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Leveraging Students' Intuitive Knowledge About the Formal Definition of a Limit

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Leveraging Students' Intuitive Knowledge About the Formal Definition of a Limit

By

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Committee in charge:

Professor Alan H. Schoenfeld, Chair
Professor Andrea A. diSessa
Professor Ole H. Hald

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Abstract

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This dissertation explores the roles of students' intuitive knowledge in learning formal mathematics. The formal definition of a limit, or the epsilon-delta definition, is a critical topic in calculus for mathematics majors' development. It is typically the first occasion when students engage with rigorous, formal mathematics. Research has documented that the formal definition is a roadblock for most students in calculus, but has also de-emphasized the productive role of their prior knowledge and sense making processes. The *temporal order of delta and epsilon* has been suggested as a conceptual obstacle for students in understanding the structure of the formal definition. The dissertation investigates the nature of and the degree to which the temporal order of delta and epsilon is a difficulty for students. The fine-grained analysis of semi-structured interviews with elementary calculus students reveals a large repertoire of reasoning patterns about the temporal order. A microgenetic study of one student shows the diversity of knowledge resources and the complex process of reasoning. *Knowledge in Pieces* (diSessa, 1993) and *Microgenetic Learning Analysis* (Parnafes & diSessa, 2013, Schoenfeld, Smith & Arcavi, 1993) provide frameworks to explore the details of the structure of students' prior knowledge and their role in learning the topic. The study offers and examines the impact of an instructional treatment called the Pancake Story, designed specifically to productively link to students' intuitive knowledge. Leveraging the notion of *quality control*, the instructional treatment offers an alternative to the idea of *functional dependence* in reasoning about the temporal order of delta and epsilon. A detailed case study shows the process by which a student incorporates resources from the story into her existing knowledge about the temporal order. The findings in this dissertation support the claim that understanding the process of learning requires serious accounting for student's prior knowledge.

This dissertation is dedicated to the memory of Randi A. Engle

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CHAPTER 1: INTRODUCTION AND OVERVIEW

In 2012, the President's Council of Advisors on Science and Technology (PCAST) called for 1 million *additional* college graduates in Science, Technology, Engineering, and Mathematics (STEM) fields based on economic forecasts (Executive Office of the President, PCAST, 2012). Within STEM, the number of mathematics graduates is very low. For example, UC Berkeley Common Data Set for 2012–2013 (University of California, Berkeley, 2012) reported that mathematics accounted for 3% of the degrees conferred, whereas engineering and the biological sciences accounted for 12% and 13% respectively.[1]

Calculus is the first opportunity for students to engage with theoretical mathematics and to make the transition into advanced mathematical thinking. Calculus courses often act as a gatekeeper into mathematics and other STEM majors. However, exemplary mathematics programs use them as the primary source for recruiting mathematics majors (Tucker, 1996).

The formal definition of a limit at a point, typically referred to as the epsilon-delta definition, is an essential topic in mathematics majors' development, which is often introduced in calculus. I introduce it here briefly for reference. I explore the conceptual underpinnings and conjectured student's learning trajectories for the formal definition in a later chapter.

The epsilon-delta definition of a limit, hereafter the formal definition, says that the limit of a function $f(x)$ as x approaches a is L—*written as follows*—

$$\lim_{x \to a} f(x) = L$$

if and only if, for every number $\varepsilon > 0$, there exists a number $\delta > 0$, such that all numbers x that are within δ of a (but not equal to a), yield $f(x)$ values that are within ε of the limit L. This is often written as "for every number $\varepsilon > 0$, there exists a number $\delta > 0$ such that if $0 < |x - a| < \delta$ then $|f(x) - L| < \varepsilon$".[2] Informally, one might say, "If L is the limit, then for however close you want $f(x)$ to be to L, I will be able to constrain the x-values so that it is." I return to this intuitive idea shortly.

The formal definition provides the technical tools for demonstrating how a limit works and introduces students to the rigor of calculus. Yet research shows that thoughtful efforts at instruction at most leave students—including intending and continuing mathematics majors—confused or with a procedural understanding about the formal definition (Cottrill, Dubinsky, Nichols, Schwingendorf, Thomas and Vidakovic, 1996; Oehrtman, 2008; Tall & Vinner, 1981).

Although studies have documented that the formal definition is a roadblock for most students, little is known about how students actually attempt to make sense of the topic, or about the details of their difficulties. Most studies have not prioritized students' sense making processes and the productive role of their prior knowledge (Davis & Vinner, 1986; Przenioslo, 2004; Williams, 2001). This may explain the reported minimal success of their instructional approaches (Cottrill et al., 1996; Davis & Vinner, 1986; Tall & Vinner, 1981; Williams, 2001). Thus, understanding the difficulty in the teaching and learning of the formal definition warrants a closer look—with a focus on student cognition and with attention to students' prior knowledge.

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[1] Stanford University reported similar numbers with 3.35% for mathematics and 14.97% for engineering (Stanford University, 2012).

[2] For the complete definition and an example of an epsilon-delta proof, see Appendix A.

It also calls for a theoretical and analytical framework that focuses on understanding the nature and role of students' intuitive knowledge in the process of learning.

Knowledge in Pieces (KiP) as a theoretical framework (diSessa, 1993; Smith et al., 1993) and a related analytical framework, Microgenetic Learning Analysis (MLA) (Parnafes & diSessa, 2013; Schoenfeld, Smith and Arcavi, 1993) share the focus of characterizing the nature and structure of students' intuitive knowledge, and documenting moment-by-moment or micro changes in learning—a microgenesis analysis on knowledge and learning. Originally developed in the context of physics education, the framework has been proven useful to provide detailed accounts of student cognition in mathematics (Campbell, 2011; Wagner, 2006; Pratt & Noss, 2002). This dissertation extends the application of KiP and MLA to the investigation of student cognition of the formal definition of limit.

A small number of studies has begun to explore more specifically students' understanding of the formal definition (Boester, 2008; Knapp and Oehrtman, 2005; Roh, 2009; Swinyard, 2011; Swinyard & Larsen, 2012). These studies suggest that students' understanding of a crucial relationship between two quantities featured in the formal definition, epsilon (ε) and delta (δ), warrants further investigation. Davis and Vinner (1986) used the term *temporal order* to describe the relationship (p. 295). Davis and Vinner (1986) used the phrase *temporal order* to describe the relationship between ε and N in the formal definition of a limit of a sequence, but it also describes the relationship between ε and δ. For the rest of the dissertation I use the phrase *the temporal order* to refer to the temporal order of delta and epsilon, unless otherwise specified.

Epsilon and delta in the formal definition follow the sequential order epsilon first, then delta. The authors find that students often neglect the important role of the temporal order. Swinyard (2011) posits that the relationship between the two quantities was one of the most challenging aspects of the formal definition for the students in his case study. Knapp and Oehrtman (2005) and Roh (2009) document this difficulty for advanced calculus students. This difficulty was also prevalent among the majority of calculus students who struggled with the formal definition in Boester (2008). While studies have shown the existence and prevalence of this difficulty, little is known about why this important relationship is difficult for students.

The relationship between the quantities epsilon and delta in the definition can be described using the idea of quality control in manufacturing an item. The conceptual structure at issue can be described as follows: given a permissible error in the measurement of the output (ε), one determines a way to control the input to achieve that result. One does so by determining the permissible error in the measurement of the input (δ) based on the given parameter for the output (ε).[3] In this way the error bounds follow the following sequential order, error bound for the output, then the error bound for the input. This is because the error bound for the output is given. In some ways, the error bound for the input could be seen as being *dependent* on the given error bound for the output. Epsilon can be seen as the error bound of the output whereas delta is the error bound for the input. Therefore, the epsilon and delta follow the order of epsilon first, and then delta; delta can thus be seen as depending on epsilon.

A working hypothesis for this dissertation is that relevant and important intuitive knowledge resides in everyday understanding of quality control. Therefore, I have developed an instructional analogy called the Pancake Story, in which the idea of quality control is

[3] It is atypical to associate the term "error" to the input because errors usually occur in the output. However, I used the word error to describe deviations in the input and the output in order to conceptualize both epsilon and delta as error bounds.

exemplified in the context of pancake making (for the complete story, see Appendix B). The story was designed to access students' intuitive knowledge and to assist students in making connections between their prior knowledge and the formal definition.

This dissertation explores how calculus students make sense of the formal definition of limit in relation to their intuitive knowledge. Specifically, it investigates students' understanding of the temporal order of delta and epsilon, an instrumental relationship that sets the structure of the formal definition. Through a fine-grained analysis of student interviews, this study addresses the following questions:

1. How do students make sense of the temporal order of delta and epsilon?
2. How does the Pancake Story influence students' understanding of the temporal order?

My hypothesis, consistent with the literature, is that many students would *not* conclude that epsilon came first. Students would have a large repertoire of ideas about the temporal order, and that many of these ideas suggest that delta came first. I anticipate that students would take up the intuitive ideas from the story. However, the process of aligning their prior knowledge with productive resources from the story would be complex (and interesting!).

This study is one of the first microgenetic studies of a topic at the heart of formal mathematical thinking. It focuses on the detailed characterization of the accessible resources students can bring to the topic. A detailed understanding of the complexity of student cognition can assist in making this historically challenging topic more accessible for students. One can then develop instruction that is analogous to or even more productive than the current instructional treatment, but importantly, one that attends to students' prior knowledge and allows them to make connections to the formal definition of a limit.

Overview of Dissertation

The dissertation was organized around two strands of analysis. The first strand of analysis aims to uncover the structure of knowledge and the process of sense making about the temporal order of delta and epsilon as a result of the students' prior instruction. The second strand of the analysis focuses on the influence of the Pancake Story on students' understanding. It seeks to elaborate on the process of incorporating productive ideas from the story and aligning them with prior knowledge. The first strand of analysis is mainly explored empirically in Chapter 5 and 6, and Chapter 7 and 8 undertake the second strand of analysis. Chapters prior to the analysis chapters motivate and set up the discussion; chapters coming afterward tie together the main findings and discuss their implications.

Chapter 2 elaborates on Knowledge in Pieces as a theoretical framework of this dissertation. It discusses some of the theoretical assumptions that guide the interpretation of data in the analysis. This theoretical framework also provides lens through which I review the literature in Chapter 3.

The literature review in Chapter 3 argues two main points. First, the review motivates students' understanding of the temporal order of delta and epsilon as a topic of investigation. Second, it argues for the particular methodology used in this dissertation: a microgenetic study of students' thinking focusing on students' prior knowledge, including their intuitive knowledge.

Chapter 4 specifies the source of data for the analysis. It discusses the population of students whose knowledge is discussed in this dissertation. The students' gender, ethnic and racial background, and academic majors are highlighted to create a multidimensional representation of who the students were. This chapter also reviews the design and revisions of the interview protocol and the Pancake Story.

Chapter 5 takes a first look at students' sense making by investigating their *reasoning patterns* about the temporal order. I define a reasoning pattern as a common way that a group of students form a justification for the topic at hand. The chapter includes the way that I identify reasoning patterns in the data. This chapter also includes documentation of the claims students make about the temporal order based on their prior instruction.

In Chapter 6, I present a microgenetic study of a student's sense making process of the temporal order. I describe the way that the student negotiated and navigated through different *knowledge resources* before settling with the claim that epsilon came first. I define a knowledge resource as an idea consisting of a single or a collection of interrelated knowledge elements with a utility in a particular context. Particular methodology in identifying knowledge resources is included in this chapter.

In Chapter 7, I describe productive intuitive ideas from the story that students took up. I document (new) reasoning patterns that emerged after students engaged with the story. I also report the changes that I observed in students' claim about the temporal order after engaging with the story.

The last analysis chapter, Chapter 8, focuses on a (different) student aligning productive resources from the story with her prior knowledge to make sense of the temporal order. This chapter focuses on which knowledge resources the student took up, and how she negotiated them with some of her prior knowledge. It also discusses some of the affordances of the Pancake Story for the student's sense making process.

The dissertation closes with a summary and a discussion of the findings from the four analytic chapters in Chapter 9. The discussion places the findings in the context of the broader literature discussed in Chapter 3. Theoretical, methodological and practical implications for the study wrap up the presentation of this study.

CHAPTER 2: THEORETICAL AND ANALYTICAL FRAMEWORK

This chapter explains the theoretical perspective I take in this dissertation. Knowledge in Pieces influenced the design of the study and the analysis. In this chapter I discuss the theoretical framework for the study, and some of the relevant theoretical assumptions about knowledge and learning. This chapter also elaborates on the construct *knowledge resource,* whose identification is one of the foci of the analysis in Chapter 6. Lastly, the theoretical perspective is also used as a lens through which I review the literature in the next chapter.

Knowledge in Pieces (KiP)

The Knowledge in Pieces (KiP) theoretical framework (diSessa, 1993; Smith et al., 1993) argues that knowledge can be modeled as a system of diverse elements and complex connections. The nature of the elements, their diversity and connections are typical interests for studies using this framework. Uncovering the fine-grained structure of knowledge is a major focus of investigation, and characterizing knowledge using generic ideas like "concept" or "theory", or the commonly used idea of "misconceptions" is viewed as uninformative and unproductive (Smith et al., 1993).

Theoretical Assumptions About Knowledge

Context specificity. One of the main principles of KiP is that knowledge is context specific (diSessa & Wagner, 2005, diSessa & Sherin, 1998, Smith et al., 1993). This means that the productivity of a piece of knowledge is highly dependent on the context in which it is used. In analysis, evidence for context-specificity of knowledge can be seen in the productivity of a piece of knowledge in one context but not in another. For example, the knowledge that "multiplication makes a number bigger" is productive in the context of multiplication with numbers larger than one. The knowledge is not productive in the context of multiplication with real numbers. Context variation can happen as a result of change in literal problem context, the passage of time or simply as knowledge is assessed more or less carefully.

KiP assumes that each piece of prior knowledge in the students' conceptual ecology (diSessa, 2002) is productive in *some* context. In contrast to studies that focus on identifying students' misconceptions, KiP focuses on building new knowledge on students' prior knowledge, instead of focusing on efforts to "replace" students' misconception (Smith et al., 1993). This means analysis focuses on ways that students build on their prior ideas while suspending judgment about their initial correctness.

Productivity and diversity of intuitive knowledge. KiP also posits that students have a lot of intuitive ideas that can and should be leveraged in instruction. Given that KiP was developed in the context of physics, it is not surprising that KiP assumes that students have a lot of intuitive ideas about physics originating from their everyday experience. While KiP has been used in the context of mathematics education (Campbell, 2011; Pratt and Noss, 2002; Wagner, 2006), it is an empirical question the degree to which intuitive ideas are prevalent in the way that students think about mathematics. The cited studies provide some evidence that suggest that the assumption might also hold in mathematics. Detailed study of students' use of intuitive knowledge in mathematics, especially its productive use of it, seems to be an excellent endeavor at this time.

Activation and Stability of Knowledge

Activation. KiP models the activation of knowledge elements in terms of *priority*. Knowledge elements are more likely to be activated if it had high *cueing priority* and/or *reliability priority.*

High cueing priority means that an element is more likely to be activated when other elements that are consistent with it are already activated. For example, the idea of functional dependence might have a high cueing priority when students are talking about functions and relationships between variables x and y. The opposite happens when other elements that contradict it are cued.

High reliability priority means that a knowledge element is more likely to stay activated because it has been proven useful in the particular context in the past. For example, calculating a limit by evaluating the value of the function at the point of interest or near the point of interest can have a high reliability priority because the method has reliably led students to the correct answer about the limit in the homework.

Stability. Context specificity also implies that contexts can also make the use of a piece of knowledge become unstable. diSessa (2004) considers this a dynamic aspect of contextuality. For example, a student might understand a particular idea in one context but a slight change in context might lead to a student thinking that the idea is no longer applicable.[4] diSessa (2004) posits that instability can be driven by a change in the presentation, language or modality of subject's expression (language, drawing or gesture).

At the same time, a piece of knowledge can become stably used (in a particular context) over time for a variety of different reasons. One possibility is that the idea develops a high reliability priority with respect to the particular context at issue, or even in a range of similar contexts. For example, returning to the high reliability of function evaluation to calculate limit, perhaps over time this method proves productive in calculating limit in multiple contexts. Then the idea of functional evaluation to calculate limit might become stable in the sense that it is regularly used in a range of contexts, and comes to feel "necessary," over time.

In addition to persistent high reliability, the knowledge element can also become stable because it is a "part of conceptual systems that contain many useful elements whose breadth and utility are not immediately apparent [to the analyst]." (Smith et al., 1993, p. 152). The authors provide an alternative explanation to what many would call deeply entrenched misconceptions. The strength of a particular knowledge might come from its connections to useful elements that might not be clear to the researcher. Thus stability can also come from connections to other stable knowledge elements.

Development of Knowledge

With these assumptions about knowledge, studies using KiP investigates the development of knowledge as knowledge elements being refined, elaborated, or incorporated with others to become a new conception. This means that new elements might be generated as a result of new experiences or that their priorities within the network might change. That is, certain elements become more or less important based on feedback from activation in the contexts. Next I discuss the construct of *reasoning patterns* and *knowledge resource* as a grain size of knowledge element that I use in this dissertation.

Reasoning Patterns and Knowledge Resources

Reasoning Patterns

Two of the four analysis chapters focus on the identification of *reasoning patterns* for the temporal order of delta and epsilon. I define a reasoning pattern as the essential common core of

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[4] Many in the field would consider this an issue of transfer, but KiP views this as issues of contextuality. For a full discussion of this topic see diSessa and Wagner (2005).

reasoning found in a range of students concerning the justification for a particular claim. This means that a group of student argued for the same claim by attending to particular information and attaching similar meanings to the information. For example, many students argued that epsilon depended on delta by attending to the statement "if $0 < | x - a | < \delta$ then $| f(x) - L | < \varepsilon$" and saying that the statement meant that delta implied epsilon. That is considered a "reasoning pattern" because a group of students attended to the same statement, attached the same meaning, to make the same conclusion.

Reasoning patterns are context specific. That is, if a piece of knowledge is assessed in multiple ways, a reasoning pattern in specific to the way that it is being assessed in the current situation. This means that attending to the same information and attaching the same meaning in one context is not necessarily the same reasoning pattern in a different context.

Since a reasoning pattern is a justification, then it is made up of several smaller knowledge elements. In this dissertation, the finer grain size of knowledge that I consider is what I call *knowledge resources*. Below, I include details about this construct, and elaborate on methods of identifying them in Chapter 6.

Knowledge Resources

I define *knowledge resource* as an idea consisting of a single or a small collection of interrelated knowledge elements with a particular utility in a particular context. A knowledge resource can be mathematical by nature, like a *functional dependence,* or something more informal or intuitive, like *proportional variation* (a small change in the independent variable leads to a small change in the dependent variable). In my analysis I aim to identify a particular type of knowledge resources. I focus on resources that describe the nature of, or roles in relationships between quantities. So for example, I would focus on resources like *functional dependence*, which explains the nature of a particular kind of relationship between two quantities, and de-emphasize the details of students' understanding of numbers.

KiP's assumptions about knowledge apply to knowledge resources. I assume that knowledge resources by nature are not random and a student's conceptual ecology includes ideas that are useful in various contexts. Knowledge resources are context specific. Their use and articulation might be different depending on the contexts in which they are cued.

Knowledge resources are viewed neutrally in terms of correctness. While knowledge resources themselves are not correct or incorrect, their application, with other ideas in a particular context, might be. For example, *functional dependence* as an idea is neutral in terms of correctness. However, when it is applied in a particular way to describe the relationship between epsilon and delta (see later analysis) then that would become incorrect.

A knowledge resource does not get "abandoned" arbitrarily. They might no longer be articulated, but a resource would not typically just disappear from a student's conceptual ecology. Instead, resources can gradually decrease in their (cueing) priority. Sometimes some knowledge resources are taken to be obvious and understood, so there is no need for it to be articulated.

A methodological orientation related to this theoretical assumption is that the analysis should aim to stay accountable to the dynamics of development of thinking and to the various contexts in which the knowledge resource is activated. That is, the analysis should optimally understand the activation of knowledge resources in various contexts by exploiting any relevant data that might inform the aim.

Knowledge resources can interact with one another. The nature of this interaction can be supportive or competitive. Interaction between knowledge resources is defined as a simultaneous activation of different knowledge resources, which influences the (cueing) priority (Kapon & diSessa, 2012; diSessa, 1993) of the relevant resources. For example, in a supportive interaction between a knowledge resource A and B, the activation of A increases the (cueing) priority of knowledge resource B. In a case when the two resources compete, the activation of A decreases the (cueing) priority of knowledge resource B.

One well-documented example of a knowledge resource is phenomenological primitives (p-prims) (diSessa, 1993). P-prims are small self-explanatory knowledge elements that describe and are established out of everyday experience. As described in diSessa (1993), p-prims are *phenomenological* in that the explanation are drawn from the behavior of things that people experience and observe. They are *primitives* in that they are self-explanatory, and so they are, in that sense, atomic level knowledge structures. When people are asked about a particular p-prim, they usually respond, "That's just the way things are." For example, Ohm's p-prim is the idea that more effort begets more result. It is the knowledge that allows a person to "know" that when s/he throws the ball harder, it would go further.

Pratt and Noss (2002), inspired by KiP, posit p-prim-like resources, which they call naïve knowledge, to describe the notion of randomness in mathematics. They found children use four separable resources to describe randomness: unpredictability (one does not know what is going to come out), unsteerability (no external agent are acting to cause the phenomenon), irregularity (lack of regular pattern in the phenomenon) and fairness (all parts of the system have to be equally likely).

I included the example of p-prims (diSessa) and naïve knowledge (Pratt and Noss) to exemplify well-documented knowledge resources. While some of the resources I identify in this dissertation might be p-prim like in size and function (or actual p-prims), the focus on knowledge resources is their utility in relating quantities.

CHAPTER 3: LITERATURE REVIEW

As noted in the introduction, there are two main goals for this literature review. The first is to motivate students' understanding of the temporal order of delta and epsilon as a topic of investigation. The second is to establish the importance of microgenetic studies to understand the structure of knowledge and the process of understanding the temporal order in the formal definition. In this review I also include the relationship of the Pancake Story to the literature about students' understanding of the formal definition.

Motivating Students' Understanding of the Temporal Order of Delta and Epsilon

> Several education researchers and curriculum committees have concluded that carrying formal limit proofs forward throughout an introductory calculus sequence might be successful in preparing a small number of the most talented students for further studies in advanced mathematics, but it leaves the vast majority of students with little more than a procedural understanding and an impression of mathematics as personally incomprehensible (Oehrtman, 2008, p. 67).

This quote, from Oehrtman's report in the Mathematical Association of America's *Notes* from 2008, summarizes the general opinion about the teaching of the formal definition in calculus. Teaching the formal definition is difficult and the resulting understanding is often limited. The research community continues to investigate the nature of this difficulty. Following are suggested sources of difficulty of learning the formal definition of a limit.

Tension Between Dynamic and Static Conception of Limit

Early work around students' understanding of limit posits that students cannot move past their "dynamic" understanding of limit—limit is the number that the function approaches but never reaches (e.g., Cottrill et al., 1996; Williams, 1991). These studies argue that students lack an understanding of the more "static" conception of limit, which is more closely related to the formal definition.

Many studies assert that students' "dynamic conception" is problematic and would lead to other misconceptions (Parameswaran, 2006; Przenioslo, 2004; Williams, 2001; Tall and Vinner, 1981). Thus, in many of these studies, any discussion about students' understanding of the formal definition was mostly about it being the ideal conception that needed to *replace* the problematic "dynamic" conception. The following quote exemplifies the kinds of conclusions these studies make about students' understanding of the formal definition:

> Students would not notice a contradiction between [the formal] definition and his or her other, more "private" conceptions, and worse, would not try to confront the two parts of his or her knowledge. More importantly still, for the majority of students the definition was not the most significant element of the image /…/ This could be a consequence of unsatisfactory understanding and inability to interpret the very formulation of the definition. Students appeared to lack a sense of the role of definitions in mathematics in general, and were convinced that their various associations determined the meaning of the concept." (Przenioslo, 2004, p. 129)

These studies provide little insights into ways of successfully assisting students with the formal definition.

Other studies focus on ways to build the formal definition on students' dynamic conception (Boester, 2008; Roh, 2008; Swinyard, 2011; Swinyard & Larsen, 2012). Cottrill et al., (1996) argued that the dynamic conception of limit is a necessary stage before students can understand the formal definition. The authors theorize stages of development of understanding the limit concept. They assert that understanding the concept of a limit of a function at a point involves the development of the dynamic processes of x approaching a and $f(x)$ approaching b, coordinating the two process of "approaching," encapsulating each into *object* and finally applying the (logical) quantification *schema*[5] in order to be able to apply the formal definition to specific situations. The authors could not empirically validate the later stages of development they theorized in their study. Swinyard and Larsen (2012) provide an elaboration of ways students can go through the later stages of development and learn the formal definition from the dynamic conception. The study was done in the context of *guided reinvention* of the formal definition (Gravemeijer & Doorman, 1999). The authors offered one possible pathway of learning the formal definition.

Boester (2008) and Roh (2008) both argue for the development of the formal definition alongside the dynamic conception. Boester (2008) found that students in his study developed the dynamic and static conceptions separately and over time merged the two conceptions. He found that some students make the static conception dynamic by allowing both epsilon and delta to decrease to zero. Roh (2008) found that some of the students in her study used dynamic language to describe a limit, but were able to identify the correct definition by attending to the uniqueness of the limit.[6]

In sum, studies that build on students' dynamic understanding show more progress with assisting students to understand the formal definition. From the Knowledge in Pieces perspective, this is not surprising. Learning happens as a result of building on students' prior knowledge, and not replacing students' misconceptions. The main principle behind constructivism is the presumed continuity between students' new form of understanding and its precursor form. Ignoring students' dynamic conception, or worse attempting to replace it with the formal definition proves unproductive in helping students' make sense of the formal definition of a limit.

Inequalities with Absolute Values

Several studies have documented students' struggle with the absolute value notation with inequalities in the formal definition (Boester, 2008; Fernández, 2004; Oehrtman, 2008). Fernández (2004) found that students in her study did not know how to interpret the inequalities algebraically or geometrically. Boester (2008) found that one of the challenges in learning the definition for students in his study was that many of them saw absolute value as an operation to make numbers positive instead of seeing it to signify a range of values. In fact, many students in the study simply ignored the use of the absolute value.

In the lesson that Fernández (2004) developed as part of a classroom instructional treatment, Fernández used an interval notation instead of absolute value notation to emphasize the

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[5] I later return to this issue of logical quantification as a source of difficulty in learning the formal definition.

[6] Roh (2008) studies the limit of a sequence, not the limit of a function. I included it in the discussion because the two formal definitions share analogous structure.

connection between the algebraic and geometric interpretation of the intervals. In the lesson she and the students generalized the game of coming up with delta for any given epsilon into a version of the formal definition using the set notation to describe the intervals. What she calls version 1 of the definition says:

Given $\lim_{x \to a} f(x) = L$.
To ensure we can get $f(x)$ within a distance ε of L, we need
to find a distance δ around a so that
if x lands within δ of a, this implies $f(x)$ lies within ε of L
OR
if $x \in (a - \delta, a + \delta)$, then $f(x) \in (L - \varepsilon, L + \varepsilon)$.

She found that afterwards it was easy to relate Version 1 and its interval notation to the usual definition and its absolute value notation. She reported that avoiding the version with the absolute value notation "proved a big step in raising students' comfort level for studying this concept" (p. 50). The author had students' answered some formal assessment items and allowed them to use either version of the formal definition. Forty of the 48 students preferred using Version 1, and the majority of her students was able to provide a correct solution to prove the limit of a linear function at a point at the end of the semester.

Oehrtman (2008) and Fernández (2004) also discussed another issue with the inequalities in the definition. The absolute value of the difference between x and a is greater than zero ($0 < |x - a| < \delta$), whereas the one for the difference between $f(x)$ and L is not ($|f(x) - L| < \varepsilon$).[7] Fernández (2004) reported that students did not understand why that was true. Oehrtman (2008) argues that not recognizing the distinction between the way x approaches a and the way $f(x)$ approaches L—as expressed by these inequalities—contributes to students' misconceptions about limit. Fernández (2004) found that changing the definition by using set notation proved helpful in helping students understand the distinction. The students naturally got into a discussion about a function that was not defined at *a,* but whose limit existed at *a,* to discuss the importance of the 0 in the delta inequality. They also discussed the constant function to discuss why there was not 0 in the epsilon inequality.

In sum, while students' struggle with absolute values and inequalities is well documented, for now the change in notation and directed discussion using illustrative examples seem to help some group of students. As Boester (2008) asserted, more research needs to be done with students' conception of absolute values in inequalities.

Logical Quantification and the Meaning of Epsilon and Delta

The literature on students' understanding of the formal definition agrees that the use of quantifiers in the definition adds another layer of complexity to the formal definition (Boester, 2008; Cottrill et al., 1996; Przenioslo, 2004; Roh, 2009; Swinyard, 2011; Swinyard & Larsen, 2012, Tall & Vinner, 1981). Students have to make sense of the statement "*for every* number $\varepsilon >$ 0," and the statement "*for every* number $\varepsilon > 0$, *there exists* a number $\delta > 0$." The first statement stipulates the desired arbitrary closeness between $f(x)$ and L. The second statement sets the structure for the formal definition. "For every number $\varepsilon > 0$, there exists a number $\delta > 0$" says

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[7] $|x - a|$ is larger than zero because of the stipulation that the value of the function when $x = a$ is irrelevant for the limit.

that delta is determined by the given epsilon. The statement sets up the temporal order of delta and epsilon. Students need to grapple with both the use of quantifiers and the logic connecting them along with the different variables within the definition.

"*For every* number $\varepsilon > 0$." Davis and Vinner (1986) included the role of the word "any" or "every" in the definition as one of the reasons for why the theory of limit succeeds. Roh (2009) found that the arbitrariness of epsilon was difficult for students in her study to learn. Two of the four categories of students' understanding she found did not account for the arbitrariness of epsilon. Przenioslo (2004) found that some students interpret the phrase "for any" or "an arbitrary epsilon" to mean "for one arbitrarily chosen epsilon."

Swinyard (2011) and Swinyard and Larsen (2012) also found that defining what it meant for the function to be "infinitely close" to the limit was one of the main cognitive challenges in *reinventing* the formal definition for their participants. In the definition this is accounted for by the arbitrariness of epsilon. In the teaching experiment, the authors asked students to define the meaning of a limit at infinity as a scaffold to gain what they call the "arbitrary closeness perspective." This proved helpful for the two students in their case study to understand this idea in the formal definition. The nature of students' understanding of this aspect of the definition warrants further investigation. However, there is another aspect of the definition that the literature seems to suggest to take precedence to the arbitrariness of epsilon: the *temporal order* of delta and epsilon.

The temporal order of delta and epsilon. Studies that focus on the development of students' understanding of the formal definition (Roh, 2009; Swinyard, 2011; Swinyard & Larsen, 2012) suggest that students understand the temporal order of delta and epsilon prior to tackling the arbitrariness of epsilon. In the analogous context of limit of a sequence, Roh (2009) found that students who focused on the arbitrariness of epsilon, prior to understanding the relationship between epsilon and N (the delta in the context of a sequence) had an understanding of the definition that diverged from the correct version of the formal definition. The students made epsilon arbitrarily small before determining the appropriate N. This seems consistent with Przenioslo's (2004) finding about "one arbitrarily chosen epsilon."

In Swinyard (2011), the pair of students struggled with adopting "the y-first perspective" to account for the temporal order of delta and epsilon in reinventing the formal definition. They had to deal with this cognitive struggle before they dealt with the arbitrariness of epsilon. The author reported that given the students' "inclination to reason from the x first perspective posed a significant challenge" to reinvent the formal definition. The author, through a series of questions and attempts to leverage the benefit of zooming along the y-axis—to analyze the local behavior of the function—was able to shift the students' attention to the y-axis. The author finally introduced the context of limit at infinity to help the students to adopt the y-first perspective. In sum, a great deal of effort was made to have students adopt this perspective, and it happened before the students began to deal with the arbitrariness of epsilon.

Davis and Vinner (1986) used the term temporal order to describe the sequential order of ε and N in the definition of a limit of a sequence. They included the temporal order of ε and N as one of the aspects of the definition that their participants often "neglected." Knapp and Oehrtman (2005) provide an example of a group of students who neglected the temporal order of delta and epsilon and illustrate its implication to the students' proof. Students in their study argued that "First we resolved δ then we go on to resolve ε." I found similar accounts from students in the data in Boester (2008).

Boester (2008) posits that the logical quantification, including the temporal order is difficult for students. The study did not focus on students' understanding of the temporal order. However, I found that 5 of the 8 students in the focus group struggled with the logical quantification. All five students (Lisa, Jason, Erica, Harriet and Daniel) who struggled with the logical quantification stated that epsilon depended on delta. For example, Erica on her second interview said, "As delta gets smaller, epsilon gets smaller, too" (line 14, p. 92).

The different studies seem to suggest that the idea of *functional dependence* (y depends on x) might have been a dominant resource in the students' knowledge structure. This was a common justification found in Adiredja and James (2013).[8] The case study with seven students show that all five students who struggled with the temporal order argued that epsilon depended on delta because $f(x)$ depended on x and delta was like x and epsilon was like y. The authors also found that many students used their interpretation of the statement "if $0 < |x - a| < \delta$ then $|f(x) - L| < \varepsilon$" to conclude that the delta had to be satisfied first before epsilon. This finding is consistent with what the students said in Boester (2008) and Knapp and Oehrtman (2005).

Przenioslo (2004) also argues for the dominance of *functional dependence* when she found that many students reversed the order of the quantifiers in the definition. That is, instead of evaluating the statement "*for every* number $\varepsilon > 0$, *there exists* a number $\delta > 0$," students considered the statement "*there exists* a number $\delta > 0$ *for every* number $\varepsilon > 0$." The second statement suggests that delta does not depend on epsilon at all, whereas the first one delta does depend on epsilon. Moreover, the second statement follows the usual order of functional dependence. Thus, while different studies have offered examples of students' struggling with this aspect of the definition, we still know very little about the nature of this difficulty and how to assist students to make sense of it.

The meaning of delta and epsilon. It is reasonable to argue that students' understanding the meaning of epsilon and delta shapes their understanding of the temporal order. Fernández (2004) reported that many of the students in her study asked the question, "What are epsilon and delta?" The study documents that question as one of the issues students brought up in learning the formal definition. The study did not specifically discuss the impact of her lesson on students' understanding of this aspect of the definition.

The students in Boester (2008) and Knapp and Oehrtman (2005) who struggled with the temporal order of delta and epsilon might have treated delta and epsilon as errors ($|x - a|$ and $|f(x) - L|$, respectively). Interpreting the two utterances in terms of error seem to reveal a particular understanding of the formal definition. "First we resolved δ then we go on to resolve ε" might mean that the students would first address the error in the input then continue on to address the error in the output. "As delta gets smaller, epsilon gets smaller, too" might mean as the error in the input gets smaller, then the error in the output also gets smaller. Without knowing the details of the utterance I can only speculate on the meaning of these utterances. However, it suggests that the meaning students' attribute to the epsilon and delta should be considered in interpreting their understanding of the temporal order.

Summary

There is ample evidence in the literature to conclude that learning the formal definition of a limit is difficult. While students' understanding of the formal definition can develop alongside

[8] Adiredja & James (2013) is a pilot study for this dissertation. The data from that study was reanalyzed and its findings are included in Chapter 5.

their dynamic conception of limit, it would not be productive, and even futile to attempt to replace students' dynamic conception with the static conception.

Recent research on students' understanding of the formal definition suggests that the challenge lies in the logical quantification in the formal definition. The two main obstacles are the arbitrariness of epsilon and the temporal order of delta and epsilon. There is evidence to suggest that it is important for students to understand the temporal order of delta and epsilon prior to engaging with the arbitrariness of epsilon.

Little is known about the nature of this difficulty. The latest findings seem to suggest potential resources to investigate. Specifically, the functional dependence relationship and the meaning students attribute to delta and epsilon seem to have impact on the way that students conceptualize the temporal order. This dissertation explores this hypothesis.

Perspective on Students' Prior Knowledge

In this section, I aim to motivate the use of microgenetic methods to explore the development of students' knowledge structure and the role of students' intuitive knowledge in that development. Uncovering the structure of students' knowledge requires a constructive perspective on students' prior knowledge and a fine-grained method sensitive to subtle changes in understanding.

Misconceptions vs. Resources

Many studies of students' understanding of limit focus on identifying students' misconceptions about the topic and emphasize their negative role in learning (Bezuidenhout, 2001; Davis & Vinner, 1986; Jordaan, 2005; Parameswaran, 2007; Szydlik, 2000). While part of this pattern is historical—early studies of students' understanding of concepts focused on the identification of misconceptions—its presence in the last decade warrants a discussion.

Smith, diSessa and Roschelle (1993), in their review the broader literature on student misconceptions, argue that learning necessitates the transformation of prior understanding. Focusing on students' misconceptions as mistakes obscures the knowledge resources students have that might be productive for learning. They assert, "Learning difficult mathematical and scientific concepts will never be effortless, but neither will it be possible at all without the support, reuse and refinement of prior knowledge (p. 153).

Misconceptions are often overgeneralizations of knowledge to an inappropriate context. This idea of context specificity is not new to the limit literature. Cornu (1991) incorporates the idea of context specificity in his use of Guy Brousseau's definition of *epistemological obstacles*. Brousseau defines epistemological obstacles as "knowledge which functions well in a certain domain of activity and therefore becomes well-established, but then fails to work satisfactorily in another context where it malfunctions and leads to contradictions" (p. 159). From KiP's perspective, a piece of knowledge that is reliable over time becomes well-established (high *reliability priority*), which explains why students would cue them in contexts where it seems applicable.

Williams (1991) also noted the idea of context specificity, despite his unfavorable opinion of it. He wrote, "Students tended to accept as true many different statements of limits: some true, some false, and some incomplete. As an example, ST held five of the six questionnaire statements as being true throughout the study, and at the last session believed all six were true 'depending on to what function they were applied.' Thus, it is not so much that she believed all six as that she thought their truth depended on the situation in which they were applied." Knowledge is context specific! The author was hoping that the students in the study would "pick a best statement or model of limit and to appreciate that such a statement would provide a maximally useful and correct set of implications" (p. 232). If students have not seen the utility of the formal definition in most the contexts in which they work with limits, then why would they have any reason to replace their "misconception?"

Several studies in the recent years took on a much more favorable perspective of students' prior knowledge. Roh's (2009) epsilon strip activity attempts to assist students in understanding the formal definition of a limit of a sequence by leveraging students' ability to reason with counting. As mentioned earlier, Roh also found ways that students can build knowledge about the formal definition on top of their dynamic conception of limit (by focusing on the uniqueness of limit). Swinyard (2011) and Swinyard and Larsen (2012) also focused on developing an understanding of the formal definition from the students' prior knowledge. Their teaching experiment started with students generating examples of a limit of a function.

Boester (2008) and Oehrtman (2009) specifically explore the role of intuitive knowledge in students' understanding of a limit. Oehrtman (2009) explores dominant conceptual metaphors that students use in explaining different concepts in calculus. Oerhtman (2009) found that the approximation metaphor is the most common metaphor generated by students to make sense of the tasks in the study. [9] The resemblance of the approximation metaphor to the structure of the epsilon-delta definition leads the author to suggest its potential utility in instruction.

Boester (2008) designed an intervention that leverages students' intuitive knowledge. He used the Bolt Problem as a grounding metaphor for the logical quantification in the formal definition (see Appendix C for the complete problem). The Bolt Problem presents a context in which students make four-inches bolts with different degree of accuracy. The problem asks students "How do we create bolts that we know will be of a length that falls within our target range?" The ideal response to this question is that "for any bolt length tolerance (or output range), there is a raw materials tolerance (or input range), and if the amount of raw materials we put into the machine falls within the raw materials tolerance, then we will get a bolt with a length that falls within our bolt length tolerance" (p. 50). He used this as both an assignment in his classroom study, as well as a problem to discuss during student interviews. While the focus of the study was not on the success of the Bolt Problem, Boester found that the context of bolt making was productive for students to learn the quantification. Students in his study was able to explain the Bolt Problem by the end of the study, even thought some students were not able to explain the formal definition completely.

Both Boester (2008) and Oehrtman (2009) attend to students' prior knowledge, in particular their intuitive knowledge. The intervention used in this dissertation is a combination of the approximation metaphor and the Bolt Problem. Like the Bolt Problem, the Pancake Story leverages students' intuition about quality control. It embodies the approximation metaphor in that it explicitly talks about error and error bound and the refinement of errors over time. This dissertation explores the degree to which the intervention in this study assists students to make sense of the formal definition, and the temporal order more specifically.

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[9] The spontaneous use of approximation ideas for limits entails an approximation of an unknown quantity. For each approximation there's an *error* = |*unknown quantity-approximation*| and so the range for the actual value is *approximation – bound* < *unknown quantity* < *approximation* + *bound*. Accuracy is measured by size of error and a good approximation method allows for minimizing error. An approximation method is precise if there is not a significant difference among the approximations after a certain point of improving accuracy.

Structure of Knowledge: "Concepts" vs. Systems of Knowledge

In this section I contrast studies that use Tall and Vinner's (1981) framework of *concept image* and *concept definition* with studies that are consistent with the Knowledge in Pieces framework. I selected Tall and Vinner's (1981) framework in particular because of its influence in the post-secondary mathematics education research and its similarity to other studies that characterize students' knowledge using a general model of students' understanding (e.g., Williams, 1991).

Tall and Vinner's (1981) framework de-emphasizes the specificity in the structure of students' knowledge. It defines concept image as "the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes," whereas concept definition is defined as "the form of words that the student uses for his own explanation of his (evoked) concept image" (p. 152). The dynamics of the structures (i.e., the way they change over time) are not the focus of the framework.

Studies that use this framework typically focus on identifying different concept images that students have about limit (e.g., Jordaan, 2005; Roh 2009; Roh, 2008; Parameswaran, 2006; Przenioslo, 2004). Except for Roh (2008, 2009), most of the studies focus on documenting students' concept images, highlighting their limitation and how little of a role the formal definition play in students' concept image. Since the focus of most of these studies is in documenting students' concept images, ways to refine these images or ways that students can begin to incorporate the formal definition into their concept image and definition are rarely discussed.

Those studies stand in contrast with studies that use or consistent with the Knowledge in Pieces' framework. Most of the studies that use KiP focus on characterizing moment-by-moment changes in the structure of knowledge as a result of learning (e.g., Campbell, 2011, Wagner, 2006). Consistent with Knowledge in Pieces's model of knowledge resource, Schoenfeld, Smith and Arcavi (1993) conceptualize student knowledge as a knowledge system with different components.[10] The study focuses on micro-changes in a student's knowledge structure about linear functions. The study is illustrative of the kinds of study of knowledge that investigate the details of students' knowledge structure. Its focus on the details of the knowledge structure can reveal very useful information about the process of learning.

The main finding of Schoenfeld et al. (1993) is that new knowledge elements cannot be learned unless they are consistent with the student's fine-grained structure. The study illustrates this point with a case of a student learning properties of linear functions. What appeared to be a trivial error in reading the coordinate of points to calculate a slope, ended up being a symptom of a non-normative fine-grained structure of knowledge for thinking about the Cartesian plane and slope of a line. Not realizing the underlying issue at the time, the experimenter helped the participant to correct the coordinates. Although she was able to compute the correct slope, she kept returning to the same error in the following weeks. The issue was not resolved until after student and experimenter uncovered and refined the student's understanding of the Cartesian plane. The findings from this study suggest that new knowledge from instruction can be fleeting unless it is reconciled with the fine-grained structure of the knowledge of the student. This is informative for any studies that consider a learning intervention.

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[10] Schoenfeld et al. (1993) and the Knowledge in Pieces framework share core principles about knowledge, in addition to sharing similar methodology. For example, Schoenfeld et al. (1993) also focus on contextuality and cueing priority of knowledge elements.

In sum, we have reached the limit of usefulness of broadly identifying students' general models of understanding about the formal definition and limit. There is a need for studies that focus on the *process* of learning, and investigate the underlying structure of students' knowledge. In the next section I provide an illustrative example of a fine-grained study of a development of student knowledge structure from the literature. I emphasize the importance of the study's perspective in recognizing the productivity of students' prior knowledge.

Why Perspective Matters

Williams (2001) investigated the predication structure of students' conception of limits. The predication structure of limit establishes the meaning of limit within the broader set of meanings. The study presented students with 10 different statements about limits and asked students to judge what was similar or different about pairs of statements. The idea was to elicit constructs that students used to think about limits thereby uncovering the structure of knowledge about limit. For example, a student predication structure might involve the notion of *closeness* and it might be associated in the predication structure with the notion of *truth*.[11] Here the construct of closeness and truth were evoked in the student predication structure about limits. As a result the study would infer that the students conceptualize limit with the notion of closeness. The study presents two cases of a student and the development of their predication structure.

Through the fine-grained analysis of the structure of students' knowledge, the author shows refinements of students' knowledge structure over time. For example, one student, Gerry, started with the conception that limit was the value that the function got close to, and to find the limit he would evaluate the function at the point of interest. By the fifth interview session, Gerry took up the *sandwiching* idea, that a limit was the value that the function approached from both sides of *a*. By the end the author shows, using the student's predication structure, that Gerry focused more on the sandwiching idea and evaluating the function at a point was seen as appropriate only for continuous functions.

A second student, Jacob also showed refinement in his understanding of a limit. Jacob started with the idea that a function never reached its limit and that he would find a limit by finding the value that the function grew toward or got close to. As the weeks progressed, Jacob began to de-emphasize the idea of a reaching a limit and plugging in values that were far from the point of interest. By the end, he disassociated evaluating function at a point with computing limits, and prioritized thinking in terms of intervals (instead of plugging in points to find the limit). Thus the methodology used in this study was able to reveal refinements of knowledge elements in the structure.

The conclusion that the author made about students' understanding illustrates the importance of perspective in studying students' understanding about a topic. I include one of the author's final comments about the two students' understanding.

> Moreover, even though the experimental sessions were specifically designed to create some cognitive discord with this notion, both Gerry and Jacob *continued* to believe in it and still claimed it was their fundamental way of understanding limits. In this, they were not alone; nine of the 10 subjects in the study also remained convinced that this dynamic view of a limit was essentially *correct*. The remaining student could see it as problematic but had no competing scheme with which to *replace* it. In general, absent the mental action of iteratively

[11] The details of the construction of this predication structure were omitted from the paper. So I took the idea of association or implication between these notions at face value.

> choosing points and evaluating the function, *students seem to have very little with which to frame a theory of limits*! (emphasis added, Williams, 2001, p. 364).

The fixation with the dynamic vs. static view of a limit, and the idea of "replacing" misconceptions inhibited the author's ability to look at potential connections between the students' understanding and the static conception of a limit. For example, the notion of sandwiching can be built upon to discuss the use of symmetric intervals in the formal definition. The preference of thinking in terms of interval is crucial in understanding the formal definition. Despite their not being the goal of the study, these are important future direction of research around students' understanding of the formal definition.

Williams' (2001) study nicely summarizes the main points I am arguing in the second part of this chapter. It illustrates the utility of a fine-grained analysis of structure of students' knowledge. The methodology employed in that study allowed the author to document the subtle refinements of knowledge over time. At the same time it highlights the importance of constructive perspective on students' knowledge. The perspective the author took with respect to students' prior knowledge being misconceptions that needed to be replaced prevented him from recognizing and discussing the potential productivity of the students' knowledge in learning the formal definition of a limit.

Summary

The two parts of this literature review provide the broader context for this dissertation. The first part reviews open questions in the literature around the nature of the difficulty in learning the formal definition of a limit. Students' understanding of both the arbitrariness of epsilon and the temporal order of delta and epsilon warrant further investigation. There is some evidence in the literature that suggest that attending to the temporal order of delta and epsilon prior to making sense of the arbitrariness of epsilon is productive. Thus this dissertation focuses on students' understanding of the temporal order of delta and epsilon in the definition.

The second part of the review focuses on constructive perspectives on students' prior (and intuitive) knowledge in learning. Building off Boester's (2008) Bolt Problem and Oehrtman's (2009) approximation metaphor, this study uses an intervention that leverages students' intuitive knowledge about quality control that embodies the notion of approximation. More importantly, this study investigates the development of the structure of students' knowledge using a microgenetic method to detect subtle refinements of knowledge. Guided by the Knowledge in Pieces's perspective, this study investigates productive roles that prior knowledge can play in the process of learning.

CHAPTER 4: METHODS

This dissertation explores the roles of students' prior knowledge in their understanding the temporal order of delta and epsilon in the definition of limit. The study investigates the nature of and the degree to which the temporal order is a difficulty for students. It focuses on uncovering the details of the structure of students' knowledge about the temporal order of delta and epsilon.

The goal of providing a detailed account of knowledge structures and learning processes influenced methodological decisions for this dissertation. I used a relatively small number of research subjects, given the depth and detail of analysis that I was to conduct. That goal also favored the use of videotaped individual interviews. Interviews provide an opportunity for students to give an account of their understanding. I used individual, and not paired or group interviews because I was interested in individual student sense-making. Video recordings helped reveal nuances in the students' utterances by capturing gestures, gazes and body language.

The study prioritized the use of semi-structured interviews over clinical interviews because it also aimed to investigate the effectiveness of an instructional treatment. The protocol was designed so that it was possible to compare responses from all the students. Both the Pancake Story and the interview protocol were revised as a result of multiple rounds of pilot study. The Pancake Story was designed to leverage students' intuitive understanding of *quality control* as was said in Chapter 1. The details of the story are discussed in this chapter.

The three sections of this chapter elaborate on the details of the methodology employed in this chapter. The first section explores the understanding of the formal definition of a limit, including important ideas involved in the definition. The second section includes what was involved in data collection, including the development of the protocol and the Pancake Story. I close the chapter with an overview of the analysis methods. The details of the analysis methods are discussed in each analysis chapter.

Brief Analysis of the Mathematical Territory

Conceptual Territory

In the late 17th century, a limit was thought of as a bound that a number got close to but never reached. It was not until around the 18th century that the foundations of Calculus began to be developed, much through the instigation of Lagrange (Grabiner, 1983). Around this time Weierstrass and Cauchy introduced the rigorous treatment of a limit in analysis, via the formal definition of a limit (Klein, 1972; Grabiner, 1983; Pourciao, 2001). In 1817, Bernard Bolzano introduced the epsilon-delta definition.

Changing attitudes toward rigor were influenced by a number of factors, including an increasing interest in the foundations of calculus and the recognition by many mathematicians that further mathematical progress required developing approaches beyond existing methods (Grabiner, 1983). It is worth mentioning that part of the reason for the increase in foundations was the need of teaching. Grabiner writes, "Because teaching forces one to ask basic questions about the nature of the most important concepts, the change in the economic circumstances of mathematicians—the need to teach—provided a catalyst for the crystallization of the foundations of the calculus out of the historical and mathematical background" (p. 189).

The epsilon-delta definition of a limit, hereafter the formal definition, says that the limit of a function $f(x)$ as x approaches a is L and we write,

$$\lim_{x\to a} f(x) = L$$

if and only if, for every number $\varepsilon > 0$, there exists a number $\delta > 0$ such that all numbers x that are within δ of a yield $f(x)$ values that are within ε of the limit L. This is often written as "for every number $\varepsilon > 0$, there exists a number $\delta > 0$ such that if $0 < |x - a| < \delta$ then $|f(x) - L| < \varepsilon$".[12] This means that the limit of the function $f(x)$ as x approaches a is L, if and only if for however close one wants the values of $f(x)$ to be to L, there is a way to control how close x has to be to a in order to guarantee the desired closeness. The previous sentence is not a common description of the formal definition seen in most calculus textbooks. I include a graphical representation in the case of a linear function, and for a particular epsilon because the majority of the students in this study discussed the formal definition in that context (see Figure 4.1).

For students in calculus, the formal definition is typically used to either prove that the limit is some value L or that the limit does not exist. This involves knowing or at least having a conjecture for L. A more intuitive definition of a limit, where the limit L is the number that the function approaches as x approaches a, is embedded in the definition through $0 < |x - a| < \delta$ then $|f(x) - L| < \varepsilon$, which says that keeping x close to a would make $f(x)$ close to L.

The definition of $f(x)$ being continuous at $x = a$ is

$$\lim_{x \to a} f(x) = f(a)$$

This means that the limit of the function, L exists and it equals $f(a)$, which is also defined. Thus in practice, the formal definition is often used to prove general properties of continuous functions as opposed to proving the value of the limit of a function at a particular point.[13]

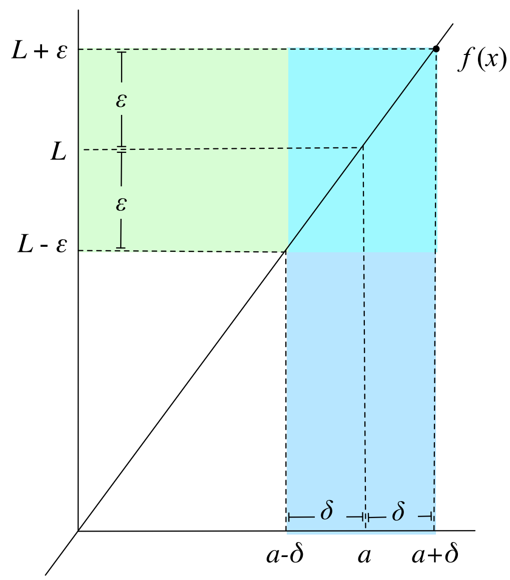

Figure 4.1. Graphical representation for the formal definition of a single ε of a linear function.

[12] For the remainder of the dissertation, I refer to the first part of the statement (for every number $\varepsilon > 0$, there exists a number $\delta > 0$), as the "for-all" statement, and the later part (if $0 < |x - a| < \delta$ then $|f(x) - L| < \varepsilon$), the "if-then" statement for brevity.

[13] In Real Analysis, many textbooks have students work with the formal definition of a limit of a sequence, first, before working with the epsilon-delta definition. The idea is to help students develop intuition for the structure of the formal definition from the simpler relationship between ε and N.

There are several elements to understand about the formal definition. In addition to understanding the behavior a function $f(x)$ at a particular point, students need to understand the meaning of the quantities epsilon and delta. Epsilon is the *bound* for the error between the function values and the limit, L, which is noted by the absolute value of the difference between $f(x)$ and L. In this way, epsilon constrains the acceptable range of $f(x)$ around L. Delta is the *bound* for the error between the input values x and the value a, which the function approaches. This is noted by the absolute value of the difference between x and a. In this way, delta is the constraint that controls the distance between x and a. If the limit exists, then choosing x values that are within the controlled distance to a would result in $f(x)$ values that are within epsilon away from L.

The statement "for every number $\varepsilon > 0$, there exists a number $\delta > 0$," or the "for-all" statement, sets the fundamental relationship between epsilon and delta and the structure of the formal definition. Logically "for every number $\varepsilon > 0$" means that one starts with the constraint epsilon. Then for that particular epsilon one must be able to determine the corresponding number delta. As I explain in Chapter 1, delta and epsilon follow the temporal order of epsilon then delta—epsilon comes first, and delta follows. In instruction, students are often told that delta depends on epsilon.[14] This statement sets up the structure of the argument for the definition where the closeness between the function and the limit (ε) is controlled by the closeness between x the values and a (δ).

According to the for-all statement, epsilon is an arbitrary number. The statement of the definition has to be true for every number epsilon. Sometimes "for every number" is written as "for any given number epsilon." Epsilon is any given number because the limit is presumed to be L, and so our function would be arbitrarily close to the limit L. "For *every* number epsilon greater than zero" is a systematic way to make the function to be as close as possible to L. Instead of using any particular epsilon, stating for *every* number epsilon means that we can make epsilon as small as possible, thus making our function as close as possible to the limit. In order to be able to vary epsilon, it is important for students to differentiate epsilon from the quantity $f(x)-L$. Epsilon is a constraint for that value, and this constraint can vary.

The last part of the definition, if $0<|x-a|<\delta$ then $|f(x)-L|<\varepsilon$, or the "if-then" statement, sets a condition for delta: all values of x that are within delta distance from a, will yield function values that are within epsilon distance from the limit. Embedded in this statement is another detail about limits. The inequality $0<|x-a|$ determines that the relevant values of x are those that are *close* to a, not a itself. That is, we are only concerned about the behavior of the function *near* a not at a.[15] Absolute value is used to set a symmetric interval around a and L.

Beginning Hypotheses About Early Stages of Understanding of Relevant Parts of the Formal Definition

Most traditional instruction about the formal definition in calculus focuses on verifying that the limit exists, particularly with linear and quadratic functions. Students are often introduced to the idea of working backwards, by breaking down $|f(x)-L|<\varepsilon$ to find the delta expression (see part 1 of the solution for the example in Appendix A). Studying and replicating arguments of this type often results in some procedural knowledge of writing proofs using the formal definition

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[14] A discussion about the constant function and its relationship to the temporal order is included in the discussion of the interview protocol in this chapter.

[15] The value of $f(a)$ affects the continuity of the function, not the existence of the limit at $x=a$.

(Oerhtman, 2008). I conjecture that the resulting students' understanding of the meaning of the formal definition would be limited. Below I present a hypothesis of early stages of students' learning trajectory based on the literature and my experience teaching this topic in calculus.

I posit that students in the early stages of understanding the formal definition would treat delta and epsilon as the actual difference between x and a and $f(x)$ and L, respectively. So instead of treating delta and epsilon as bounds ($0<|x-a|<\delta$ and $|f(x)-L|<\varepsilon$), they would argue that $|x-a|=\delta$ and $|f(x)-L|=\varepsilon$.

I also anticipate that most students would de-emphasize the importance of the statement "for every number $\varepsilon > 0$, there exists a number $\delta > 0$." Many would interpret that purpose of the statement was to setup delta and epsilon as quantities that were greater than zero, without imposing any relationship between them. Many of them would overlook the importance of the phrase "for every," which said that epsilon was an arbitrarily given number (Swinyard, 2011).

Missing these important elements, most students would then conclude that delta came first. Relying mostly on the symbols from if-then part, many students would mistakenly interpret this statement to mean that a statement about delta implied a statement about epsilon. Thus delta implied epsilon (Boester, 2008; Knapp & Oehrtman, 2005).

Students in the early stages of learning the definition would often believe that the limit was not known when working with the formal definition. For this reason, many would use the intuitive notion of limit—the limit is the value that $f(x)$ approaches as x approaches a—in interpreting the formal definition of a limit, and the temporal order.

Data Collection

Participants

Participants in this study were undergraduate students at a large public research university. Twenty-five students participated in the study, 7 in a pilot version and 18 in the full study. I recruited participants from students enrolled in more advanced calculus courses. Each of these students has received some form of instruction on the formal definition during their first semester calculus course. The breakdown of the students' racial and ethnic background and their academic majors are presented below (Table 4.1 & 4.2). This information was drawn from students' background survey that was administered at the end of the interview (see Appendix D).

Table 4.1
Racial and Ethnic Demographics of Participants

Race/Ethnicity	Number of students
African American	1
Asian/ Asian American[a]	11
Hispanic/ Latina/o	6
Middle Eastern	0
Native American or Alaska Native	0
Native Hawaiian or Pacific Islander	1
White (Non-Hispanic)	5
Other	1
Total	25

Note. [a]Includes Filipina/o.

Table 4.2
Students' Academic Majors

Academic Major	Number of students[a]
Architecture	1
Business	2
Computer Science	5
Chicano Studies	1
Economics[b]	2
Engineering[c]	7
Mathematics[d]	4
Molecular and Cell Biology	2
Nutritional Science	1
Political Science	1
Undeclared	2
Total	28

Note. [a]Some students have two majors so the total number exceeds 25.
[b]Includes Political Economy
[c]Includes Mechanical, Energy, and Electrical Engineering and Computer Science
[d]Includes Applied Mathematics

As the table shows, participants of the study were racially diverse. The student who chose the "other" category self-identified as North African. These students had a variety of different majors, ranging from different types of engineering to nutritional science. Many of these students were participants of a program at the university that focuses on supporting STEM majors. This explains the substantial representation of science and technical majors.

I include Table 4.1 to also give a better representation of the students whose knowledge I discuss in this dissertation. Knowledge is influenced by language, and given that many of these students' home languages are not English, accounting for students' race and ethnic background allowed me to stay mindful of any other knowledge resources outside of past instruction that might be relevant (e.g., home language). Moreover, there is a presumption that when we talk about students in research, the "generic" student is male and White unless otherwise stated..

Therefore, to challenge that assumption I made two deliberate choices in the presentation of data. First, I chose pseudonyms for students that were similar in gender and had similar origin to their actual name. If a non-European student had a European name, I did not change it to reflect their race. Similarly, a gender-neutral name was replaced by another gender-neutral name. In the presentation of data, I include the students' pseudonyms to help provide a better representation of the student. Simon was African American. Sheila, Silvia, Jane, Julia, Ryan, Patricia, Chen, Aruna, Erin, David, and Jacob were Asian. Katrina, Jose, Roberto, Guillermo, Sophia, and Adriana were Hispanic/Latina/o. Spencer was Native Hawaiian or Pacific Islander. Brian, Milo, Veronica, Dean and Adam were White. Lastly, Anwar was the North African student.

Second, I include a more elaborate description of the students that I used as case studies in Chapter 6 and 8. Chapter 6 discusses a case of a White male student, Adam. Chapter 8 discusses a case of a Chicana (Hispanic female) student, Adriana. I discuss the comparison of the two students and its implications in the discussion chapter. Thus, while this dissertation does not focus on issues of power and race, I aim to be mindful of students' race and background when I discuss their knowledge.

Study Procedure

The eighteen students for the latest iteration of the study were randomly split into two groups, 12 students in the Pancake Story group and 6 in the comparison group. The Pancake Story group was introduced to and engaged with the Pancake Story as an instructional intervention. Details about the story and its development are discussed after the next section.

The comparison group read a page about the formal definition that was adapted from Stewart's *Calculus*, 7th edition (see Appendix I). Re-reading the textbook seemed like a reasonable representation of what many students would do when they were confused about a topic. The selected text also included a discussion about the relationship between epsilon and delta.

The comparison group was used to make sure that changes that occurred after the pancake story were not a result of students' simply being re-asked the questions. Showing the effectiveness of the pancake story over re-reading the textbook is not the main goal of this study. Instead this study focuses on explaining the how and the why the pancake story works.

Audio and Video Recordings

I collected video and audio recordings of these interviews using one video camera and an audio recorder. The camera was pointed at an angle that faced the interviewee and captured the side or the back of the interviewer. The positioning of the camera maintained the focus on the student but was still able to capture gestures and gazes between the interviewer and the student. This provided additional information about the student utterances (Parnafes & diSessa, 2013; Jordan and Henderson, 1995). Even though the interaction (cf. Jordan and Henderson, 1995) was not the focus of investigation, I stayed mindful of the subtle ways that the interaction contributed to the conversation that occurred.

Attention was placed on the artifacts produced during the interview (e.g., drawing or writing). To maintain continuity between the written artifacts and the video, the camera zoomed in when a student was producing the artifact (Hall, 2007). This was only possible when a research assistant was available. Otherwise the camera maintained the broader frame capturing the student and the interviewer.

Transcripts and Segmenting of Episodes

I used Ochs (1979) for guidelines in transcriptions. Transcripts were organized by turns, marked by pauses and/or changes in speaker. They included non-verbal behaviors, including relevant gazes, laughter and gestures, which were placed immediately after the utterance. Any actions involving written artifacts were described in words, and the artifacts would be presented along with the transcript. The transcripts were presented in columns where the columns were the turn number, the speaker and the utterance. I used modified orthography (e.g., yah-see? wanna, gonna, cus) instead of pure orthography to stay close to the actual utterance of the students.

For the two case study chapters (Chapter 6 and 8), the transcript for the relevant episode was segmented according to the student's claim about the temporal order. The goal of segmenting the episode was to break down the larger data set into smaller chunks to help explore the relevant resources the student might have cued. A segment started with a student's response about the temporal order, e.g., epsilon depended on delta. The segment ended when the student indicated that the explanation or justification for the relationship was done. It could also end when the discussion is interrupted by a discussion of an unrelated topic or when the interviewer asked an unrelated question.

Materials

Interview protocol. The main source of data for this study was individual student interviews, each lasting approximately 2 hours. Studies of student cognition have historically relied on data from student interviews as it provides opportunity for students to give a detailed account of their understanding (diSessa, 1993; Wagner, 2006). The protocol that I used in the study went through two iterations. It was originally developed following the recommendations of diSessa (2007) and Ginsburg (1997). Since then it has been revised to better elicit student understanding of the topic of interest. The protocol used in the latest iteration of the study can be found in Appendix E.[16] The interview started with six (moderately easy) tasks adapted from Szydlik (2000) to help students refresh their memory about limits, and to provide context for the discussion about the formal definition (see Appendix H).

The protocol began with a series of questions exploring students' knowledge about limits more generally and about the formal definition specifically. While the focus of this dissertation is to explore student understanding of the relationship between delta and epsilon within the formal definition, the theoretical framework prioritizes students' prior knowledge and their role in learning. Thus, the first two parts of the interview protocol explored students' knowledge about limit and the formal definition more generally. For example, students' conception of delta and epsilon, as well as different parts of the definition were informative of their understanding of the main topic of the dissertation: the temporal order of delta and epsilon.

Students were presented with the formal definition in Appendix A *without* the example. I used the version of the formal definition in Appendix A because it is a commonly used definition for limit. I did not, for example, use Version 1 from Fernández (2004), despite its utility in helping students understand the formal definition. The focus of the study is to understand how students make sense of the formal definition using a commonly used definition.

The analysis chapters focus on students' responses to the questions about the temporal order (questions 15, 17-19 and 38, 40-42). Questions 15, 17, 18 and 19 were dedicated to exploring students' understanding of the temporal order. Questions 38, 40, 41 and 42 explored students' understanding of the temporal order *after* engaging with the Pancake Story (or reading the text).

From the KiP perspective, knowledge is context specific and the degree of alignment of students' conceptions across different contexts is diagnostic of the state of their conceptions (diSessa & Sherin, 1998). The different ways of asking about the same relationship serve as different contexts to explore the students' understanding of the temporal order. It also assessed the stability of the student's understanding.

I asked about the temporal order of delta and epsilon in four contexts: dependence, sequential order, set-ness and the order of ε, δ, x and $f(x)$. The following were the actual questions from the protocol:

15. In the definition, with epsilon and delta, what depends on what, if anything you think? Delta depends on epsilon? Epsilon depends on delta? They depend on each other? Or they do not depend on each other? And why?
17. In the definition, between epsilon and delta, which one do you think comes first and which one do you figure out as a result? And why?

[16] The protocol used in the pilot studies can be found in Appendix F and G for comparison.

18. In the definition, between epsilon and delta, which one do you think is set? Epsilon? Delta? Both? Or neither? And why?
19. How would you put the four variables, epsilon, delta, x and $f(x)$ in order, in terms of which comes first in the definition? And why?

To reflect the normative temporal order of delta and epsilon, a student would answer the questions in the following way. The student would say that delta depended on epsilon, epsilon came first, and epsilon was set first, and provide an order where epsilon came before delta in the sequence of four "variables."

Delta's depending on epsilon is usually the way the relationship between the two variables is described in introductory calculus classes. Most of the examples' using linear and quadratic functions allows for an explicit description of delta in terms of epsilon, e.g., for any linear function, delta equals epsilon divided by the slope of the line.[17] Epsilon and delta also follow the sequential order of epsilon first then delta. This is the way Davis and Vinner (1986) described the temporal order in their study. Question 19 was inspired by a student's putting the four variables in order without being prompted during the pilot study. It was another way of asking the sequential order of the two variables.

Some students interpreted two questions about the temporal order differently than the question originally intended. The first one was question 18, which asked which of the two variables is set. This question came from the pilot study. Adam, who ended up being the case study of Chapter 6, claimed that delta depended on epsilon *if* epsilon was set. He ended up spending some time discussing which of epsilon and delta was set first. I used this as another context for students to discuss the temporal order. The question in the protocol was written as "set" instead of "set first." Four of the 25 students interpreted the question to ask if epsilon or delta was a set number, not to ask which was set first. This was a reasonable interpretation. I still analyzed the students' responses and I discuss the implication of this differing interpretation on the findings at the end of Chapter 5.

Question 16 was also a question about the temporal order. It asked students about which between x and $f(x)$ they were trying to control. The question was designed to explore if students would recognize the goal of controlling the x values using delta, for a given constraint in the output (epsilon). The responses varied greatly across students. A lot of students ended up confused with the idea of controlling x or $f(x)$. I let the student answer the question according to what they thought it was asking and received incomparable responses.[18] I decided not to analyze students' responses to this question in the analysis chapter.

Instructional intervention: The Pancake Story. The story and the questions following the story were designed to assist students in making sense of the formal definition. The Pancake Story (see below) was partly inspired by the Bolt Problem (Boester, 2008, see Appendix C). In the summer of 2010, I was planning to videotape my lesson on the formal definition as part of a

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[17] One might argue that a constant function $f(x)=k$ serves as a counterexample to the assertion that delta is always dependent on epsilon. For that function, for every number epsilon, any number delta would satisfy the definition. While any number delta would satisfy the definition, it does not change the fact that the goal is still to find a number delta based on a number epsilon, with which one would start. In this way delta depends on epsilon.

[18] I found that students had different ideas about control regardless of how the question was asked. Different version of the protocols aimed at getting at this issue but three revisions did not end with a good version of the question.

research project. I met with Andy diSessa to discuss the possibility of using the Bolt Problem in my lesson. He suggested using pancake making as a more accessible context. He argued that the process of creating bolts using machines would make the process less transparent for students. I explained the context of the Bolt Problem as a way to help students make sense of the formal definition. Some students understood the idea behind the analogy.

It was not until I brought up the context of pancakes that the majority of the class had an “A-ha” moment. Students from that class explained that the context resonated with their experience in making pancakes. It is also worth noting that different cultures have their own version of a “pancake.” Since then I created a story that illustrates the key elements in the formal definition using the context of pancake making. Just like the protocol, the story went through several revisions as a result of multiple rounds of pilot study (for comparison see the Pancake Story in previous versions of the protocol in Appendix F and G). This is the version used in this study.

The Pancake Story

You work at a famous pancake house that's known to make pancakes with 5” diameter. To make the perfect 5” pancake you would use exactly 1 cup of batter. On your first day of work your boss told you that it is practically impossible for you to be able to use exactly one cup to make the perfect 5 inches given how many and how fast you will be making these pancakes. So for now, since you’re new, as long as your pancakes are anywhere within ½” from the 5”, he won’t fire you. Your job is then to figure out the maximum you can be off from the 1 cup to still make pancakes that meet your boss’ standard. Specifically, given that your boss gave you the ½ an inch error bound for the size, you need to figure out the error bound for the batter so that your pancakes won’t be off more than the given error bound.

According to the work manual, there are two steps to do this. Based on the error bound for the size, you first need to guess an error bound for the amount of batter. THEN, you have to check to see if using any amount of batter that is within the error bound from the 1 cup would make pancakes that are within the given error bound from the 5”.

For example, suppose based on the ½ inch error bound, you guessed ⅙ of a cup error bound for the amount of batter. Then you check to see if using any amount of batter that is within ⅙ of a cup from the 1 cup, so between ⅚ and 1⅙ of a cup would make pancakes with size somewhere between 4½” and 5½”, that is within the ½” error bound from 5”.

Over time, your boss expects you to be even more precise. So instead of ½” error bound from 5”, he says he wants you to make pancakes that are within some ridiculously small error bound from 5”, but you don’t know what it’s going to be. This means while he started by asking you to be within ½,” later he might want ¼” or $^{1}/_{1000}$” from 5”. Your job then becomes for however close your boss wants the pancake to 5”, you need to figure out the maximum you can be off from 1 cup of batter such that if you use any amount of batter that is within that error bound from the 1 cup then your actual pancakes will still be within whatever error bound your boss gives you from the 5”.

Now, you don’t want to spend time each morning to recalculate everything. So you will try to come up with a way to calculate an error bound for the batter based on whatever the given error bound for the size.

The design of the story focused on assisting students to make sense of the temporal order as one of the underlying structures of the formal definition. To do so involved opening up the space to discuss many of the issues mentioned above in the conceptual trajectory. First, the story was broadly designed to help students to describe relationships within the formal definition by providing an accessible language for students to use, by tapping into students' familiar experiences. In addition to using the context of pancake making, the story also taps into the experience of working for a boss and creating something with specifications—the idea of quality control. Thus, the story provides access into ideas within the formal definition by using accessible language.

The story was specifically designed to assist students to clarify the meaning of delta and epsilon by differentiating error from error bound. Pilot studies documented the prevalent use of the functional dependence between $f(x)$ and x to determine the dependence between epsilon and delta (Adiredja & James, 2013). The story offers the language of error and error bound to help students differentiate between ε and $|f(x)-L|$ and δ and $|x-a|$, and their respective relationships.

Error in the output would depend on the error in the input, but the boss in the story gives the error bound in the size to be used to find the error bound in the amount of batter. In this way, the story also highlights the fact that epsilon is a given quantity. The boss gives the acceptable error bound for the pancake size. For many students in the pilot, this idea proved helpful in discussing the issue of the temporal order, if not to introduce a conflict for the claim that delta came first. The decreasing error bounds illustrate the arbitrariness of epsilon.[19]

The story also focuses on the logic behind the statement "for every number "$\varepsilon > 0$, there exists a number $\delta > 0$," and more importantly differentiating it from the statement "if $0<|x-a|< \delta$ then $|f(x)-L|< \varepsilon$." Differentiating the error from error bound and recognizing the givenness of epsilon would support this delineation by attaching meaning and actions to the words in those sentences. More specifically, the story treats the two statements as two separate steps in the worker's manual. First, the employee is given the specification to find the error bound in the input. Second, the employee is to check whether using any amount of batter within the discovered bound, would make pancakes within the boss' specification. The story also includes an example of the whole process with a particular error bound. The hope was that this would ultimately assist students to recognize the appropriate relationship between delta and epsilon as a structure underlying the formal definition.

An important disclaimer about the story: the story was not designed to be *solved* mathematically. Attempting to construct an epsilon-delta proof for the story would be challenging, particularly given that there was no constraint on the thickness of the pancakes. The numbers used in the story was specifically selected to connect with the limit of a linear function, $f(x)=3x+2$ at $x=1$, fully understanding that a function that describes the relationship between the amount of batter and the diameter of the pancake is not linear.[20] The goal of the story is not to teach students how to construct an epsilon-delta proof, but to provide them with productive intuition to understand the meaning of the formal definition, and the temporal order more specifically.

Overview of Analysis Methods

This dissertation explores the roles of prior knowledge in understanding the temporal order of delta and epsilon. The next four chapters present the empirical analysis investigating the issue.

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[19] Alan Schoenfeld made the suggestion of including a number of decreasing epsilons to illustrate the arbitrariness of epsilon.

[20] One might also argue that 1-cup of batter would make an extremely thick 5 inch pancakes. While typical pancake recipes would call for ½-cup of batter, I decided to use 1-cup for simplicity.

Chapter 5 explores students' claim about the temporal order before the story, and their reasoning patterns. Chapter 6 presents a microgenetic study of Adam in making sense of the temporal order without the story. Chapter 7 focuses on students' reasoning patterns after the engaging with the story, and their responses to the temporal order questions. Chapter 8 presents another microgenetic study of Adriana in changing her claim about the temporal order by using the story.

The analysis methods in Chapter 5 and 7 are similar. The focus of the analysis is on students' responses to the four temporal order of delta and epsilon questions. I place them into categories of delta first, epsilon first or no order. I assigned a score to each response (0, 2, 1, respectively). Then I counted the number of questions each student answered with epsilon first to give a broad characterization of the group performance on the questions. Lastly, each of these chapter documents justifications that students provided in answering the temporal order questions. Chapter 7 discusses improvements in students' understanding of the temporal order after engaging with the story.

Chapter 6 and 8 are both a detailed case study of one student moment-by-moment decision-making (microgenetic) in learning. The analysis is a fine-grained analysis of a learning episode with a focus on theory, with consideration of any relevant data to explore claims about the process of learning (Microgenetic Learning Analysis, Parnafes & diSessa, 2013). *Competitive argumentation* (Schoenfeld, Smith and Arcavi, 1993, VanLehn, Brown and Greeno, 1984) guides the interpretation of students' utterances in both chapters. However, the two chapters have different goals and that influenced the analysis methods for each chapter.

Chapter 6 seeks to document Adam's process of reasoning using his prior knowledge. I identified *knowledge resources* that Adam used to construct his argument. Competitive argumentation was used to identify these resources and to finalize the model of his argument in each segment of the transcript. *Counter-models* were used as an explicit illustration of competitive argumentation in finalizing the model for Adam's argument.

Chapter 8 explores the influence of the story on Adriana's reasoning about the temporal order. While I documented uses of knowledge resources in this chapter, it was not the main goal of the analysis. This chapter explores some of the influential ideas that Adriana took up from the story. The chapter focuses on explaining the process by which she developed her claim about the temporal order. In particular I pay particular attention to the interaction between the productive resources from the story and Adriana's existing prior knowledge.

CHAPTER 5: STUDENTS' CONCEPTION OF THE TEMPORAL ORDER OF DELTA AND EPSILON

This analysis chapter explores the claim from the literature that students struggle to understand the temporal order of delta and epsilon within the formal definition. In this chapter I focus on the first research question for the dissertation. Specifically, this chapter explores the following questions:

1. What claims do students make about the temporal order of delta and epsilon?
2. How do students reason about the temporal order of delta and epsilon?

The findings in this chapter are enriched and elaborated by the findings in the next chapter, which offers a detailed case study of the ways a student reasons about the temporal order of delta and epsilon.

Analysis Methods

The first part of the analysis categorized students' response to each question about the temporal order. The three categories were: epsilon first, delta first or no order. Students responded to four questions related to the temporal order. I asked about the temporal order of delta and epsilon in four contexts: dependence, sequential order, set-ness and the order of ε, δ, x and $f(x)$. The following were the actual questions:

1. In the definition, with epsilon and delta, what depends on what, if anything you think? Delta depends on epsilon? Epsilon depends on delta? They depend on each other? Or they do not depend on each other? And why?
2. In the definition, between epsilon and delta, which one do you think comes first and which one do you figure out as a result? And why?
3. In the definition, between epsilon and delta, which one do you think is set? Epsilon? Delta? Both? Or neither? And why?
4. How would you put the four variables, epsilon, delta, x and f (x) in order, in terms of which comes first in the definition? And why?

Each way of asking the question was considered a context. The response to each question was given a score from 0 to 2 (delta first=0, no order=1, epsilon first=2). No order received a score of 1 to recognize that the student is a step closer to recognizing that epsilon could come first compared to students who believed that delta came first. For example, in the context of dependence, epsilon and delta depending on each other counts as a no order response.

The sum of the scores ranged from 0 to 8 and students' total scores placed them along a continuum between the claim of delta first and epsilon first. For students from the pilot study, scoring 2 on all the questions that were asked would lead to a total score of 8. In the first round of pilot study, students were asked only one question about the temporal order (question 1, above). In the second round of pilot study, students were asked three of the four questions (questions 1, 2 and 4). In those cases, the total would be normalized to 8 based on the number of available questions.

The second part of the analysis identified *reasoning patterns* from students' justifications for the temporal order. I define a *reasoning pattern* as the essential common core of reasoning found in a range of students concerning the justification for a particular claim. To identify reasoning patterns, I started by recording students' justification for each of the temporal order question. A justification included details about what the student attended to and the meaning they attached to it. I first sorted justifications according to the temporal order they supported: epsilon first, no order or delta first. At times a student started with one claim for the temporal order, but changed their mind afterwards. In this case, the justification for each claim was recorded as two different justifications and was sorted accordingly. Some students provided contradicting justifications to support the claim that there was no order. In this case, I treated the two justifications as one reasoning pattern.

The catalogue of reasoning patterns was developed through an iterative process of open coding (Glaser & Strauss, 1967). In this part of the study, I did not try to delve deeply into the reasons behind student statements. I relied as much as possible on the particular thing the student said and attended to. For example, if a student were to say that epsilon depended on delta because he or she used delta to find epsilon, I recorded it as a reasoning pattern without investigating where the student could have gotten that idea. The student might have gotten the idea from the if-then statement, but unless the student explicitly attended to it, I did not include it as part of the reasoning pattern. The goal of the analysis was to show the diversity in justifications for the temporal order, and not to come up with an exhaustive list of justifications for any student in calculus.

I. Students' Responses About the Temporal Order of Delta and Epsilon

The table below shows how each student in the study answered each question about the temporal order. The table is split into two. The top half includes students from the current study and the bottom half are students from the pilot study (Adiredja & James, 2013).[21] Red shading denotes delta first response. Yellow denotes no order response. Green denotes epsilon first response. Blue denotes questions that were not asked. The use of color was meant to help the reader get an overall sense of students' responses across the different questions.

Table 5.1.
Students' Responses to Each Question About the Temporal Order

Student	Dependence	Sequential	Set	Order	Total
Chen	0	0	0	0	0
Sheila	0	0	0	0	0
Spencer	0	0	0	0	0
Veronica	0	0	0	0	0
Patricia	0	0	0	0	0
Julia	0	0	1	0	1
Aruna	0	0	1	0	1
Jane	1	0	1	0	2
Milo	0	0	2	0	2
Jose	1	2	0	0	3
Katrina	0	2	1	0	3
Simon	1	0	2	0	3
Ryan	0	2	2	0	4
Guillermo	2	0	0	2	4
Silvia	1	2	1	0	4
Bryan	0	1	2	2	5
Roberto	2	2	1	2	7

(Table continues)

[21] The data from Adiredja & James (2013) was re-analyzed using the methods used in this chapter.

Table 5.1. (continued)

Students' Responses to Each Question About the Temporal Order

Student	Dependence	Sequential	Set	Order	Total
Erin	2	2	2	2	8
David	0	N/A	N/A	N/A	0
Jacob	0	N/A	N/A	N/A	0
Anwar	0	0	N/A	0	0
Sophia	0	0	N/A	0	0
Adriana	0	0	N/A	2	3
Adam	2	2	2	N/A	8
Dean	2	N/A	N/A	N/A	8

Note: The table is sorted from the lowest to highest total. It is grouped by current study and pilot study.

Whereas in the pilot study (see bottom 7 rows) I mostly found consistency of responses across the contexts, the latest iteration of the study shows that students' conception of the temporal order was more unstable across contexts. Some students were consistent across all questions. But the majority of students answered with epsilon first in some context, but answered with delta first in others. For example, Katrina claimed that epsilon came first by recalling an epsilon-delta proof. However, when she was asked to order x, $f(x)$, ε and δ, she put delta first because "the definition says that if you have delta then you have epsilon."

During the pilot study, I did not ask the students all of the questions. Adam and Dean scored an 8 without answering the other three questions because they normatively answered the questions that were asked, and was able to explain the formal definition accurately. To assist in parsing the table above, I charted the number of questions that students answered with epsilon first (score=2).

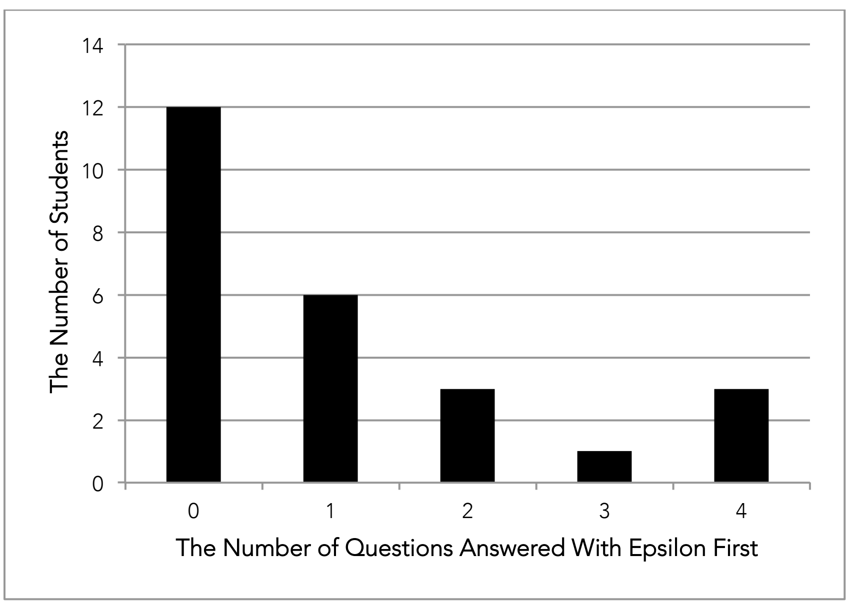

Figure 5.1. The distribution of students in answering the four temporal order of delta and epsilon questions with epsilon first.

Forty eight percent (48%) of students (12/25) answered none of the questions with epsilon first, while only 12% of students (3/25) answered with epsilon first on all the questions. The percentages of the rest are as follows: 1 question-24%, 2 questions-12% and 3-questions-4%. This chart shows that the majority of students in the study struggled with the temporal order of delta and epsilon.

II. Reasoning Patterns for the Temporal Order

The table below shows the different reasoning patterns that emerged from the data. As noted, each reasoning pattern is a type of justification students provided to support their claim about the temporal order. The table is organized by the temporal order claim for which the students used the justification. I include the number of students who used each reasoning pattern. The total number of student counts exceeds 25 because some students included more than one justification per question.

Table 5.2.
Students' Reasoning Patterns About the Temporal Order Questions.

The temporal order	Reasoning pattern	Number of students
Delta comes first, or ε depends on δ, or δ is set first	1) Because the statement, "for every $\varepsilon > 0$, there exists $\delta > 0$" means that there needs to be a delta (greater than zero) for the epsilon to exist.	4
	2) Because of a procedural understanding of a limit. That is, find x values close to a and check the $f(x)$ values.	6
	3) Because the if-then statement suggests that delta needs to be satisfied first then epsilon. *Note*: students might be reading the first-part of the definition, but their focus is on satisfying the delta inequality to satisfy epsilon *Variation*: the if then statement says if delta then epsilon	11
	4) Because we use delta to find epsilon.	3
	5) Because delta is related to x and epsilon is related to $f(x)$ and since $f(x)$ depends on x epsilon depends on delta. *Variation 1:* Epsilon depends on delta because $f(x)$ depends on epsilon and x depends on delta and $f(x)$ depends on x. *Variation 2:* Epsilon depends on delta because output depends on input, and delta constrains our input and epsilon constrains our output.	11
	6) Epsilon is not set because epsilon is arbitrary. So delta is set first.	4
	7) Because x and a are known, but not L. So we can use the delta inequality but not the epsilon inequality.	7

(Table continues)

Table 5.2. (continued)

Students' Reasoning Patterns About the Temporal Order Questions.

The temporal order	Reasoning pattern	Number of students
Delta comes first, or ε depends on δ, or δ is set first	8) Because the definition follows the order x then get delta then $f(x)$ and then epsilon. Notes: This is different from focusing on the if-then because students do not interpret the if-then question but just follow the location of each variable. *Note:* Students may look at the if-then statement but focus on the order of the quantities.	6
	9) Because of recall from the epsilon-delta proof procedure, the answer is epsilon over some number.	2
	10) Because of recall from epsilon-delta proof procedure, we start with the delta inequality and it will come out in the epsilon inequality.	2
No order, or ε and δ are dependent on each other	11) Because we have to find both of them. *Variation 1:* We are not given both epsilon and delta. *Variation 2:* If one is set, the other one is also set	7
	12) Because the for-all statement says delta depends on epsilon and the if-then statement says epsilon depends on delta	2
	13) Because the if-then statement suggests that they depend on each other.	3
	14) Because if the limit exists then as delta gets smaller epsilon gets smaller and if the limit doesn't exist then delta getting smaller has no effect on epsilon	1
	15) Because of recall from the epsilon-delta proof procedure, of getting a number times delta is less than epsilon	1
Epsilon comes first, or δ depends on ε, or ε is set first	16) Because the definition reads for every number ε, there exists a number δ, such that if $0<\|x-a\|<\delta$ then $\|f(x)-L\|<\varepsilon$, (a normative reading of the statement).	1
	17) Because of the spatial location of the variables, starting with for all.	1
	18) Epsilon is given. *Variation:* Epsilon comes first and then you find delta.	4
	19) Because of the statement for all epsilon there exists a delta	2
	20) Because of recall from the epsilon-delta proof procedure, we break down the epsilon inequality to get it to look like the delta inequality or the answer is epsilon over some number.	5

(Table continues)

Table 5.2. (continued)
Students' Reasoning Patterns About the Temporal Order Questions.

The temporal order	Reasoning pattern	Number of students
Epsilon comes first, or δ depends on ε, or ε is set first	21) Since you know a and $f(x)$ you can find L, then you can set epsilon, and find delta	1
	22) Because of a counterexample where the limit does not exist and thus for a given epsilon there is no delta.	1
	23) Epsilon is set and that constrains the output which then constrains the input. *Variation*: Because we want epsilon to be really small because we want $f(x)$ to be very close to L we would want delta to be really small because we want x to be close to a.	2
	24) Because epsilon no longer depends on delta since the if-then statement is about x and $f(x)$	1

Themes in the types of reasoning patterns

I found quite a large number of reasoning patterns for the temporal order across the four contexts. This shows the diversity of ways of reasoning about the temporal order. Quite a number of reasoning patterns (8/24) relied on an interpretation of different parts of the statement of the formal definition (e.g., Reasoning Pattern 1, 13, 16). Almost as common was those that involved a recall of the proof procedure from instruction (Reasoning Pattern 9, 10, 15, 20). Notice that even though students attended to the same procedure, sometimes they concluded a different temporal order of delta and epsilon. Some reasoning patterns relied on a more intuitive understanding of a limit, where one would select values of x close to a to determine the limit (e.g., Reasoning Pattern 2, 7). Some students justified their claim using physical location of the different variables in the statement of the definition (e.g., Reasoning Pattern 8, 17). Thus, while many of these reasoning patterns might have originated from instruction, others were students' interpretation of the formal definition during the interview.

Common reasoning patterns in different contexts

One of the most common reasoning patterns, with 11 students using it as a part of their reasoning was the same justification found in Adiredja and James (2013): epsilon depends on delta because delta is related to x and epsilon is related to $f(x)$ and since $f(x)$ depends on x epsilon depends on delta (Reasoning Pattern 5). Bryan provided a very clear example of this reasoning pattern. He argued, "[Epsilon depends on delta] because delta is the independent variable which would be x in the f of x equals y [$f(x)=y$] relationship /…/ that spits out *y,* which is our epsilon [dependent variable] /…/ because delta is like x and epsilon is y." Bryan was drawing on his knowledge of functional relationships and applying that to the delta epsilon relationship. And like most students who used the functional dependence idea, Bryan treated delta like x and epsilon like y.

Another reasoning pattern that was equally as common (11 students) relies on an interpretation of the if-then statement. Reasoning pattern 3 says that the if-then statement suggests that delta needs to be satisfied first before epsilon can be satisfied. So delta comes first. For example, Ryan said, "For every number epsilon there is a number delta such that if the delta thing is satisfied then the epsilon is satisfied /…/ the delta has to happen for the epsilon to be satisfied. Because it goes, if this, then the epsilon is satisfied. Delta needs to be satisfied before the epsilon can be." Ryan was reading the whole statement of the definition, but he clearly focused on the if-then part of the statement. He then concluded that delta came first in the temporal order. This reasoning pattern is an example of one that relied on an interpretation of part of the statement of the definition. Next I explore another common reasoning pattern, which relied on students' intuitive understanding of a limit.

Seven of the 25 students argued that since the *x* and *a* were known then they could use those to find delta, whereas the limit was unknown so they could not find epsilon (Reasoning Pattern 7). Veronica argued, "Um, I would say delta [is set first] because the delta equation includes *a* whereas the components of the epsilon equation include L and you may or may not know what the limit is yet because you might be solving for the limit. But they give you *a* so I would assume that would be a better tool to use to solve." Veronica treated the delta and epsilon inequalities ($0 < |x-a| < \delta$ then $|f(x)-L| < \varepsilon$) as equations ($|x-a| = \delta$ then $|f(x)-L| = \varepsilon$). This was quite common among the students I interviewed. Doing so led her to conclude that with *x* and *a* known, she could find the delta, whereas the existence of the limit was in question. So far I have explored common reasoning patterns that support the claim that delta comes first. I explore a common one students used to argue that there was no order for epsilon and delta.

Seven students focused on the fact that they needed to find both epsilon and delta to conclude that neither was set, so there was no order (Reasoning Pattern 11). For example, Roberto argued that neither epsilon nor delta were set because "you have to sort of find them or figure them out." Silvia expressed a similar opinion, "neither set because you have to solve for both of them." These students attended to whether epsilon and delta could be set, instead of which of the two was set first. I return to his subtlety in the discussion.

It is worth noting most students who concluded that epsilon came first recalled parts of the proof procedure (Reasoning Pattern 20). They were able to infer the temporal order appropriately from the proof. The question then becomes, what was it about the proof procedure that allowed many students to the correctly infer the order?

Ten students recalled, without prompting, parts of the epsilon-delta proof from prior instruction. Five of them concluded the appropriate temporal order of delta and epsilon, but the other five did not. In fact, many of them recalled the same procedure, attended to the same information and concluded a different temporal order of delta and epsilon. For example, both Veronica and Katrina recalled that the "delta" would "come out" from the epsilon inequality. But Veronica concluded that delta came first, while Katrina concluded epsilon came first!

Veronica said, "I'm thinking delta [comes first] because for some reason I feel like because these [$0 < |x-1| < \delta$ then $3|x-1| < \varepsilon$] look kinda similar, like you can take /…/ this equation with delta and plug it in for the epsilon equation. So I'm thinking maybe you should check out delta first possibly." Katrina explained, "Oh, the one that comes first is epsilon and you figure out delta because you're gonna take this f of x minus L [$|f(x)-L|$] is less than epsilon and you're gonna manipulate it, and then you'll get it to look like x minus a [$|x-a|$] and depending on that, you know what delta is." The two students were both examining the two inequalities and trying to manipulate one to look like the other. This is something that is commonly talked about in calculus classes, and students spontaneously produced during the interview. However, Veronica concluded that delta came first while Katrina concluded that epsilon came first.

The goal of this comparison is not to compare the students' ability, but to make the point that the difference in interpretations by the two students warrants a deeper analysis to explore what was truly underlying these conclusions, and the ways in which these justifications arose. The next analysis chapter goes into the details of the way one student used the proof procedure as a resource to think about the temporal order. The student attended to the same information but changed his mind about the temporal order several times during the interview.

Discussion

I found that students struggled with the temporal order of delta and epsilon within the formal definition. Twelve of the 25 students in this study were not able to answer *one* question about the temporal order correctly. The methods that I employed in this chapter reveal the variability of student conceptualization of and reasoning about the temporal order. Ten students received a total score of 0 across the four different contexts and three students scored 8, but the majority of students were somewhere in between. The fact that some students scored 2 in one context but 1 or 0 in others shows that student knowledge about the temporal order was not quite stable across the different contexts. This highlights the importance of assessing student knowledge in multiple contexts in research and practice.

With respect to students' justifications, "functional dependence between x and $f(x)$," and "delta is with x; epsilon is with y" remain the most common reasoning patterns for the temporal order in the latest iteration of the study. I discussed the nature of that reasoning and its implication in Adiredja and James (2013). However, the current study also found another common reasoning pattern that relied on an interpretation of the if-then statement in the definition. In Adiredja and James (2013) we found that most of what we called "knowledge resources" were mathematical in nature; we hypothesized that either this indicated lack of access into the formal definition using intuitive knowledge or it was a product of using too large of a grain size to find intuitive knowledge resources. The findings from this study suggest that it might be both.

The findings from this chapter confirm that students use their interpretation of mathematical statements and previous experiences with mathematics to make sense of the temporal order. For example, many of the reasoning patterns I found relied on students' interpretation of the if-then statement in the definition. At the same time, a microgenetic case study of Adam in the next chapter reveals that most of what Adiredja and James called "knowledge resource" was reasoning patterns. I show in the next chapter that a reasoning pattern is a result of the use of various knowledge resources, thus a reasoning pattern is larger in grain size. However, these reasoning patterns are useful to identify knowledge resources. For example, the case study explores the way that the student, Adam interpreted the inequality $3|x-1|<\varepsilon$ from an epsilon-delta proof he recalled. Sometimes Adam read the inequality to say that epsilon must be greater than three times the interval around 1. Other times he read it as saying three times the interval around 1 must be smaller than epsilon. And depending on his reading, he drew different conclusions about the temporal order. The next chapter looks into the underlying knowledge resources that influence the way he read the inequality. The findings in Chapter 6 can also illuminate what happened with Veronica and Katrina earlier.

I note one potential limitation of the current study. Four of the 19 students (Jane, Katrina, Roberto and Silvia) who were asked the set question did not interpret the question as I intended. Instead of focusing on which of the two quantities had to be set first, they were focused on whether epsilon and delta could be set. I recognize that this was a reasonable interpretation. I still coded them as "no order" for consistency instead of creating a new category for them. One option that I could have done, but did not do, was to not code their response at all, and normalize their scores much like I did with the students in the pilot study who was not asked the question. I did not do so, because I do believe that ultimately this would not dramatically change the general finding in this chapter: that a lot of students struggled with the temporal order and they used a very diverse set of their reasoning patterns to justify their claim.

CHAPTER 6: REVISIONS OF THE TEMPORAL ORDER CLAIM BEFORE THE PANCAKE STORY

This analysis chapter focuses on a case of a student, Adam, who used a diverse set of knowledge resources to make sense of the temporal order of delta and epsilon. Adam was selected for the case study because despite being a high-performing student and being able to recall the procedure of an epsilon-delta proof from memory, he was unsure about the temporal order of delta and epsilon. Adam self-identified as a White, Non-Hispanic student. He was an intended mathematics major who took first-semester calculus in high school and received a five on his AP Calculus AB and BC. In the span of 17 minutes, Adam changed his claim 7 times and ultimately arrived at the correct conclusion about the temporal order before discussing the Pancake Story. Adam was one the few students who revised his claim about the temporal order without discussing the story. Adam was very articulate in the way that he reasoned about the temporal order. Adam's ways of reasoning can illuminate his selection of resources and the interaction between these knowledge resources.

Goals and Foci for Analysis

The main goal for this chapter is to understand the *process* by which Adam made sense of the temporal order of delta and epsilon. More specifically, this chapter has a number of specific goals related to the content of the formal definition:

1. This chapter aims to uncover formal mathematical resources and intuitive knowledge resources and their interaction in the development of Adam's claim.[22]

2. This chapter aims to investigate the details of the knowledge resource, **functional dependence** and its influence in making sense of the temporal order.

3. This chapter also aims to investigate the richness of the formal definition of a limit in terms of the amount of resources students might bring to it, and how the richness might have influenced Adam's sense making.

The analysis focuses on Adam's response to the temporal order questions (questions 15 to 19 in the protocol). Hence it focuses on the episode that starts with Adam's response to question 15 and continues to his last words about question 19 (turns 286–415, see Appendix J for half of the full transcript). I broke down the episode into segments based on the changes in his claims about the temporal order. Each segment was categorized based on Adam's current claim about the temporal order. Adam answered the temporal order question of delta and epsilon with respect to dependence (which depended one which?), sequential order (which came first?), and which one was set first. He discussed the temporal order in those three contexts, but he opted not to respond to the last context where he needed to order the four variables. His discussion of the contexts was not linear. For example, he discussed the dependence between epsilon and delta, but in the middle of that discussion he would explore which one was set first. In this way, the analysis is not organized by the type of question for the temporal order, but by the temporal order claim that Adam made.

There are two parts of the analysis. The first part documents the changes of Adam's claim across the nine segments during the 17-minute episode. This gives the overall picture of Adam's sense making process. The second part of the analysis is a microgenetic learning analysis (Parnafes & diSessa, 2012) of Adam's sense making in four of the ten segments (Segments 3(a),

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[22] Adam's claim about the temporal order changes over the 17-minute episode, but I treat them as intermediate states of Adam's overall claim about the temporal order.

3(b), 4 and 5). It focuses on moment-by-moment (micro) changes in the deployment of Adam's knowledge during the learning episode. Not coincidentally, a diverse set of knowledge resources was involved in these segments. The first part of the analysis provides context for the second part of the analysis, which explores the details of Adam's transitional claim in each segment. It documents the relevant knowledge resources and their interaction that might have led to the overall change of Adam's claim.

I. Overall Change in Claim

Adam's claim about the relationship between delta and epsilon changed several times during the interview. Initially he argued that epsilon depended on delta (delta first) because "delta is giving you an interval for *x*, and then, /…/ epsilon is evaluating *x* and subtracting the limit" (turns 288–290). By the end he argued that delta depended on epsilon (or epsilon first)[23] because "epsilon's [set] first and you break down epsilon… and you find delta" (turns 393–411).

Figure 6.1 shows the changes in Adam's claim about the temporal order in between those two segments. Each box represents a segment, and the shade characterizes the overall nature of the argument about the temporal order. Pink is for delta first. Yellow is for no order. Green is for epsilon first. Despite the color code, Adam was not sure about his claim at each segment. In fact, Adam often used hedging language like "kind of," "sort of", "I think" to qualify his statements.

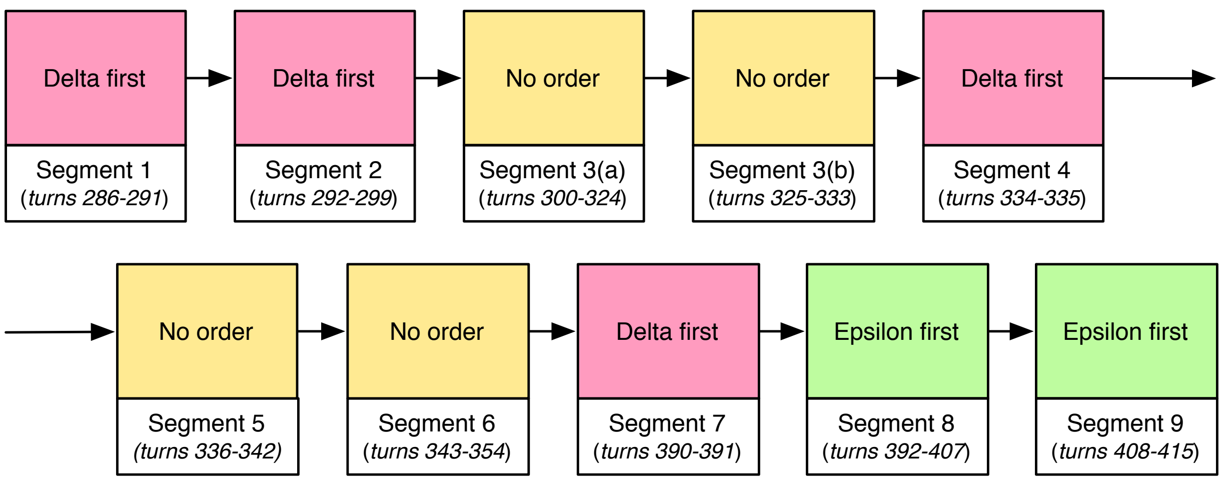

Figure 6.1. Adam's progression of claims about the temporal order of delta and epsilon during the 17-minute episode.

As documented in Figure 6.1, Adam went back and forth on his claim about the temporal order. Adam spent the majority of the time discussing the dependence between epsilon and delta (segments 1-6). The question about the sequential order was discussed in Segment 7 and 8, and he discussed which of epsilon and delta was set first in Segment 5 and 9.

Adam started with the claim that epsilon depended on delta by focusing on the idea of *functional dependence.* By Segment 3, he changed to epsilon and delta's being dependent on each other. He explained why epsilon depended on delta in 3(a), and why delta depended on

[23] I used the phrase "comes first" to indicate the temporal order. In the data, most of the discussion was about the issue of dependence between delta and epsilon and Adam did not use the phrase "epsilon comes first," until I specifically asked him about the sequential order in Segment 7 and 8.

epsilon in 3(b). In this segment he began to recall parts of an epsilon delta proof. He used the inequality $3\delta<\varepsilon$ to help him determine the temporal order. I separated the segment into two because there were a lot of ideas embedded in each of the segments. In Segment 4, he abandoned the claim that epsilon had an influence on delta, only to return to it in Segment 5.

At the end of Segment 5, Adam said that epsilon would depend on delta provided that epsilon was set (turn 340), but he concluded that the two were dependent on each other. When asked if epsilon was set, he said that both variables were "sort of" set independently but their relationship was that they were dependent on each other (turn 354 in Segment 6). When we returned to the topic in Segment 7, Adam focused on his recall of the procedural steps of an epsilon-delta proof. The procedure started with setting up the delta inequality, then simplifying the epsilon inequality to find the "delta expression," which Adam called "breaking down" the epsilon inequality. In the next turn, Adam changed his mind and said that epsilon came first because he would break down the epsilon inequality first (Segment 8). Adam described in Segment 9, that the process was like, "fine tuning" the size of the output using epsilon (turn 410), and that's why epsilon was set first.

Having documented the changes in Adam's claim about the temporal order, in the second part of the analysis, I explore reasons behind some of the changes that occurred. It starts with an analysis of Segment 3(a) and 3(b) and ends with an analysis of Segment 5 (turns 300–341). As I said, Segment 3(a) and 3(b) were rich with ideas and justifications for why epsilon and delta depended on each other. There was a change in claim in Segment 4. In Segment 5, Adam settled back with the claim that the two depended on each other after re-establishing epsilon's influence on delta. Adam's claim was still not stable as Adam returned to delta first in Segment 7. We begin to see stability of Adam's claim in Segment 8 and 9. Thus, part II of the analysis focuses on the beginning of the change (Segment 3(a) to 5) and investigate Adam's selection of resources and the interaction between these knowledge resources in those segments.

II. Adam's Transitional Claims and Knowledge Resources

Adam cued a variety of knowledge resources in Segment 3(a) and (b), including some that were also cued in the first two segments. In 3(a) and 3(b) Adam also recalled several potentially productive resources from instruction, but they competed with Adam's existing resources. Segment 4 and 5 show that Adam struggled with the claim that epsilon could influence delta. At the end of Segment 5, Adam sorted out the competing resources using the inequality $3\delta < \varepsilon$ that he generated from his recall of the proof procedure. He concluded that epsilon influenced delta because delta had to "conform" to the size of epsilon, provided that epsilon was set.

The analysis in this section aims to explore some of the issues for Adam and investigate ways that he resolved them. Before I start a discussion about the data, I describe the structure of this section and the details of the methodology I used in this chapter.

Analysis Methods

For each segment, I present the relevant turns in the transcript. In the presentation of the transcript, partial repeats, "like," "ums," and "uh-huhs" were removed and replaced by an ellipsis (/…/). When it was helpful to maintain the student's flow of argument, I combined multiple turns that would be otherwise be broken by the "ums" and "uh-huhs." When the interviewer interjected a clarifying question or a different idea, I presented the multiple turns as they were. I included hedges like, "sort of" or "kind of" because it was informative of Adam's certainty of the claim he made.

Below the transcript, I present a summary of Adam's argument with minimal analysis of the correctness of Adam's argument. This is to assist the reader in making sense of the mathematics behind his utterance. Then I present my analysis of the argument. The analysis focuses on uncovering the meaning behind different parts of Adam's argument and the way that Adam constructed his argument. The final product is a model of Adam's argument. I close the section with a list of counter-models, and an argument for why each counter-model was less likely to be a valid interpretation of the data. Figure 6.2 below shows the structure for the analysis for each segment of data.

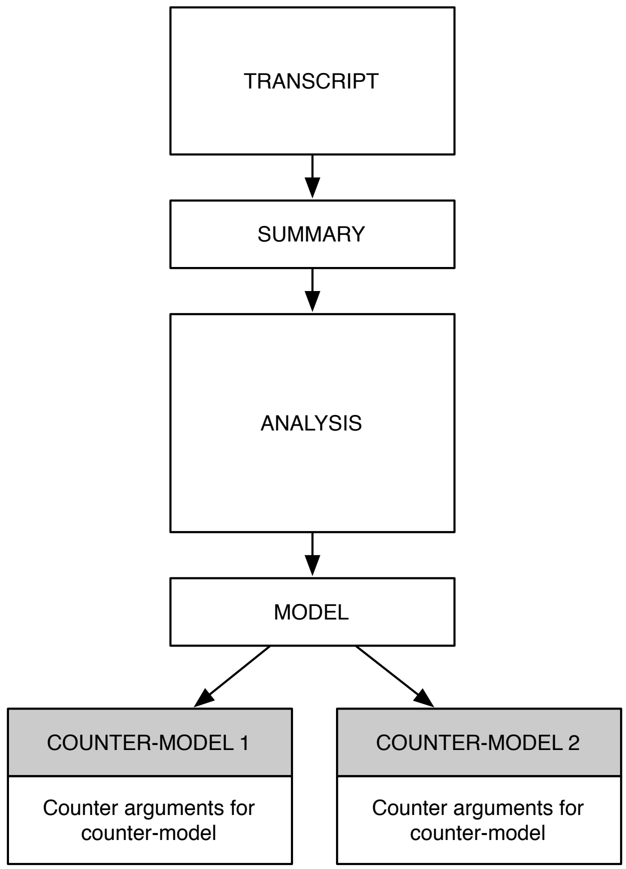

Figure 6.2. The structure for each segment of the analysis.

I employed "competitive argumentation" (Schoenfeld, Smith and Arcavi, 1993, VanLehn, Brown and Greeno, 1984) as the main methodology to analyze the data in this section. Competitive argumentation holds an interpretation of data accountable to empirical evidence, theory and existing literature. In this section competitive argumentation was used in two ways. First, I used it to identify *knowledge resources*. As a reminder, a knowledge resource is an idea consisting of a single or a collection of interrelated knowledge elements with a utility in a particular context. I specifically identified knowledge resources that related quantities, like **functional dependence** (see Table 6.1 below for a glossary for all the relevant resources for the analysis). In this chapter knowledge resources are bolded in the text.

Table 6.1.
Glossary of Knowledge Resources

Knowledge resource	Description
Absolute condition	It refers to an antecedent to a consequence that is necessary and sufficient. In mathematics, the P in the statement, P if and only if Q is as an absolute condition. This resource is commonly misapplied to a one-way conditional statement, if P then Q. Students often assume that with a conditional statement, the inverse of the conditional statement are also true for the statement. It may have stemmed from experiences with statements like "If you finish your homework, then you can play videogames," which usually means, "If you haven't finished your homework, then you cannot play videogames."
Determining	If A determines B, then A uniquely establishes B. While determining is more specific than constraining, when A determines B, A also constrains B. This resource stipulates that determining involves a *determiner* and a *determined*. When A determines B then A is the *determiner* and B is the *determined.* A can be a determiner of other quantities, in addition to B. For example, epsilon can determine delta, but it can also determine the range of acceptable output values. Indicators of this resource include the phrase "must be," and the idea of constraining. For example, if a student say "B must be greater than A," then A is the determiner and B is the determined. Likewise, if a student says, "Delta constrains the size of the interval," then delta is a determiner of the interval. I assert that the distinction between *determining* and *constraining* is inconsequential for purposes of the analysis in this chapter.
Domain constraint for a limit	A limit only considers values of x near or close to a.
Dynamic definition of a limit	A limit is a number that $f(x)$ approaches, as x approaches a.
Function slots	This resources uses the stipulation that a function is a relationship between x and $f(x)$ or y. This resource supposes that when two quantities share a functional relationship, one quantity is the x and the other is the $f(x)$ or the y.

(Table continues)

Table 6.1.
Glossary of Knowledge Resources

Knowledge resource	Description
Functional dependence	In an input-output relationship, the input directly determines the output. The relationship has a clear direction that the input determines the output, but not vice-versa. *Functional dependence* is a particular kind of *determining*. In the analysis, I differentiate between *functional dependence* and *determining* because of the specific nature and prevalent use of functional dependence in mathematics. The resource likely stemmed from students' experiences learning about functions. The y or $f(x)$ is often treated as the dependent variable, and x the independent variable.
Givenness	A characteristic of being specified in advanced. A *given* quantity is one whose properties are previously stipulated, and therefore its determination is not relevant for further examination. Any mention of a quantity being previously set is a version of this resource. Students often assume that the variable x or the domain of any function is a given.
Proportional variation	A small change in the independent variable leads to a small change in the dependent variable. This may be a mathematical application of the more physically intuitive Ohm's p-prim (diSessa, 1993). Ohm's p-prim says bigger effort begets bigger result, and consequently smaller effort begets smaller result.
Quality control	The idea of controlling the input values of a function in order to meet a given specification (e.g., error bound) for the output values. This resource involves the resource **givenness** of the constraint for the desired output, but it also emphasizes the modification of the input to satisfy the given constraint.

The first step in identifying a knowledge resource is to uncover the meaning behind relevant parts of Adam's argument. To help interpret the meaning of the idea in the particular context, I looked for other instances in the full transcript where Adam expressed a similar idea. I did not assume that the two ideas would be consistent. In fact analyzing their consistency helped to interpret the use of an idea in the particular context. At times I also relied on existing literature to help interpret the idea that Adam used.

A theoretical assumption about knowledge resources, and a methodological orientation assisted in the identification of knowledge resources. The neutrality assumption—knowledge resources are not correct or incorrect—distinguished knowledge resources from larger ideas that were results of the use of several knowledge resources. For example, as I alluded to at the end of Chapter 5, students' reasoning patterns could not be knowledge resources because those reasoning patterns could be correct or incorrect. Unpacking the ideas behind these reasoning patterns helped identify the knowledge resources.

The methodological orientation of staying accountable to the dynamics of development of thinking across contexts was also helpful in identifying knowledge resources. I strive to understand when a knowledge resource is activated and when it is not, and why. For example, suppose one knowledge resource was used in one segment but not in another. First I asked myself the question, was the resource really not activated in the other segment? If so, why was it not? I exploit all the available data to answer those questions. The "analysis" section for each segment focuses on identifying knowledge resources, assisted by the theoretical assumption and the methodological orientation.

I also used competitive argumentation to construct a model of Adam's argument, including the role for each knowledge resource in the particular context. Competitive argumentation is illustrated explicitly with the use of *counter-models*.[24] The counter-models serve as competing hypotheses for the way that Adam put together the different knowledge resources that I identified in the "analysis" section. The methodological orientation of focusing on the dynamics of the development of thinking was useful in generating counter-models and in deciding the most likely to be valid model of Adam's argument.

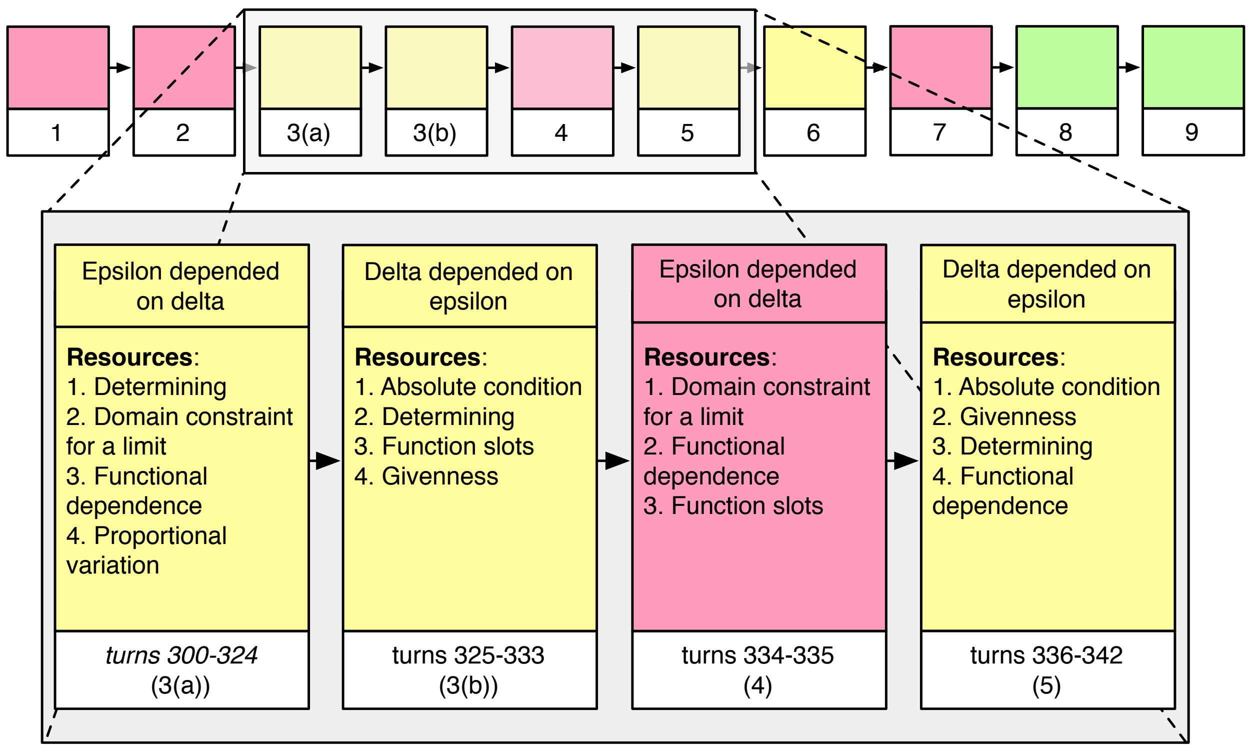

Figure 6.3. Knowledge resources Adam cued in each of the four segments.

In summary, this analysis section seeks to investigate the role of intuitive and formal mathematical knowledge resources in the sense making process. Figure 6.3 above provides a preview of the analysis. As a reminder, pink is for delta first. Yellow is for no order, and green is for epsilon first. As we continue with the analysis of the four segments, I invite the reader to

[24] The counter-model is also a helpful way to present the data. It allows the reader to focus on the model and the supporting explanation before they are problematized by competitive argumentation. Then the reader can consider the validity of the counter-models.

keep in mind the three specific aims of the analysis: to determine the student's repertoire of resources, to establish the dominance of **functional dependence** and to explore the richness of the formal definition as a topic and its implication. I revisit these three aims in the discussion of the chapter. We start with the first focus segment next.

Segment 3(a): The Determiner vs. the Determined and Functional Dependence

312	Adam	/…/ [Y]ou're saying delta [*points at* $0<\|x-a\|<\delta$] must be greater than the input /…/ subtracted by what you're centered around /…/. So you're saying that /…/ the interval around a number *a* [*points at* $0<\|x-a\|<\delta$] must be less than delta. So you're saying, the input cannot get outside of this [*unspecified*] region. /…/ This interval /…/ cannot get exceedingly big.
314-316	Adam	And then for epsilon, you're evaluating *x* around *a,* and then you're subtracting 1. When you plug in /…/ *a* for *x*. So what winds up happening is you're seeing how big the difference is between a number near *a* and the *a* itself.
317	Int.	So how does that say that epsilon depends on delta?
318	Adam	It's because your input, your delta is influencing your input and then epsilon must be greater than your input minus your input of *a,*
319	Int.	Okay. Or your output [*correcting*].
320	Adam	Your output [*in agreement*].
321	Int.	And since, so since output [*points at* $\|f(x)-L\|<\varepsilon$] depends on input [*points at* $0<\|x-a\|<\delta$]..
322	Adam	Yes.
323	Int.	Epsilon depends on delta.
324	Adam	Yes.

Summary of argument. Adam initially said that delta was determined by the difference between *x* and *a* ("delta must be greater than the input /…/ subtracted by what you're centered around," turn 312). But ultimately he concluded that delta was influencing the input (turn 318) because it constrained the interval for *x* ("the interval around a number *a* [*points at* $0<|x-a|<\delta$] must be less than delta," turn 312). Epsilon evaluated the *x* values that were within the constraint (turns 314-316). Adam attended to the two inequalities ($0<|x-a|<\delta$ and $|f(x)-L|<\varepsilon$) to make his claim. However, Adam read the two inequalities differently. He read the delta inequality as saying that delta constrained the *x* values. He read the inequality to say that "epsilon *must be* greater than [the output for] your input minus [the output for] your input of *a*" (turn 318). Adam suggested a role for delta when he said that the input could not get outside of the region specified by delta (turn 312). And he prescribed a relationship between epsilon and the output when he said that epsilon must be greater than the difference in outputs. Adam also agreed with the suggestion that epsilon depended on delta because output depended on input.

Analysis. Adam treated the two variables epsilon and delta differently. The way he read the epsilon and delta inequality suggests his treatment of each variable and the relevant knowledge resources. Adam read the delta inequality ($0<|x-a|<\delta$) in two ways. Initially he read it as an inequality that would allow him to determine delta from the difference between *x* and *a* (first line in turn 312). Here, delta would be **determined** by the difference between x and a. But for the remainder of the segment, he switched to delta constraining the interval around a. Delta made sure that the interval around a to not be too large ("This interval cannot get exceedingly big," turn 312). He also said, "the input cannot get outside this region" (turn 312). Then delta **determined** the acceptable x values. Thus, delta played the role of the **determiner**. It is unclear at this point what he meant by the phrase "the interval around a must be less than delta." The meaning of this is revealed in the next segment.

Adam's switch in his interpretation of the inequality was supported by the flexibility of the relationship between two sides of an inequality. The inequality $|x-a|<\delta$ within the inequality $0<|x-a|<\delta$, just like any other inequalities could be interpreted in two ways: either the left hand side determined the right or vice versa. Adam's initial interpretation of the difference between x and a determined the delta was consistent with the left determining the right. This is consistent with a very common way students spatially reason with equations or inequalities: the right hand side is always the "answer," or the thing to be calculated (for a discussion of potential causes for this interpretation see Carpenter, Franke and Levi, 2003, pp. 22-23). Despite this common tendency, Adam switched his interpretation. I revisit this potential inconsistency between delta as **a determiner** and delta as **a determined** in the next episode.

Delta also made sure that only values of x that were close to a to be considered. Presumably this was the goal behind Adam's assertion that the interval around a "cannot get exceedingly big." This is evidence for the **domain constraint for a limit** resource. I posit that the notion of **proportional variation** was also used in this segment: a small variation in x leads to a small variation in $f(x)$. The goal of making sure that the variation in $f(x)$ was kept small was part of the reason why Adam said that the interval could not get exceedingly big.

Epsilon, on the other hand, plays the role of the one to be **determined** by the difference of the function values. Adam read the epsilon inequality ($|f(x)-L|<\varepsilon$) as suggesting steps for calculation. First, evaluate the function at two different points, at a number near a and at a.[25] Then calculate their difference as a comparison value for epsilon. He said, "[E]psilon *must be* greater than [the output for] your input minus [the output for] your input of a" (turn 318). Thus, epsilon was **to be determined** by calculating the difference between the outputs.

Different parts of the statement of the definition supported Adam's interpretation of epsilon and the way he read the epsilon inequality. The $f(x)$ within the epsilon inequality suggests some form of function evaluation (or plugging things in for some students). This might explain why Adam said for epsilon, one evaluates the input values (turns 314-316). Some form of evaluation did need to happen, but Adam thought that the evaluation would *determine* the number epsilon. This made epsilon the quantity **to be determined**. This is further supported by the implication structure of the if-then statement. The implication structure demands a verification of the epsilon inequality as a result of choosing x values that satisfy the delta inequality (turn 316). However, instead of verifying that all the output values were within epsilon distance from the limit, Adam concluded that epsilon had to be greater than the difference between the output and the limit. His interpretation that epsilon was **to be determined** was also supported by the tendency to interpret

[25] The way Adam read the epsilon inequality as $|f(x)-f(a)|<\varepsilon$ suggests an assumption that $L=f(a)$. However, during the interview Adam was discussing the formal definition in the context of the limit of a linear function, which was continuous. Hence, I did not consider this as a "misconception" as listed in Davis and Vinner (1986).

the right hand side of an inequality as the answer. Thus, in this segment for Adam, delta was a **determiner** (of the interval) and epsilon is the **determined** (by the difference in output values).

Now that we know how Adam conceptualized epsilon and delta, the remaining question is how did Adam connect epsilon and delta to deduce the temporal order? **Functional dependence** connected epsilon and delta in the following way. Delta **determined** the small interval of x values around a. The **functional dependence** took in x values and produced $f(x)$ values, which **determined** the number epsilon. Again, epsilon was **determined** by the difference in output of the function at a and at a number near a. As documented in the previous chapter, **functional dependence** between the input and the output, together with delta is similar to x and epsilon is similar to y (**function slots**) was a very common reasoning pattern to argue that epsilon depended on delta.

Note that Adam did not apply the **function slots** resource to epsilon and delta like many students. That is, he did not use delta and epsilon to fill the x and y or $f(x)$ slots. In this context, it seems that Adam cued the **functional dependence** with a productive interpretation of the delta (a constraint for the interval), and a particular interpretation of epsilon (greater than the difference of output values). Delta's constraining the input is different from delta's being similar to x. And epsilon's being greater than the difference between the output at a and at a number near a is more specific than epsilon's being similar to y. Thus, we have a different model than "delta is with x, epsilon is with y and since y depends on x, epsilon depends on delta."

Model. Adam focused on the role of delta and epsilon in the definition. Delta played the role of the **determiner**. Delta served the function of focusing on x values that were close to a (**domain constraint for a limit**), and made sure that the interval was not so large as to produce too large of a difference in the output (**proportional variation**). Epsilon on the other hand played the role of the one to be **determined**. The apparent role of delta and epsilon led him to read the inequalities in a particular way. Adam read $0<|x-a|<\delta$ as saying the interval around a number a must be less than delta, and he read $|f(x)-L|<\varepsilon$ as saying epsilon must be greater than the difference between the output of a number near a and the limit. **Functional dependence** connected the known **determiner**, delta, to the unknown to be **determined,** epsilon, via input-output relationship. In this segment **functional dependence** supported the resources delta as the **determiner** of the acceptable input and epsilon as **determined** by the difference of $f(x)$ and L, for x near a. Thus the resources for this segment include: **determining**, **domain constraint for a limit**, **proportional variation** and **functional dependence**.[26]

Potential counter-model 1. Adam used the **functional dependence** of the output on the input as the main resource to determine the dependence between delta and epsilon. Adam needed to relate epsilon and delta to the input x and the output $f(x)$ to utilize that argument. His interpretation of the two inequalities provided him a way to related delta to the input and epsilon to the output. Delta constrained the range of inputs, and epsilon was a comparison value for the difference between the output values at a and near a. Thus, **function slots** put delta in the slot of x and epsilon in the slot of $f(x)$.

It is true that the way Adam read the inequalities related epsilon to the output and delta to the input, and the **functional dependence** connected the input with the output. I have argued that

[26] The analysis in Segment 5 reveals that in this segment, Adam likely cued the **dynamic definition of a limit** in Segment 2 right before this Segment. The resource is consistent with the other resources cued in this segment. Thus, it is likely that **dynamic definition of a limit** was also used in this segment, without an explicit mention to it.

Adam's interpretation of epsilon and delta was more specific than just an application of the **function slots** to epsilon and delta. A more important question to consider is *how* did Adam come up with the specific interpretation of the inequalities? Suppose Adam believed that epsilon and delta played a different role. Epsilon was a **determiner** of the difference in the output, and delta was **determined** by the difference between x and a. Then Adam would read the epsilon inequality to say that the difference between the output and the limit must be less than epsilon, and delta must be a number greater than the difference between x and a. This would also imply that epsilon was known and delta was not. This would have resulted in a closer to normative interpretation of the formal definition, but we did not see that. Moreover, in that case **functional dependence** would not be applicable with the known output and the unknown input. This emphasizes the central role **determining** played as resource in this segment.

Potential counter-model 2. Adam interpreted the temporal order of delta and epsilon through the structure of the if-then statement in the definition. Since delta spatially came before the epsilon, then he inferred via the principle that the input came before the output that epsilon depended on delta. This model would need an argument for how Adam felt justified in ignoring the other symbols in the if-then statement. That is, how could if $0<|x-a|<\delta$ then $|f(x)-L|<\varepsilon$ be read as saying if δ then ε? While this was a common interpretation of the if-then statement with other students, in the next segment we see that this was not how Adam interpreted the if-then statement.

Segment 3(b): Epsilon Is Set and Determines the Input

325	Int.	So what about delta depending on epsilon?
326	Adam	Um [*long pause*] actually, /…/ it's more of uh—Because the epsilon can only be, is a set number and that, the difference, the outputs can only be a certain length apart [*points at* $|f(x)-L|<\varepsilon$]. That sort of limits *also* how far the inputs can be, how far apart [*gestures a horizontal interval with his* hands] the input can be from what you're trying to find the limit as x—the input approaches some number.

Summary of argument. Before providing his argument for delta depending on epsilon, Adam very briefly reconsidered his claim that delta depended on epsilon ("actually /…/ it's more of uh—"). He then explained that delta depended on epsilon because epsilon was a set number, which constrained the range of acceptable outputs. The output could only be a certain distance from the limit, which also constrained how far the input could be from a.

Analysis. Adam started this segment by briefly reconsidering his claim about the temporal order. The next segment reveals that Adam might have thought to say that it was more that epsilon depended on delta. In this segment we begin to see some evidence for Adam's uncertainty with the temporal order. At the same time, his brief reconsideration also provided some indications of the stability of his claim that epsilon depended on delta. But Adam temporarily suspended that thought and answered the question. He provided an explanation for why delta depended on epsilon. I explore some of the resources that came up.

Some productive resources, like the **givenness** of epsilon came up in this segment. While it is likely that some of them might have come from Adam's prior instruction, I focus on the way they played out in Adam's reasoning. What was the role of epsilon and delta in Adam's argument in this segment? What was the nature of the relationship between epsilon's role in constraining the output and its role in constraining the inputs? Ultimately, how did Adam connect delta with epsilon? I explore these questions below.

I start by uncovering the meaning behind the parts of Adam's argument as in the analysis of the earlier segment. However, one resource, **absolute condition** required analysis of earlier parts of the interview where Adam spoke about his general perspective on the definition. I assert that this resource had much to do with the way Adam connected delta and epsilon in this segment. I explore the resources related to epsilon, then go into the details concerning the **absolute condition** resource. Lastly, I discuss the role of delta in this segment.

Competing characteristics of epsilon. From the beginning of the interview, Adam believed that he would be *given* an arbitrary number epsilon ("you're gonna be given like a number epsilon, it's gonna be a general epsilon," turn 80). Moreover, epsilon also played a role of a **determiner**. Early in the interview I asked Adam about the meaning of epsilon, and he said that epsilon was a number that "[made] sure that the difference between the actual limit and numbers [$f(x)$'s] near it [was] exceedingly small" (turn 190). Adam might have alluded to this role of epsilon in this segment, but he did not emphasize the constraining characteristic of epsilon. Adam said that the epsilon was set—he cued the **givenness** of epsilon—but he only said "*and* that, the difference, the outputs can only be a certain length apart." He did not say "epsilon is a set number *and so* the outputs can only be a certain length apart." This way of describing epsilon de-emphasized the constraining role of epsilon when compared to the way he described it before using phrases like "must be" (turn 318) or "made sure" (turn 190). While the **givenness** of epsilon and the outputs' being **determined** were cued together, there was a slight modification in the language that Adam used to describe those resources.

Adam also started his argument with "the epsilon can only be," and later said, "the *outputs* can only be." On the one hand we can treat this as a coincidental repeated use of the phrase "can only be." But I posit that the repeated use, along with the modification in the language noted earlier, suggest a shift in Adam's treatment of epsilon from its being a **determiner**. In the next segment, Adam summarized epsilon as the difference between $f(x)$ and $f(a)$, which suggests the application of **function slots** to epsilon (i.e., epsilon was y or $f(x)$). This also suggests that epsilon would be **determined** by the difference between $f(x)$ and $f(a)$. Thus, in this segment we begin to see a tension between the characteristic of epsilon being a given number (**givenness**) and a **determiner** of the desired outputs, and epsilon's being **determined** by the difference between $f(x)$ and $f(a)$ (**function slots**). In sum, concerning resources related to epsilon, Adam cued the **givenness** of epsilon, and epsilon was a **determiner** of the desired outputs, even though it was transitioning to inaccurately become the difference in output values.

Emergence of the absolute condition resource. Adam said that epsilon "[also] limits how far the inputs [could] be." The connection between epsilon and delta exists in the statement, "That sort of limits *also* how far the inputs can be." Epsilon's constraining the range of output values and *also* the input values might be rooted in the way Adam viewed the goal with the formal definition. Adam asserted that the goal was to find *an interval* of x values that would satisfy the epsilon and the delta inequality. I argue that Adam's interpretation stemmed from Adam's treating the delta inequality as an **absolute condition**. Below I explore the details concerning the resource **absolute condition** by elaborating Adam's goal of finding an interval that satisfied the two inequalities in the formal definition. He also revisited this idea of finding an interval later in Segment 5.

Adam elaborated on the goal of finding an interval of x values when he explained the meaning of the statement "the limit of $3x+2$ as x approaches 1 is 5."

> [T]here's an interval /…/ around 1 that, such that if you *plug in* the numbers on the interval /…/ you'll get a number less than epsilon /…/, but it'll also satisfy the other part that /…/ if you *plug in* the interval /…/ for x minus 1, you'll get a number less than delta (turn 126)

Adam asserted that the limit existed if he could find an interval of x values, such that if he were to plug in values from that interval into $f(x)$ in $|f(x)-5|$, it would have resulted a number less than epsilon. At the same time, if the same values were to be used as x in $|x-1|$,[27] then it would have also resulted a number less than delta. It seems clear that for Adam this *interval* was not equivalent to the one defined by the delta inequality. Adam seems to believe that the inequality was an additional condition to be verified. He discussed this idea several times during the interview (turns 44-54, 77-82, 107-118, 125-130, 213-233).

He reiterated the same idea when he explained the meaning of the if-then statement. Adam interpreted "if $0<|x-a|<\delta$ then $|f(x)-L|<\varepsilon$" as two conditions for the *interval* to satisfy:

> The way I'm thinking is instead of saying, 'if this [$0<|x-a|<\delta$] is true then this [$|f(x)-L|<\varepsilon$] must be true,'/…/, I'm thinking for this [$|f(x)-L|<\varepsilon$] to be true then both of these [$0<|x-a|<\delta$ and $|f(x)-L|<\varepsilon$] must be true (turn 230).

This differs from the normative interpretation of a logical implication, where the antecedent's, $0<|x-a|<\delta$ being false is irrelevant to the truth of the logical implication. If we simplify the statement "if $0<|x-a|<\delta$ then $|f(x)-L|<\varepsilon$" as "if P then Q," then Adam was saying that for Q to be true, he would not check if Q was true, while assuming that P was true. Instead, Adam asserted that for Q to be true, both P and Q had to be true. The implication was that if P were false, then Q would also be false. Adam was aware of a normative interpretation of the logical implication, but he opted for a different interpretation in the context of the formal definition. Thus, Adam believed that for the epsilon inequality to be satisfied, the delta inequality had to also be satisfied. In other words, Adam treated the delta inequality as **an absolute condition**.

There is not enough evidence to argue for some hypothesis concerning the origin of the idea, but the literature on students' understanding of conditional statement suggests that this was a common overgeneralization. Students often infer from the truth of P then Q, the truth of the inverse. That is, if $\sim P$ then $\sim Q$ is also true (Hoyles and Kuchemann, 2002). The statement, "if you finish your homework, then you can play videogames," usually means to a child that the inverse of the statement is also true: "if you don't finish your homework, then you cannot play videogames." It would be reasonable for the child to generalize that in order to be able to play video games, they must have finished their homework. This generalization is consistent with Adam's assertion that for "if $0<|x-a|<\delta$ then $|f(x)-L|<\varepsilon$" to be true then the delta inequality had to be satisfied.

Interpreting Adam's argument in that light, we can now interpret the statement, "That sort of limits *also* how far the inputs can be." The epsilon's being given and the outputs' being

[27] Similarly to what I suggested in Segment 3(a), Adam seemed fluent in thinking about function evaluation in terms of intervals. He performed operations on intervals without considering actual x values. For example, in turn 78, he used end points as proxies for the whole interval, "[W]e can find an interval sufficiently small enough that if you plug *it* [the interval] in for x, the end points will always be less than a number delta."

constrained might be a proxy for the epsilon inequality's being satisfied. Since the delta inequality was **an absolute condition**, then the delta inequality had to be satisfied. The **givenness** of epsilon also provided an order in which the two inequalities were to be satisfied.

The delta inequality's being satisfied meant that the input was also constrained. It is unclear what would constrain the input values. Normatively, delta would play that role. That was what Adam said in the previous segment. While Adam did not explicitly mention delta in this segment, his argument for delta depending on epsilon suggests a particular interpretation of delta. Adam only mentioned constrained input values in connecting his argument to delta. This suggests that Adam thought of delta as the input or x values. That is, Adam applied **function slots** to delta (i.e., delta was the x). Recall that Adam was about to apply the **function slots** resource to epsilon.[28]

Model. Adam argued that delta depended on epsilon with the **givenness** of a **determiner** epsilon. Epsilon was a **determiner** of the range of acceptable outputs. Epsilon was in transition between being a **determiner** in the previous segment and a **determined** in the segment after this one. Delta on the other hand became input values in this segment. Adam applied **function slots** to delta (delta is the x). Adam's treating the delta inequality as an **absolute condition** provided a connection between the output values and the input values. That is, the delta inequality had to be satisfied in order for the epsilon inequality to be satisfied. The **givenness** of epsilon provided the order in satisfying the two inequalities. Thus the resources for this segment include: **givenness** of epsilon, epsilon **determined** the range of outputs, partial application of **function slots** to delta but not yet epsilon, and treating delta as an **absolute condition**.

Potential counter-model. Adam showed productive understanding of the formal definition. He understood that epsilon was given (**givenness**) and that it **determined** the range of acceptable outputs. Without explicitly saying it, Adam argued that from the given epsilon, he would find a delta that would constrain the range of input values. In this way delta played a role of a **determiner** of the acceptable range of input values.

This model differentiated between delta and the range of inputs. While Adam treated delta as a **determiner** in the previous segment, the lack of mention of delta in this segment and his treating delta as the actual interval reject that differentiation. Moreover, this model also missed the subtlety of the relationship between the constrained outputs and the constrained inputs. The relationship was not well defined for Adam. In fact, even at the end of the interview, Adam did not end up justifying the temporal order from a normative interpretation of the statement "for every number $\varepsilon > 0$, there exists a $\delta > 0$." For the most part Adam focused on the if-then statement in the definition and the **givenness** of epsilon.

In fact earlier part of the interview shows that Adam's interpretation of the statement "for every number $\varepsilon > 0$, there exists a $\delta > 0$" would be inconsequential. Adam believed that the statement served two purposes. First, the statement gave delta and epsilon names, and second, the statement set both epsilon and delta to be greater than zero, *without* any relationship or order between the two (turn 194). So even if Adam used his interpretation of that statement, it would not have necessarily influenced the ordering.[29] We now explore Adam's hesitation with his claim

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[28] In the next segment, Adam applied **function slots** to both epsilon and delta.

[29] It is interesting to contrast the first part of the definition with the if-then statement. It seems that in terms of access, the if-then statement is more accessible to understand for most students. There is a lot more productive prior knowledge that could be applied in interpreting this part of the definition. The literature has noted that the quantifiers can be a roadblock for students.

in the beginning of this segment, which resulted in a change in claim about the temporal order in the next segment.

Segment 4: Epsilon Really Doesn't Have Any Effect

327	Int.	Can you sort of just restate what you just said?
328 –334	Adam	So epsilon is the difference between the output and f of x [$f(x)$]. So a number near a and then f of a [$f(a)$]. And then delta is *the interval* around a, so the epsilon sort of influences how far the delta can be.. from it. Even though it has *no direct connection* /…/, delta must.. be within a certain /…/ distance from the center /…/ Actually it really doesn't have any effect, I don't think, because— It's more that epsilon depends on delta than delta depends on epsilon.

Summary of argument. Adam was attempting to restate his previous argument in 3(b): "the output can only be a certain length apart and that sort of limits also how far the inputs can be." He asserted that epsilon *was* the difference between $f(x)$ and $f(a)$, and delta *was* the interval around a. He said that epsilon "sort of influences" how far the interval could be from a, albeit with "no direct connection." Then Adam changed his mind and said that epsilon had no effect on delta, and so epsilon depended on delta instead. He attempted to provide a counter argument for why epsilon had no effect on delta, but he ended up concluding his thought.

Analysis. This segment focused the waning influence of epsilon on delta. This resulted in Adam's changing his mind and asserting that it was more that epsilon depended on delta. I discuss what might be behind the changing influence of epsilon and the alternative claim about the temporal order that Adam ended up prioritizing.

Adam revised the role and meaning of both epsilon and delta in this segment. Whereas in 3(a) Adam explicitly mentioned that epsilon **determined** the acceptable range of outputs, and alluded to the same idea in 3(b), by this segment epsilon became a difference between the output $f(x)$ and the limit. That is, epsilon was the difference between the output at a, and near a (**domain constraint for a limit**). Adam also explicitly said that delta was the interval around a, which was markedly different from delta **determining** the range of acceptable x values from 3(a). It is clear that in this segment he applied **function slots** to delta and epsilon, where delta was the x and epsilon was the $f(x)$.

Adam did not explicitly focus on the way that epsilon influenced delta in the summary he provided. With the revisions on interpretations of the role and meaning of epsilon and delta, Adam summarized the relationship as "epsilon *sort of* influences delta." The use of the phrase "sort of" might be a reflection of the tenuous nature of epsilons' influence on delta for Adam. Notice also that the term "influence" is a much more general term in describing a relationship between two quantities than "limit," as used in 3(b). The use of the more general term might have also deemphasized the constraining or **determining** aspect of epsilon on delta.

He also did not cue the **givenness** of epsilon in this segment. I asserted that in 3(b) that the **givenness** of epsilon provided an order in satisfying the two inequalities. Not attending to these resources might have contributed to the ambiguity of *how* delta depended on epsilon. This lack of a *direct* connection from epsilon to delta might have contributed to the uncertainty of the notion that delta depended on epsilon.

Adam also had the competing claim from Segment 3(a), where he cued **functional dependence**. In multiple instances throughout the interview, **functional dependence** assisted

Adam in establishing a direct connection between epsilon and delta via function evaluation (e.g., turns 289–290). Early in the interview Adam explained this idea explicitly:

> … this one [*points at* $|f(x)-L|<\varepsilon$] is dependent on this one [$0<|x-a|<\delta$] because in this one [*points at* $0<|x-a|<\delta$] you're choosing the *x*, this [*points at* $|f(x)-L|<\varepsilon$] is evaluating the function at *x* (turn 238).

While Adam provided a more nuanced argument in 3(a), his use of functional dependence in other context, as exemplified above, can help inform what might be a competing argument for Adam.

In Segment 3(a), Adam cued **functional dependence** independently from **function slots**. In this segment the activation of **function slots**—delta was the interval and epsilon was the difference in output values—might have increased the priority of **functional dependence**. In other words, having revised the meaning of epsilon and delta to be a difference in $f(x)$ and $f(a)$ and an interval of *x* values, respectively, the two should then follow the relationship of $f(x)$ depending on *x*. Thus, epsilon depended on delta. In particular, faced with the ambiguity of the way that epsilon influenced delta, it is likely that Adam prioritized this argument and concluded that epsilon depended on delta.

Model. The lack of direct [functional] relationship describing the way that epsilon influenced delta, and a revision on the meaning and role of epsilon and delta contributed to the change of claim in this segment. The removal of the **determining** resource and the **givenness** of epsilon might have further obscured the way that epsilon influences delta. **Function slots**, which associated epsilon with $f(x)$ values and delta with *x* values might have increased the priority of **functional dependence** between *x* and $f(x)$ from earlier segments. Together they might have contributed to Andrew prioritizing his earlier claim that epsilon depended on delta. While applying **function slots**, Adam still prioritized values of *x* that are close to *a* (**domain constraint for a limit**). Thus the resources for this segment include: **functional dependence**, **function slots**, and **domain constraint for a limit**. It is likely that Adam used **proportional variation**, but he did not explicitly mention that resource.

Potential counter-model 1. The main issue in this segment was Adam's interpretation of delta and epsilon instead of the lack of direct relationship. Correcting his interpretation of the meaning and the role of epsilon and delta would have resolved the issue.

Suppose Adam returned to the interpretation that epsilon constrained the range of acceptable output values, and delta was a constraint for the interval of *x* values. He would still need an idea to connect epsilon and delta. In Segment 3(b), the **givenness** of epsilon helped determine the order by which epsilon and delta were satisfied. Also notice that in that segment, Adam also applied **function slots** to delta, and partially to epsilon ("how far the input can be from [*a*]" and "difference in outputs"). What led him to the correct ordering was, again, his attending to the **givenness** of the **determiner** epsilon, which also made delta the **determined**. In this segment Adam did not prioritize those resources.

The model then predicts that knowing a more direct way that epsilon influences delta might have led Adam to a different conclusion. Suppose Adam knew that delta was proportional to epsilon,[30] then Adam might have stayed with the claim that delta was also dependent on epsilon. The next segment supports this argument. In the next segment Adam used a direct relationship suggested by the inequality $3\delta < \varepsilon$, along with the **givenness** of epsilon to conclude the temporal order of delta and epsilon.

Segment 5: The Size of the Interval Must Be Smaller Than the Epsilon

336 Adam Because, [*looks at the definition*] you can always find, you may be able to find an *interval* such that uh… [*looks up to the ceiling and smiles*] I don't know actually. I actually don't know. Cus if you go back to thinking this idea [*circles* $3\delta < \varepsilon$ in Figure 6.4],

338 Adam then you get the idea that the size of the interval *must be* smaller than the number epsilon. That [the] size of the interval times three [3δ] must be smaller than epsilon. So the radius times three [3δ] I should say, of the interval must be smaller than epsilon.

340 Adam So it influences delta cus if epsilon is set then.. delta.. has to be a certain size of radius in order to… conform to the size of epsilon but I don't know how I, I prove, well not prove, show that *that* [the limit] exists using the definition.

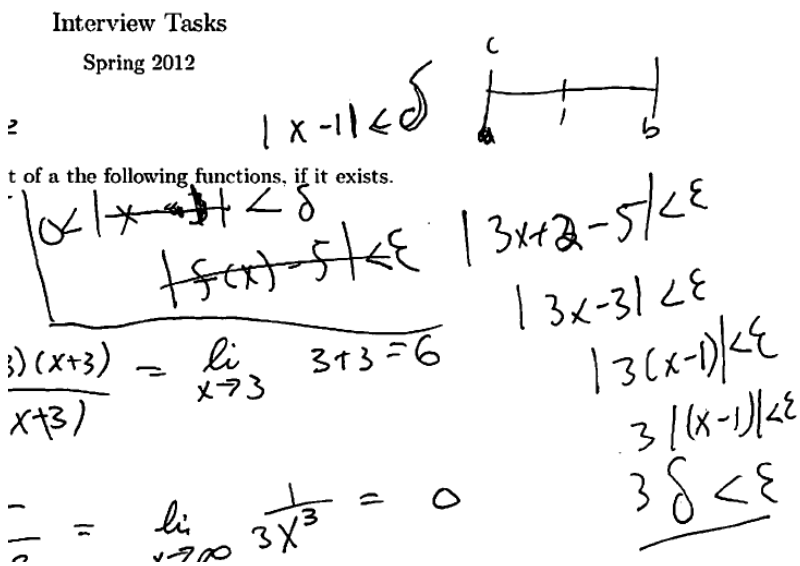

Figure 6.4. [Top right] Adam's recall of the "proof" for the limit of $f(x) = 3x + 2$ as x goes to 1.

Summary of argument. Adam started this segment by attempting to provide a counterargument for delta depending on epsilon. He mentioned that one could always find "an

[30] In practice, it is rare that one would define a direct functional relationship between delta and epsilon for the formal definition. Though in most first semester calculus, this might be the impression that many students have.

interval," but then he stopped himself. He focused instead on the inequality $3\delta < \varepsilon$.[31] The inequality said that three times the size of the interval (delta) must be smaller than the number epsilon. Unlike in the previous segment where he said that delta was the interval itself, here Adam specified that delta was the radius of the interval. He also said that if epsilon was set, then delta had to conform to the size of the epsilon. He re-established the influence of epsilon on delta and returned to his previous claim that epsilon and delta depended on each other. However, Adam still felt like he did not know how to prove the limit existed using the definition.

Analysis. This segment followed immediately after Segment 4. Adam started by providing an argument against his previous claim that delta depended on epsilon. His statement, "you can always find, or you may find an interval such that…" was his reading of part of the statement of the definition (turn 336). He was reading "there exists a $\delta > 0$ such that…" in the definition. Like I discussed in Segment 3(b), Adam believed that for the limit to exist then he needed to be able to find an interval of *x* values that would satisfy the epsilon and delta inequalities. Here, Adam did not finish his thoughts, but he was reading that statement while arguing that epsilon did not have any influence on delta. Thus, with the statement Adam might have wanted to say that he could always find an interval that would satisfy the epsilon inequality, but *not* the delta inequality (delta as **an absolute condition**). The cueing of this resource is not surprising, considering in 3(b) he cued this resource to argue for the same claim, that delta could depend on epsilon. The consistency of the interpretation with what comes next provides additional warrant.

While Adam was trying to figure out if such an interval—one that did not satisfy the delta inequality—existed, his attention was taken by the inequality $3\delta < \varepsilon$ from his proof for the limit of $f(x)=3x+2$ as *x* approached one (see Figure 6.4). He said, "I actually don't know. Cause if you go back to thinking this idea [*circles* $3\delta < \varepsilon$], then you get the idea that the size of the interval *must be* smaller than the number epsilon" (turn 336-338). For some reason, the size of the interval's being smaller than the number epsilon from the inequality $3\delta < \varepsilon$, challenged the possibility of his always being able to find an interval of *x* values that did not satisfy the delta inequality. I use Adam's argument in Segment 2 to help uncover the way that the inequality addressed the possibility of finding such an interval.

In Segment 2, Adam explained the way he inferred the existence of a limit of a function through the if-then statement. He did so while discussing the dependence of epsilon and delta. Several turns prior, Adam explained that the dependence between delta and epsilon depended on the function (turns 292–299). Adam explained why with the function, $f(x)=3x+2$, epsilon and delta depended on each other:

> In this case [*the limit of* $f(x)=3x+2$ *as x approaches* 1], they do depend on each other because this [function] actually <u>works out good</u>, because when you break it [the epsilon inequality] down, you find out three delta [3δ] must be less than epsilon. So the interval times three [3δ] must be less than epsilon (turn 304).

Adam's statement in turn 304 was similar to his statement in turn 338. Here, in Segment 5, he said, "You get the idea that the size of the interval *must be* smaller than the number epsilon. That

[31] $\delta=\varepsilon/3$ would have been the optimal solution for the proof of the limit, but Adam focused, instead on the inequality, $3\delta < \varepsilon$. Even toward the end of the interview, Adam wrote the relationship as an inequality $\delta < \varepsilon/3$, instead of an equation.

[the] size of the interval times three [3δ] must be smaller than epsilon" (turn 338). The difference was that delta in turn 338 was the size of the interval, not the interval itself. This similarity motivates the use of Segment 2 to help interpret what Adam was thinking in Segment 5.

The phrase "works out good" suggests the connection between the dependence of epsilon and delta and the existence of the limit.[32] When Adam said, "works out good," he meant that with the function, $f(x)=3x+2$, as x approached 1, $|f(x)-L|$ would get smaller as a result of $|x-a|$'s getting smaller. Just several turns prior, he explained, "If the limit exists then /…/ as this [*points at* $0<|x-a|<\delta$] gets smaller, this [*points at* $|f(x)-L|<\varepsilon$] /…/, the difference is gonna get smaller. But if the limit doesn't exist /…/ then as this [*points at* $0<|x-a|<\delta$] gets smaller, this [*points at* $|f(x)-L|<\varepsilon$] isn't gonna change, this [*points at* $0<|x-a|<\delta$] isn't gonna help [*inaudible*]" (turns 292–294).[33] Earlier in the interview, he wrote a limit symbol in front of $|x-a|$ and $|f(x)-L|$ with x approaching a (see the end of Appendix J for the first page of the written artifact). With the limit symbol, $|x-a|$ and $|f(x)-L|$ could get smaller as x approached a. The use of the limit symbol in front of $|x-a|$ and $|f(x)-L|$, and the focus on the idea of $f(x)$'s getting closer to L as x's getting closer to a, serve as evidence for Adam's use the **dynamic definition of a limit** to infer this relationship between delta and epsilon.

Therefore Adam inferred that the limit existed from the inequality $3\delta < \varepsilon$ (this was the case that "works out good"). Certainly, Adam already knew that the limit existed from calculating the limit at the beginning of the interview, but $3\delta < \varepsilon$ *cued* this knowledge for Adam in this segment. When the limit existed, according to Adam, then there had to be "an interval" of x values that satisfied both the epsilon and delta inequality (turn 126, see Segment 3(b)). Therefore, there was at least one function, whose limit existed, where it was *not* possible to find an interval where the epsilon inequality was satisfied but not the delta inequality. Adam found his counterexample.

Having addressed his counterargument—there was at least one case where he could not find an interval that did not satisfy the delta inequality—Adam returned to thinking about the "influence" of epsilon on delta (turn 340). Recall that in the last segment Adam was also lacking a direct [functional] relationship to show that epsilon influenced delta. In this segment the inequality could provide such a relationship depending on how it was read (turn 338). The **determining** resource specified a way to read the inequality. Epsilon returned to its role as the **determiner**, and for the first time, it **determined** delta ("[The] size of the interval times three [3δ] must be smaller than epsilon," turn 338).

Adam re-asserted the **givenness** of epsilon, which he just used in Segment 3(b), to further establish the influence of epsilon on delta as suggested by the way he read the inequality. He said, "*if epsilon is set* then.. delta.. has to be a certain size of radius in order to… conform to the size of epsilon" (turn 340). Adam re-asserted the **givenness** of epsilon and the fact that epsilon was the **determiner** of delta. The two resources established the influence of epsilon on delta.

Adam's use of the word "conform" might suggest a different resource that was involved. Delta's conforming to the size of epsilon could be interpreted to mean that the input needed to be

[32] Another interpretation is that Adam was arguing for the claim that epsilon and delta depended on each other by strictly using the inequality, $3\delta < \varepsilon$. He rejected it when I offered it as an interpretation of what he said (turn 310). I discuss this in the counter-model for the argument in this segment.

[33] Notice Adam's normative interpretation of the if-then statement in this context. In stipulating the existence of a limit, he correctly constructed the negation of the if-then statement as *P* but not *Q*. This contrasts with his understanding of the if-then statement in Segment 3(b).

modified in order to accommodate a specification on the output: **quality control**. However, there was no mention of x (input) or $f(x)$ (output) when Adam was discussing the inequality $3\delta < \varepsilon$.[34] Adam only mentioned the size of each of these "quantities" and their relationship to each other, without explicitly mentioning its constraining properties on the output and the input. So I posit that Adam did not cue **quality control** but instead focused on the **functional dependence** of delta on epsilon, which was suggested by the way that he read the inequality.

Model. Adam revisited the inequality $3\delta < \varepsilon$ to resolve previous conflicts. First, he used it to address the issue that he could always find an interval that would satisfy the epsilon inequality but not the delta inequality (delta inequality as **an absolute condition**). The fact that the limit of the function existed, as suggested by $3\delta < \varepsilon$, provided Adam a counterexample for the claim that such an interval would always exist.

Adam also used the inequality to address the lack of direct [functional] relationship from the previous segment (**functional dependence**). The inequality suggested to Adam that epsilon would **determine** delta in that three times the radius of the interval must be smaller than epsilon. The **givenness** of epsilon, which was cued in 3(b) was reasserted to establish the influence of epsilon on delta. Having addressed the previous conflicts, Adam returned to his previous claim that delta could also depend on epsilon, and concluded that they depended on each other. Therefore the resources for this segment include: **absolute condition**, **givenness** of epsilon, the idea of **determining** as applied to delta and epsilon, **functional dependence**.

Potential counter-model 1. The main resource in this segment was the idea of **determining**. Adam used that resource to interpret the inequality $3\delta < \varepsilon$, saying that three times the radius of the interval must be less than epsilon. Epsilon determined delta. Thus, delta depended on epsilon.

This would not be a sufficient explanation because I argued that part of the reason for Adam to attend to the inequality was to challenge the **functional dependence** and **function slots** argument from Segment 4. So that argument needed to be addressed in this segment. Moreover, this interpretation completely ignored the first statement of this segment about the idea of finding "an interval" to satisfy the inequalities (the delta inequality as an **absolute condition**). Adam attended to this resource to come up with the counterexample. Adam also did not use the **determining** resource directly to establish the temporal order. He used it to read the inequality to establish a functional dependence relationship, and later to re-establish the influence of epsilon on delta.

Potential counter-model 2. Adam already knew that delta depended on epsilon from the inequality $3\delta < \varepsilon$. There are always two ways to read any inequalities: the left hand side is less than the right hand side or the right hand side is more than the left hand side. So the two sides are always dependent on each other. In fact, that seems to be what Adam was saying in Segment 2: "[T]hey depend on each other cause the one must be 3 times smaller than the other" (turn 306).

Adam could have read the same inequality to conclude that epsilon should conform to the delta. He could have done so by concluding that the size of the epsilon must be greater than three times the radius of the interval. This would have been inconsistent with the **givenness** of epsilon. We did not see Adam reversing the order. Adam also mentioned specifically that delta was not given a number of times (e.g., turns 108, 148). Moreover, Adam rejected this model earlier in the interview (Segment 2). I had asked him if the reason epsilon and delta depended on each other

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[34] It is also notable that at this point, Adam *did not* take the next logical step and say that $\delta < \varepsilon/3$. This would have further supported the idea that epsilon **determined** delta. However, in Segment 8, he did (turn 397). In Segment 8, Adam argued that epsilon came first.

was just because of the inequality. He said no, and elaborated on his argument. His elaboration was actually the beginning of Segment 3(a) in this chapter.

It is worth noting that having addressed the issue was not sufficient for Adam to abandon the claim that epsilon depended on delta. In the discussion after Segment 5, which I do not explore in detail, Adam wrestled with the issue of whether epsilon was set. Ultimately, relying on his experience of "breaking down epsilon" (simplifying the epsilon inequality to find the "delta expression"), Adam concluded that epsilon had to come first and so delta depended on epsilon. I close the analysis section with a full narrative summary from the four segments.

Full Narrative Summary

Figure 6.5 shows the resources that Adam used in Segments 3(a), 3(b), 4 and 5. In Segment 3 Adam claimed that epsilon and delta depended on each other. In 3(a) Adam explained that epsilon depended on delta because delta **determined** the input and epsilon **was determined** by the difference in the output and *L*. Those resources assisted Adam in reading the epsilon and delta inequalities. **Functional dependence** supported the directionality of the relationship via the input output relationship. **Domain constraint for a limit** and **proportional variation** provided details for the meaning Adam attached to the delta inequality: delta made the interval small so that the epsilon was small. In 3(b) Adam argued that delta could also depend on epsilon by relying on the **givenness** of **the determiner** epsilon. This resource interacted with Adam's goal of finding an interval to satisfy the delta inequality as an **absolute condition**. In finding this interval, Adam would start with a **given** epsilon to find the *interval* delta. Adam applied **function slots** to delta (delta was the interval of x values).

In the process of restating his explanation for why delta depended on epsilon, Adam ended up concluding that epsilon depended more on delta than the other way around. First, the de-emphasis on **givenness** of epsilon and the **determining** resource led to the weakening of the influence of epsilon on delta. Second, the application of **function slots** to both epsilon and delta and the lack of direct [functional] relationship from epsilon to delta might have prioritized Adam's argument using **functional dependence** from before. The **function slots** resource, which associated epsilon with $f(x)$ values at a and near a (**domain constraint for a limit** and likely **proportional variation**) and delta with x values might have cued the **functional dependence** between x and $f(x)$. This might have provided a more convincing argument in the face of the ambiguous influence of epsilon on delta.

In fact, in Segment 5 Adam switched back to claiming that delta could also depend on epsilon by re-establishing the influence of epsilon on delta. He first addressed the possibility of always finding an interval that did not satisfy the delta inequality (**absolute condition**), with a counterexample of a function whose limit existed. He used the **dynamic definition of a limit** to determine whether the limit existed. He then re-established the influence of epsilon by re-asserting the **givenness** of epsilon and its role as a **determiner** of delta as suggested by the inequality $3\delta < \varepsilon$.

Figure 6.5 summarizes the activation of different knowledge resources throughout the four segments. Grey signifies activation of the resource, white is no activation, and shaded is a consistent resource but no explicit mention of the resource. One interesting pattern, which can be seen in the figure, is that the use of knowledge resources was not consistent across segments. Some resources were dominant in one segment but became less dominant in the next segment (e.g., **givenness** played a significant role in Adam's argument in Segment 3(b), was deemphasized in Segment 4, only to return in Segment 5).

Seg.	CLAIM	KNOWLEDGE RESOURCE							
		Absolute condition	Proportion-al variation	Determin-ing	Function slots	Functional Dependence	Given ness	Dynamic definition of a limit[35]	Domain constraint for a limit
3(a)	No order (delta first)								
3(b)	No order (epsilon first)								
4	Delta first								
5	No order (epsilon first)								

Figure 6.5. The progression of activation of knowledge resources across the segments.

Discussion

In the interview, Adam started with the claim that epsilon depended on delta (delta first) and ended with delta depended on epsilon (epsilon first). Despite knowing how to reproduce parts of epsilon-delta proofs from memory, when asked about the relationship between delta and epsilon, Adam wavered (Part I). While this pattern is consistent with the finding from Chapter 5 that the temporal order was challenging for students, this analysis chapter investigates the details of the struggle. The analysis of Segments 3 to 5, in Part II, explores the process of Adam's reasoning by uncovering some of the knowledge resources that were involved. I now discuss the three specific aims of the analysis: the student's repertoire of resources, the richness of the formal definition as a learning context and the prevalence of the **functional dependence**.

Revealing Resources: Mathematical and Intuitive Knowledge Resources

The fine-grained analysis was able to reveal a number of key resources in Adam's reasoning. Some of them were closely tied to the concept of limit, like **domain constraint for a limit** and **dynamic definition of a limit**. Others were more general mathematical resources, like **functional dependence** and **function slots**. The analysis also revealed several informal/intuitive knowledge resources, like **absolute condition**, **givenness** and the idea of **determining**.

Progression and influence of the determining resource

The **determining** resource describes the role of one quantity in establishing and constraining another. We see in the data that the **determining** resource played a consistent and significant role in Adam's reasoning about the temporal order. In particular this resource was cued in discussing the role of epsilon and delta in the formal definition. Table 6.2 documents the activation of this

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[35] Dynamic definition of a limit was cued in Segment 2 to determine the existence of a limit. While the resource was consistent with Adam's argument in Segment 3(a), it was not explicitly mentioned. In Segment 5, while the conclusion of the limit of the function existed used the resource, Adam had made that conclusion before. The exposition for Segment 5 included it to elaborate on the meaning behind the inequality $3\delta < \varepsilon$.

resource with the meaning of epsilon and delta across the segments.

Table 6.2.
The Progression of the Meaning of Delta and Epsilon vis-à-vis the Determining Resource

	Segment	Operational definition	Turn	**Determining** resource
Delta	3(a)	constrains the x values to be close to a and made sure the interval is not too large	312	determiner (of the input)
	3(b)	constrains the interval; but it is also an unknown	326	determiner (of the interval) & determined (delta is an unknown)
	4	is the interval or the x values	332	determiner (it is the x values in a functional dependence)
	5	is the radius of the interval	338	determined (by epsilon through $3\delta < \varepsilon$)
Epsilon	3(a)	must be larger than the difference in the outputs	318	determined (by the outputs)
	3(b)	is set and constrains the difference in the outputs and also the input x from a	326	determiner (of the output and input)
	4	is the difference in the outputs or the $f(x)$ values	328–330	determined (it is the y values in a functional dependence)
	5	is set and must be larger than three times delta ($3\delta < \varepsilon$)	338	determiner (of delta)

The **determining** resource specified a way to read the different inequalities. Adam used the resource to interpret the inequalities in the if-then statement. Adam's interpretation of the statement $0<|x-a|<\delta \Rightarrow |f(x)-L|<\varepsilon$, and $3\delta < \varepsilon$ was influential in the development of his claim about the temporal order. In Segment 3(a) and (b), the if-then statement directly influenced Adam's transitional claims about the temporal order. For example, in Segment 3(a) where delta was a **determiner** and epsilon the **determined**, Adam interpreted the if-then statement to mean that the known delta constrained the x values and the epsilon inequality evaluated the x's to compute the unknown epsilon. Thus, epsilon depended on delta.

In Segment 4, the de-emphasis on this resource, specifically the **determining** role of epsilon reduced the influence of epsilon on delta, which led to the change of claim. In Segment 5, the resource was supported by the **givenness** of epsilon to re-establish the influence of epsilon on delta. It also assisted Adam in reading the inequality $3\delta < \varepsilon$. The inequality asserted that three times the radius of the interval must be less than epsilon. He could have read the inequality in the other direction, but just like in Segment 3(a), the resource helped Adam read the inequality in a particular direction.

One might argue that this resource was directly cued by the wording of the question in the interview about dependence between epsilon and delta. Is it not the case that the issue of dependence is about which determine which? Yes, but the quantity that the **determiner** determined matters. Epsilon's being a **determiner** could mean that epsilon determined delta, but it could also mean that it determined the output. In this way the resource can be attached to different quantities. Thus, **determiner** and **determined** can be attached to different quantities, just like other resources. The resource can be attached to epsilon and delta (epsilon is the determiner and delta is the determined), or it can also be attached to delta and the interval (delta is the determiner and the interval is the determined).

The dual role of delta being both a **determiner** and a **determined** might add another layer to the complexity of understanding the formal definition. On the one hand, according to the temporal order, delta is **determined** by epsilon. On the other hand, in the delta inequality, delta is the **determiner** of the size of the interval—we only consider x values that are within delta of a. While Table 6.2 shows Adam switching back and forth on the roles of both delta and epsilon, according to the definition epsilon can only take on one role. Epsilon is strictly a **determiner** in the definition. It is a determiner of delta and the determiner of the acceptable distance from the limit. Delta on the other hand, normatively holds the dual role, a subtlety that students need to recognize.

Interaction between resources

A claim about the temporal order was not based on a single knowledge resource. A claim was based on several resources that Adam put together in a particular way. We see different kinds of interaction between the resources across the four segments. Sometimes a resource can connect one resource with another. For example, in Segment 3(a), delta **determined** the acceptable input, and epsilon **determined** the acceptable difference between the output and the limit. The two would not have been related had it not been for **functional dependence** between input and output. The interaction between these resources made up Alan's argument for that segment.

A resource can also impose an order where there was ambiguity associated with another resource. In Segment 3(b), there was ambiguity as a result of treating the delta inequality as an **absolute condition.** The **absolute condition** provided Adam with a goal of finding an interval to satisfy both the epsilon and delta inequality. The **givenness** of epsilon provided a reasonable ordering to satisfy the two inequalities. Since epsilon was **given**, it was reasonable for Adam to satisfy the epsilon inequality before the delta inequality.

Lastly, a resource can either bolster or weaken another resource. In Segment 4, the application of **function slots** to delta and epsilon likely increased the (cueing) priority of **functional dependence**. At the same time, this highlighted the lack of functional relationship between the **determiner,** epsilon, and the **determined,** delta. This contributed to Adam's questioning of the influence of epsilon on delta, which temporarily changed his claim about the temporal order. In Segment 5, we see the **givenness** of epsilon supporting the role of epsilon as the **determiner** of delta.

Post-analysis notes about identification of knowledge resources

Identifying knowledge resources is a difficult endeavor, but the various methodological orientations and theoretical assumptions assisted with the effort. Competitive argumentation as a methodology was helpful in problematizing the existence of a more obscure knowledge resource and also in ensuring the activation of other resources. For example, competitive argumentation assisted in identifying the **determining** resource. Initial stage of analysis identified **functional dependence** as the main resource in Segment 3(a), but several rounds of competitive argumentation led to the initial claim being a counter-model (seeCounter-model 1). As discussed in 3(a), the **determining** resource helped Adam interpret the inequalities, which in turn helped him decide the temporal order.

The theoretical assumptions about the resources also helped the identification process. The neutrality of a knowledge resource was a productive assumption. Initial stages of the analysis labeled larger claims and interpretations as knowledge resources. The resources were the neutral ideas underneath the claims and interpretations. **Function slots** was identified from the claims "delta is *x*" and "epsilon is *y*." The reason that they were not resources was that "delta is *x*" and "epsilon is *y*," in addition to being a larger claim, they were not neutral in terms correctness. That is, "delta is *x*" and "epsilon is *y*" could be seen as an incorrect. The incorrectness can come as a result of attaching resources to other quantities. For example, students attached delta to *x* and epsilon to *y*. The stipulation that two quantities sharing a functional relationship, one is the *x* and the other is the $f(x)$ is neutral, which made it a knowledge resource called **function slots.**

Absolute condition was also identified as a result of the neutrality assumption. **Absolute condition** was identified from the claim "conditional statements are bi-conditional." The claim that "conditional statements are bi-conditional" was incorrect and thus could not be a knowledge resource. The resource underneath the claim was **absolute condition***,* or students' knowledge about what it meant for a condition to be necessary and sufficient (absolute). The definition provided in Table 6.1 shows that the resource **absolute condition** could often lead to the assertion that "conditional statements are bi-conditional."

The **determining** resource was identified from Adam's interpretation of inequalities in the if-then statement. His interpretation of the if-then statement was a reasoning pattern, and unpacking the ideas behind his interpretation of the statement led to the **determining** resource. As noted above, the process of competitive argumentation also helped in identifying this particular resource.

The methodological orientation of striving to understand *when* and *why* a knowledge resource is and is not used was also helpful in identifying the resources. Figure 6.5 is a concretization of the methodological orientation. The figure in some way kept the analysis accountable to track the activation of resources across the segments. For example, with **absolute condition***,* the methodological orientation questioned if the resource was activated in other segments. Its activation in Segment 5 was not immediately obvious. When I realized that Adam was trying to find an interval, I had to consider the **absolute condition** resource. That introduced a new piece of information to triangulate.

The orientation also assisted in the revision of Figure 6.5. Initially I had claimed the activation of **dynamic definition of a limit** in Segment 5. This raised the question of when else did Adam use that resource, and what happened in between its activations? I concluded that Adam did not determine the existence of the limit of the function using the definition in Segment 5. He did that in Segment 2. In Segment 5, he used the assertion that the limit existed, to counter the claim that he could always find an interval that did not satisfy the delta inequality. Since it was activated it Segment 2, then I had to consider the possibility of its activation in Segment 3(a). Based on its consistency and immediacy of its activation between Segment 2 and 3(a), I concluded that the resource was present but not explicitly attended to.

Potential limitation of counter-models

There is a potential limitation to the use of counter-models in identifying resources. As the analyst is the one generating the counter-models, then it is limited by the extent to which the person is able to generate counter-models that would be illuminating of the thought process of the research subject. The challenge is that another analyst might read the data and come up with another counter-model.

For this reason collaborating with other researchers can be helpful in making sure that no important counter-models have been overlooked. At the same time, the result of the analysis is *a model*. In that way, a model can always be refined and elaborated. Creating a model that is most informative is more productive than finding the "final" model.

The finality of the model also partly depends on the goal of the analysis. For example the goal of the analysis in this chapter is to identify *some* of the more dominant resources to illustrate the process of Adam's sense making. It was not to come up with an exhaustive list of intuitive knowledge students have about the temporal order. That is not to say that such endeavor would be unproductive, but it would require a more extensive analysis with a different data source.[36] So keeping in mind the goal of the analysis, while continuing to engage with other researchers about the data can increase the productivity of this methodology.

Lastly, a certain level of healthy skepticism is recommended. I caution against being too comfortable with one's model of a student thinking. As I reported earlier, I was convinced of theCounter-model 1 in Segment 3(a) for a number of months. It was not until I untied myself from that one interpretation that I was able to recognize other resources that could potentially be involved.

Changing Levels of Specificity of Knowledge as an Indicator for Source of Difficulty

Changing levels of specificity of delta and epsilon

We see Adam moved through different levels of specificity in thinking about different aspects related to the temporal order. For example, one of the key players in determining the temporal order was the meaning of delta and epsilon. Depending on the segment, the level of specificity of delta and epsilon changed (see Table 6.2 above). For example, from 3(b) to 4, delta and epsilon underwent similar transformation. Delta went from being the **determiner** of the acceptable interval to becoming the interval itself. Epsilon's role went from the **determiner** of the acceptable difference in the outputs, to being the difference itself. How can we explain the changes in the meaning of epsilon and delta? Did Adam turn the inequality into an equation? Why was the word radius only used to describe delta in Segment 5?

I argue that these changes reflect changing levels of specificity in Adam's use of the terms epsilon and delta, and it was less of a misunderstanding or instability of the meaning of epsilon and delta for Adam. These changes illuminate the process by which Adam developed the claim about the temporal order. Some of these characterizations of delta and epsilon were incorrect (e.g., delta was just *not* the interval) but they did not stem from a misunderstanding.

Moving between different levels of specificity about epsilon and delta served a particular purpose for Adam. For example, in 3(a), Adam treated delta as a **determiner** and epsilon as the one **determined** to read the inequalities to mean that delta determined the appropriate interval for the input, and epsilon determined the acceptable difference in the outputs. In Segment 4, Adam's conceptualization of epsilon and delta (as $f(x)$ and x values respectively) might have been a reflection of Adam's prioritizing **functional dependence**, or it might have increased the priority of the **functional dependence** resource.

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[36] Studies whose focus is in identifying knowledge elements typically employ clinical interview method (e.g., diSessa, 1993; Wagner, 2010), which this study did not fully employ.

Source of difficulty: hyper-richness of context with high level of cognitive load

The literature on the teaching and learning of the formal definition of a limit has long argued that the definition is complex. Studies have pointed to different aspects of it as potential source of difficulty: quantifiers, absolute values and issues pertaining to the backward functional relationship. Thus, it is reasonable to conclude that one difficulty with the definition is that reasoning about the formal definition probably requires students to sustain a high level of cognitive load (Chandler and Sweller, 1991). There was some evidence for Adam's struggle with quantifiers—he interpreted the first part of the definition as simply giving epsilon and delta a name and setting it larger than zero. Related to quantifiers, some studies argue that students have difficulty negating the statement of the definition, and there was some evidence that Adam might have had similar issue (e.g., what it meant to satisfy the if-then statement).[37]

The analysis in this chapter presents a hypothesis on the source of difficulty at a finer level of detail. I argue that Adam struggled in navigating through and aligning the different knowledge resources. diSessa's (2007) calls this the *hyper-richness hypothesis.* Certain learning goal might cue a large repertoire of prior knowledge that can be applied to it. diSessa posits that one of the downsides of an example or a learning context's being hyper-rich is that students might experience difficulty in deciding which prior knowledge is most relevant in the particular context.

I claim that the temporal order and the formal definition are hyper-rich. For example, consider the way Adam read epsilon and delta. Epsilon and delta as concepts are hyper-rich. Adam was aware of different characteristics and roles of epsilon and delta. Some of these characteristics competed with the temporal order (see characteristics of epsilon section in Segment 3(b)). Depending on what the argument he was making, he attended to different aspects of epsilon and delta (Table 6.2). Thus, when Adam glossed over the delta inequality and concluded that delta was the interval, I posit that he was not ignoring the symbols.[38] Instead, he did so to support an argument that he was trying to make. His argument in each segment utilizes a particular aspect of epsilon and delta. For example, Adam's argument in Segment 4 using **functional dependence** was consistent with epsilon's being $|f(x)-f(a)|$, whereas his argument in Segment 3(b) and 5 uses epsilon's **givenness** and constraining role.

I posit that the complexity of the formal definition magnifies the complexity of dealing with its component parts. That is, while individual ideas, like quantifiers, can be difficult, some of these concepts might be more challenging to deal with correctly when working with a concept as complex as the formal definition. Students might need to navigate through different aspects of the concept to support their argument. The changing levels of specificity of student thinking might be a good indicator of the hyper-richness phenomenon.

According to the model presented in the analysis, Adam resolved his difficulty by addressing conflicts on the level of individual resources (e.g., lack of **functional dependence** in Segment 4). If we assumed that Adam's struggle was with the definition of epsilon and delta or with the quantifiers, addressing those might not have necessarily addressed some of the underlying issues for Adam. Moreover, the hyper-richness hypothesis also challenges the notion that had Adam just stuck with a normative definition of epsilon and delta then he would not have been confused. I argue that changing levels of specificity was necessary and was a by-product of the hyper-richness of this context.

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[37] It was unclear if Adam struggled with absolute values, though his facility with thinking in terms of interval suggests otherwise.

[38] Other students also loosely associated delta with x and epsilon with y.

Prevalence of the Functional Dependence Resource

The previous analysis chapter clearly documented how **functional dependence**, along with delta is similar to x and epsilon is similar to y was a very common *reasoning pattern* across students in this study. In this chapter, we see that the resource played an important and consistent role in the four segments. Adam's argument initially in Segment 3(a) appeared similar to the common reasoning pattern documented in chapter 5 (see Counter-model 1). However, a closer analysis revealed a slightly different use of the resource. The **functional dependence** was used to connect the input to the output, but not in the way it typically does in the common reasoning pattern. Adam used the **determining** resource to read the epsilon and delta inequalities to describe the relationship between epsilon and delta to $f(x)$ and x values. Moreover, in Segment 4, the data suggests that the lack of having **functional dependence** to describe the way that epsilon influenced delta might have led Adam to challenge his own claim that delta depended on epsilon. We see Adam "addressed" this issue in Segment 5 when he re-established the **functional dependence** using the inequality $3\delta < \varepsilon$.

The analysis of this chapter also suggests that students—likely as a result of instruction—might be predisposed toward explicit functional relationships. Students might see a relationship between two variables that cannot be expressed as an equation (or inequality) as tenuous and thus less preferred. In Segment 4, Adam challenged epsilon's influence on delta and favored the **functional dependence** between x and $f(x)$ to described the temporal order. He returned to the appropriate temporal order with an explicit **functional dependence** relationship to describe epsilon's influence on delta, as suggested by the inequality $3\delta < \varepsilon$.

One hypothesis of the study is that the fact that epsilon is a **given** quantity might challenge students' reliance on the **functional dependence** relationship between x and $f(x)$. That is, the **givenness** resource might reduce the prevalence of the **functional dependence** resource. The **givenness** of epsilon was a productive resource that helped Adam conclude the correct temporal order of epsilon and delta, even with the prevalence of the **functional dependence** resource. We see in the analysis that the **givenness** of epsilon did not *replace* the common unproductive **functional dependence** between epsilon and delta. The **givenness** resource helped Adam reorganize his knowledge (Smith et al, 1993). The temporal order claim changed as a result of interactions of various knowledge resources. The **givenness** of epsilon played a significant role in some of those interactions.

In one case, the **givenness** of epsilon interacted with the goal of finding an interval of x values that satisfied the epsilon and delta inequalities (delta as an **absolute condition**). The **givenness** of epsilon set the order in verifying the two inequalities (see Segment 3(b)). In Segment 5, the **givenness** of epsilon was also cued to support the influence of epsilon in **determining** delta. Recall that Adam already stated that epsilon **determined** delta, but in stating his final justification, Adam included the **givenness** of epsilon to support his argument.

The analysis also shows that the **givenness** of epsilon does not always compete with **functional dependence.** On the one hand, one might argue that Adam's argument in Segment 3(a), which used the **functional dependence,** and his argument in 3(b), which focused on the **givenness** of epsilon were contradicting each other. Thus, the two resources were conflicting. On the other hand, in Segment 5, the **givenness** of epsilon and the **functional dependence** of *delta* on epsilon (inferred through $3\delta < \varepsilon$) were cued together to support the claim that delta depended on epsilon. Adam did not switch his temporal order claim between Segment 4 and 5 by replacing **functional dependence** with the **givenness** of epsilon. Instead, Adam inferred another functional dependence relationship, which was *consistent* with the **givenness** resource.

That is certainly not the only way for students to productively use the **functional dependence** resource. In Chapter 8, we see Adriana's repurposing the **functional dependence** relationship to describe the relationship between errors instead of between error bounds as a result of discussing the Pancake Story. The common thread between the two students was that the change in conception happened as a result of students' reorganizing their prior knowledge, and not by replacing a "misconception." Specifically both Adam and Adriana looked for a context in which to productively use **functional dependence**, and they also specified the meaning and role of delta and epsilon and de-emphasized the claim "delta is *x*, and epsilon is *y*."

CHAPTER 7: REASONING ABOUT THE TEMPORAL ORDER WITH THE PANCAKE STORY

This chapter explores the influence of the Pancake Story on students' understanding of the temporal order of delta and epsilon. The goal of the analysis is to document changes in students' responses to the four questions about the temporal order after engaging with the Pancake Story. It also explores some of the more common reasoning patterns that emerged after discussing the story. The chapter closes with discussions of remaining challenges regarding the story that students reported.

Analysis Methods

The first part of the analysis is similar to the analysis done in Chapter 5. To document changes in students' responses to the temporal order questions, I categorized students' response to each question about the temporal order after they engaged with the story. As a reminder, I asked about the temporal order of delta and epsilon in four contexts: dependence, sequential order, set-ness and the order of ε, δ, x and $f(x)$. The following were the actual questions:

1. In the definition, with epsilon and delta, what depends on what, if anything you think? Delta depends on epsilon? Epsilon depends on delta? They depend on each other? Or they do not depend on each other? And why?
2. In the definition, between epsilon and delta, which one do you think comes first and which one do you figure out as a result? And why?
3. In the definition, between epsilon and delta, which one do you think is set? Epsilon? Delta? Both? Or neither? And why?
4. How would you put the four variables, epsilon, delta, x and f (x) in order, in terms of which comes first in the definition? And why?

The three categories were: epsilon first, delta first or no order. Students responded again to the four questions related to the temporal order. The response to each question was given a score from 0 to 2 (delta first=0, no order=1, epsilon first=2). The sum of the scores ranged from 0 to 8 and their total score placed them along a continuum between the claim of delta first and epsilon first. For students from the pilot study, scoring 2 on all the questions that were asked would lead to a total score of 8. In the first round of pilot study, students were asked only one question about the temporal order (question 1, above). In the second round of pilot study, students were asked three of the four questions (questions 1, 2 and 4). In those cases, the total was normalized to 8 based on the number of available questions.

I also documented the changes in the number of questions students answered with epsilon first across the four questions. I grouped students based on the number of epsilon first responses they provided. Figure 7.1 presents these numbers in a bar graph. I also included the number of epsilon first responses for the comparison group in the graph. The comparison group in this study read a section from a textbook about the formal definition, which also talked about the temporal order, instead of discussing the Pancake Story. Figure 6 is presented along with a modified Figure 5.1 from Chapter 5 to help the reader make the comparison with students' responses prior to the intervention. The student score that was described in the previous paragraph detected more subtle changes in student understanding about the temporal order that was not captured by the comparison of the number of questions students answered with epsilon first.

I documented new categories of *reasoning patterns* for students' responses to the temporal order questions that emerged as a result of engaging with the story. As noted in Chapter 2, I define a *reasoning pattern* as the essential common core of reasoning found in a range of students concerning the justification for a particular claim. The methods by which I identify reasoning patterns are the same as in Chapter 5. I look for core ideas shared in students' justification for the temporal order. The difference is that many of the reasoning patterns in this chapter use the language of the Pancake Story. New reasoning patterns were identified.

I. Changes in Students' Responses About the Temporal Order

The table below shows how each student in the study answered each question about the temporal order after the story. The top half of the table includes students from the current study and the bottom half includes students from the two rounds of pilot study. The color was again meant to help the reader get an overall sense of the responses across the different questions.

Table 7.1.

Students' Responses to Each Question About the Temporal Order After the Pancake Story

Student	Dependence	Temporal	Set	Order	Total
Erin	2	2	2	2	8
Sheila	2	2	2	2	8
Simon	2	2	2	2	8
Bryan	2	2	2	2	8
Roberto	2	2	2	2	8
Ryan[39]	2	2	2	2	8
Spencer	1	2	2	2	7
Chen	2	2	1	2	7
Julia	1	1	1	1	4
Aruna	1	1	1	0	3
Jane	0	0	2	0	2
Silvia	0	0	1	0	1
Dean	2	NA	NA	NA	8
David	2	NA	NA	NA	8
Jacob	2	NA	NA	NA	8
Adam[40]	2	NA	NA	2	8
Adriana	2	2	NA	2	8
Sophia	1	2	NA	2	7
Anwar	1	1	NA	0	3

Note. The table is sorted from the lowest to highest total. It is grouped by current study and pilot study.

Similarly to Table 5.1 in Chapter 5, the table above shows that students' responses were not stable across contexts after the story. Whereas 7 of the 12 students in the latest iteration of the study (top rows) argued consistently across contexts, the rest did not. For example, Aruna

[39] Ryan responded with delta first on all of the questions after the story. His justification was that he was given the error in size and he would determine the error in the batter. Other students used this reasoning pattern to conclude that delta depended on epsilon. Ryan miss-mapped the error in the batter to epsilon and the error in size to delta. At the end of the interview, Ryan said that he knew that it was wrong but said that he was too tired to fix it. I gave it 2's instead of 0's because I attribute the miss-mapping to the length of the interview and not lack of understanding.

[40] Adam's interview went long so that I decided to ask two of the four questions at the end of the interview. Adam was part of the pilot study.

claimed that epsilon and delta were dependent on each other and so there was no particular order by which they were set or which one came first between the two. However, in ordering the four quantities, she argued that epsilon had to be last, so delta came first.

Findings from Chapter 5 can help contextualize the findings in this chapter and assist in observing the changes in students' responses to the temporal order questions. The first comparison is between the numbers of questions students answered with epsilon first (score of 2) (see Figure 7.1 and 7.2 below). The two figures compare the number of questions that students answered with epsilon first before and after the story. Both figures include all students who participated in the study during its multiple iterations. The responses for the comparison group (who read the text instead of the story) are shown as separate bars in the figure.

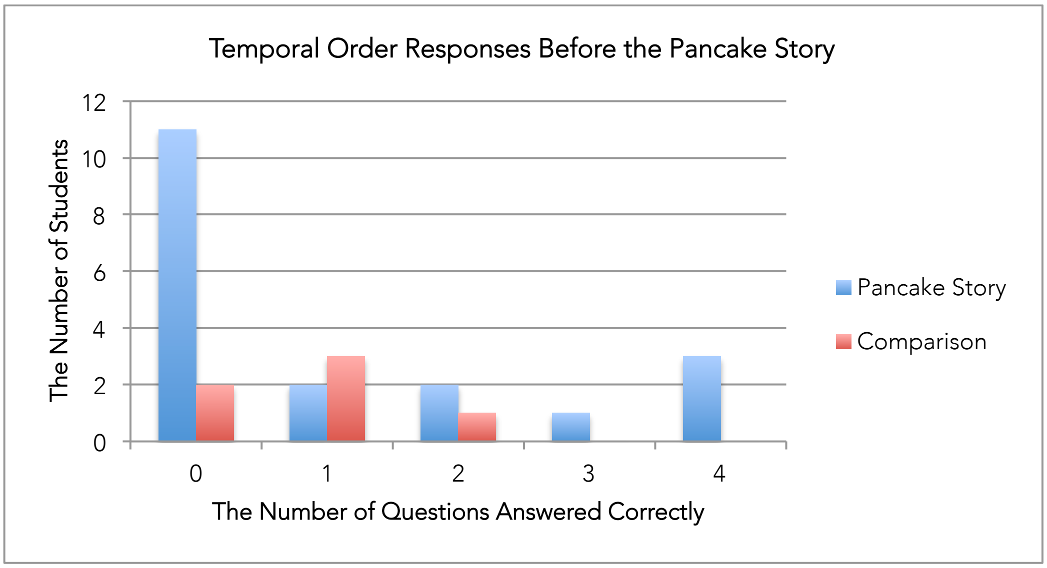

Figure 7.1. The distribution of students in answering four questions about the temporal order with epsilon first.

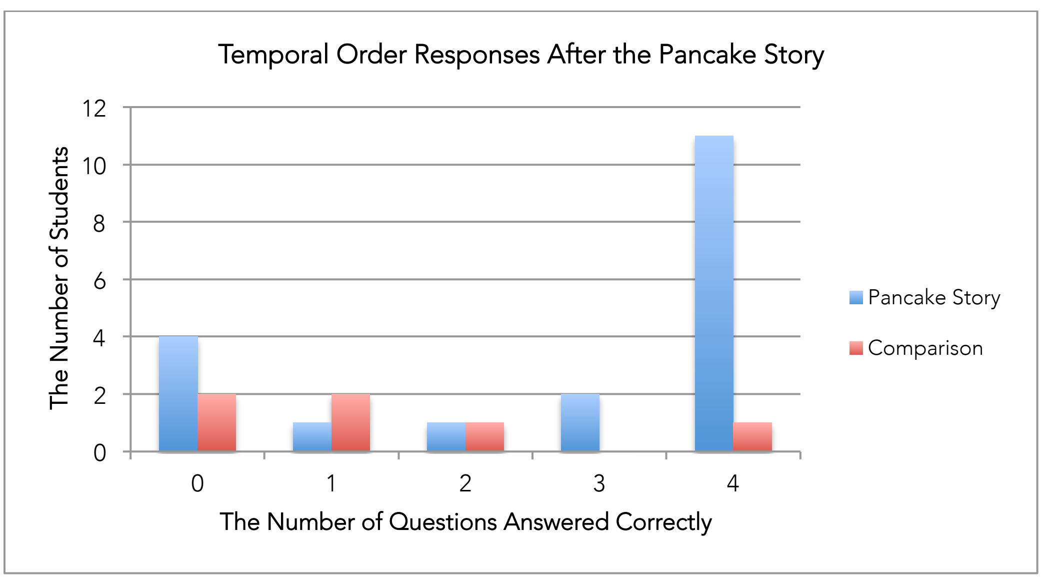

Figure 7.2. The distribution of students in answering four questions about the temporal order with epsilon first *after* the story.

Comparing the two figures, we observe that there was a significant shift in students' responses to the temporal order questions. In the beginning of the interview 11 of 19 students (58%) did not answer *one* question about the temporal order with epsilon first, and only 3 of 19 students (16%) answered all four questions with epsilon first. After the story, 4 of 19 students (21%) answered none of the questions with epsilon first, and 11 of 19 students (58%) answered all four questions with epsilon first. The comparison group did not reflect a similar pattern of change.

Table 7.2 documents the change in the total scores for the responses before and after the story. It reflects more subtle changes in the students' reasoning about the temporal order. Blue marks an increase, red marks a decrease and white marks an unchanged score.

Table 7.2.
Total Scores Before and After the Intervention

Group	Student	Pre Score	Post Score	Change
Current Study	Sheila	0	8	8
	Spencer	0	7	7
	Simon	3	8	5
	Ryan	3	8	5
	Chen	0	5	5
	Jane	1	5	4
	Bryan	5	8	3
	Aruna	1	3	2
	Julia	0	2	2
	Roberto	7	8	1
	Erin	8	8	0
	Silvia	5	2	-3
Pilot Study	David	0	8	8
	Jacob	0	8	8
	Adriana	0	8	8
	Sophia	0	7	7
	Anwar	0	3	3
	Adam	6	8	2
	Dean	8	8	0
Comparison group	Patricia	0	8	8
	Katrina	3	5	2
	Jose	1	2	1
	Milo	2	2	0
	Veronica	0	0	0
	Guillermo	4	1	-3

Note. The table is sorted according to the net change in score from highest to lowest.

The majority of students who discussed the Pancake story increased in scores (16 of 19, 84%). Two of the students who started with 8 stayed at 8. There was one student whose score decreased after engaging with the story. We explore her justifications and the implication of this decrease in the discussion. The total score of some students in the comparison group also changed. Three of the six students increased in scores, two stayed the same, and one score decreased. The number of students in the comparison group was lower to start, so a comparison of proportion of students whose total score increased might not be informative. However, considering the number of students who answered with epsilon first might help put things in context. As noted earlier, the dramatic shift from 0 question answered to epsilon first to 4 questions answered with epsilon first, did not happen with the comparison group.

In summary, there is evidence that the Pancake Story had a positive influence on students' conception of the temporal order. The question then becomes, how can we begin to account for the success of the Pancake Story? We now explore reasoning patterns across students and some of the additional resources the story might have provided for students.

II. Reasoning Patterns for the Temporal Order After the Pancake Story

In this section I explore different reasoning patterns that emerged from the different justifications students provided in answering the temporal order questions. Table 7.3 shows the catalogue of reasoning patterns that emerged and the number of students who used the particular reasoning pattern. The table is organized by the temporal order claim for which the students used the justification. The total number of students was more than 19 (those who engaged with the story) because some students included more than one justification per question, and some students provided no justification. Star marks a new reasoning pattern compared to the ones reported in Chapter 5.

Table 7.3.
Students' Reasoning Patterns About the Temporal Order Questions After the Pancake Story

The temporal order	Reasoning pattern	Number of students
Epsilon comes first, or $\boldsymbol{\delta}$ depends on $\boldsymbol{\varepsilon}$, or $\boldsymbol{\varepsilon}$ is set first	**1. Because one finds $\boldsymbol{\varepsilon}$ first.** Students argued that epsilon could be found first. Students said that they could do this because they knew $f(x)$ and L. **Example:** *"Because you know what f (x) is and I wanna say you know what L is so you find epsilon [first]"* (Roberto)	2
	2*. Because one chooses $\boldsymbol{\varepsilon}$ first. The focus was on the fact that one could choose some epsilon, and that was done first. **Example:** *"Epsilon [is set first] because they say for every number epsilon, which I think means choose whatever epsilon I want"* (Sheila).	3

(Table continues)

Table 7.3. (continued)

Students' Reasoning Patterns About the Temporal Order Questions After the Pancake Story

The temporal order	Reasoning pattern	Number of students
Epsilon comes first, or $\boldsymbol{\delta}$ depends on $\boldsymbol{\varepsilon}$, or $\boldsymbol{\varepsilon}$ is set first	**3. Because $\boldsymbol{\varepsilon}$ is given.** The givenness of epsilon was used directly to determine the order. That is, the fact that epsilon was given directly implied that epsilon came first. *Variation 1*: One determines the batter error from the given size error. *Variation 2*: Epsilon is set first because epsilon is given in the story. **Example:** *"[Epsilon comes first because] you could say that epsilon's given, you figure out delta"* (Adam).	11
	4*. Because $\boldsymbol{\varepsilon}$ is given in the story, and the goal is to satisfy that output constraint by constraining the input. This pattern highlighted the notion of quality control, which included the role of constraining the input in order to satisfy a given constraint in the output. **Example:** *"[Delta depends on epsilon] cus you have an error level that you want, and so you set delta so that you can achieve that level, that epsilon level"* (Jacob).	7
	5*. Because the goal is to minimize the distance between the function and the limit. Instead of just focusing on satisfying some constraint, this pattern incorporated the idea of minimizing the difference between the function and the limit. **Examples:** *"[E]psilon [is] set first because you're given in the story /…/ the pancake diameter and you want to minimize it. So that's what you're trying to do, you want to minimize the difference between the function and the limit"* (Erin).	3
	6*. Because $\boldsymbol{\varepsilon}$ is given, and $\boldsymbol{\delta}$ is not $\boldsymbol{\lvert x-a \rvert}$ and $\boldsymbol{\varepsilon}$ is not $\boldsymbol{\lvert f(x)-L \rvert}$. In the language of the story, some students made a distinction between errors and error bounds. **Example:** *"[C]us I thought that the epsilon and the delta were the errors but they're the error bounds. And if the epsilon is already set then you would have to change your delta"* (Adriana).	3

(Table continues)

Table 7.3. (continued)
Students' Reasoning Patterns About the Temporal Order Questions After the Pancake Story

The temporal order	Reasoning pattern	Number of students
Epsilon comes first, or δ depends on ε, or ε is set first	**7. Because of the statement for all ε>0 there exists δ >0.** A few students read this statement as ε>0 implies δ>0. **Example: "***Because from this definition, for every number epsilon is greater than zero, I want to say delta depending on epsilon because if there's an epsilon greater zero, then there's a delta that's also greater than zero"* (Chen).	3
	8. Because the ε-δ proof procedure starts by simplifying the epsilon inequality to "find" the delta inequality.[41]	3
No order, or ε and δ are dependent on each other	**9*. Because on the one hand, δ depends on ε because ε is given and one finds delta to satisfy ε. On the other hand, δ implies ε because $f(x)$ depends on x.** Students concluded that there was no order from the contradicting justifications (#3 above and #11 below). **Example:** *"They depend on each other because if the delta value changes then epsilon value changes. /…/ I see now more that delta also depends on epsilon if you're given epsilon. /…/ since I said that pancake size was epsilon and the batter was delta, then if the pancake size was a certain size first, then you know your delta size from that"* (Jane).	3
	10*. Because it depends on which was given. If epsilon were given, then epsilon would come first. If delta were given then delta would come first. **Example:** *"Depends on the situation, and what I was given. Sometimes you might be given epsilon, sometimes delta"* (Aruna).	4
	11. Because both ε and δ are arbitrary numbers. **Example: "***Neither are set [first] because the epsilon and delta changes"* (Silvia).	2

(Table continues)

[41] Students described the ε-δ proof procedure, and since this was also a reasoning pattern from Chapter 5, I did not include an example given space constraint in the table.

Table 7.3. (continued)

Students' Reasoning Patterns About the Temporal Order Questions After the Pancake Story

The temporal order	Reasoning pattern	Number of students
Delta comes first, or ε depends on δ, or δ is set first	**12. Because batter makes pancakes ($f(x)$ depends on x), and δ is the amount of batter and ε is the size (δ is x and ε is y).** **Example:** *"The error in the batter causes the error in the pancake size. So delta comes first"* (Silvia).	4
	13. Because the process in the formal definition ends with checking if $\lvert f(x)-L\rvert<\varepsilon$. So ε comes last. Variation: Students attended to the if-then statement and said that one needs to satisfy the delta inequality before the epsilon. **Example: "***Your epsilon is always last. To see if you are going to be within that epsilon"* (Aruna).	3
	14*. Because working backwards from a given ε implies that the given ε is the intended result. **Example:** *"I still think that epsilon is the result /.../ if you're trying to make your way backwards then the given [epsilon] would be the result"* (Julia).	1

In Chapter 5, I recorded 25 different reasoning patterns for the temporal order. After the story, I found 14 different reasoning patterns. The number of reasoning patterns decreased after students engaged with the story. Seven of the reasoning patterns (1, 3, 7, 8, 11, 12 and 13) were a repeat from Chapter 5. While they were reported earlier, the number of students who used each one and the way it was articulated was different. For example, Reasoning Pattern 12, which relied on the functional dependence between x and $f(x)$, was one of the most common reasoning patterns before the story (11/25 students). It was not as common after the story (4/19 students), and they also explained the idea using the language of the story (e.g., error in the batter and size). Reasoning Pattern 3, which relied on the givenness of epsilon, was used by 4 of 25 students before the story, but afterwards it became the most common reasoning pattern used by students (11/19 students). Students also used the language of the story in their justification.

Seven new reasoning patterns also emerged after students engaged with the story, showing that the story provided additional resources for students to think about the temporal order. I included an example for each reasoning pattern in the table to give a sense for which ideas from the story students took up. One of the most noticeable patterns is that all 19 students used the idea of epsilon being given (*givenness* of epsilon) as part of their justification for the temporal order, after the story. Reasoning patterns 3, 4, 5, 6 used the idea that epsilon was given (in the story or in general) to argue that epsilon came first. Reasoning patterns 9, 10 and 14 also used the fact that epsilon was given, despite them arguing for a different temporal order of delta and epsilon. All 19 students used at least one of those reasoning patterns. The majority of the time, students used the givenness of epsilon along with another justification to justify their claim about the temporal order. The next several sections explore different ways that students used the idea of givenness of epsilon: 1) to determine the temporal order directly, 2) to determine the temporal order by treating it like a constraint for the output, and 3) to reconcile conflicting ideas from before the story.

Givenness of Epsilon Directly Determines the Temporal Order

In this section I focus on a reasoning pattern that used the givenness of epsilon to directly determine the temporal order (Reasoning Pattern 2). This is the most common reasoning pattern that students provided (11/25 students). Seven of the 11 students (Adam, Dean, Erin, Ryan, Sophia, Spencer and Sheila) who used Reasoning Pattern 2, argued that epsilon came first because they were to figure out delta from a given epsilon. For example, Erin said, "Epsilon comes first /…/ [because] epsilon is the pancake diameter error bound and that's what you're given, and that's set first. And it's up to me to figure out what the error bound for the batter, delta." A couple of these students (Adam and Dean) alluded to the idea of quality control (pattern 4), but they did not emphasize the importance of controlling the input to achieve the desired result. These students focused more on the givenness of epsilon and finding delta from that (but did not elaborate on the role of delta to control the input). Dean explained how the story made this process more obvious to him:

> And when you specifically say um your boss gives you an error range in the *y* values, you know, in your pancake size, and you have to figure out what values of *x* correspond to those you know, correspond to being sufficient to meet those error requirement, it just makes it more obvious that you know you have these conditions for *y* you need to figure out *x*. So that, to me that made it more obvious <u>you're given epsilon, you need to figure out delta</u>.

Dean explained how the story provided the intuition for him to attend to the condition on the *y* values (epsilon) to determine the appropriate *x* values. I explore the notion of quality control in the next section.

Four of the 11 students (Chen, Erin, Jane, and Roberto) used the givenness of epsilon in the context of which between epsilon and delta was set first. These students argued that since epsilon was given in the story, then epsilon was set first. For example, Roberto, explained how he changed his mind about the temporal order with this justification. He said, "I changed my mind now. Um. I said neither before, but now, I'm just thinking back to the story. Epsilon is set. Epsilon is set. Yeah. You still kinda figure… No, yeah, epsilon is set. Yeah I guess it was given, so it's set." He made a reference to the givenness of epsilon from the story and deduced that then it was set. This explanation is representative of the other three students who used this justification.

Some students used similar idea of the givenness of epsilon determines the order, but they also argued that if delta were given then that would change the order. Aruna, Anwar, Jane, and Spencer used this justification to conclude that there was no order to delta and epsilon (Reasoning Pattern 10). This reasoning pattern focuses on the idea that at times epsilon would be given, but other times, delta could be given as well. For example, Aruna explained:

> Like for example, if I was given the pancake story, /…/ the boss gave me an upper limit of like five inches, I can be five inches away, or half inch away from 5-inch pancake. But like, if my boss gave me like, I have to be within this much of 1 cup of batter then my diameter pancake would have been the dependent /…/ I would have figured out the epsilon *after* I

> figured out my delta. Because my delta he would have said, oh you can only be one sixteenth of a cup off. Then if I measure the size of my pancakes, if I follow within that range then I would be able to figure out the maximum I can be off from, like because I use one, one sixteenth (1 1/16) cup of pancake, the maximum diameter that I would get, or the minimum diameter I would get from using that cup of batter.

Depending on which quantity the boss decided to constrain, she would use that information to figure out the other quantity. If, like the pancake story, she was given epsilon (an upper bound for the size), then she would figure out delta (the maximum she could be off for the batter). But if the boss gave her a constraint on the batter (delta) then she would figure out the epsilon after she figured out her delta. Spencer expressed a similar idea to Aruna. When asked about the dependence between epsilon and delta, he responded, "So *if* you're given epsilon then delta depends on epsilon." When asked if he was given epsilon, he said, "It depends, yeah from here (the statement of the definition) it's not given." He reported that he could go either way. So while the givenness idea is salient for many students, some students were not convinced that epsilon was the only quantity given in the definition.

[The Given] Epsilon Is a Constraint

In this section I explore reasoning patterns that used epsilon, or the givenness of epsilon as a constraint for the output, to determine the temporal order. This is a different use from what I discussed in the previous section, where students mostly focused on the givenness of epsilon. Here, students attached meaning to epsilon that it was a constraint to be satisfied.

Quality control. I specifically designed the idea of quality control into the Pancake Story, as an intuitive idea for students to potentially use to make sense of the temporal order. Reasoning 4 uses the idea of quality control, and it was the next most common reasoning patterns (7/19 students used it). As I alluded to earlier, the notion of quality control is that for a given constraint in the output, the goal is to find a constraint for the input so that the set of inputs yield outputs within the given constraint. In this way, quality control also incorporates the use of the givenness of epsilon. Jacob provided a succinct explanation of quality control. He said, "[Delta depends on epsilon] cus you have an error level that you want, and so you set delta so that you can achieve that level, that epsilon level." He emphasized the idea of setting delta in order to achieve the given constraint.

Adriana who initially said that epsilon depended on delta, explained how she changed her mind:

> Because I was given an epsilon and that's kinda like the main goal. The main goal is to get the pancake, /…/ and they gave me a constraint. And then they didn't give me an error bound for the batter or for the *a* or *x*. But I know I want to make it small so that it's within the error bound, the epsilon. So then I would kinda base my delta on what was epsilon.

Adriana treated creating pancakes within the given error bound as the goal. She would make the error bound for the batter small so that the outcome is within the given error bound epsilon. She noted the givenness of epsilon, as well as quality control in her justification. Bryan did not change his mind. His order of the four variables, and thus the temporal order of delta and epsilon stayed the same before and after the story. However his explanation changed. He explained, "So we're gonna have epsilon first, as a requirement. And then we're gonna have, two, our delta second because those are our bounds to be within the requirement and then three, we're gonna have x, which is what we actually, what we do, or what we scoop. And then four, we're gonna have f of x [$f(x)$] which is the outcome, what happens."

When I asked him if anything changed since we discussed the story, he responded, "Yeah exactly what changed, the reason I gave you for this [before] was the literal interpretation of the sentence [of the definition]. The reasoning I gave you for this [now] was an understanding of how to make a pancake and to meet the requirements." Before the story, Bryan relied on the spatial location of the different variables in the statement of the definition. He read epsilon, delta, x and $f(x)$, as they appeared in the statement. It seems that the notion of quality control, and his experience of making pancakes became salient for Bryan in reading the story. In the next chapter, I explore in greater detail the role of quality control and other knowledge resources in assisting a student, Adriana to sort through different claims about the temporal order. For now, I focus on another way that students used epsilon as a constraint, and also incorporate more ideas to support their claim about the temporal order.

Minimizing the error. Some students (Adriana, Erin and Sophia) who used the notion of quality control also included an idea of minimizing the difference between the output and the limit. The idea of minimizing is important because of the consideration of multiple errors. Whereas in the discussion above, most students focused on one or two numbers as a constraint, the idea of minimizing suggests that the constraint might be decreasing. This is a specific idea designed into the story, where ultimately the boss would not give a specific error bound for the pancake, but wanted the employer to get increasingly more precise. Sophia changed her mind about the temporal order to delta depending on epsilon. This was the explanation she provided right before she changed her mind, and she incorporated the idea of minimizing the error in the output as the goal:

> Ok, so you, you, you figure out the error on the batter based on the error in the pancake size that you're given, like the maximum allowed error of the pancake size that you're given, so in that way it is dependent the other way around, /…/ your batter error is dependent on the pancake error. Because your pancake error is like, is like given to you and then it's like what you're looking to *minimize* so you're trying to create a batter error that will facilitate that.

Sophia thought of epsilon as the maximum allowed error of the pancake size, but seemed to at times interchange it with just the error ("pancake error, is like given to you and then it's like what you're looking to minimize"). On the one hand, she recognized that epsilon was given, and that she was trying to minimize it. At the same time, she might have confused epsilon as the error in the pancake size. Nonetheless, she still used the idea of epsilon being given, and that she was trying to minimize the error. Certainly this provides an opening for her to think about the arbitrariness of epsilon, perhaps after she sort through the meaning of epsilon.

Erin spoke about this idea of minimizing the error (not quite error bound) but in the context of the function and the limit. Her claim about the temporal order did not change, but she explained what might have changed in her thinking about it. She explained,

> I'm kind of more sure epsilon set first, because you're given in the story that, you're given that the pancake diameter and you want to minimize it. So that's what you're trying to do, you want to *minimize* the difference between the function and the limit.

Similar to Sophia, Erin's language seemed a bit vague (Was she trying to minimize the pancake diameter?). The phrase "you're given that the pancake diameter" could be interpreted to mean that she was thinking that epsilon was the pancake diameter, which would be erroneous. She could also be thinking that given an error bound for the diameter, the diameter had a given constraint. The word "it" was also unclear. However, she clarified that ultimately what she wanted to minimize was the difference between the function and the limit. But just like Sophia, it's unclear whether she grasped the arbitrariness of epsilon and its role in minimizing.

Adriana attended to the idea that she wanted to be really close to the limit and the role that epsilon played in that. After the story she argued that delta came first before epsilon. She explained why she changed her mind,

> That changed because like the, the ultimate goal is to get close to L or to get to the limit. So we want like, we want a goal to be within and we don't want like 100, so we want it to be very very close so we said like within a half. Um so then that's how we would figure out like delta it's not gonna be like a hundred pounds of batter, we want it to be closer we want it to be within 5 inches of diameter, or close to 5 inches in diameter.

Adriana attended to the notion of quality control. She treated epsilon as a constraint for the output but she also related it back to the idea that she wanted to get "very very close" to the limit. Instead of considering multiple epsilons, she considered what would be unreasonable for epsilon. She was alluding to incorporating the arbitrariness of epsilon, albeit limitedly.

Effects of Givenness of Epsilon on Conflicting Ideas From Before the Story

The argument that the temporal order was dependent on which of epsilon and delta was given shows that while the idea of givenness of epsilon was salient to students, it was not sufficient for students to conclude that epsilon came first. In this section I contrast Reasoning Pattern 6, 9 and 12 to illustrate different ways that the givenness of epsilon interacted with the most common reasoning patterns from before the story: imposing the functional dependence between $f(x)$ and x onto epsilon and delta, and interpreting the if-then statement to suggest that delta had to be satisfied first.

Error vs. error bound to address functional dependence. In Chapter 5, I reported quite a number of students who broadly related delta and epsilon to x and y values. This led many of them to conclude epsilon depended on delta either through the functional dependence of x and y or the if-then statement. After the story, it seems that a few students (Adriana, Chen, Simon, and Sheila) made the distinction between delta and epsilon and the x and y values (Reasoning Pattern 6). It was not sufficient to change their claim about the temporal order. But it allowed many of them to see that epsilon no longer depended on delta. Then, with the givenness of epsilon, many of them changed their mind. Sheila provided an example. She said,

Int.	With epsilon and delta what depends on what if anything?
Sheila	[*pause*] Uh, delta depends on epsilon [*laughs*].
Int.	Delta depends on epsilon.

Sheila	Yeah.
Int.	It seems like you changed your mind.
Sheila	Uh-hm.
Int.	Why?
Sheila	Uh, because earlier I said epsilon depends on delta because I think of epsilon as an error instead of an error bound.
Int.	Hm, ok.
Sheila	Yeah.
Int.	So how did that, how did you figure out that epsilon's not an error, not an error but an error bound?
Sheila	Because the story said guess an error bound. So like I just, from that point I just realized there's an error and there's an error bound like they're different.

Prior to the story, Sheila argued that delta was a bound for x and epsilon was a bound for y and the if-then statement suggested delta implied epsilon. So delta used to come first for Sheila. After the story, she made a distinction between the error and error bound. In the segment above, she explained how she used to think. She did not explain how making the distinction led her to conclude that delta depended on epsilon. What she said right before I asked her the question about the dependence relationship might clue us to another justification that might have been at play.

Before she and I discussed the dependence between delta and epsilon, Sheila explained how the story helped her understand that the goal in the definition was to find delta. She explained,

> "Because [the story] is giving us a strong correlation between the epsilon and delta, like, it says bluntly like, oh the boss want you to make cake with [this] epsilon. So I have to find the delta so I can satisfy the condition that the boss gave."

This is the notion of quality control from the story. So it seems that together they might have led her to conclude that delta depended on epsilon.

The next chapter gets into more details about how distinguishing between error and error bound led a student to change her claim about the temporal order. I explore the process by which the student, Adriana, distinguished between the error and error bound. We also see how unstable that distinction was. For now, I share what Adriana ultimately concluded about the error and error bound, and its effect on the temporal order:

> Um, see cus I was looking at it like /…/ the f of x [$f(x)$] depends on the x and that's how I was like saying that epsilon depends on delta because epsilon is related to the f of x [$f(x)$]/…/. But that's just saying the error of the L and the f of x [$f(x)$] depends on the a and x but that's not to say that epsilon depends on delta.

She said that initially she relied on the functional dependence between x and $f(x)$ to determine the temporal order. The story helped her to notice that the functional dependence described a relationship between the $f(x)-L$ and $x-a$, not between epsilon and delta. How she came to that conclusion is one of the foci of the analysis in the next chapter. But just like Sheila, Adriana ultimately used quality control and the givenness of epsilon to change her claim. Next, I discuss students who took up the idea of givenness and/or quality control from the story, but whose idea about functional dependence got validated by the story.

Batter makes pancakes (functional dependence). One of the most common justifications provided by students for the temporal order before the story was using the functional dependence between x and $f(x)$. Typically, students also loosely associated delta with x and epsilon with $f(x)$. After the story, four students (Adriana, Julia, Ryan and Silvia) held on to this reasoning pattern (Reasoning Pattern 12). These students thought that delta and epsilon were either error in the batter and pancake size, respectively, or they were the actual batter and the pancake size. Then many said that batter made pancakes or that an error in the batter led to an error in the pancake size. So epsilon depended on delta. Silvia provided a clear example of this reasoning pattern. I asked her with epsilon and delta, which depended on which. She responded:

> I think I said I switched this last time too, but now I think epsilon depends on delta. I think before I said delta depends on epsilon, well like with the pancake story, epsilon is like the pancake size and delta is the batter so pancake error is gonna change, pancake error is gonna change with the batter error so epsilon is gonna change with the delta.

She connected her reasoning to the pancake story. She ultimately mapped the epsilon and delta to be the error in the pancake size and the error in the batter, and argued that when delta changes, then the epsilon would to. Silvia was the only student who after the story answered fewer questions with epsilon first. Her total score also decreased. Before the story, she argued that if epsilon changed then delta would to, and vice versa so they depended on each other. Now, it seems that the intuition of making pancakes became really salient to her and she went with epsilon was going to change with delta.

When asked in the context of which came first, Silvia was consistent and provided very similar justification. She said,

> I think epsilon [comes first], or, [*pause*] I'm thinking ok, well if I relate it to the batter story then the delta would change and then because your delta changes your epsilon changes, because delta is like your batter size if that error changes then your error for the pancake will change too.

Ryan expressed a similar idea. He argued, “Like the story it says that if one goes up the other has to go up the errors, /…/ if the delta goes up or maybe I think maybe epsilon depends on delta.” Ryan, albeit seemingly less sure, argued that if one error went up, the other had to as well. So epsilon was dependent on delta.

Julia and Adriana both used the functional dependence more explicitly. Adriana, responded with the dependency question with:

> [B]ut more like whatever you're getting like f of x [$f(x)$] is always gonna depend on what x you're inputting it /…/ mostly whatever you putting in for x will determine what you get for f of x [$f(x)$]. So I, I still say the same thing like delta depends on epsilon

Julia revealed the use of this idea after she answered which of the two variables were set first. In that context she used the givenness of epsilon so it was set. But when I asked her how did that fit with her previous claim that delta came first in the question before. She provided her reason:

"The reason? Normally for me, thinking that delta is *x*, and so *x* comes first then you think about the *y*." Here it seems that the functional dependence between *x* and *y* undermined the givenness.

No order settles conflicting justifications. Some students, instead of prioritizing the givenness of epsilon or functional dependence concluded that there was no order to epsilon and delta. Anwar, Jane and Sophia recognized the givenness of epsilon. At the same time they held on to the functional dependence idea (Reasoning Pattern 9). Sophia used the two justifications to clearly explained why epsilon and delta depended on each other:

Sophia	I still believe that, like they depend on each other in both ways, both influence each other. But I mean, I feel like definitely like you're, the error that you have, <u>how much batter you use determine how big your pancake is</u> so how big the *x* values that you put into your function or how small they are, that distance does affect how your function how close your function goes to the full limit, so I feel like that's why I feel like your delta, your epsilon depends on your delta a lot
Int.	But delta doesn't depend on epsilon
Sophia	Delta /I think
Int.	/Because before you were saying they depend on each other
Sophia	Yeah, um, in a way I feel like they do depend on each other because you're looking for a desired epsilon like <u>if you're working backwards you'd want a desired pancake size so you're looking for desired error to get that desired pancake size and to get that you need your desired batter size or batter amount.</u> So in that way you could see how epsilon could, could influence delta because epsilon is what you're looking to please, looking to fit to.

Much like students who used the functional dependence idea, Sophia stated that batter made pancakes. At the same time, she recognized the epsilon as a constraint. With this she concluded that they depended on each other. Jane provided a very similar explanation, stating that after the story she was able to better see that "delta also depends on epsilon if you're given epsilon." She still argued that epsilon depended on delta because batter made pancakes and "pancake size was epsilon and the batter was delta." This example highlights the role of interpretation of delta and epsilon play in influencing the take up of ideas from the story, which I explore next.

Different interpretations of epsilon and delta. Contrasting Reasoning Pattern 6, 9 and 12 seems to suggest that the quantities that students attributed to epsilon and delta might have influenced the ideas they took up from the story. Students who relied on the relationship between pancake batter and the size of the pancake (functional dependence) attributed the amount of batter and the size of pancake to delta and epsilon, respectively. For example, Silvia said, "Well, like with the pancake story, epsilon is like the pancake size and delta is the batter." Some students thought that epsilon and delta were errors, and this too, led many of them to conclude that any error in the batter would yield an error in the size. But many of them distinguished error from error bound and changed their mind about the temporal order. It is possible that doing so might have allowed them to prioritize ideas from the story, like the givenness of epsilon or the notion of quality control.

Some students seem to change what they thought to be epsilon and delta as they provided their justification for the temporal order. For example, earlier when I discussed the idea of minimizing the difference between the function and the limit, I mentioned that both Sophia and Erin were both vague in what they said about the two quantities. They used maximum error or error bound at one point, and error at another. When Sophia concluded that epsilon and delta depended on each other, it seems that she treated epsilon and delta as errors. I explore this back and forth students did with error and error bound in the next chapter.

Reasoning Pattern 6, 9 and 12 show interesting ways that students differ in the way they conceptualized epsilon and delta in the story. However, it is interesting that whichever interpretation students had about epsilon and delta, all of them used the givenness of epsilon. This suggests that the notion of givenness was not tied to what epsilon was for students. This affirms that givenness might be a knowledge resource as suggested in Chapter 6.

It is worth noting that students were not necessarily consistent with their mapping of quantities during the interview. After discussing the story, I asked students to map each quantity from the statement of the definition to the pancake story. This answer did not necessarily reflect their interpretation of epsilon and delta when they were discussing the temporal order. So that is why in this section, I focused more on ways that students used their interpretation during the discussion about the temporal order.

Summary of Findings

The first analysis section shows that the Pancake Story had a positive influence on students' conception of the temporal order between epsilon and delta. The majority of students (16 of 19) increased in total score in answering the temporal order question across the four contexts. Moreover, the number of students who answered none of the questions with epsilon first decreased from 11 to 4, and those who answered all of them with epsilon first increased from 3 to 11. The findings from the first section also show that some students' knowledge of the temporal order was still consistent across contexts.

The second analysis section explores the reasoning patterns for the temporal order across the four contexts after the story. After engagement with the story, the number of reasoning patterns decreased from 25 to 14. Half of the reasoning patterns documented in this chapter are new while the other half are similar to those reported in Chapter 5. While I did not include the details of the use of reasoning pattern across the contexts, some students did use similar justification across contexts. More specifically, students took up the idea of givenness of epsilon. All 19 students used it in their justification for the temporal order.

Students used the idea of givenness of epsilon in different ways. Some students used it to directly determine the temporal order. Most students combined the givenness of epsilon with other ideas, like quality control or minimizing the difference between the function and the limit. The givenness of epsilon also competed with other ideas students started with, and they were sort out in different ways. Distinguishing error from error bound helped some students to prioritize other ideas from the story, while some students focused on the relationship between batter and pancake size. I also explored interpretation of epsilon and delta in the story as one factor in the take up of ideas from the story.

Discussion

The general finding of this chapter is that students took up a number of productive ideas from the story to shape their justification for the temporal order. I have shown that there was a general trend of movement in a positive direction in terms of students' conception of the temporal order. Some improvements in the comparison group suggests that additional time spent on focusing on the temporal order and reading the text can help move their conception forward. The findings about the types of reasoning patterns students used uncovered some of the affordances provided by the Pancake Story in reasoning about the temporal order.

That there was a decrease in the number of reasoning patterns after the story is interesting. The story was designed to have students to attend to particular ideas and they did, albeit with different specificities. Students took up really good ideas from the story but in different ways. For example, the givenness of epsilon was taken up by all the students and used in very creative ways. Some salient ideas remained (e.g., functional dependence). Some students were able to use the story as a resource, including the givenness of epsilon, to sort it out, but others did not. This is a different story of change from a story of replacing students' conception. The story did not discourage students' prior knowledge, as half of the reasoning patterns found in this chapter existed before (productive or unproductive). The intent was for the story to help students reorganize their prior knowledge and possibly add some new knowledge, and that was what happened.

With the case of Adam in Chapter 6, I suggested a potential tension between the givenness of epsilon and the idea of functional dependence. I suggested that the givenness of epsilon might be helpful in problematizing the functional dependence idea for epsilon and delta. The findings from this chapter give more insight into the process that these two seemingly competing ideas get sorted out by students. In this chapter, we saw some students using the givenness of epsilon to determine the temporal order once they sorted out the difference between error and error bound. But we found that some students were just comfortable having the two side-by-side and concluding that there was no order. As I discussed earlier, students' conception of epsilon and delta might be an influential resource to sort this out.

I note that students were still able to move forward with their conception of the temporal order, despite not having addressed these issues in their understanding (e.g., error vs. error bound, functional dependence). This could be explained partly by context specificity of knowledge (quite a number of students used givenness of epsilon to answer the set question). The hyper-richness argument from Chapter 6 can also be helpful here. Given the hyper-richness of the formal definition, Adam's conception of epsilon and delta moved between different levels of generality. Sometimes they were very specific (delta was the radius of the interval), and sometimes they were imprecise (delta had to do with inputs). This was a byproduct of considering different plausible arguments for the temporal order. It is possible that some of the reasoning patterns provided students with a broader frame to think through the temporal order without necessarily having to sort through every aspect of the definition. In instruction, this process can be seen as getting students to think through the big idea of the problem, before going into the details of the problem.

Silvia was an interesting case. She was a student who came in with quite a few productive resources, and after the story her score decreased because she took up an intuitive idea from the story. Some might argue, and say, "Aha! Look at how intuition gets in the way of productive knowledge!" I believe this would be shortsighted. I view Silvia's case as a case *for* intuitive knowledge, instead of against it. She did come in with some productive knowledge, but it was fragile on the face of a more intuitive idea. We can try to introduce information to students, but ultimately it has to make sense to them for it to be used by students consistently. I interpreted the problem with Silvia to be a mapping problem. The issue was that she thought epsilon and delta as pancake size and amount of batter. Looking for ways to make more salient the meaning of epsilon and delta in the story for students is a useful and productive direction to take in the future. Silvia could also be noise in the data. One student in the comparison group also decreased in score. But the case of Silvia does bring up very interesting discussion points about the role intuition in the learning of mathematics.

The discussion concerning epsilon being a constraint, in particular the idea of minimizing the errors was important. Focusing on epsilon as a constraint can serve as a potential springboard to discuss the arbitrariness of epsilon. As preliminary evidence for this claim, I include a discussion at the end of my interview with Dean about why no particular epsilon was given at the end of the story. Dean had just made an assertion about limit that with limit one was concerned with constraining the *y*-values and that was why epsilon was important. He then reflected on where epsilon came from. I answered that in reality we were not given any particular epsilon, then I asked him why.

Int.	In this story, at the end you're not given a value [for epsilon], why is that? Why in the story were you not given epsilon? You were initially given epsilon equals 1/2 but why were you not given epsilon at all by the end. What did your boss want, such that you're not given it?
Dean	Oh, your boss wanted your pancakes to be as close as possible. So if you choose one arbitrary number then you're not gonna get as close as possible to *y*. So the idea between epsilon and delta is that you can choose any epsilon that's close to your y value, and you will, the result will be a delta that is close to your *x* value.
Int.	Does that make sense?
Dean	Yep it does. Cool, cool! Wonder how long will it take to forget this. Probably won't actually now. You might have taught me something today.

It would be difficult to claim that Dean arrived at the arbitrariness of epsilon from focusing on epsilon being a constraint. However, focusing on epsilon as the output constraint can lead students to ask questions about properties of epsilon, like what happened with Dean. This then became an opportunity to leverage other aspects of the story, which led to Dean making the connection between the arbitrariness of epsilon with the closeness to the limit.

The question that remains now is, “What about the story helped the students, and what was the process of change like for many of these students?” In this chapter, I identified some productive reasoning patterns, but how did students take that up? How easy was it for students to change their claim after being exposed to some of the productive resources from the story? Was it a straightforward process or was there a negotiation of resources? Throughout this chapter I alluded to Adriana as an interesting case to explore more deeply. The next chapter explores all these questions with the case of Adriana.

CHAPTER 8: USING THE PANCAKE STORY TO MAKE SENSE OF THE TEMPORAL ORDER

This chapter focuses on the case of Adriana, to illuminate the process by which a student made sense of the temporal order using the Pancake Story. Adriana was an intending mathematics and Chicano studies major. She was a Hispanic/Latina student who self-identified as Chicana. She took first semester calculus in high school and received an A, and retook the course in college and also received an A. But despite her background, even after engaging with the story she still initially argued that epsilon depended on delta.

Adriana was a representative case in the following ways. Like many students, she used the justification "epsilon depended on delta because $f(x)$ depended on x." She also took up the idea of *givenness* of epsilon and *quality control* from the story. She was one of the students who sorted out the difference between error and error bound in making sense of the temporal order. Adriana's case can illuminate the process by which students take up these ideas. To contextualize this case with the findings in other chapters, Adriana's total score rose from 2 to 8. She answered one question with epsilon first before the story, and ultimately answered all temporal order questions with epsilon first, after the story. This chapter explores the role that the Pancake story played in the development of Adriana's claim about the temporal order.

The main goal of this analysis is to uncover the process by which Adriana changed her claim about the temporal order while discussing the dependence between epsilon and delta. First I show that Adriana's conception of the temporal order was stable across the three contexts in the interview (Section I). This shows that the change in Adriana's claim was stable. Then I explore how Adriana made a distinction between an error and an error bound (Section II). This distinction becomes influential in Adriana's discussion about the temporal order. The last section is the main episode where Adriana discussed the temporal order in the context of dependence (Section III). She recalled the distinction between the error and the error bound, but this conflicted with many productive resources she drew from the story. The narrative in this chapter does not track the chronological progression in the transcript (e.g., the episodes in section I happened last in the interview). The arrows in Figure 8.1 show the chronological progression in the transcript. The figure also labels each section of the analysis and shows their connection.

The main episode of interest was a learning episode instigated by a conflict between some of the productive resources from the story and Adriana's prior knowledge. This occurred in the discussion of the first temporal order question. Section III focuses on Adriana's response to the temporal order question in the context of dependence. Each section analyzes one episode of interaction. Each episode is made up of several thematic segments. That is, each segment focuses on a particular theme, issue or a claim that the student was making.

The discussion at the end of the chapter focuses on the process of learning for Adriana, and the role of the pancake story plays in that process. So while some knowledge resources are identified, the identification of such resource is not the main goal of this chapter.[42] As such the model and counter-model methodology are not explicitly used in this chapter. However, *competitive argumentation* still guides the analysis. I still put any model I construct of Adriana's argument through the process of argumentation, and in some sections this is made explicit. The transcript that is presented was modified to facilitate reading. To do so many hedges, and uh-huh's and um-hm's from the interviewer were removed.

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[42] For the glossary of all the *knowledge resources,* see Table 6.1 in Chapter 6, pp. 41-43.

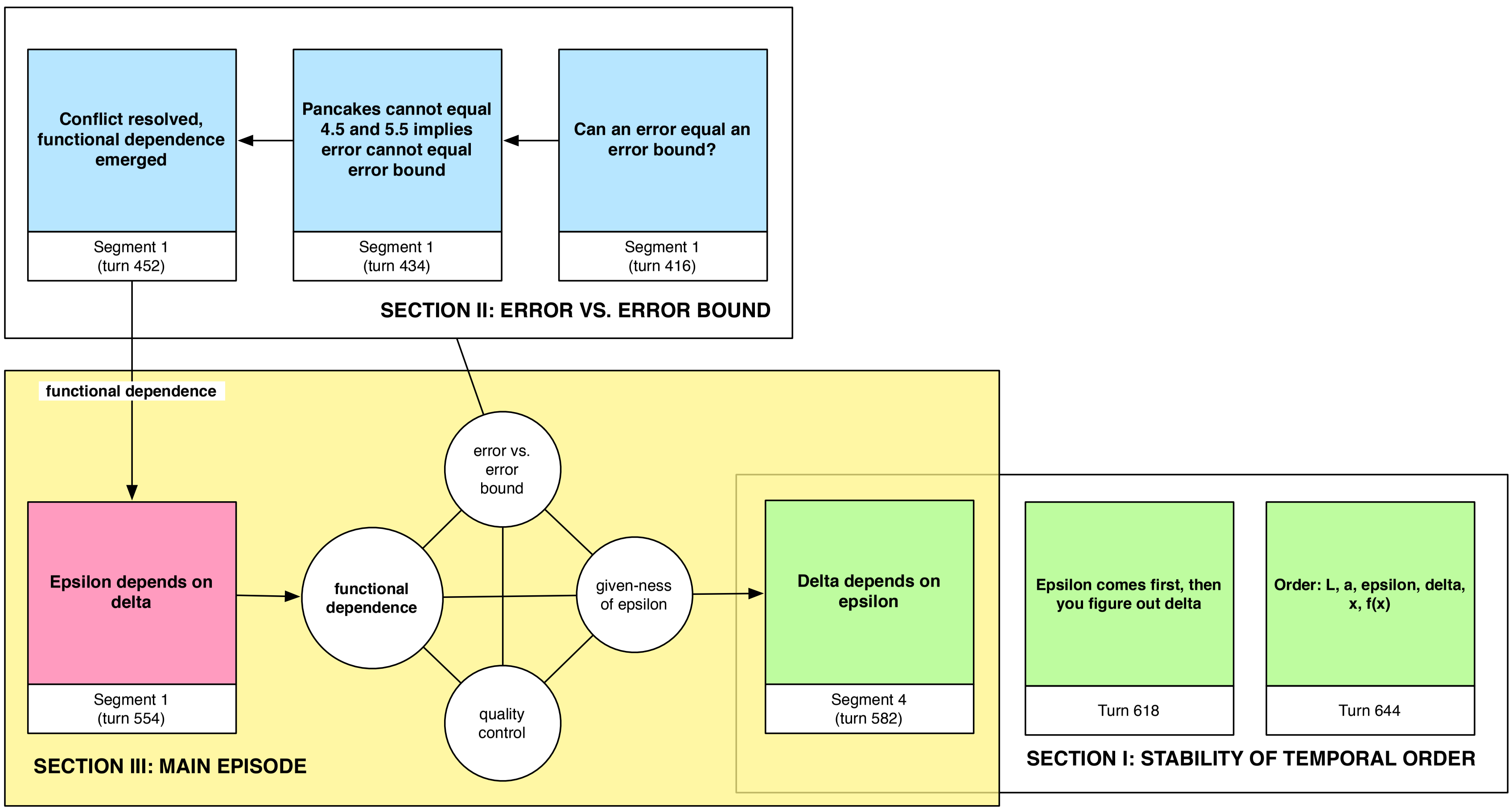

Figure 8.1. The relationships between the three analysis sections. The arrows show the chronological progression in the transcript.

I. The Stability of the Temporal Order Claim Across Contexts

After the story, Adriana initially argued that epsilon depended on delta by using the *functional dependence* argument.[43] However, after some discussion she changed her claim to delta depending on epsilon (epsilon first). She also answered the other two temporal order questions in a similar way. She argued that epsilon came first and she would figure out delta as a result. Last, she ordered the quantities in the definition in the order L, a, ε, δ, x and $f(x)$ (epsilon first). She thought it was important to include the L and the a in the ordering. Below we see that one of the ideas that drove the change in the temporal order was the change in the meaning of epsilon and delta for Adriana.

Dependence

Adriana argued that delta depended on epsilon by first acknowledging her mistake of thinking about epsilon as errors. She then prioritized productive resources from the story.

577	Int.	So, do they depend on each other, is it just one way now?
578-580	Adriana	Um, see cus I was looking at it like /…/ the f of x [$f(x)$] depends on the x and that's how I was like saying that epsilon depends on delta because epsilon is related to the f of x [$f(x)$]/…/. But that's just saying the error of the L and the f of x [$f(x)$] depends on the a and x but that's not to say that epsilon depends on delta.
581	Int.	Ok, so?
582	Adriana	So, I think that delta depends on epsilon now [*laughs*]. Just cus if it's given like this [*unclear*] and you're trying to aim at getting /…/ within a certain error bound, then you're gonna try to manipulate your entries /…/ to be within a certain error bound [*gestures a small horizontal interval with her palms*]

Toward the end of the interview, Adriana prioritized the *givenness* of constraint (epsilon) and the notion of *quality control* to determine the temporal order. Recall that the *givenness* is a characteristic of a quantity whose existence is granted and thus its origin would not be questioned. *Quality control* is the idea of modifying the input of a function in order to achieve a desired output. Adriana seems to have taken the role of the manipulator of the input in order to be within a prescribed error bound for the output. The *functional dependence* from before was used to describe the relationship between the errors of the outputs and the inputs. She also noted her mistake in treating epsilon and delta as errors and using the *functional dependence* to describe their relationship. I present a closer analysis in the last section in this chapter.

Sequential Order

Adriana argued that epsilon came first and that she would figure out delta as a result. Again, she explained that she used to think of epsilon and delta as errors.

> Yeah I think I did change because I was thinking that whatever change I was making here ($0<|x-a|<\delta$), so I was thinking of it as errors. So I was thinking of it like whatever change I was making the x and the a, would affect the $f(x)$ and the L (turns 614–616).

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[43] For the relevant part of Adriana's full transcript and written artifact, please see Appendix K.

She explained that she used to think about the sequential order of epsilon and delta using the idea of *functional dependence*. However, this involved treating epsilon and delta as the errors, $f(x)-L$ and $x - a$. Below she used the idea that getting arbitrarily close to the limit as the justification for epsilon coming first in the definition.

> That changed because like the, the ultimate goal is to get close to L or to get to the limit. So we want like, we want a goal to be within, we don't want like a hundred [pounds of batter] close. So we want it to be very very close. So we said like within a half [inch]. Um, so then that's how we would figure out like delta, it's not gonna be like a hundred pounds of batter, we want it to be closer if we want it to be within 5 inches of diameter, or close to 5 inches in diameter (turns 618–622).

She used the same ideas as she did in the context of dependence (*the givenness of epsilon and quality control*). Here she also emphasized the requirement that epsilon and delta be a small quantity. She wanted it to be "very very close." Thus that was why she said that she would use half an inch error bound. Delta's being small made sure that the pancakes would be "close to 5 inches in diameter." At the same time, Adriana also exemplified the process with familiar situations. She said, "It's not gonna be like a hundred pounds of batter." She was invoking familiar imagery of making pancakes and amounts of batter to make sense of the order.

Adriana also began using formal mathematics language to describe the goal with the formal definition. Instead of saying that the goal was to make pancakes "within an error bound," like she did in the previous context, she said that her goal was to "get to the limit." Thus, in this particular context she argued that epsilon came first by cueing many of the same knowledge resources as she did in the context of dependence. She explained that the change in conception was a result of not treating epsilon and delta as errors.

Order of Quantities

Adriana put the quantities in the following order: L, a, ε, δ, x and $f(x)$. So she put the epsilon first. In this context, she, again, emphasized the importance of getting close to the limit. And that she changed her mind based on changing the way she conceptualized epsilon and delta.

> Well first I would put your ultimate goal as the first one, which is the L. And then what you're trying to get to look at to get to your L is a, which is why I chose those two first. And then I put epsilon next because we're trying to get really close to L /…/. So I chose epsilon next, and then delta is kinda similar to epsilon. We want something that's really close to a so we put /…/ like a constraint on it too so we're not going too wide. [Epsilon's first] because we're trying to get close to our ultimate goal, L. Yeah, so then delta. And then x next because that's what we're gonna try to get close to delta, or within delta to get an f of x [$f(x)$] that will give us /…/ a result that is within epsilon (turns 644–656).

To determine the order between epsilon and delta, Adriana focused on getting the function close to the limit. She emphasized that delta was a constraint that kept the input x really close to a. She explained that some of the changes happened because she confused x with a. Early in the interview she said that a approached x, instead of the other way around.

Below she explained the other changes she made. Some of the changes occurred because she no longer thought of epsilon and delta as $f(x)-L$ and $x-a$, respectively. To follow what Adriana said below, it might be helpful to consider that the order changed from $f(x)$, x, a, L, ε, δ at the beginning of the interview. It changed to L, a, ε, δ, x and $f(x)$ by the end. Adriana wrote the new order below the old order during the interview. I include Adriana's writing below the transcript.

> I think these two [*circles* $f(x)$ *and* x *from the old order and* L *and* a *from the new one*] I changed them because I thought that these [*points* at x and a] were each other. So I thought we were trying to get close to x [instead of a]. And then these [*circles* ε and δ] changed because I thought /…/ these [ε and δ] were the difference [$f(x)-L$ and $x-a$]. So that's why I put them last. Cus I was thinking of it like, oh analyze our errors at the end. And then these [*circles* x *and* $f(x)$] I put them last because I realized these [*circles* x *and* $f(x)$] are what we can control based on all of these [*circles* L, a, ε, δ]. Well this, this [*points at* x] is what we can control. Once we have all of this [L, a, ε, δ] laid out then I can start picking x's that are close to a and then f of x [$f(x)$] will be what I get. And then I can compare them to what I have [*circles* L, a, ε, δ] (turns 662–672).

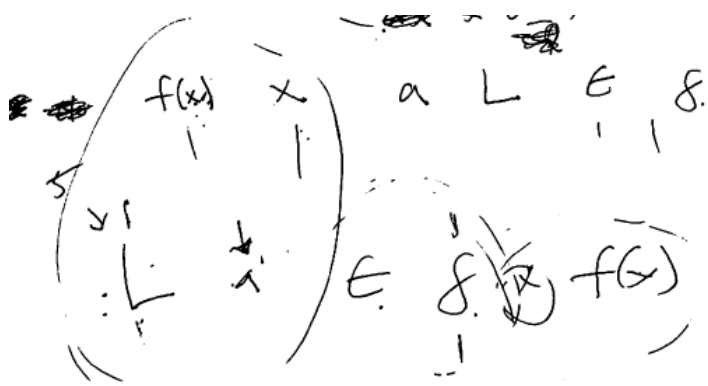

Figure 8.2. Adriana's old and new ordering of the six quantities.

Adriana changed her ordering partly because she had previously confused x for a. She also used to end the ordering with ε and δ because she thought those were the errors, and errors were analyzed at the end. She, again, focused on getting the function close to the limit. These change of meaning of epsilon and delta and the goal of approaching the limit were the same ideas she used in the other two contexts.

At the same time, in this segment she revealed an aspect of her understanding of the definition that she had not revealed before. She concluded that she had control over x based on knowing the limit, L, a, ε and δ. She understood that the values of the limit and of a were set. Epsilon was given, and delta was determined from epsilon. In analyzing the error, she only had control over the x values she was inputting, to check if the $f(x)$ was within the prescribed bound. She did not mention this idea in the other contexts, presumably because this context asked her to make sense of the relationships between all the quantities, and not just epsilon and delta.

Summary

Adriana's claim about the temporal order was stable across the three different contexts at the end of the interview. She concluded in all three contexts that epsilon came first. Moreover, she also relied on very similar resources. The *quality control* resource was reframed in terms of getting the function close to the limit L. More noticeably, her justification in all contexts involves the correction that epsilon and delta were no longer errors. She did not specify what they were, but she emphasized that they were no longer errors. That seems to play a significant role in her changing her claim about the temporal order.

We also begin to see the influence of the Pancake Story on Adriana's conceptualization of the temporal order. In the context of dependence she invoked a familiar image of controlling the input values to meet the prescribed range of output values. In the context of sequential order, the familiar context of the story allowed her to come up with examples to illustrate her point. I return to this discussion at the end of the chapter. The next section explores how and when she first made the distinction between error and error bound, which seemed to drive the changes in her understanding of the temporal order.

II. The Distinction Between Error and Error Bound

In the previous section we see that Adriana justified the change in temporal order by changing the meaning of epsilon and delta. The Pancake Story and the proceeding questions were designed to make the distinction between error and error bound salient for students. For example, one of the questions after the story asks if students think that there is a difference between an error and an error bound.[44] This was Adriana's response:

> Well I think the error bound is kinda like the max /…/, the minimum and the maximum that you can get, and, and error would be /…/ anything in between there, it could be /…/ the actual bound or anything in between (turn 333).

Adriana thought of error bound as the minimum and the maximum that she was allowed to get. For example, if $-0.5<f(x)-L<0.5$, then -0.5 and 0.5 are the error bounds for the quantity $f(x)$ from the limit, L. An error could be anything in between the two bounds. She also mentioned that an error could be "the actual bound," as well. This was going to be a source of conflict that would ultimately allow her to distinguish between $x-a$ and delta, and $f(x)-L$ and epsilon.

She focused on the "less than" symbol (<) in the inequalities $0<|x-a|<\delta$ and $|f(x)-L|<\varepsilon$ as the part of the definition that was conflicting for her. She said,

> The biggest thing that I'm like unclear about, is like, how come the delta and the epsilon are greater than this [*circles* $|x-a|$] this error or this [*circles* $|f(x)-L|$] difference or whatever. Like, that's what I don't understand (turns 406–408).

The fact that epsilon was greater than $f(x)$ minus L and the delta was greater than the x minus a, were problematic for her. Before she focused on this issue, when asked what epsilon and delta were in the story, Adriana said that epsilon and delta were "the error of the diameter of the pancake" (turn 385) and "the error of the batter" (turn 381). This suggests that Adriana conceptualized epsilon and delta as the errors, $|f(x)-L|$ and $|x-a|$ leading up to this point. The inequality strongly suggests that epsilon and delta were not errors because of the inequality sign. Conceptualizing epsilon and delta as errors fuels the conflict in the next segment. The nature of the confusion with the inequality also becomes clear in the next two segments.

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[44] Adriana did not use the terms error and error bound before discussing the story. She spoke of epsilon as margin for the difference between $f(x)$ and L. The story introduced these terms to her.

Segment 1: Can Errors Equal Error Bounds?

Adriana pointed to the inequalities as the source of the conflict in her understanding. In this segment to explore the issue, Adriana tried to see if delta could equal the error. I, the interviewer, mistakenly thought that she was not attending to the difference between an error and an error bound. I attempted to remind her of the distinction she had made between the two, earlier (turn 333, above). Since this was not the issue, it did not help her confusion. She restated her understanding about the difference between error and error bound at the end.

416	Adriana	But I don't, yeah, that's the thing that I don't understand, like, what are these, the epsilon and delta if they're not exactly that error, then, what are they?
417	Int.	Ok, um, let's see… So if they're not exactly the error what are they?
418	Adriana	Um-hm.
419	Int.	Ok, um.
420	Adriana	Well I guess I mean I guess it could be, yeah cus if I was thinking of error here [*points at a region near a on the x axis*] like I said the error was half [1/2] and tried to make my delta half [1/2] but then I would make it be equal to [*points to the sign < in the delta inequality*].
421	Int.	Uh-huh.
422	Adriana	And since it's like less than, I don't…
423	Int.	I see, um [long pause]. You mentioned earlier that... there is a difference between... there's a difference between error bound and error?
424	Adriana	Yeah
425	Int.	Right? Did the story talk about error or error bound?
426	Adriana	Um well, it gave us an error bound for, for the pancake and then it asked us to be, to guess a bound for the batter
427	Int.	Um-hm.
428	Adriana	Um and then it just wanted our errors to be within those bounds
429	Int.	Um-hm.
430	Adriana	So it kinda talked about both.
431	Int.	Ok, so how does that relate to epsilon and delta and this notion of less than thing.
432	Adriana	Um, this error [points at $\lvert x-a \rvert$] is less than the error bound [*points at delta*]. Well if I call this [*points at delta*] the error bound, I would have called the epsilon and delta the bound, like the highest it could be that we want our error to be less than that, that bound.

Adriana made a deliberate move in questioning the meaning of epsilon and delta at the beginning of the segment (turn 416). Her confusion was whether error could equal the error bound, not so much if there was a difference between the two. From 426–432, Adriana accurately described the meaning of error bounds and their role in the story. The error bound for the diameter of the pancake was given, and she was asked to guess an error bound for the batter (turn 426). Then she was to make sure that the errors were within those bounds (turn 428). However, this did not address the issue that she brought up in the beginning: what was epsilon and what was delta?

She tried to use an example from the story to illustrate the conflict (turn 420). In the story the initial error bound for the pancake diameter was half an inch. Here, it seems that she tried to apply it as an error bound for the delta. Mistakenly using the ½ as delta did not affect the real issue: making the error and the delta both one-half conflicted with the less than symbol. When I encouraged her to use the distinction between error and error bound to sort out the inequality issue, she returned to her conception of error bound as the maximum for the error. This still did not address the issue. In the next excerpt she still focused on whether $f(x)-L$ could equal 4.5 and 5.5 (referring to the inequality $4.5<f(x)-L<5.5$, which she would later revise to be $-0.5<f(x)-L<0.5$).

Segment 2: Pancake Sizes' Not Equaling 4.5 and 5.5 Inches Implies Errors Cannot Equal Error Bounds

Adriana was in the process of figuring out if $|x-a|$ and $|f(x)-L|$ could equal δ and ε, respectively. She went back to the story to confirm that the story asked for pancake sizes that were between 4.5 and 5.5 inches, but not equal to either of those numbers. From there she inferred that the error could not equal the error bound.

434	Adriana	[*Reading the pancake story*] Oh ok, so, so it does say like we want the pancake to be between 4.5 and 5.5. So within an inch of bound. So it doesn't say like it *can* be 4.5 and 5.5 so
435	Int.	Alright.
436	Adriana	So I guess that's what this [$|x-a|<\delta$] would be like, it [$|x-a|$], it has to be less than that [δ], like if the delta or no like if the epsilon [points at ε in $|f(x)-L|<\varepsilon$] or whatever, we wanted it to be within a *half* or whatever.
437	Int.	Um-hm.
438	Adriana	Like it would be, let me write this down [*writes* $4.5<f(x)-L<5.5$]. So it would want it [to] be like greater than 4.5 or less than 5.5 but it doesn't say like it can be 4.5.

Adriana productively mapped different quantities from the definition to the pancake story. She returned to the fact that the pancake diameter had to be between 4.5 inches and 5.5 inches and not equal to the boundary values. She correctly mapped the idea of "between" to the use of the inequality. But she confused the goal of wanting the error to be within ½ inch with the goal of wanting the epsilon to be within ½ inch. She concluded that $4.5<f(x)-L<5.5$.

It seems that Adriana deduced that the error could not equal the error bound by noting that the story asked for a pancake that were between 4.5 and 5.5 inches, but not equal to those numbers (turn 434). She inferred this from the story because it did not say that the pancakes could equal those sizes. Whereas in the previous segment Adriana was saying that the errors needed to be within the error bound, here she focused on the *pancakes* being within an inch of bound (turn 434). It seems that in that turn Adriana used the term bounds but she was not referring to an error bound. So how did Adriana make the jump from the bounds for pancakes to the error bound, epsilon? Understanding this shift is crucial in interpreting the change that occurred in Adriana's conceptualization of the temporal order at the end of this chapter.

Conceptualizing error bound. Adriana's inscription can help illuminate a subtlety in the way that she thought about error bounds. She inaccurately wrote $4.5 < f(x) - L < 5.5$. The story illustrated that when the error bound was ½ inch, the pancake had to have a diameter between 4.5 and 5.5 inches. Thus, the inequality should have been either $4.5 < f(x) < 5.5$ or $-0.5 < f(x) - L < 0.5$. Adriana eventually corrected the inequality to $-0.5 < f(x) - L < 0.5$, but not until much later in the interview (turn 676, about 40 minutes from the segment above). I offer two interpretations for Adriana's mistake and the potential implications, which may have informed the way she deduced the error bound from the inequality.

I argue that Adriana meant to write $4.5 < f(x) < 5.5$ but made a mistake in including the L in the inequality. Adriana was definitely focusing on the quantities 4.5 and 5.5 in the inequality (turns 434 and 436). So the outer parts of the inequality clearly were not a mistake. She inferred the ½ error bound from looking at the two bounds for the size of the pancakes (turns 436-438).

Knowing that the ideal pancake size was 5 inches, then the lower and upper bound of the pancake size reflected the ½ inch error bound. Since the size could not equal the lower or the upper bound, then the error had to be less than the ½ inch error bound. In later segments we see that Adriana would refer to this inequality whenever she spoke about error bounds. It is worth noting that the inequality $4.5 < f(x) - L < 5.5$ explicitly focuses on the range of possible output values, but not the error bound, 0.5. As noted, this carries implication for the temporal order later in Section III.

The alternate interpretation for Adriana's mistake is that she meant to write $-0.5 < f(x) - L < 0.5$ and so 4.5 and 5.5 *were* the error bounds for the pancake size. One might argue that this would be consistent with Adriana's definition of error bound ("the error bound is kinda like the max /…/, the minimum and the maximum that you can get," turn 333). The 4.5 and 5.5 were the minimum and the maximum values and thus they were the error bounds. However, this would imply that Adriana confused error bounds with the bounds for the size of the pancakes.

Consider some of the ways that Adriana had used the word bounds. In turn 428, she said, "[The story] just wanted our errors to be within those bounds [for pancake and batter]." Contrast this with the way it was written in the story that the boss wanted the pancakes to be within ½ from 5. Errors cannot really be within an error bound. It could be less than the error bound.

She was using "within" when she meant between. When Adriana talked about error bounds, Adriana did have the idea of minimum and maximum and a range of values. However, the 4.5 and 5.5 were not mistakes because she would appropriately infer the bound from the minimum and the maximum values. For example, in turn 434 Adriana mentioned the pancakes needed to be "within an inch of bound." If she was confused, she could have said that the pancakes needed to be "within a half an inch bound," like she did in turn 436, but she did not. If she were in fact was talking about the error bound for the pancake in turn 434, the fact that it said "within one" instead of "within a half" would also have conflicted with her. Thus, it is more likely that the mistake was in including the L in the inequality.

In sum, by focusing on the pancake story Adriana determined that the error could not equal the error bound. The story said that the pancakes needed to be between 4.5 and 5.5 inches. The inequality $4.5 < f(x) - L < 5.5$ focused on the range of acceptable pancake sizes (output values), but I argue that she appropriately deduced the ½ error bound from the maximum and the minimum value. From there she inferred that since the pancake sizes could not equal their maximum or minimum values, then neither could the errors.[45]

Segment 3: The Emergence of Functional Dependence

In between the previous segment and this segment, I asked Adriana what the quantities $f(x)-L$ and $x-a$ were in the story. She correctly mapped them to the error in the pancake size and the batter, respectively. The discussion presented thus far occurred during a discussion about the meaning of the "if-then" statement in the story. In this segment, she summarized the meaning of the "if-then" statement. In the process, she also summarized her previous confusion with error bounds.

452	A	Um so I think this is the, like these delta and the epsilon they're the error bounds of what I want, relating back to the story, like f of x [$f(x)$] minus L, since that's the error of the pancake size. So say we want this [*points at* $f(x)-L$] to be as close to 5 as possible or yeah, that was the point, but we want it to be within 1/2 bound [*points at* $4.5<f(x)-L<5.5$]. So this [*points at 4.5*] is the smallest it could be, this [*points at 5.5*] is the biggest it could be, but we want it to be *within* that range [*points at the inequality*]. So that's why I was looking for to see if it said that it *can be* equal to 4.5
453	I	Um-hm.
454	A	But it's not. So that's why it doesn't have the equal sign. Um, so yeah. So we want it to be within these two quantities [*refers to 4.5 and 5.5*] or whatever
455	I	Um-hm.
456	A	And then we would have to, well cus it says like this one [*points at* $0<\|x-a\|<\delta$] first, right? So this one [$\|x-a\|<\delta$] would be like the batter /…/ if this [$\|x-a\|$] was the, the error of batter [*writes error of batter below* $\|x-a\|$ *in the definition*], then /…/ this [*points at* $4.5<f(x)-L<5.5$], this is a result of how much batter [*points to* $0<\|x-a\|<\delta$] we're using. So size is a result of the batter that we're using.
457	I	Ok.
458	A	But it goes back to this, [*points to "for every number ε>0, there exists a number δ>0"*] too. Cus if there's an epsilon [*circles* $f(x)-L$ *in* $4.5<f(x)-L<5.5$] which means like there's gonna be an error bound for here [*points back and forth at* 4.5 *and* 5.5] then there has to be an error bound for this one [*points back and forth at the 0 and δ in* $0<\|x-a\|<\delta$], but this one [*points at* $0<\|x-a\|<\delta$] is the one that manipulates that one [*points* $4.5<f(x)-L<5.5$].

[45] Adriana's inferring error bounds from the range of acceptable values might have been seen as a limitation of her understanding. One might argue that the story contributed in enforcing a misconception about an understanding of error bounds as lower and upper bound. While there were other ways to accurately infer error bounds, there was nothing wrong with the way she inferred error bounds from the range. Not only was it a productive use of an error bound (to determine acceptable values), she was able to correctly infer the appropriate error bound from the range of values.

Confirming the interpretation of the previous segment, Adriana inferred the error bound of ½ inch from the inequality $4.5 < f(x) - L < 5.5$. She differentiated the error bound from the bounds for the size of the pancakes, and invoked a notion of range of values (turn 452). She specified that having an epsilon meant that there would be an error bound at the maximum and minimum size of the pancakes. In turn 458, she pointed back and forth between the 4.5 and 5.5 to find the error bound. She did not treat the delta inequality any differently because the inequality with the 0 being a "lower bound" might have fit with her understanding of error bound. All of this further support the interpretation that she inferred the error bound appropriately from the inequality, despite having written an erroneous inequality.

Adriana's argument in turns 456 and 458 focused her on the *functional dependence* relationship. "Batter makes pancakes" or *functional dependence* was a common idea taken up by students from the story (a part of a common Reasoning Pattern in Chapter 7). Unlike those other students, Adriana did not claim that epsilon depended on delta in this segment, but she interpreted the "for-every" statement non-normatively.

She argued that the statement meant that if there was an epsilon, if epsilon existed, then there had to be a delta that caused it (turn 458). This is different from one can find a delta for a given epsilon. This was actually quite common interpretation among students before and after engaging with the story (see Chapter 5 and 7). Here, we see evidence of Adriana using *functional dependence* to interpret the meaning of the "for every" statement. While the goal of being arbitrarily close to the limit was cued in this segment (turn 452), unlike what happened at the end of the interview (Section I of this chapter), here, that goal had not yet emphasize the appropriate relationship between epsilon and delta. It did not seem that Adriana was focusing on that relationship in this segment.

I argue that the focus of the inequality $4.5 < f(x) - L < 5.5$ on the range of acceptable output values might have also contributed to the use of the *functional dependence* argument. While Adriana could appropriately infer the error bound using the inequality $4.5 < f(x) - L < 5.5$, its focus on the range of acceptable sizes of the pancakes might have unintentionally also prioritized the *functional dependence* relationship. The *functional dependence* relationship was appropriate to describe the error relationship and the "if-then" statement, but not to describe the "for every" statement, which prescribes the appropriate temporal order.

Summary

Adriana had a very particular (intuitive) way of conceptualizing error bounds. She inferred the error bound from the range of acceptable values. She knew the relationship between the error and error bound. The issue in this episode was whether an error could equal an error bound. Adriana reread the story to reestablish the goal of creating pancakes with sizes within a specified range, or *between* two particular sizes. She then inferred that the error also could not equal the error bound. While she could appropriately infer the error bound from the inequality, its focus on the range of output values might have contributed to her prioritizing the *functional dependence* idea at the end of the episode. While it did not materialize to her claiming that epsilon depended on delta, it did influence the way she inferred the "for-every" statement.

III. Change in The Temporal Order

I document Adriana's stable conception of the temporal order in the first section. In the second section I explored the potential origin of the distinction between error and error bound. The two sections provide contexts for the discussion where I show how Adriana navigated through conflicts between productive resources from the story and her prior knowledge. The distinction between an error and an error bound played a role in the development of her claim about the temporal order. She moved from initially using the *functional dependence* argument to prioritizing productive resources from the story to argue for the correct temporal order.

There are four segments to this section. The first segment shows Adriana's attempt to incorporate some of the productive resources from the story, which did not succeed; she ended up focusing on the *functional dependence* idea. The second segment shows that the distinction between error and error bound ended up confusing Adriana. The third segment shows Adriana's first attempt at resolving the conflict. In the last segment Adriana settled the conflict by explicitly reorganizing her prior knowledge. To help the reader read the transcript, I highlighted productive statements in blue, erroneous statements in red and the temporal order claim in black.

Segment 1: Incorporation of Productive Resources From the Story

Aspects of Adriana's prior knowledge were reintroduced into the discussion. At the same time productive resources from the story began to emerge. Adriana prioritized her prior argument that epsilon depended on delta because $f(x)$ depended on x. The segment started with the interviewer asking Adriana about the dependence between epsilon and delta. She started with the claim that epsilon depended on delta. She prioritized the idea of *functional dependence* between x and $f(x)$ over other ideas from the story in justifying her claim. In addition to the emergence of productive ideas from the story, there was a back-and-forth in the claim about the temporal order for which Adriana was arguing.

547 I Ok, with epsilon delta what depends on what if anything?

548 A Um, the [*pause, stares at the wall*] **del-, the epsilon depends on the delta**.

549 I Did you change your mind? [*Adriana is still staring at the wall*] Actually, you- you said the same thing.

550 A [*Looks at the interviewer]* Yeah. So yeah, I think the **delta depends on the epsilon** cus that's=

551 I =Did anything change?

552 A Um, I think /.../ in this case.. I think it can- /.../ **they can kinda depend on**=

553 I =Each other?

554 A Both, yeah in a sense because, but more **whatever you're getting, like $f(x)$ is always gonna depend on what x you're inputting it.** But then, **if you want to get something that's within delta** [*marks a small interval on the x axis with two fingers*] you need to see if /.../ for example here [*points to the pancake story*] our **epsilon here was already set**, then that [*points back and forth between* 4.5 *and* 5.5 *in the inequality* $4.5<f(x)-L<5.5$] kind of depended on what we were putting in for x [*points at the same interval around x on the graph*] but.. but mostly **whatever you're putting in to your x is gonna determine what you get for $f(x)$** [*pause*]. So I'm still saying the same thing like **delta depends on epsilon** but=

555 I =Delta depends on epsilon? Or epsilon depends..

556 A No, yeah, **epsilon depends on delta**

557 I Um-hm.

558 A But, /…/ **if epsilon's already set then you'll manipulate your** /…/ **delta so it's within an error bound** and /…/ then continue to manipu- wait [*long pause*] wait, so you're… hm.

It seems clear that in this segment Adriana prioritized the *functional dependence* argument from before the story ("but mostly whatever you're putting in for x is gonna determine what you get for $f(x)$," end of turn 554). Adriana's statement, "delta so it's within an error bound" (turn 558) suggests that Adriana might have returned to thinking of delta as an error in x. In this way, Adriana's argument was still like that of many students at the beginning of the interview. This typical argument suggests the use of the following knowledge resources: *functional dependence* (between x and $f(x)$), *function slots* associating epsilon with $f(x)$ and delta with x. Notice also, that she pointed to the inequality $4.5 < f(x) - L < 5.5$ to talk about epsilon.

She also recognized the possibility that delta could also depend on epsilon and so the two could have been dependent on each other (turn 552). Turns 554 and 558 could be interpreted as Adriana's justification for delta depending on epsilon. She said, "[H]ere [*points to the pancake story*] our epsilon here was already set, then that [*points back and forth between the* 4.5 *and* 5.5 *in the inequality* $4.5 < f(x) - L < 5.5$] kind of depended on what we were putting in for x [*points at the same interval around x on the graph*]" (turn 554). In turn 558, she said, "if epsilon's already set then you'll manipulate your /…/ delta so it's within an error bound." At first glance these two turns seem to be arguing for different things.

In turn 558 Adriana seems to be using the notion of *quality control.* The *givenness* of epsilon was interpreted as a constraint, and the error was controlled to be within an error bound to satisfy the constraint. The "it" in the statement was referring to delta, and Adriana mistakenly treated delta as an error ("you'll manipulate your /…/ delta so [delta]'s within an error bound"). According to the story delta was supposed to be the error bound. In exactly two turns Adriana would confirm this interpretation. She would explain that she thought of delta as errors (turns 560–564). Moreover, Adriana said, "so it's within *an* error bound," not *the* error bound. Later segments show that when she referred to epsilon while talking about delta, she would refer to it as "*the* given error bound" (turns 568–576). So Adriana was saying that if a constraint were given, then she would manipulate her error in the input to be within an error bound.

The justification in turn 554 is more challenging to interpret. While it is clear that she took up the *givenness* of epsilon from the story ("here, the epsilon was already set"), how this argument supported the claim that delta depended on epsilon was less clear. Because it seems that the statement "that [the inequality] depended on what we were putting in for x" would suggest that epsilon depended on delta, the opposite of what she was arguing. Adriana also started this by saying that "if you want to get something that's within delta."

I took "if you want something within delta" as evidence for Adriana's use of *quality control*. Comparing her use of the phrase "within delta" in other parts of the transcript suggests that in this segment, Adriana was arguing that to satisfy the given constraint, epsilon, one would want to x to be close to a, or "within delta" (turns 538, 640, 656). In turn 538, she used the same phrase in describing the goal with the formal definition after the story. She said,

> Yeah so we're only concerned with *x's that are within the delta* so that we're only getting answers that are within the epsilon. /…/ [S]o we want the errors to be like really close to *a* so that we're only looking at numbers close to the *a*. So that we're getting a really small error to get like the perfect pancake (turn 538).

She also described a similar idea of constraining the x to be within delta in later parts of the interview (turns 640, and 644–656). So it is not likely that Adriana's "want[ing] to get something that's within delta" was a reflection of confusion. She was not confusing the goal of making sure that the function was within epsilon from the story, and misapplied it to delta. She instead was arguing that in order to satisfy the given constraint, one would want x to be close to a. However, when she referred to the inequality and said that it kind of depended on the x she was inputting (turn 554), we see evidence for the way she prioritized the *functional dependence* at the end of the episode in the previous section ("This [*points to* $4.5 < f(x) - L < 5.5$], this is a result of how much batter [*points to* $0 < |x - a| < \delta$] we're using. So size is a result of the batter that we're using…," turn 456).

The statement "that [the inequality] depended on what we were putting in for x, but... but mostly whatever you're putting in to your x is gonna determine what you get for $f(x)$" in turn 554 provides an opening for conflict. I argue that the pause on 554 and the long pause in 558 suggest that Adriana might have realized a conflict in her explanation. The "but" at the end of turn 554 suggests that Adriana was trying to provide an explanation that would counter her previous statement. That is, whereas earlier in turn 554 she was arguing for delta depending on epsilon, at the end she was supposed to argue for epsilon depending on delta. But her justification ended up being the same: the output depended on the input.

Adriana did not use the phrase "error bound" until turn 558. This is significant because had she continued with the conception that delta and epsilon were errors, they would follow the two relationships she described. The error in the output could depend on the error in the input. At the same time, given an acceptable error in the output (normatively set by a given error bound), one would want the error in the input to be small to satisfy the given constraint. So the error in the input could also depend on the error in the output. However, the use of the phrase "error bound" might have reminded her of a difference between an error and an error bound. She might have recalled the conversation that I analyzed in Section II. In fact Adriana start the next segment differentiating error from error bound.

In sum, Adriana argued that epsilon depended on delta because $f(x)$ depended on x and delta and epsilon were (errors) associated with x and $f(x)$, respectively. At the same time, she also recognized the possibility that they might also depend on each other. She used quality control to argue the necessity to constrain x using delta to satisfy the given constraint, epsilon. However, a conflict emerged when she used the phrase error bound.

Segment 2: Distinguishing Error From Error Bound Instigates a Conflict

In this segment, Adriana remembered the delta and epsilon were error bounds, not errors, a distinction she made earlier (Section II). This led to a revision of the use of the productive resources from the previous segment. At the same time it also led to a conflict that she could not resolve. I included turn 558 from the previous segment to provide continuity.

558	A	But, /…/ **if epsilon's already set then you'll manipulate your** /…/ **delta so it's within an error bound** and /…/ then continue to manipu-. Wait [*long pause*] wait, so you're… Hm.
559	I	What's happening?
560–564	A	Oh cus /…/ I thought that **the epsilon and the delta were the errors** but **they're the error bounds.** And if **the epsilon is already set** then you would have to change

your delta. Yeah, /…/ I guess **if your epsilon is already set** then your **delta would depend on epsilon** [*silence*]

565 I What just happened?

566 A Uh, [*laughs*] well cus /…/ just looking at this [*points back and forth between the pancake story and the inequality 4.5<f(x)-L<5.5*] if I said **epsilon was an error bound** and if **they already give me an error bound**, **I want… my result to be within this error bound here** [*circles the inequality 4.5<f(x)-L<5.5*] **then.. I would try to manipulate my errors here** [*points to a small range on the x axis on the graph*] **to be within a smaller error bound** [*points at delta in the delta inequality in the definition*], **which would be, delta would be** [*quietly*] **the biggest it can be** [long pause] Huh.. [*looks at the interviewer and smiles*] I'm confused.

In this segment, Adriana seems to be prioritizing the quality control argument to support the claim that delta depends on epsilon. She was using many of the same resources from the previous segment: *givenness* of epsilon ("they already give me an error bound," turn 566), *quality control* ("I want my result to be within this error bound then I would try to manipulate my errors here to be within a smaller error bound," turn 566). But the first thing she did in this segment was to remember that epsilon and delta were not errors.

Adriana reiterated the *givenness* of epsilon. In turns 560–564, Adriana inferred the temporal order directly from the givenness of epsilon. She argued, "If the epsilon is already set then you would have to change your delta. Yeah, /…/ I guess if your epsilon is already set then your delta would depend on epsilon [*silence*]." This was a common justification used by many of the students I interviewed, but Adriana followed it with a pause. The phrases "I guess" and "would" at the end of turns 560–564 ("delta would depend on epsilon") suggests that the temporal order was an implication from her newfound realization that epsilon and delta were error bounds. Since this argument was newly constructed, she might have needed time to align it with her previous argument.

In comparison to what Adriana said in turn 558 in the previous segment, her explanation of quality control in turn 566 was more complete and explicit. Not only did she attend to the difference between error and error bounds, doing so also allowed her to be more explicit about roles of delta and epsilon in the process. Epsilon as an error bound was used as a constraint for the outputs ("I want… my result to be within this error bound here [*circles the inequality* $4.5< f(x)-L<5.5$]"). Again, she focused on the range of acceptable outputs to infer the error bound. Delta as an error bound was used as a way to manipulate the errors in the input to achieve the desired result ("I would try to manipulate my [input] errors here to be within a *smaller* error bound /…/ delta would be the biggest it can be"). The *givenness* of epsilon helped to determine the order.

Adriana could also have been attempting to incorporate the arbitrariness of epsilon in this segment. The phrase, "I would try to manipulate my errors here to be within a *smaller* error bound" suggests this kind of attempt. Analysis of other parts of the transcript reveals that the use of the phrase "smaller error bound" might have been Adriana's attempt to vary epsilon. Adriana used the phrase "a smaller error bound" on two other turns during the whole interview (turns 482 and 684). In both instances, Adriana was explaining the need for the phrase "for every number epsilon" in the definition. When discussing the story, she explained, "Cus /…/ maybe the boss will ask you later to make it *a smaller error bound.* But in terms of the task right there it's one half an inch" (turn 482). She was explaining the part of the story where the boss would later change the constraint on the pancake diameter to be more precise.

Much later, I asked her the same question. She said, "Like I said, the only thing I can think of is, if we want *a smaller error bound* here [epsilon inequality] then we would want to manipulate the smaller error bound for here [delta inequality]" (turn 684). Here, she explained the process with more details. Adriana explained that if she wanted a smaller error bound for the output (a smaller epsilon), then she would use a smaller error bound for the input as well (a smaller delta). So when she was saying "smaller" she was comparing the epsilon and the delta to the ones she used in the previous iteration. This could also explain what she meant by "continue to manipu[late]" in turn 558. She might have started with the idea of refining *errors* in turn 558, but by the end of turn 566 she was moving towards the idea of multiple epsilons.

In sum, Adriana distinguished errors from error bounds, and used the *givenness* of epsilon and a very detailed account of *quality control* to argue that delta depended on epsilon. She even incorporated the role of varying epsilon in her explanation of quality control. Many of these ideas were perhaps new to her, but instead of simply acknowledging its novelty, she explicitly said that she was confused. The question is, what is the source of her confusion?

I posit that the confusion stems from Adriana's focus on the inequality $4.5 < f(x) - L < 5.5$ to determine error bound. On the one hand, that inequality had embedded in it the error bound of ½. On the other hand, what were salient in the inequality were the lower and upper bound for the size of the pancakes. So while the error bound could be inferred, the inequality showed a range of output values. Adriana acknowledged the difference between error and error bound, but the representation focused on a range of output values. This was problematic when she was also juggling *functional dependence* as a justification for the temporal order. She created a distinction between error and error bound, but that was not enough to move away from the *functional dependence* idea. The error bound was still very much tied to a range of $f(x)$ values, which she knew to be dependent on x values. She then had to reconcile the fact that the range of acceptable output values were specified, but at the same time they depended on x values she was inputting.

Segment 3: Givenness of Epsilon as the Source of Confusion

In this segment, Adriana made her first attempt at explaining the nature of her confusion. The givenness of epsilon confused her. In the process of explaining her confusion she cued many of the productive resources from the story. At the end, instead of corroborating her confusion, she ended up supporting the claim that delta depended on epsilon. But a fleeting description of delta might indicate another source of confusion.

567	I	Why are you confused?
568–576	A	Because if epsilon did depend on delta then, then I could change it here [*points at the inequality* $4.5 < f(x) - L < 5.5$] or I mean, I'm confused because **they gave me an epsilon** [*points at the inequality* $4.5 < f(x) - L < 5.5$]. And **it's already set**. And **they didn't give me a delta** so in that sense it didn't depend on delta… But then **the delta, I would want it to be really close to..** or I would **want my error bound to be really small to accommodate** /…/ **the error bound that was already given** or the epsilon that was already given to me. So [*quietly*] **epsilon could depend on delta**? I mean, **delta could depend on epsilon**, or **does depend on epsilon**...

Adriana cued many of the same knowledge resources from previous segment: *givenness* of epsilon ("they gave me an epsilon"), and *quality control* ("want my error bound to be really small to accommodate /…/ the error bound that was already given"). But Adriana also applied the *givenness* idea to delta. That is, she emphasized the non-*givenness* of delta to support her claim that delta depended on epsilon. She attempted to explain her confusion twice in this segment, but ended up reiterating resources from the story and became more convinced that delta could depend on epsilon. There was also a shift in the description of delta. In previous segments Adriana attempted to describe the relationship of delta with other quantities ("so [delta] is within an error bound," 558, "delta is the biggest [the error bound] can be," 566). Here she started describing delta in a similar way ("delta to be really close to..."). She ultimately simplified the description to say that she wanted delta to be really small, thereby focusing more on the delta's role as a bound.

This episode confirms the confusion posited in the previous segment. Adriana was confused by the fact that epsilon was given. She said, "I'm confused because they gave me an epsilon and it's already set." When she said this, she was pointing at the inequality that focused on a range of $f(x)$ values. The fact that epsilon was already given conflicted with the idea that $f(x)$ values depended on the x values. The range of acceptable $f(x)$ values was given, yet it was also depended on the x values used.

The brevity of the first sentence of turn 568 makes it difficult to interpret. When she said, "epsilon depended on delta" in that turn, it could mean that the goal of the process was for the output values to be within epsilon. "If epsilon depended on delta then I could change it here [in the inequality]" could mean that given that the story specified a range of acceptable output values, then for the values to be in that range, Adriana could simply select an output value to was within that range of acceptable values. There is little supporting evidence for this, but I offer it as one interpretation. In any case, it is clear that the broader conflict was between her use of *functional dependence* and *givenness* of epsilon.

The sentence "I would want [my delta] to be close to…" also suggests that another part of the confusion was sorting out the different conceptions of delta. Adriana's wanting the delta to be close to something is analogous to when she wanted delta to be within an error bound (turn 558). Both instances treated delta as an error. Delta returning to being an error might have also increased the cueing priority of the *functional dependence* argument. But by saying that she wanted her error bound to be really small in turns 568–576, she removed one layer of complexity of delta (what quantity it determined). Doing so might have allowed her to focus on the temporal order suggested by the *quality control*. She wanted her error bound for the input to be really small to accommodate the given epsilon.

I argue that in this segment Adriana also used the *proportional variation* resource.[46] She wanted to make the error bound for the input small in order to accommodate the given error bound for the output. Notice that she was no longer talking about a smaller error bound as she did in the previous segment. Other instances in the transcript where she used the word "small" with delta were mostly in attempts to make delta small so that x was close to a. This also suggests another knowledge resource: *domain constraint for a limit.* That is, with limits, Adriana would focus on x values that were close to a.

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[46] Adam used the same resource in Chapter 6. This resource means that a small change in the input leads to a small change in the output.

In summary, in this segment Adriana pointed to the givenness of epsilon as a source of conflict between the story and her prior knowledge. The confusion might have stemmed from the inequality for the error bound, epsilon. She was still adjusting to the idea that delta was no longer an error bound, but she ended up cueing many of the productive resources from the story to support the claim that delta depended on epsilon.

Segment 4: Previous Resources Aligned, and Productive Resources Prioritized

In this segment we see that, after prioritizing the resources from the story, Adriana concluded that delta depended on epsilon. She did so after explaining how she repurposed the *functional dependence* argument from the first segment to describe the relationship between errors. Adriana's description of error bounds still stemmed from a range of acceptable values. She explicitly used language from story to explain why her reasoning changed in this segment.

577	I	So, do they depend on each other, is it just one way now?
578–580	A	Um, see cus I was looking at it like /…/ **the f of x [$f(x)$] depends on the x** and that's how I was like saying that epsilon depends on delta because **epsilon is related to the f of x** [$f(x)$]/…/. But that's just saying **the error of the L and the f of x [$f(x)$] depends on the a and x** but that's not to say that epsilon depends on delta.
581	I	Ok, so?
582	A	So, I think that delta depends on epsilon now [*laughs*]. Just cus if it's given like this [*reference unclear*] and **you're trying to aim at getting /…/ within a certain error bound**, then you're gonna **try to manipulate your entries /…/ to be within a certain error bound** [*gestures a small horizontal interval with her palms*]
583	I	Ok. Alright, so and so you changed your mind it seems? Um, so how did that happen? Why did you change your mind?
584–588	A	Because I was **given an epsilon** [*points at the inequality* $4.5<f(x)-L<5.5$] and that's kinda like the main goal. **The main goal is to get the pancake, /…/ and they gave me a constraint** /…/ and /…/ they **didn't give me an error bound for the batter** or for like the a or x, they didn't give me an error bound. But I know I **want to make it small so that it's within the error bound, the epsilon**. So then **I would kinda base my delta on what was epsilon**.

If one considers this segment in isolation, it seems to suggest a nice success story. Adriana realized and corrected her mistake in using *functional dependence* to determine the temporal order for epsilon and delta. She realized that she had loosely associated epsilon with the function, $f(x)$, and recognized that that was false. She then used *functional dependence* to describe the dependence between the errors. Once that happened, it seems that the productive resources from the story became more salient. With those resources prioritized, Adriana concluded that epsilon depended on delta. However, considering this segment in relation to the prior segments and episodes reveal a more nuanced story of learning.

In the episodes and segments leading up to this one, Adriana never once stated that epsilon is related to $f(x)$.[47] There was more to Adriana's statement about epsilon's being related to the

[47] This was confirmed by a search of the full transcript for the phrase "epsilon is."

function. Really early in the interview, Adriana did say that epsilon was the difference between the function and the limit. But closer to this segment, Adriana had been inferring the error bound, epsilon from the inequality $4.5<f(x)-L<5.5$. In fact, this inequality was what Adriana referred to whenever she spoke about error bound. Phrases like "You aim at getting within a certain error bound" and "manipulating your entries to within a certain error bound" show that epsilon—and possibly delta—was still conceptualized as stemming from a range of values (turn 582). So in this way, her way of conceptualization of error bound was still the same as before. Epsilon was an error bound that could be inferred from a range of $f(x)$ values.

More than just correcting a mistake, Adriana *repurposed* the *functional dependence* between epsilon and delta to describe the relationship between the errors (turns 578–580). This was a very important and productive move. She attended to a context in which the *functional dependence* idea could be used productively—the relationship between errors—and it helped her align her prior knowledge with productive resources from the story.

The move also further differentiated the conception of an error from the conception of an error bound. By using *functional dependence* to describe a relationship between the errors, and not the error bounds, she opened up the possibility of the relationship between the error bounds to be described using other kinds of relationships (e.g., *quality control*). Thus she distinguished the nature of the *relationship* between the errors and that between the error bounds in addition to their not being equal.

The move of attributing *functional dependence* as the relationship between the errors also addressed the conflict with the fact that epsilon was given (*givenness* of epsilon from earlier). To Adriana, the inequality, $4.5<f(x)-L<5.5$ represented a range of $f(x)$ values as well as the existence of an error bound (the 4.5 and 5.5). The move examined the ties between the errors and error bounds that existed through the inequality. It specified the functional dependence to only describe the relationship of $f(x)-L$—the inside of the inequality—with $x-a$ (turn 580). No longer tied to $f(x)$ values, the error bounds (the 4.5 and 5.5) were freed to take up any other characteristic and relationship. Now the *givenness* of epsilon was no longer a conflict, and she could also take up the *quality control* relationship that was consistent with the *givenness* of epsilon (turns 584–588). The flow of Adriana's argument and the lack of pauses suggest that this particular shift might have addressed the conflict that she had before.

In addition to focusing on the *givenness* of epsilon and the notion of *quality control*, Adriana also cued other resources that were consistent with those resources (turns 584–588). Adriana cued the *givenness* of epsilon ("they gave me a constraint") alongside the non-*givenness* of delta ("they didn't give me an error bound for the batter"). Adriana was unique in the way that she inferred the idea that delta was not given in the story by comparing it to epsilon. This was a productive inference as it supported her interpretation of the appropriate temporal order.

Quality control was cued with *proportional variation* ("I want to make the [delta] small so that [the output] is within epsilon"). This contrasts with other students who did not include the "small-ness" of delta in talking about the goal of satisfying an output constraint by controlling the input. As Adriana started to do in Segment 3, she reestablished the role of delta as a bound by focusing on its "small-ness." The *proportional variation* was also used in Segment 3, where she focused on the making sure that delta was even smaller—perhaps compared to epsilon—to meet the requirement of a small epsilon.

She also noticeably switched her language when explaining the reason for the change in her thinking. When I first asked the temporal order question, she did not mention pancakes or batter in her explanation. She did use the terms "error" and "error bound," which were terms introduced in the story that she did not use before.[48] However, in explaining the change in her thinking, she used parts of the story more explicitly. She referred to satisfying epsilon as the goal of making pancakes within a specified constraint. Delta was the error bound for the batter, and that was not a quantity that was given. This suggests that ideas from the story solidified her thinking in productive ways.

In sum, Adriana corroborated the difference between an error and an error bound that she had previously established, before she prioritized the productive resources from the story. Focusing on the *functional dependence* relationship to describe the relationship between the errors but not the error bounds was a move to align productive resources with her prior knowledge. She then prioritized the idea of *givenness* of epsilon and *quality control* to conclude that delta depended on epsilon. She still conceptualized epsilon and delta as stemming from a range of acceptable values.

Summary of Analysis

Adriana changed her temporal order claim from epsilon depended on delta to delta depended on epsilon. She initially focused on the *functional dependence* idea, likely because she treated epsilon and delta as errors instead of error bounds. Productive ideas from the story, in particular the *givenness* of epsilon, confused Adriana. The confusion was caused by her way of determining error bounds from the inequality $4.5 < f(x) - L < 5.5$, which showed a range of $f(x)$ values. The focus of this inequality on the range of output values might have prioritized the *functional dependence* relationship. Adriana specified *functional dependence* to describe the relationship between the errors, the $f(x) - L$, which freed up the relationship between the error bounds for *quality control*. With the error bounds no longer being attached to *functional dependence*, she could then prioritize the *givenness* of epsilon from the story. By cueing other consistent resources, like the non-*givenness* of delta and *proportional variation*, she concluded that delta depended on epsilon.

Discussion

The findings from this chapter elaborate on the findings from Chapter 7. The pancake story provides productive resources for students to reason about the temporal order of delta and epsilon. Whereas in Chapter 7 the benefits are seen through students' justifications, in this chapter we see the ways that the story influence and interact with Adriana's thinking about the temporal order. Adriana took up many of the productive resources from the story, but she needed to do some work to align them with her existing prior knowledge. The findings from this chapter provide evidence to support many of the theoretical assumptions of Knowledge in Pieces. They also illustrate different ways that the Pancake Story assists students in making sense of the temporal order, and in understanding the formal definition more broadly. I elaborate on each of these points below.

Illustrations of Theoretical Assumptions

The first section of the analysis and Adriana's repurposing the use of the *functional dependence* resource in a more productive context, nicely illustrates the importance of context for knowledge. In the first section of the analysis we see how Adriana's conception of the temporal order was stable across contexts. In this way, Adriana's understanding about the

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[48] As we see in part II of the analysis, the distinction between the two was not immediately clear to Adriana.

temporal order was ideal because after engaging with the story, her conception was stable across the three contexts.[49] She consistently argued that epsilon came first across the three contexts. She did so by focusing on the same productive ideas, some of which were resources introduced by the story (e.g., getting arbitrarily close to the limit, *quality control, givenness* of epsilon and distinguishing error from error bound). At the same time the particular language to describe these ideas and the details of her arguments were different. For example, in the context of sequential order, she created familiar examples to stabilize her understanding of the need to use a small delta ("we're not gonna use 100 lbs."). On the other hand, in ordering the four variables, she also concluded that *x* was the only variable she could control. This described the relationship of the four variables in addition to specifying the temporal order. So the ideas might have been similar but the particular details and implication of their use varied by context.

Adriana's productive use of the *functional dependence* resource nicely illustrates the context specificity of knowledge. *Functional* dependence was productive to describe the relationship between errors but not the relationship between error bounds. Not only did Adriana find a productive context for the knowledge resource, doing so also further established the difference between an error and an error bound. This ultimately helped Adriana prioritize the productive resources.

The way Adriana reorganized her knowledge illustrates the process of learning as theorized by Knowledge in Pieces. During the discussion about the temporal order, Adriana did not learn the correct order by replacing her "misconceptions" with the correct conception. She initially supported her claim using the *functional dependence* argument for the temporal order. This supposed "misconception" was not replaced but was reprioritized. Adriana found a context in which the knowledge was productive (*functional dependence* for errors). Since she was focusing on the relationship between the error bounds, then that relationship became less prioritized. Instead the resource *quality control* along with *givenness* of epsilon became prioritized.

Adriana's case also illustrates how learning did not take place by simply having the right ideas at one's disposal. Adriana knew about the productive resources from the story and a way to productively put them together (Segment 2 in the last section). However, that did not change her claim immediately. Learning happened for Adriana as she aligned different productive resources from the story (e.g., *givenness* of epsilon) with her existing prior knowledge (e.g., *functional dependence*). That is, the conflict between her prior knowledge and the productive resources needed to be addressed before she could prioritize the productive ideas that she learned from the story.

Adriana's case also shows the non-linearity of the process of learning. Her understanding an idea in one context did not guarantee that it would be used in another context. We see this with Adriana's distinction of error and error bound. Adriana made the distinction between error and error bound using the story (Section II). She was able to articulate the difference and the relationship between them in talking about the "if-then" statement in the story, and concluded that epsilon could not equal the error. However, later, when discussing the dependence between epsilon and delta, it took her time before she remembered that epsilon and delta were error bounds and not errors. The distinction that she made in section II was new and might not have been stable across contexts yet. In Section III, Adriana elaborated the distinction by differentiating the nature of the relationships between the errors and the error bounds. This was

[49] Not everyone in the study achieved such stability with his/her understanding, though many did (see table 7.1 in Chapter 7).

when she was able to prioritize the resources from the story. Thus at the end, Adriana incorporated that knowledge (about the distinction) in a new context, which potentially solidified the distinction. Notice that from the last segment in section III that distinction remained with Adriana and was used consistently across contexts (Section I).

The Role of the Pancake Story

The pancake story provided a number of productive resources for Adriana to think about temporal order. The language of the story was designed to be more accessible to students to communicate ideas from the definition. Adriana took up ideas such as the *givenness* of epsilon, *quality control,* error and error bound, which were designed to be salient in the story. It is clear that making the distinction between error and error bound also played a significant role in Adriana's final response to the temporal order question in the last episode. I now explore the details of these affordances of the Pancake Story.

Benefits of accessible concepts through an accessible language. At the end of the final episode, Adriana's language completely shifted to the language of the story. This shift in language was quite common among all the students I interviewed. Twelve of the 19 students incorporated words like "error" and "error bound," and nine students specifically mentioned "pancakes" and "batter" in their justification for the temporal order. First, the take up of the language from the story is important because access into formal mathematics through everyday language was part of the design of the story. At the same time, the productive mapping that Adriana made between the formal definition and the story highlights another aspect of access into the topic of formal definition provided by the story.

Adriana was able to use the story to stabilize and gain access into conceptualizations of different parts of the definition. The story made salient the *givenness* of epsilon, an important aspect of the definition, which was not immediately obvious to Adriana and many students in the study. The error bound for the pancake was given, but Adriana inferred the givenness of epsilon from that. Adriana also used the story to stabilize the distinction between $f(x)-L$ and epsilon, an aspect of the definition with which many students struggled. This happened as she distinguished error from error bound. This also helped Adriana deepen her conceptualization of epsilon and delta. In certain segments, she even made productive inferences from the story that helped her make sense of the formal definition. She could only do this after she successfully mapped the different parts of the story to the definition. I return to this idea shortly.

Distinguishing error from error bound. Adriana used the story to distinguish the meaning of errors and error bounds, and the resulting distinction played a significant role in the change of her conception of the temporal order. In Section II of the analysis, Adriana focused on the "between-ness" of the range of acceptable pancakes to deduce that the error could not equal to the error bound. In this way, the story served as a stepping-stone to creating the conceptual distinctions between error and error bound. In one of the segments in Section III, Adriana began to shift her conceptualization of delta using the story. She moved from focusing on delta's relationship to other quantities (e.g., delta is close to a) to delta's being a small bound.

The distinction between the error and the error bound also became relevant in answering the temporal order question. She elaborated on the distinction by distinguishing the relationship between the errors from the relationship between the error bounds. This last distinction happened as a result of a productive resource that was made salient by the story. The *givenness* of epsilon conflicted with *functional dependence* and she needed to reconcile them. It was in reconciling the two resources that she was able to further distinguish the errors and the error bounds, and prioritized *quality control* to describe the relationship between the error bounds. She also used other resources to support her claim.

Productive inferences. Some of these resources came as a result of Adriana's productive inferences from the story. For example, Adriana re-read the story to decide if the error could equal the error bound. Adriana focused on the goal of creating pancakes with sizes between 4.5 and 5.5 inches. Since the story did not say that the pancake sizes could *equal* 4.5 or 5.5, she inferred that the errors also could not equal the error bound. This was the first distinction she made between the error and error bound. The story did not explicitly say that the two could not be equal. Another example was when she focused on the non-*givenness* of delta to support her claim about the temporal order. She inferred that the error bound for the batter (delta) was not given. From that she concluded that epsilon could not depend on delta. The story did not emphasize that delta was not given, but she inferred as much and made productive conclusion from it. Both of these examples show a different kind of interaction with the story. While she incorporated many productive resources from the story, she also made productive inferences using resources from the story to support her claim.

The analysis in this chapter suggests that the pancake story can serve as a rich learning space for students. It is a space that can be used by students to make sense of the temporal order and the formal definition more broadly. In this chapter we explore the process for Adriana in making sense of the temporal order of delta and epsilon using the story. We see the change in Adriana's conception of the temporal order as she aligned productive resources with her existing prior knowledge. The case also nicely illustrates many of the affordances of the story for making sense of the temporal order. Specifically, the case illustrates the way that the story honors students' prior knowledge and leverages students' intuitive knowledge that are productive in learning the formal definition.

CHAPTER 9: DISCUSSION, CONCLUSION AND IMPLICATION

This chapter connects the findings from the previous analysis chapters with the broader literature. It also discusses connections among the findings from the different chapters to provide additional insight into the process of making sense of the temporal order for students. I start with a summary of the findings and revisit to the two research questions I posed in Chapter 1:

1. How do students make sense of the temporal order of delta and epsilon?
2. How does the Pancake Story influence students' understanding of the temporal order?

I close with some implications for research and the teaching of the formal definition of a limit.

Summary of Findings

Chapter 5 documents students' struggle with the temporal order of delta and epsilon. The majority of students in the study were not able to answer one question about the temporal order correctly. This confirms the findings from the literature that students have difficulty in making sense of the temporal order. Moreover, students' claims about the temporal order were not stable across contexts.

Students used a variety of reasoning patterns to justify their claim about the order. Most of these patterns came from interpretations of the statement of the definition and students' (partial) recall of the epsilon-delta proof procedure. Two reasoning patterns were most common among the students. First, many students concluded that epsilon depended on delta as a result of using the knowledge resource *functional dependence* because delta was linked with x and epsilon was linked with $f(x)$. Second, many students argued that delta came first because the if-then statement of the definition said that delta had to be satisfied first.

The microgenetic analysis of Adam's learning episode in Chapter 6 reveals details about the process of making sense of the temporal order. The chapter reveals various knowledge resources that were at play across the three focus segments (for the glossary of all the knowledge resources, see Table 6.1 in Chapter 6, pp. 41–43). Some resources were mathematical (*functional slots, functional dependence, proportional variation*) and others were more intuitive (*absolute condition, determining, givenness, quality control*). A couple of resources were specific to the topic of limit (*domain constraint for a limit, dynamic definition of a limit*). The priorities of each of these resources changed as Adam considered different arguments about the temporal order. See Figure 6.5 for a summary of the activation of knowledge resources (p. 60).

Adam's interpretation of the meaning and the roles of delta and epsilon varied as his understanding of the temporal order developed. For example, epsilon started as a constraint for a range of acceptable output values, but it became the difference in $f(x)$ and $f(a)$ in the next segment. The *determining* resource was cued consistently across the segments and it helped define the role of epsilon and delta (for a summary, see Table 6.2, p. 59). The changes in the specifics of delta and epsilon might be an indication of the formal definition as a *hyper-rich* learning context that also demands high level of cognitive work.

Chapter 7 measures the impact of the Pancake story on students' understanding of the temporal order of delta and epsilon. There was a general shift for students toward being able to answer the majority of the questions about the temporal order questions correctly. There was some evidence for the stability of students' claim about the temporal order after the story.

The number of reasoning patterns decreased after students engaged with the story. In particular the number of unproductive reasoning patterns decreased after students engaged with the story. The story did not suppress students' prior knowledge. Some of the reasoning patterns from before the story reappeared, including a few of the unproductive ones. All the students took up the idea of the *givenness* of epsilon from the story, though they used it in very different ways to support their claim. The resource *quality control* was also taken up by many of the students to make sense of the temporal order.

The case study in Chapter 8 illustrates how one student, Adriana aligned productive knowledge resources from the story with her prior knowledge. She effectively used the *givenness* of epsilon and *quality control* to determine the temporal order. She distinguished between error bounds from errors and repurposed *functional dependence* to describe the dependence of the errors. She also used the language of the story to explain the changes in her reasoning. The data suggests that her claim about the temporal order was stable by the end of the interview. I now discuss these findings vis-à-vis the two research questions this dissertation aims to answer.

A Return to the Research Questions

Students' Understanding and Sense-making of the Temporal Order of Delta and Epsilon

Students clearly struggled with the temporal order of delta and epsilon. Students had a lot of ideas about the temporal order. We see this through the large number of reasoning patterns documented in Chapter 5. Adam's and Adriana's case in Chapter 6 and 8 also document the diversity of ideas they had about the temporal order.

As I argued in the literature review in Chapter 3, perspective on the diversity of prior knowledge is very important. Discounting the table of reasoning patterns in Chapter 5 as misconceptions, and not a transitional form of an understanding would be shortsighted. These reasoning patterns reveal useful information about students' understanding of the temporal order and the formal definition more generally.

Consider a common reasoning pattern that Knapp and Oehrtman (2005) also found in their study: "First we resolved δ then we go on to resolve ε." Many students based that claim on their interpretation of the if-then statement. This tells us that students tend to focus on the if-then statement of the definition, over the for-every statement of the definition, to establish the relationship between epsilon and delta. This points to the necessity of making salient the first part of the definition in instruction.

At the same time the reasoning pattern also shows that students had minimal issue in interpreting the if-then part of the statement. The idea of verifying the delta inequality, then moving on to verifying the epsilon inequality seems accessible for students, and it is a part of the normative understanding of the formal definition. It is equally important to recognize what students understand, as well as what they have not yet understood.

The functional dependence argument was a very common reasoning pattern. The reasoning pattern suggests that many students linked delta and epsilon to x and $f(x)$ values, respectively. How and why students linked epsilon and delta to x and $f(x)$ values needed further exploration. In fact, each documented reasoning pattern in Chapter 5 says something about the way that students make sense of the temporal order of delta and epsilon and the formal definition more generally. This was one of the goals of the analysis in Chapter 6: to unpack some of the reasoning patterns to reveal the underlying knowledge resources.

The analysis reveals a number of knowledge resources that Adam drew upon in discussing the temporal order of delta and epsilon. Adam, like many students from Chapter 5, at some point said that delta was the interval of x values, and epsilon was the difference between $f(x)$ and $f(a)$ (*function slots*). We learn that he did so to accommodate an argument that uses *functional dependence* resource. However, as Adam considered his other arguments about the temporal order, the meaning and role of epsilon and delta changed accordingly. His use of *function slots* and *functional dependence* suggest students' general predisposition toward explicit functional relationship to describe a relationship between two quantities.

Adam's case also shows that the complexity of the formal definition as a concept might amplify the complexity of dealing with its components (e.g., inequalities or conditional statements). The changes in specificities of epsilon and delta across the segments reflect Adam's struggle in considering competing arguments about the temporal order. At some point, Adam even rejected a normative interpretation of a conditional statement to favor the *absolute condition* resource (see Segment 3(b) in Chapter 6). Yet he was able to use the normative interpretation of the conditional statement in a different segment (see Segment 5 in Chapter 6).

The analysis also reveals that delta normatively took on a role of a *determiner* (of the acceptable interval for *x*), and a *determined* (by epsilon). Delta's dual role contrasts with epsilon's consistent role as a *determiner* of delta and the acceptable range of output values. I posit that the dual role of delta might be another layer of complexity that students need to uncover and understand. Students' understanding of role of delta in the definition warrants further investigation.

In addition to being complex, the formal definition is also a *hyper-rich* learning context (diSessa, 2002). Students had a lot of ideas that appeared equally productive to them. Deciding which knowledge resource to prioritize at any given point was challenging, particularly when many ideas were present. Adam and Adriana had various—at times competing—arguments about the temporal order. For example, both Adam and Adriana had productive ideas about the formal definition. Adam recalled an epsilon-delta proof, and Adriana took up many productive ideas from the story. It was not immediate that they gave priority to these productive ideas, particularly when they had other ideas that were also viable to them.

In sum, I found that students struggled with the temporal order of delta and epsilon. However, the struggle could not simply be characterized as a struggle in understanding a single topic, (e.g., a reversal of function process, or absolute value), as many in the literature suggest. Each of the components of the formal definition is complex, but the formal definition as a topic is conceptually difficult and it increases the difficulty of dealing with the individual component.

These findings may seem to suggest that any effort to teach the formal definition would be futile. Fortunately, the Pancake Story proved to be helpful in assisting students to make sense of the temporal order. What have we learned about ways that students could come to understand the temporal order of delta and epsilon from the success of the story and other analyses in this dissertation?

Progress with the Temporal Order of Delta and Epsilon

"A given error bound" and "quality control" are productive intuitive resources. A working hypothesis for this dissertation is that relevant and important intuitive knowledge resides in everyday understanding of *quality control*. The take-up of this intuitive idea by many students suggests that it was made salient by the story and it was accessible for students. At the same time the fact that it was a common reasoning pattern that was used to argue for the correct temporal order shows its relevance to the understanding of the formal definition more broadly. I posit that the notion of *quality control* can also help organize the different ideas in the formal definition and reduce some of the complexity.

The analysis also suggests that the story made the *givenness* of epsilon more salient to students. The boss' giving a constraint on the pancake size was an accessible idea for students to recognize the *givenness* of epsilon in the definition. The reasoning patterns in Chapter 7 suggest that some students used the particular knowledge resource to determine the temporal order directly. Others combined it with other resources like *quality control* to conclude the correct temporal order. A deeper analysis of the case of Adriana reveals that this resource instigated a very important conflict that drove the change in her conception of the temporal order.

Students' conception of epsilon and delta is crucial in understanding the temporal order. The story was designed to help students make meaning of epsilon and delta. The story's use of the terms "error" and "error bounds" was intended to help students create a distinction between δ and $|x-a|$, and between ε and $|f(x)-L|$. Three students used a reasoning pattern that directly mentions this distinction (Epsilon comes first because ε is given and δ is not $|x-a|$ and ε is not $|f(x)-L|$). Most of the students who used the idea of *quality control* also thought of epsilon and delta as error bounds or constraints. Moreover, in the case of Adriana, creating the distinction was crucial for her to move toward the correct temporal order. These findings suggest that distinguishing errors from error bounds might be productive in giving students access into conceptualizing and specifying the meaning of delta and epsilon.

The process of creating the distinction between error and error bound, or between δ and $|x-a|$, and between ε and $|f(x)-L|$ was not trivial. While Adriana seemed to have a good grasp of the difference earlier in the interview, it took some work for her to use it productively in the context of discussing the temporal order. Similarly, the case of Adam also shows that even when a student knows that epsilon was a constraint for the acceptable output values, different arguments might favor a particular interpretation of delta and epsilon. Thus, the relationship between students' conception of epsilon and delta and their understanding of the temporal order is not one directional. They inform and support each other.

Everyday language can provide access to concepts. As I argue at the end of Chapter 8, the Pancake Story's everyday language provided access into concepts that otherwise would be challenging for students. Adriana was able to make sense of various quantities within the definition using the language of the story. For example, at one point she used a familiar example of not wanting to make the error bound for the batter to be one hundred pounds to emphasize the need for a small delta. Moreover, Adriana also made productive inferences from the story. For example, she inferred that since the story did not give an error bound for the batter, then delta was also not given. She was able to correctly map the error bound for the batter to delta. She then used it, along with other productive resources to conclude the temporal order in the context of the formal definition.

Productive resources are not sufficient. Adriana's case also illustrates a very important point about learning from instruction. Her case clearly shows that knowing productive resources is not sufficient to change a student's claim about the temporal order. Adriana needed to align all the productive resources with her prior knowledge (Schoenfeld et al, 1993; Smith et al., 1993). Adriana had the exact productive argument from the story about the temporal order based on *quality control* and *givenness* of epsilon, but it was not until after she found a productive context to apply the *functional dependence* resource that she was able to prioritize the argument.

Adam's case supports this claim. Adam had many productive resources from instruction (e.g., knowledge about epsilon-delta proof procedure, his facility in working with intervals). He concluded the correct temporal order for delta and epsilon using productive resources a number of times during the episode. However, competing resources emerged as he discussed the temporal order. It took him some time to align all the different resources before he prioritized some of the more productive resources and concluded the correct temporal order for delta and epsilon. Both Adam's and Adriana's case confirm that learning happens as a result of reorganizations of prior knowledge, along with additions of knowledge elements from instruction, not by replacement of misconceptions (Smith et al, 1993).

Procedural understanding of the epsilon delta proof can be a productive resource. Adam and many other students relied on their recall of an epsilon-delta proof to arrive at the correct temporal order. This suggests that the procedural understanding of the epsilon-delta proof that Oehrtman (2008) lamented might be a productive first step in learning the formal definition. The prevalence of this knowledge in students' reasoning patterns to conclude the correct temporal order shows the productivity of this resource for many students (see Table 5.2 in Chapter 5, p. 34). The iterative process between a procedural and a conceptual understanding has been suggested in the broader mathematics education literature (e.g., Rittle-Johnson, Siegler and Alibali, 2001).

Limitations and Recommendations

The study presented in this dissertation is not without its limitations. I document them here and discuss some ways to address them in future studies. The first limitation is the length of the interviews I conducted with students. Quite a few students reported that after the first hour thinking about the formal definition, they were quite drained. The extreme case of this was the student, Ryan, who opted not to change answers that did not make sense to him because he was tired from thinking. Future studies of the formal definition, given its cognitive demand, should consider conducting the interviews over two or more days.

The changes in understanding reflected in this study were of students who had previously received instruction on the formal definition of a limit. In this way, the students' prior knowledge, however limited it might be, should be considered in interpreting the success of the pancake story. For example, it just wouldn't be the case that all calculus students would have knowledge about the epsilon-delta proof procedure. There was a student that I interviewed who had not received instruction on the formal definition. I included him in the study because there was very little difference in the way that he initially made sense of the formal definition compared to other students who had seen it before. His case suggests that there are potential benefits of using the story with students who have not seen the formal definition before.

I also did not analyze the reasoning patterns of the comparison group who read a page from the textbook. This has the potential of contextualizing the findings in this study. This was the original design of the study that was not carried out in the analysis. At the same time, given the small number of students in the comparison group, it might have been difficult to infer reasoning patterns from the six students.

Implications for Research and Practice

The Pancake Story as a Discussion Tool

The findings in this dissertation show that the Pancake Story can help leverage many productive intuitive ideas to help students learn the formal definition of a limit. While the focus of the discussion in this dissertation is on the influence of the story on students' understanding of the temporal order, at the end of Chapter 7, I include a brief excerpt where a discussion about the arbitrariness of epsilon happened as a result of understanding that epsilon was a constraint from the story. It suggests that the story has the potential of enlightening students about different aspects of the formal definition, not just the temporal order.

The goal of the story is not to teach the formal definition, per se. While it was used as an instructional treatment in the study, the students in this study had seen the formal definition of a limit before. I imagine the story would be helpful as a way to discuss the formal definition. Students can gain access into the various concepts involved in the formal definition through the story's use of everyday language and intuitive ideas, as many students did in this study. The story can be used as a tool to help students make sense of some of the subtle parts of the formal definition in conjunction with other means of understanding the formal definition.[50]

Students' Understanding of the Arbitrariness of Epsilon

Several studies in the literature (Roh, 2009; Swinyard, 2011; Swinyard & Larsen, 2012) suggest that understanding the temporal order is a precursor to understanding the arbitrariness of epsilon. This dissertation did not specifically investigate the relationship between understanding the two topics. However, as part of the protocol, I asked students about why we needed epsilon to be arbitrary in the definition. Preliminary analysis of the students' responses to the question before and after the Pancake Story suggests that students' understanding about the arbitrariness of epsilon is limited. Most students said that they did not know why epsilon needed to be arbitrary, and I did not observe any noticeable difference in their thinking after the story. Two students guessed that the reason for epsilon's being arbitrary was to be able to include the limit of all possible *functions*. One student in the comparison group associated the arbitrariness of epsilon with the idea of smaller and smaller neighborhood around the limit.

I offer a couple of hypotheses to be considered in future studies about students' understanding of the arbitrariness of epsilon. First, the fact that most students could not provide any response to the question might be a reflection of the limited resources students had available to make sense of the arbitrariness of epsilon. Students had different parts of the statement of the definition (e.g., the if-then statement) to help them make sense of the temporal order; such help is limited for the arbitrariness of epsilon. Moreover, students need to understand epsilon before they could understand the arbitrariness of it. The findings from this dissertation show that the process of understanding the meaning and the role of epsilon in the definition was not trivial. This fact might also contribute to students' difficulty with the arbitrariness of epsilon.

The second hypothesis is that the fact that students learn the temporal order before they learn the arbitrariness of epsilon might be a result of the limited resources hypothesis. Considering the limited resources to help make sense of the idea, understanding the temporal order might serve as a motivation, or perhaps a way to enter into a discussion about the arbitrariness of epsilon. Just like students in Swinyard's (2011) study, having understood the importance on focusing on the y values (*y-first conception*), students might then be able to begin to explore the idea of how to conceptualize being as close as possible to the limit. The nature of the order of learning the two ideas within the definition merits further investigation.

Reducing the Cognitive Demand of the Written Definition

The findings from this study also suggest that one way to assist students in understanding the formal definition of a limit is to reduce the cognitive demand of the written definition. Fernández (2007) offered a suggestion with her version of the formal definition and her students found that useful. I speculate that the changes she made helped reduce some notational burdens, which allowed students to focus on some of the conceptual subtleties of the definition.

Building on this idea, I suggest incorporating the word "given" into the definition to include the intuitive idea of *givenness*. The word "constraint" or "bound" can also attach some meaning

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[50] Wolfram Demonstrations Project (demonstrations.wolfram.com) offers different modules that can show a graphical representation of the formal definition.

to epsilon and delta that distinguished them from the errors $|x-a|$ and $|f(x)-L|$ might provide additional support for students. One version of the definition might say:

> The limit of a function $f(x)$ as x approaches *a is L* if and only if, for any *given* constraint (or error bound) on the output, $\varepsilon > 0$, there exists a constraint on the input, $\delta > 0$, such that if $0 <| x - a |< \delta$ then $| f(x) - L |< \varepsilon$.[51]

It remains to be seen if these modifications embody some of the intuitive ideas expressed in the story, and/or if it has impact on students' understanding of the formal definition.

Microgenetic Study and Methods

I set out to take an extremely close look at students' understanding of the formal definition. The goal was to understand the knowledge resources students have about the formal definition and the process of their development in learning. The analysis in this dissertation is a first look into ways that students put these resources together. Considering the findings in this dissertation, I argue that this type of study is important particularly for understanding students' understanding of persistently difficult topic in mathematics.

The combination of competitive argumentation (Schoenfeld, Smith and Arcavi, 1993, VanLehn, Brown and Greeno, 1984) and the use of the *counter-models* is a productive analysis method to identify the different knowledge resources. It was also helpful in constructing a model for Adam's argument about the temporal order. Together with the theoretical assumptions about knowledge resources, and some methodological orientations, the analysis reveals the complexities and subtleties in the use of the knowledge resources. The methodology used in this dissertation was able to detect some the subtle changes in students' reasoning, and reject interpretations that did not capture the full complexity of the process, however reasonable they might appear.

Given the complexity of Adam and Adriana's sense making, I do not claim generality in the way that the students put together the resources at their disposal. The goal of this type of study is to uncover the complexities of students' thinking. Students, like Adam and Adriana think in very particular ways, and use knowledge resources in unique ways. While common resources might emerge, the goal is to understand the specifics of their use. Generality might happen at the theoretical level. That is, the knowledge resources identified in this study might be found in learning episodes in other contexts. Future studies can examine the degree to which the knowledge resources found in the analysis of this dissertation are used by other students, and in other topics in mathematics.

Concluding Thoughts

The formal definition of a limit is complex, and it requires a great deal of focus and attention for students to understand. It is challenging by way of quantifiers and inequalities, but also by the amount of information that students need to keep track of as they make sense of it. The abundant relevant prior knowledge provides an additional challenge to the process of making sense the temporal order. At the same time, students' existing prior knowledge includes many productive knowledge resources that can be leveraged in instruction.

I posit that unveiling students' knowledge resources should be the work of both instruction

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[51] I found that the absolute value notation $|x-a|$ and $|f(x)-L|$ are helpful for students to conceptualize them as errors and compare them to epsilon and delta.

and research. The more we successfully unpack the process, the closer we are to assisting students in organizing the different resources that they might have about the topic. Any instructional approach has to be sufficiently flexible to incorporate—or better, leverage—productive knowledge resources in students' prior knowledge.

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APPENDIX A: The Formal Definition of Limit at a Point and an Example

Definition. Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then we say that the **limit of** $f(x)$ **as** x **approaches** a **is** L, and we write

$$\lim_{x \to a} f(x) = L$$

if for every number $\epsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if} \quad 0 < |x - a| < \delta \quad \text{then} \quad |f(x) - L| < \epsilon$$

Example. Prove that $\lim_{x \to 1} 3x + 2 = 5$

Solution:

1. *Preliminary analysis of the problem (guessing a value for* δ*).*

Let ϵ be a given positive number. We want to find a number δ such that

$$\text{if} \quad 0 < |x - 1| < \delta \quad \text{then} \quad |(3x + 2) - 5| < \epsilon.$$

But $|(3x + 2) - 5| = |3x - 3| = |3(x - 1)| = 3|x - 1|$. Therefore, we want

$$\text{if} \quad 0 < |x - 1| < \delta \quad \text{then} \quad 3|x - 1| < \epsilon$$

that is,

$$\text{if} \quad 0 < |x - 1| < \delta \quad \text{then} \quad |x - 1| < \frac{\epsilon}{3}.$$

This suggests that we should choose $\delta = \epsilon/3$.

2. *Proof (showing that this* δ *works).* Given $\epsilon > 0$, choose $\delta = \epsilon/3$. If $0 < |x - 1| < \delta$, then

$$|(3x + 2) - 5| = |3x - 3| = |3(x - 1)| = 3|x - 1| < 3\delta = 3\left(\frac{\epsilon}{3}\right) = \epsilon.$$

Thus

$$\text{if} \quad 0 < |x - 1| < \delta \quad \text{then} \quad |(3x + 2) - 5| < \epsilon.$$

Therefore, by the definition of a limit,

$$\lim_{x \to 1} 3x + 2 = 5.$$

APPENDIX B: The Pancake Story

You work at a famous pancake house that's known to make pancakes with 5” diameter. To make the perfect 5” pancake you would use exactly 1 cup of batter. On your first day of work your boss told you that it is practically impossible for you to be able to use exactly one cup to make the perfect 5 inches given how many and how fast you will be making these pancakes. So for now, since you’re new, as long as your pancakes are anywhere within ½” from the 5”, he won’t fire you. Your job is then to figure out the maximum you can be off from the 1 cup to still make pancakes that meet your boss’ standard. Specifically, given that your boss gave you the ½ an inch error bound for the size, you need to figure out the error bound for the batter so that your pancakes won’t be off more than the given error bound.

According to the work manual, there are two steps to do this. Based on the error bound for the size, you first need to guess an error bound for the amount of batter. THEN, you have to check to see if using any amount of batter that is within the error bound from the 1 cup would make pancakes that are within the given error bound from the 5”.

For example, suppose based on the ½ inch error bound, you guessed ⅙ of a cup error bound for the amount of batter. Then you check to see if using any amount of batter that is within ⅙ of a cup from the 1 cup, so between ⅚ and 1⅙ of a cup would make pancakes with size somewhere between 4½” and 5½”, that is within the ½” error bound from 5”.

Over time, your boss expects you to be even more precise. So instead of ½” error bound from 5”, he says he wants you to make pancakes that are within some ridiculously small error bound from 5”, but you don’t know what it’s going to be. This means while he started by asking you to be within ½,” later he might want ¼” or $^{1}/_{1000}$” from 5”. Your job then becomes for however close your boss wants the pancake to 5”, you need to figure out the maximum you can be off from 1 cup of batter such that if you use any amount of batter that is within that error bound from the 1 cup then your actual pancakes will still be within whatever error bound your boss gives you from the 5”.

Now, you don’t want to spend time each morning to recalculate everything. So you will try to come up with a way to calculate an error bound for the batter based on whatever the given error bound for the size.

APPENDIX C: The Bolt Problem

from Boester (2008)

Suppose we run a bolt manufacturing company. We have lots and lots of contracts, with lots and lots of different companies. As you might expect, everybody's needs are a little different. Bolts that we provide for home construction have to be of good quality, whereas bolts that we provide for NASA to be used on the space shuttle have to be of exceptional quality. For the sake of simplicity, let's look at only one variable that goes into the quality of our bolts: length. Bolts for home construction that are supposed to be, say, four inches long can be a little more or a little less than four inches. But bolts for the space shuttle that are supposed to be four inches long have to be within a much smaller target range in order to be acceptable. The length of the bolt depends directly on how much raw material we put into the bolt making machine (assuming that the diameter of the bolts for the home and for NASA are the same), according to some function.

How do we create bolts that we know will be of a length that falls within our target range?

APPENDIX D: Student Background Survey

Student Background Survey

Name: _______________________

1. How much do you agree with these statements?

	Strongly agree	Agree	Disagree	Strongly disagree	N/A
I'm certain I can master the skills taught in a math class.					
I'm certain I can figure out how to do the most difficult class work in a math class.					
I can do almost all the work in a math class if I don't give up.					
Even if the work is hard, I can learn it.					
I can do even the hardest work in a math class if I try.					

2. When **and** where did you take first semester calculus class? ____

3. Please rate your first semester calculus class:

Very challenging	Challenging	Moderately challenging	Not challenging at all	N/A

4. What grade did you receive for it? If you took the AP test, please indicate what score you received **and** for which course (AB or BC).

____

4. When did you take Real Analysis? ____

5. Please rate your Real Analysis class:

Very challenging	Challenging	Moderately challenging	Not challenging at all	N/A

6. What grade did you receive for it? ____

7. What is your class standing? ____

8. What is your intended major? ____

9. What is your self-identified gender? (Select all that applies)

Male	Female	Transgendered	Other: ___________	Decline to state

10. How would you describe your race/ethnicity? (Please circle all that applies)

African American	Asian /Asian American	Hispanic/ Latino	Middle Eastern	Native American or Alaska Native	Native Hawaiian or Pacific Islander	White (Non-Hispanic)	Other	Decline to state

APPENDIX E: Interview Protocol

PART I: TASKS AND DEFINITION OF LIMIT

The Task [see attached].

"The reason why we're here today is that I want to talk to you about limits and specifically its formal definition or the epsilon delta definition. Turns out that the research literature on limit agrees that learning and teaching limit and the formal definition is difficult. To this day I'm still struggling to find the best way to present the material. So today I want to hear how you make sense of limit and the formal definition. Even if things don't make sense for you, it's still helpful. In a way, your confusion or getting things wrong is still really helpful to me in figuring out how and why this is difficult. Now there will be times when I will be asking you the same question multiple times or similar questions, this is not suggesting that you're wrong. It's just my way of making sure that I really understand you. The ultimate goal is for me to design a better instruction in the future.

With that I have some problems that I want you to take a look and use to refresh your memory. I want you to try to think out-loud as you're looking at this problem (see Task Sheet).

Do you have any questions about the purpose of this interview or what I am expecting from you?"

Definition of Limit

16. Now that you've solved those problems, I want you to think about 'what is a limit' to you. So what do you think limit is? "What does it mean for say example 1 that the limit is 5?"

17. Why do you think a limit of certain function exist while others' do not?
    Follow up: Can you give an example where a function whose limit DNE?

PART II: EPSILON DELTA DEFINITION OF LIMIT

Epsilon Delta Definition of Limit

So there's this thing called the epsilon delta definition of limit. Here you have the statement (see Formal Definition of Limit of a Function)

A. General questions about epsilon delta.

18. Before we get into what the statement means, what do you think this statement is for? OR The purpose for it OR Why do we have it?

19. What do you think we're trying to do with the definition? Is there anything we're trying to figure out or trying to show is true?

20. Can you please try your best to explain what this definition says in the context of example 1?

21. Is there a way that you can show what this definition says graphically?
    If already drawn: Can you explain what you drew, there?

22. Is this related at all to what you said about what limit was to you? You said [insert response here]. Are they related? If so, how?

B. Specific questions about parts of the definition

23. What do you think epsilon is?

24. What about delta?
    Follow up: Can you show where epsilon and delta are on the graph?

25. What about "for all epsilon greater than zero, there exists delta greater than zero"? What is that for?

26. 0<|x-a|<delta? What do you think that means?

27. |f(x)-L|<epsilon? What do you think that means?

28. "If 0<|x-a|<delta then |f(x)-L|<epsilon." Do you have a sense as to what's the if/then for?

29. Why do you think that here we have the greater than zero, but not there?

C. Specific questions about epsilon and delta. Now I am going to ask you some specific questions about epsilon and delta, and after each question I am going to ask how sure you are of your answer.

30. In the definition, with epsilon and delta, what depends on what, if anything you think? Delta depends on epsilon? Epsilon depends on delta? They depend on each other? Or they do not depend on each other? And why?
    Follow up: Where did you get that from? OR How does that relate to your idea that ____ depends on _____?
    Rating: If you were to rate how sure you are from 1 to 5, 1 being a guess and 5 being no doubt in your mind, how would you rate what you said about ________depending on __________?

31. In the definition, between x and f(x), which one do you think, you are trying to control? And why?
    Follow up: If f(x), what is the role of epsilon and delta, if any?
    Follow up: If x, what is the role of epsilon and delta, if any?
    Rating: If you were to rate how sure you are from 1 to 5, 1 being a guess and 5 being no doubt in your mind, how would you rate what you said about ________is the thing you're trying to control and the roles of delta and epsilon?

32. In the definition, between epsilon and delta, which one do you think comes first and which one do you figure out as a result? And why?
    Rating: If you were to rate how sure you are from 1 to 5, 1 being a guess and 5 being no doubt in your mind, how would you rate what you said about ________coming first?

33. In the definition, between epsilon and delta, which one do you think is set? Epsilon? Delta? Both? Or neither? And why?
    Rating: If you were to rate how sure you are from 1 to 5, 1 being a guess and 5 being no doubt in your mind, how would you rate what you said about ________being set?

34. How would you put the four variables, epsilon, delta, x and f(x) in order in terms of which comes first in the definition? And why?
    Follow up: Why did you order it that way?
    Follow up: In terms of the process within the definition, how would you put the four variables in order?
    Rating: If you were to rate how sure you are from 1 to 5, 1 being a guess and 5 being no doubt in your mind, how would you rate what you said about the order?

35. Option (a): Some students in the past have told me that epsilon comes first because it is given to you. What do you say to that?
    Option (b): Some students in the past have told me that delta comes first because delta is related to x, and x comes first. What do you say to that?
    Follow up: How does it fit with what you said earlier?

36. For some reason the definition starts with "For every number epsilon>0." A couple of questions about this:
    (a) Can you think of a reason or reasons why [we/they] start with epsilon?
    (b) Why do you think [we/they] need "for every" epsilon?

D. The Pancake Story. At this point I will share with you a Pancake Story. You have it in front of you. Feel free to follow along and take notes, as you'd like.
OR
D. Standard Text. At this point I would like you to read the following text taken from a textbook, and afterwards I am going to ask you some questions about the formal definition again.

Pancake story.

You work at a famous pancake house that's known to make pancakes with 5" diameter. To make the perfect 5" pancake you would use exactly 1 cup of batter. On your first day of work your boss told you that it is practically impossible for you to be able to use exactly one cup to make the perfect 5 inches given how many and how fast you will be making these pancakes. So for now, since you're new, as long as your pancakes are anywhere within ½" from the 5", he won't fire you. Your job is then to figure out the maximum you can be off from the 1 cup to still make pancakes that meet your boss' standard. Specifically, given that your boss gave you the ½ an inch error bound for the size, you need to figure out the error bound for the batter so that your pancakes won't be off more than the given error bound.

According to the work manual, there are two steps to do this. Based on the error bound for the size, you first need to guess an error bound for the amount of batter. THEN, you have to check to see if using any amount of batter that is within the error bound from the 1 cup would make pancakes that are within the given error bound from the 5".

For example, suppose based on the ½ inch error bound, you guessed ⅙ of a cup error bound for the amount of batter. Then you check to see if using any amount of batter that is within ⅙ of a cup from the 1 cup, so between ⅚ and 1⅙ of a cup would make pancakes with size somewhere between 4½" and 5½", that is within the ½" error bound from 5".

Over time, your boss expects you to be even more precise. So instead of ½" error bound from 5", he says he wants you to make pancakes that are within some ridiculously small error bound from 5", but you don't know what it's going to be. This means while he started by asking you to be within ½," later he might want ¼" or $^{1}/_{1000}$" from 5". Your job then becomes for however close your boss wants the pancake to 5", you need to figure out the maximum you can be off from 1 cup of batter such that if you use any amount of batter that is within that error bound from the 1 cup then your actual pancakes will still be within whatever error bound your boss gives you from the 5".

Now, you don't want to spend time each morning to recalculate everything. So you will try to come up with a way to calculate an error bound for the batter based on whatever the given error bound for the size.

37. As an employee what is your job?
    Follow up: Is there a specific quantity you're trying to figure out?

38. There are four quantities that are changing in the story, what might they be?

39. Which quantity or quantities are you given?

40. Which quantity or quantities are you trying to figure out? Why?

41. Does the story provide steps for you to do that? If so, what?

42. Between, the error bound for the batter and the error bound for the size, which one comes first in the story and which do you figure out as a result?
    Follow up: How does it compare to the first step in the manual?

43. How do you think the error bound for the batter and the error bound for the size are dependent on each other, if at all? Why?
    Follow up: How does it compare to what you said in the previous question?

44. How do you think the *error* in the batter and the error in size of the pancake are dependent on each other, if at all? Why?

45. Do you think there is a difference between an error and an error bound in the story? If so, what?

46. Why were you not given a particular error bound for the size towards the end?

47. How do you deal with the fact that your boss might give you different error bounds in the size, in finding the error bound for the batter?

48. Do you think the story had anything to do with the formal definition? If so, how?

49. If you try to relate this story to the formal definition:
    (a) What do you think are the following quantities: x? f(x)? a? L? |x-a|? |f(x)-L|? epsilon? delta?
    (b) "For all epsilon there exists a delta," what do you think that is in the story?
    Follow up: What was epsilon again? AND/OR What was delta?
    (c) What about the "if-then" statement?

(d) Why "for all epsilon"?

50. Is there anything that is still unclear to you at this point about the story?

E. Post story sense making.

At this point I will be asking some final questions. You answered a lot of these already but I want to ask them again just in case any of them has changed since we last talked about it. Fee free to say the same thing if it has not.

51. (a) Can you try and explain what the formal definition is saying in the context of example 1?
(b) Did your explanation change at all and if it did, why?

52. (a) What is it that you're trying to do within the formal definition? Is there anything we're trying to figure out?
(b) Did your explanation change at all and if it did, why?

53. (a) With epsilon and delta, what depends on what, if anything?
(b) Did you change your mind? If you did, why?
Rating: If you were to rate how sure you are from 1 to 5, 1 being a guess and 5 being no doubt in your mind, how would you rate what you said about ________depending on __________?

54. (a) In the definition, between x and f(x), which one do you think, you are trying to control? And why?
Follow up: If f(x), what is the role of epsilon and delta, if any?
Follow up: If x, what is the role of epsilon and delta, if any?
Rating: If you were to rate how sure you are from 1 to 5, 1 being a guess and 5 being no doubt in your mind, how would you rate what you said about ________is the thing you're trying to control and the roles of delta and epsilon?
(b) Did you change your mind? If you did, why?

55. (a) Between epsilon and delta, which one comes first and which one do you figure out as a result?
(b) Did you change your mind? If you did, why?
Rating: If you were to rate how sure you are from 1 to 5, 1 being a guess and 5 being no doubt in your mind, how would you rate what you said about ________coming first?

56. (a) In the definition, between epsilon and delta, which one do you think is set? Epsilon? Delta? Both? Or neither?
(b) Did you change your mind? If you did, why?
Rating: If you were to rate how sure you are from 1 to 5, 1 being a guess and 5 being no doubt in your mind, how would you rate what you said about ________being set?

57. (a) How would you put the four variables, epsilon, delta, x and f(x) in order in terms of which comes first?
(b) Did you change your mind? If you did, why?
Rating: If you were to rate how sure you are from 1 to 5, 1 being a guess and 5 being no doubt in your mind, how would you rate what you said about the order?

58. (a) Why does the definition start with epsilon?
(b) Why do we need "for every number epsilon?"
(c) Did you change your mind? If you did, why?

59. (a) What is a limit to you?
(b) Did it change at all and if it did, why?

60. In what ways, if at all, does the pancake story influence how you think about the formal definition?

61. Does the formal definition meant to help you compute the limit? Why or why not?

62. Well given what you said, what does this have anything to do with limits?

Administer Background Survey

APPENDIX F: Interview Protocol for Pilot Study II

PART I: TASKS AND DEFINITION OF LIMIT

The Task [see attached].

"The reason why we're here today is that I want to talk to you about limits and specifically its formal definition or the epsilon delta definition. Turns out that the research literature on limit agrees that learning and teaching limit and the formal definition is difficult. To this day I'm still struggling to find the best way to present the material. So today I want to hear how you make sense of limit and the formal definition. Even if things don't make sense for you, it's still helpful. In a way, your confusion or getting things wrong is still really helpful to me in figuring out how and why this is difficult. Now there will be times when I will be asking you the same question multiple times or similar questions, this is not suggesting that you're wrong. It's just my way of making sure that I really understand you. The ultimate goal is for me to design a better instruction in the future.

With that I have some problems that I want you to take a look and use to refresh your memory. I want you to try to think out-loud as you're looking at this problem (see Task Sheet). Do you have any questions about the purpose of this interview or what I am expecting from you?"

Definition of Limit

1. Now that you've solved those problems, I want you to think about 'what is a limit' to you. So what do you think limit is? "What does it mean for say example 1 that the limit is 5?"

2. Why do you think a limit of certain function exist while others' do not?
   Follow up: Can you give an example where a function whose limit DNE?

PART II: EPSILON DELTA DEFINITION OF LIMIT

Epsilon Delta Definition of Limit

So there's this thing called the epsilon delta definition of limit. Here you have the statement (see Formal Definition of Limit of a Function)

A. General questions about epsilon delta.

3. Before we get into what the statement means, what do you think this statement is for? OR The purpose for it OR Why do we have it?

4. What do you think we're trying to do with the definition? Is there anything we're trying to figure out or trying to show is true?

5. Can you please try your best to explain what this definition says in the context of example 1?

6. Is there a way that you can show this graphically?
   If already drawn: Can you explain what you drew, there?

7. Is this related at all to what you said about what limit was to you? You said [insert response here]. Are they related? If so, how?

B. Specific questions about parts of the definition

8. What do you think epsilon is?

9. What about delta?
   Follow up: Can you show where epsilon and delta are on the graph?

10. What about "for all epsilon greater than zero, there exists delta greater than zero"? What is that for?

11. 0<|x-a|<delta? What do you think that means?

12. |f(x)-L|<epsilon? What do you think that means?

13. “If 0<|x-a|<delta then |f(x)-L|<epsilon.” Do you have a sense as to what’s the if/then for?

14. Why do you think that here we have the greater than zero, but not there?

C. Specific questions about epsilon and delta.

15. In the definition, with epsilon and delta, what depends on what, if anything you think? Delta depends on epsilon? Epsilon depends on delta? They depend on each other? Or they do not depend on each other? And why?
    Follow up: Where did you get that from? OR How does that relate to your idea that ____ depends on _____?
    Rating: If you were to rate how sure you are from 1 to 5, 1 being a guess and 5 being no doubt in your mind, how would you rate what you said about ________depending on __________?

16. In the definition, between x and f(x), which one do you think, you are trying to control? And why?
    Follow up: If f(x), what is the role of epsilon and delta, if any?
    Follow up: If x, what is the role of epsilon and delta, if any?
    Rating: If you were to rate how sure you are from 1 to 5, 1 being a guess and 5 being no doubt in your mind, how would you rate what you said about ________is the thing you’re trying to control and the roles of delta and epsilon?

17. In the definition, between epsilon and delta, which one do you think comes first and which one do you figure out as a result? And why?
    Rating: If you were to rate how sure you are from 1 to 5, 1 being a guess and 5 being no doubt in your mind, how would you rate what you said about ________coming first?

18. In the definition, between epsilon and delta, which one do you think is set? Epsilon? Delta? Both? Or neither? And why?
    Rating: If you were to rate how sure you are from 1 to 5, 1 being a guess and 5 being no doubt in your mind, how would you rate what you said about ________being set?

19. How would you put the four variables, epsilon, delta, x and f(x) in order in terms of which comes first in the definition? And why?
    Follow up: Why did you order it that way?
    Follow up: In terms of the process within the definition, how would you put the four variables in order?
    Rating: If you were to rate how sure you are from 1 to 5, 1 being a guess and 5 being no doubt in your mind, how would you rate what you said about the order?

20. Option (a): Some students in the past have told me that epsilon comes first because it is given to you. What do you say to that?
    Option (b): Some students in the past have told me that delta comes first because delta is related to x, and x comes first. What do you say to that?
    Follow up: How does it fit with what you said earlier?

21. For some reason the definition starts with “For every number epsilon>0.” A couple of questions about this:
    (a) Can you think of a reason or reasons why [we/they] start with epsilon?
    (b) Why do you think [we/they] need “for every” epsilon?

Pancake story.

You work at a famous pancake house that's known to make pancakes with 5” diameter. To make the perfect 5” pancake you would use exactly 1 cup of batter. On your first day of work your boss told you that it is practically impossible for you to be able to use exactly one cup to make the perfect 5 inches given how many and how fast you will be making these pancakes. So for now, since you’re new, as long as your pancakes are anywhere within ½” from the 5”, he won’t fire you. Your job is then to figure out the maximum you can be off from the 1 cup to still make pancakes that meet your boss’ standard. Specifically, given that your boss gave you the ½ an inch error bound for the size, you need to figure out the error bound for the batter so that your pancakes won’t be off more than the given error bound.

According to the work manual, there are two steps to do this. Based on the error bound for the size, you first need to guess an error bound for the amount of batter. THEN, you have to check to see if using any amount of batter that is within the error bound from the 1 cup would make pancakes that are within the given error bound from the 5”.

For example, suppose based on the ½ inch error bound, you guessed ⅙ of a cup error bound for the amount of batter. Then you check to see if using any amount of batter that is within ⅙ of a cup from the 1 cup, so between ⅚ and 1⅙ of a cup would make pancakes with size somewhere between 4½” and 5½”, that is within the ½” error bound from 5”.

Over time, your boss expects you to be even more precise. So instead of ½” error bound from 5”, he says he wants you to make pancakes that are as close as possible to 5”. This means while he started by asking you to be within ½ ” later he might want ¼” or 1/1000” from 5”. Your job then becomes for however close your boss wants the pancake to 5”, you need to figure out the maximum you can be off from 1 cup of batter such that if you use any amount of batter that is within that error bound from the 1 cup then your actual pancakes will still be within whatever error bound your boss gives you from the 5”.

Now, you don’t want to spend time each morning to recalculate everything. It would be nice if you can come up with a way to calculate an error bound for the batter based on whatever the given error bound for the size.

D. Questions about the Pancake Story.

22. As an employee what is your job?
    Follow up: Is there a specific quantity you’re trying to figure out?

23. There are four quantities that are changing in the story, what might they be?

24. Which quantity or quantities are you given?

25. Which quantity or quantities are you trying to figure out? Why?

26. Does the story provide steps for you to do that? If so, what?

27. Between, the error bound for the batter and the error bound for the size, which one comes first in the story and which do you figure out as a result?
    Follow up: How does it compare to the first step in the manual?

28. How do you think the error bound for the batter and the error bound for the size are dependent on each other, if at all? Why?
    Follow up: How does it compare to what you said in the previous question?

29. How do you think the error in the batter and the error in size of the pancake are dependent on each other, if at all? Why?

30. Do you think there is a difference between an error and an error bound in the story? If so, what?

31. Why were you not given a particular error bound for the size towards the end?

32. How do you deal with the fact that your boss might give you different error bounds in the size, in finding the error bound for the batter?

33. Do you think the story had anything to do with the formal definition? If so, how?

34. If you try to relate this story to the formal definition:
    (a) What do you think are the following quantities: x? f(x)? a? L? |x-a|? |f(x)-L|? epsilon? delta?
    (b) “For all epsilon there exists a delta,” what do you think that is in the story?
    Follow up: What was epsilon again? AND/OR What was delta?
    (c) What about the “if-then” statement?
    (d) Why “for all epsilon”?

35. Is there anything that is still unclear to you at this point about the story?

E. Post story sense making.

At this point I will be asking some final questions. You answered a lot of these already but I want to ask them again just in case any of them has changed since we last talked about it. Fee free to say the same thing if it has not.

36. (a) Can you try and explain what the formal definition is saying in the context of example 1?
    (b) Did your explanation change at all and if it did, why?

37. (a) What is it that you're trying to do within the formal definition? Is there anything we're trying to figure out?
    (b) Did your explanation change at all and if it did, why?

38. (a) With epsilon and delta, what depends on what, if anything?
    (b) Did you change your mind? If you did, why?
    Rating: If you were to rate how sure you are from 1 to 5, 1 being a guess and 5 being no doubt in your mind, how would you rate what you said about ________depending on __________?

39. (a) In the definition, between x and f(x), which one do you think, you are trying to control? And why?
    Follow up: If f(x), what is the role of epsilon and delta, if any?
    Follow up: If x, what is the role of epsilon and delta, if any?
    Rating: If you were to rate how sure you are from 1 to 5, 1 being a guess and 5 being no doubt in your mind, how would you rate what you said about ________is the thing you're trying to control and the roles of delta and epsilon?
    (b) Did you change your mind? If you did, why?

40. (a) Between epsilon and delta, which one comes first and which one do you figure out as a result?
    (b) Did you change your mind? If you did, why?
    Rating: If you were to rate how sure you are from 1 to 5, 1 being a guess and 5 being no doubt in your mind, how would you rate what you said about ________coming first?

41. (a) In the definition, between epsilon and delta, which one do you think is set? Epsilon? Delta? Both? Or neither?
    (b) Did you change your mind? If you did, why?
    Rating: If you were to rate how sure you are from 1 to 5, 1 being a guess and 5 being no doubt in your mind, how would you rate what you said about ________being set?

42. (a) How would you put the four variables, epsilon, delta, x and f(x) in order in terms of which comes first?
    (b) Did you change your mind? If you did, why?
    Rating: If you were to rate how sure you are from 1 to 5, 1 being a guess and 5 being no doubt in your mind, how would you rate what you said about the order?

43. (a) Why does the definition start with epsilon?
    (b) Why do we need "for every number epsilon?"
    (c) Did you change your mind? If you did, why?

44. (a) What is a limit to you?
    (b) Did it change at all and if it did, why?

45. In what ways, if at all, does the pancake story influence how you think about the formal definition?

46. Does the formal definition meant to help you compute the limit? Why or why not?

47. Well given what you said, what does this have anything to do with limits?

Administer Background Survey

APPENDIX G: Interview Protocol for Pilot Study I

PART 1: TASKS AND HOW TO FIND LIMIT

The Task [see attached].

"The reason why we're here today is that I want to talk to you about limits and specifically its formal definition or the epsilon delta definition. Turns out that the research literature on limit agrees that learning and teaching limit and the formal definition is difficult. To this day I'm still struggling to find the best way to present the material. So today I want to hear how you make sense of limit and the formal definition. Even if things don't make sense for you, it's still helpful. In a way, your confusion or getting things wrong is still really helpful to me in figuring out how and why this is difficult. Now there will be times when I will be asking you the same question multiple times or similar questions, this is not suggesting that you're wrong. It's just my way of making sure that I really understand you. The ultimate goal is for me to design a better instruction in the future.

With that I have some problems that I want you to take a look and use to refresh your memory. I want you to try to think out-loud as you're looking at this problem.

Do you have any questions about the purpose of this interview or what I am expecting from you?"

How to Find Limit

1. Now that you've done these problems, can you help me understand how do you go about finding limits generally?

2. "By the way, how do you know that the limit is not some other number?" or "How do you know that the limit exists and it's this number?"

3. Do you have other ways to determine that the limit is in fact that number?

Definition of Limit

1. "At this point I want you to think about 'what is a limit' to you. So what do you think limit is? "What does it mean for say example 1 that the limit is 5?"

2. "If you were to explain this to someone who's never taken calculus before. What would you say limit is?"

3. "What about if you're trying to convince your 1B instructor that you deserve to be in his/her class? What would you say?"

4. Can you explain why you would explain it in that way to the non-calculus student vs. the teacher?

5. Is there a difference between how to find a limit and what limit is for you?

PART 2: EPSILON DELTA DEFINITION OF LIMIT

Epsilon Delta Definition of Limit

"So there's this thing called the epsilon delta definition of limit. Here's you have the statement.

A. The purpose of epsilon delta.

1. "Before we get into what the statement means, can you tell me what this statement is for?" OR The purpose for it OR Why do we have it?
2. How do you go about doing that with the statement? Say, what would be your first step?

B. Exploring parts of the statement:

1. What is epsilon?

2. What is delta?

3. For all epsilon greater than zero, there exists delta greater than zero?

4. 0<lx-al<delta? What does that mean?

5. lf(x)-Ll<epsilon? What does that mean?

6. “If 0<lx-al<delta then lf(x)-Ll<epsilon.” Why the if/then?

C. Meaning making.

1. Can you please try your best to explain what the statement says?

2. Is this related at all to what you said about what limit was to you? You said [insert response here]. Are they related? If so, how?

3. Is there a way that you can show this graphically?

4. What does it mean to satisfy this statement?

5. [Follow-up] Can you tell me a situation where this statement would be false?

6. With epsilon and delta, what depends on what, if anything? Delta depends on epsilon? Epsilon depends on delta? They depend on each other? Or they do not depend on each other? WHY?

7. With x and y, what depends on what, if anything? x depends on y? y depends on x? They depend on each other? Or they do not depend on each other? WHY?

8. Of the four variables, epsilon, delta, x and y, which one(s), if any, do you have control over (i.e., you can change), and which one(s), if any, are you trying to control? WHY?

9. Option (a): Some students in the past have told me that what you’re trying to control is epsilon because you’re trying to get your f(x) close to L. What do you say to that?

   Option (b): Some students in the past have told me that what you’re trying to control is delta because epsilon is given to you. What do you say to that?

10. The definition is written in such a way where it starts with “For all epsilon>0.” Sometimes people say, for any epsilon greater than zero. Or for a given epsilon greater than zero. Why do we say this?

E. Pancake story.

You work at a famous pancake house that's known to make a specific size pancake, say 5 inches in diameter. To make it perfectly you would use exactly 1 cup of batter. Since you're new and will be making lots of pancakes very quickly, your boss allows for a certain amount of error in the pancake size, say plus or minus 1/2 an inch. Your job is to figure out how much off you can be from 1 cup of batter in order to make sure that your pancake is still within the error that your boss gave you.

Over time, your boss expects you to be even more precise. So instead of giving you a specific amount of error from 5 inches, he says he wants the pancake to be as close as possible to 5. Your responsibility is to figure out how far off you can be from 1 cup in order to make sure your pancake is as close as possible to 5 inches.

So now that you’ve heard the story, I want to ask you some questions about it.

1. In this story, what is your responsibility?

2. What amount are you trying to figure out?

3. What do you have control over as an employee?

4. What you are trying to control?

5. What are you given as you work there, (i.e., what is out of your control)?

6. If we try to relate this story to the formal definition, what do you think [x, y, f(x), delta, epsilon, a, L, if/then statement, why for all there exists, why absolute value?

F. Post story sense making.

At this point I will be asking some final questions. You answered a lot of these already but I just want to make sure that I really understand you.

1. Can you try and interpret what the formal definition is saying? Or what it means?

2. [I want to briefly ask you some questions about a written proof here, and ask you some questions about why they do the steps.]

3. With epsilon and delta, what depends on what, if anything? [Did you change your mind, why?]

4. Of the four variables, epsilon, delta, x and y, which one(s) do you have control over (i.e., you can change), and which one(s) are you trying to control? [Did you change your mind, why?]

5. Why are we given epsilon? [If so, is getting as close as possible to L and assumption or a goal?]

6. Does the formal definition meant to help you compute the limit? Why or why not?

Well given what you said, what does this have anything to do with limits?

APPENDIX H: Interview Task

Interview Tasks

Fall 2011

Name: ____________________________

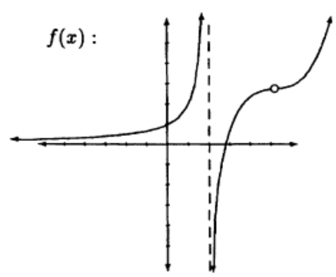
INSTRUCTION: Please find the limit of a the following functions, if it exists.

1. $\lim_{x\to 1} 3x+2$

2. $\lim_{x\to 3} \dfrac{x^2-9}{x-3}$

3. $\lim_{x\to\infty} \dfrac{\ln x}{x^3}$

4-6 Let f be the function whose graph is presented below.



4. $\lim_{x\to 2} f(x)$

5. $\lim_{x\to 1} f(x)$

6. $\lim_{x\to 5} f(x)$

APPENDIX I: Standard Text for Comparison Group
(adapted from Stewart, *Calculus*, 7th Edition, p. 73)

Since $|x - a|$ is the distance from x to a and $|f(x) - L|$ is the distance from $f(x)$ to L, and since ε can be arbitrarily small, the definition of a limit can be expressed in words as follows:

> $\lim_{x \to a} f(x) = L$ means that the distance between $f(x)$ and L can be made arbitrarily small by taking the distance from x to a sufficiently small (but not 0).

Alternatively,

> $\lim_{x \to a} f(x) = L$ means that the values of $f(x)$ can be made as close as we please to L by taking x close enough to a (but not equal to a).

We can also reformulate Definition 2 in terms of intervals by observing that the inequality $|x - a| < \delta$ is equivalent to $-\delta < x - a < \delta$, which in turn can be written as $a - \delta < x < a + \delta$. Also $0 < |x - a|$ is true if and only if $x - a \neq 0$, that is, $x \neq a$. Similarly, the inequality $|f(x) - L| < \varepsilon$ is equivalent to the pair of inequalities $L - \varepsilon < f(x) < L + \varepsilon$. Therefore, in terms of intervals, Definition 2 can be stated as follows:

> $\lim_{x \to a} f(x) = L$ means that for every $\varepsilon > 0$ (no matter how small ε is) we can find $\delta > 0$ such that if x lies in the open interval $(a - \delta, a + \delta)$ and $x \neq a$, then $f(x)$ lies in the open interval $(L - \varepsilon, L + \varepsilon)$.

APPENDIX J: Half of Adam's Full Transcript and Written Artifacts

(up to question question 21 on the protocol in Appendix F)

Line	Speaker	Utterance
1	Int.	What do you think limit is? What does it mean to say in example 1 or problem 1 that the limit is 5?
2	Adam	Alright so the limit is 5 is, as you approach a number of some function, a number on a func- a graph of a function you're gonna [phone rings] you're gonna approach a number on the function as you approach a number that you're plugging in, in this case it'd be x so if you have f(x) and you approach the number [gesture approaching with left hand] that you want the the limit of, from either side you're gonna keep come- keep coming closer and closer to the number you actually, when you get when you plug it in
3	Int.	Can you use the numbers in that particular problem and the actual function?
4	Adam	So.. What? Can you restate that?
5	Int.	Oh can you , can you say what you said again but instead of using like a general number or function, actually…
6	Adam	So this one is as you approach 1 on the function from both the left and right. So if you come from this side of 0 as you approach 1 you're gonna start approaching the number 5 and as you approach the number 1 from, on the function from the right or from a number greater than 1 such as 2, you're gonna approach a number that's 5 because the limit is continuous and there's no discontinuity there.
7	Int.	ok, so, you- you talk about this approaches stuff. Um, I wanna know what's approaching 1 and what's approaching 5?
8	Adam	it's the, for what approaches 1 it's what the input is
9	Int.	what the input?
10	Adam	yeah so in this case x is your input, so as x approaches 1 you get 5 which is your output.
11	Int.	ok, so the input approaches one, the x approaches 1 and what happens?
12	Adam	then the output approaches 5.
13	Int.	the output approaches 5. okay, alright that's fair enoguh, um alright, uh.. du du du what does it mean for a limit to not exist
14	Adam	it means uh that there's some remova-, non removable discontinuity so.. like if you approach a number from one side so like in number 4
15	Int.	um hm
16	Adam	you asked just a random funiton, you don't really give us the function you just give us the graph. What if the function approaches 2, if you approach, if you put oh, put in a number for x close to 2 but is on the left you're gonna get a massive number that's positive.
17	Int.	um hm
18	Adam	and then it's going to to go off and it's just, it's gonna be become infinite as you get closer to 2. if, like if,you keep going and take the limit from that one side,
19	Int.	um hm
20	Adam	but if you do it, you going from the right, you're gonna get a negative number and as you approach 2 from the right it's gonna become negative and infinite,

21	Int.	um hm
22	Adam	so it does not exist because the 2, the approach from the both sides don't meet in the middle.
23	Int.	ok, do-so do they have to?
24	Adam	can you restate that please?
25	Int.	do you have to meet in the middle or..?
26	Adam	sort of
27	Int.	what is this thing that you're talking about as they//
28	Adam	//the limits have to agree from the left perspective and the right perspective and in this case they don't
29	Int.	alright, um, how are you feeling by the way? I mean with this whole thing, this whole interview thing, I'm sensing you're a little, are you nervous at all?
30	Adam	It's just weird explaining this to you [laughs]
31	Int.	Ok, so here's the other thing. What we have to get used to here is I'm no longer your teacher.
32	Adam	I know
33	Int.	Right? Which is you know takes some getting used to and so I really, um, what I'm interested in is what you're thinking about. I really-really don't care if it's right or wrong, right? So, um and if you don't know something please let me know instead of like. And If you're trying to figure something, like, I don't know but this is what I'm thinking right now, please say something like that, yeah? Alright, so let's move on. Um, so the bulk of this interview you can use the back of this like scratch paper if you ever want to.
34	Adam	Sweet [laughs]
35	Int.	Um, the bulk of this interview is gonna be spent talking about the followingdefinition of the limit alright? And so this is what it is, you have it right here [gives student the formal definition sheet]. Actually you can use this as well as scratch paper. Um, so here you have, here you have this statement. Um, before we get into what this statement means can you tell me what this statement is for, or the purpose of it, or why do we have it?
36	Adam	The formal definition is, every math has a formal definiton [laughs].
37	Int.	Okay.
38	Adam	I mean a formal way, it's a proof, it's a way to go about proving the limit exists instead of just stating that they exist you know?
39	Int.	Okay.
40	Adam	Cus if you make a statement without proof then what is that statement actually mean?
41	Int.	Okay.
42	Adam	And, uh, these, this, this, you know, it's also helps with everyone knowing exactly what it means, you know like, there's no argument over the definition, the formal definition.
43	Int.	Ok, so you said something about um *proving* that the limit exists instead of just *saying* that the limit exists, um, how would you go about doing that for this statement? Like, what, yeah.
44	Adam	Alright so the way if, the way you prove it is, uh you show that for some limit is, the input x approaches *a* you get a number L

45 Int. um hm

46 Adam and that's what that's saying, but if for every number greater than epsilon, for every number epsilon which is greater than zero, sorry, there is a number delta greater than zero. So what they're saying is if you take the difference, the magnitude of or actually it's just the difference in this case between x and a in which you're taking the limit of, I- I mean as x approaches, will be less than delta but greater than zero, so be some number. Doesn't matter really what number, but exists if then if you choose that, you can prove that when you plug in x for that number that minus the L that it will be less than a certain number epsilon, so.. uh let's see it's been a while since I've done this, anything like this um. Alright so it's gonna be like, this is just gonna be, um *a* is just gonna be a number and then you're gonna say, does the limit as you know x approaches *a*

47 Int. um hm

48 Adam of this, um it's gonna be less than d, well delta. Well delta's gonna shrink as x approaches *a* because once you hit *a*, *a-a* is gonna be 0

49 Int. um hm

50 Adam but on the other side you're greater than zero so it's technically not gonna be *a*

51 Int. okay

52 Adam so and then on for this you have f(x)-L, whoops [writing], it's gonna be less than epsilon and this is again when it approaches *a.* But at *a*, um at *a*, x here becomes L because you plug in thethe input x into the function and you get L and L minus L is 0 but according to this x cannot be *a* cause zero is greater, I mean zero is not greater than zero, so you're gonna be a little off for like each thing it's gonna be miniscule but it's still gonna be a little off. But that's number gonna be smaller than another number delta

53 Int. um hm

54 Adam and such that that delta happens that if you plug in for, uh if you choose a number smaller than, uh, delta, that delta and you'll, you plug that number that you plugged in here they get a lot smaller than delta, you'll get the func- the input of that function minus L plus the actual, meaning the limit exists.

55 Int. ok, alright. So your [inaudible] so what I'm hearing from you that you're talking about, you have this number x that is um slightly off than *a.*

56 Adam Yeah.

57 Int. And the reason for that is because, because of the greater than zero=

58 Adam =in the formal definition, yes

59 Int. Oh okay, what do you mean by formal definition?

60 Adam In this definition it states that x cannot be *a,* technically, because 0 is greater than, x minus a, the absolute value is greater than zero,

61 Int. Uh-hm

62 Adam but if *a* if x is equal to a you get zero=

63 Int. =equals zero, ok so according to the definition it has to be greater than zero and you're saying that this number and then when you plug it into your f(x), that what, when subtracting L, that number will be less than epsilon.

64 Adam The absolute value, yes.

65 Int. The absolute value will be less than epsilon.Um, I'm curious about what you wrote here because you wrote um a limit as x goes to *a* of x minus *a*. What does that mean?

66 Adam That would mean as x approaches *a* but technically that's not supposed to be there.

67 Int. Uh-hm.

68 Adam but uh,I was thinking, it's- it's not gonna be exactly *a*, you get closer and closer to *a* and this number 's gonna shrink

69 Int. um hm

70 Adam and it's gonna be less than this number delta,

71 Int. um hm

72 Adam but it still has to be greater than zero.

73 Int. I see. So, so is it fair to say that this definition essentially, these two parts the, you're essentially taking the limit as x goes to *a* of each one of these things [referring to |x-a|<delta and |f(x)-L|<epsilon]?

74 Adam Umm

75 Int. Is that how you're thinking if that's not what you're thinking about it then it's not.

76 Adam That's technically how I'm thinking about it, yeah.

77 Int. Ok, ok, alright, um, so here's a question, what do you think we're trying to do with this formal definition? Is there anything that we're trying to show is true? Or we're trying to find? Or anything like that?

78 Adam We're trying to show that for a a- number epsilon given to us, which is just stated as epsilon, we can find an interval sufficiently small enough that if you plug it in for x, the endpoints will always be less than a number delta, and then that proves that the limit exists.

79 Int. For, for every number epsilon, I'm gonna try to work through what you said, um, or you, or you might. Why don't you try to restate what you just said?

80 Adam I know what I said, I'm wondering if I said it backwards... No I said that, I think I said it the right way. Alright so for every, you're gonna be given like a number epsilon it's gonna be a general epsilon

81 Int. um hm

82 Adam cus yeah. So as, um, for every, you'll find intervals such that, um in that interval of x-a it'll be less than delta but such that, that will also exist if you plug in the interval for this (refers to 0<|x-*a*|<delta) it will be less than epsilon.

83 Int. ok, alright, um, so sorry did you answer the question what is that you're trying to find?

84 Adam Oh, you're trying to prove that the limit exists.

85 Int. Right but how- is there anything specific that you're trying to find?

86 Adam Um, I have no idea what you're asking me. Well, I don't know what exactly how to answer that, I mean.

87 Int. Ok, that's fine but there's this thing about, uh, you said for a given number epsilon you're trying to find this interval, you said, right?

88 Adam Yeah

89 Int. And then, if you, and then what happens after you find that interval?

90	Adam	That interval if you subtract it from *a*, and you take the absolute value it will be less than delta
91	Int.	And that's it?
92	Adam	Well you're trying to find delta using that
93	Int.	Ok. So the end result, so I'm hearing is the end result then now that the differenc- when you subtract *a* it will be less than delta?
94	Adam	Yes, but then that proves the limit exists.
95	Int.	Oh so, when you show that difference is less than delta then that proves the limit exists.
96	Adam	Yes [pauses]. Yeah.
97	Int.	Ok, alright
98	Adam	Feeling I'm doing this wrong.
99	Int.	What- what makes you think that you're doing this wrong?
100	Adam	Mainly, I haven't done a formal epsilon delta proof
101	Int.	um hm
102	Adam	in, in what? Two, three years now?
103	Int.	Yeah, it's fine.
104	Adam	and [laughs]
105	Int.	That's ok. I mean it might come back to you as we're talking about it more. Um, ok, is there a way you can show, oh wait, sorry, um. So can you try your best to, so we've sorta spoken about, we've spoken about this statement more generally. Can we go back to this statement right here like the fact that the limit is 5?
106	Adam	um hm
107	Int.	Can you explain what this is saying what this statement is saying, using this function?
108	Adam	[inaudible] Ok, so alright um x minus *a* and *a* is gonna be [writing]. And um, so if we believe that the limit is 5, [um hm] as x approaches 1 in this case then there must be the intervals such that if you take x, the, like endpoint of that interval minus 3 and take the absolute value you'll get a number less than delta, which isn't given, and then if you plug in the input of that interval you'll become significantly small, sufficiently small enough that if you take the absolute value, you'll get a number less than epsilon. So what ends up happening is this interval- you're trying to find like, when you're- when I stated that you're supposed to find an interval
109	Int.	hm
110	Adam	in this case you just go. If the function which in this case is [inaudible] like 3x+2
111	Int.	um hm
112	Adam	minus 5 is less than epsilon [writing]
113	Int.	um hm
114	Adam	that's equal to [writing]. And then um this is just [writing], but so you're trying to get, it's this, if 3 times this, which is uh, function, it's not a function it's more an equation, if the equation is less than epsilon but uh, this is set so this interval is for this would be like a number around 1 because unless this [x-1?] is close to zero, this is gonna be a bigger number than epsilon really.
115	Int.	ok

116	Adam	so like if you have a number like 1.11111111111111 or like 1.0000000000000000001 this can be a tiny number
117	Int.	um hm
118	Adam	and it'll be less than epsilon and if you plug that into here, it'll also be a tiny number, when you take the absolute value, if you take three times this number and plug in [looks perplexed/confused]... Um you take this number and plug in here it will be less than that delta, if you knew like an interval from this case it's centered arou-around 1 and you'll have a number like a and b
119	Int.	so your number is centered around 1?
120	Adam	yeah
121	Int.	I see.
122	Adam	Op, yeah
123	Int.	So, where did that 3 come from?
124	Adam	Oops, that- that should be, no idea why I wrote 3 [laughs], it should have been 1 cus a is 1 in this case and I should have used a different letter than a let's call it c
125	Int.	So can you say that one more time, so in- in sort of in summary what does it mean for the limit to of 3x+2 is 5 as x approaches 1 according to the formal definition?
126	Adam	So, it's that there is, around, there's an interval such that around 1 that such that if you plug in the numbers on the interval
127	Int.	um hm
128	Adam	that you'll get a number less than epsilon which is like, but it'll also satisfy the other part that uh if you plug in the interval you'll get you know for x minus 1, you'll get a number less than delta.
129	Int.	Ok, are you ok with that?
130	Adam	yes [laughs]
131	Int.	Um is this related at all to what you said limit was to you? Um because you said something about, you know as x approaches 1 the output which is 5.
132	Adam	Yes and no, it's um , it can be because the interval is like going to be small and it'll be as if it's approaching 1.
133	Int.	um hm
134	Adam	but then you also have this like delta, which you know, must be greater than this part of the definition you know x-a and I don't really think of that when I think of limit
135	Int.	um hm
136	Adam	even though it's part of the formal definition, it's, even though it's significant I just, it's sort of like forget about it [laughs]
137	Int.	ok, that's fair. Um, so I'm hearing from you, as you're explaining this um, so it seems like these two things are something to, to satisfy=
138	Adam	yes
139	Int.	you have to satisfy the f(x) minus L less then epsilon and at the same time the x minus a less than delta also
140	Adam	yeah
141	Int.	and it's basically trying to, trying, are you trying to find the x that that would satisfy the two or?

142 Adam um, you’re trying to, find, yeah an interval around the *a* that'll satisfy the two

143 Int. ok so yo're tyrying to find an interval around the 1 in this case

144 Adam in this case yes, the 1.

145 Int. Right. and then what do you do with that interval, are you? Oh so you take x's from that interval or?

146 Adam Mainly you want to make sure the endpoints work. So it's as if uh you don't actually ever equate it. You make sure that um so in this case its like you can if you can- you can realize that this is just, um x minus 1's here [points at |x-1|<delta] and x minus 1's here [points at 3|(x-a)|<epsilon] so this is like you can also say 3 delta is less than epsilon

147 Int. ok

148 Adam so it's, you have to make sure that the delta.. let me make sure I'm saying this right. I think I am. So you're trying to make it satisfy the delta and epsilon equation, but you also are trying to satisfy, like as if you combine them, so you want to make sure where, this is, if you find a delta you multiply by 3 it's gonna be smaller than epsilon still, but you're never given a like an absolute delta you're just trying to find a, uh, like sort of like an equation,

149 Int. uh huh

150 Adam such that this works.

151 Int. Ok, alright. Is there anything else you want to add about what you think the formal definition says?

152 Adam No [laughs].

153 Int. ok. Alright, so we've, we've gone overlike things kinda spoken generally, um so I want to ask you specific questions about different parts of the definition.

154 Adam Alright

155 Int. What is epsilon?

156 Adam Epsilon is a just a number that you use, to make sure that f(x) minus L, the absolute value is less than that number epsilon

157 Int. Ok, sorry, say that one more time my mind was somewhere else

158 Adam epsilon's just a number

159 Int. um hm

160 Adam and you're using it to make sure that f(x) minus L, the absolute value is just less than some certain nu- number and it must be greater than zero so you call it epsilon, cus, yeah.

161 Int. um, I forgot to ask you, the reason why my mind was somewhere else, I forgot to ask you, can you show this definition graphically? Like in a graph?

162 Adam [Looks away] Yeah, I can.

163 Int. Ok, can you try that?

164 Adam It's like [starts drawing]

165 Int. you can feel free to use this function or general function

166 Adam a general function will be easier [laughs]

167 Int. ok, that's fine.

168	Adam	um it doesn't really matter you can just choose an *a* um, let's call it 3 and then what it is is you have a function and [draws] the limit cus it's continuous would be, [draws] so this would be. The- you've like, the interval, which is here uh less than delta, so this is gonna be a small interval even though it's not [small] on the graph
169	Int.	It's fine. It's zoomed in.
170	Adam	So it's gonna be, this is gonna be an interval from c to b [draws bracket to indicate interval on the graph]
171	Int.	uh huh
172	Adam	and it's gonna be, doesn't actually matter if it's closed or open. And then from that you're gonna find, you subtract the function, I mean the function at that value the point minus L you get a number less than epsilon so the limit would be this, so any, if, for in between here you're gonna find a bunch of numbers tha, that if you evaluate them there and you subtract the actual limit you'll find a number that's less than epsilon.
173	Int.	what is the actual limit in that case?
174	Adam	in this case it, it'd be, as you approach 3 it can be 5
175	Int.	ok
176	Adam	*a* or whatever you wanna call it. So and just as this interval allows you to be extremely close to the actual limit in this case 5
177	Int.	um hm
178	Adam	such that if you actually plug in 3 for the limit you get 5 but if as long as you're within this interval it's gonna be, the numbers is gonna be extremely close and you-, I guess you could call it the error, would be exceedingly small.
179	Int.	Ok, alright. Ok, so um so you mentioned the word error?
180	Adam	Yeah
181	Int.	Um, error in what?
182	Adam	The difference between the actual func-, the l-, what you think or in this case is the limit,
183	Int.	um hm
184	Adam	and what you're getting from points near the limit.
185	Int.	ok, um, ok. Does that change what you think epsilon is at all or?
186	Adam	No
187	Int.	Ok, so you think epsilon is just some number, to make sure that the difference, sorry I'm tryin to rephrase,
188	Adam	Epsilon is just well it's a small number,
189	Int.	yeah, it's a small number,
190	Adam	but uh it had- it it measu-, it makes sures that the difference in between the actual limit and you know numbers near it is exceedingly small.
191	Int.	Okay, what about delta?
192	Adam	Delta is another small number such that the interval, it makes the interval, sm-small but big enough so you can actually, it's not just a point but it's uh, that you get numbers that are close to the limit.
193	Int.	Ok, um why do we have this part "For all epsilon, for every number epsilon greater than zero there is a delta greater than zero" why do we have that you think? Why do you think we have that?

194 Adam Because, uh, if you have to have the, if there's epsilon then there is a delta because one it, it puts restrictions that epsilon and delta cannot be zero and that, if you believe that that one exists it has to satisfy this [circles 0<|x-a|<delta] and this [circles|f(x)-L|<epsilon]. It can't just satisfy one or the other. That uh, but mainly it's just restrictions on the number not being zero that you [inaudible] and then it just, it also uh limits confusion such that you don't have like mess up, you know it's saying there must be an epsilon and a delta it cannot be some random what what people wanna call them.

195 Int. So you said two things so one is saying that the epsilon and the delta both, cannot be both cannot be zero

196 Adam Yeah.

197 Int. Ok.

198 Adam And its uniformity, it's you wanna, you're stating them beforehand what you- the numbers are gonna be called that you're using to,

199 Int. I see.

200 Adam to evaluate.

201 Int. So instead of calling it a and b or c and d you wanna call it epsilon and you wanna call it call it delta=

202 Adam =yes.

203 Int. So it's a matter of naming what it is.

204 Adam Yes.

205 Int. Ok, so what you do you think means, just this part [points to the part in the definition] um x minus a in absolute value is greater than zero less than delta?

206 Adam So it means um, that that part is saying that around *a*, x around *a*, if you get x close to *a* it will be, if you take the actual value so you can be less than *a* or greater than *a* slightly it will be less than a number delta so its, it's actually just showing part of the interval is, *a* is less than delta but it's greater than zero but if you remove absolute value then you have, then you have to split you know, if you remove the absolute value it becomes less than delta but then you have a negatve x minus *a* is less than delta... plus *a*, sorry, and the you have x minus *a* is less than delta so, it gives you an interval,

207 Int. Uh-hm.

208 Adam such that if you take the difference of the two numbers there'll be, and the absolute, if you take the absoulte value it'll be less than a number delta.

209 Int. Ok, what about the f(x) minus L less than epsilon? What do you think that means?

210 Adam it's um when you start, when you think, you start like approachin it's gonna come exceedingly close and if you take the absolute value of the function approaching it and you subtract it from what you believe is the actual limit,

211 Int. um hm

212 Adam um, you can, uh it's supposed to be less than the uh epsilon but again it's sorta like an interval cus it can be greater if you take out the absolute value it can be greater than or less than like, cus it's in absolute value so it's saying the error is less than, but if it's to make sure it's not negative so it's not less than zero.

213 Int. Okay. Ok, so now we're looking at this statement as a whole. Do you have a sense, or what do yout think it, the-the if then is for, you know, the if this then that like, what is that for, you think?

214 Adam It's uh, it's like causation. If this exists then this also has to exist for this to be true.

215 Int. So it's a causation thing?

216 Adam like, you have to prove. It's not causation, it's uh. For this to be true, if this [delta inequality] is true then this [epsilon inequality] has to be true.

217 Int. ok, um so that's a little different from what you were saying before because before you were saying these two things are just conditions two conditions to satisy

218 Adam yeah

219 Int. so how does this, if this is true then this has to be true play a part in what you said before?

220 Adam um [pauses]

221 Int. Or does it? I mean it doesn't have to be.

222 Adam Because uh, it does because if it doesn't satisfy both like I was thinking before hand, then one of these is not true and the limit doesn't exist.

223 Int. Okay.

224 Adam So the way I was thinking of it is in saying one has, if one is true then the other one has to be true for it to exists. I was thinking for this to exist, both of them must be true, so both equations must be satisfied.

225 Int. So for the- for the limit to be that, but both equation have to be satisfied. But then at the same time if one is true then the other one has=

226 Adam to be true for the limit to exist.

227 Int. So are these now two separate conditions?

228 Adam No [laughs]. They're-

229 Int. I'm trying to understand how you reconcile the two.

230 Adam They're, they're pretty much the same, it's just uh. The way I'm thinking is instead of saying, if this is true then this must be true for this to be true. I'm thinking for this [|f(x)-L|<epsilon] to be true then both of these [0<|x-a|<delta and |f(x)-L|<epsilon] must be true so there must be this, this [maybe the 0<|x-a|<delta], this, they're pretty much stating this and giving this to you,

231 Int. Um-hm.

232 Adam So this is true, like you can always find something for this to be true but then this also has to be true for this to exist.

233 Int. so if they're saying that you can always pick something for this [delta] to be true

234 Adam Yeah.

235 Int. So how does that work with what you said about you have to satisfy both, well I mean if you can always find something then=

236 Adam =You can't always you can find something for this to be true but you know like x minus a less to be less than delta but that might not always work for this part now the function of x minus L evaluated at f(x)-L is gonna be les than epsilon, that might not always be true from the interval.

237	Int.	Ok, that makes sense to me. Um, ok, so then do you need to satisfy both or do you just need to satisfy this one? Do you just need to satisfy the f(x) minus L less than epsilon?
238	Adam	Uh [pause] no, because they're dependent so they both have to satisfy. Like, this one [points at 0<|x-a|<delta], well, this one, if this one [0<|x-a|<delta] is satisfied, this one [points at |f(x)-L|<epsilon] may not be satisfied, even though this one [points at |f(x)-L|<epsilon] is dependent on this one because in this one [points at 0<|x-a|<delta] you're choosing the x,
239	Int.	Hm.
240	Adam	this [points at |f(x)-L|<epsilon] is evaluating the function at x,
241	Int.	Ok.
242	Adam	so what ends up happening is uh, you then, this one like, you, even if you just solve this one [|f(x)-L|<epsilon] to be true, uh, you work backwards[points back to 0<|x-a|<delta] and that's what, technically you work backwards and for it to work but uh, you find, you know, this [unclear where he's pointing] is true and then you can find the interval, so you must, you're sort of combining them into one is satisfying is how I'm thinking about it.
243	Int.	What do you, you said this thing about working backwards?
244	Adam	yeah
245	Int.	like what does that mean?
246	Adam	because uh
247	Int.	did you do that there [points at his work]
248	Adam	yeah I uh cus you, you plugged it in and then you work all the way down through here [points at his work of working out the epsilon expression] and if it's less than 3 times the absolute value of x minus 1
249	Int.	um hm
250	Adam	cus 3 is always positive
251	Int.	um hm
252	Adam	but you realize that x minus 1 is always less than delta
253	Int.	um hm
254	Adam	so you can always say 3 delta is less than epsilon
255	Int.	um hm
256	Adam	and then it's work-, I think of it as working backwards because you plug this [points at possibly at 3delta<epsilon] in and then you found there's this [0<|x-1|<delta], this is saying this interval is less than delta well, that interval times 3 is always gonna be less than epsilon so you work backwards to prove that it exists even though you didn't technically find an absolute interval
257	Int.	so when you said working backwards, what are you working towards?
258	Adam	you're working towards to find an equation, sort of like this [points at 3|x-1|<epsilon, 3delta<epsilon]
259	Int.	what does that tell you?
260	Adam	[long pause] it tells you, well this [3delta<epsilon] tells you that the difference between the interval and *a*, so the number surrounding *a* is less than, times 3, is less than epsilon.
261	Int.	ok
262	Adam	just a number.

263 Int. ok, alright. Um, alright. So uh one um thing, one last question with the parts, why is there a zero here [delta inequality] but not a zero there [epsilon inequality] do you think? Why do you think that is? … I mean=

264 Adam because um if, the like, if the limit does exist than this [f(x)-L<epsilon] is going to be exceedingly small and it may end up like flattening out

265 Int. um hmn

266 Adam so if you have a flat function like just a constant number then um it doesn’t matter where you approach from what side you approach it from you're gonna get the same number so if you have like y equals 3 it

267 Int. um hm

268 Adam it's just gonna be a flat line

269 Int. um hm

270 Adam and if you plug in any number on the line you're gonna get 3 and then if you take the limit which is, which, which would be 3 and you plug a number in you're gonna get zero

271 Int. I see

272 Adam but that interval is still gonna be bigger than zero from this idea

273 Int. but the interval is still gonna be bigger than zero…

274 Adam yeah it's not gonna just be a point

275 Int. oh ok because [point at 0<|x-a|<delta] you're talking about interval whereas over here [|f(x)-L|<epsilon] you're not talking, you're talking about

276 Adam you're talking about, you're comparing a value

277 Int. which value?

278 Adam on the interval=

279 Int. oh you're comparing the values of the [the function for] x from the interval and the limit.

280 Adam yeah

281 Int. ok, fair enough, ok now I have a specific questions about epsilon and delta. How are you doing? Doing alright? We'll take a break in about 10 15 minutes

282 Adam this reminds me how much I hated epsilon [laughs]

283 Int. [laughs] oh no, bringing back nightmares, I'm just checking to make sure um

284 Adam it wasn't nightmares, it's just mainly when I solved them it was like I god, why?

285 Int. Ok, alright well we'll try to make this quick and pain- as painless as possible but alright so specific questions for epsilon delta, in the definiton with epsilon and delta, um what depends on what if anything, you think? Does delta depend on epsilon, epsilon depend on delta, they depend on each other, they do not depend on each other and why?

286 Adam Um delta, no, epsilon sorta depends on delta.

287 Int. Epsilon depends on delta…

288 Adam Because, um delta is giving you an interval for x,

289 Int. Uh-hm.

290 Adam And then like epsilon is evaluating x and subtracting the limit,

291 Int. Uh-hm.

292	Adam	So, um, they- they depend on each other a little but not like completely it's, a weak I'd say it's more of a weak, um connection because, uh if the limit exists then there's gonna be some sort of you know, if, as this [points at 0<|x-a|<delta] gets, you know, smaller, this [points at |f(x)-L|<epsilon] is getting, the difference is gonna get smaller,
293	Int.	Uh-hm.
294	Adam	but if the limit doesn't exist like it did in 4 [problem 4 from the beginning of the interview], where it, it, you know, it doesn't exist then as this [points at 0<|x-a|<delta] gets smaller this [|f(x)-L|<epsilon] isn't gonna change, this isn't [delta expression] gonna help [inaudible].
295	Int.	Uh-hm.
296	Adam	So it's gonna, still not exist even though the function is getting close to that point [gestures his two hand coming together horizontally] whereas if you have the, like I said, the [function that] exists like here then it keeps getting numbers closer [gestures his two palms coming together horizontally] to the actual value, the limit.
297	Int.	Ok, so, so you're saying they s- sor- depend on each other but it's,
298	Adam	//Sometimes.
299	Int.	//a weak connection.
300	Adam	Yeah, it depends on the function, I'd say.
301	Int.	It depends on the function, so let's just use number one for example.
302	Adam	Ok.
303	Int.	In that sense
304	Adam	So in this sense they do depend on each other because this actually works out good because when you break it down, you find out 3 delta must be less than epsilon. So the interval times 3 must be less than epsilon.
305	Int.	Okay so that shows that delta=
306	Adam	that delta depends on epsilon, well they depend on each other cause the one must be 3 times smaller than the other
307	Int.	Uh-hm. So you can always, ok. Um, so let me just make sure um, so can you say why um, so since you say the depend on each other can you say why epsilon depends on delta?
308	Adam	Why?
309	Int.	Yeah. Or is it just because of that equation?
310	Adam	No, it's not just because of this equation, it's because delta is um, you're saying delta [points at the delta inequality] must be greater than the input minu- subtracted by what you're centered around [gestures a small horizontal interval with his palms].
311	Int.	Uh-hm.
312	Adam	So you're saying that delta must be, the interval around a number *a* [points at delta inequality] must be less than delta, so you're saying, um the input cannot get outside of this region, it cannot be getting, well not region, [but] this interval it, it cannot get exceedingly big.
313	Int.	Uh-hm.
314	Adam	And then for epsilon you're evaluating x around *a,*
315	Int.	Uh-hm.

316 Adam — and then you're subtracting 1. When you plug in x for, when you plug in *a* for x, so what winds up happening is you're seeing how big the difference is between a number near *a* and the *a* itself.

317 Int. — Uh-hm. So how does that say that epsilon depends on delta?

318 Adam — It's, uh, because your input, your delta is influencing your input and then epsilon must be greater than your input minus your input of *a,*

319 Int. — ok. or your output [correcting]

320 Adam — your output [in agreement].

321 Int. — And since, so since output [points at epsilon inequality] depends on input [points at delta inequality]..

322 Adam — Yes.

323 Int. — epsilon depends on delta…

324 Adam — Yes.

325 Int. — So what about delta depending on epsilon?

326 Adam — Um [long pause] actually, it, it uh, it's more of uh, because the epsilon can only be, is a set number and that the difference, the output can only be a certain length apart, that sort of limits also how far the inputs can be, how far apart the input can be from what you're trying to find the limit as x- the input approaches some number.

Int. — Can you sort of just restate what you just said?

328 Adam — Um, so epsilon is the difference between the output and,

329 Int. — Uh-hm.

330 Adam — uh, uh f of x [f(x)] so a number near *a* and then f of *a* [f(*a*)].

331 Int. — Uh-hm.

332 Adam — and then delta is the interval around *a,* so um, the epsilon sort of influences how far the delta can be..

333 Int. — um hm

334 Adam — from it? Even though it has no direct connection it's uh, delta must.. be within a certain length, um not length uh distance from the center however.. doesn't... Actually it really doesn't have any effect, I don't think, um, because, it's more that epsilon depends on delta than delta //depends on epsilon

335 Int. — //depends on epsilon.

336 Adam — Because, you can always find, you may be able to find an interval such that uh, [stops himself], actually, I actually don't know. Because if you go back to thinking this idea [circling the $3\delta < \varepsilon$ in Figure 5],

337 Int. — Uh-hm.

338 Adam — then you get the idea that the size of the interval must be smaller than the number epsilon. That size of the interval times three [3δ] must be smaller than epsilon. So the radius times three [3δ] I should say, of the interval must be smaller than epsilon.

339 Int. — Uh-hm.

340 Adam — so it influences delta cus if epsilon is set then delta has to be a certain size of radius in order to, um, conform to the size of epsilon but I don't know how I, I prove, well not prove, show that that exists using the definition.

341 Int. — I see. So you're saying something about the epsilon being set, so you're saying, IF the epsilon is set ..

342	Adam	Yes .
343	Int.	then you know, delta has to conform to this epsilon. Um, so is, so is epsilon set?
344	Adam	[It normally asks for delta].
345	Int.	One of them is set?
346	Adam	Normally one of them is set, it's not a given number but you're given the equation.
347	Int.	So which one do you think it is that's set?
348	Adam	Epsilon I think.
349	Int.	Okay. Why do you think that is?
350	Adam	[pauses] Because it's a function minus the limit so what ends up happening is, um, you, it's the function minus the limit so you just, of x, so if you keep it generalized which you do when you subtract the limit, you're, you're gonna be it's gonna be exceedingly small and approaching zero [points to epsilon inequality in the definition] but then you can pull out [points at the work for 3delta<epsilon], you can pull it out so you get delta times a number normally
351	Int.	um hm
352	Adam	so you get but um, but delta is kinda also set because um, if x minus *a* but, um x is just your input and *a* is the number you're taking the limit as x approaches, so they're both actually kinda set
353	Int.	Uh-hm.
354	Adam	So, [long pause] I'd say they're both set actually just in the fact that uh, they're set like sort of independently but the relationship is dependent on each other in a way.
355	Int.	They depend on each other, ok. Ok, um, I forgot to ask, where is, where is delta in your, in your picture here? If you don't mind using the red pen. Like where is delta and where is epsilon?
356	Adam	Delta would be this interval so it would be, um, delta [marks what would be the interval from 3 to 3+delta on y=5].
357	Int.	Ok.
358	Adam	then epsilon in this case would be the difference in the numbers, that'd be the rise so it's the difference between let's say [draws two dotted lines one at y=5 and another right above] if you like this, it'd be [marks the vertical interval as epsilon],
359	Int.	Ok.
360	Adam	even though it's uh [laughs] not to scale.
361	Int.	I understand. It's fine. So does that change what- what you- what you say deltas and epsilons are, or does that fit with what you said deltas and epsilons are?
362	Adam	I say It fits, with what I said deltas and epsilons are.
363	Int.	So how does that- can you explain again how you- what,
364	Adam	cus
365	Int.	epsilon is?
366	Adam	delta limits the size of the interval and this is your interval and delta, this is is like the radius in the interval,
367	Int.	Uh-hm.
368	Adam	so delta must be greater than the radius of the interval is what this says
369	Int.	hm

370 Adam so the radius is this, delta is gonna be this. And epsilon is the difference between the function evaluated minus the actual limit

371 Int. um hm

372 Adam which is less than epsilon, this would be you're evaluating as it approaches this, the difference between the actual limit and the number, the input close to the input you're trying to get to, so it isn't gonna be, the difference between them is gonna be exceedingly small but it's gonna be epsilon, a number epsilon so greater than that.

373 Int. ok, alright, thank you for that. So let's see in the definition with x and y what depends on what if- if anything? x depends on y, y depends on x, they depend on each other they do not depend on each other?

374 Adam so you're saying

375 Int. between x and f of x I guess

376 Adam so you're saying x, instead of using f and x, use y

377 Int. right, or between x and f of x, it seems kinda silly but

378 Adam which depends on?

379 Int. what, yeah.

380 Adam Um, y or f(x) depends on x.

381 Int. Ok. Um, ok, how do you feel, do you want to take a break? Or do you…

382 Adam Um, take a break

383 Int. Let's take a break.

384 Int. So, let's return to the beautiful epsilon delta, several more questions here. Um, so in the definition between x and y which one comes first and which do you try to figure out as a result?

385 Adam Um, I'd say you try to figure out sort of concurrently

386 Int. between x an y between x and f(x)

387 Adam yeah because what you do is, um you're trying to get the interval be smaller than delta [points at 0<|x-a|<delta] but you also want it such that the output so f(x) minus the uh, minus the limit, or L in this case [points to |f(x)-L|<epsilon] will be less than epsilon so, you go down through like this [points to work of breaking down epsilon] so

388 Int. so, so how does that, figuring it out concurrently? I mean, and no one comes first, really?

389 Adam yeah cus like you find out that this like delta [points to |x-1|<delta] and then you break it down [points to the work of breaking down |3x+2-5|<epsilon], and you keep breaking it down, and you find out that, like normally there's a relationship sorta like this [points at 3delta<epsilon]

390 Int. ok, um so what about epsilon and delta? Which one do you think comes first, if at all? and which one do you figure out as a result?

391 Adam um, I'd say delta [points at |x-1|<delta] comes first mainly but like you end up figuring out [points to the break down work] they're both..., you find, you figure out they're both like sorta dependent on each other [circles the general area of 3delta<epsilon] so they don't really... which one comes first is, you know the size of the intervals [points back to |x-1|<delta] like, but then you and then you break down epsilon [points at work] and you know you get another, like I said, you get like another relationship like this [3delta<epsilon] and uh, but i'd say you find pretty much an epsilon and a- i mean you find an epsilon delta, i'd say you find the delta first.

392 Int. so why do you say the delta comes first?

393 Adam I mean the epsilon's sorta first

394 Int. oh ok

395 Adam because um, uh you're, your, cus you break down the epsilon

396 Int. um hm

397 Adam and then it ends up that you normally have this sort of uh um relationship in between the delta and epsilon [points at 3delta<epsilon]. So it ends up like if your epsilon is sorta like set and you solve for it [points at breaking down work] you have this relationship such that you know like this delta is less than 1/3 epsilon and then you got to make sure that works with uh the equation for delta [points back to |x-1|<delta]

398 Int. you said a couple things here. So you said delta is 1/3 epsilon

399 Adam in, for yes, in this like equation

400 Int. ok, so is that important that delta is 1/3 epsilon?

401 Adam when you do multivariable it is [laughs]

402 Int. oh ok why?

403 Adam uh, why multivariable

404 Int. no no like why is it important

405 Adam just because uh like you wanna make sure that while it satifies that this works [points at [3delta<epsilon] it also satisfies this [|x-1|delta]

406 Int. the x minus 1 less than delta ok, um so uh let me ask you something here, because you're now saying that the epsilon is set but then earlier you said that the epsilon and the delta are both set

407 Adam yeah

408 Int. so

409 Adam so

410 Int. which one's set first? I mean, are they both set at the same time?

411 Adam um, trying to remember, last time I did this was multivariable [laughs] I'd say epsilon's set first because um like they're both set because I mean the delta is apparently not, is gonna be be greater than zero, you know that much, and then like you're just sorta like I guess you'd call it like fine tuning the size and you do that using epsilon. so epsilon is set because of the function, and then you just find, you know, a delta for the, a certain size

412 Int. so does that change your answer about which between epsilon and delta which depends on which

413 Adam yeah, I'd say so

414 Int. so what is it now?

415	Adam	I say delta more depends on epsilon than epsilon delta
416	Int.	delta depends more on epsilon, ok. Um, ok, uh let me throw a monkey wrench into this [student laughs] um, you said this [students work of breaking down the epsilon inequality] was working backwards
417	Adam	yeah
418	Int.	right, so how does that fit into epsilon being set if this [points at student work on arriving at 3delta<epsilon] is working backwards?
419	Adam	because if you look back to the definition you're saying if this [delta inequality] then this [epsilon inequality]
420	Int.	um hm
421	Adam	if you have epsilon set and you solve for this [points at |3x+2-5|<epsilon] you end up getting delta [points at 3delta<epsilon] which is supposed to be less than the uh the interval [points at [0<|x-a|<delta] so then it's sorta working backwards because you solve for this [points at |3x+2-5|<epsilon] and then you find delta [3delta<epsilon] and it's, you know, you know this relationship [3delta<epsilon] which is then this [|x-1|<delta]
422	Int.	um hm
423	Adam	so it's working backwards because it's as if this [delta inequality] then this [epsilon inequality], well you take this [the epsilon equation] and get to this [delta inequality].
424	Int.	um hm, ok. Um, so so there are four variables in there, there's x there's f(x) there's epsilon and there's delta
425	Adam	yeah four
426	Int.	can you put them in order at all like in terms of process in the formal definitoin
427	Adam	no
428	Int.	no
429	Adam	not even gonna try
430	Int.	not even gonna try ok. So here are some things, so it seems like we finally agree, or you say
431	Adam	[laughs] we agree
432	Int.	no, no, no, you say that um
433	Adam	I've made up my mind
434	Int.	you've finally made up your mind that delta depends on epsilon
435	Adam	yeah
436	Int.	and you believe epsilon is set first
437	Adam	right
438	Int.	and so, so in a sense it's like epsilon set first so you're trying to figure out delta as a result, is that ok?
439	Adam	yeah,
440	Int.	so some students in the past have told me that what you're trying to figure out instead of delta is epsilon. So you're trying to figure out epsilon because you're trying to get your f(x) close to L
441	Adam	yeah
442	Int.	what do you say to that? You see that that's the opposite of what you're saying
443	Adam	yes
444	Int.	so what do you say to that

445	Adam	um, so they're saying exactly that you're given delta in a way, well no, they're saying you're trying to find epsilon
446	Int.	um hm instead of trying to find delta they're trying to find epsilon
447	Adam	but they don't really specifically say that you're trying to find delta- delta is set, they're saying, right?
448	Int.	yeah, they're just saying that I'm just gonna figure out what epsilon is
449	Adam	um well um really epsilon is set but it's a relationship to delta in a way
450	Int.	um hm
451	Adam	so it's uh the interval is, I'm saying you're saying while you're trying to make it within a small distance from the actual limit [points at the epsilon inequality] so then you're finding uh you're finding the, you're, they're saying you're finding an epsilon. well, really I'm just I'm just thinking that you're taking this relationship [circles the epsilon inequality] and then you're going backwards [points at the delta inequality] and finding a delta so, they're opposite of me and um
452	Int.	what do you think?
453	Adam	I think.. Uh.. well by the way you put it I'd say that uh either one of us is wrong or we're both stating the same thing just in a very very [inaudible] unsimilar ways, different ways simply because they don't say delta is set, they just said you're trying to find the epsilon.
454	Int.	so what if they say delta is set?
455	Adam	Then I'd say we're both, one of us is wrong [laughs]
456	Int.	Can you figure out, like..
457	Adam	Which, Who's wrong? Um, not really cus if they believe that than um, they probably have a good argument and I can make an arugment
458	Int.	So let's make that argument. Their only argument is that they're trying to make f(x) close to L. So they're trying to figure out what my epsilon is.
459	Adam	So me I would make the argument for the limit to exist the f(x)-L must be close to, f(x) must be close to L
460	Int.	uh huh
461	Adam	so then you're finding a small interval [points at the delta inequality] such that that [points at the epsilon inequality] is true
462	Int.	ok but so it's basically
463	Adam	You're arguing, you're presenting a different argument than what they're saying
464	Int.	yeah
465	Int.	Can you argue against them?
466	Adam	um not really because cus what they're saying is, um.. let's see, could I? um.. trying to get, trying to solve.. yeah um I can make a small argument I don't know how feasible it would be but I could argue that uh that yes, you're trying to make f(x) close to L but by this equation [the epsilon inequality] you're saying, you're stating that f(x) must be close to L and what you're trying to do is, they're trying to like, do the opposite, which is um f(x) must be close to L so there must be an interval such that that is true whereas um what they're trying to do is they're trying they're trying to do the same thing is that they're trying to minimize um how close it is,
467	Int.	um

468	Adam	but like
469	Int.	is that ok, is that not ok?
470	Adam	it's, how should I put this? it seems, it doesn't seem natural, it's like yes you're trying to make it as close as possible but, if that's true then by, why not just say like "as the difference approaches zero" like in the definition as, like that's how I see it as like, um you have, they're arguing that uh, you're trying to find an epsilon [points at the epsilon inequality] because it's,cus you're trying to make that difference as small as possible but I, I'm thinking more of it's like you're trying to get this within a certain difference [epsilon inequality] so you must find an interval such that it is within that certain interval.
471	Int.	ok
472	Adam	so there's no real counterargument just that
473	Int.	you just disagree with them
474	Adam	yeah

Definition. Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then we say that the **limit of** $f(x)$ **as** x **approaches** a **is** L, and we write

$$\lim_{x \to a} f(x) = L$$

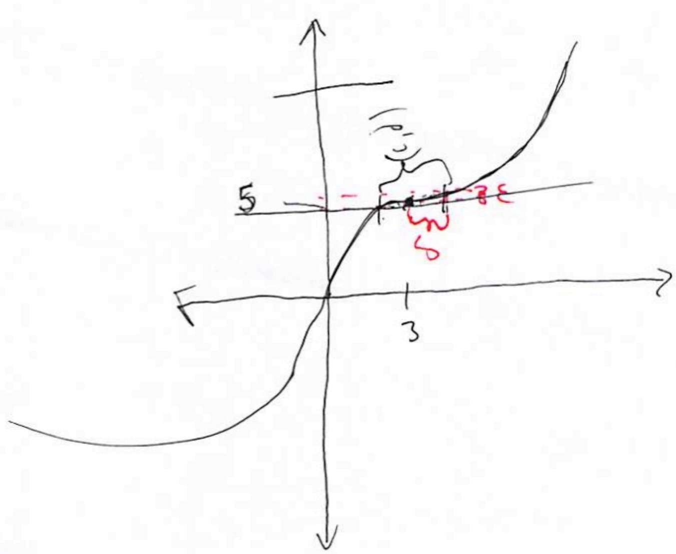
if for every number $\epsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if} \quad 0 < |x - a| < \delta \quad \text{then} \quad |f(x) - L| < \epsilon$$

$0 < |x-a| < \delta$

$|f(x)-L| < \epsilon$

$0 < |x-a| < \delta$



Interview Tasks

Spring 2012

Name: [redacted]

INSTRUCTION: Please find the limit of a the following functions, if it exists.

1. $\lim_{x\to 1} 3x+2$ $\quad 3(1)+2=5$

2. $\lim_{x\to 3} \frac{x^2-9}{x-3} = \lim_{x\to 3} \frac{(x-3)(x+3)}{(x+3)} = \lim_{x\to 3} \quad 3+3=6$

3. $\lim_{x\to\infty} \frac{\ln x}{x^3} = \lim_{x\to\infty} \frac{\frac{1}{x}}{3x^2} = \lim_{x\to\infty} \frac{1}{3x^3} = 0$

$|x-1|<\delta$

$|3x+2-5|<\varepsilon$

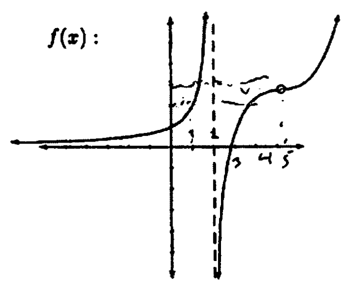
$|3x-3|<\varepsilon$

$|3(x-1)|<\varepsilon$

$3|(x-1)|<\varepsilon$

$3\delta<\varepsilon$

4-6 Let f be the function whose graph is presented below.



4. $\lim_{x\to 2} f(x)$ DNE

5. $\lim_{x\to 1} f(x)$ ≈ 1

6. $\lim_{x\to 5} f(x)$ ≈ 1.75

APPENDIX K: Relevant Part of Adriana's Transcript and Written Artifacts
(From turn 405 to turn 674)

405 Int. Ok, um let's ok, what about this if then statement what is this [points at if-then statement] saying right here?
406 Adriana Um [9 sec pause] ok if like the only, the biggest thing that I'm like unclear about
407 Int. Hm
408 Adriana Is like why like how come the delta and the epsilon are, are greater than this [circles |x-a|] this error or this [circles |f(x)-L|] difference or whatever. Like that's what I don't understand
409 Int. Um hm
410 Adriana Um, but from what I like, from my understanding
411 Int. Um hm
412 Adriana It's the if there's a like a difference here [points at the region on the x axis near a] that. Yeah this, 'the less than' thing throws me off cus I don't know [inaudible].. but if like there's and error within here [points to a region around a] then there has to be an error here [points at a region near L] too but
413 Int. So this is, this is a statement about the errors?
414 Adriana Yeah
415 Int. Ok, um,
416 Adriana But I don't, yeah, like that's the thing that I don't understand, like, what are these like the epsilon and delta like if they're not exactly that error, then like, what are they?
417 Int. Ok, um, let's see… So if they're not exactly the error what are they?
418 Adriana Um hm
419 Int. Ok, um
420 Adriana Well I guess I mean I guess it could be like, uh hm. Yeah cus if I was thinking of error here [points at a region near a on the x axis] like I said the error was 1/2 and tried to make my delta 1/2 but then I would make it be like equal to [points to the sign < in the delta inequality]
421 Int. Uh huh
422 Adriana And since it's like less than, I don't…
423 Int. I see, um. You mentioned earlier that... there is a difference between... there's a difference between error bound and error?
424 Adriana Yeah
425 Int. Right? Did the story talk about error or error bound?
426 Adriana Um well it gave us an error bound for, for the pancake and then it asked us to be, to guess a bound for the batter
427 Int. Um hm
428 Adriana Um and then it just wanted our errors to be within those bounds
429 Int. Um hm
430 Adriana So it kinda talked about both
431 Int. Ok, so how does that relate to epsilon and delta and this notion of less than thing

432	Adriana	Um, this error [points at |x-a|] is less than the error bound [points at delta]. Well if I call this [points at delta] the error bound, I would have called the epsilon and delta the bound, like the highest it could be that we want our error to be less than that, that bound
433	Int.	Um hm.
434	Adriana	[Reading the pancake story] Oh ok, so so it does say like we want the pancake to be between 4.5 and 5.5 so within an inch of bound so it doesn't say like it *can* be 4.5 and 5.5 so
435	Int.	Alright
436	Adriana	So I guess that's what this would be like, it, it has to be less than that, like if the delta or no like if the epsilon or whatever we wanted it to be within a half
437	Int.	Um hm
438	Adriana	Whatever like it would be, let me write this down. [writes 4.5<f(x)-L<5.5] So it would want it be like greater than 4.5 or less than 5.5 but it doesn't say like it can be 4.5
439	Int.	Hm
440	Adriana	So I think that's what that means?
441	Int.	So I forgot to ask one thing, what is f(x)-L
442	Adriana	f(x)
443	Int.	I'm sorry, what is x-a?
444	Adriana	Oh
445	Int.	Do you think, in the story?
446	Adriana	It's the error of the batter
447	Int.	It's the error of the batter
448	Adriana	Yeah
449	Int.	And f(x)-L?
450	Adriana	Is the error of the pancake size
451	Int.	Ok, I see, ok, um alright, so can you sort of because it seems like you've, it seems like you've talked a little bit about this stuff so can you say more, can you say again what this [if-then statement] is about?
452	Adriana	Um so I think this is the, like these delta and the epsilon they're the error bounds of like what I want, relating back to the story so like f(x) minus L since that's the error of the pancake size so like say we want this [points at f(x)-L] to be as close to 5 as possible or yeah that was the point but we want it to be within 1/2, bound [points at the inequality 4.5<f(x)-L<5.5] so this [points at 4.5] is the smallest it could be this [points at 5.5] is the biggest it could be, but we want it to be within that range [points at the inequality] so that's why I was looking for to see if it said that it *can be* equal to 4.5
453	Int.	Um hm
454	Adriana	But it's not so that's why it doesn't have the equal sign. Um so yeah so we want it to be within these two quantities [refers to 4.5 and 5.5] or whatever
455	Int.	Um hm

456 Adriana And then we would have to, well cus it says like this one [points at |x-a|<delta] first, right? So this one [points at the delta inequality] would be like the batter if, if our bound for yeah so if this was the the error of batter [writes error of batter below |x-a|], um then then we would get [points at the 4.5<f(x)-L<5.5] like, then this [points at the delta inequality then quickly back to 4.5<f(x)-L<5.5], this is a result of how much batter [points at the delta inequality] we're using. So size is a result of the batter that we're using.

457 Int. Ok

458 Adriana But it's like it goes back to this [points to for every number epsilon, there exists a delta] too cus if there's an epsilon [places finger on epsilon in the statement] which means like there like there's gonna be an error bound for here [points at 4.5 and 5.5] then there has to be an error bound for this one [refers to the delta inequality], but this one [points to the delta inequality] is the one that like manipulates that one [points 4.5<f(x)-L<5.5]

459 Int. Ok so so can you say one more time like what is this about again?

460 Adriana Oh, this is

461 Int. Like what is it saying?

462 Adriana Oh this is saying that if there is, basically if you're not like hitting a exactly

463 Int. Um hm

464 Adriana If you're not hitting a exactly you're pick another x that um and then so that means there's gonna be an, there's going to be an error here

465 Int. Um hm

466 Adriana Then you're not going to hit this exactly, the limit exactly so you're gonna get something close to it so that means there has to be an error here. So because there's an error here you're gonna get an error here

467 Int. Um hm, ok, um ok. Why do we need for every number epsilon?

468 Adriana Hm.

469 Int. In the story?

470 Adriana Um, [pause] um I think just cus it's um just cus we will have an error here so if we can't get 5 inches exactly like if we have to have a bound 4.5 to 5.5 like if we're trying to get anywhere within that then that means we're using, like we're not using exactly 1 cup of batter so we're, we're like we're either using more or less of 1 cup

471 Int. Um hm

472 Adriana So if we're getting more or less than the 5 inches which is epsilon, um then we're gonna get a number that's like more or less ofthe 1 cup so it's kinda like um. Ok hold on, see if you understand.

473 Int. Um hm

474 Adriana If we're not hitting 5 exactly then we're not hitting 1 exactly here so if we have an error down here then we're gonna have, er with the pancakes then we're gonna have an error with the batter for every number epsilon which is like an error with the pancakes, we're gonna have um a delta

475 Int. Oh I see, so I, I'm not asking about the whole statement, I'm saying about why not just for one number epsilon, why do we have for every number epsilon?

476	Adriana	Um, because like oh I guess kinda how I said like, the batter and the pancake was kinda like a proportional thing I think that's why. Like if you're choosing different, for every, for every manipulation that you're making you're gonna have a specific answer for it
477	Int.	Is epsilon changing?
478	Adriana	If you're changing, oh. No?
479	Int.	In the story is epsilon changing?
480	Adriana	Uh no it's, well it's eventually it will but not, not as [inaudible]
481	Int.	Not as of?
482	Adriana	Cus it says like maybe, maybe the boss will ask you later to make it a smaller error bound but in terms of the task right there it's 1 1/2 an inch
483	Int.	I see
484	Adriana	But epsilon isn't changing in this case
485	Int.	/Ok
486	Adriana	/Like if I said epsilon was the error bound then that's not what's changing, what's changing is the error but um yeah I don't know
487	Int.	Ok, ok at this some point I will be asking some final questions, you answered a lot of these already but I wanted to ask again as some might have changed since the last time we talked about it. Feel free to say the same thing if it has not ok?
488	Adriana	Ok
489	Int.	Can you try to explain what the formal definition is saying using example 1 or um basically, this thing, can you say what the formal definition?
490	Adriana	Um
491	Int.	So
492	Adriana	Um, do you want me to use this to explain that?
493	Int.	Yeah
494	Adriana	Oh um. Um ok so I wouldn't know how to use this part though but so 3x +2 is our f so that's our function um and if we're looking at like um all x that is approaching so the limit of x as, the limit as x approaches 1 of this, this function or whatever at a, 1 is [muttering to self, inaudible]
495	Int.	Feel free to explain what, so let's do this, why don't we try to explain um what that means. I'm curious as to what you think that is saying now, after that, after we've gone over the story
496	Adriana	Um so like ok so I'll try to ok. Um so like if f is a function so f can be like any function
497	Int.	Um hm
498	Adriana	And it says it's defined on an open interval that contains a so yeah a is just a point in this like domain I guess
499	Int.	/Um hm
500	Adriana	/Or whatever. Um and it says except possible at a itself so sometimes a won't be defined but the point is a is on this interval
501	Int.	Um hm

502	Adriana	Um so then we're trying to see, we're trying to see that as x, like our interval is our x, we're trying to go from, if we're well I guess it's kinda backwards too but like if we're looking at as our x axis or whatever as it's approaching a, like or we could actually hit a or whatever like if that's on our function, like where, we're trying to see the limit of it so we're trying to see where it's where it's at at that point a. Um, I can explain it with a picture or something. Uh, um, ok so um um ok so I'm just gonna yeah so if 3x+2 is our function and it says like it's defined on an open interval so it's going fromnegative infinity to infinity or whatever but we just want to know where it is close to 1 so we're just concerned about this a, this a point that is in this interval so as like our x axis or like whatever like our quantities that we're trying to look at we just want to see like where it's at close to 1
503	Int.	Um hm
504	Adriana	And so as x approaches 1 or at 1 um we get 5 or whatever
505	Int.	Um hm
506	Adriana	So I guess yeah. Um, yeah so I think I think a is just where we're concerned about getting close to. So even like, if, like if this was our equation for our pancake thing and we're trying to get close to 1 cup of batter or whatever we would just plug in 1 cup of batter to get to this
507	Int.	Um hm
508	Adriana	Number but if we can't say it's not defined at 1 or we're not getting the right result, we would look at numbers that are really close to 1
509	Int.	Um hm
510	Adriana	But I um, yeah
511	Int.	So that's what the formal definition is saying?
512	Adriana	Yeah. So yeah. So I think I think the point of this is that we're looking at the number a so that's what we're concerned about is looking at the function cus if it's like defined on an open interval than this could be a lot of things but we're just looking at this specific number a cus that's what we're concerned about. Um and then, yeah, but sometimes it says it's probably not defined at a itself
513	Int.	Um hm
514	Adriana	So we just want to see what this function looks like or does or
515	Int.	Um hm
516	Adriana	Yeah, like close to a
517	Int.	Um hm, so I didn’t' hear any epsilon or delta or anything like that
518	Adriana	Oh I said that I didn't know how to
519	Int.	You didn't know how to do it ok alright, that's fine. Um, did your explanation change at all and if it did, why?
520	Adriana	Um it changed from last time just cus I thought we were looking at x and trying to pick a point a to be close to x but now that I think about it, well I guess I just read this better today we're looking mor at a than anything so a, I would probably switch these so a would be there and trying to get close to x and yeah so we're trying to see where, we're trying to get close to a and if there's no a close if there's no a, like if it's not defined at a we're trying to find and x that's really close to a
521	Int.	Um hm

522	Adriana	Um so in that sense it did change
523	Int.	How did that change? Or why did that change?
524	Adriana	I just read it better, cus I was looking at it backwards
525	Int.	Hm. Ok, uh what is that you're trying to do within the formal definition? Is there anything we're trying to figure out?
526	Adriana	Um, we're trying to figure out what let's see if, we're trying to figure out like what. What I'm thinking about is like a function is like, not just the function for itself, so like applying it to something. So if we’re looking at um, like this pancake story or whatever like if we're looking at that then we want to have uh like a result so if we're, if we're plugging in like these numbers of batters we want to see what we're getting like how big our diameter is for our for um for our pancake. Um, and then like not just that so we're, so we're concerned with something that's close to that 1 cup of batter so if we can't get that exact one we're just concerned with what it looks like more or less than that 1 that 1 cup of batter to see how that affects our pancake. Um, and then we wanna see we wanna make sure that we're looking at those errors um just so we're not picking super random numbers like we have a constraint so we want to stay within that to use that the errors and the error bounds to help us get closer and closer to that a or that 1 cup of batter. To get closer and closer to our like perfect pancake
527	Int.	Can you explain that just one more time? So that I just make sure that I understand you?
528	Adriana	Ok. Um. Ok so if we're looking at, cus like I said, if we're just looking at random functions then it could be anything but if we’re trying to apply it to something um like let's say we have a function for getting pancakes or whatever
529	Int.	Um hm
530	Adriana	Like for making pancakes and we have a specific function and we're trying to get, we're trying to get really, we're trying to get a which is our 1 cup of batter but at a we're not getting our like what we want which is our, our limit, we're not getting our 5 inch perfect pancake
531	Int.	Um hm
532	Adriana	So we're looking at xs which is like other quantities of batter
533	Int.	Um hm
534	Adriana	So but we want it to obviously be really really close to a, which is our 1 cup, we're only concerned with numbers close to a
535	Int.	Um hm
536	Adriana	Um to get something our f of x really really really close to our 5 inch um pancake and. Oh and we want we're concerned with the errors and the epsilon and delta so that we can we're not picking like random random numbers we're staying within a bound that's really close to *a* and [writing] so this would be at at the most this would be delta or whatever
537	Int.	Ok
538	Adriana	Can't draw a delta sign. Yeah so we're only concerned with x's that are within the delta so that we're only getting answers that are within the epsilon. This whole thing. Um [inaudible] um, where was I? Oh yeah so we want the errors to be like really close to a so that we're only looking at numbers close to the a. So that we're getting a really small error to get like the perfect pancake

539	Int.	Um hm. Ho- did that change at all?
540	Adriana	From?
541	Int.	From before
542	Adriana	Yes
543	Int.	And how did that change?
544	Adriana	Oh just cus I actually can see something that it's applied to and not just like this random function that I'm using
545	Int.	Um hm
546	Adriana	Um, but yeah I think when you're, when you know you're trying to get something specific then that's when you'll like try different uh, that's when you know you're trying to get really really close to that, that number and it's not just this point a on the graph or whatever
547	Int.	Ok, with epsilon delta what depends on what if anything?
548	Adriana	Um, the del-, the epsilon depends on the delta
549	Int.	Did you change your mind? Actually, you said the same thing
550	Adriana	Yeah. So yeah, I think the delta depends on the epsilon cus=
551	Int.	Did anything change?
552	Adriana	Um, I think well, I mean in this case.. I think it can- it caaan they can kinda depend on
553	Int.	Each other
554	Adriana	Both, yeah in a sense because, but more like whatever you're getting like f(x) is always gonna depend on what x you're inputting it but then if you want to get something like that's clos- like within delta you need to see if you, like if for example here [*points to the pancake story*] like that our epsilon here was already like set [*points back and forth between the .5's in the inequality .5<f(x)-L<.5*] then that kind of depended on what we were imple- putting in for x [*points at a region around x on the graph*] but.. but mostly whatever you putting in for x will determine what you get for f(x). So I I still say the same thing like delta depends on epsilon but
555	Int.	Delta depends on epsilon? Or epsilon depends?
556	Adriana	No, yeah, epsilon depends on delta
557	Int.	Um hm
558	Adriana	But, you like if if epsilon's like already set then you'll manipulate your, your delta so it's like within an error bound and then um and then continue to manipu-. Wait [long pause] wait, so you're. Hm.
559	Int.	What's happening?
560	Adriana	Oh cus I'm thinking like cus I thought that the epsilon and the delta were the errors but they're the error bounds
561	Int.	Uh huh
562	Adriana	And if the epsilon is already set then you would have to change your delta
563	Int.	Uh huh
564	Adriana	Yeah, so I mean, I guess if your epsilon is already set then your delta would depend on epsilon [pauses]
565	Int.	What just happened?

566	Adriana	Uh, [laughs] like well cus just just looking at this [points back and forth between the pancake story and the inequality .5<f(x)-L<.5] um if I said like epsilon was an error bound and if they already give me an error bound like I want.. my result to be within this error bound here [circles the inequality .5<f(x)-L<.5] then like.. then I would try to manipulate um my errors here [points to a small range on the x axis on the graph] to be within a smaller error bound [points at delta in the delta inequality in the definition] which would be, delta would be [quietly] the biggest it can be... Huh.. I'm confused.
567	Int.	Why are you confused?
568	Adriana	Because if epsilon did depend on delta then I could change it here [*points at the inequality .5<f(x)-L<.5*] or I mean like. I'm confused because they gave me an epsilon [*points at the inequality .5<f(x)-L<.5*]
569	Int.	Um hm
570	Adriana	And it's already set
571	Int.	Um hm
572	Adriana	And they didn't give me a delta so in that sense it didn't depend on delta…
573	Int.	Um hm
574	Adriana	But then the delta, I would want it to be really close to.. or I would want my error bound to, to be really small to, to, to like accommodate or whatever the error bound that was already given or the epsilon that was already given to me so
575	Int.	Um hm
576	Adriana	[*Quietly*] Epsilon could depend on delta? I mean, delta could depend on epsilon, or does depend on epsilon...
577	Int.	So, do they depend on each other, is it just one way now?
578	Adriana	Um, see cus I was looking at it like the x or the f(x) or the yeah, the f(x) depends on the x and that's how I was like saying that epsilon depends on delta because epsilon like is related to the f(x) or whatever
579	Int.	Um hm
580	Adriana	But that's just saying the error of the the L and the f(x) depends on the a and x but that's not to say that epsilon depends on delta
581	Int.	Ok, so?
582	Adriana	So, I, I think that delta depends on epsilon now [*laughs*]. Just cus if it's given like this [*unclear what she's referring to, possibly the story*] and you're trying to aim at getting like a certain, within a certain error bound then you're gonna try to manipulate your entries or whatever to be within a certain error bound [*gestures a small horizontal interval with her palms*]
583	Int.	Ok. Alright, so and so you changed your mind it seems? Um, so how did that happen? Why did you change your mind?
584	Adriana	Because I was like given an epsilon [*points at the inequality .5<f(x)-L<.5*] and that's kinda like the main goal
585	Int.	Um hm
586	Adriana	The main goal is to get the pancake, like that's the main goal and they gave me like a constraint or whatever
587	Int.	Um hm

589	Int.	Ok, of the four variables epsilon delta x and y, which one or which ones are you trying to control?
590	Adriana	Wait, what's y?
591	Int.	Oh sorry, epsilon delta, x and f(x), which one or which ones are you trying to control?
592	Adriana	Um, I'm trying to control x and f(x), wait x
593	Int.	So
594	Adriana	Yeah
595	Int.	Just x
596	Adriana	Just x
597	Int.	Did you change your mind?
598	Adriana	Um, I think I did.
599	Int.	And if you did? Why?
600	Adriana	I think I said that we're trying to manipulate delta and epsilon but that's because I was thinking of delta and epsilon as the error
601	Int.	Um hm
602	Adriana	And not the error bound
603	Int.	Yeah, you, yeah you said epsilon and delta to get the best estimate but now you're thinking that you're trying to control x. Um, what why did you change your mind?
604	Adriana	Um because that's ultimately what we're manipulating in the, like that's the only thing we can manipulate in the whole thing cus that's like what we're inputting, that's the only thing we can
605	Int.	Um hm
606	Adriana	Like manipulate
607	Int.	So between epsilon and delta, which one comes first and which one do you figure out as a result?
608	Adriana	Uh which one do you figure out first?
609	Int.	No no which one comes first and which one do you figure out as a result of that one
610	Adriana	Oh epsilon comes first and then delta you figure out
611	Int.	Um hm, um did that change?
612	Adriana	Um, I think so
613	Int.	You think so? /Um
614	Adriana	/I don't remember. Yeah I think I did change because I was thinking like that whatever whatever um change I was making here so I was thinking of it as errors so I was thinking of it like whatever change I was making the x and the a
615	Int.	Um hm
616	Adriana	Would affect the f(x) and the L
617	Int.	Yeah you said delta first and the epsilon as a result because x to get f(x) you control the input first then the output, yeah so that changed. How did that change?
618	Adriana	That changed because like the, the ultimate goal is to get close to L or to get to the limit
619	Int.	Um hm

620	Adriana	So we want like, we want a goal to be within in we don't want like a 100 close, so we want it to be very very close so we said like within a half
621	Int.	Um hm
622	Adriana	Um so then that's how we would figure out like delta it's not gonna be like a hundred pounds of batter, we want it to be closer if we want it to be within 5 inches of diameter, or close to 5 inches in diameter
623	Int.	How would you put the four variables in order like epsilon delta x and f(x) in order in terms of which comes first
624	Adriana	Hm, you said epsilon, delta, f(x)?
625	Int.	x, um epsilon and delta and before you had the a and the L in there as well, you ordered it that way it seems. I'm wondering how would you put them in order now?
626	Adriana	Ok so now I would put them in, wait we don't have the a and the L?
627	Int.	I don't know if you
628	Adriana	Wait you said
629	Int.	Yeah I said the four variables but last time you added the a and the L
630	Adriana	Um, I would put the L first and then the a
631	Int.	Um hm
632	Adriana	And then epsilon and then hm. [pause] Ok I would put I'll just put delta and then x and f(x)
633	Int.	Um hm so as you can see it changes quite a bit so why did that change?
634	Adriana	Um, because like L is our ultimate goal or whatever, that's what we're trying to get to, a is what we're trying to input to get to L, like this would be the perfect number
635	Int.	Um hm
636	Adriana	Epsilon would be how close we're trying to get to L or within how much can we get to L
637	Int.	Um hm
638	Adriana	Delta like see I was confused, I paused for delta cus I was thinking of if you could just find an estimate
639	Int.	Um hm
640	Adriana	Like if you have no idea, but I suppose you could just find an estimate or a constraint for um for how close you want to get or how yeah, how close of an input you want to put that close to a and then x would be anything within delta like anything within that point that you want to plug in and f(x) would be kinda like your result of what you're getting
641	Int.	Um hm
642	Adriana	After you're estimating
643	Int.	Um hm so you explained each one but I'm not sure why you ordered them in that way?
644	Adriana	Oh um, cus if you're well first I would put your ultimate goal as the first one which is the L. And then what you're trying to get to to look at to get to your L is a which is why I chose those two
645	Int.	Um hm
646	Adriana	First
647	Int.	Um hm

648	Adriana	And then I put like epsilon next because we're trying to get really close to L
649	Int.	Um hm
650	Adriana	And we like we don't wanna go this open interval, whatever, so I chose epsilon next and then delta kinda similar to epsilon we want something that's really close to a so we put a gap or not like a gap like a constraint on it too so we're not going to wide
651	Int.	Um hm, but why epsilon first? Instead of delta first?
652	Adriana	Um because we're trying to get close to our ultimate goal, L
653	Int.	Ok
654	Adriana	Yeah so then delta
655	Int.	Um hm
656	Adriana	And then x next because that's what we're gonna try to get close to delta or like within delta to get an f(x) that will give us like an estimate or a result that is within epsilon
657	Int.	Um hm
658	Adriana	Or just give it [inaudible]
659	Int.	So did you say why it changed?
660	Adriana	Oh um
661	Int.	Why or how it changed?
662	Adriana	I think these two [circles f (x) and x from old order and L and a from the new one] I changed them because I thought that these [points at x and a] were each other.
663	Int.	Oh right right, ok
664	Adriana	So I thought we were trying to get close to x [instead of a].
665	Int.	And
666	Adriana	And then these [circles ε and δ] changed because I thought this was the difference, so I thought these [circles ε and δ] were the difference so I put them last
667	Int.	Oh I see
668	Adriana	It's kinda like, cus I was thinking of it like, oh analyze our /errors at the end.
669	Int.	/errors. I see I see ok
670	Adriana	And then these [circles f (x) and x] I put them last because I realized these [circles f (x) and x] are what we can control based on all of these [circles L, a, ε, δ].
671	Int.	Hm, I see
672	Adriana	Well this, this [points at x] is what we can control. Once we have all of this [L, a, ε, δ] laid out then I can start picking x's that are close to a and then f of x [f (x)] will be what I get. And then I can compare them to what I have [circles L, a, ε, δ]
673	Int.	Um, why does the definition start with epsilon?
674	Adriana	Cus that's that's what we're trying to get close to the limit or like, but if the limit's our goal we're trying to get really close to the limit so it starts with epsilon to show that we're, how close we want to get

Definition. Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then we say that the **limit of $f(x)$ as x approaches a is L**, and we write

$$\lim_{x \to a} f(x) = L$$

if for every number $\epsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if} \quad 0 < |x - a| < \delta \quad \text{then} \quad |f(x) - L| < \epsilon$$