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Two fundamental methods of solving point-source problems in discrete-space radiative transfer theory

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TWO FUNDAMENTAL METHODS OF SOLVING POINT-SOURCE PROBLEMS

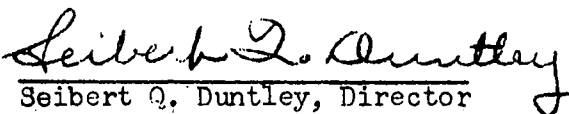IN DISCRETE-SPACE RADIATIVE TRANSFER THEORY

Rudolph W. Preisendorfer

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TABLE OF CONTENTS

Two Fundamental Methods of Solving Point-Source Problems
in Discrete-Space Radiative Transfer Theory

Rudolph W. Preisendorfer

INTRODUCTION

The analytical techniques of discrete-space radiative transfer theory [1-6] are here employed to develop two fundamental methods of determining the light field generated by a point source within a given optical medium. Before outling the methods of solution it may be helpful for the general reader to briefly review some specific examples of point source problems, and the motivations for studying them on a discrete-space level.

Examples of Point Source Problems and their Physical Origin

Consider a natural aerosol or hydrosol in which there is a highly directional source such as a searchlight. The radiant flux from the searchlight is initially directed in some small solid angle about a given direction. The scattering mechanisms in the medium initiate a multiple scattering process, the end result of which is a diffuse steady state radiance distribution about each point of the medium in which the source is located.

One form of the point source problem requires the detailed
calculation of these diffuse steady state radiance distribu-
tions generated by the directional point source. Minor
variants of this problem allow for the possibility of less
directional point sources, such as flares.

The ultimate uses to which the solution of the point
source problem may be directed are manifold: the resolution
of problems of underwater or aerial visibility, television,
and photography, given the inherent optical properties of the
source of radiant flux, and the properties of the optical
targets. Conversely, the solution of the point source problem
may be used to resolve basic research problems which require a
determination of the inherent optical properties of a medium,
given the radiometric response of the medium to a particular
point source.

Another example of the point source problem arises when
it is required to determine the transient (non steady state)
light field generated by powerful pulse-emitting light sources
(either isotropic or anisotropic) within a given medium. Still
another example arises in physical and biological · · radiant
emission phenomena occurring in localized portions of the at-
mosphere and the sea. As a final example which at first sight

may appear not to be of the point source type, we cite the
problem of determining the perturbation of the natural or
artificial light field in a medium by the introduction of
foreign bodies into the field.

Since the formulations below do not commit themselves
to any specific type of medium, all of the above transfer
problems may be solved within a general scattering-absorbing
medium. Furthermore, since the response of a general medium
to a set of point sources is the sum of the responses to the
individual sources, the present solution methods can in
principle be used to determine the effects of extended plane
source and volume sources. Thus the present results subsume
those of the plane-source methods [6], which in turn were
generalized cubic lattice methods.[5]

Motivations for the Discrete-Space Formulation

The motivations for the discrete-space approach to
radiative transfer problems have been stated and illustrated
in detail in the earlier papers of this series.[1] However,
it would perhaps be of interest to briefly point out the
special merits of the discrete-space formulations in the

present case.

The general point source problem in radiative transfer theory is the most difficult of the standard problems encountered in the theory because of the greater number of parameters required in its formulation and solution. Thus, in plane-parallel media with a given incident plane radiance distribution at its upper boundary, three parameters: depth z , and direction angles (θ, ϕ) are needed to specify the radiance distribution. In the case of a general point source within the medium, three parameters (x, y, z) are needed at the very outset to locate a point in the medium relative to the source, and then two more (θ, ϕ) to specify the radiance distribution at that point.

In order to reduce this relatively unmanageable number of parameters, when using the continuous approach, both the optical properties of the medium and the inherent radiance distribution of the point source invariably undergo vitiating assumptions such as the adoption of isotropic sources, isotropic scattering, and homogeneous properties, all resulting in an emasculated radiative transfer problem which is indistinguishable from the more simple varieties of certain heat transfer and neutron transfer problems. Now while such

procedures may point up certain basic processes underlying all transfer problems, they nevertheless do an injustice to the photon transfer process: the very features inherent in it, which distinguish it from these other transfer processes, are indiscriminately hammered smooth in order to obtain a mathematically tractable--but in the last analysis meaningless-- formulation.

To be specific, we are referring to such questions as the following which still have not been satisfactorily answered, by the continuous methods, for the photon transfer case: What effect on the light field results from, (a) changing the scattering law from the isotropic extreme to the high forward scattering extreme actually found in natural aerosols and hydrosols? (b) from arbitrarily changing the directional structure of the inherent radiance distribution of the point source? (c) from the introduction of various types of inhomogeneities into the optical medium? (d) from the introduction of foreign objects into the light field? (e) from changes in the shape and size of the medium?

The discrete-space (or finitary) formulations can go beyond the continuous=space formulations by providing relatively simple and direct answers to these questions (See, e.g.

references 1-6). Unlike the continuous space formulation, which must be re-cast with fundamental changes for each new problem, the discrete space formulation uses basically only one equation to answer each of the preceding five quite dissimilar questions (see Equation (2) below); this equation in various simple guises takes the form of an irreducibly simple algorithm suitable for use in large scale automatic computers.

Now in order to avoid misunderstanding it should be pointed out immediately that, in order to reap this seemingly incredible analytical harvest, the discrete (or finitary) theory just as the continuous (or infinitary) theory, must offer up a sacrifice of fidelity somewhere along the way. Just where this sacrifice occurs and what form it takes constitutes the fundamental distinction between the finitary and infinitary formulations. Briefly, the fidelity-reducing procedures in the infinitary theory center on the radiative processes, leaving the underlying space of the optical medium intact; those of the finitary procedures center on the underlying space of the optical medium, leaving intact for exploration the rich variety of radiative processes on the space.

The Basic Ideas Behind the Two Methods

The first step in either method is the transformation of
the given continuous geometrical setting X to its discrete
counterpart X_n . As will be shown below, every continuous
setting X may be replaced by (or imbedded within) a cubic
lattice setting X_n , a relatively simple form of discrete
space, for which a complete theory has been developed (references
2 and 5).

The first of the two methods presented below is referred to
as the Iteration Method; the second is referred to as the
Categorical Analysis Method. Each is, in a certain sense, the
antithesis of the other.

In the Iteration Method, the imbedding cubic lattice
is viewed as an irreducible unit. Equations are drawn from
the basic theory of reference 1.which describe the response
of X_n at an arbitrary point ($x,\ y,\ z$) to an impinging
radiance distribution at another arbitrary point ($x',\ y',\ z'$).
The equation governing this method is an elementary matrix-
operator form which may be solved by the simple expedient of
matrix iteration, i.e., by repeated raising of a single matrix
to integral powers and summing over the resultant set of

iterants, a method eminently suitable for use on large-scale automatic computers.

The Categorical Analysis Method, on the other hand, views the imbedding cubic lattice not as an irreducible unit, but rather as a set of plane lattices (the monolayers) arranged in a definite way--much as a crystal lattice can be viewed as an assembly of monomolecular layers. The response of the cubic lattice to an impinging radiance distribution is then reducible to the superposition of the responses of its monolayer components. The Categorical Analysis Method further views each of the mono- layers as a reducible collection of linear lattices (the monolines)-- again following the crystal lattice analogy which sees a mono- molecular layer as a collection of interacting strings or chains of atoms. Thus the response of a monolayer in the cubic lattice may be found by the suitable superposition of the responses of its monoline components. Finally, the monoline response to an incident radiance distribution may be found by using the methods of linear lattices (reference 4) which is developed directly from the given data of its "atomic" (individual point) components. The response of the monobloc (the imbedding cubic lattice) may then be viewed in the final analysis as the suitable superposition of the re- sponses of its individual component points to incident radiance distributions. The final response formula for this method is

exprissible in compact form, suitable for use on large scale

automatic computers. Except for a few geometrical details

peculiar to the monobloc geometry, the requisite procedures in

the Categorical Analysis method have already been developed in

an earlier work of this series (reference 6).

Critique of the Two Methods

The two methods studied below, though based on the same

point of departure (Equation (2)), are fundamentally distinct

in their practical realizations.

The Iteration Method, as we have seen, attacks the multiple

scattering problem directly; and through the brute strength of

automatic computers, forges a path from the given source N^o to

the required goal N . It may be likened to the sculpturing of

a statue from a single monobloc of marble. There is a minimum

of preparatory work for the computer; and once the computer is

programmed, it methodically hews its way toward the solution

until its memory capabilities and free computation channels are

saturated. By its very nature (an _infinite_ series of iterations)

the iteration method can never be fully completed. However, as

will be shown below, the radiometric distance from N^o to N

can essentially be covered in most practical instances; and

the residual distance, after the computations have ground to

a halt, is estimable in each case.

The Categorical Analysis Method, on the other hand, does

not attack the problem directly. There is a relatively intri-

cate and detailed preliminary breakdown of the multiple

scattering problem by the analyst before the computer even

sees the problem in program form. Furthermore the problem,

which is eventually insuperable in monobloc form, is chopped

into such sizes and forms (no matrix larger than 9 x 9 to invert,

no matter what the size of X_n) that the computer can build

each piece in a <u>finite</u> amount of steps and time. In the statue

analogy used above, the present method forms the head, body,

arms and legs. separately and then assembles them one by one

into the final whole, rather than hew them from a single monobloc.

Thus the Categorical Analysis Method is <u>fully solvable</u> by present

day computer systems. However, the price for this is paid by the

relatively large number of preliminary steps of analysis required

before the computer is set into action.

FORMULATION AND SOLUTION OF THE GENERAL PROBLEM

The Optical Medium

Let X represent a bounded subset of Euclidean three-space E_3 with an orthogonal cartesian coordinate frame and an arbitrary but given metric on $X \times X$. The associated phase space $\Phi = X \times \Xi$, where Ξ is the unit sphere in E_3 is the domain of definition of the radiance function N. Let α, σ and n be the volume attenuation, volume scattering, and index of refraction functions defined, respectively, on X, $X \times \Xi \times \Xi$, and X. (Both time and wavelength domains of N, α, σ, and n are implicitly carried along in the notation.) The ordered quintuple (Φ, N, α, σ, n) is an optical medium, which forms the starting point of the present discussion.

Let the boundary and initial values of N on Φ be given. Further, an emission radiance function N_η, is presumed given on Φ. If n is not a constant function on X then define a new radiance function $G = N/n^2 c$ on Φ, and replace N by G. Because of this, no loss in generality is engendered if, at the outset, we assume n a constant function on X and drop it from notation (see reference

7). Under these given conditions, the problem is to determine N on $\oint$.

<p style="text-align:center">The Associated Discrete Setting</p>

Because of the boundedness of X , there is a parallelepiped in E_3 , of the form:

$$\left\{ (x,y,z) : |x|, |y| \leqslant c , \quad a \leqslant z \leqslant b \,;\, a, b, c \text{ integers} \right\}$$

which properly contains X . This parallelepiped may be partitioned into a set of n cubes of unit volume, where $n = (2c+1)^2 (b-a+1)$. The set of centerpoints of the cubes of this partition determine the associated discrete-space X_n .

The methods of reference 1 are then ready for application to X_n ; assume a particular set of radiance and eclipse conventions. To fix ideas, assume for the remainder of this work that the radiance and eclipse conventions of reference 5 are in force, and that the Σ and A functions for X_n have been defined on X_n in accordance with the procedures set forth in that reference. The procedure may be generalized to non-stratified media, if necessary, by simply replacing

everywhere the single " z " coordinate by an (x , y , z)
triple. Finally, the discrete source functions $N^0_{k\iota}$ on $X_x \equiv$
associated with the given continuous emission function N_η ,
are obtained by setting $N^0_{k\iota} = N_\eta (x_\iota, \bar{s}_{k\iota}) \Delta_k D_\iota$
where x_ι is the center of the ιth cube in X_η , and
$\bar{s}_{k\iota}$, Δ_k , and D_ι are defined in reference 5.

For reasons which will become clear in the discussion of
the Iteration Method, it will not be necessary to explicitly
consider unbounded subsets of E_3 in the present work. We
note that the theory of the only important type of unbounded
media, namely the stratified plane-parallel type, has already
been considered in detail and fully worked out in reference 5.

The general problem has now been formulated and the
geometric setting of its discrete counterpart determined. The
only remaining preliminary is to formally state the problem
and establish the existence of its basic solution in the discrete
context. This basic solution will then form the nucleus of the
two fundamental methods under present study which show how to
realize the basic solution in practical terms.

The Point Source Problem

Given: The discrete optical medium (Φ_n, N_+, A, Σ) where the discrete phase space Φ_n is manufactured from the continuous optical medium $(\Phi, N, \alpha, \sigma)$ (in which n is a constant function on X). Boundary and source conditions on the radiance function N on $\Phi_n = X_n \times \Xi$ are obtained in accordance with the procedures outlined above and are summarized by the discrete source vector N^o.

Required: The values of the components of the $1 \times n^2$ specific radiance vector $N_+ = [N_{ij} : 1 \leq i, j \leq n]$. Using the basic terminology and concepts set forth in reference 1, the requisite form of the local interaction principle is†

$$N_+ = N_- \Sigma + N^o \Sigma^o, \tag{1}$$

This determines the required N_+ in terms of the given source vector N^o. Here Σ is the $n^2 \times n^2$ scattering matrix associated with X_n. In the present general formulation the vector N^o has $\sum_{i=1}^{n} \mathcal{A}_i$ components, where $\mathcal{A}_i \geq 0$ is the number of incident source directions at $x_i \in X_n$. Hence Σ^o is an $\left(\sum_{i} \mathcal{A}_i \right) \times n^2$ matrix whose elements, as those

of Σ , are determined by assigning values to the local scattering function: $\Sigma(x_i; \mathcal{S}'; \mathcal{S})$. The assignation is arbitrary except, of course, that the usual local conservation requirement is satisfied.[1] The present problem is then to determine N_+ (or, equivalently, $N_- = N_+ M$, where M is the permutation matrix of reference 1).

Basic Solution of the Point Source Problem

We recall first of all that the matrix operator Σ is a norm-contracting operator on the vector space $V_n{}^2$ of all ordered n^2-tuples of real numbers (reference 1). The norm on $V_n{}^2$ is defined by the rule: $|V| = \sum_{i=1}^{n^2} |\mathcal{V}_i|$, for each $V = [\mathcal{V}_1, \ldots, \mathcal{V}_{n^2}] \in V_n{}^2$. It is easy to check that the operation $|\cdot|$ (which is a function on $V_n{}^2$ into the set of non-negative real numbers has the properties of a norm (positive-definite, homogeneous; triangle inequality holds). This particular norm defined on $V_n{}^2$ is called the radio-metric norm. The continuous analog of this norm is the operation $\int_{\Phi} [\cdot] d\nu$ on the set of all radiance functions on Φ . The physical significance of the norm is that the number $\frac{1}{\nu} \int_{\Phi} N \, d\nu = U(\Phi)$ is the radiant energy content of Φ where ν is the speed of light in Φ . An exactly

similar interpretation holds for the discrete form of the
radiometric norm.

It may be well to repeat the argument, sketched in
reference 1, which shows that Σ on $V_n{}^2$ is norm-contracting,
for this fact is crucial in the existence proof of the solution
of (1).

Now the $1 \times n^2$ field vector N_- is related to the
specific radiance vector N_+ by means of an $n^2 \times n^2$
permutation matrix M (reference 1): $N_- = N_+ M$. It is
clear, because of the fact that M simply rearranges components,
that M preserves the norm of an element of $V_n{}^2$, i.e.,
$|VM| = |V|$. Hence if Σ is norm-contracting, so will
$M\Sigma$ be norm-contracting. This fact is useful in the present
solution procedure because (1) may then be cast into the following
form:

$$N_+ = N_- M (M\Sigma) + N^o \Sigma^o ,$$

by making use of the idempotency of M (reference 1). Further,
from the fact that $N_+ = N_- M$ we have

$$N_+ [I - M\Sigma] = N^o \Sigma^o ,$$

where I is an $n^2 \times n^2$ identity matrix.

We next show that Σ is norm-contracting. It is sufficient for our purposes to let N_- be any field radiance vector in $V_n{}^2$. Recall that (pp. 31, 32 reference 1) N_- and Σ have been so constructed that we may write

$$N_- \Sigma = [N_-(x_1), \ldots, N_-(x_n)] \begin{pmatrix} \Sigma(x_1) & & \bigcirc \\ & \ddots & \\ \bigcirc & & \Sigma(x_n) \end{pmatrix},$$

It follows that

$$N_- \Sigma = [N_-(x_1)\Sigma(x_1), \ldots, N_-(x_n)\Sigma(x_n)].$$

From the defined form of $\Sigma(x_i)$ (Equation (14), reference 1) and the definition of $S(x_i, \xi_j)$ (Equation (11), reference 1) it follows that

$$|N_-\Sigma| < \gamma|N_-| ,$$

where $\gamma = \max\left\{ S(x_i, \xi_{ji}) : 1 \leq i, j \leq n \right.$. Clearly $0 < \gamma < 1$ for every non-conservative X_n (which has been assumed throughout the present series of papers) i.e., for every X_n in which $0 < A(x_i, \xi)$ for each $\xi \in \Xi'$ at every $x_i \in X_n$.

For the reasons outlined in reference 1, it follows that

the matrix $I - M\Sigma$ has an inverse whenever Σ is

norm-contracting, and hence that (1) possesses a unique non-

trivial solution:

$$N_+ = N^0 \Sigma^0 [I - M\Sigma]^{-1}$$

(2)

whenever N^0 is not the zero vector in V_n^2 .

Hence the linear transformation Σ on V_n^2 maps the

non-negative cone of V_n^2 (the physical specific and field

radiance vectors) into itself, and is norm-contracting thereby

assuring the system (1) of a unique generally non-trivial

solution (2).

The stage is now set for the presentation of the two

fundamental methods. Part A is devoted to the Iteration

Method, Part B to the Categorical Analysis Method.

PART   A

ITERATION METHOD

THE BASIC ITERATION FORMULA

Starting with the norm-contracting operator $M\Sigma$ we
form the $n^2 \times n^2$ matrix $I - M\Sigma$, and recall (reference
1) that its inverse $[I - M\Sigma]^{-1}$ has the series repre-
sentation:

$$[I - M\Sigma]^{-1} = I + (M\Sigma) + (M\Sigma)^2 + \cdots \quad (3)$$

Thus if we set

$$N_+^0 = N^0\Sigma^0, \quad (4)$$

and in general define

$$N_+^{j+1} = N_+^j (M\Sigma) , \quad j = 0, 1, \cdots, \quad (5)$$

then (2) may be written:

$$N_+ = \sum_{j=0}^{\infty} N_+^j . \quad (6)$$

Equation (6) is the desired <u>basic iteration formula</u>.

The $|\times \cap^2$ radiance vector N_+^j is generally a specific radiance vector and, as such, is therefore subject to the eclipse and local direction space conventions extant in its associated discrete space $X_\cap$. The physical significance of the jth iterant N_+^j of N_+^0 is easy to state: N_+^j represents radiance on $X_\cap$ which, relative to the radiant flux comprising N_+^0 , has been scattered at most j times. Hence (6) represents a scattering-order decomposition of the radiance field over $X_\cap$.

Equation (6) forms the core of the Iteration Method. By starting with the given source vector N^0 and the given source scattering matrix Σ^0, the jth-order scattering component of N_+ is determined from the known $(j-1)$ component according to the rule (5). The required specific radiance vector is then the sum of the infinite series of $|\times \cap^2$ vectors, as shown in (6). There are several interesting features inherent in (6) which, if judiciously employed, considerably simplify the practical details of the computation procedure leading to (6). We will now consider the most important of these features.

A TIME-DEPENDENT INTERPRETATION OF THE ITERATION FORMULA

In any discrete space with a reasonable number of points
(of the order of two to fifty) the associated scattering matrix
Σ , while unwieldy for hand or desk computation, is still
within manageable proportions for a large-scale automatic com-
puter. For larger spaces ($n > 50$) the resultant weight
of matrix operations will stagger even the best of present day
computers. However, there are two features of the Iteration
Method which, in many practical situations, effectively reduce
the computation load--or at least forestall it for a while.
The first of these is, for lack of a more suggestive name, the
time-dependent property of the Iteration Method in (6). The
second is the convergence property of (6) which can effectively
reduce the infinite series to a finite series for many practical
situations. These two features will now be illustrated in detail.

For simplicity, and to fix ideas with a minimum of geometric
detail, we shall illustrate these features by means of a plane
lattice X_n . The plane lattice will be defined by setting
$a = b = 0$ in the general expression for a cubic lattice (see
above). Hence $n = (2c+1)^2$ in this case. Two types of plane
lattice will be used. They may be distinguished by the structure
of the local direction space assigned to each. The first type,

which we shall designate by the name <u>square grid</u>, has assigned

to each point a four element local direction space $\equiv'$

(Figure 1). In terms of the elements of $\equiv$ in $\mathbb{E}_3$,

these are (in ad hoc notation): $\xi_1 = (1,0,0)$, $\xi_2 = (0,1,0)$,

$\xi_3 = -\xi_1$, $\xi_4 = -\xi_2$. The second special plane lattice

will be designated by the name <u>square lattice</u>; it has an eight

element local direction space $\equiv'$ assigned to each point

(Figure 2):

$$\equiv' = \left\{ \xi_1 , (\xi_1 + \xi_2)/\sqrt{2} , \xi_2 , (\xi_2 + \xi_3)/\sqrt{2} , \right.$$
$$\left. \xi_3 , (\xi_3 + \xi_4)/\sqrt{2} , \xi_4 , (\xi_4 + \xi_1)/\sqrt{2} \right\}$$

The Square Grid

Let the source vector N^0 on the square grid have

precisely one nonzero component of unit magnitude in the direc-

tion ξ_1 , at point $x_1 = (0,0,0)$ (Figure 1a). Now by the

construction of the local direction space on X_n , and by

virtue of the standard restriction convention of reference 1

((ii) on page 28), it follows that x_1 may interact <u>directly</u>

only with its four <u>immediate neighbors</u> $x_1 + \xi_i$, $1 \le i \le 4$

in X_n . In general, any point $x_j \in X_n$ may then interact

directly only with its four <u>immediate neighbors</u> $x_j + \xi_i$,

$1 \le i \le 4$. Thus it follows that we may reduce the $1 \times n^2$

vectors N_+ and N_- to $1 \times 4n$ vectors, and Σ reduces to a $4n \times 4n$ matrix. This contraction results from weeding out the permanently zero elements of the vectors $N_\pm$ and those of the matrix Σ which result from the adoption of the present restriction convention.

Now by hypothesis, N_+^0 has only one nonzero component, so that the $1 \times 4n$ vector $N^0 \Sigma^0 = N_+^0$ has at most four nonzero components. These are the specific radiance components at point x_1, and are shown schematically in Figure 1b. The remaining $4n - 4$ components of N_+^0 are zero, and are thus not explicitly shown on the grid.

According to the definition (5), the specific radiance vector N_+^1 is obtained from N_+^0 by multiplication by the $4n \times 4n$ matrix $(M\Sigma)$. It follows from the structures of $(M\Sigma)$, N_+^0 and the present restriction conventions that N_+^1 has at most sixteen additional nonzero components than N_+^0. These are shown schematically in Figure 1c.

The pattern forming is now clear: Each point of the grid which has nonzero vector components in the jth iterant N_+^j, $j = 0, 1, \cdots$, generally "infects" its four immediate neighbors so that they have nonzero radiance components in N_+^{j+1}. The region of radiometrically active points in X_n is then an

"expanding" region with increasing j . The important feature
to observe is that no matter how extensive X_n , the iteration
scheme initially, and for relatively low order iterants, does not
take place over the entire grid. Therefore if (6) is a relatively
rapidly converging series, it may be possible in some cases to
restrict the iteration to a relatively small region   about the
irradiated point. The details of the convergent process will be
considered below. For the present we continue to explore some
further (apparent) time-dependent properties of the iteration scheme.

An examination of the Figures 1a - 1e shows that the dis-
turbance initiated by N^o , i.e., the subregion of X_n
associated with nonzero N_+^j components, "travels" outward
from X_1 in the form of a diamond-shaped wave front. By concen-
trating on only the successive values of N_+^j we are in effect
observing the history of a transient light field induced by a
unit pulse source on X_n at the time $t = -1$, which is shut
off for all $t \geq 0$. In the terminology and general point of
view of reference 8, the diamond-shaped region is the character-
istic spheroid associated with the local source X_1 . If we
define "time" t by the value of j in N_+^j , then the local
source initially emits flux at time $t = j = 0$ (after being
irradiated at time $t = -1$). In this framework, time increases
by unit jumps, so that the speed of light on X_n is taken as

unity. Each point of X_n is eventually (i.e., for some sufficiently large t) a local source and, for all subsequent time, it becomes the center of a characteristic spheroid expanding about it. Thus, any given point of the grid is irradiated again and again during alternate time instants. To round out the geometric description of the expanding light field in X_n , an illustration of the <u>characteristic ellipsoid</u> associated with two points (source-receiver pair) is given in Figure 1f. This shows what subregion of X_n is redirecting scattered flux from the source to the receiver point at any given instant.

By the norm-contracting property of Σ in any non-conservative X_n , we conclude that $|N_+^j| \longrightarrow 0$ as $j \longrightarrow \infty$. Hence, as the characteristic spheroid of $\mathcal{X}_1$ expands, the residual radiant energy of the initial irradiating pulse distributed within the spheroid monotonically decreases to zero with increasing scattering order j . In fact, according to the results of reference 1 (page 44) $|N_+^j| < \gamma^j |N_+^0|$, so that the radiometric norm of N_+^j decreases toward zero, at least as fast as an exponential function, with "time" j .

Now if the source were to remain steady for all integral $t \geqq 0$, then at any time $t = k \geqq 0$, the specific radiance

vector on X_n would be none other than $\sum_{l}^{k} N_+^j$, whose

radiometric norm builds up eventually to its "steady state"

limit $|N_+| = \left| \sum_{l}^{\infty} N_+^j \right|$. Adopting this time-dependent

point of view, we may then postulate several intermediate source

possibilities between the two extremes considered so far (unit

pulse, steady source). For example, the incident source vector

N_+^o may be of unit magnitude for the times $t = -1 , \cdots, t = k-1$,

and then zero for all $t \geq k$. In such a case the radiance

vector on X_n at integral times $t \geq k$ would be

$$N_+(t) = \left(\sum_{j=0}^{k} N_+^j \right) (M\Sigma)^{t-k}$$

This follows from the fact that the 1 x $4n$ vector $\sum_{o}^{k} N_+^j$ may

serve as a new N^o for X_n . Still further possibilities

exist, but the preceding examples will suffice for the present

purposes of illustrating the time-dependent interpretation of

the Iteration Method.

We now go on to consider a final example of the time-

dependent interpretation. This example is designed to show the

effect of the structure of the local direction space $\overline{\overline{}}'$ on

the characteristic spheroids and ellipsoids generated within X_n .

(For a complete discussion of these latter concepts, see reference

8.)

The Square Lattice

The square lattice (Figure 2) occupies the same point set of E_3 as does the square grid defined above. However, the local direction space $\equiv'$ associated with the points of the square lattice has twice as many elements as that of the square grid. Thus the square lattice allows a richer radiometric interaction among its points, which manifests itself in $1 \times 8\eta$ vectors and $8\eta \times 8\eta$ scattering matrices. This may be seen in the series of Figures 2a - 2e. The characteristic spheroid of the local source X_i, now has the general outlines of a square. A diagram of a typical characteristic ellipsoid is given in Figure 2f. Unlike the square grid, once a point (with the exception of X_i) in the square lattice is "lit up", it remains "lit up" ever afterward in a radiance field induced by a unit pulse source irradiating X_i, at time $t = -1$. In the square grid, and under the same source condition, a point alternately assumed the receiver-emitter role.

GENERALIZATIONS OF THE METHOD

True Time-Dependent Scattering Problems

The preceding interpretations can be formally cast into a time-dependent version of the local interaction principle with general <u>time-varying</u> local scattering function $\Sigma(x_i, t; \bullet; \bullet)$ on $\Xi' \times \Xi'$ and time-dependent local absorption function $A(x_i, \bullet)$ on Ξ' for each $x_i \in X_n$ and t in some <u>discrete</u> time domain. The time-dependent scattering function now generally defines an $n^2 \times n^2$ time-dependent scattering matrix $\Sigma(t)$ which in all other respects is identical with its steady state counterpart Σ . For example, on a square grid the solution vector $N_+(t)$, $t = 0, 1, \ldots$, would now be represented by

$$N_+(t) = \sum_{j=0}^{t} N_+^j ,$$ (7)

where

$$N_+^{j+1} = N_+^j M \Sigma(j) , \quad j = 0, 1, \cdots .$$ (8)

Here, for simplicity, the incident unit source N^o is assumed to be applied only to $x_1 \, \varepsilon \, X_n$ and only at $t = -1$, thereby giving rise to a characteristic spheroid in X_n with x_1 as center.

By successively using (7) and (8) in some automatic computer programmed to perform and repeat the operations defined in these equations, the successive values of the components of the $1 \times 4n$ vector N_+^j, $j = 0, 1, \cdots$, may be obtained and recorded. The range of choices of the elements $\Sigma \, (x_i \, , t \, ; \, \zeta_{\alpha i} \, ; \, \zeta_{i \beta})$ of $\Sigma(t)$ is practically limitless. Thus on the one hand, X_n may be characterized as conservative, isotropic and homogeneous, with steady state isotropic scattering; on the other hand we may choose the Σ values so as to actually simulate the movement of an opaque, reflecting object in a quasistationary light field within X_n which may now be selected as a non-conservative, anisotropic, inhomogeneous medium with anisotropic time-dependent scattering. Both these extremes, it is quite clear, are handled with equal ease by the pair of equations (7), (8).

Radiative Transfer on Extended Discrete Spaces

The theory of radiative transfer on discrete spaces has been explicitly limited to <u>finite</u> discrete spaces (see reference 1). In such spaces the number of points in X_n is finite and, as a set, X_n is bounded in E_3. The principal reason for restricting X_n in this way stems from the intuitionistic limitations on the mathematics of general infinite unbounded sets of points (reference 2): Roughly speaking, the intuitionistic mathematics permit only constructable infinities such as those comprising the set of all integers or rationals, or some subsets of the real numbers. Up to the present section of this work, however, even though they were available, there was no need for the extended discrete spaces X_n (the cubic lattice of reference 5 is essentially a finite X_n), that is, spaces which contain a constructable, unbounded, infinity of points. Furthermore, no computer is available which could first of all store the infinity of elements of the scattering matrix Σ associated with X_n, let alone invert the related matrix $I - M\Sigma$ as required in the formal solutions for the specific radiance vector N_+ on X_n.

The observations on the time-dependent character of the scattering-order decomposition, especially those which observed that at any finite stage j of the iterations in (5), there were at most a finite number of components of N_+ which were not zero, lead us to conclude that at any finite stage of the iterations only a finite portion of an extended space is active radiometrically. It follows that only a <u>finite</u> portion of the scattering matrix Σ of the extended space X_n need be used in the computation, and hence that a computer with only a finite memory will be needed.

This observation coupled with the one in reference 1 concerning the estimate of the difference $\left| N_+ - \sum_{j=0}^{k} N_+^j \right|$ (expressions (37), (38) reference 1) leads to an explicit means of carving a finite discrete space out of an extended space which, for all practical purposes, is equivalent to the extended space. Suppose we agree to stop the iteration procedure when the norm of the difference between N_+ and the truncated solution $\sum_{o}^{k} N_+^j$ is less than $\epsilon \left| N_+^o \right|$, $0 < \epsilon < 1$. Now if the space X_n is non-conservative with a γ-factor defined by $0 < \gamma = \lim_n \gamma_n < 1$ where γ_n is associated with an ascending sequence $\{X_n\}$ of discrete spaces such that $X_n \longrightarrow X$ (the original continuous space) then we can estimate the number of iterations which yield a relative error less than ϵ. For, from (38) of reference 1,

this number is $[k]$,where

$$k = \frac{\ln[\epsilon(1-\gamma)]}{\ln\gamma} - 1 , \qquad (9)$$

and where $[k]$ means the smallest integer larger than k .

Thus we need only work with a subset of X which has a temporal radius (reference 8) equal at most to $[k]$, which then gives rise to a strictly finite discrete space X_n .

We now conclude the discussion of the Iteration Method with the development of two conservation theorems which are of intrinsic interest in discrete-space theory and supply valuable numerical checks in the computation procedure.

TWO CONSERVATION THEOREMS

Conservation theorems in the present theory are those which may be deduced from the local interaction principle with the explicit help of the local conservation property (equation (11), reference 1). We exhibit two conservation theorems on a general X_n which are perhaps the most important from both a theoretical and computational point of view. These are the local and global divergence theorems. They are precise analogs of those in continuous case, and follow immediately from the local interaction principle. In order to set the stage for their derivation, we introduce the following definitions.

If $N_-(x_i)$ is the local field radiance vector at $x_i \, \varepsilon \, X_n$ and $\{ N_{ki} : 1 \leqslant k \leqslant n \}$ the set of its components, then in analogy to the definition,

$$ h(x) = \int_{\equiv} N(x, \xi) \, d\Omega(\xi) $$

of scalar irradiance in the continuous theory, we define:

$$ h_i = \sum_{k=1}^{n} N_{ki} \quad , \quad 1 \leqslant i \leqslant n . \tag{10} $$

Now in the continuous theory the scalar irradiance function
is related to the radiant density function μ (radiant energy
per unit volume) by:

$$h = \mu v ,$$

where v is the speed of light. In the present general space
X_n we have set $v = 1$. Hence in (10), we have $\mu_i = h_i$
(numerically). In analogy to the continuous case, we set

$$U(X_n') = \sum_{x_i \,\varepsilon\, X_n'} u_i , \tag{11}$$

i. e., $U(X_n')$ is the radiant energy content of the subset
$X_n' \subset X_n$ obtained by summing over the radiant densities
associated with each element x_i of X_n' .

Finally, if $N_{ij} : 1 \leqq j \leqq n$ are now the components of
the local specific radiance vector $N_+(x_i)$ at x_i , then we set:

$$h_{*i} = \sum_{j=1}^{n} N_{ij} , \quad 1 \leq j \leq n , \tag{12}$$

which is clearly analogous to

$$h_*(x) = \int_{\equiv} N_{*}(x,\xi)\, d\Omega(\xi)$$

in the continuous case, where N_* is the path function.

Now starting with the local interaction principle:

$$N_{ij} = \sum_{k=1}^{n} N_{ki}\, \Sigma(x_i;\, \xi_{ki};\, \xi_{ij}) + \sum_{k=1}^{\Delta i} N_{ki}^{0}\, \Sigma^{0}(x_i;\, \xi_{ki}^{0};\, \xi_{ij}),$$

and summing over all j , we have:

$$h_{*i} = \sum_{k=1}^{n} N_{ki} \left[\sum_{j=1}^{n} \Sigma(x_i;\, \xi_{ki};\, \xi_{ij}) \right]$$

$$+ \sum_{k=1}^{\Delta i} N_{ki}^{0} \left[\sum_{j=1}^{n} \Sigma^{0}(x_i;\, \xi_{ki}^{0};\, \xi_{ij}) \right]$$

$$= \sum_{k=1}^{n} N_{ki}\, S(x_i;\, \xi_{ki}) + \sum_{k=1}^{\Delta i} N_{ki}^{0}\, S^{0}(x_i;\, \xi_{ki}^{0}),$$

still following the notation of reference 1. Now <u>if X_n is</u>

<u>isotropic</u>, so that S and S^o do not depend on the direction

of incidence ζ_{ki} or ζ^o_{ki} , respectively, the preceding

relation may be written:

$$h_{*i} = h_i \, S(x_i) + h^o_i \, S^o(x_i) \qquad (13)$$

which, in the event that $h^o_i = 0$, the scalar irradiance induced

by N^o_{ki} at point x_i , reduces to the useful equation

involving h_{*i}, h_i , and $S(x_i)$:

$$\boxed{\frac{h_{*i}}{h_i} = S(x_i)} \qquad (14)$$

It may be of interest to compare (14) with its continuous

counterpart in reference 9, which states that

$$\frac{h_{*}(x)}{h(x)} = \mathcal{A}(x)$$

where $\mathcal{A}$ is the volume total scattering function in the

continuous context.

Local And Global Divergence Relations

The first conservation theorem is obtained by assuming that

X_n is locally isotropic and source-free at x_i and making

use of the preceding definitions, and (11) of reference 1. The

result is the local divergence relation:

$$\text{I.} \qquad h_{*i} - h_i = -A(x_i)\, h_i ,$$

<div style="text-align:right">(15)</div>

where $A(x_i)$ is the ζ-independent value of the local absorp-

tion function at $x_i \in X_n$. This is the discrete counterpart

to the continuous-space divergence relation:

$$\nabla \cdot \underline{H}(x) = -a(x)\, h(x) ,$$

The global divergence relation follows from assuming X_n

to be isotropic, homogeneous and source-free over some subset

X'_n of X_n . From this it follows that, by summing (15)

over X'_n and making use of (11),

$$\text{II.} \qquad \widehat{P}(X'_n, -) = A\, U(X'_n) ,$$

<div style="text-align:right">(16)</div>

where A is the constant value of $A(\cdot,\cdot)$ on $X_n' \times \Xi$, $U(X_n')$

is the radiant energy content of X_n' , and

$$\bar{P}(X_n', -) = \sum_{x_i \in X_n'} (h_i - h_{*i}) \tag{17}$$

is the net inward flux to X_n' . The reader may wish to compare

this with the continuous-space counterpart in reference 10:

$$\bar{P}(S, -) = a \nu\, U(M) .$$

This completes the present discussion of the Iteration

Method of solving the point source problem.

PART B

THE CATEGORICAL ANALYSIS METHOD

GENERAL OUTLINE OF THE CATEGORICAL ANALYSIS METHOD

There are twelve preparation stages in the solution of the point source problem in an arbitrary discrete space X_n using the Categorical Analysis Method. We now explicitly outline the procedures required to realize each of these stages. The first three stages form the conceptual core of the Categorical Analysis Method and therefore will be developed in minute detail. The remaining nine stages are simply the higher-dimensional analogs of the stages developed in reference 6 for the plane-source problem and are mainly mechanical in nature. For this reason, these latter nine stages are not developed in detail in the present work.

The first stage of the present method is devoted to the derivation of the Ψ-operator equation from the basic solution (2). This Ψ-operator equation is a three-dimensional analog of the one-dimensional Ψ-operator equation of the plane-source method of reference 6. However, since it plays a crucial role in the present method, it will be well to give its formulation in detail.

The second stage of the method defines the required analysis of the imbedding cubic lattice (the monobloc) into its component parts: monobloc into monolayers, monolayers into monolines, mono-lines into points of X_n. The monobloc, monolayer, and monoline are the catagories of X_n.

The third stage develops the three sets of principles of invariance which govern the various R and T operators associated with each of the three categories. The remaining nine stages are devoted to the methodical repetition of the basic themes of references 5 and 6: representation of the global and local Ψ-operators in terms of the monobloc R and T operators; representation of the monobloc R and T operators in terms of monolayer operators; representation of monolayer R and T operators in terms of monoline operators; representation of monoline R and T operators in terms of the given A and Σ functions on X_n; and finally, the synthesis of the categorical representations into a compact solution formula.

STAGE ONE: THE Ψ-OPERATOR

Introduction of the Global and Local Ψ-Operators

The starting point for the derivation of the Ψ-operator equation is equation (2), the general representation of the 1 x n^2 specific radiance vector N_+ in terms of the $n^2 \times n^2$ matrix of the general discrete optical medium (Φ_n, N_+, A, Σ), the $1 \times (\sum_{i=1}^{n} A_i)$ source vector N^0, and the $(\sum_{i=1}^{n} A_i) \times n^2$

source scattering matrix Σ^o .

In order to give a concrete instance of the Ψ-operator and also to tie in the results of the present work with those of the earlier papers of this series, we specifically adopt the twenty-six component local direction space and the radiance restriction conventions of reference 5. It follows that, for the remainder of this work, $N+$ and N^o are 1 x 26n vectors, and that Σ^o and Σ are 26n x 26n matrices, where $n = (2c+1)^2 (b-a+1)$. (See the section above, entitled: "The Associated Discrete Setting.") Consequently, the matrix

$$\Psi = \Sigma^o [I - M\Sigma]^{-1} \tag{18}$$

defined in (2) is a 26n x 26n matrix.

Our immediate task is to reduce the operator equation (2):

$$N+ = N^o \Psi \tag{19}$$

to one which relates the 1 x 26 specific radiance vector $N(x_j)$ associated with point x_j in X_n to the 1 x 26 source radiance

vector $N^o(x_i)$ irradiating an arbitrary point x_i in X_n ,
assuming no other sources are present in X_n . The final
result of the reduction will then be the <u>basic operator equation</u>:

$$N(x_j) = N^o(x_i) \, \Psi(x_i \,;\, x_j),$$

                                                                    (20)

where $\Psi(x_i \,;\, x_j)$ is a 26 x 26 matrix, the requisite Ψ -
operator, and where x_i , and x_j are ordered triples of real
numbers denoting points in $X_n \subset E_3$.

A special form of this operator plays an important role in
the following theory, and is defined by setting $x_i = x_j$. The
resultant operator $\Psi(x_i \,;\, x_i)$ is called the <u>local Ψ -operator.</u>
Occasionally, when it is required to explicitly distinguish between
the two, we call $\Psi(x_i \,;\, x_j)$ the <u>global</u> Ψ -operator whenever
$x_i \neq x_j$.

Linearity of the global Ψ -operator implies that, for an
arbitrary set of sources on X_n at points x_i , $1 \leq i \leq n$,
we may write:

$$N(x_j) = \sum_{i=1}^{n} N^o(x_i) \, \Psi(x_i \,;\, x_j),$$

                                                                    (21)

Derivation of the Basic Operator Equation

The derivation of (20) from (19) consists simply of a pair of matrix extraction operations. To see this, recall * that the 1 x 26 $\cap$ vectors N_+ and N^o may be expressed in general as an ordered set of $\cap$ vectors each of twenty-six components:

$$N_+ = \left[N(x_1) , \ldots, N(x_j) , \ldots , N(x_n) \right],$$ (22)

$$N^o = \left[N^o(x_1) , \ldots , N^{\overset{o}{}}(x_i) , \ldots , N^o(x_n) \right].$$ (23)

In order to extract the 1 x 26 vectors $N(x_j)$ and $N^o(x_i)$ from N_+ and N^o we introduce the general 26$\cap$ x 26 contracting matrix $C(x_k)$ of the form:

$$C(x_k) = \begin{pmatrix} O \\ I \\ O \end{pmatrix},$$ (24)

---

* Reference 1, pp. 30, 31; note that for simplicity here we have dropped the + subscripts on the component vectors of N_+ ..

where I is a 26 x 26 identity matrix beginning at row number $26(K-1)+1$ of $C(x_K)$. The remaining entries in $C(x_K)$ are occupied by the O scalar. It follows that

$$N(x_j) = N_+ \, C(x_j) \, , \tag{25}$$

$$N^o(x_i) = N^o \, C(x_i) \, . \tag{26}$$

The contracting matrix $C(x_K)$ is a generalized form of the contracting matrix C' used in reference 5. (See Equation (24) of that reference.)

The present source hypothesis (namely that only the single arbitrary point $x_i \in X_n$ is irradiated by a nonzero radiance vector) is analytically described by:

$$N^o = \left[0, \dots, 0, N^o(x_i), 0, \dots, 0 \right] \, , \tag{27}$$

where we have set the 1 x 26 vectors $N^o(x_K) = 0$, for $K \neq i$. It follows from this particular form of N^o, that we may write:

$$N^o \Psi = N^o \, C(x_i) \, C^T(x_i) \, \Psi \, , \tag{28}$$

where $C^T(x_i)$ is the transpose of $C(x_i)$. Hence, from (19):

$$N_+ = N^0 C(x_i) \, C^T(x_i) \, \Psi \, .$$

(29)

Postmultiplying each side of (29) by the contracting matrix $C(x_j)$:

$$N_+ \, C(x_j) = N^0 C(x_i) \, C^T(x_i) \, \Psi \, C(x_j) \, .$$

This now reduces to (20) by virtue of (25), (26) and the following definition of $\Psi(x_i \, ; \, x_j)$:

$$\boxed{\Psi(x_i \, ; \, x_j) \equiv C^T(x_i) \, \Psi \, C(x_j)}$$

(30)

This completes the derivation of the required basic Ψ-operator equation (20).

First Decomposition of the Global Ψ -Operator

Following the pattern set in reference 6, we now decompose $\Psi(x_i ; x_j)$ into a 3 x 2 block matrix. The decomposition is induced by a specific decomposition of $N(x_j)$ and $N^o(x_j)$ into subvectors associated with the partition: $\left\{ \Xi'_+ , \Xi_0 , \Xi_- \right\}$ of the twenty-six component local direction space Ξ' adopted in the present study. See reference 5 for a complete description of Ξ' and this partition.

Thus we first write:

$$N(x_j) = \left[N_+(x_j) , N_-(x_j) \right] \qquad (31)$$

where $N_+(x_j)$ now stands for the 17-component radiance vector associated with Ξ_+, and $N_-(x_j)$ the 9-component radiance vector associated with Ξ_- .

Furthermore:

$$N_+(x_j) = \left[N'_+(x_j) , N_0(x_j) \right] , \qquad (32)$$

where $N_+^{'}(x_j)$ is the 9-component radiance vector associated with $\equiv_+^{'}$, and $N_o(x_j)$ is the 8-component radiance vector associated with $\equiv_o$.

A similar partition of $N^o(x_i)$ into subvectors is possible:

$$N^o(x_i) = \left[N_+^o(x_i)^{'}, \; N_o^o(x_i), \; N_-^o(x_i) \right],$$

(33)

where the subvectors, from left to right, have 9, 8, and 9 components, and are associated with $\equiv_+^{'}$, $\equiv_o$ and $\equiv_-$, respectively. It turns out that we require the partition (31) of $N(x_j)$, and the partition (33) of $N^o(x_i)$. Hence (20) becomes

$$\left[N_+(x_j), \; N_-(x_j) \right] = \left[N_+^o(x_i)^{'}, \; N_o^o(x_i), \; N_-^o(x_i) \right] \Psi(x_i; x_j),$$

which induces the following cleavage of the matrix representation

of the global Ψ -operator:

$$\Psi(x_i; x_j) = \begin{array}{c} (9) \\ \\ (8) \\ \\ \\ (9) \end{array} \begin{bmatrix} \Psi_{++}(x_i; x_j) & \Psi_{+-}(x_i; x_j) \\ \\ \Psi_{0+}(x_i; x_j) & \Psi_{0-}(x_i; x_j) \\ \\ \Psi_{-+}(x_i; x_j) & \Psi_{--}(x_i; x_j) \end{bmatrix} \qquad (34)$$
$$\qquad\qquad (17) \qquad\qquad\qquad (9)$$

The dimensions of the block matrices are fully defined by those of

the subvectors of $N(x_j)$ and $N^0(x_i)$. For example, $\Psi_{0+}(x_i; x_j)$

is an 8 x 17 matrix, and $\Psi_{+-}(x_i; x_j)$ is a 9 x 9 matrix. The

dimensions are indicated on the side and bottom of (34). Equation

(34) is the required first decomposition of the global Ψ -operator;

it begins to show the power inherent in the Categorical Analysis

Method: the matrices are dimensionally independent of the size of

X_n . It will turn out that 9 x 9 is the upper bound on the sizes

of matrices required to be inverted by this method applied to any X_n .

STAGE TWO: CATEGORICAL ANALYSIS OF THE MONOBLOC

The $\not{z}$ -Oriented Monobloc

In the section above on the formulation and solution of the
general problem, the location space X of the given continuous
optical medium was imbedded in a certain parallelepiped. The
associated discrete space X_n was formed from this parallele-
piped by the set-formation rule:

$$X_n = \left\{ (x,y,z) : |x|, |y| \leqslant c , a \leq z \leq b ; x,y,z,a,b,c, \text{ integers} \right\}.$$

$$(35)$$

X_n is the imbedding cubic lattice, or _monobloc_ for short.
When defined in this way with an obvious favoritism shown for
the $\not{z}$ -axis, we will say that X_n is   _$\not{z}$-oriented_. The
two other possibilities, namely an x and a y -oriented
monobloc may do just as well for our present purposes. However,
we adopt the $\not{z}$ -oriented monobloc in order to maintain contact
with the notations established earlier in the series and in other
works which use the $\not{z}$ -axis as the axis along which depth or
altitude is customarily measured in real media.

The Monolayers and Monolines

A <u>monolayer in X_n at level z</u>   is a subset $X_n(z)$ of X_n defined by

$$X_n(z) = \left\{ (x', y', z') \in X_n : z' = z \right\},\qquad (36)$$

A <u>monoline in X_n at levels y and z</u>   is a subset $X_n(y, z)$ of X_n defined by

$$X_n(y, z) = \left\{ (x', y', z') \in X_n : y' = y , z' = z \right\}\qquad (37)$$

The generic term covering the notion of monobloc, monolayer and monoline will henceforth be <u>category</u>.   A graphical representation of typical catagories is shown in Figure 3.

A monolayer at level z is evidentally the set union of all monolines at levels y with fixed z :

$$X_n(z) = \bigcup_{|y| \leqslant c} X_n(y, z)\quad ,$$

and X_n itself is the set union of all monolayers in X_n :

$$X_n = \bigcup_{z=a}^{b} X_n(z) \, .$$

Hence X_n is viewable as the set union of all monolines in X_n :

$$X_n = \bigcup_{z=a}^{b} \left[\bigcup_{|y| \leq c} X_n(y,z) \right] \, .$$

A monoline, of course, is simply a form of linear lattice, the general theory for which was worked out in reference 4. It will be possible to characterize the response of a monolayer in terms of the responses of its monoline subsets making use of the principles of invariance on a linear lattice and plane lattice. The response of the entire monobloc, as noted earlier, would then be some suitable superposition of the responses of its monolayer subsets. The form of this superposition is governed basically by the principles of invariance on the monobloc. We now give the detailed statements of the principles of invariance on the various categories.

STAGE THREE: PRINCIPLES OF INVARIANCE ON THE CATEGORIES

The principle of invariant imbedding (Equation (47),

reference 2) forms the starting point of derivation of all the

forms of the principles of invariance on the present categories.

The principles have already been derived in detail several times

in simpler contexts. For example, reference 4 gives a derivation

of the principles for a linear lattice, and reference 5 gives

their derivation for the extended cubic lattice considered there.

The pattern of derivation is now so well established that we may

safely leave the details of the present derivations to the reader.

Thus we will merely state the requisite forms of the <u>two main</u>

<u>principles</u> (I, II) for each of the three categories. The re-

maining two principles (III, IV) in each of the three categorical

cases are derived in the usual manner (See, e.g., references 4, 5).

Principles for the Monobloc

Let (x, y, z) denote a point in X_n : let $a - 1 \leq z' < < z < z'' \leq b + 1$, then

$$\text{I} \quad N_+(x, y, z) = N'_+(\bar{x}, \bar{y}, z'') \, T(\dot{\bar{x}}, x; \bar{y}, y; z'', z)$$
$$+ \; N_-(\bar{x}, \bar{y}, z-1) \, R(\bar{x}, x; \bar{y}, y; z, z'') \tag{38}$$

$$\text{II} \quad N_-(x, y, z) = N_-(\bar{x}, \bar{y}, z') \, T(\bar{x}, x; \bar{y}, y; z', z)$$
$$+ \; N'_+(\bar{x}, \bar{y}, z+1) \, R(\bar{x}, x; \bar{y}, y; z, z') \tag{39}$$

These principles and their operators apply to the set of monolayers in X_n beginning with $X_n(z'+1)$ and ending with $X_n(z''-1)$ (Figure 4). The following _summation convention_ has been adopted: If f and g are real or matrix valued functions defined on X_n , then we write:

$$\sum_{|x| \leq c+1} \sum_{|y| \leq c+1} f(x, y, z_1) \, g(x, y, z_2) \not\equiv f(\bar{x}, \bar{y}, z_1) g(\bar{x}, \bar{y}, z_2) . \tag{40}$$

Thus the bars over a symbol for a real number denote the

summation over the range of that real variable as defined in

the formulae (40). Thus the x and y variables range

from $-(c+1)$ to $(c+1)$, and z lies in the interval

$[a-1, b+1]$. The reason for allowing the range of x,

y, and z to exceed the geometric limits of the monobloc

by one unit all around the monobloc is to allow a compact formu-

lation of the boundary source conditions within the principles.

The precedent for this convention has been established in

references 4 and 5, and a study of the convention in these

simpler contexts will clarify its usage here.

The R and T operators in (38) and (39) above are the

standard operators for the monobloc. Because of the z -

orientation convention and the adoption of the twenty-six

component local direction space Ξ', the R and T

operators are matrices of the same general dimensions as those

in reference 5. For example the matrix $T(x', x ; y', y ; z', z)$

is the standard transmittance operator for downward flux, since

$z' < z$; hence it is a 9 x 9 matrix. Further $R(x', x ; y', y ;$

$z', z)$ is the standard reflectance operator for downward

flux with dimensions 9 x 17. A similar set of observations

hold for the other operators R and T for the upward flux.

Finally, the zero and identity operator properties hold for the
present R and T operators just as in the extended cubic
lattice case. (See equations (16) - (20), reference 5.)

The physical processes summarized by these standard R
and T operators are exactly analogous to their extended cubic
lattice counterparts developed in reference 5. As an example,
consider the 9 x 9 matrix $T(x', x \; ; \; y', y \; ; \; z', z''-1)$, $z' < z''$.
Because of the z -oriented monobloc convention, this matrix
refers to the transmittance of the subset (a slab) of X_n
consisting of the $z'' - z' - 1$ monolayers beginning with $X_n (z'+1)$
and ending with $X_n (z''-1)$. (See Figure 4.) Thus if this
subset could be isolated from the remaining points of X_n,
and if it is irradiated by the radiance distribution emitted by
point (x', y', z') in $X_n (z')$ i.e., by the 1 x 9 vector
$N_- (x', y', z')$, then the resultant transmitted radiance
distribution in the form of the 1 x 9 vector $N_- (x, y, z''-1)$
emitted by a point on $X_n (z''-1)$ (the lowest monolayer of the
slab) would be:

$$N_- (x, y, z''-1) = N_- (x', y', z') \; T(x', x \; ; \; y', y \; ; \; z', z''-1) .$$

In general, if all of the points on the upper boundary of
this isolated slab are irradiated by downward flux from the
points of $X_n(z')$, the resultant transmitted radiance
$N_-(x,y,z''-1)$ at $(x,y,z''-1) \varepsilon X_n(z''_{-1})$ would be

$$N_-(x,y,z''-1) = N_-(\bar{x},\bar{y},z') \, T(\bar{x},x;\bar{y},y;z',z''-1).$$

This is the first term in (39).

Finally, to complete this example, we observe that the
present subslab, consisting of the layers $X_n(z'+1), ..., X_n(z''-1)$,
cannot in practice be isolated from its complement in X_n .
Hence the complete response $N_-(x,y,z''-1)$ must take
cognizance of the flux impinging on the slab which is emitted by
the points in monolayer $X_n(z'')$; this flux is represented
by the 1 x 9 vector $N_+'(x',y',z'')$. When this additional
irradiation is taken into account, the result is the main principle
of invariance II for the slab comprised of the monolayers $X_n(z'+1)$
to $X_n(z''-1)$.

These remarks and observations should help the reader to
form his own additional observations on the remaining operators
and principles of invariance above, and to understand the material
presented below.

Principles for a Monolayer

To prepare the way for the principles of invariance associated with the monolayer $X_n(z)$, $a \leq z \leq b$, we must introduce several new radiance vectors. These are sub-vectors of the 1 x 8 vector $N_0(x, y, z)$ defined above (see Equation (32)). Specifically, we require the following partition of $N_0(x, y, z)$:

$$N_0(x,y,z) = \left[N_{0+}(x,y,z), \; N_{0-}(x,y,z) \right], \qquad (41)$$

where $N_{0+}'(x,y,z)$ is a 1 x 5 vector and $N_{0-}(x,y,z)$ is a 1 x 3 vector. Further, we partition $N_{0+}(x,y,z)$ thus:

$$N_{0+}(x,y,z) = \left[N_{0+}'(x,y,z), \; N_{00}(x,y,z) \right], \qquad (42)$$

where $N_{0+}'(x,y,z)$ is a 1 x 3 vector and $N_{00}(x,y,z)$ is a 1 x 2 vector. The geometric significance of these vectors is clarified by Figure 5. Their exact definitions may be phrased by means of the elements of $\Xi_0 \subset \Xi'$:

Recall that (reference 5) the subset $\equiv_o$ of elements of

$\equiv'$ consists of the members:

$$\equiv_o = \left\{ \xi_{13}, \xi_{12}, \xi_{11}, \xi_{10}, \xi_{14}, \xi_{15}, \xi_{16}, \xi_{17} \right\}.$$

Then <u>redefine</u>[*] the old N_o-vector as follows:

$$N_o(x_i) = \left[N(x_i, \xi_{13}), N(x_i, \xi_{12}), \ldots, N(x_i, \xi_{17}) \right], \qquad (43)$$

where the order of components is governed by the order of

appearance of the ξ_i in $\equiv_o$ above, and where $x_i \in X_n(z)$,

---

[*] This new arrangement of the components of $N_o(x_i)$ is adopted in order to have (41) and (42) partition into the indicated subvectors. Thus N_o and N_{o+} correspond exactly to N and N_+ as given in (5) and (6) of reference 5. A simple permutation matrix relates the old and new arrangements of N_o.

Therefore:

$$N_{o+}(x_i) = \left[N(x_i, \zeta_{13}), N(x_i, \zeta_{12}), N(x_i, \zeta_{11}), N(x_i, \zeta_{10}), N(x_i, \zeta_{14}) \right] \quad (44)$$

$$N'_{o+}(x_i) = \left[N(x_i, \zeta_{13}), N(x_i, \zeta_{12}), N(x_i, \zeta_{11}) \right] \quad (45)$$

$$N_{0-}(x_i) = \left[N(x_i, \zeta_{15}), N(x_i, \zeta_{16}), N(x_i, \zeta_{17}) \right] \quad (46)$$

$$N_{oo}(x_i) = \left[N(x_i, \zeta_{18}), N(x_i, \zeta_{14}) \right] \quad (47)$$

Now within $X_n(z)$, the vectors $N_{o+}(x_i)$ and $N'_+(x_i)$ represent the upward radiance vectors; $N_{0-}(x_i)$ is the downward radiance vector, and $N_{oo}(x_i)$ is the horizontal radiance vector. This choice of terminology for $N_{o\pm}(x_i)$ and $N_{oo}(x_i)$ attempts to follow the general terminology established earlier in the series, and serves to assign intuitively meaningful names that can be used in following the text. With these notational conventions, the requisite principles of invariance for the monolayer $X_n(z)$, $a \leq z \leq b$, may now be written down. Let $-(c+1) \leq y' < y < y'' \leq (c+1)$, and let $(x, y, z) \in X_n$, then:

$$I_{\not z} \qquad N_{o+}(x,y,z) = N'_{o+}(\bar{x}, y'', z)\, T_{\not z}(\bar{x}, x\,;\, y'', y) \qquad (48)$$

$$+\; N_{o-}(\bar{x}, y-1, z)\, R_{\not z}(\bar{x}, x\,;\, y,\, y'')$$

$$II_{\not z} \qquad N_{o-}(x,y,z) = N_{o-}(\bar{x}, y', z)\, T_{\not z}(\bar{x}, x\,;\, y', y) \qquad (49)$$

$$+\; N'_{o+}(\bar{x}, y+1, z)\, R_{\not z}(\bar{x}, x\,;\, y,\, y')$$

The bars over the x-symbols denote the summation conven-
tion adopted above. In practice, when no confusion can result,
it is permissible to drop all references to the subset Ξ_o of
Ξ' and to the fixed $\not z$ associated with the monolayer
under consideration. Thus, (48) and (49) may be written more
succinctly as:

$$I'_{\not z} \qquad N_+(x,y) = N'_+(\bar{x}, y'')\, T(\bar{x}, x\,;\, y'', y) \qquad (48')$$

$$+\; N_-(\bar{x}, y-1)\, R(\bar{x}, x\,;\, y, y'')$$

$$II'_{\not z} \qquad N_-(x,y) = N_-(\bar{x}, y')\, T(\bar{x}, x\,;\, y', y) \qquad (49')$$

$$+\; N'_+(\bar{x}, y+1)\, R(\bar{x}, x\,;\, y, y')$$

$$-(c+1) \le y' < y < y'' \le (c+1).$$

However, when the R and T operators and radiance distributions in adjacent monolayers $X_n(z \pm 1)$ to $X_n(z)$ are to be explicitly considered, it will be necessary to use the set (48), (49). The principles (48) - (49) and their operators apply to the subset of the monolayer $X_n(z)$ beginning with the monoline $X_n(y'+1, z)$ and ending with $X_n(y''-1, z)$.

For any $X_n(z)$, $a \leq z \leq b$, the operator $T_z(x', x; y', y)$ where $y' < y$, is the <u>standard transmittance operator</u> for downward flux within the monolayer $X_n(z)$, and is a 3 x 3 matrix. The operator $R_z(x', x; y', y)$ where $y' < y$ is the <u>standard reflectance operator</u> for downward flux on within monolayer $X_n(z)$, and is a 3 x 5 matrix. Similar definitions and dimensions hold for the operators R_z and T_z associated with upward flux. The discussion of the physical significance of these operators is analogous in all respects to that for the standard monobloc operators. For example $R_z(x', x; y', y)$, $y' < y$ is the standard reflectance operator for the set of $(y-y')$ monolines in $X_n(z)$ beginning with $X_n(y', z)$ and ending with $X_n(y-1, z)$.

Principles for a Monoline

Let $X_n(y, z)$ be a monoline in X_n . Then at any
point $(x, y, z) \in X_n(y, z)$ we have two directions of flow
within $X_n(y, z)$: in the direction ξ_{10} the upward
direction (+), and ξ_{14} the downward direction (−). (See
Figure 6.) Using the components of $N_{oo}(x_i)$ in (47) which
describe the specific radiances at x_i in $X_n(y, z)$,
the requisite forms of the two main principles of invariance for
the monoline are:

$$\text{I}_{yz} \quad N(x, y, z, \xi_{10}) = N(x'', y, z, \xi_{10}) \, T_{yz}(x'', x) \tag{50}$$

$$+ \; N(x-1, y, z, \xi_{14}) \, R_{yz}(x, x'')$$

$$\text{II}_{yz} \quad N(x, y, z, \xi_{14}) = N(x', y, z, \xi_{14}) \, T_{yz}(x', x) \tag{51}$$

$$+ \; N(x+1, y, z, \xi_{10}) \, R_{yz}(x, x')$$

When no confusion with other monolines is possible, and the

monoline $X_n(y, \not z)$ under study is clearly identified, all

specific references to y , $\not z$, ξ_{10} and ξ_{14} may be dropped,

so that the principles may take on the relatively compact forms:

$$I'_{y\not z} \qquad N(x,+) = N(x'',+)\, T(x'',x) + N(x-1)\, R(x,x'') \qquad (50')$$

$$II'_{y\not z} \qquad N(x,-) = N(x',-)\, T(x',x) + N(x+1,+)\, R(x,x') \qquad (51')$$

where we have set

$$N(x, y, \not z, \xi_{10}) \equiv N(x,+) \qquad (52)$$

$$N(x, y, \not z, \xi_{14}) \equiv N(x,-) \qquad (53)$$

These abbreviated forms for the principles of invariance on

$X_n(y, \not z)$ are now <u>precisely</u> the forms of the principles of

invariance for a general linear lattice developed in reference 4.

Observe, however, that if the interaction of two adjacent mono-

lines is under consideration, it will be necessary to use the

complete statements (50), (51). The principles (50), (51) and

their R and T operators apply to the subset of the monoline

$X_n (y, z)$ beginning with the point $(x'+1, y, z)$ and ending

with $(x''-1, y, z)$.

On the Use of the Categorical Statements of the
Principles of Invariance

The principles (38), (39) for the monobloc, (48), (49) for

a monolayer, and (50), (51) for a monoline are now used to find

the associated <u>complete</u> reflectance and transmittance operators

from the standard operators for each category. The general out-

line for this procedure has been given in reference 6 in sufficient

detail so that the reader may readily construct the presently

required complete operators after the appropriate sections of

reference 6 have been studied (See Equations (10) - (16), reference

6.) These complete operators will be required in STAGE FOUR to

TWELVE below.

STAGES FOUR TO TWELVE IN THE CATEGORICAL ANALYSIS METHOD

We now outline the procedures that must be followed to realize

the last nine stages of the present method. Since every one of

these stages has a lower-dimensional analog whose details have been

fully covered in preceding papers of this series, there is no longer

any need to enter into any further detailed discussion of the stages

except to cite the appropriate reference and perhaps to make an

occasional remark and establish notation.

STAGE FOUR: REPRESENTATION OF THE GLOBAL $\mathcal{Y}$ -OPERATOR

IN TERMS OF THE LOCAL $\mathcal{Y}$ -OPERATOR AND THE COMPLETE

OPERATORS FOR A MONOBLOC

References: (a) Equations (17) - (32), reference 6.

(b) TABLE I, reference 6.

Remarks: (a) The summation convention now replaces the straight

matrix multiplication of reference 6.

Notation: (a) Set $a = 1$, $b = n$. Complete transmittance operator

for upward flux: $\mathcal{T}(x', x; y', y; z'', z, o)$

Complete reflectance operator for upward flux:

$\mathcal{R}(x'_{,}x ; y'_{,}y_{,} z_{,}z'_{,}o)_{,}$complete reflectance operator for downward flux:
$Q(x'_{,}x ; y'_{,}y; z_{,}z''_{,}n^{+1});$ complete transmittance operator for

downward flux: $\mathcal{T}(x'_{,}x ; y'_{,}y; z'_{,}z_{,}n^{+1})$.

STAGE FIVE: COMPLETE MONOBLOC OPERATORS FROM STANDARD
MONOBLOC OPERATORS

References: (a) Equations (53) - (66), reference 6.

(b) TABLE III, reference 6.

STAGE SIX: STANDARD MONOBLOC OPERATORS FROM COMPLETE
MONOLAYER OPERATORS

References: (a) Equations (45) - (48), reference 5.

(b) Equations (49) - (78), reference 5.

(c) Equations (33) - (41), reference 6.

Remarks: (a) The present counterparts to the recurrence relations

of reference 5 must be developed. These show how the

Standard Operators for a monobloc are built up start-

ing from the complete monolayer operator. The model

for this occurs in equations (45) - (48), reference 5.

(b) The next step is to link together the complete

monolayer operators for adjacent monolayers. The

appropriate $\mathcal{S}$ -operators for this task must be

defined. The model for this procedure occurs in

equations (49) - (78), reference 5, and (33) $\rightarrow$ (41),

reference 6.

(c) Observe how the radiant flux generally fans out from

<u>one point</u> (x, y, z) on $X_n(z)$ and strikes the point

$(x, y, z \pm 1)$ <u>and its eight neighbors</u> on $X_n(z \pm 1)$,

respectively. Thus the reflectance response (say)

of a monolayer $X_n(z)$ to the radiance vector

$N_+^{\prime}(x, y, z+1)$ (say) must be synthesized by an

appropriate linear combination of (generally) nine

complete $\mathcal{R}$ operators for $X_n(z)$.

<u>STAGE SEVEN</u>: THE LOCAL $\mathcal{Y}$ -OPERATOR IN TERMS OF

COMPLETE MONOLAYER OPERATORS

<u>Reference</u>: (a) Equations (42) - (52), reference 6.

(b) TABLE II, reference 6.

STAGE EIGHT: COMPLETE MONOLAYER OPERATORS FROM

STANDARD MONOLAYER OPERATORS

References: Repeat Stage Five above for the monolayer context.

STAGE NINE: STANDARD MONOLAYER OPERATORS FROM

COMPLETE MONOLINE OPERATORS

References: Repeat Stage Six above for the Monolayer Context.

STAGE TEN: COMPLETE MONOLINE OPERATORS FROM

STANDARD MONOLINE OPERATORS

References: Repeat Stage Eight for the monoline context.

STAGE ELEVEN: DETERMINATION OF THE STANDARD

MONOLINE OPERATORS

References: (a) Equations (32) - (39), reference 4.

(b) Equations (83) - (95), reference 5.

Remark: (a) Observe that these operators are determined by

direct appeal to the Σ and A functions on X_h .

.

(b) The Σ and A functions are determined for

the present cubic lattice in accordance with

the procedure set forth in equations (83) - (95)

of reference 5.

STAGE TWELVE: SYNTHESIS OF SOLUTION

Reference: (a) SUMMARY, p 45, reference 6.

Remark: (a) The method of tabular representations established

in reference 6 should be adopted in the present

method.

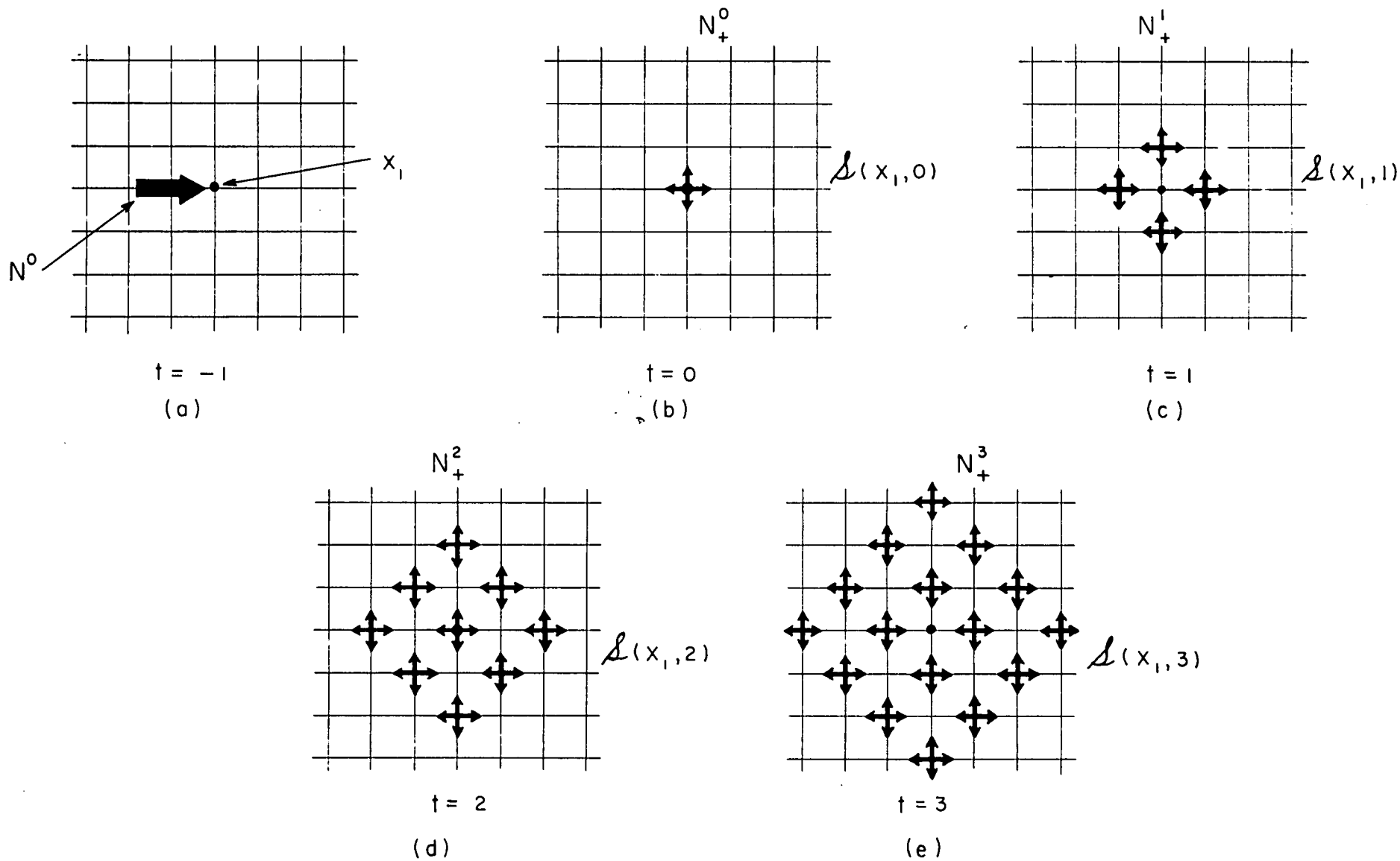THIS COMPLETES THE DISCUSSION OF THE CATEGORICAL ANALYSIS METHOD.

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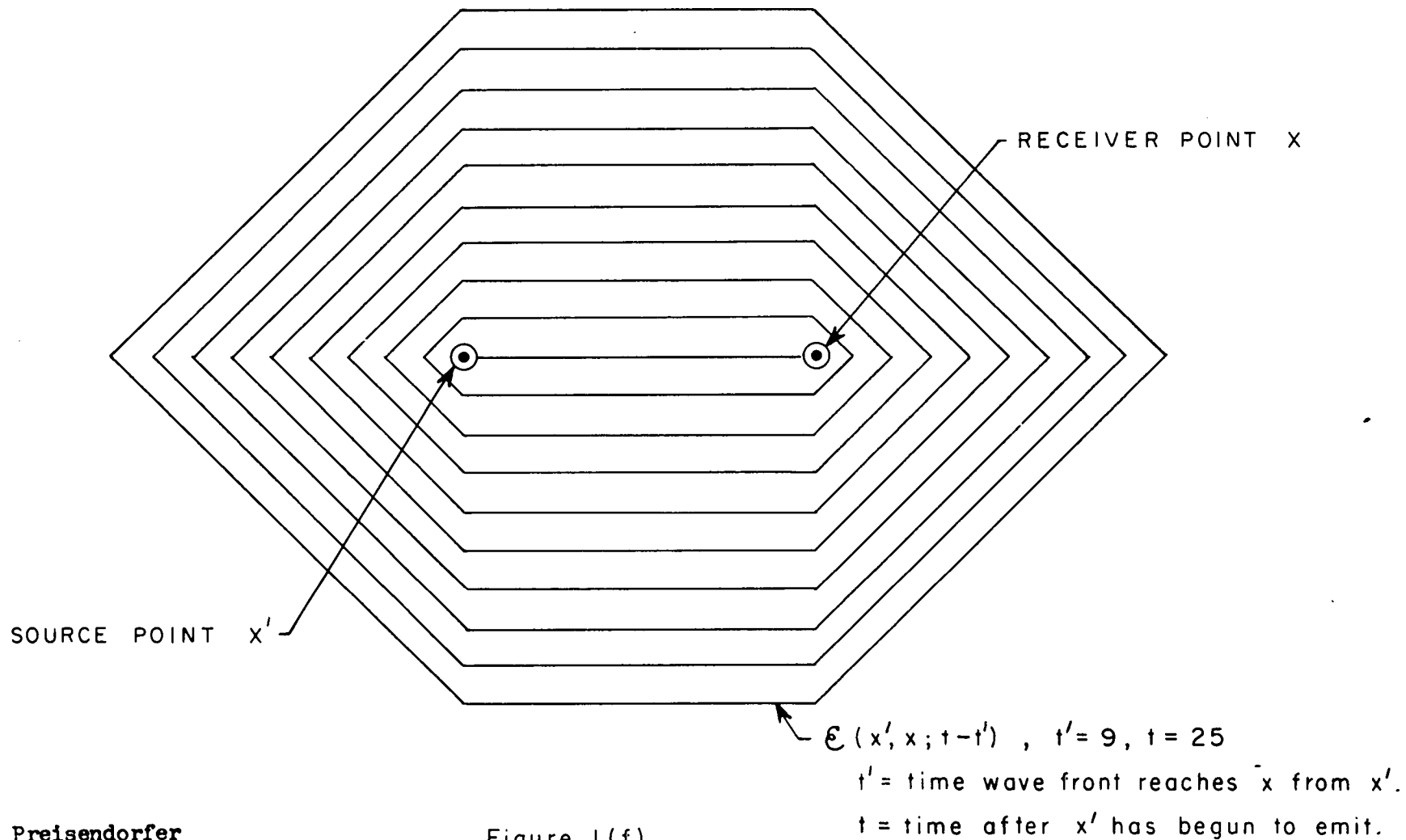   graphy, University of California, La Jolla, California (1959).

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7. Preisendorfer, R. W., <u>A Mathematical Foundation for Radiative Transfer Theory</u>, J. Math. and Mech. <u>6</u>, 685-730 (1957).

8. _____ <u>Temporal Metric Spaces in Radiative Transfer Theory. III Characteristic Spheroids and Ellipsoids,</u> SIO Ref. 59-10, Scripps Institution of Oceanography, University of California, La Jolla, California (1959).

9. _____ <u>On the Direct Measurement of the Total Scattering Function</u>, SIO Ref. 59-41, Scripps Institution of Oceanography, University of California, La Jolla, California (1959).

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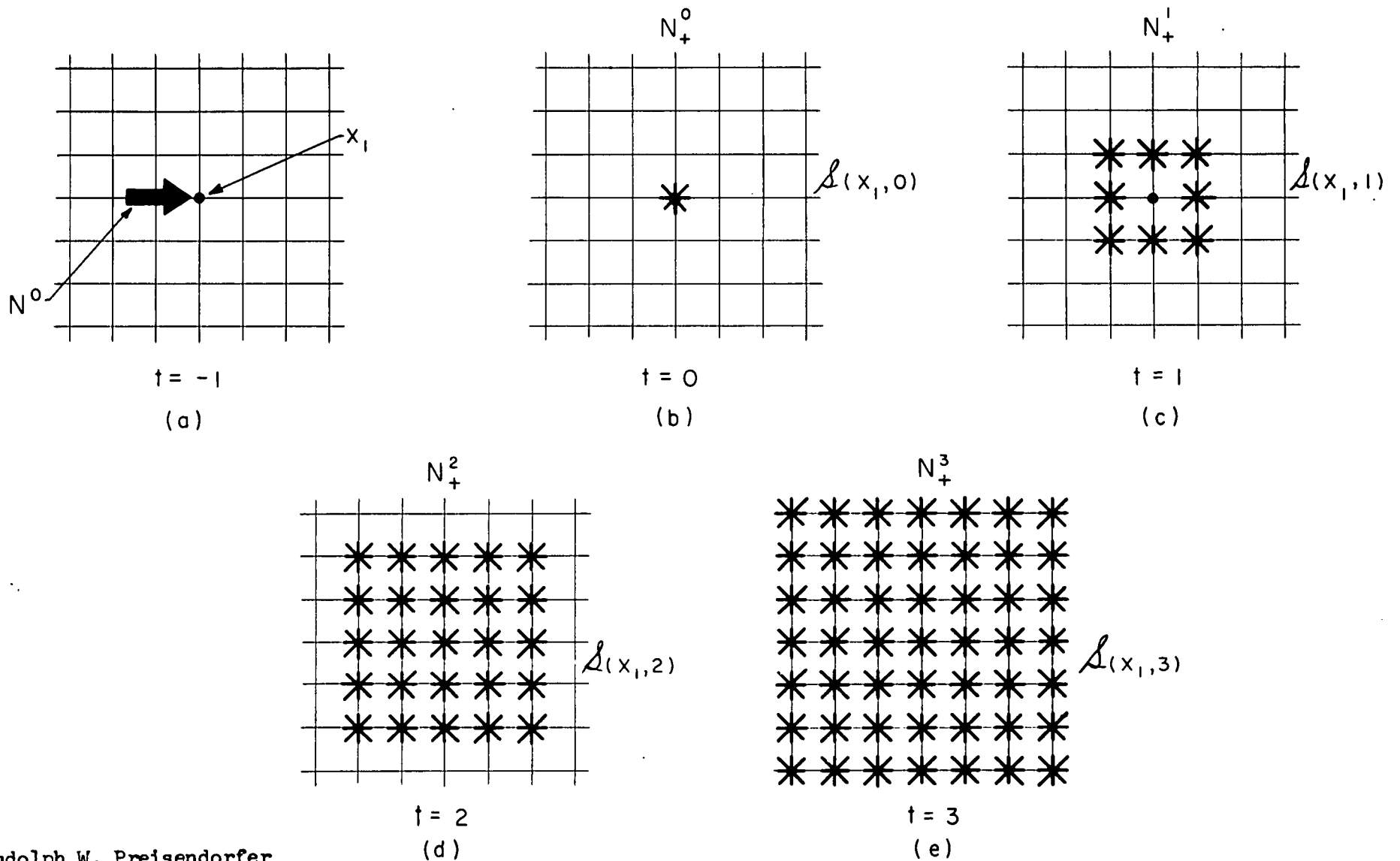
CHARACTERISTIC SPHEROIDS OF A POINT SOURCE IN A SQUARE GRID

N^0

N_+^0

N_+^1

x_1

N^0

$\mathscr{S}(x_1, 0)$

$\mathscr{S}(x_1, 1)$

t = -1

(a)

t = 0

(b)

t = 1

(c)

N_+^2

N_+^3

$\mathscr{S}(x_1, 2)$

$\mathscr{S}(x_1, 3)$

t = 2

(d)

t = 3

(e)

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SIO Ref. 59-71

Figure 1

CHARACTERISTIC ELLIPSOIDS OF A SOURCE-RECEIVER PAIR
IN A SQUARE GRID

RECEIVER POINT X

SOURCE POINT X'

$\mathcal{E} (x', x ; t-t')$, $t'= 9$, $t = 25$

$t' =$ time wave front reaches $\bar{x}$ from x'.

$t =$ time after x' has begun to emit.

Figure I(f)

CHARACTERISTIC SPHEROIDS OF A POINT SOURCE IN A SQUARE LATTICE

$t = -1$

(a)

N_+^0

$\mathcal{L}(x_1, 0)$

$t = 0$

(b)

N_+^1

$\mathcal{L}(x_1, 1)$

$t = 1$

(c)

N_+^2

$\mathcal{L}(x_1, 2)$

$t = 2$

(d)

N_+^3

$\mathcal{L}(x_1, 3)$

$t = 3$

(e)

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Figure 2

CHARACTERISTIC ELLIPSOIDS OF A SOURCE-RECEIVER PAIR

IN A SQUARE LATTICE
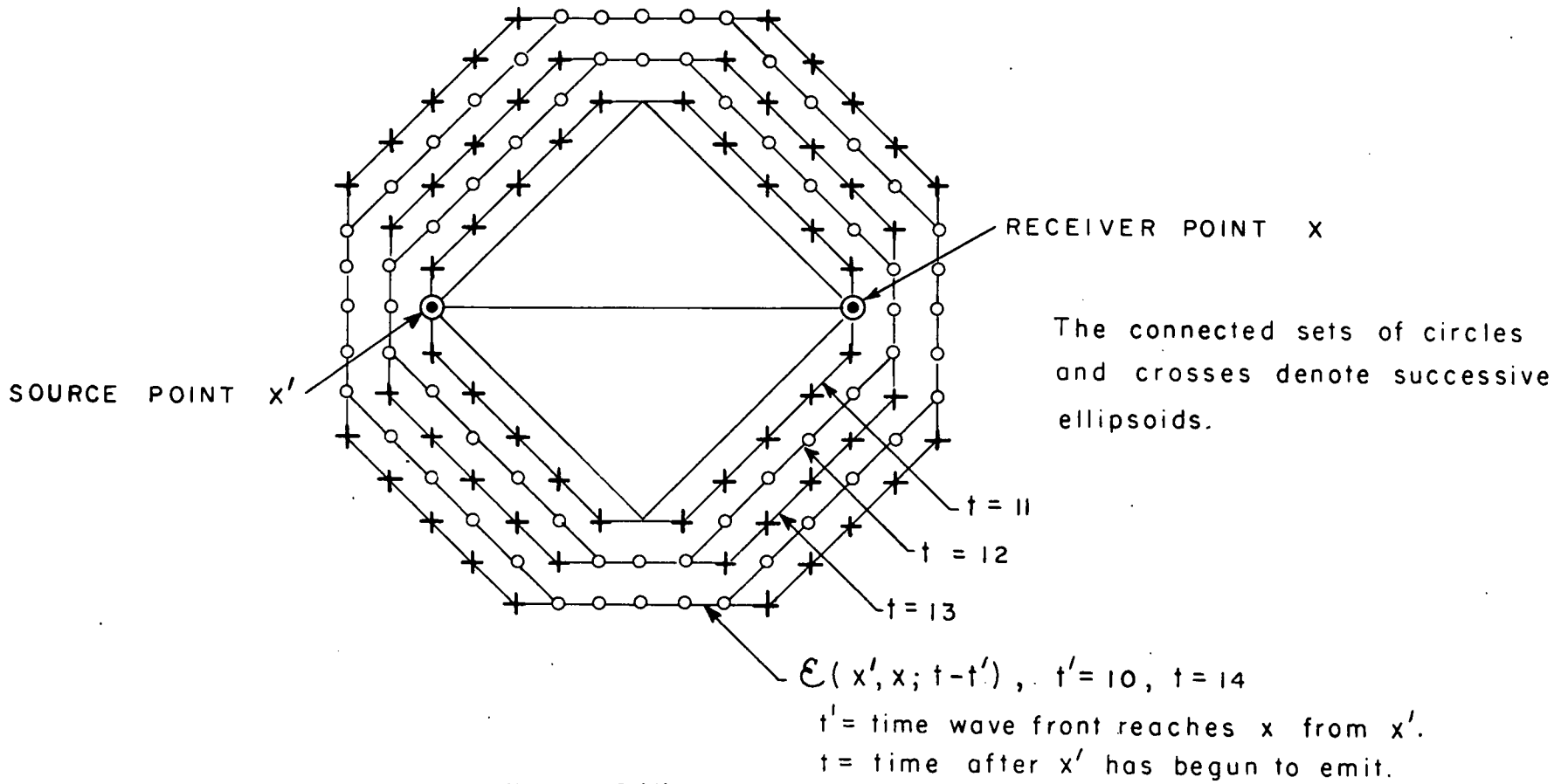
RECEIVER POINT X

The connected sets of circles
and crosses denote successive
ellipsoids.

SOURCE POINT x′

t = 11

t = 12

t = 13

$\mathcal{E}(x′, x; t-t′)$, $t′ = 10$, $t = 14$

$t′ =$ time wave front reaches x from x′.
t = time after x′ has begun to emit.

Figure 2 (f)

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$X_n(Z)$ monolayer

$X_n(Y,Z)$ monoline

X_n monobloc

k

i

j

$X_n(X,Y,Z) \equiv (X,Y,Z) \equiv x_i$ point

i

j

i

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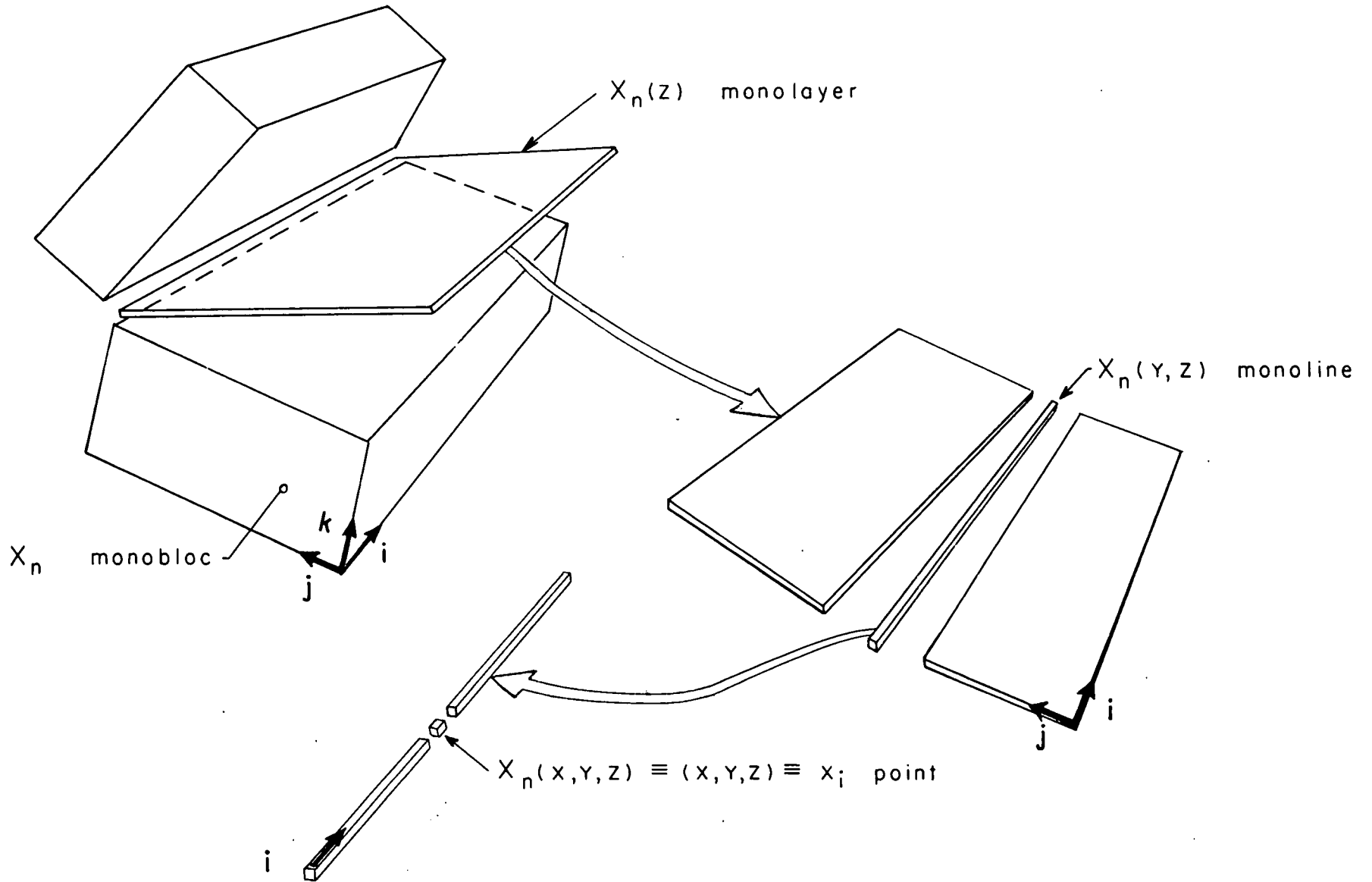
Figure 3

GEOMETRIC SETTING FOR THE PRINCIPLES OF INVARIANCE
ON THE MONOBLOC

Figure 4

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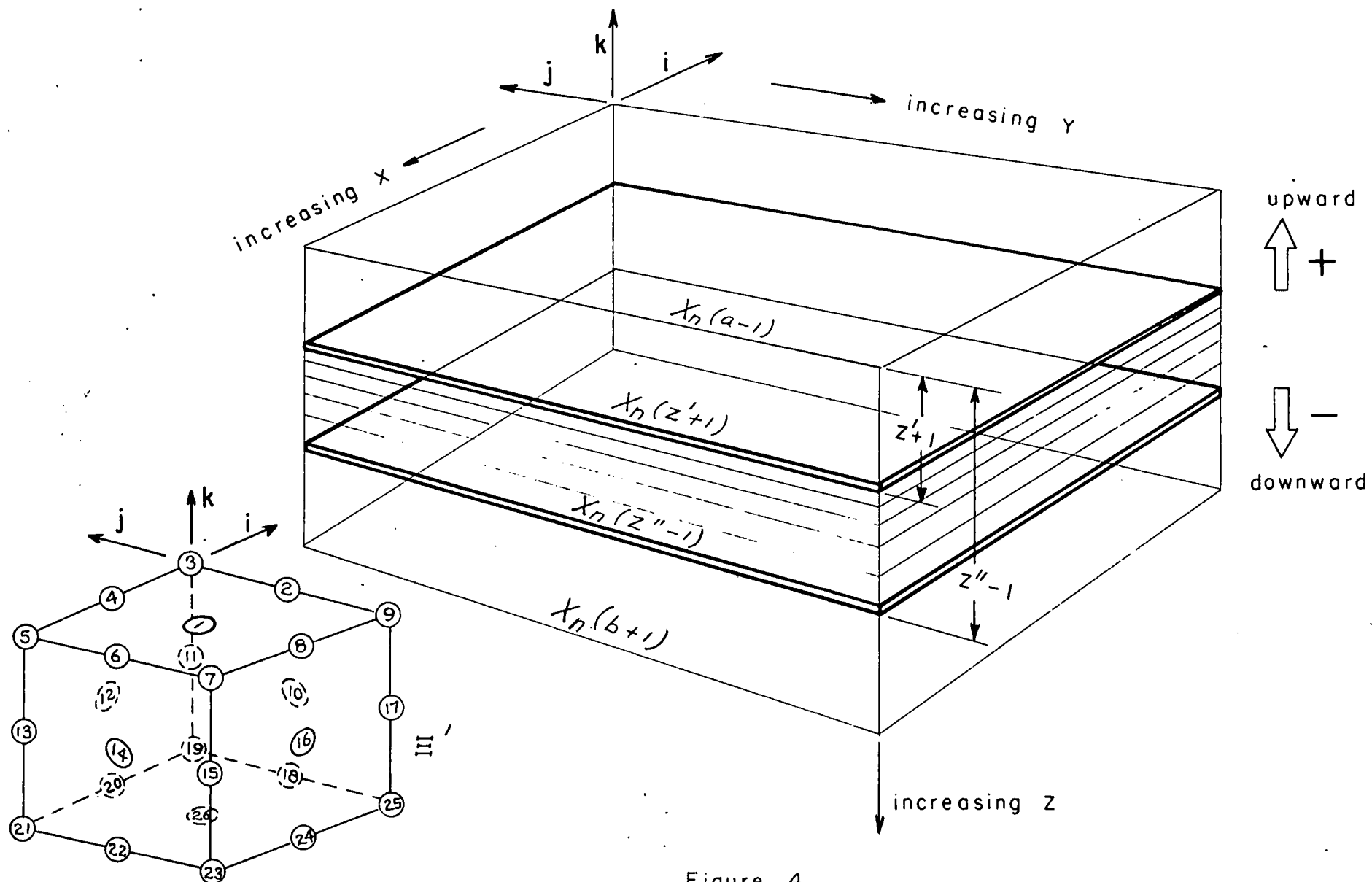
GEOMETRIC SETTING FOR THE PRINCIPLES OF INVARIANCE
IN A MONOLAYER

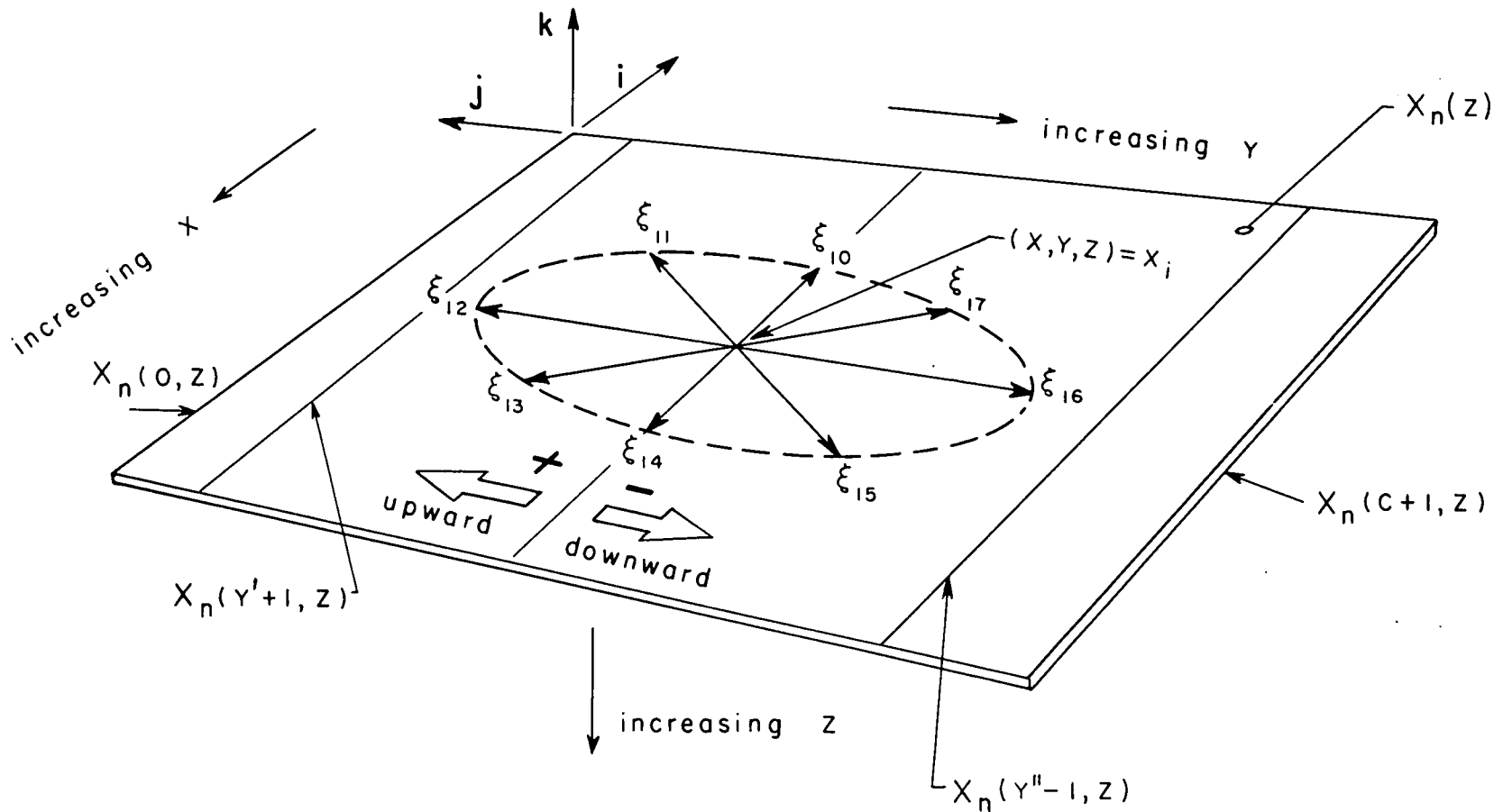Figure 5

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GEOMETRIC SETTING FOR THE PRINCIPLES OF INVARIANCE
IN A MONOLINE

Figure 6