institute. The basis [for these studies] was provided by the aerologic ascents, internationally organized by Hergesell, and to which the newly developed methods of diagnosis were applied. We hoped to be able to elucidate the processes in the atmosphere through the use of series of sequential complete diagnoses, and then to apply the dynamic-thermodynamic methods

of prognosis. In this manner the nature of the cyclones had to emerge eventually, whether they be wave-like or not.

We had progressed with this comprehensive project just far enough to feel that everything was moving forward very well, and we were at the point of expanding the work in breadth and depth to be able to tackle the main problems in earnest, when the war made all expansion of the work impossible. Much of the staff of the Institute were drafted; of ten doctoral students five were killed. With the help of two female doctoral students and the two Norwegian Carnegie assistants it was possible to work out some additional synoptic charts in abbreviated form without the planned expansions. Regarding the mathematical tasks that opened up so richly in connection with the wave problems: the work was brought to a complete halt by missing mathematical assistance.

In these circumstances, and without yet having produced results of some importance in any of the main goals, in 1917 I moved to Bergen on the west coast of Norway together with J. Bjerknes and H. Solberg, then my Carnegie assistants. Fortunate circumstances made it possible to resume work here, although now in a considerably different manner, concentrating on practical weather prediction rather than on theoretical prognosis. During the war, the Norwegian Navy had in operation a dense series of observational stations all along the coast, and this series became the basis for a promptly improvised meteorological observation network. We had to do without aerologic observations. In compensation, the observation network had the density required by theory. However, the two assistants who had come along from Leipzig, soon to be joined by T. Bergeron as equally worthy collaborator, were used to think in three dimensions, even when observations were available only from the Earth's surface. To make up for the missing aerologic ascents we developed an *indirect aerology*: we succeeded to derive extensive conclusions about the conditions and processes in the free atmosphere from the observations at the base, and thus a series of important results emerged from the studies of the young meteorologists, now working in a practical and empirical fashion.

A phenomenon that had already attracted our attention during the Leipzig period was the appearance of *lines of convergence*, which emerged as geometrically necessary consequences on the synoptic charts of streamline contours for the Earth's surface. From the ascent data we had not been able determine how they continued at higher elevations. The first doctoral student who had joined the Institute, Mr. H. Pezold, had received the task to investigate the lines of convergence in greater detail. He had already collected some relevant material, and had chosen a particular example, where at the Earth's surface a moving temperature discontinuity accompanied a moving wind-field discontinuity. He intended to study the phenomenon following the pattern of the works of Köppen or Durand-Gréville, but according to the expanded three-dimensional program that had become possible thanks to the improved aerologic data. He was just starting the project when he was drafted for war service. The investigation had to be suspended. He lost his life in 1916 at Verdun. J. Bjerknes then took up his work, and wrote already in 1917 a short notice concerning these lines of convergence.

Thanks to the dense network of observations and to the indirect aerology we succeeded in Bergen to move from the *geometric* concept of lines of convergence to the *physical* concept of the *fronts*, that is, *when the field of air motion is drawn relative to the moving front, the geometric line of convergence becomes congruent with the physical front*. The front is the line of intersection between a moving inclined atmospheric boundary – the frontal plane – and the Earth's surface. There is no significant mixing across the frontal plane; thus, it separates *air masses*, which differ in temperature and humidity, and content of dust etc.

By precisely tracking these fronts from one observational time to the next we succeeded to track the movement of physically differing air masses. This became of crucial importance for the further development of the methods for practical weather prediction. The work on prognosis gradually took on the following form: 1. *The determination of the displacements of the physically differing air masses*; 2. *the determination of the conditions which describe the state of the air masses in their new location*. This was no less than the realization of the program for weather prediction proposed in 1904, founded upon the principles of mechanics and physics (21), with the exception that for the daily weather service rapid estimates must replace the more complex calculations. By now everything is sufficiently worked out so that in time the rather crude estimates can be replaced by more and more precise calculations, the more so since in this area there has appeared an extraordinarily remarkable study by L. F. Richardson (36).

The daily work on prediction according to the new methods further led to the fundamental result that most of the newly formed cyclones originate somewhere along a front. A wandering bulge forms thereby at the respective front, the point of which constitutes the cyclone center. This can only mean that the concerned part of the discontinuity surface has entered wave motion. Thus, *the type of waves that generate cyclones was now found, and the progressive development from wave to eddy motion could be tracked*.

The intense demands of the weather service left but little time for the young meteorologists to prepare their findings for publication. The most important empirical results of the first years [in Bergen] were presented between 1919 and 1922 (37, 38, 39). These results are summarized in Chapters XVIII and XIX of the present treatise, albeit with elaborations and corrections.

On these empirical studies there followed a theoretical one of my own (40), from which is taken a large portion of the material for Chapter s IV and XII, including the fundamentals of the wave theory of the cyclones.

From now on there arose in the Bergen School a division of labor, focused on empirical and theoretical work, respectively. The leading empiricists stayed in Bergen, where natural conditions had proved exceptionally favorably for their research [cyclones are common], but the theoreticians work since 1926 at the University of Oslo.

The most important empirical studies moving the science forward are [several works of J. Bjerknes and T. Bergeron, published between 1924 and 1932] (41-46). Their results also are presented in the Chapters XVIII and XIX. The articles (38), (43) and (45) contain forerunners to the cellular planetary circulation system, which appears in the present treatise for the first time, constructed by J. Bjerknes (Section 173).

After the elaboration of the prognosis methods in the sense of the program of 1904 could be viewed as solved in preliminary fashion, theoreticians focused on the main goal of moving the solution of the cyclone problem forward as far as possible. Gradually it became clear that a systematic reworking of the general fundamentals of physical hydrodynamics had to occur before this problem could be tackled efficiently. Contributions from myself [in this area] [included four papers on wave motion, solar hydrodynamics, atmospheric disturbance equations and the linearisation of Lagrangean and Eulerean hydrodynamic equations] (47-50).

The quasi-static method used in (47) appears in Chapter X in a greatly improved form, provided by Solberg.

The article (48) was a work of opportunity. During 1924, when I lectured at the California Institute of Technology on physical hydrodynamics (through the initiative of Hale and Millikan), I had plenty of opportunity to discuss phenomena on the Sun with the astronomers of the Mt. Wilson Observatory. The article was published as a first attempt to attack physical-hydrodynamical problems of cosmic physics. I did not consider the elastoid stability (see Section 46, Chapter XI) of rotating fluid masses in the article, as my attention was drawn to this subject only much later. Another missing item making the article less than complete is the fact that one cannot find on the Sun (or on Earth) a pressure distribution that could maintain a permanent meridional circulation symmetric about the rotation axis. Consequently there has to arise a circulation broken up into cells, similar to the one postulated by J. Bjerknes for the general circulation of the terrestrial atmosphere (Section 173), only that on the Sun one must find cyclonic motion where on Earth one finds anticyclones, and vice versa. The fact that the underlying general circulation becomes somewhat more complicated, however, does little to change the assumptions for the attempt made in the article, to explain a number of sunspot phenomena from a common [hydrodynamic] point of view.

The articles (49) and (50) are the basis for Chapters III and VII.

Solberg, through his integrative efforts, achieved important results in his article [on atmospheric disturbance equations] (51). Here, for the first time, appear the *cellular inertial waves*, albeit in a somewhat veiled form owing to the way the problem is posed. From the existence of these waves, Solberg deduced that the earlier theories of tides (Section 122) are incomplete, and also that it will be impossible to solve the problem of cyclones using the extant tidal theories for template, that is, according to the quasi-static method (as we had initially hoped). However, the formal solution of the disturbance problem, which is the focus of the article, has the form of an integral that is difficult to

discuss, and the solutions provided by elementary functions did not yet show the characteristics of cyclonic waves (Section 141).

Subsequently, the nature of the cellular internal waves was clarified in the article by V. Bjerknes and H. Solberg, published in 1929 (52), and Solberg expanded his integrations solvable with elementary functions to the case of compressible isothermal layers, with the result that waves similar to cyclones emerged (53). The final result of the various mathematic Investigations on waves is presented in the Chapters VIII-XI, XIII, XIV. In this final work, the assistance of Mr. C. L. Godskes was invaluable, as already stated in the preface [to the treatise].

Originally all the articles published since the year 1912 were planned to serve as preparatory to the Volume III of the treatise "Dynamic Meteorology and Hydrography." However, they go well beyond the scope of the present volume. Therefore, in the present treatise, there appears everything that could be identified as "physical hydrodynamics"; complemented by the more obvious applications and findings in the field of meteorology, for the sake of association, but without becoming engaged with the adjoining technical meteorological methods of analysis of weather maps and of weather prediction.

References

- (1) V. Bjerknes, 1933. C. A. Bjerknes, sein Leben und seine Arbeit. Julius Springer, Berlin.
- (2) V. Bjerknes, 1900-02. Vorlesungen über hydrodynamische Fernkräfte nach C. A. Bjerknes' Theorie. Band I, Theorie, Band II, Versuche.
- (3) V. Bjerknes, 1898. Zur Theorie gewisser Vektorgroßen. Videnskabselskabets Skrifter, Kristiania (Oslo).
- (4) V. Bjerknes, 1898. Über die Bildung von Zirkulationsbewegungen und Wirbeln in reibungslosen Flüssigkeiten. Videnskabselskabets Skrifter, Kristiania (Oslo).
- (5) L. Silberstein, 1896. Über die Entstehung von Wirbelbewegungen in einer reibungslosen Flüssigkeit. Berichte der Universität Krakau.
- (6) V. Bjerknes, 1904. Om en speciel form av de hydrodynamiske bevægelsesligninger. Videnskabselskabets Forhandling, Kristiania.
- (7) V. Bjerknes, 1905. Recherches sur les champs de force hydrodynamiques. Acta Mathematica, Bd. 30, Stockholm.
- (8) V. Bjerknes, 1906. Fields of Force. Lectures given at Columbia University, 1905. New York.

- (9) A. P. Wills, 1906. A variation of the Bjerknes analogy between hydrodynamic and electric or magnetic fields. *Physical Review*, v. 26, New York.
- (10) V. Bjerknes, 1909. Die Kraftfelder. Serie "Die Wissenschaft", Heft 28. Braunschweig.
- (11) V. Bjerknes, 1909. On the absolute measurement of electromagnetic quantities. *Phys. Review*, New York. (French: *Journal de Physique*, Paris 1909.)
- (12) V. Bjerknes, 1922. Zur Berechnung der auf Tragflächen wirkenden Kräfte. In: Kármán and Levi-Civita, "Vorträge aus dem Gebiet der Hydro- und Aerodynamik. Innsbruck, 1922.
- (13) V. Bjerknes, 1898. Über einen hydrodynamischen Fundamentalsatz und seine Anwendung besonders auf die Mechanik der Atmosphäre und des Weltmeeres. *Vetenskapsakademiens Handlingar*, Stockholm.
- (14) V. Bjerknes, 1900. Das dynamische Prinzip der Zirkulationsbewegungen in der Atmosphäre. *Meteorol. Zeitschrift*.
- (15) V. Bjerknes, 1901. Zirkulation relativ zu der Erde. Öfversikt av *Vetenskapsadademiens Förhandlingar*, Stockholm 1901. *Meteorol. Zeitschrift* 1902.
- (16) Fridtjof Nansen, 1902. The oceanography of the North Polar Basin. The Norwegian North Polar Expedition, vol. III, Kristiania (Oslo), p. 369.
- (17) J. W. Sandström, 1901. Über die Beziehung zwischen Temperatur und Luftbewegung in der Atmosphäre unter stationären Verhältnissen. *Öf. Svensk. Vet. Ak. Förh.* 58, Stockholm.
- (18) J. W. Sandström, 1902. Über die Beziehung zwischen Luftbewegung und Druck in der Atmosphäre unter stationären Verhältnissen. *Ibid.* 59.
- (19) J. W. Sandström, 1902. Anwendung von Prof. Bjerknes' Theorie. *Svensk. Vet. Ak. Handl.* 33, Stockholm.
- (20) J. W. Sandström, 1907. Über die Temperaturverteilung in den allerhöchsten Luftschichten. *Ark. för Mat., Astr. och Fysik*, 3, Stockholm.
- (21) V. Bjerknes, 1904. Das Problem von der Wettervorhersage, betrachtet vom Standpunkt der Mechanik und der Physik. *Meteorol. Zeitschrift*.
- (22) V. Bjerknes and J. W. Sandström, 1906. Hilfsgrößen zur Berechnung der Druckverteilung in der Atmosphäre an den internationalen Tagen 1900-1903. *Beiträge zur Physik der freien Atmosphäre*.

- (23) V. Bjerknes and various collaborators: Dynamic meteorology and hydrography.
- (24) V. Bjerknes and J. W. Sandström, 1910. Part I. Statics. Washington, D.C., 1910. German edition, Braunschweig 1911.
- (25) V. Bjerknes, Th. Hesselberg, O. Devik, 1911. Part II. Kinematics. German edition, Braunschweig 1912.
- (26) “Veröffentlichungen des Geophysikalischen Instituts der Universität Leipzig.”
- (27) First series: Synoptische Darstellungen atmosphärischer Zustände über Europa (10 issues published, 1913-1920).
- (28) Second series: Spezialarbeiten aus dem Geophysikalischen Institut (still being published).
- (29) von Ficker, 1910. Die Ausbreitung kalter Luft in Rußland und Nordasien. Sitzungsberichte der Wiener Akademie 119, Wien 1910.
- (30) H. Helmholtz, 1888. Über atmosphärische Bewegungen. Berl. Sitzungsber. 1888, Werke Bd. III, p. 289.
- (31) H. Solberg, 1923. Sur le frottement dans les couches basses de l’atmosphère. Förh. 17de skand. naturforskarmöte. Göteborg.
- (32) V. Bjerknes, 1917. Ein prognostisches Prinzip der dynamischen Meteorologie. Berichte der math.-phys. Klasse der sächsischen Gesellsch. der Wiss. Leipzig 1917.
- (33) J. W. Sandström, 1916. Meteorologische Studien im Schwedischen Hochgebirge. Göteborgs Kungl. Vetenskaps- och Vitterhetssamhälles Handlingar. Göteborg.
- (34) V. Bjerknes, 1916. Über thermodynamische Maschinen, die unter der Mitwirkung der Schwerkraft arbeiten. Abhandl. der math.-phys. Klasse der sächsischen Gesellsch. der Wiss. Leipzig 1916.
- (35) V. Bjerknes, 1916. Wellenbewegungen in kompressiblen schweren Flüssigkeiten. Erste Mitteilung. Ibid. Leipzig 1916.
- (36) L. F. Richardson, 1922. Weather Prediction by Numerical Process. Cambridge.
- (37) J. Bjerknes, 1919. On the structure of moving cyclones. Geofysiske Publikationer, Vol I, No. 2. Kristiania.
- (38) J. Bjerknes and H. Solberg, 1921. Meteorological conditions for the formation of rain. Ibid. Vol. II, No. 3.

- (39) J. Bjerknes and H. Solberg, 1922. Life cycle of cyclones and the polar front theory of atmospheric circulation. Ibid. Vol. III. No. 1.
- (40) V. Bjerknes, 1921. On the dynamics of the circular vortex with application to the atmosphere and atmospheric vortex- and wavemotions. Ibid. Vol. II, No. 4.
- (41) J. Bjerknes, 1924. Diagnostic and prognostic application of mountain observations. Ibid. Vol. III, No. 6.
- (42) T. Bergeron and G. Swoboda, 1924. Wellen und Wirbel an einer quasi-stationären Grenzfläche über Europa. Veröff. des Geophys. Inst. Leipzig.
- (43) T. Bergeron, 1928. Über die dreidimensional verknüpfende Wetteranalyse. Erster Teil. Geof. Publ. Vol. V, No. 6, Oslo.
- (44) J. Bjerknes, 1930. Practical examples of polar-front analysis over the British Isles in 1925-1926. Geophysical Memoirs No. 50. London.
- (45) T. Bergeron, 1930. Richtlinien einer dynamischen Klimatologie. Meteor. Zeitschr. 1930, p. 246.
- (46) J. Bjerknes, 1932. Exploration de quelques perturbations atmosphériques à l'aide de sondages rapprochés dans le temps. Geof. Publ. Vol. IX, No. 9.
- (47) V. Bjerknes, 1923. On quasi-static wave motion in barotropic fluid strata. Geof. Publ. Vol. III, No. 3, Oslo.
- (48) V. Bjerknes, 1926. Solar hydrodynamics. Astrophysical Journal, Chicago.
- (49) V. Bjerknes, 1926. Die atmosphärischen Störungsgleichungen. Beitr. zur Physik der freien Atmosphäre, Bd. XIII.
- (50) V. Bjerknes, 1929. Über die hydrodynamischen Gleichungen in Lagrangescher und Eulerscher Form, und ihre Linearisierung für das Studium kleiner Störungen. Geof. Publ. Vol. V, No. 11, Oslo.
- (51) H. Solberg, 1928. Integrationen der atmosphärischen Störungsgleichungen. Geof. Publ. Vol. V, No. 9, Oslo.
- (52) V. Bjerknes and H. Solberg, 1929. Zellulare Trägheitswellen und Turbulenz. Videnskabsakademiets avhandlingar, I, No. 7, Oslo.
- (53) H. Solberg, 1930. Das Zyklonenproblem. Verh. III Internat. Kongreß für techn. Mechanik, Stockholm 1930.

Equations referred to in text

(a) $dC/dt =$ three terms with integrals see p. 138 5B

(b) $dC/dt =$ one term concerning velocity only see p. 138 5A

(c) equation 7a. p. 139.

Below: facsimile of the pages where these equations are discussed.

Appendix: pages 137-139 of Bjerknes et al. 1933; referred to in the autobiographical chapter “Bibliography with Historial Explanations.”

Gleichwertige und für viele Zwecke bequemere Formen dieser Sätze ergeben sich, wenn wir aus den Gleichungen (1) Linienintegrale für geschlossene, bewegte materielle Kurven bilden. Durch skalare Multiplikation der Gleichungen (1) mit dem Vektorlinienelemente $\delta \mathbf{r}$ einer solchen Kurve und Integration längs der ganzen Kurve ergibt sich:

$$(3A) \quad \left\{ \begin{array}{l} \int \dot{\mathbf{u}} \cdot \delta \mathbf{r} = - \int s V p \cdot \delta \mathbf{r} \\ \quad \quad \quad = N(s, -p) , \end{array} \right. \quad (3B) \quad \left\{ \begin{array}{l} \int q \dot{\mathbf{u}} \cdot \delta \mathbf{r} = - \int q V \varphi \cdot \delta \mathbf{r} \\ \quad \quad \quad = N(q, -\varphi) . \end{array} \right.$$

[137]

wo wir in der zweiten Zeile die Linienintegrale der zweifach lamellaren Vektoren $s \nabla p$ und $q \nabla \varphi$ nach 5 (14) durch Solenoidzahlen ausgedrückt haben.

Die Gleichungen (3) geben nicht wesentlich mehr als das schon in den Gleichungen (2) Enthaltene. Weil aber die Kurve eine materiell bewegte Kurve ist, können wir die Integrale links nach den Formeln 19 (12) und (19) umformen. Indem wir uns der Definitionen $\bar{\mathbf{v}} = q \mathbf{v}$ und

$$(4A) \quad C = \int \mathbf{v} \cdot \delta \mathbf{r}, \quad (4B) \quad \bar{C} = \int \bar{\mathbf{v}} \cdot \delta \mathbf{r}$$

erinnern, erhalten wir:

$$(5A) \quad \frac{dC}{dt} = - \int s \nabla p \cdot \delta \mathbf{r}, \quad (5B) \quad \left\{ \begin{array}{l} \frac{d\bar{C}}{dt} = - \int q \nabla \varphi \cdot \delta \mathbf{r} \\ - \int \frac{1}{2} v^2 \nabla q \cdot \delta \mathbf{r} \\ - \int e \bar{\mathbf{v}} \cdot \delta \mathbf{r}, \end{array} \right.$$

wo $e = \text{div } \mathbf{v}$ ist.

Dank der Relation 19 (12) hat der Satz (5A), der sich auf Geschwindigkeiten bezieht, seine einfache Form behalten, während der auf Bewegungsgrößen bezogene Satz (5B) komplizierter ist, da hier drei partielle Zirkulationsbeschleunigungen auftreten, nämlich:

$$(6) \quad \frac{d\bar{C}}{dt} = \left(\frac{d\bar{C}}{dt} \right)_a + \left(\frac{d\bar{C}}{dt} \right)_b + \left(\frac{d\bar{C}}{dt} \right)_c,$$

gegeben durch die Integrale:

$$(6a) \quad \left(\frac{d\bar{C}}{dt} \right)_a = - \int q \nabla \varphi \cdot \delta \mathbf{r} = \int \varphi \nabla q \cdot \delta \mathbf{r},$$

$$(6b) \quad \left(\frac{d\bar{C}}{dt} \right)_b = - \int \frac{1}{2} v^2 \nabla q \cdot \delta \mathbf{r} = \int q \nabla \left(\frac{1}{2} v^2 \right) \cdot \delta \mathbf{r},$$

$$(6c) \quad \left(\frac{d\bar{C}}{dt} \right)_c = - \int e \bar{\mathbf{v}} \cdot \delta \mathbf{r}.$$

Das zweite Integral (6a) und (6b) wird durch eine partielle Integration längs der geschlossenen Kurve erhalten. Die relativen Größenordnungen der beiden Integrale (6a) und (6b) hängen von den Wertunterschieden ab, welche die auf die Masseneinheit bezogene potentielle Energie φ bzw. die kinetische Energie $\frac{1}{2} v^2$ in den verschiedenen Punkten der geschlossenen Kurve aufweisen können. Nun nimmt φ im Schwerfeld für jeden Meter Höhe um $10 \text{ m}^2/\text{sec}^2$, für 1000 m Höhe um $10000 \text{ m}^2/\text{sec}^2$ und für 10000 m Höhe (Höhe der Troposphäre) um $100000 \text{ m}^2/\text{sec}^2$ zu. Gleichzeitig verändert sich die Luftgeschwindigkeit, von ganz bodennahen Schichten abgesehen, auf 100 m Höhe nur selten um mehr als ein paar m/sec. Wenn man von Orkanen und Überorkangeschwindigkeiten absieht (30 m/sec oder mehr), so kommen innerhalb der ganzen Troposphäre keine größeren kinetischen Energien pro

Masseneinheit als $900 \text{ m}^2/\text{sec}^2$ vor. Gewöhnlich wird deshalb (6b) neben (6a) bei atmosphärischen Bewegungen nur sehr wenig und bei ozeanischen Bewegungen noch weniger zu bedeuten haben. Noch bedeutungsloser ist das Integral (6c), das von der Volumänderungsgeschwindigkeit ϵ der bewegten Teilchen abhängt; denn schon in der kompressiblen Luft und noch mehr in dem fast inkompressiblen Meereswasser verlaufen diese Volumänderungen mit äußerster Langsamkeit.

Bei der Diskussion atmosphärischer oder ozeanischer Bewegungen sind wir daher berechtigt, in erster Annäherung von den Gliedern (6b) und (6c) abzusehen. Wir gehen deshalb zu der Diskussion der beiden folgenden formal parallelen Sätze über, von denen der erste exakt und der zweite angenähert gültig ist:

$$(7A) \quad \frac{dC}{dt} = N(s, -p).$$

(A). Die Beschleunigung der Zirkulation C (Geschwindigkeitszirkulation) einer geschlossenen materiellen Kurve ist gleich der Zahl der isobar-isosteren Solenoide $(s, -p)$, welche die Kurve umschließt, und hat die Richtung von dem Volumen-ascendenten ∇s zu dem Druckgradienten $-\nabla p$.

$$(7B) \quad \frac{d\bar{C}}{dt} \approx N(q, -\varphi).$$

(B). Die Beschleunigung der Zirkulation $\bar{C}$ (Bewegungsgrößezirkulation) einer geschlossenen materiellen Kurve ist angenähert gleich der Zahl der äquidens-äquipotentiellen Solenoide $(q, -\varphi)$, welche die Kurve umschließt, und hat die Richtung von dem Dichteascendenten ∇q zu dem Potentialgradienten $-\nabla \varphi$.

Der exakte Satz (A) bezieht sich auf die Zirkulation im gewöhnlichen Sinne des Wortes (Geschwindigkeitszirkulation) und wegen der STOKESschen Identität auch auf das entsprechende Wirbelfeld. (7A) gibt deshalb das allgemeine Gesetz der Wirbelbildung in reibungslosen Flüssigkeiten allgemeinsten Art und enthält als Spezialfall die HELMHOLTZsche Wirbelerhaltung. Diese trifft für die autobarotropen Flüssigkeiten zu, wo die Solenoidzahl innerhalb jeder beliebigen geschlossenen Kurve identisch Null ist, $N(s, -p) = 0$, weil in diesem Falle jede isobare Fläche immer zugleich äquisubstantiell ist. Dann ist:

$$(8) \quad \frac{dC}{dt} = 0, \quad \text{d. h. } C = \text{konst.}$$

D. h. in der autobarotropen Flüssigkeit hat jede geschlossene materielle Kurve zeitlich unveränderliche Zirkulation, so daß Wirbellinien und Wirbelröhren materiell erhalten bleiben und die letzteren zeitlich unveränderliche Wirbelstärke haben, 19 (13) und (14). Das Wirbeln $\text{curl } \mathbf{v}$ selbst ändert sich deshalb im umgekehrten Verhältnis zu dem während der Bewegung zeitlich veränderlichen Querschnitt der Wirbelröhre.