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Finsler bordifications of symmetric and certain locally symmetric spaces

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Abstract

We give a geometric interpretation of the maximal Satake compactification of symmetric spaces $X = G/K$ of noncompact type, showing that it arises by attaching the horofunction boundary for a suitable G-invariant Finsler metric on X. As an application, we establish the existence of natural bordifications, as orbifolds with corners, of locally symmetric spaces which are orbifold quotients X/Γ by arbitrary uniformly weakly regular subgroups $\Gamma < G$. These bordifications result from attaching Γ-quotients of suitable domains of proper discontinuity at infinity. We further prove that such bordifications are compactifications in the case of weakly regular conical antipodal ($=\tau_{mod}$-RCA) subgroups, equivalently, Anosov subgroups. We show, conversely, that τ_{mod}-RCA subgroups are characterized by the existence of such compactifications. As one of the applications of our methods we give a positive answer to a question of Peter Haïssinsky on convergence group actions for torsion-free hyperbolic groups.

Contents

1 Introduction

The goal of this paper is two-fold:

1. We give a geometric interpretation of the *maximal Satake compactification* $\overline{X}^S_{max}$ (see [BJ, Chapter 2]) of symmetric spaces $X = G/K$ of noncompact type. (G is the connected component of the isometry group of X.) We prove that this compactification is G-equivariantly homeomorphic, as a manifold with corners, to a *regular Finsler compactification* $\overline{X}^{\bar\theta}$ of X, obtained by adding to X points at infinity represented by *Finsler horofunctions*. These horofunctions arise as limits, modulo additive constants, of distance functions

$$d^{\bar\theta}_x = d^{\bar\theta}(x, \cdot)$$

where $d^{\bar\theta}(x, y)$ is a certain G-invariant Finsler distance on X associated with a regular direction $\bar\theta$ in the model spherical chamber σ_{mod} of X. It turns out that the particular choice of $\bar\theta$ is irrelevant, as long as it is an interior point of σ_{mod}. Such *horofunction boundary* constructions of compactifications of metric spaces are quite standard. The novelty is finding the *right* metric on X which yields the maximal Satake compactification.

Theorem 1.1. *For every regular type* $\bar\theta \in \mathrm{int}(\sigma_{mod})$,

$$\overline{X}^{\bar\theta} = X \sqcup \partial_\infty^{\bar\theta} X$$

is a compactification of X *as a* G*-space which satisfies the following properties:*

(i) There are finitely many G*-orbits* $S_{\tau_{mod}}$ *indexed by the faces* τ_{mod} *of* σ_{mod}*. (*$X = S_\emptyset$*.)*

(ii) The stratification of $\overline{X}^{\bar\theta}$ *by* G*-orbits is a* G*-invariant manifold–with–corners structure.*

(iii) There is a K*-equivariant homeomorphism of* $\overline{X}^{\bar\theta}$ *to the closed unit ball in* X *centered at the fixed point of* K *with respect to the dual Finsler metric* $d^*_{\bar\theta}$ *on* X*. In particular,* $\overline{X}^{\bar\theta}$ *is homeomorphic to the closed ball.*

(iv) The compactification $\overline{X}^{\bar\theta}$ *is independent of the regular type* $\bar\theta$ *in the sense that the identity map* id_X *extends to a natural homeomorphism of any two such compactifications.*

(v) There exists a G*-equivariant homeomorphism of manifolds with corners between* $\overline{X}^{\bar\theta}$ *and the maximal Satake compactification* $\bar{X}^S_{max}$ *which yields a natural correspondence of strata.*

Remark 1.2. (i) We also give a geometric interpretation of the points in $\partial_\infty^{\bar\theta} X$ as strong asymptote classes of Weyl sectors, see Remark 3.26.

(ii) The strata $S_{\tau_{mod}} \subset \overline{X}^{\bar\theta}$ are disjoint unions of *small strata* X_τ, where the τ's are simplices of type τ_{mod} in the Tits building of X, i.e. elements of the partial flag manifold $\mathrm{Flag}_{\tau_{\mathrm{mod}}}(\mathrm{X}) = G/P_{\tau_{mod}}$. Note that the full flag manifold $\mathrm{Flag}_{\sigma_{\mathrm{mod}}}(\mathrm{X}) \cong G/B \cong \partial_{F\ddot{u}} X$ is the Fürstenberg boundary of X. Each small stratum is naturally identified with a symmetric subspace of X, namely the cross section of a parallel set. A subset $S \subset \partial_\infty^{\bar\theta} X$ is called *saturated* if it is a union of small strata.

(iii) The Finsler view point had emerged in several instances during our earlier study [KLP1, KLP2, KLP3] of asymptotic and coarse properties of regular discrete isometry groups acting on symmetric spaces and euclidean buildings. For instance, the notion of *chamber* or *flag convergence*, see [KLP1, §7.2] and [KLP2, §5.3], is a special case of the Finsler convergence at infinity considered in this paper. Furthermore, the *Morse Lemma* proven in [KLP3] can be rephrased to the effect that regular quasigeodesics in symmetric spaces and euclidean buildings are uniformly close to Finsler geodesics. In the same vein, *Morse subgroups* $\Gamma < G$ can be characterized as Finsler quasiconvex.

(iv) The maximal Satake compactification is known to carry a G-invariant real-analytic structure, see [BJ].

Remark 1.3. After finishing this work we learnt about the recent work of Anne Parreau [P] where she studies the geometry of CAT(0) model spaces, i.e. of symmetric spaces of noncompact type and euclidean buildings, from a very natural perspective, regarding them as metric spaces with a vector valued distance function with values in the euclidean Weyl chamber Δ (called Δ-distance in our paper). Among other things, she shows that basic properties of CAT(0) spaces persist in this setting, notably the convexity of the distance, and develops a comparison geometry for the Δ-distance function. Furthermore, she proves that the resulting Δ-valued horofunction compactifications of model spaces are naturally homeomorphic to their maximal Satake compactifications.

2. Our main application of Theorem 1.1 is to discrete subgroups $\Gamma < G$. Recall that if X is a negatively curved symmetric space, then the locally symmetric space X/Γ (actually, an orbifold) admits the standard bordification

$$X/\Gamma \hookrightarrow (X \cup \Omega(\Gamma))/\Gamma$$

where $\Omega(\Gamma) \subset \partial_\infty X$ is the domain of discontinuity of Γ. The quotient $(X \cup \Omega(\Gamma))/\Gamma$ is an orbifold with boundary $\Omega(\Gamma)/\Gamma$. Furthermore, a subgroup Γ is *convex cocompact* if and only if $(X \cup \Omega(\Gamma))/\Gamma$ is compact.

In our earlier papers [KLP1, KLP2, KLP3], we introduced several classes of discrete subgroups Γ of semisimple Lie groups G, generalizing the notions of discreteness and convex cocompactness in rank 1. These classes are defined relative to faces $\tau_{mod} \subseteq \sigma_{mod}$, equivalently, with respect to conjugacy classes of parabolic subgroups of G. The most important (for the purposes of this paper) of these classes are:

1. τ_{mod}-regular and *uniformly τ_{mod}-regular* discrete subgroups Γ of higher rank Lie groups G; in the rank 1 case these conditions amount to discreteness of the subgroup.

2. τ_{mod}-RCA subgroups: These are subgroups of $\Gamma < G$ which are τ_{mod}-regular and their limit sets $\Lambda_{\tau_{mod}}(\Gamma) \subset \mathrm{Flag}_{\tau_{\mathrm{mod}}}(\mathrm{X})$ are *conical* and *antipodal*.

3. τ_{mod}-URU subgroups: These are τ_{mod}-uniformly regular (finitely generated) undistorted subgroups $\Gamma < G$.

4. τ_{mod}-asymptotically embedded subgroups $\Gamma < G$: These are intrinsically Gromov hyperbolic, τ_{mod}-regular subgroups of G, whose limit sets $\Lambda_{\tau_{mod}}(\Gamma)$ are antipodal and equivariantly homeomorphic to the Gromov boundary of Γ.

Remark 1.4. (i) Our regularity conditions capture the asymptotic behavior of divergent sequences in discrete subgroups $\Gamma < G$ with respect to the strata of the Finsler ideal boundary of X (in the τ_{mod}-regular setting) and of the visual ideal boundary of X (in the τ_{mod}-uniformly regular setting). For instance, let $\Lambda(\Gamma) \subset \partial_\infty X$ be the limit set of Γ, i.e. the accumulation set at infinity of the orbits $\Gamma x \subset X$ in the visual compactification $\overline{X}$ of X. Similarly, let $\Lambda^{\bar\theta}(\Gamma, x) \subset \overline{X}^{\bar\theta}$ be the accumulation sets of the orbits $\Gamma x \subset X$ in the Finsler compactifications $\overline{X}^{\bar\theta}$. Then Γ is τ_{mod}-regular iff $\Lambda^{\bar\theta}(\Gamma, x)$ is contained in the closure of the (big) stratum $S_{\tau_{mod}} \subset \partial^{\bar\theta}_\infty X$. (This is independent of x.) Accordingly, $\Lambda_{\tau_{mod}}(\Gamma)$ is the set of simplices $\tau \in \mathrm{Flag}_{\tau_{\mathrm{mod}}}(\mathrm{X})$ such that $\Lambda^{\bar\theta}(\Gamma, x) \cap \overline{X}_\tau \neq \emptyset$ for one (equivalently, any) $x \in X$. Similarly, Γ is uniformly regular iff $\Lambda(\Gamma)$ consists only of regular points of $\partial_\infty X$.

(ii) The classes 2, 3 and 4 are higher-rank analogues of convex cocompact subgroups of rank 1 Lie groups, reflecting various aspects of "geometric finiteness" of $\Gamma < G$.

In [KLP2] we also gave a (the first) definition of Anosov subgroups $\Gamma < G$ which avoids the language of geodesic flows; these are the (P-)Anosov subgroups of G defined earlier in [L, GW]. In [KLP2, KLP3] we proved that the classes 2, 3 and 4 coincide and are equal to the class of (τ_{mod}-)Anosov subgroups. In the regular case $\tau_{mod} = \sigma_{mod}$, we will refer to (uniformly) σ_{mod}-regular and σ_{mod}-RCA subgroups simply as *(uniformly) regular* and *RCA*.

Our applications of Finsler compactifications to discrete groups establish the existence of natural *bordifications*, as orbifolds with corners, of locally symmetric spaces which are orbifold quotients X/Γ by *arbitrary* uniformly τ_{mod}-regular subgroups $\Gamma < G$. We further prove that such bordifications are *compact* in the case of *uniformly regular conical subgroups* (when $\tau_{mod} = \sigma_{mod}$) and, in full generality, for τ_{mod}-RCA subgroups.

We now state our results first for uniformly regular conical subgroups and then in general.

Theorem 1.5. *Let $\Gamma < G$ be a uniformly regular subgroup.*

(i) There exists a natural saturated Γ-invariant open subset $\Omega_{Th}(\Gamma) \subset \partial_\infty^{\bar\theta} X$ such that the action

$$\Gamma \curvearrowright X \cup \Omega_{Th}(\Gamma) \subset \overline{X}^{\bar\theta} \tag{1.6}$$

is properly discontinuous. The quotient

$$(X \cup \Omega_{Th}(\Gamma))/\Gamma$$

provides a real-analytic bordification of the orbifold X/Γ as an orbifold with corners.

(ii) If, in addition, the chamber limit set $\Lambda_{ch}(\Gamma) \subset \partial_{F\ddot{u}} X$ is conical, then the action (1.6) is also cocompact. In particular, this provides a real-analytic compactification of the orbifold X/Γ as an orbifold with corners. The boundary part of this orbifold is the quotient $\Omega_{Th}(\Gamma)/\Gamma$.

The domain $\Omega_{Th}(\Gamma)$ at infinity results from the Finsler ideal boundary $\partial_\infty^{\bar\theta} X$ by removing a suitable *thickening* of the chamber limit set $\Lambda_{\sigma_{mod}}(\Gamma)$, compare (6.20).

This theorem is a combination of Theorems 6.21 and 7.6, and Corollaries 6.22 and 7.8.

Remark 1.7. While the compactification $\overline{X}^{\bar\theta}$ is independent of $\bar\theta$ as long as the latter is regular, the subset $\Omega_{Th}(\Gamma)$ depends on the choice of the regular type $\bar\theta$, which, in this theorem, has to be an *almost root type*, see Definition 6.7. Different root types yield in general different domains $\Omega_{Th}(\Gamma)$. For instance, if $\Gamma < PSL(2,\mathbb{R}) = G$ is a cocompact Fuchsian subgroup and $\Gamma' < G \times G$ is the image of Γ under its diagonal embedding into $G \times G$, then the two different root types of $G \times G$ yield two different (but homeomorphic) compactifications of $(\mathbb{H}^2 \times \mathbb{H}^2)/\Gamma'$, namely $(\mathbb{H}^2 \times \overline{\mathbb{H}^2})/\Gamma'$ and $(\overline{\mathbb{H}^2} \times \mathbb{H}^2)/\Gamma'$.

We now turn to the general case of uniformly τ_{mod}-regular subgroups $\Gamma < G$.

The group W which appears below is the Weyl group of X, the map $\iota : \sigma_{mod} \to \sigma_{mod}$ is the opposition involution of the model spherical Weyl chamber of W, the subgroup $W_{\tau_{mod}} < W$ is the stabilizer of the face $\tau_{mod} \subset \sigma_{mod}$. We refer the reader to section 8.4 for the precise definitions of thickenings. For now, the reader can think of $\mathrm{Th} \subset W$ as an auxiliary combinatorial datum, a "thickening" of the neutral element inside W, which is used to define the *Finsler thickening* $\mathrm{Th}^{\bar\theta}(\Lambda_{\tau_{mod}}(\Gamma))$ of the limit set $\Lambda_{\tau_{mod}}(\Gamma) \subset \mathrm{Flag}_{\tau_{\mathrm{mod}}}(\mathrm{X})$, a certain Γ-invariant saturated compact subset of $\partial_\infty^{\bar\theta} X$.

The following result is a combination of Theorems 9.18 and 10.20.

Theorem 1.8. *Let $\Gamma < G$ be a uniformly τ_{mod}-regular subgroup. Then:*

(i) For each balanced $W_{\tau_{mod}}$-invariant thickening $Th \subset W$, the action

$$\Gamma \curvearrowright X \cup \Omega^{\bar\theta}_{Th}(\Gamma) := \overline{X}^{\bar\theta} - Th^{\bar\theta}(\Lambda_{\tau_{mod}}(\Gamma))$$

is properly discontinuous. The quotient

$$\left(X \cup \Omega^{\bar\theta}_{Th}(\Gamma)\right)/\Gamma \tag{1.9}$$

provides a real-analytic bordification of the orbifold X/Γ.

(ii) If, in addition, Γ is τ_{mod}-RCA, then $\left(X \cup \Omega^{\bar\theta}_{Th}(\Gamma)\right)/\Gamma$ is compact. In particular, this provides a real-analytic compactification of the orbifold X/Γ as an orbifold with corners. The boundary part of this orbifold is the quotient $\Omega^{\bar\theta}_{Th}(\Gamma)/\Gamma$.

Thus, our Theorems 1.5 and 1.8 establish the existence of natural compactifications (as orbifolds with corners) for the locally symmetric spaces X/Γ by attaching Γ-quotients of suitably chosen saturated domains in the Finsler ideal boundary of X.

More abstractly, we say that a discrete subgroup $\Gamma < G$ is *S-cocompact* if there exists a Γ-invariant saturated open subset $\Omega \subset \partial_\infty^{\bar\theta} X$ such that Γ acts properly discontinuously and cocompactly on $X \cup \Omega$. (Note that no regularity is assumed in this definition. For instance, all uniform lattices $\Gamma < G$ are S-cocompact with $\Omega = \emptyset$.) Theorem 1.8 shows that τ_{mod}-RCA subgroups of G are S-cocompact with $\Omega = \Omega^{\bar\theta}_{Th}(\Gamma)$.

In section 11 we also prove the converse to the last theorem:

Theorem 1.10. *Uniformly τ_{mod}-regular S-cocompact subgroups of G are τ_{mod}-RCA.*

Combining the last two theorems, we obtain:

Corollary 1.11. *A uniformly τ_{mod}-regular subgroup $\Gamma < G$ is τ_{mod}-RCA if and only if it is S-cocompact.*

Our cocompactness results thus provide a precise higher-rank analogue of the characterization of convex cocompact subgroups of rank 1 Lie groups in terms of compactifications of the corresponding locally symmetric spaces.

In section 10.4 we use our proof of cocompactness part of Theorem 1.8 to verify a conjecture by Peter Haïssinsky on cocompactness properties for convergence group actions on compact metrizable spaces in the case of torsion free hyperbolic groups.

While proving Theorem 1.10, we establish yet another coarse-geometric characterization of τ_{mod}-RCA subgroups of G as uniformly τ_{mod}-regular subgroups in G which are *coarse retracts*, see sections 2.7 and 11 for the details. This theorem is a higher-rank analogue of the characterization of quasiconvex subgroups of Gromov-hyperbolic groups as coarse retracts. Restricting to the regular case, our Theorem 1.10 proves that the *antipodality* condition in the definition RCA is redundant in the context of uniformly regular subgroups:

Theorem 1.12. *Uniformly regular conical subgroups of G are RCA.*

Remark 1.13. We note that the existence of an orbifold-with-boundary compactification of locally symmetric quotients by Anosov subgroups of some special classes of simple Lie groups (namely, $Sp(2n,\mathbb{R}), SU(n,n), SO(n,n)$) appears in [GW].

Remark 1.14. Except for the application of our cocompactness argument to convergence actions (Theorem 10.22), all results in this version of our preprint were already contained in its second version. Shortly after that version, the preprint [GGKWa] appeared, also addressing the compactification of locally symmetric spaces. The main claims (Theorems 1.1 and 1.2) of [GGKWa] are weaker than our Theorem 1.8, as compactifications modelled on maximal Satake are only obtained for certain classes of Anosov subgroups. Also, a characterization of Anosov subgroups in terms of the cocompactness of actions (compare our Corollary 1.11) is not provided. Moreover, the proof of compactness given in [GGKWa] was wrong, the erroneous homological argument being later replaced by a dynamical argument based on our techniques from [KLP1], see [GKWb, Lemma 4.12 in §4.3]. However, the argument in [GKWb] still remains incomplete as no proof is provided for the manifold-with-corners structure claimed in Theorem 1.1, see e.g. the lack of details in the proof of Lemma A.9.

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2 Preliminaries

2.1 Notations and definitions

We note that for Hausdorff paracompact topological spaces (and in this paper we will be dealing only with such topological spaces), Alexander-Spanier and Čech cohomology theories are naturally isomorphic, see [Sp, Ch. 6.9]. Therefore, in our paper, all cohomology is Alexander-Spanier-Čech with field coefficients (the reader can assume that the field of coefficients is $\mathbb{Z}_2$). For manifolds and CW complexes, singular and cellular cohomology is naturally isomorphic to the Čech cohomology. We will use the notation H^*_c for cohomology with compact support. As for homology, we will use it again with field coefficients and only for locally-finite CW complexes, where we will be using singular homology and singular locally finite homology, denoted H^{lf}_*. By Kronecker duality, for each locally-finite CW complex X,

$$(H^{lf}_k(X))^* \cong H^k_c(X), \quad k \geqslant 0.$$

We refer the reader to [J] for the definitions of manifolds and orbifolds with corners. The only examples of orbifolds with corners which appear in this paper are the *good* ones, i.e., quotients of manifolds with corners by properly discontinuous group actions.

In the paper we will use the notion of *dynamical relation* between points of a topological space Z, which is an open subset of a compact metrizable space, with respect to a topological action $\Gamma \curvearrowright Z$ of a discrete group. The reader will find this definition in [F], [KLP1] and [KL1]. We write

$$\xi \overset{\Gamma}{\sim} \xi'$$

if the points ξ and ξ' are dynamically related with respect to the action of the group Γ, and

$$\xi \overset{(\gamma_n)}{\sim} \xi'$$

if ξ is dynamically related to ξ' with respect to a sequence $\gamma_n \to \infty$ in Γ, cf [KLP1, §2.1]. An action is properly discontinuous if and only if no points of Z are dynamically related to each other, see [F], [KL1].

2.2 Basics of symmetric spaces and their discrete isometry groups

In the paper we assume that the reader is familiar with basics of symmetric spaces of noncompact type (denoted by X throughout the paper), their isometry groups $G = \mathrm{Isom}(X)$, visual boundaries and Tits boundaries. We refer the reader to our earlier papers [KLP1, KLP2, KLP3, KL1] for the review of these. We also refer the reader to the same papers for the notions of τ_{mod}*-regular* and *uniformly* τ_{mod}*-regular* subgroups $\Gamma < G$, defined with respect to faces τ_{mod} of model spherical Weyl chamber σ_{mod} of X. These notions of regularity present a higher rank strengthening of the discreteness condition for Γ: In the case of rank 1 symmetric spaces, the regularity of a subgroup is equivalent to its discreteness.

In the same papers we introduced several properties of τ_{mod}-regular subgroups $\Gamma < G$ which generalize equivalent definitions of convex cocompact subgroups in rank 1 Lie groups. We will mostly use in this paper the τ_{mod}*-RCA* property, where R stands for *regular*, C stands for *conical* and A stands for *antipodal*; these properties describe the geometry of the limit sets of Γ and the dynamics of Γ on these limit sets. We note that the class of τ_{mod}-RCA subgroups is proven in [KLP2] to be equal to the class of P-Anosov subgroups $\Gamma < G$, where the parabolic subgroup $P < G$ is the stabilizer of a face of type τ_{mod}. We will refer to σ_{mod}-regularity and the σ_{mod}-RCA property as regularity and RCA.

We recall (see [KLP1, KL1]) that *regular sequences* in X are divergent sequences $x_n \to \infty$ satisfying the property that for some, equivalently, every base-point $p \in X$, the sequence of Δ-valued distances

$$d_\Delta(p, x_n) \in \Delta$$

diverges from the boundary of Δ. Accordingly, a sequence $g_n \to \infty$ in G is *regular* if the sequence $(g_n x)$ is regular for some (equivalently, every) $x \in X$. A subgroup $\Gamma < G$ is regular if each infinite sequence in Γ is regular. In the paper we will be mostly using the stronger *uniform regularity* condition, see §6.3.

Below are some standard notations and notions that we will use throughout the paper:

1. X will always denote a symmetric space of noncompact type, G its group of isometries and $K < G$ a maximal compact subgroup, the stabilizer of a base point in X which will be denoted o or p.

2. xy will denote the oriented geodesic segment in X connecting a point x to a point y; similarly, $x\xi$ will denote the geodesic ray from $x \in X$ asymptotic to the point $\xi \in \partial_\infty X$.

3. F_{mod} will denote the model maximal flat for X, whose (finite) Weyl group will be denoted W. It is the stabilizer of the origin $0 \in F_{mod}$, viewing F_{mod} as a vector space. We will use the notation a_{mod} for the visual boundary of F_{mod}; we will identify a_{mod} with the unit sphere in F_{mod} equipped with the angular metric. The sphere a_{mod} is the *model spherical apartment* for the group W.

4. $\Delta = \Delta_{mod} \subset F_{mod}$ will be the *model euclidean Weyl chamber* of W; its visual boundary is the model spherical Weyl chamber σ_{mod}. We let $\iota : \sigma_{mod} \to \sigma_{mod}$ denote the *opposition involution*, also known as the *standard involution*, of σ_{mod}; it equals $-w_0$, where $w_0 \in W$ is the element sending σ_{mod} to the opposite chamber in the model apartment a_{mod}.

5. R will denote the root system of X, $\alpha_1, ..., \alpha_n$ will denote simple roots with respect to Δ.

6. $\bar{\rho}$ will denote a *root type* in σ_{mod}, i.e., $\bar{\rho}$ is the direction of the coroot $\alpha^\vee$ of a root $\alpha \in R$. For instance, the coroot vector of the highest root always determines a root type $\bar{\rho}$. For simply-laced irreducible root systems, σ_{mod} contains exactly one root type, while for non-simply-laced ones, σ_{mod} contains two root types.

7. $\overline{X} = X \sqcup \partial_\infty X$ will denote the *visual compactification* of X with respect to its Riemannian metric, equipped with the visual topology, and $\partial_{Tits} X$ the Tits boundary of X, which is the visual boundary together with the Tits metric $\angle_{Tits}$. The Tits boundary carries a natural structure as a piecewise spherical simplicial complex.

8. $\angle$ will denote the angle between vectors in a euclidean vector space, respectively, the angle metric on spherical simplices.

9. For each face τ_{mod} of σ_{mod} one defines the *flag manifold* $\mathrm{Flag}_{\tau_{\mathrm{mod}}}(\mathrm{X})$, which is the set of all simplices of type τ_{mod} in $\partial_{Tits} X$. Equipped with the *visual topology*, $\mathrm{Flag}_{\tau_{\mathrm{mod}}}(\mathrm{X})$ is a homogeneous manifold homeomorphic to G/P, where P is a parabolic subgroup of G stabilizing a face of type τ_{mod}. The full flag manifold $G/B = \mathrm{Flag}(\sigma_{mod})$ is naturally identified with the Fürstenberg boundary $\partial_{F\ddot{u}} X$ of X.

10. For a point $x \in X$, $\Sigma_x X$ denotes the *space of directions* at x, i.e., the unit sphere in the tangent space $T_x X$. Similarly, for a spherical building B or a subcomplex $\mathrm{C} \subset \mathrm{B}$, and a point $\xi \in \mathrm{C}$, we let $\Sigma_\xi \mathrm{C}$ denote the space of directions of C at ξ.

11. There is a *logarithm map* $\log_x$ mapping $\partial_\infty X$ homeomorphically to $\Sigma_x X$ which sends each ray $x\xi$ to its initial direction in $\Sigma_x X$. This map endows $\Sigma_x X$ with the structure of a thick spherical building, since the ideal boundary of X has one. If τ is a simplex in $\partial_{Tits} X$, then τ_x will denote the image of τ under $\log_x$.

12. For a simplex τ in $\partial_{Tits} X$, the *star* $\mathrm{st}(\tau)$ *of* τ, is the union of all chambers of $\partial_{Tits} X$ containing τ. We will use the notation $\mathrm{int}(\tau)$ for the open simplex, which is the complement in τ to the union of its proper faces.

13. For a subset $Y \subset X$ we let $\partial_\infty Y$ denote the *visual boundary* of Y, i.e., its accumulation set in the visual boundary of X. A set $Y \subset X$ is said to be *asymptotic* to a subset $Z \subset \partial_\infty X$ if $Z \subset \partial_\infty Y$.

14. For a subset $Y \subset \overline{X}$ we will use the notation $CH(Y)$ for the *closed convex hull* of Y, which is the smallest closed convex subset C of X such that the closure of C in $\overline{X}$ contains Y. Note that $CH(Y)$ exists if $Y \cap X$ is nonempty.

15. For a subset $Z \subset \partial_\infty X$ we let $V(x, Z) \subset X$ denote the closed convex hull of $\{x\} \cup Z$. In the special case when $Z = \tau$ is a simplex in $\partial_{Tits} X$, $V(x, \tau)$ is the *Weyl sector in* X *with tip* x *and base* τ. A Weyl sector whose base is a chamber in $\partial_{Tits} X$ is a *(euclidean) Weyl chamber* in X.

16. Two Weyl sectors $V(x_1, \tau)$ and $V(x_2, \tau)$ are *strongly asymptotic* if for any $\epsilon > 0$ there exist points $y_i \in V(x_i, \tau)$ such that the subsectors $V(y_1, \tau)$ and $V(y_2, \tau)$ are ϵ-Hausdorff close.

17. A sequence $x_i \in V(x, \tau)$ (where τ has the type τ_{mod}) is τ_{mod}*-regular* if it diverges from the boundary of $V(x, \tau)$, i.e., from the subsectors $V(x, \tau')$ for all proper faces τ' of τ. We refer the reader to [KLP2] for the more general notion of τ_{mod}-regular sequences in X, which are not necessarily contained in sectors.

18. $\theta : \partial_{Tits} X \to \sigma_{mod}$ will denote the *type map*, i.e. the canonical projection of the Tits building to the model chamber.

19. $d_\Delta(x, y)$ is the Δ-valued distance function on X. For distinct points $x, y \in X$ we let $\theta(xy) \in \sigma_{mod}$ denote the *type of the direction* of the oriented segment xy.

20. For distinct points $x, y \in X$ and $\xi \in \partial_\infty X$ we let $\angle_x(y, \xi)$ denote the angle between the geodesic segment xy and the geodesic ray $x\xi$ at the point $x \in X$.

21. We will always use the notation $\tau, \hat\tau$ to indicate that the simplices $\tau, \hat\tau$ in $\partial_{Tits} X$ are opposite (antipodal). Each simplex, of course, has a continuum of antipodal simplices.

22. Simplices $\tau, \hat\tau$ are called x*-opposite* if the Cartan involution fixing x sends τ to $\hat\tau$.

23. $P(\hat\tau, \tau)$ will denote the parallel set of two antipodal simplices $\tau, \hat\tau$ in $\partial_{Tits} X$; it is the union of all flats f in X of dimension $\dim(\tau) + 1$, whose ideal boundaries contain both τ and

$\hat{\tau}$. We will use the notation $T(\hat{\tau}, \tau)$ for the group of transvections in X along the flat f: This group is the same for all flats parallel to f and depends only on $\tau, \hat{\tau}$. We denote by

$$H = H(\hat{\tau}, \tau) = P_{\hat{\tau}} \cap P_\tau < G \tag{2.1}$$

the intersection of the parabolic subgroups of G fixing the simplices $\hat{\tau}, \tau$. The subgroup H preserves the parallel set $P(\hat{\tau}, \tau)$.

24. The parallel set $P(\hat{\tau}, \tau)$ splits isometrically as the direct product $CS(\hat{\tau}, \tau, p) \times f$, where f is one of the flats as above and $CS(\hat{\tau}, \tau, p) \subset X$ is a symmetric subspace containing the point $p \in P(\hat{\tau}, \tau)$. We let $s(\hat{\tau}, \tau)$ denote the ideal boundary of f; it is the intersection of all apartments in $\partial_{Tits} X$ containing $\hat{\tau} \cup \tau$.

25. b_η will denote the Busemann function (defined with respect to the usual Riemannian metric on X) associated with a point η in the visual boundary of X.

26. d will denote the standard distance function on X, $B(x, R)$ the closed ball of radius R centered at $x \in X$, Hb_η a closed horoball in X, which is a sublevel set $\{b_\eta \leqslant t\}$ for the Riemannian Busemann function b_η.

27. For a convex Lipschitz function $f : X \to \mathbb{R}$, we will denote by $\mathrm{slope}(f, \xi)$ the *asymptotic slope* of f along one (equivalently, any) geodesic ray $x\xi$ asymptotic to ξ,

$$\mathrm{slope}(f, \xi) = \lim_{t \to \infty} \frac{f(r(t)) - f(r(0))}{t}$$

where $r : [0, \infty) \to x\xi$ is the arc-length parameterization of $x\xi$, see [KLM]. The function $\mathrm{slope}(f, \cdot)$ on $\partial_\infty X$ is continuous and Lipschitz with respect to the Tits metric.

If f is the supremum of a family of uniformly Lipschitz convex functions f_ι, $f = \sup_\iota f_\iota$, then

$$\mathrm{slope}(f, \cdot) = \sup_\iota \mathrm{slope}(f_\iota, \cdot). \tag{2.2}$$

The asymptotic slopes of Busemann functions are given by

$$\mathrm{slope}(b_\xi, \cdot) = -\cos \angle_{Tits}(\xi, \cdot) \tag{2.3}$$

for $\xi \in \partial_\infty X$.

28. For a chamber σ in $\partial_{Tits} X$ we let N_σ denote the associated *horocyclic subgroup*, the unipotent radical of the Borel subgroup of G stabilizing σ. Similarly, for a simplex τ in $\partial_{Tits} X$ we we let N_τ denote the associated horocyclic subgroup, the unipotent radical in the parabolic subgroup of G stabilizing τ, see [KLP2, §2.4.4]. Elements of N_τ preserve the strong asymptote classes of geodesic rays $x\xi$, $\xi \in \mathrm{int}(\tau)$ and hence the strong asymptote classes of sectors $V(x, \tau)$.

2.3 Some point set topology

Let Z and Z' be first countable Hausdorff spaces, and let $O \subset Z$ and $O' \subset Z'$ be dense open subsets. Let $f : Z \to Z'$ be a map such that $f(O) \subseteq O'$, and suppose that f has the following partial continuity property: If (y_n) is a sequence in O which converges to $z \in Z$, then $f(y_n) \to f(z)$ in Z'. In particular, $f|_O$ is continuous.

Lemma 2.4. *Under these assumptions, the map f is continuous.*

Proof. The lemma follows from a standard diagonal subsequence argument. ☐

Let (A_n) be a sequence of subsets of a metrizable topological space Z. We denote by $\mathrm{Acc}((A_n))$ the closed subset consisting of the accumulation points of all sequences (a_n) of points $a_n \in A_n$.

We say that the sequence of subsets (A_n) *accumulates at* a subset $S \subset Z$ if $\mathrm{Acc}((A_n)) \subseteq S$.

If Z is compact and $C \subset Z$ is a closed subset, then the sequence (A_n) accumulates at S if and only if every neighborhood U of C contains all but finitely many of the subsets A_n.

2.4 A transformation group lemma

Let K be a compact Hausdorff topological group, and let $K \curvearrowright Y$ be a continuous action on a compact Hausdorff space Y. We suppose that there exists a *cross section* for the action, i.e. a compact subset $C \subset Y$ which contains precisely one point of every orbit.

Consider the natural surjective map

$$K \times C \xrightarrow{\alpha} Y$$

given by the action, $\alpha(k, y) = ky$. We observe that Y carries the quotient topology with respect to α, because $K \times C$ is compact and Y is Hausdorff. The identifications by α are determined by the stabilizers of the points in C, namely $\alpha(k, y) = \alpha(k', y')$ iff $y = y'$ and $k^{-1}k' \in \mathrm{Stab}_K(y)$.

Consider now two such actions $K \curvearrowright Y_1$ and $K \curvearrowright Y_2$ by the same group with cross sections $C_i \subset Y_i$, and suppose that

$$C_1 \xrightarrow{\phi} C_2$$

is a homeomorphism.

Lemma 2.5. *If ϕ respects point stabilizers, i.e.* $\mathrm{Stab}_K(y_1) = \mathrm{Stab}_K(\phi(y_1))$ *for all $y_1 \in C_1$, then ϕ extends to a K-equivariant homeomorphism $\Phi : Y_1 \to Y_2$.*

Proof. According to the discussion above, the stabilizer condition implies that there exists a bijection $\Phi : Y_1 \to Y_2$ for which the diagram

$$\begin{array}{ccc} K \times C_1 & \xrightarrow{id_K \times \phi} & K \times C_2 \\ \downarrow \alpha_1 & & \downarrow \alpha_2 \\ Y_1 & \xrightarrow{\Phi} & Y_2 \end{array}$$

commutes. Since the α_i are quotient projections, it follows that Φ is a homeomorphism. $\square$

2.5 Thom class

In this section H_*^{lf} denotes locally finite homology with $\mathbb{Z}_2$-coefficients.

Lemma 2.6 (Thom class). *Let $F \xrightarrow{\iota} E \to B$ be a fiber bundle whose base B is a compact CW-complex and whose fiber F is a connected m-manifold (without boundary). Suppose that there exists a section $s : B \to E$. Then the map*

$$\underbrace{H_m^{lf}(F)}_{\cong \mathbb{Z}_2} \xrightarrow{\iota_*} H_m^{lf}(E)$$

induced by an inclusion of the fiber is nonzero.

Proof. By thickening the section, one obtains a closed disk subbundle $D \to B$. Then we have the commutative diagram:

$$\begin{array}{ccc} H_m^{lf}(F) & \xrightarrow{\iota_*} & H_m^{lf}(E) \\ \downarrow j & & \downarrow \\ H_m(D_F, \partial D_F) & \xrightarrow{\iota'_*} & H_m(D, \partial D) \end{array}$$

The map j is an isomorphism. By Thom's isomorphism theorem, the map ι'_* is injective. It follows that the map ι_* is injective. $\square$

2.6 The horoboundary of metric spaces

We refer the reader to [G], [Ba, ch. II.1] for the definition and basic properties of horofunction compactification of metric spaces. In this section we describe these notions in the context of *nonsymmetric metrics*, compare [W2].

Let (Y, d) be a metric space. We allow the distance d to be *non-symmetric*, i.e. we only require that it is positive,

$$d(y, y') \geqslant 0 \text{ with equality iff } y = y',$$

and satisfies the triangle inequality

$$d(y, y') + d(y', y'') \geqslant d(y, y'').$$

The symmetrized distance

$$d^{sym}(y, y') := d(y, y') + d(y', y)$$

is a metric in the standard sense and induces a *topology* on Y. One observes that d is continuous, and the distance functions

$$d_y := d(y, \cdot)$$

are 1-Lipschitz with respect to d^{sym}. These functions satisfy the inequality

$$-d(y', y) \leqslant d_y - d_{y'} \leqslant d(y, y'). \tag{2.7}$$

Let $\mathcal{C}(Y)$ denote the space of continuous real valued functions, equipped with the topology of uniform convergence on bounded subsets. Moreover, let

$$\overline{\mathcal{C}}(Y) := \mathcal{C}(Y)/\mathbb{R}$$

be the quotient space of continuous functions modulo additive constants. We will denote by $[f] \in \overline{\mathcal{C}}(Y)$ the equivalence class represented by a function $f \in \mathcal{C}(Y)$, and our notation $f \equiv g$ means that the difference $f - g$ is constant.

We consider the natural map

$$Y \longrightarrow \overline{\mathcal{C}}(Y), \quad y \mapsto [d_y]. \tag{2.8}$$

It is continuous as a consequence of the triangle inequality. This map is a topological embedding provided that Y is a geodesic space; see [Ba, Ch. II.1], where this is proven for symmetric metrics, but the same proof goes through for nonsymmetric metrics as well.

We identify Y with its image in $\overline{\mathcal{C}}(Y)$ and call the closure $\overline{Y}$ the *horoclosure* of Y, and $\partial_\infty Y := \overline{Y} - Y$ the *horoboundary* or *boundary at infinity*, i.e. we have the decomposition

$$\overline{Y} = Y \sqcup \partial_\infty Y.$$

We note that the horoclosure $\overline{Y}$ is Hausdorff and 1st countable since the space $\overline{\mathcal{C}}(Y)$ is.

The functions representing points in $\partial_\infty Y$ are called *horofunctions*. We write

$$y_n \to [h]$$

for a divergent sequence of points $y_n \to \infty$ in Y which converges to a point $[h] \in \partial_\infty Y$ represented by a horofunction h, i.e. $d_{y_n} \to h$ modulo additive constants, and say that (y_n) *converges at infinity*. Each horofunction is 1-Lipschitz with respect to the symmetrized metric.

If the metric space (Y, d^{sym}) is proper (which will be the case in this paper since we are interested in symmetric spaces), then Arzelà-Ascoli theorem implies that the closure $\overline{Y}$ and the boundary $\partial_\infty Y$ at infinity are *compact*. In this case, $\overline{Y}$ is the *horofunction compactification* of Y.

Suppose that

$$G \curvearrowright Y$$

is a d-isometric group action. Then the embedding (2.8) is equivariant with respect to the induced action on functions by $g \cdot f = f \circ g^{-1}$. For every Lipschitz constant $L > 0$, the subspace of L-Lipschitz functions $\overline{Lip}_L(Y, d^{sym}) \subset \overline{\mathcal{C}}(Y)$ is preserved by the action and contains, for $L = 1$, the closure $\overline{Y}$. We equip G with the topology of uniform convergence on bounded subsets, using the symmetrized metric d^{sym} for both. Then the action $G \curvearrowright \overline{Lip}_L(Y, d^{sym})$ is continuous. In particular, the action

$$G \curvearrowright \overline{Y}$$

is continuous. We will use this fact in the situation when G is the isometry group of a Riemannian symmetric space. In this case the topology of uniform convergence on compact subsets coincides with the Lie group topology.

An *oriented geodesic* in (Y, d) is a "forward" isometric embedding $c : I \to Y$, i.e. for any parameters $t_1 \leqslant t_2$ in I it holds that

$$d(c(t_1), c(t_2)) = t_2 - t_1.$$

In particular, c is continuous with respect to the symmetrized metric d^{sym}. The metric space (Y, d) is called a *geodesic* space, if any pair of points (y, y') can be connected by an oriented geodesic from y to y'.

If (Y, d) is a geodesic space, then the horofunctions arising as limits of sequences along geodesic rays are called *Busemann functions*, and their sublevel and level sets are called *horoballs* and *horospheres*. We will denote by Hb_b a horoball for the Busemann function b, and more specifically, by $Hb_{b,y}$ the horoball of b which contains the point y in its boundary horosphere.

In the situations studied in this paper, all horofunctions will turn out to be Busemann functions, cf. section 3.2.3.

2.7 Some notions of coarse geometry

Definition 2.9. A correspondence $f : (X, d) \to (X', d')$ between metric spaces is *coarse Lipschitz* if there exist constants L, A such that for all $x, y \in X$ and $x' \in f(x), y' \in f(y)$, we have

$$d'(x', y') \leqslant Ld(x, y) + A.$$

Note that if (X, d) is a geodesic metric space, then in order to show that f is coarse Lipschitz it suffices to verify that there exists a constant C such that

$$d'(x', y') \leqslant C$$

for all $x, y \in X$ with $d(x, y) \leqslant 1$ and all $x' \in f(x), y' \in f(y)$.

Two correspondences $f_1, f_2 : (X, d) \to (X', d')$ are said to be *within distance $\leqslant D$ from each other*, $\text{dist}(f, g) \leqslant D$, if for all $x \in X, y_i \in f_i(x)$, we have

$$d'(y_1, y_2) \leqslant D.$$

Two correspondences f_1, f_2 are said to be within finite distance from each other if $\text{dist}(f_1, f_2) \leqslant D$ for some D.

A correspondence $(X, d) \to (X, d)$ is said to have *bounded displacement* if it is within finite distance from the identity map.

Definition 2.10. A coarse Lipschitz correspondence $f : (X, d) \to (X', d')$ is said to have a *coarse left inverse* if there exists a coarse Lipschitz correspondence $g : X' \to X$ such that the composition $g \circ f$ has bounded displacement.

By applying the Axiom of Choice, we can always replace a coarse Lipschitz correspondence $f : (X, d) \to (X', d')$ with a coarse Lipschitz map $f' : (X, d) \to (X', d')$ within bounded distance from f. With this in mind, if a coarse Lipschitz correspondence $f : (X, d) \to (X', d')$ admits a coarse left inverse, then f is within bounded distance from a quasiisometric embedding $f' : (X, d) \to (X', d')$. However, the converse is in general false, even in the setting of maps between finitely-generated groups equipped with word metrics.

We now specialize these concepts to the context of group homomorphisms. We note that each continuous homomorphism of groups with left-invariant proper metrics is always coarse Lipschitz. Suppose in the remainder of this section that Γ is a finitely generated group and G is a connected Lie group equipped with a left invariant metric.

Definition 2.11. We say that for a homomorphism $\rho : \Gamma \to G$, a correspondence $r : G \to \Gamma$ is a *coarse retraction* if r is a coarse left inverse to ρ. A subgroup $\Gamma < G$ is a *coarse retract* if the inclusion map $\Gamma \hookrightarrow G$ admits a coarse retraction.

Similarly, we say that a homomorphism $\rho : \Gamma \to G$ admits a *coarse equivariant retraction* if there exists a coarse Lipschitz retraction $r : G \to \Gamma$ such that

$$r(hg) = r(h)r(g), \quad \forall h \in \rho(\Gamma).$$

Accordingly, a subgroup $\Gamma < G$ is a *coarse equivariant retract* if the inclusion homomorphism $\Gamma \hookrightarrow G$ admits a coarse equivariant retraction.

More generally, given an isometric action of $\rho : \Gamma \curvearrowright X$ on a metric space X, we say that a coarse retraction $r : X \to \Gamma$ is a *coarse equivariant retraction* if

$$r(\gamma x) = \gamma r(x), \quad \forall \gamma \in \Gamma, \quad x \in X.$$

In the case when $X = G/K$ is the symmetric space associated with a connected semisimple Lie group G, a homomorphism $\Gamma \to G$ admits a coarse equivariant retraction iff the isometric action of Γ on X defined via ρ admits a coarse equivariant retraction. Similarly, a subgroup $\Gamma < G$ is a coarse retract iff the orbit map $\Gamma \to \Gamma \cdot x \subset X$ admits a coarse left-inverse.

3 Finsler compactifications of symmetric spaces

Let $X = G/K$ be a symmetric space of noncompact type.

3.1 Finsler metrics

3.1.1 The Riemannian distance

We rewrite the Riemannian distance d^{Riem} on X in a way which motivates our later definition of Finsler distances.

Consider an oriented segment xy in X. As a 1-Lipschitz function, every Busemann function b_ξ has slope $\leqslant 1$ along xy, and we have the inequality

$$d^{Riem}(x,y) \geqslant b_\xi(y) - b_\xi(x). \tag{3.1}$$

Observe that equality holds, i.e. b_ξ has slope $\equiv 1$ along xy, iff $x \in y\xi$. In particular, we obtain the representation

$$d^{Riem}(x,y) = \max_{\xi \in \partial_\infty X} \big(b_\xi(y) - b_\xi(x)\big) \tag{3.2}$$

for the Riemannian distance.

3.1.2 Certain Finsler distances

Now we fix a regular type $\bar\theta \in \mathrm{int}(\sigma_{mod})$ and restrict only to Busemann functions b_ξ which are centered at ideal points of this type, $\theta(\xi) = \bar\theta$. For a chamber $\sigma \subset \partial_\infty X$, we denote by $\theta_\sigma \in \sigma$ the unique point of type $\bar\theta$.

There is a sharper bound for the slopes of Busemann functions of type $\bar\theta$ along an oriented segment, which depends on the type of the segment:

Lemma 3.3. *The slope of a Busemann function b_{θ_σ} along a non-degenerate oriented segment xy is bounded above by $\cos\angle(\theta(xy), \iota\bar\theta)$, with equality in some point, equivalently, along the entire segment, iff $x \in V(y,\sigma)$.*

Proof. The slope of $b_{\theta_\sigma}|_{xy}$ in an interior point $z \in xy$ equals $\cos\angle_z(x,\theta_\sigma)$, and is hence maximal if the angle is minimal. The angle is minimal iff the directions $\overrightarrow{zx}$ and $\overrightarrow{z\theta_\sigma}$ lie in a common chamber of the space of directions $\Sigma_z X$, equivalently, iff x is contained in the euclidean Weyl chamber $V(z,\sigma)$, and the angle then equals

$$\angle(\theta(zx), \bar\theta) = \angle(\theta(yx), \bar\theta) = \angle(\iota\theta(xy), \bar\theta) = \angle(\theta(xy), \iota\bar\theta).$$

In this case, the slope is maximal along the entire segment and $x \in V(y,\sigma)$. ☐

In analogy with (3.2) we define the *$\bar\theta$-Finsler distance* $d^{\bar\theta} : X \times X \to [0,+\infty)$ by

$$d^{\bar\theta}(x,y) := \max_\sigma \big(b_{\theta_\sigma}(y) - b_{\theta_\sigma}(x)\big) \tag{3.4}$$

where the maximum is taken over all chambers $\sigma \subset \partial_\infty X$. The triangle inequality is clearly satisfied. Positivity follows from the fact that $\mathrm{diam}(\sigma_{mod}) \leqslant \frac{\pi}{2}$ and the assumption that $\bar\theta$ is regular. The $\bar\theta$-distance is symmetric iff $\iota\bar\theta = \bar\theta$.

According to the lemma, we have the inequality, analogous to (3.1),

$$d^{\bar{\theta}}(x,y) \geqslant b_{\theta_\sigma}(y) - b_{\theta_\sigma}(x) \tag{3.5}$$

with equality iff $x \in V(y,\sigma)$.

It follows that we can write the $\bar{\theta}$-distance in the form

$$d^{\bar{\theta}}(x,y) = l(d_\Delta(x,y))$$

with the linear functional $l = -b_{\iota\bar{\theta}}$ (normalized at the origin) on $\Delta_{mod} \subset F_{mod}$.

Inequality (3.5) also implies that the $\bar{\theta}$-distance restricts on maximal flats $F \subset X$ to analogously defined metrics, because only the Busemann functions centered at the visual boundary of the flat enter into the maximum. We have that

$$d^{\bar{\theta}}(x,y) = \max_{\sigma \subset \partial_\infty F} \big(b_{\theta_\sigma}(y) - b_{\theta_\sigma}(x)\big) \tag{3.6}$$

for points $x, y \in F$, where the maximum is taken over the finitely many chambers in the visual boundary of the flat. The restriction of $d^{\bar{\theta}}$ to maximal flats is thus the translation invariant metric associated to the *polyhedral norm* on F_{mod} given by

$$\|\cdot\|_{\bar{\theta}} = \max_{w \in W}(l \circ w^{-1}) = \max_{w \in W}\big(b_{w\bar{\theta}} - b_{w\bar{\theta}}(0)\big).$$

The $\bar{\theta}$-distance is then the path metric associated to the G-invariant Finsler metric on X induced by this norm.

We note that the $\bar{\theta}$-distance as well as its symmetrization are equivalent, as metrics, to the Riemannian distance.

In order to describe *geodesics*, we analyze when equality holds in the triangle inequality.

Lemma 3.7. *The equality*

$$d^{\bar{\theta}}(x,z) + d^{\bar{\theta}}(z,y) = d^{\bar{\theta}}(x,y)$$

holds iff there exists a maximal flat $F \subset X$ containing the points x, y and z, and a pair of opposite chambers $\sigma_\pm \subset \partial_\infty F$ such that

$$z \in V(x,\sigma_+) \cap V(y,\sigma_-).$$

Proof. Let σ_- be a chamber such that $x \in V(y,\sigma_-)$. Then $b_{\theta_{\sigma_-}}$ has maximal slope along xy. From

$$d^{\bar{\theta}}(x,z) + d^{\bar{\theta}}(z,y) \geqslant \big(b_{\theta_{\sigma_-}}(z) - b_{\theta_{\sigma_-}}(x)\big) + \big(b_{\theta_{\sigma_-}}(y) - b_{\theta_{\sigma_-}}(z)\big) = \big(b_{\theta_{\sigma_-}}(y) - b_{\theta_{\sigma_-}}(x)\big) = d^{\bar{\theta}}(x,y)$$

it follows that $b_{\theta_{\sigma_-}}$ has maximal slope also along the segments xz and zy, which implies that $z \in V(y,\sigma_-)$.

Let F be the maximal flat containing $V(y,\sigma_-)$, and let $\sigma_+ \subset \partial_\infty F$ be the chamber opposite to σ_-. Similarly, $b_{(\iota\theta)_{\sigma_+}}$ has maximal slope along the reversely oriented segment yx and it follows that $z \in V(x,\sigma_+)$. ☐

From the lemma one sees that $(X, d^{\bar\theta})$ is a *geodesic* space. All Riemannian geodesics are $d^{\bar\theta}$-geodesics, but not vice versa. In view of Lemma 3.7, the $d^{\bar\theta}$-geodesics can be described as follows.

For a $d^{\bar\theta}$-geodesic $c : I \to X$ there exists a (in general non-unique) pair of opposite chambers $\sigma_\pm \subset \partial_\infty F$ such that

$$c(t_\pm) \in V(c(t_\mp), \sigma_\pm))$$

for all $t_- < t_+$ in I, i.e. *c drifts* towards σ_+ and away from σ_-. In particular, $d^{\bar\theta}$-geodesics are contained in maximal flats. More precisely, a $d^{\bar\theta}$-geodesic segment is contained in the singular flat which is the intersection of all maximal flats containing the endpoints. (This is no longer true for singular types $\bar\theta$. There, the geodesics are contained in certain parallel sets.)

3.2 Finsler compactifications

3.2.1 Definition

If one applies the horoboundary construction, cf. section 2.6, to the Riemannian distance d^{Riem} on X, one obtains the visual compactification

$$\overline{X} = X \sqcup \partial_\infty X. \tag{3.8}$$

The ideal boundary points are represented by Busemann functions, i.e. the horofunctions are in this case precisely the Busemann functions.

We define the *Finsler compactification of type* $\bar\theta$ or $\bar\theta$*-compactification* of X as the compactification

$$\overline{X}^{\bar\theta} = X \sqcup \partial^{\bar\theta}_\infty X. \tag{3.9}$$

which one obtains when applying the horoboundary construction to the Finsler distance $d^{\bar\theta}$. Our next goal is to describe horofunctions in $\partial^{\bar\theta}_\infty X$ in terms of Riemannian Busemann functions on X.

3.2.2 Certain mixed Busemann functions

According to the definition of the $d^{\bar\theta}$-distance, see (3.4), we have that

$$d^{\bar\theta}_x = d^{\bar\theta}(x, \cdot) = \max_\sigma \big(b_{\theta_\sigma} - b_{\theta_\sigma}(x)\big).$$

For a simplex $\tau \subset \partial_\infty X$, we put

$$b^{\bar\theta}_{\tau,x} := \max_{\sigma\supset\tau} \big(b_{\theta_\sigma} - b_{\theta_\sigma}(x)\big) \tag{3.10}$$

where the maximum is taken only over the chambers which contain τ as a face.

The functions $b^{\bar\theta}_{\tau,x}$ for simplices $\tau \subset \partial_\infty X$ and points $x \in X$ will turn out to be the horofunctions for the $\bar\theta$-compactification, i.e. the functions which represent the Finsler boundary points at infinity. We will now study their properties.

The Busemann functions b_{θ_σ} for $\sigma \supset \tau$ are invariant under the horocyclic subgroups $N_\sigma \supset N_\tau$ (cf. [KLP2, §2.4.4]). As a consequence, the functions $b^{\bar\theta}_{\tau,x}$ are invariant under N_τ.

Let f be a minimal singular flat asymptotic to τ, $\partial_\infty f \supset \tau$. Then $\partial_\infty f = s(\tau,\hat\tau)$ for a face $\hat\tau$ opposite to τ. The Busemann functions b_{θ_σ} for $\sigma \supset \tau$ are affine linear along f and coincide up to additive constants. Therefore $b^{\bar\theta}_{\tau,x}|_f$ coincides with them up to additive constants and is itself affine linear.

More precisely, let $T(\tau,\hat\tau) \subset G$ denote the subgroup of transvections along f. Then there is a surjective homomorphism $\psi_\tau : T(\tau,\hat\tau) \to \mathbb{R}$, independent of f, such that

$$(b_{\theta_\sigma} \circ t^{-1})|_f = b_{\theta_\sigma}|_f + \psi_\tau(t)$$

for $\sigma \supset \tau$ and $t \in T$, and hence

$$(b^{\bar\theta}_{\tau,x} \circ t^{-1})|_f = b^{\bar\theta}_{\tau,x}|_f + \psi_\tau(t) \tag{3.11}$$

If $\sigma \supset \tau$ is a chamber such that $x \in V(y,\sigma)$, then

$$b^{\bar\theta}_{\tau,x}(y) = b_{\theta_\sigma}(y) - b_{\theta_\sigma}(x),$$

cf. Lemma 3.3. Thus,

$$b^{\bar\theta}_{\tau,x} = b_{\theta_\sigma} - b_{\theta_\sigma}(x) \tag{3.12}$$

on $V(x,\sigma_-)$, where σ_- denotes the chamber x-opposite to σ. With the behavior (3.11) under translations, it follows that (3.12) remains valid on $T \cdot V(x,\sigma_-) = V(x, CH(\sigma_- \cup \tau))$.

Let now F be a maximal flat through x asymptotic to τ, $F \supset V(x,\tau)$. Note that the union of the cones $V(x, CH(\sigma_- \cup \tau))$, as σ_- runs through the finitely many chambers in $\mathrm{st}(\tau_-) \cap \partial_\infty F$, equivalently, as σ runs through the chambers in $\mathrm{st}(\tau) \cap \partial_\infty F$, equals F. We therefore obtain that

$$b^{\bar\theta}_{\tau,x}|_F = \max_{\tau \subset \sigma \subset \partial_\infty F} \big(b_{\theta_\sigma}|_F - b_{\theta_\sigma}(x)\big). \tag{3.13}$$

Thus the restriction of $b^{\bar\theta}_{\tau,x}$ to a maximal flat asymptotic to τ is the maximum of finitely many affine linear functions.

The following result will be used to distinguish the functions $b^{\bar\theta}_{\tau,p}$ from each other. Let $\hat\tau$ be the simplex p-opposite to τ, and let $CS(p) = CS(\tau,\hat\tau,p)$ denote the cross section of the parallel set $P(\tau,\hat\tau)$ through p (cf. [KLP2, §2.4.1]).

Lemma 3.14. *$b^{\bar\theta}_{\tau,p}|_{CS(p)}$ has a unique maximum in p.*

Proof. Let $p \neq q \in CS(p)$. We need to find a chamber $\sigma \supset \tau$ such that

$$b_{\theta_\sigma}(q) > b_{\theta_\sigma}(p).$$

The latter holds if (and only if)

$$\angle_p(q,\theta_\sigma) > \frac{\pi}{2}. \tag{3.15}$$

We choose an apartment a_p in the space of directions $\Sigma_p P(\tau, \hat{\tau}) \subset \Sigma_p X$ which contains the direction $v := \overrightarrow{pq}$. As every apartment in $\Sigma_p P(\tau, \hat{\tau})$, it also contains the singular sphere $s_p := \log_p s(\tau, \hat{\tau})$. It suffices to find a chamber $\sigma_p \supset \tau_p$ in a_p such that

$$\angle_p(v, \theta_{\sigma_p}) > \frac{\pi}{2}. \tag{3.16}$$

because then there exists a chamber $\sigma \supset \tau$ such that $\sigma_p = \log_p \sigma$, and (3.15) follows.

Let $\bar{\theta}' \in \mathrm{int}(\tau_{mod})$ denote the nearest point projection of $\bar{\theta} \in \mathrm{int}(\sigma_{mod})$ to τ_{mod}. We denote by θ'_{τ_p} the point (direction) of type $\bar{\theta}'$ in τ_p. Furthermore, we let $v_- \in a_p$ be the antipode of v in a_p. Then inequality (3.16) follows from

$$\angle_{\theta'_{\tau_p}}(v, \theta_{\sigma_p}) = \pi - \angle_{\theta'_{\tau_p}}(v_-, \theta_{\sigma_p}) > \frac{\pi}{2}. \tag{3.17}$$

The last inequality is satisfied if $\sigma_p \subset a_p$ is the chamber whose space of directions $\Sigma_{\theta'_{\tau_p}} \sigma_p$ contains the direction $\overrightarrow{\theta'_{\tau_p} v_-}$. To see this, we note that $\Sigma_{\theta'_{\tau_p}} \sigma_p$ decomposes as the spherical join of the sphere $\Sigma_{\theta'_{\tau_p}} \tau_p$ and a simplex σ_{τ_p} of diameter $\leqslant \frac{\pi}{2}$ (isometric to a Weyl chamber for the spherical building $\Sigma_{\tau_p}(\Sigma_p X)$, which may be reducible but has no sphere factor). Since the direction $\overrightarrow{\theta'_{\tau_p} \theta_{\sigma_p}}$ is perpendicular to τ_p, it lies in σ_{τ_p}, which yields the non-strict inequality. The strict inequality holds because $\bar{\theta}$ is regular, and hence $\overrightarrow{\theta'_{\tau_p} \theta_{\sigma_p}}$ lies in the interior of σ_{τ_p}. ☐

Based on these properties, we can now distinguish the functions $b^{\bar{\theta}}_{\tau,p}$ from each other. (Recall that the notation $f \equiv g$ means that $f - g$ is a constant.)

Lemma 3.18. *$b^{\bar{\theta}}_{\tau,p} \equiv b^{\bar{\theta}}_{\tau',p'}$ iff $\tau = \tau'$ and the sectors $V(p, \tau)$ and $V(p', \tau')$ are strongly asymptotic.*

Proof. Suppose that $b^{\bar{\theta}}_{\tau,p} \equiv b^{\bar{\theta}}_{\tau',p'}$. We first show that then $\tau = \tau'$.

By our assumption, the difference $b^{\bar{\theta}}_{\tau,p} - b^{\bar{\theta}}_{\tau',p'}$ is in particular bounded, and hence for every point $q \in X$ also $b^{\bar{\theta}}_{\tau,q} - b^{\bar{\theta}}_{\tau',q}$ is bounded. We choose q inside a maximal flat F which is asymptotic to both simplices τ and τ'. We know from (3.13) how the restrictions of $b^{\bar{\theta}}_{\tau,q}$ and $b^{\bar{\theta}}_{\tau',q}$ to F look: In particular, the asymptotic slope $(\mathrm{slope}\, b^{\bar{\theta}}_{\tau,q})|_{\partial_\infty F}$ attains the maximal value 1 precisely in the points opposite to θ_σ for $\tau \subset \sigma \subset \partial_\infty F$. Since $(\mathrm{slope}\, b^{\bar{\theta}}_{\tau,q})|_{\partial_\infty F} = (\mathrm{slope}\, b^{\bar{\theta}}_{\tau',q})|_{\partial_\infty F}$, it follows that $\mathrm{st}(\tau) \cap \partial_\infty F = \mathrm{st}(\tau') \cap \partial_\infty F$ and hence $\tau = \tau'$.

The assertion in the case $\tau = \tau'$ follows from Lemma 3.14. ☐

Consequently, the functions $b^{\bar{\theta}}_{\tau,p}$ modulo additive constants one-to-one correspond to strong asymptote classes of Weyl sectors in X.

3.2.3 Points at infinity and topology at infinity

Every function $b^{\bar{\theta}}_{\tau,p}$ represents a Finsler boundary point at infinity:

Lemma 3.19. *Let $x_n \to \infty$ be a sequence in the Weyl sector $V(p, \tau)$ such that $d(x_n, V(p, \partial\tau)) \to +\infty$. Then*

$$d^{\bar{\theta}}_{x_n} - d^{\bar{\theta}}_{x_n}(p) \to b^{\bar{\theta}}_{\tau,p} \tag{3.20}$$

uniformly on compacta.

Proof. Step 1. Let $\hat{\tau} \subset \partial_\infty X$ be the simplex p-opposite to τ. We first prove the assertion on the parallel set $P(\hat{\tau}, \tau)$.

On the Weyl cone $V(x_n, \mathrm{st}(\hat{\tau})) \subset P(\hat{\tau}, \tau)$ it holds that $d^{\bar{\theta}}_{x_n} = b^{\bar{\theta}}_{\tau, x_n}$, compare the equality case in (3.5) and (3.12), whence

$$d^{\bar{\theta}}_{x_n} - d^{\bar{\theta}}_{x_n}(p) = b^{\bar{\theta}}_{\tau, x_n} - b^{\bar{\theta}}_{\tau, x_n}(p) = b^{\bar{\theta}}_{\tau, p}.$$

From our assumption that $d(x_n, V(p, \partial\tau)) \to +\infty$, it follows that $V(x_n, \mathrm{st}(\hat{\tau}))$ contains balls $B(p, r_n) \cap P(\hat{\tau}, \tau)$ with radii $r_n \to +\infty$. Thus, (3.20) holds on $P(\hat{\tau}, \tau)$.

Step 2. We extend this to X using the action of the horocyclic subgroup N_τ, relying on the invariance

$$b^{\bar{\theta}}_{\tau, p} \circ u = b^{\bar{\theta}}_{\tau, p}$$

of mixed Busemann functions under isometries $u \in N_\tau$. By step 1, it holds for $(u, y) \in N_\tau \times P(\hat{\tau}, \tau)$ that

$$\underbrace{d^{\bar{\theta}}_{ux_n}(uy)}_{d^{\bar{\theta}}_{x_n}(y)} - d^{\bar{\theta}}_{x_n}(p) \to \underbrace{b^{\bar{\theta}}_{\tau, p}(uy)}_{b^{\bar{\theta}}_{\tau, p}(y)},$$

and the convergence is locally uniform in (u, y). Note that

$$d^{\bar{\theta}}(x_n, ux_n) = d^{\bar{\theta}}(u^{-1}x_n, x_n) \to 0$$

for $u \in N_\tau$ because $d(x_n, V(p, \partial\tau)) \to +\infty$, and the convergence is locally uniform in u. Hence

$$\sup_X |d^{\bar{\theta}}_{ux_n} - d^{\bar{\theta}}_{x_n}| \to 0$$

due to the triangle inequality, compare (2.7), and it follows that

$$d^{\bar{\theta}}_{x_n}(uy) - d^{\bar{\theta}}_{x_n}(p) \to b^{\bar{\theta}}_{\tau, p}(uy)$$

locally uniformly in (u, y), i.e.

$$d^{\bar{\theta}}_{x_n} - d^{\bar{\theta}}_{x_n}(p) \to b^{\bar{\theta}}_{\tau, p}$$

locally uniformly on X, as claimed. ☐

We want to show that, vice versa, the functions $b^{\bar{\theta}}_{\tau, p}$ represent all Finsler boundary points.

We fix a base point $o \in X$ and denote by K the maximal compact subgroup of G fixing o. For a discussion of the concept of Δ-distance, see [KLP2, 2.1].

Lemma 3.21. *Let $x_n \to \infty$ be a divergent sequence in X. Then, after passing to a subsequence,*

(i) there exists a face type $\tau_{mod} \subseteq \sigma_{mod}$ such that the sequence of Δ-distances

$$\delta_n := d_\Delta(o, x_n) \to \infty \tag{3.22}$$

in the model euclidean Weyl chamber $\Delta = V(0, \sigma_{mod})$ is contained in a tubular neighborhood of the sector $V(0, \tau_{mod})$ and drifts away from its boundary $V(0, \partial\tau_{mod})$,

$$d(\delta_n, V(0, \partial\tau_{mod})) \to +\infty,$$

(ii) and, as a consequence, there exists a sequence of simplices $\tau_n \in \mathrm{Flag}_{\tau_{\mathrm{mod}}}(\mathrm{X})$ and a bounded sequence of points

$$p_n \in CS(\hat{\tau}_n, \tau_n, o) \subset P(\hat{\tau}_n, \tau_n),$$

where $\hat{\tau}_n$ denotes the simplex o-opposite to τ_n, such that $x_n \in V(p_n, \tau_n)$.

Proof. Property (i) can clearly be achieved by passing to a subsequence.

Let $\sigma_n \subset \partial_\infty X$ be chambers such that $x_n \in V(o, \sigma_n)$, and let $\tau_n \subseteq \sigma_n$ denote their faces of type τ_{mod}. Moreover, let $\hat{\tau}_n \subset \partial_\infty X$ denote the simplices o-opposite to τ_n. There exist unique points $p_n \in CS(\hat{\tau}_n, \tau_n, o) \subset P(\hat{\tau}_n, \tau_n)$ such that x_n lies in the minimal flat $f(\hat{\tau}_n, \tau_n, p_n)$ containing the sector $V(p_n, \tau_n)$. The sequence (p_n) is bounded, because the sequence (δ_n) is contained in a tubular neighborhood of the sector $V(0, \tau_{mod})$. Furthermore, $x_n \in V(p_n, \tau_n)$ for large n, because (δ_n) drifts away from $V(0, \partial\tau_{mod})$. This yields (ii). ☐

Since $\mathrm{Flag}_{\tau_{\mathrm{mod}}}(\mathrm{X})$ is compact, one can further strengthen property (ii) by passing to a subsequence once more and achieve that the sequences (τ_n) and (p_n) converge. These are then the data which characterize the convergence at infinity:

Proposition 3.23 (Convergence at infinity). *Let $x_n \to \infty$ be a divergent sequence in X. Then*

$$x_n \to [b^{\bar{\theta}}_{\tau,p}]$$

with $p \in CS(\hat{\tau}, \tau, o)$, where $\hat{\tau}$ denotes the simplex o-opposite to τ, if and only if, without passing to a subsequence, properties (i) and (ii) hold for large n with $\tau_{mod} = \theta(\tau)$, $\tau_n \to \tau$ and $p_n \to p$.

Proof. Given properties (i) and (ii), we can write $\tau_n = k_n\tau$ and $\hat{\tau}_n = k_n\hat{\tau}$ with $k_n \to e$ in K and $\hat{\tau}$ o-opposite to τ. The sequence of points

$$k_n^{-1}x_n \in V(\underbrace{k_n^{-1}p_n}_{\to p}, \tau) \subset P(\hat{\tau}, \tau)$$

is contained in a tubular neighborhood of the sector $V(o, \tau)$ and drifts away from its boundary, $d(k_n^{-1}x_n, V(o, \partial\tau)) \to +\infty$. Lemma 3.19 yields, combined with (2.7), that

$$d^{\bar{\theta}}_{k_n^{-1}x_n} - d^{\bar{\theta}}_{k_n^{-1}x_n}(k_n^{-1}p) \to b^{\bar{\theta}}_{\tau,p}$$

uniformly on compacta. It follows that also

$$d^{\bar{\theta}}_{x_n} - d^{\bar{\theta}}_{x_n}(p) \to b^{\bar{\theta}}_{\tau,p},$$

because $d^{\bar{\theta}}_{k_n^{-1}x_n} = d^{\bar{\theta}}_{x_n} \circ k_n$ and $|d^{\bar{\theta}}_{k_n^{-1}x_n} - d^{\bar{\theta}}_{x_n}| \to 0$ uniformly on compacta. Thus, $x_n \to [b^{\bar{\theta}}_{\tau,p}]$.

Conversely, suppose that $x_n \to [b^{\bar\theta}_{\tau,p}]$. Then, if (δ_n) is not contained in a tubular neighborhood of $V(0,\tau_{mod})$, or if it is contained in a tubular neighborhood of $V(0,\tau_{mod})$ but does not drift away from $V(0,\partial\tau_{mod})$, then a subsequence of (δ_n) is contained in a tubular neighborhood of another sector $V(0,\nu_{mod})$, $\nu_{mod} \neq \tau_{mod}$, and drifts away from $V(0,\partial\nu_{mod})$. It follows that (x_n) subconverges to a boundary point $[b^{\bar\theta}_{\nu,q}]$ with $\theta(\nu) = \nu_{mod}$, a contradiction, since $[b^{\bar\theta}_{\nu,q}] \neq [b^{\bar\theta}_{\tau,p}]$. Thus, property (i) holds with $\tau_{mod} = \theta(\tau)$.

Regarding property (ii), it follows from the proof of the previous lemma, that for sufficiently large n there exist simplices $\tau_n \in \mathrm{Flag}_{\tau_{\mathrm{mod}}}(\mathrm{X})$ and points $p_n \in CS(\hat\tau_n,\tau_n,o)$ such that $x_n \in V(p_n,\tau_n)$ and the sequence (p_n) is bounded in X. Suppose that $\tau_n \not\to \tau$ or $p_n \not\to p$. Since $\mathrm{Flag}_{\tau_{\mathrm{mod}}}(\mathrm{X})$ is compact, we can then pass to a subsequence such that $\tau_n \to \tau'$ and $p_n \to p' \in CS(\hat\tau,\tau,o)$ with $(\tau',p') \neq (\tau,p)$. According to the first part of the proof, this implies that (x_n) accumulates at $[b^{\bar\theta}_{\tau',p'}]$. However, $[b^{\bar\theta}_{\tau',p'}] \neq [b^{\bar\theta}_{\tau,p}]$ as a consequence of Lemma 3.14, and we obtain a contradiction. Thus, property (ii) holds with $\tau_n \to \tau$ and $p_n \to p$. ☐

Remark 3.24. (i) The assumption that $p \in CS(\hat\tau,\tau,o)$ where $\hat\tau$ is o-opposite to τ is a normalization of p. It can be arranged in a unique way by replacing p while keeping the strong asymptote class of the sector $V(p,\tau)$ unchanged.

(ii) In the case $\tau_{mod} = \sigma_{mod}$ the condition simplifies: It holds that $x_n \to [b_{\theta_\sigma}]$ if and only if $d(\delta_n, V(0,\partial\sigma_{mod})) \to +\infty$ and $x_n \in V(o,\sigma_n)$ with a sequence of chambers $\sigma_n \to \sigma$.

Corollary 3.25. *Every Finsler boundary point at infinity is represented by a function $b^{\bar\theta}_{\tau,p}$.*

Proof. By the lemma and the proposition, every divergent sequence in X subconverges to a point at infinity represented by a function $b^{\bar\theta}_{\tau,p}$. Hence there are no other points at infinity. ☐

Remark 3.26. (i) Together with Lemma 3.18 we conclude that Finsler boundary points at infinity one-to-one correspond to strong asymptote classes of Weyl sectors in X.

(ii) Since all horofunctions are of the form $b^{\bar\theta}_{\tau,p}$ and arise as limits of sequences along Weyl sectors, and in particular as limits of sequences along Finsler geodesic rays, it follows that all horofunctions are *Busemann functions*, as defined in section 2.6.

Lemma 3.27. *Let (x_n) and (x'_n) be sequences in X which are bounded distance apart and converge at infinity, $x_n \to [b^{\bar\theta}_{\tau,p}]$ and $x'_n \to [b^{\bar\theta}_{\tau',p'}]$. Then $\tau = \tau'$.*

Proof. By our assumption, the differences of distance functions $d^{\bar\theta}_{x_n} - d^{\bar\theta}_{x'_n}$ are uniformly bounded independently of n. It follows that also $b^{\bar\theta}_{\tau,p} - b^{\bar\theta}_{\tau',p'}$ is bounded, which implies that $\tau = \tau'$. ☐

Remark 3.28. Note that, unlike in the case of the visual boundary at infinity, the limit points $[b^{\bar\theta}_{\tau,p}]$ and $[b^{\bar\theta}_{\tau',p'}]$ do in general not coincide.

Note that the continuous extension of the G-action on X to the Finsler compactification $\overline{X}^{\bar\theta}$ is given by

$$g \cdot [b^{\bar\theta}_{\tau,p}] = [b^{\bar\theta}_{\tau,p} \circ g^{-1}] = [b^{\bar\theta}_{g\tau,gp}].$$

We now extend the above discussion to sequences at infinity.

Let $\tau_{mod} \subset \nu_{mod}$ be face types. Then every boundary point of type ν_{mod} is a limit of boundary points of type τ_{mod}:

Lemma 3.29. *Let $x_n \to \infty$ be a sequence in the Weyl sector $V(p, \nu)$ such that $d(x_n, V(p, \partial\nu)) \to +\infty$ and let $\tau \subseteq \nu$ be a face. Then $[b^{\bar\theta}_{\tau,x_n}] \to [b^{\bar\theta}_{\nu,p}]$.*

Proof. According to Lemma 3.19, there exist points $y_n \in V(x_n, \tau) \subset V(p, \nu)$ such that

$$d^{\bar\theta}_{y_n} - d^{\bar\theta}_{y_n}(x_n) - b^{\bar\theta}_{\tau,x_n} \to 0$$

uniformly on compacta. Since also $d(y_n, V(p, \partial\nu)) \to +\infty$, applying Lemma 3.19 again yields that

$$d^{\bar\theta}_{y_n} - d^{\bar\theta}_{y_n}(p) \to b^{\bar\theta}_{\nu,p}$$

uniformly on compacta. It follows that $[b^{\bar\theta}_{\tau,x_n}] \to [b^{\bar\theta}_{\nu,p}]$. □

The next result partially characterizes the convergence of sequences at infinity.

Lemma 3.30. *If*

$$[b^{\bar\theta}_{\tau_n,x_n}] \to [b^{\bar\theta}_{\nu,p}]$$

and $\theta(\tau_n) = \tau_{mod}$ for all n, then $\tau_{mod} \subseteq \theta(\nu)$ and $\tau_n \to \tau \subseteq \nu$.

Proof. We may assume without loss of generality that $x_n \in CS(\hat\tau_n, \tau_n, o)$ and $p \in CS(\hat\nu, \nu, o)$ where $\hat\tau_n$ is o-opposite to τ_n and $\hat\nu$ is o-opposite to ν.

As in the proof of the previous lemma, using Lemma 3.19, we approximate the points $[b^{\bar\theta}_{\tau_n,x_n}]$ at infinity by points $y_n \in V(x_n, \tau_n)$ such that still

$$y_n \to [b^{\bar\theta}_{\nu,p}].$$

The latter holds if the growth

$$d(y_n, V(x_n, \partial\tau_n)) \to +\infty$$

is sufficiently fast. Sufficiently fast growth implies moreover that $y_n \in V(o, \mathrm{st}(\tau_n))$, and hence that there exist chambers $\sigma_n \supseteq \tau_n$ such that $y_n \in V(o, \sigma_n)$.

After passing to a subsequence, there exists a face type $\tau'_{mod} \subseteq \sigma_{mod}$ such that the Δ-distances $d_\Delta(o, y_n)$ lie in a tubular neighborhood of $V(0, \tau'_{mod})$ but drift away from $V(0, \partial\tau'_{mod})$. Invoking sufficiently fast growth again, it follows that $\tau'_{mod} \supseteq \tau_{mod}$.

Consider the faces $\tau_n \subseteq \tau'_n \subseteq \sigma_n$ of type $\theta(\tau'_n) = \tau'_{mod}$, and denote by $\hat\tau'_n$ the simplices o-opposite to τ'_n. There exists a bounded sequence (x'_n) of points $x'_n \in CS(\hat\tau'_n, \tau'_n, o)$ such that $y_n \in V(x'_n, \tau'_n)$. After passing to a subsequence once more, we may assume convergence $\tau'_n \to \tau'$ and $x'_n \to x'$. Then $y_n \to [b^{\bar\theta}_{\tau',x'}]$ by Proposition 3.23, and hence $[b^{\bar\theta}_{\tau',x'}] = [b^{\bar\theta}_{\nu,p}]$. In particular, $\tau'_{mod} = \theta(\nu)$ and $\tau' = \nu$. It follows that $\tau_n \to \tau \subseteq \nu$, i.e. the assertion holds for the subsequence.

Returning to the original sequence of points $[b^{\bar\theta}_{\tau_n,x_n}]$, our argument yields that every subsequence has a subsequence for which the assertion holds. Consequently, $\tau_{mod} \subseteq \theta(\nu)$ and the sequence of simplices τ_n can only accumulate at the face $\tau \subseteq \nu$ of type τ_{mod}. In view of the compactness of $\mathrm{Flag}_{\tau_{\mathrm{mod}}}(\mathrm{X})$, it follows that $\tau_n \to \tau$. □

Our discussion of sequential convergence implies that the Finsler compactification does not depend on the regular type $\bar{\theta}$.

Proposition 3.31 (Type independence of Finsler compactification). *For any two regular types $\bar{\theta}, \bar{\theta}' \in \mathrm{int}(\sigma_{mod})$, the identity map id_X extends to a G-equivariant homeomorphism*

$$\overline{X}^{\bar{\theta}} \to \overline{X}^{\bar{\theta}'}$$

sending $[b^{\bar{\theta}}_{\tau,p}] \mapsto [b^{\bar{\theta}'}_{\tau,p}]$ at infinity.

Proof. The extension of id_X sending $[b^{\bar{\theta}}_{\tau,p}] \mapsto [b^{\bar{\theta}'}_{\tau,p}]$ is a G-equivariant bijection $\overline{X}^{\bar{\theta}} \to \overline{X}^{\bar{\theta}'}$. The conditions given in Proposition 3.23 for sequences $x_n \to \infty$ in X to converge at infinity do not depend on the type $\bar{\theta}$, i.e. we have convergence $x_n \to [b^{\bar{\theta}}_{\tau,p}]$ in $\overline{X}^{\bar{\theta}}$ if and only if we have convergence $x_n \to [b^{\bar{\theta}'}_{\tau,p}]$ in $\overline{X}^{\bar{\theta}'}$. A general point set topology argument, see Lemma 2.4, now implies that the extension is a homeomorphism. ☐

3.2.4 Stratification and G-action

For every face type $\tau_{mod} \subset \sigma_{mod}$, we define the *stratum*

$$S_{\tau_{mod}} = \{[b^{\bar{\theta}}_{\tau,p}] : \theta(\tau) = \tau_{mod}, p \in X\}. \tag{3.32}$$

Furthermore, we put $S_{\emptyset} = X$. We define the stratification of $\overline{X}^{\bar{\theta}}$ as

$$\overline{X}^{\bar{\theta}} = \bigsqcup_{\emptyset \subseteq \tau_{mod} \subseteq \sigma_{mod}} S_{\tau_{mod}}.$$

The combination of Lemmas 3.29 and 3.30 yields for the closures of strata:

$$\overline{S}_{\tau_{mod}} = \bigsqcup_{\nu_{mod} \supseteq \tau_{mod}} S_{\nu_{mod}} \tag{3.33}$$

In particular, there is one open stratum $S_{\emptyset} = X$ and one closed stratum $S_{\sigma_{mod}} = \partial_{F\ddot{u}} X$, and the latter is contained in the closure of every other stratum.

There is the natural fibration

$$S_{\tau_{mod}} \longrightarrow \mathrm{Flag}_{\tau_{\mathrm{mod}}}(\mathrm{X}) \tag{3.34}$$

by the forgetful map $[b^{\bar{\theta}}_{\tau,p}] \mapsto \tau$, and the fiber over τ is the space of strong asymptote classes of Weyl sectors $V(x, \tau)$, cf. Lemma 3.18, which is canonically identified with the cross section of any parallel set $P(\tau, \hat{\tau})$ for a simplex $\hat{\tau}$ opposite to τ.

The natural G-action on $\overline{X}^{\bar{\theta}}$ preserves each stratum, along with its fibration, and acts transitively on it.

The stabilizer of a point $[b^{\bar{\theta}}_{\tau,p}]$ is the semidirect product

$$N_\tau \rtimes \big(T(\tau, \hat{\tau}) \times K_{f(\tau,\hat{\tau})}\big)$$

where $N_\tau \subset P_\tau$ is the horocyclic subgroup, $\hat{\tau}$ denotes a simplex opposite to τ, $T(\tau, \hat{\tau})$ is the group of transvections along the singular flat $f(\tau, \hat{\tau})$, see section 3.2.2, and the compact subgroup $K_{f(\tau,\hat{\tau})}$ is the pointwise stabilizer of $f(\tau, \hat{\tau})$.

The following observation will be very useful to us:

Lemma 3.35. *For every open subset $O \subset \overline{X}^{\bar{\theta}}$ intersecting the closed stratum, $O \cap \partial_{F\ddot{u}} X \neq \emptyset$, the G-orbit is the entire space, $G \cdot O = \overline{X}^{\bar{\theta}}$.*

Proof. Every stratum contains the closed stratum in its closure, and G acts transitively on every stratum. ☐

In addition to the "big" strata $S_{\tau_{mod}}$, we define for every simplex $\tau \subset \partial_\infty X$ the "small" stratum

$$X_\tau = \{[b^{\bar{\theta}}_{\tau,p}] : p \in X\}. \tag{3.36}$$

The strata X_τ for the simplices $\tau \in \mathrm{Flag}_{\tau_{\mathrm{mod}}}(\mathrm{X})$ are the fibers of the fibration (3.34).

Note that X_τ is canonically identified with every cross section $CS(\hat{\tau}, \tau, p)$ for every simplex $\hat{\tau}$ opposite to τ and every point $p \in P(\hat{\tau}, \tau)$.

The closures of the small strata are given by

$$\overline{X}_\tau = \bigsqcup_{\nu \supseteq \tau} X_\nu \tag{3.37}$$

The stratum closure $\overline{X}_\tau$ is canonically identified with the Finsler compactification of X_τ with respect to the natural induced regular Finsler metric $d^{\bar{\theta}_{\tau_{mod}}}$ on X_τ, where the regular type $\bar{\theta}$ defines the regular type $\bar{\theta}_{\tau_{mod}} \in \Sigma_{\tau_{mod}} \sigma_{mod}$ for the Coxeter complex of X_τ; namely, $\bar{\theta}_{\tau_{mod}}$ is the point corresponding to the simplex $CH(\bar{\theta} \cup \tau_{mod}) \subset \sigma_{mod}$ of dimension $1 + \dim(\tau_{mod})$.

Note that for different simplices τ_1, τ_2 of the same type τ_{mod}, it holds that

$$\overline{X}_{\tau_1} \cap \overline{X}_{\tau_2} = \emptyset.$$

There is the following relation between flag convergence in the sense of [KLP2] and Finsler convergence, which also justifies [KLP2, Remark 5.8]:

Lemma 3.38. *A sequence (x_n) τ_{mod}-flag converges, $x_n \to \tau \in \mathrm{Flag}_{\tau_{\mathrm{mod}}}(\mathrm{X})$, if and only if it accumulates in $\overline{X}^{\bar{\theta}}$ at the small stratum closure $\overline{X}_\tau$.*

Proof. This follows from the definition of flag convergence and Proposition 3.23. ☐

3.2.5 Compactification of maximal flats and Weyl sectors

Let $F \subset X$ be a maximal flat. Applying the horoboundary construction to the restricted Finsler distance $d^{\bar{\theta}}|_{F \times F}$, one obtains the $\bar{\theta}$-compactification

$$\overline{F}^{\bar{\theta}} = F \sqcup \partial^{\bar{\theta}}_\infty F$$

of F. By analogy with Corollary 3.25, the Finsler boundary points at infinity are represented by the mixed Busemann functions

$$b^{F,\bar\theta}_{\tau,p} := \max_{\sigma\supset\tau}\big(b_{\theta_\sigma} - b_{\theta_\sigma}(p)\big)$$

for Weyl sectors $V(p,\tau) \subset F$, cf. (3.10). We note that

$$b^{F,\bar\theta}_{\tau,p} = b^{\bar\theta}_{\tau,p}|_F,$$

see (3.13). Specializing the discussion of sequential convergence in X to F, see the proof of Corollary 3.25, we obtain the following version of Proposition 3.23. (Note that the visual boundary $\partial_\infty F$ is a *finite* simplicial complex.)

Lemma 3.39 (Convergence at infinity for maximal flats). *Suppose that $x_n \to \infty$ is a sequence in F which converges at infinity, and let $o \in F$ be a base point. Then:*

(i) There exists a unique face $\tau \subset \partial_\infty F$ such that the sequence (x_n) is contained in a tubular neighborhood of the Weyl sector $V(o,\tau)$ and τ_{mod}-regular for $\tau_{mod} = \theta(\tau)$.

(ii) There exists a convergent sequence $p_n \to p$ of points in the orthogonal complement $f^\perp_{\tau,o}$ through o of the minimal singular flat $f_{\tau,o}$ containing $V(o,\tau)$, such that $x_n \in V(p_n,\tau)$.

(iii) It holds that

$$x_n \to [b^{F,\bar\theta}_{\tau,p}]$$

in the Finsler compactification $\overline{F}^{\bar\theta}$.

The convergence at infinity of divergent sequences in F is the same intrinsically and extrinsically, i.e. sequences $x_n \to \infty$ in F converge in $\overline{F}^{\bar\theta}$ iff they converge in $\overline{X}^{\bar\theta}$. We thus have the natural topological embedding

$$\overline{F}^{\bar\theta} \longrightarrow \overline{X}^{\bar\theta}$$

extending the inclusion map and sending $[b^{F,\bar\theta}_{\tau,p}] \mapsto [b^{\bar\theta}_{\tau,p}]$, compare also Lemma 3.14.

For every face $\tau \subset \partial_\infty F$, we define the stratum

$$S^F_\tau = \{[b^{F,\bar\theta}_{\tau,p}] : p \in F\} \subset \partial^{\bar\theta}_\infty F. \tag{3.40}$$

It is canonically identified with the cross section $f^\perp_{\tau,o}$ mentioned in the lemma. Moreover, for every face type $\tau_{mod} \subset \sigma_{mod}$ we define the stratum

$$S^F_{\tau_{mod}} = \{[b^{\bar\theta}_{\tau,p}] : \theta(\tau) = \tau_{mod}\} = \bigsqcup_{\theta(\tau)=\tau_{mod}} S^F_\tau$$

analogous to (3.32). Then $S^F_{\tau_{mod}} = S_{\tau_{mod}} \cap \partial^{\bar\theta}_\infty F$, and the S^F_τ are the fibers for the restricted fibration (3.34). As in (3.37), the closures of strata decompose as:

$$\overline{S}^F_\tau = \bigsqcup_{\partial_\infty F \supseteq \nu \supseteq \tau} S^F_\nu \tag{3.41}$$

Let $T_F \subset G$ denote the subgroup of transvections along F. We regard it, intrinsically, also as the group of translations of F. Unlike for the visual boundary, the induced action of

$$T_F \curvearrowright \partial_\infty^{\bar\theta} F$$

on the Finsler boundary is nontrivial. The action preserves each stratum S_τ^F, and T_F acts transitively on it.

The discussion for Weyl sectors is analogous.

Let $V(o,\tau) \subset X$ be a Weyl sector. Again, sequential convergence at infinity for divergent sequences in $V(o,\tau)$ is the same intrinsically and extrinsically, and we have the natural topological embedding

$$\overline{V(o,\tau)}^{\bar\theta} \longrightarrow \overline{X}^{\bar\theta}$$

extending the inclusion map and sending $[b_{\tau,p}^{V(o,\tau),\bar\theta}] \mapsto [b_{\tau,p}^{\bar\theta}]$.

The ideal points in $\partial_\infty^{\bar\theta} V(o,\tau) \subset \partial_\infty^{\bar\theta} X$ are the points $[b_{\nu,p}^{\bar\theta}]$ for the Weyl sectors $V(p,\nu) \subset V(o,\tau)$. We have analogously defined strata $S_\nu^{V(o,\tau)} \subset \partial_\infty^{\bar\theta} V(o,\tau)$ for the faces $\nu \subseteq \tau$, and the decomposition

$$\overline{S}_\nu^{V(o,\tau)} = \bigsqcup_{\tau \supseteq \nu' \supseteq \nu} S_{\nu'}^{V(o,\tau)} \tag{3.42}$$

of their closures.

If $\tau' \subset \tau$, then $\overline{V(o,\tau')}^{\bar\theta} \subset \overline{V(o,\tau)}^{\bar\theta}$. An ideal point $[b_{\nu,p}^{\bar\theta}] \in \partial_\infty^{\bar\theta} V(o,\tau)$ with $V(p,\nu) \subset V(o,\tau)$ belongs to $\partial_\infty^{\bar\theta} V(o,\tau')$ iff $V(p,\nu) \subset V(o,\tau')$.

Furthermore, regarding the intersection of compactified sectors, we obtain:

Lemma 3.43. *For any two simplices $\tau_1, \tau_2 \subset \partial_\infty X$, it holds that*

$$\overline{V(o,\tau_1)}^{\bar\theta} \cap \overline{V(o,\tau_2)}^{\bar\theta} = \overline{V(o,\tau_1) \cap V(o,\tau_2)}^{\bar\theta} = \overline{V(o,\tau_1 \cap \tau_2)}^{\bar\theta}. \tag{3.44}$$

Proof. Suppose that $[b_{\nu,p}^{\bar\theta}] \in \overline{V(o,\tau_1)}^{\bar\theta} \cap \overline{V(o,\tau_2)}^{\bar\theta}$. Then $\nu \subset \tau_1 \cap \tau_2$. There are sectors $V(p_i,\nu) \subset V(o,\tau_i)$ such that $[b_{\nu,p}^{\bar\theta}] = [b_{\nu,p_i}^{\bar\theta}]$. They are contained in the parallel set $P(\nu,\hat\nu)$ for the simplex $\hat\nu$ o-opposite to ν, because the sectors $V(o,\tau_i)$ are contained. Consequently, the minimal flats f_{ν,p_i} containing the sectors $V(p_i,\nu)$ are parallel. Since $[b_{\nu,p_1}^{\bar\theta}] = [b_{\nu,p_2}^{\bar\theta}]$, the sectors $V(p_i,\nu)$ are strongly asymptotic and the flats f_{ν,p_i} must coincide. We may therefore assume without loss of generality that $p_1 = p_2 = p$. But then $V(p,\nu) \subset V(o,\tau_1) \cap V(o,\tau_2) = V(o,\tau_1 \cap \tau_2)$ and hence $[b_{\nu,p}^{\bar\theta}] \in \overline{V(o,\tau_1 \cap \tau_2)}^{\bar\theta}$. This shows the inclusion

$$\overline{V(o,\tau_1)}^{\bar\theta} \cap \overline{V(o,\tau_2)}^{\bar\theta} \subseteq \overline{V(o,\tau_1 \cap \tau_2)}^{\bar\theta}.$$

The reverse inclusion is clear. ☐

In the case of the model euclidean Weyl chamber Δ, we will use the following notation. For a face type $\tau_{mod} \subseteq \sigma_{mod}$ we define the stratum

$$S_{\tau_{mod}}^{\Delta} = \{[b_{\tau_{mod},\delta}^{\Delta,\bar\theta}] : \delta \in \Delta\} \subset \partial_\infty^{\bar\theta} \Delta.$$

Its closure is

$$\overline{S}^{\Delta}_{\tau_{mod}} = \bigsqcup_{\nu_{mod} \supseteq \tau_{mod}} S^{\Delta}_{\nu_{mod}},$$

cf. Lemmas 3.29 and 3.30.

3.2.6 K-action

Let $o \in X$ be the fixed point of K. Let $V = V(o, \sigma) \subset X$ be a euclidean Weyl chamber.

We recall some basic facts about the action $K \curvearrowright X$:

(i) V is a *cross section* for the action, i.e. every K-orbit intersects V exactly once.

(ii) *Stabilizers:* The fixed point set in V of any element $k \in K$ is a Weyl sector $V(o, \tau)$, where $\emptyset \subseteq \tau \subseteq \sigma$ is the face fixed by k. In other words, if k fixes a point $p \in V(o, \sigma)$, then it fixes the smallest Weyl sector $V(o, \tau)$ containing it. (Here, $V(o, \emptyset) := \{o\}$.)

We now establish analogous properties for the action of K on the compactification.

Lemma 3.45 (Cross section). $\overline{V}^{\bar\theta} \subset \overline{X}^{\bar\theta}$ *is a cross section for the action of* K *on* $\overline{X}^{\bar\theta}$.

Proof. Since $K \cdot \overline{V}^{\bar\theta}$ is compact and contains $K \cdot V = X$, and since X is dense in its compactification, it holds that $K \cdot \overline{V}^{\bar\theta} = \overline{X}^{\bar\theta}$.

We have to verify that the K-action does not carry different points of $\overline{V}^{\bar\theta}$ to each other. Suppose that

$$k \cdot [b^{\bar\theta}_{\tau,p}] = [b^{\bar\theta}_{\tau',p'}] \tag{3.46}$$

for $k \in K$ and Weyl sectors $V(p, \tau), V(p', \tau') \subset V$. Then, in particular, $k\tau = \tau'$. Since σ_{mod} is a cross section for the action of K on $\partial_\infty X$ (in fact, for the action of G), this implies that $\tau = \tau'$ and $k\tau = \tau$.

It follows that k fixes the sector $V(o, \tau)$ pointwise, and hence also the minimal (singular) flat $f_{\tau,o}$ containing it. Moreover, it preserves the parallel set $P(\tau, \hat\tau) = P(f_{\tau,o})$. Here, $\hat\tau \subset \partial_\infty X$ denotes the simplex o-opposite to τ. Note that $p \in V \subset P(\tau, \hat\tau)$

Condition (3.46) is then equivalent to

$$kf_{\tau,p} = f_{\tau,p'}$$

where $f_{\tau,p}, f_{\tau,p'} \subset P(\tau, \hat\tau)$ denote the flats parallel to $f_{\tau,o}$ through p and p', equivalently, the minimal flats containing the sectors $V(p, \tau)$ and $V(p', \tau)$. Since V is a cross section for the action of K on X, it follows that $f_{\tau,p} \cap V = f_{\tau,p'} \cap V$ is fixed pointwise by k. The intersection $f_{\tau,p} \cap V$ is nonempty and contains a sector $V(q, \tau)$. Hence, $[b^{\bar\theta}_{\tau,p}] = [b^{\bar\theta}_{\tau,q}]$ is fixed by k. □

Lemma 3.47 (Stabilizers). *Let* $k \in K$ *and* $V(p, \tau) \subset V$*. The following are equivalent:*

(i) k *fixes* $[b^{\bar\theta}_{\tau,p}] \in \overline{V}^{\bar\theta}$.

(ii) k *fixes* $V(p, \tau)$ *pointwise.*

(iii) k *fixes pointwise the smallest Weyl sector* $V(o, \nu)$ *containing* $V(p, \tau)$.

Proof. The proof of the previous lemma shows in particular the equivalence of (i) and (ii). Conditions (ii) and (iii) are equivalent, because the fixed point set of k on V is a sector $V(o,\nu)$, namely for the face ν, $\varnothing \subseteq \nu \subseteq \sigma$, which is the fixed point set of k on σ. $\square$

Let K_τ denote the stabilizer in K of the simplex τ, and put $K_\varnothing = K$.

Corollary 3.48. *The points in the compactified euclidean Weyl chamber* $\overline{V(o,\sigma)}^{\bar\theta}$ *fixed by* K_τ *are precisely the points in*

$$\overline{V(o,\tau)}^{\bar\theta},$$

and the points with stabilizer equal to K_τ *are precisely the points in*

$$\overline{V(o,\tau)}^{\bar\theta} - \bigcup_{\varnothing\subseteq\nu\subsetneq\tau} \overline{V(o,\nu)}^{\bar\theta}.$$

4 Coxeter groups and their regular polytopes

4.1 Basics of polytopes

We refer the readers to [Gr] and [Z] for a detailed treatment of polytopes. In what follows, V will denote a euclidean vector space, i.e. a finite-dimensional real vector space equipped with an inner product (x,y). We will use the notation V^* for the dual vector space, and for $\lambda \in V^*$ and $x \in V$ we let $\langle\lambda, x\rangle = \lambda(x)$. The inner product on V defines the inner product, again denoted (λ,μ), on the dual space.

A *polytope* B in V is a compact convex subset equal to the intersection of finitely many closed half-spaces. Note that we do not require B to have nonempty interior. The *affine span* $\langle B\rangle$ of B is the intersection of all affine subspaces in V containing B. The topological frontier of B in its affine span is the boundary ∂B of B. A *facet* of B is a codimension one face of ∂B.

Each polytope B has a *face poset* $\mathcal{F}_B$. It is the poset whose elements are the faces of B with the order given by the inclusion relation. Two polytopes are *combinatorially isomorphic* if there is an isomorphism of their posets. Such an isomorphism necessarily preserves the dimension of faces. Two polytopes B and B' are *combinatorially homeomorphic* if there exists a (piecewise linear) homeomorphism $h : B \to B'$ which sends faces to faces.

Given a polytope B whose dimension equals $n = \dim(V)$, the *polar* (or *dual*) polytope of B is defined as the following subset of the dual vector space:

$$B^* = \{\lambda \in V^* : \lambda(x) \leqslant 1, \forall x \in B\}.$$

Thus, $\lambda \in B^* \subset V^*$ implies that the affine hyperplane $H_\lambda = \{\lambda = 1\}$ is disjoint from the interior of B. Moreover, $\lambda \in \partial B^*$ iff H_λ has nonempty intersection with B. Each face φ of B determines the *dual face* φ^* of B^*, consisting of the elements $\lambda \in B^*$ which are equal to 1 on the entire face φ. This defines a natural bijection between the faces of B and B^*:

$$\star : \varphi \mapsto \varphi^*.$$

Under this bijection, faces have complementary dimensions:

$$\dim(\varphi) + \dim(\varphi^*) = n - 1.$$

The bijection $\star$ also reverses the face inclusion:

$$\varphi \subset \psi \iff \varphi^* \supset \psi^*.$$

In particular, the face poset of ∂B^* is dual to the face poset of ∂B. If W is a group of linear transformations preserving B, its dual action

$$w^*(\lambda) = \lambda \circ w^{-1}$$

on V^* preserves B^*. Naturality of $\star$ implies that it is W-equivariant.

A polytope B is called *simplicial* if its faces are simplices. It is called *simple* if it has a natural structure of a manifold with corners: Each vertex v of B is contained in exactly d facets, where d is the dimension of B. Equivalently, the affine functionals defining these facets in $\langle B \rangle$ have linearly independent linear parts. For each simplicial polytope, its dual is a simple polytope, and vice versa.

Lemma 4.1. *Two polytopes are combinatorially isomorphic if and only if they are combinatorially homeomorphic.*

Proof. One direction is clear. Suppose that $c : \mathcal{F}_B \to \mathcal{F}_{B'}$ is an isomorphism of posets. Using this bijection we will define a homeomorphism $h : B \to B'$, sending each face F to $c(F)$, by induction on skeleta. We let $h : B^0 \to (B')^0$ be equal to c restricted to the vertex sets.

Suppose, inductively, that we constructed a homeomorphism h on k-skeleta of our polyhedra, sending each F to $c(F)$. We extend h to the $(k+1)$-dimensional skeleton as follows. Given a $(k+1)$-dimensional face F of B, we already have a homeomorphic embedding

$$h : \partial F \to B',$$

sending faces to faces and preserving the order. Since c preserves the posets, we have that

$$h(\partial F) = \partial c(F).$$

We pick an arbitrary pair of interior points $x \in F, x' \in F' = c(F)$ and set $h(x) = x'$. Then we extend h to a PL homeomorphism $h : F \to F'$ via the *Alexander trick*, meaning that we cone off the boundary map. ☐

For simple polytopes one can make a sharper statement, see [Da]:

Theorem 4.2. *If B and B' are combinatorially isomorphic simple polytopes, then there exists a combinatorial diffeomorphism $h : B \to B'$ inducing the given combinatorial isomorphism.*

Here, a diffeomorphism of polytopes means a homeomorphism which is the restriction of a diffeomorphism defined on a larger open set.

4.2 Root systems

In this and the following sections, the euclidean vector space V is identified with the model maximal flat F_{mod} for the symmetric space X; the root system $R \subset V^*$ is the root system of X. Accordingly, the Coxeter group W defined via R is the Weyl group of X. Since the symmetric space X has noncompact type, R spans V^*, i.e. W fixes only the origin 0 in V.

Given a face τ of the spherical Coxeter complex $\partial_\infty V$, we define the root subsystem

$$R_\tau \subset R$$

consisting of all roots which vanish identically on $V(0,\tau)$.

Each root $\alpha \in R$ corresponds to a *coroot* $\alpha^\vee \in V$, which is a vector such that the reflection $s_\alpha : V \to V$ corresponding to α acts on V by the formula:

$$s_\alpha(x) = x - \langle \alpha, x \rangle \alpha^\vee. \tag{4.3}$$

The group W also acts isometrically on the dual space V^*; each reflection $s_\alpha \in W$ acts on V^* as a reflection. The corresponding wall is given by the equation

$$\{\lambda \in V^* : \langle \lambda, \alpha^\vee \rangle = 0\},$$

equivalently, this wall is $\alpha^\perp$, the orthogonal complement of α in V^*.

From now on, we fix a Weyl chamber $\Delta = \Delta_{mod} \subset V$ for the action of W on V. The visual boundary of Δ is the model spherical chamber σ_{mod}.

Notation 4.4. *We let* $[n]$ *denote the set* $\{1, ..., n\}$.

The choice of Δ determines the set of positive roots $R^+ \subset R$ and the set of *simple roots* $\alpha_1, ..., \alpha_n \in R^+$, where $n = \dim(V)$;

$$\Delta = \{x \in V : \alpha_i(x) = \langle \alpha_i, x \rangle \geqslant 0, i \in [n]\}.$$

We will use the notation $s_i = s_{\alpha_i}$ for the *simple reflections*. They generate W.

The *dual chamber* to Δ is

$$\Delta^* \subset V^*, \quad \Delta^* = \{\lambda \in V^* : (\alpha_i, \lambda) \geqslant 0, i \in [n]\}.$$

Remark 4.5. Note that there is another notion of a dual cone to Δ in V^*, namely the *root cone* $\Delta^\vee$, consisting of all $\lambda \in V^*$ such that the restriction of λ to Δ is nonnegative. The root cone consists of the nonnegative linear combinations of simple roots. The root cone contains the dual chamber but, is, with few exceptions, strictly larger.

Let B be a W-invariant polytope in V with nonempty interior. We will use the notation $\Delta_B = \Delta \cap B$, $\Delta^*_{B^*} = \Delta^* \cap B^*$.

Lemma 4.6. *Suppose that* $\lambda \in \Delta^*$ *is such that* $\lambda(x) \leqslant 1$ *for all* $x \in \Delta_B$. *Then* $\lambda \in B^*$.

Proof. Let $\lambda \in V^*$ and let $v \in \text{int}(\Delta) \subset V$. Then $\lambda|_{Wv}$ is maximal in v iff $\lambda \in \Delta^*$. The assertion follows. ☐

4.3 Geometry of the dual ball

We assume now that $B \subset V$ is a W-invariant polytope in V with nonempty interior, such that

$$\Delta_B = \{x \in \Delta : l(x) \leqslant 1\}$$

where $l = l_\Delta \in \mathrm{int}(\Delta^*)$ is a *regular linear functional.* The gradient vector of l gives a direction $\iota\bar{\theta}$, which is a regular point of σ_{mod}.

Set $l_w = w^* l = l \circ w^{-1}$, where $w \in W$. Then,

$$B = \bigcap_{w \in W} \{x \in V : l_w(x) \leqslant 1\},$$

i.e. the facets of B are carried by the affine hyperplanes $l_w = 1$ for $w \in W$.

The polytope B defines a (possibly nonsymmetric) norm on V, namely the norm for which B is the unit ball:

$$||x|| = ||x||_{\bar{\theta}} = \max_{w \in W} (l_w(x)) \,. \tag{4.7}$$

We let $\omega_1, ..., \omega_n$ denote the nonzero vertices of the n-simplex Δ_B. We will label these vertices consistently with the labeling of the simple roots: ω_i is the unique vertex of Δ_B on which α_i does not vanish. Geometrically speaking, ω_i is opposite to the facet A_i of Δ_B carried by the wall $\alpha_i = 0$.

The regularity of l implies:

Lemma 4.8. *The polytope B is simplicial. Its facets are the simplices*

$$\{x \in w\Delta : l_w(x) = 1\}.$$

For each reflection $s_i = s_{\alpha_i}$, the line segment $\omega_i s_i(\omega_i)$ is not contained in the boundary of B.

Proof. We will prove the last statement. The segment $\omega_i s_i(\omega_i)$ is parallel to the vector $\alpha_i^\vee$. If $\alpha_i^\vee$ were to be parallel to the face $l = 1$ of B, then $\langle l, \alpha_i^\vee \rangle = 0$, which implies that l is singular. ☐

Corollary 4.9. *Since the polytope B is simplicial, the dual polytope B^* is simple.*

The chamber Δ^* contains a distinguished vertex of Δ^*_{B*}, namely the linear functional $l = l_\Delta$; this is the only vertex of Δ^*_{B*} contained in the interior of Δ^*. (The other vertices of Δ^*_{B*} are not vertices of B^*.)

We now analyze the geometry of Δ^*_{B*} in more detail.

Lemma 4.10. Δ^*_{B*} *is given by the set of $2n$ inequalities $(\cdot, \alpha_i) \geqslant 0$ and $\langle \cdot, \omega_i \rangle \leqslant 1$ for $i \in [n]$.*

Proof. It is clear that these inequalities are necessary for $\lambda \in V^*$ to belong to Δ^*_{B*}. In order to prove that they are sufficient, we have to show that each λ satisfying these inequalities belongs to B^*. The inequalities $\langle \lambda, \omega_i \rangle \leqslant 1$ show that the restriction of λ to Δ_B is $\leqslant 1$. Now, Lemma 4.6 shows that $\lambda(x) \leqslant 1$ for all $x \in B$. $\square$

Close to the origin, Δ^*_{B*} is given by the n inequalities $(\cdot, \alpha_i) \geqslant 0$, while the other n inequalities are strict. Close to l, it is given by the n inequalities $\langle \cdot, \omega_i \rangle \leqslant 1$, while the other n inequalities are strict.

We define the *exterior facet* $E_i \subset \Delta^*_{B*}$ by the equation

$$\langle \cdot, \omega_i \rangle = 1,$$

and the *interior facet* $F_j \subset \Delta^*_{B*}$ as the fixed point set of the reflection s_j, equivalently, by the equation

$$(\cdot, \alpha_j) = 0.$$

For subsets $I, J \subset [n] = \{1, ..., n\}$ we define the *exterior faces*

$$E_I := \bigcap_{i \in I} E_i$$

containing l, and the *interior faces*

$$F_J := \bigcap_{j \in J} F_j$$

containing the origin. These are nonempty faces of Δ^*_{B*} of the expected dimensions, due to the linear independence of the ω_i's, respectively, the α_j's.

As a consequence of the last lemma, every face of the polytope Δ^*_{B*} has the form

$$E_I \cap F_J$$

for some subsets $\varnothing \subseteq I, J \subseteq [n]$.

We now describe the combinatorics of the polytope Δ^*_{B*}.

Lemma 4.11. *For each $i = 1, ..., n$, $E_i \cap F_i = \varnothing$.*

Proof. Suppose that $\lambda \in \Delta^*_{B*}$ is a point of intersection of these faces. Then λ is a linear function fixed by the reflection s_i and satisfying the equation $\langle \lambda, \omega_i \rangle = 1$. Then $\lambda(s_i(\omega_i)) = 1$ as well. Thus, $\lambda = 1$ on the entire segment connecting the vertices ω_i and $s_i(\omega_i)$ of B. Since λ belongs to B^*, this segment has to be contained in the boundary of B. But this contradicts Lemma 4.8. Therefore, such a λ cannot exist. $\square$

We denote by $W_J < W$ the subgroup generated by the reflections s_j for $j \in J$. The fixed point set of W_J on Δ^*_{B*} equals F_J.

Furthermore, we define ω_I as the face of B, as well as of Δ_B, which is the convex hull of the vertices ω_i for $i \in I$. The dual face ω_I^* of B^* is given, as a subset of B^*, by the equations $\langle \cdot, \omega_i \rangle = 1$. It is contained in $W_J \cdot \Delta^*_{B*}$, where we put $J = [n] - I$. Indeed, the vertices of ω_I^* are the functionals l_w for which the dual facet $l_w = 1$ of B contains ω_I, equivalently, for $w \in W_J$.

Note that W_J preserves ω_I and therefore also ω_I^* (and acts on it as a reflection group). The fixed point set of W_J on $W_J \cdot \Delta_{B*}^*$ is contained in the intersection

$$\bigcap_{w \in W_J} w\Delta_{B*}^*$$

and in particular in Δ_{B*}^*. This implies that

$$\varnothing \neq \mathrm{Fix}_{W_J}(\omega_I^*) \subset \Delta_{B*}^*.$$

Note that $E_I = \omega_I^* \cap \Delta_{B*}^*$. It follows that

$$E_I \cap F_J \supseteq \mathrm{Fix}_{W_J}(\omega_I^*) \neq \varnothing.$$

In combination with the previous lemma, we conclude:

Lemma 4.12. *For arbitrary subsets $\varnothing \subseteq I, J \subseteq [n]$, it holds that $E_I \cap F_J \neq \varnothing$ iff $I \cap J = \varnothing$.*

Next, we prove the uniqueness of the labeling of the faces.

Lemma 4.13. *If $E_I \cap F_J = E_{I'} \cap F_{J'} \neq \varnothing$, then $I = I'$ and $J = J'$.*

Proof. Since $E_I \cap E_{I'} = E_{I \cup I'}$ and $F_J \cap F_{J'} = F_{J \cup J'}$, the proof reduces to the case of containment $I \subseteq I'$ and $J \subseteq J'$.

Suppose that $j' \in J' - J$. Then, intersecting both sides of the equality $E_I \cap F_J = E_{I'} \cap F_{J'}$ with $E_{j'}$, the previous lemma yields that

$$\varnothing \neq E_{I \cup \{j'\}} \cap F_J = E_{I' \cup \{j'\}} \cap F_{J'} = \varnothing,$$

a contradiction. Thus $J = J'$, and similarly $I = I'$. □

For the n-cube $[0,1]^n$, we define similarly facets $E'_i = \{t_i = 1\}$ and $F'_j = \{t_j = 0\}$. They satisfy the same intersection properties as in Lemmas 4.12 and 4.13. Hence the correspondence

$$E_I \cap F_J \overset{c}{\mapsto} E'_I \cap F'_J$$

is a combinatorial isomorphism between the polytopes Δ_{B*}^* and $[0,1]^n$. Lemma 4.1 now yields:

Theorem 4.14. *The polytope Δ_{B*}^* is combinatorially homeomorphic to the n-cube $[0,1]^n$, i.e. there exists a combinatorial homeomorphism*

$$\Delta_{B*}^* \xrightarrow{h} [0,1]^n$$

inducing the bijection c of face posets.

4.4 Cube structure of the compactified Weyl chamber

In this section we construct a canonical homeomorphism from the Finsler compactification $\overline{\Delta}^{\bar\theta}$ of the model Weyl chamber $\Delta \subset V$, to the cube $[0, \infty]^n$. Recall that $\alpha_1, ..., \alpha_n$ are the simple roots with respect to Δ. Each intersection

$$\Delta_i = \ker(\alpha_i) \cap \Delta$$

is a facet of Δ.

For $x \in \Delta$ define

$$\vec{\alpha}(x) := (\alpha_1(x), \ldots, \alpha_n(x)) \in [0, \infty)^n.$$

This map is clearly a homeomorphism from Δ to $[0, \infty)^n$. We wish to extend the map $\vec{\alpha}$ to a homeomorphism of the compactifications.

We recall the description of sequential convergence at infinity in Δ, compare Lemma 3.39. A sequence $x_k \to \infty$ in Δ converges at infinity iff the following properties hold:

(i) By parts (i) and (ii) of the lemma, there exists a face $\tau = \tau_{mod}$ of $\sigma_{mod} = \partial_\infty \Delta$ such that for every $\alpha_i \in R_\tau$ the sequence $(\alpha_i(x_k))$ converges.

(ii) By the τ_{mod}-regularity assertion in part (i) of the lemma, for the other simple roots $\alpha_i \notin R_\tau$, we have divergence $\alpha_i(x_k) \to +\infty$.

In other words, the sequence (x_k) converges at infinity, iff the limit

$$\lim_{k \to +\infty} \vec{\alpha}(x_k) \in [0, \infty]^n$$

in the closed cube exists. Moreover, part (iii) of Lemma 3.39 combined with Lemma 3.18 implies that the extension

$$\overline{\Delta}^{\bar\theta} \xrightarrow{\vec{\alpha}} [0, \infty]^n$$

sending

$$\lim_{k \to +\infty} x_k \mapsto \lim_{k \to +\infty} \vec{\alpha}(x_k)$$

for sequences (x_k) converging at infinity is well-defined and bijective. Now, Lemma 2.4 implies that the extension is a homeomorphism. Composing with the homeomorphism

$$\kappa : [0, \infty]^n \to [0, 1]^n, \quad \kappa : (t_1, \ldots, t_n) \mapsto \left(1 - \frac{1}{t_1 + 1}, \ldots, 1 - \frac{1}{t_n + 1}\right)$$

we obtain:

Lemma 4.15. *The map $\kappa \circ \vec{\alpha}$ is a homeomorphism from $\overline{\Delta}^{\bar\theta}$ to the cube $[0, 1]^n$. It sends the compactification of each face $\overline{\Delta}_i^{\bar\theta}, i \in [n]$, to the face F_i' of the cube $[0, 1]^n$.*

For a partition $[n] = I \sqcup J$, we define $\emptyset \subseteq \tau_I \subseteq \sigma_{mod}$ as the face fixed by the reflections s_j for $j \in J$. Equivalently, the vertices of τ are the directions of the vectors ω_i for $i \in I$.

Vice versa, for a face $\emptyset \subseteq \tau = \tau_{mod} \subseteq \sigma_{mod}$, we define the partition $[n] = I_\tau \sqcup J_\tau$ such that $\tau_{I_\tau} = \tau$, i.e. I_τ indexes the vertices of τ.

Moreover, we have the sector $\Delta_I = \cap_{i \in I} \Delta_i = V(0, \tau_I) \subset \Delta$ and its compactification

$$\overline{\Delta}_I^{\bar\theta} = \bigcap_{i \in I} \overline{\Delta}_i^{\bar\theta},$$

cf. (3.44).

Recall that our vector space V is the underlying vector space of the model maximal flat $F = F_{mod}$. We can now combine the above lemma with the homeomorphism constructed in Theorem 4.14:

Theorem 4.16. *There exists a homeomorphism*

$$\overline{\Delta}^{\bar\theta} \xrightarrow{\phi} \Delta^*_{B^*} \subset B^*$$

satisfying the following:

1. For each partition $[n] = I \sqcup J$,

$$\phi(\overline{S}^{\Delta}_{\tau_I}) = E_I$$

and

$$\phi(\overline{\Delta}_J^{\bar\theta}) = F_J.$$

In particular, $\phi(0) = 0$.

2. The map ϕ preserves the W-stabilizers: $x \in \overline{\Delta}^{\bar\theta}$ is fixed by $w \in W$ iff $\phi(x)$ is fixed by w.

3. As a consequence, ϕ extends to a W-equivariant homeomorphism of the compactified model flat to the dual ball:

$$\Phi_{F_{mod}} : \overline{F}^{\bar\theta}_{mod} \to B^*.$$

Proof. Combining Theorem 4.14 and Lemma 4.15, we define

$$\phi = h^{-1} \circ \kappa \circ \overrightarrow{\alpha}.$$

$\Delta^*_{B^*}$ is a cross section for the action of W on B^*, because Δ^* is a cross section for its action on V^*. By Lemma 3.45, the compactified chamber $\overline{\Delta}^{\bar\theta}$ is a cross section for the action of W on $\overline{F}^{\bar\theta}$. We also note that for $J = [n] - I$, the fixed point sets of the subgroup $W_J < W$ in $\overline{\Delta}^{\bar\theta}$ and $\Delta^*_{B^*}$ are precisely $\overline{\Delta}_I^{\bar\theta}$ and F_I, cf. Corollary 3.48. The last assertion of the theorem follows using Lemma 2.5. ☐

Remark 4.17. One can also derive this theorem from [BJ, Proposition I.18.11]. Our proof is a direct argument which avoids symplectic geometry.

Remark 4.18. We note that the paper [KMN] computes horofunctions on finite dimensional vector spaces V equipped with polyhedral norms, but does not address the question about the global topology of the associated compactification of V. See also [Bri, W1].

Question 4.19. Suppose that $||\cdot||$ is a polyhedral norm on a finite-dimensional real vector space V. Is it true that the horoclosure $\overline{V}$ of V with respect to this norm, with its natural stratification, is homeomorphic to the closed unit ball for the dual norm? Is it homeomorphic to a closed ball for arbitrary norms?

5 Manifold with corners structure on the Finsler compactified symmetric space

In Theorem 4.16 we proved the existence of a W-equivariant homeomorphism $\Phi_F : \overline{F}^{\bar{\theta}} \to B^*$. Since B^* is a simple polytope, it has a natural structure of a manifold with corners, whose strata are the faces of B^*. Via the homeomorphism Φ_F, we then endow $\overline{F}^{\bar{\theta}}$ with the structure of a manifold with corners as well. The homeomorphism Φ_F^{-1} sends each face τ^* of B^* (dual to the face τ of B, which we will identify with the corresponding face of the Coxeter complex at infinity a_{mod}) to the ideal boundary

$$\partial_\infty^{\bar{\theta}} V(0,\tau).$$

The latter can be described as the set of strong asymptote classes of sectors $V(x,\tau)$:

$$[V(x,\tau)] = [V(x',\tau)] \iff x \equiv x' \in F/Span(V(0,\tau)),$$

see Lemma 3.18. In other words, this is the stratum S_τ^F of $\overline{F}^{\bar{\theta}}$, see (3.40). The goal of this section is to extend this manifold with corners structure from $\overline{F}^{\bar{\theta}}$ to $\overline{X}^{\bar{\theta}}$. We will also see that this structure matches the one of the maximal Satake compactification of X.

5.1 Manifold with corners

Let $\sigma \in \partial_{F\ddot{u}} X$ a chamber which we view as a point in the closed stratum of $\overline{X}^{\bar{\theta}}$. Let $o \in X$ be the fixed point of K.

Lemma 5.1. *For every neighborhood U of σ in $\overline{V(o,\sigma)}^{\bar{\theta}}$ and every neighborhood U' of the identity e in K, the subset $U' \cdot U$ is a neighborhood of σ in $\overline{X}^{\bar{\theta}}$.*

Proof. Suppose that $U' \cdot U$ is not a neighborhood. Then there exists a sequence $\xi_n \to \sigma$ in $\overline{X}^{\bar{\theta}}$ outside $U' \cdot U$. There exist chambers σ_n such that $\xi_n \in \overline{V(o,\sigma_n)}^{\bar{\theta}}$, and points $y_n \in V(o,\sigma_n)$ approximating ξ_n such that $y_n \to \sigma$. Our description of sequential convergence, cf. Proposition 3.23, implies that the sequence (y_n) is σ_{mod}-regular and $\sigma_n \to \sigma$. Hence there exist elements $k_n \to e$ in K such that $k_n\sigma = \sigma_n$. Then, due to the continuity of the K-action, the points $k_n^{-1}\xi_n \in \overline{V(o,\sigma)}^{\bar{\theta}}$ converge to σ. Hence they enter the neighborhood U, and (k_n) enters U' for large n. This is a contradiction. ☐

Suppose now that the neighborhood $U \subset \overline{V(o,\sigma)}^{\bar{\theta}}$ is sufficiently small, say, disjoint from the union of the compactified sectors $\overline{V(o,\tau)}^{\bar{\theta}}$ over all proper faces $\tau \subsetneq \sigma$. Then the stabilizer of every point in U equals the pointwise stabilizer $K_\sigma = K_F$ of the maximal flat $F \supset V(o, \sigma)$, see Corollary 3.48. We consider the bijective continuous map

$$K/K_F \times U \longrightarrow KU \subset \overline{X}^{\bar\theta}$$

given by the K-action. By the previous lemma, its image KU is a neighborhood of the closed stratum $S_{\sigma_{mod}} = \partial_{F\ddot{u}} X$. After shrinking U to a compact neighborhood of σ, the map becomes a homeomorphism. After further shrinking U to an open neighborhood, the map becomes a homeomorphism onto an open neighborhood of $\partial_{F\ddot{u}} X$.

Since U is a manifold with corners, see Theorem 4.16, and K/K_F is a manifold, we conclude with Lemma 3.35:

Theorem 5.2 (Manifold with corners). $\overline{X}^{\bar\theta}$ *is a manifold with corners with respect to the stratification by the strata* $S_{\tau_{mod}}$. *In particular, the manifold-with-corners structure is* G*-invariant.*

This means that the k-dimensional stratum of the manifold with corner structure equals the union of the k-dimensional strata $S_{\tau_{mod}}$.

5.2 Homeomorphism to ball

At last, we can now prove that the Finsler compactification of the symmetric space X is K-equivariantly homeomorphic to a closed ball. Let B^* be the dual ball to the unit ball $B \subset F_{mod}$ of the norm (4.7) on the vector space F_{mod}, defined via the regular vector $\bar\theta$. We will identify the dual vector space of F_{mod} with F_{mod} itself using the euclidean metric on F_{mod}. Hence, B^* becomes a unit ball in F_{mod} for the *dual norm*

$$||\cdot||^* = ||\cdot||^*_{\bar\theta}$$

of our original norm.

Since $B^* \subset F_{mod}$ is W-invariant, the dual norm extends from F_{mod} to a G-invariant Finsler metric on X. The latter defines a G-invariant distance function on X

$$d^*_{\bar\theta}(x, y) = ||d_\Delta(x, y)||^*_{\bar\theta},$$

cf. §3.1.2. The closed unit ball (centered at $o \in X$) for this dual norm is

$$B^*(o, 1) = \{q \in X : d^*_{\bar\theta}(o, q) \leqslant 1\}.$$

The group K preserves this dual ball since K fixes the point o.

We can now prove:

Theorem 5.3. *There exists a* K*-equivariant homeomorphism*

$$\overline{X}^{\bar\theta} \xrightarrow{\Phi} B^*(o, 1)$$

which restricts to the homeomorphism $\phi : \overline{\Delta}^{\bar\theta} \to \Delta^*_{B^*}$ *from Theorem 4.16.*

Proof. We will use Lemma 2.5 to construct Φ. In order to do so, we have to know that $\overline{\Delta}^{\bar{\theta}}$ and Δ^*_{B*} are cross sections for the actions of K on $\overline{X}^{\bar{\theta}}$ and $B^*(o,1)$, and that ϕ respects the K-stabilizers.

1. According to Lemma 3.45, $\overline{\Delta}^{\bar{\theta}}$ is a cross section for the action of K on $\overline{X}^{\bar{\theta}}$. Since K preserves the dual ball $B^*(o,1)$ and

$$\Delta^*_{B*} = \Delta \cap B^*(o,1),$$

while Δ is a cross section for the action $K \curvearrowright X$, it follows that Δ^*_{B*} is a cross section for the action $K \curvearrowright B^*(o,1)$.

2. The faces τ, $\emptyset \subseteq \tau \subseteq \sigma$, correspond to index sets J, $\emptyset \subseteq J_\tau \subseteq [n]$, where $j \in J_\tau$ iff the reflection s_j fixes τ. According to Corollary 3.48, the fixed point set of K_τ on $\overline{\Delta}^{\bar{\theta}}$ equals $\overline{V(o,\tau)}^{\bar{\theta}}$. On the other hand, the fixed point set of K_τ on Δ^*_{B*} equals the interior face F_{J_τ}. By Theorem 4.16, the homeomorphism ϕ carries $\overline{V(o,\tau)}^{\bar{\theta}}$ to F_{J_τ}. Therefore, ϕ respects the point stabilizers. ☐

5.3 Relation to the maximal Satake compactification

It turns out that the compactification $\overline{X}^{\bar{\theta}}$ constructed in this paper is naturally isomorphic to the *maximal Satake compactification* $\overline{X}^S_{max}$. To this end, we will use the *dual-cell* interpretation of the maximal Satake compactification, see [BJ, Ch. I.19]

Theorem 5.4. *There is a G-equivariant homeomorphism of manifolds with corners $\overline{X}^{\bar{\theta}} \to \overline{X}^S_{max}$ which extends the identity map $X \to X$.*

Proof. We first observe that the group K acts on both compactifications so that the cross sections for the actions are the respective compactifications of the model euclidean Weyl chamber $\Delta = \Delta_{mod} \subset F = F_{mod}$. We therefore compare the W-invariant compactifications of F_{mod}. On the side of $\overline{X}^{\bar{\theta}}$, the ideal boundary of F is the union of strata S^F_τ as in §5.2. Elements of S^F_τ are equivalence classes $[V(x,\tau)]$ of sectors $V(x,\tau)$ in F. Two sectors $V(x,\tau), V(x',\tau)$ with the same base τ are equivalent iff x, x' project to the same vector in $F/Span(V(0,\tau))$. These are exactly the strata, denoted $e(C)$, in the maximal Satake compactification of F, denoted by $\overline{F}^S_{max}$, see [BJ, Ch. I.19]: For each sector $C = V(0,\tau)$, the stratum $e(C)$ is $F/Span(C)$. We then have a W-equivariant bijection

$$h : \overline{F}^{\bar{\theta}} \to \overline{F}^S_{max}$$

defined via the collection of maps

$$[V(x,\tau)] \mapsto [x] \in e(C).$$

For $\tau = \emptyset$, this is just the identity map $F \to F$.

In order to show that this map is a homeomorphism we note that the topology on $\overline{F}^S_{max}$ is defined via roots (see [BJ, Ch. I.19]) and on the Weyl chamber Δ in F this topology is exactly the topology on $\overline{\Delta}^{\bar\theta}$ described in terms of simple roots, cf. the proof of Lemma 4.15.

Lastly, we note that the map h we described respects the stabilizers in the group K. Therefore, by Lemma 2.5, we obtain a K-equivariant homeomorphic extension

$$\overline{X}^{\bar\theta} \to \overline{X}^S_{max}$$

of h, which is also an extension of the identity map $X \to X$. Since the identity is G-equivariant, the same holds for the extension. ☐

Remark 5.5. The maximal Satake compactification is a real-analytic manifold with corners on which the group G acts real-analytically, see [BJ, Ch. I.19]. Therefore, the same conclusion holds for the compactification $\overline{X}^{\bar\theta}$.

5.4 Proof of Theorem 1.1

The theorem is the combination of the following results:

Part (i) is proven in §3.2.4, where we established that $\overline{X}^{\bar\theta}$ is a union of strata $S_{\tau_{mod}}$ each of which is a single G-orbit. Thus, G acts on $\overline{X}^{\bar\theta}$ with finitely many orbits.

Part (ii) is proven in Theorem 5.2.

Part (iii) is proven in Theorem 5.3.

Part (iv) is the content of Proposition 3.31.

Lastly, Part (v) is established in Theorem 5.4. ☐

6 Proper discontinuity: regular case

The main result of this and the following section is Corollary 7.8 proving that the quotient space of each RCA subgroup $\Gamma < G$ admits a compactification as an orbifold with corners. Our discussion parallels that in [KLP1] where we first prove the nonexistence of Γ-dynamical relation between points at infinity outside of certain thickenings of chamber limit sets and then establish cocompactness outside of the same thickenings.

Given a regular subgroup $\Gamma < G$, for each type $\bar\theta \in \sigma_{mod}$ which is sufficiently close to the root type $\bar\rho \in \sigma_{mod}$, we define a Γ-invariant *thickening* $Th_{\bar\theta}(\Lambda_{ch}(\Gamma)) \subset \partial^{\bar\theta}_{\infty} X$ of the limit set $\Lambda(\Gamma) \subset \partial_{\infty} X$ of a regular subgroup $\Gamma < G$. (Note that the limit set and its thickening live in different spaces!) We also define the complementary set

$$\Omega_{Th_{\bar\theta}}(\Gamma) = \partial^{\bar\theta}_{\infty} X - Th_{\bar\theta}(\Lambda_{ch}(\Gamma)).$$

We then prove that the action of Γ on the *Finsler–bordified symmetric space*

$$X \cup \Omega_{Th_{\bar\theta}}(\Gamma)$$

is properly discontinuous (assuming that Γ is regular) and cocompact (assuming that Γ is conical). Then the quotient orbifold

$$\left(X \cup \Omega_{Th_{\bar{\theta}}}(\Gamma)\right)/\Gamma$$

is a compactification of the locally symmetric space X/Γ as an orbifold with corners.

6.1 A metric inequality for dynamical relation

For a regular type $\bar{\theta} \in \mathrm{int}(\sigma_{mod})$, we consider the action

$$G \curvearrowright \overline{X}^{\bar{\theta}} = X \cup \partial_{\infty}^{\bar{\theta}} X$$

of the full isometry group on the Finsler compactification of type $\bar{\theta}$.

Let $g_n \to \infty$ be a σ_{mod}-regular sequence in G. After passing to a subsequence, we may suppose that we have convergence

$$g_n^{\pm 1} x \to \lambda_{\pm} \in \partial_{\infty} X$$

in the visual compactification $\overline{X}$ for some (any) point $x \in X$.

The following result is a Finsler version of [KLP1, Sublemma 6.2].

Lemma 6.1 (Dynamical relation with respect to regular sequences of isometries). *If $[b^{\bar{\theta}}_{\tau_{\pm},p_{\pm}}] \in \partial_{\infty}^{\bar{\theta}} X$ are Finsler boundary points such that*

$$[b^{\bar{\theta}}_{\tau_-,p_-}] \overset{(g_n)}{\sim} [b^{\bar{\theta}}_{\tau_+,p_+}]$$

with respect to the action of (g_n) on $\overline{X}^{\bar{\theta}}$, then

$$\mathrm{slope}(b^{\bar{\theta}}_{\tau_-,p_-}, \lambda_-) + \mathrm{slope}(b^{\bar{\theta}}_{\tau_+,p_+}, \lambda_+) \leqslant 0 \tag{6.2}$$

Proof. We denote $b_{\pm} = b^{\bar{\theta}}_{\tau_{\pm},p_{\pm}}$. By assumption, there exists a sequence $x_n \to \infty$ in X such that $g_n^{-1} x_n \to [b_-]$ and $x_n \to [b_+]$ in $\overline{X}^{\bar{\theta}}$, i.e.

$$d^{\bar{\theta}}_{g_n^{-1} x_n} \to b_- \quad \text{and} \quad d^{\bar{\theta}}_{x_n} \to b_+$$

uniformly on compacta modulo additive constants.

Fix a base point $x \in X$ and let $x_n^{\pm}(t)$ be the point at distance t from x on the segment connecting x to $g_n^{\pm 1} x$ (it is defined for sufficiently large n depending on t), and let $x^{\pm}(t)$ be the point at distance t from x on the ray $x\lambda_{\pm}$.

We consider the behavior of the convex functions $d^{\bar{\theta}}_{x_n} = d^{\bar{\theta}}(x_n, \cdot)$ along the subsegments connecting $x_n^+(t)$ to $g_n x_n^-(t)$; more precisely, we use the monotonicity of discretized slopes. We have

$$d^{\bar{\theta}}_{x_n}(x_n^+(t+1)) - d^{\bar{\theta}}_{x_n}(x_n^+(t)) \leqslant d^{\bar{\theta}}_{x_n}(g_n x_n^-(t)) - d^{\bar{\theta}}_{x_n}(g_n x_n^-(t+1)) = d^{\bar{\theta}}_{g_n^{-1} x_n}(x_n^-(t)) - d^{\bar{\theta}}_{g_n^{-1} x_n}(x_n^-(t+1))$$

for sufficiently large n depending on t. Letting $n \to +\infty$, we obtain

$$b_+(x^+(t+1)) - b_+(x^+(t)) \leqslant b_-(x^-(t)) - b_-(x^-(t+1)),$$

using that the functions $d^{\bar\theta}_{x_n}$ are uniformly Lipschitz (e.g. w.r.t. d^{Riem}), and letting $t \to +\infty$, we get

$$\mathrm{slope}(b_+, \lambda_+) \leqslant -\,\mathrm{slope}(b_-, \lambda_-),$$

as claimed. ☐

Corollary 6.3. *Under the assumptions of the lemma, at least one of the inequalities*

$$\mathrm{slope}(b^{\bar\theta}_{\tau_-,p_-}, \lambda_-) \leqslant 0 \quad \textit{and} \quad \mathrm{slope}(b^{\bar\theta}_{\tau_+,p_+}, \lambda_+) \leqslant 0$$

holds.

Remark 6.4. (i) The condition $\mathrm{slope}(b^{\bar\theta}_{\tau,p}, \lambda) \leqslant 0$ is equivalent to $\lambda \in \partial_\infty Hb_{b^{\bar\theta}_{\tau,p}} \subset \partial_\infty X$.

(ii) Since sequential convergence at infinity is independent of the regular type $\bar\theta \in \mathrm{int}(\sigma_{mod})$, the proof of the lemma implies that inequality (6.2) holds *simultaneously for all types* $\bar\theta \in \sigma_{mod}$.

We also investigate the dynamical relations between points in X and ideal points in $\partial^{\bar\theta}_\infty X$.

Addendum 6.5. *If $x_- \in X$ is dynamically related to $[b_+] \in \partial^{\bar\theta}_\infty X$ with respect to the action of (g_n) on $\overline{X}^{\bar\theta}$, then* $\mathrm{slope}(b_+, \lambda_+) \leqslant 0$.

Proof. The same argument as before yields that

$$\mathrm{slope}(d^{\bar\theta}_{x_-}, \lambda_-) + \mathrm{slope}(b_+, \lambda_+) \leqslant 0.$$

The distance function $d^{\bar\theta}_{x_-}$ is proper because the type $\bar\theta$ is regular, and hence asymptotically increasing along rays by convexity, i.e. $\mathrm{slope}(d^{\bar\theta}_{x_-}, \cdot) \geqslant 0$. The assertion follows. ☐

Remark 6.6. More precisely, one obtains that $\mathrm{slope}(d^{\bar\theta}_{x_-}, \lambda_-) = \cos\angle_{Tits}(\iota\bar\theta, \theta(\lambda_-))$ and therefore

$$\mathrm{slope}(b_+, \lambda_+) \leqslant -\cos\angle_{Tits}(\iota\bar\theta, \theta(\lambda_-)) < 0.$$

6.2 Almost root types

We let $\bar\rho \in \sigma_{mod}$ be a root type. For instance, we can take for $\bar\rho$ the direction of the coroot corresponding to the highest root $\tilde\alpha$ of the root system R. However, for instance, in the case of irreducible root systems which are not simply-laced, we have two choices of root types in σ_{mod}. The important property of the root type $\bar\rho$ is that the closed ball $\overline{B}(\bar\rho, \frac{\pi}{2})$ in the model spherical apartment a_{mod} of W is a subcomplex. In particular, its boundary sphere $S(\bar\rho, \frac{\pi}{2})$ contains no regular points. The idea is to replace $\bar\rho$ with a nearby regular type $\bar\theta$ so that the metric sphere $S(\bar\theta, \frac{\pi}{2})$ contains only *nearly singular* points.

Definition 6.7. Let $\Theta \subset \text{int}(\sigma_{mod})$ be a compact convex subset. We say that $\bar{\theta} \in \sigma_{mod}$ is a $(\bar{\rho}, \Theta)$-*almost root type* if

$$\angle(\bar{\theta}, \bar{\rho}) < d(\Theta, \partial\sigma_{mod}). \tag{6.8}$$

Let from now on $\bar{\theta} \in \text{int}(\sigma_{mod})$ denote a $(\bar{\rho}, \Theta)$-almost root type.

Lemma 6.9. *Let $\xi, \lambda \in \partial_\infty X$ be ideal points with types $\theta(\xi) = \bar{\theta}$ and $\theta(\lambda) \in \Theta$. Then*

$$\angle_{Tits}(\xi, \lambda) \neq \frac{\pi}{2}.$$

Proof. Suppose that $\angle_{Tits}(\xi, \lambda) = \frac{\pi}{2}$. Let $\rho \in \partial_\infty X$ be a point of type $\bar{\rho}$ in a common chamber with ξ. Then $\angle(\xi, \rho) = \angle(\bar{\theta}, \bar{\rho})$ and $\frac{\pi}{2} - \angle(\bar{\theta}, \bar{\rho}) \leqslant \angle_{Tits}(\rho, \lambda) \leqslant \frac{\pi}{2} + \angle(\bar{\theta}, \bar{\rho})$. Hence there exists a point $\eta \in \partial_\infty X$ with $\angle(\rho, \eta) = \frac{\pi}{2}$ at distance $\angle(\eta, \lambda) \leqslant \angle(\bar{\theta}, \bar{\rho})$. It follows that η is regular, a contradiction. ☐

The asymptotic slopes of mixed Busemann functions of almost root type, as they occur as functions representing Finsler boundary points at infinity, vanish only at nearly singular visual boundary points:

Lemma 6.10. *For the mixed Busemann functions $b^{\bar{\theta}}_{\tau,p}$ it holds that*

$$\text{slope}(b^{\bar{\theta}}_{\tau,p}, \cdot) \neq 0$$

on $\theta^{-1}(\Theta)$.

Proof. As a consequence of (2.3) and (2.2) (or (3.13)), for every visual boundary point $\xi \in \partial_\infty X$ exists a chamber $\sigma(\xi)$ such that

$$\text{slope}(b^{\bar{\theta}}_{\tau,p}, \xi) = -\cos\angle_{Tits}(\theta_{\sigma(\xi)}, \xi).$$

The assertion therefore follows from the previous lemma. ☐

Notation 6.11. *For the rest of this chapter, we let $\bar{\theta}$ be an almost root type.*

6.3 Limit sets of uniformly regular subgroups

For each subgroup $\Gamma < G$, the limit set $\Lambda(\Gamma) \subset \partial_\infty X$ of Γ is the accumulation set in $\partial_\infty X$ of one (equivalently, any) orbit $\Gamma \cdot x \subset X$. A subgroup $\Gamma < G$ is *uniformly regular* (see [KLP1]) if it is discrete and $\Lambda(\Gamma)$ consists only of regular points:

$$\Lambda(\Gamma) \subset \partial^{reg}_\infty X.$$

The *chamber limit set* $\Lambda_{ch}(\Gamma)$ consists of those chambers in $\partial_{Tits} X$ which have nonempty intersection with the limit set of Γ. In other words, $\Lambda_{ch}(\Gamma)$ is the image of $\Lambda(\Gamma)$ under the canonical projection $\partial^{reg}_\infty X \to \partial_{F\ddot{u}} X$. It is clear that $\Lambda(\Gamma)$ and, hence, $\Lambda_{ch}(\Gamma)$ are compact and Γ-invariant. The set $\theta(\Lambda(\Gamma))$ of types of limit points is a compact subset of $\text{int}(\sigma_{mod})$.

For a compact convex ι-invariant subset $\Theta \subset \text{int}(\sigma_{mod})$, a subgroup $\Gamma < G$ is Θ-*regular* if

$$\theta(\Lambda(\Gamma)) \subset \Theta.$$

6.4 Dynamical relation on almost root type Finsler compactifications

In what follows, we will assume that the discrete subgroup $\Gamma < G$ is Θ-regular and that $\bar\theta \in \mathrm{int}(\sigma_{mod})$ is a $(\bar\rho, \Theta)$-almost root type.

We apply our general observation about dynamical relations with respect to divergent sequences of isometries, see section 6.1, to the Γ-action:

Proposition 6.12 (Dynamical relation). *If $[b_\pm] \in \partial^{\bar\theta}_\infty X$ are Finsler boundary points such that*

$$[b_-] \stackrel{\Gamma}{\sim} [b_+]$$

with respect to the Γ-action on $\overline{X}^{\bar\theta}$, then there exist limit chambers $\sigma_\pm \in \Lambda_{ch}(\Gamma)$ such that at least one of the inequalities

$$\mathrm{slope}(b_-, \cdot)|_{\sigma_- \cap \theta^{-1}(\Theta)} < 0 \textit{ and } \mathrm{slope}(b_+, \cdot)|_{\sigma_+ \cap \theta^{-1}(\Theta)} > 0$$

holds.

Proof. By assumption, there exists a sequence $\gamma_n \to \infty$ in Γ such that $[b_-]$ is dynamically related to $[b_+]$ with respect to the action of (γ_n) on $\overline{X}^{\bar\theta}$. After passing to a subsequence, we have convergence

$$\gamma_n^{\pm 1} x \to \lambda_\pm \in \partial^{reg}_\infty X$$

because Γ is uniformly regular. Our assumption implies that now $\theta(\lambda_\pm) \in \Theta$.

Let $\sigma_\pm \in \Lambda_{ch}(\Gamma)$ denote the limit chambers containing the limit points $\lambda_\pm$. Corollary 6.3 yields that at least one of the inequalities $\mathrm{slope}(b_-, \lambda_-) \leqslant 0$ and $\mathrm{slope}(b_+, \lambda_+) \leqslant 0$ holds. Suppose that the former holds: $\mathrm{slope}(b_-, \lambda_-) \leqslant 0$.

According to Lemma 6.10, $\mathrm{slope}(b_-, \cdot) \neq 0$ on $\theta^{-1}(\Theta) \supset \Lambda(\Gamma)$. Hence, the strict inequality $\mathrm{slope}(b_-, \lambda_-) < 0$ holds, and moreover, since the convex set Θ is connected, that

$$\mathrm{slope}(b_-, \cdot) < 0$$

on $\sigma_- \cap \theta^{-1}(\Theta)$. □

Remark 6.13. The condition $\mathrm{slope}(b, \cdot)|_{\sigma \cap \theta^{-1}(\Theta)} \leqslant 0$ is equivalent to

$$\sigma \cap \theta^{-1}(\Theta) \subset \partial_\infty H b_b$$

in $\partial_\infty X$.

As before, we also obtain:

Addendum 6.14. *If $x_- \in X$ is dynamically related to $[b_+] \in \partial^{\bar\theta}_\infty X$ with respect to the Γ-action on $\overline{X}^{\bar\theta}$, then* $\mathrm{slope}(b_+, \cdot)|_{\sigma_+ \cap \theta^{-1}(\Theta)} < 0$.

Remark 6.15. The strict inequalities in Proposition 6.12 and Addendum 6.14 are equivalent to the non-strict inequalities, compare Lemma 6.10.

6.5 Proper discontinuity

We define the *thickening* of a chamber $\sigma \in \partial_{Fü} X$ in $\partial_\infty^{\bar\theta} X$ by

$$\mathrm{Th}_{\bar\theta}(\sigma) := \{[b] \in \partial_\infty^{\bar\theta} X \,|\, \underbrace{\mathrm{slope}(b,\cdot)|_{\sigma\cap\theta^{-1}(\Theta)} \leqslant 0}_{\Leftrightarrow\, \sigma\cap\theta^{-1}(\Theta)\subset\partial_\infty Hb_b}\} \tag{6.16}$$

and, correspondingly, the thickening of the chamber limit set

$$\Lambda_{\bar\theta}(\Gamma) = \mathrm{Th}_{\bar\theta}(\Lambda_{ch}(\Gamma)) := \bigcup_{\sigma\in\Lambda_{ch}(\Gamma)} \mathrm{Th}_{\bar\theta}(\sigma). \tag{6.17}$$

It is clearly Γ-invariant.

This construction of thickenings is analogous to the *root thickenings* defined in [KLP1], with the difference that now the thickening of the chamber limit set is defined via *almost root types* and takes place in the Finsler boundary instead of the visual boundary.

We will need:

Lemma 6.18. *Let (f_n) be a sequence of uniformly Lipschitz continuous convex functions on X which converge uniformly on compacta, $f_n \to f$, and let $\xi_n \to \xi$ be a convergent sequence in $\partial_\infty X$ such that*

$$\mathrm{slope}(f_n, \xi_n) \leqslant 0.$$

for all n. Then

$$\mathrm{slope}(f, \xi) \leqslant 0.$$

Proof. Fix a base point $o \in X$. The condition $\mathrm{slope}(f_n, \xi_n) \leqslant 0$ is equivalent to the property that $f_n \leqslant f_n(o)$ along the ray $o\xi_n$. Since the rays $o\xi_n$ Hausdorff converge to the ray $o\xi$, it follows that $f \leqslant f(o)$ along $o\xi$, i.e. $\mathrm{slope}(f, \xi) \leqslant 0$. ☐

Corollary 6.19. *$\Lambda_{\bar\theta}(\Gamma)$ is compact.*

Proof. This follows from the lemma and the compactness of $\Lambda(\Gamma)$. ☐

We now define a Γ-invariant open subset in $\partial_\infty^{\bar\theta} X$:

$$\Omega_{Th_{\bar\theta}}(\Gamma) := \partial_\infty^{\bar\theta} X - \mathrm{Th}_{\bar\theta}(\Lambda_{ch}(\Gamma)) \subset \partial_\infty^{\bar\theta} X \tag{6.20}$$

Note that it is saturated, i.e. a union of small strata.

We obtain

Theorem 6.21 (Domain of proper discontinuity). *Let $\Gamma < G$ be a Θ-regular discrete subgroup and suppose that $\bar\theta \in \mathrm{int}(\sigma_{mod})$ is a $(\bar\rho, \Theta)$-almost root type. Then the action*

$$\Gamma \curvearrowright X \cup \Omega_{Th_{\bar\theta}}(\Gamma)$$

is properly discontinuous.

Proof. According to Proposition 6.12 and Addendum 6.14, there are no dynamical relations between points outside $\mathrm{Th}_{\bar\theta}(\Lambda_{ch}(\Gamma))$. Therefore, the action is properly discontinuous, see [F] and [KL1]. ☐

Corollary 6.22. *The quotient*

$$\left(X \cup \Omega_{Th_{\bar\theta}}(\Gamma)\right)/\Gamma$$

is a bordification as an orbifold with corners of the orbifold X/Γ.

Proof. The space $X \cup \Omega_{Th_{\bar\theta}}(\Gamma)$ is an orbifold with corners according to Theorem 5.2, and the corner structure is preserved by Γ. Therefore, the quotient inherits the structure of an orbifold with corners. ☐

7 Cocompactness: regular case

7.1 Finsler Dirichlet fundamental domains

In order to prove cocompactness of the Γ-action on the bordified symmetric space $X \cup \Omega_{Th_{\bar\theta}}(\Gamma)$ we use a rather classical idea: Constructing a compact Dirichlet fundamental domain for the action. The main novelty lies in the use of a Finsler distance and Finsler Busemann functions for the construction. Our proof parallels the one in [KLP1, §8.1.3], where we were proving cocompactness in domains of proper discontinuity in root type flag manifolds by constructing Dirichlet fundamental domains in those flag manifolds.

Pick a point $o \in X$ not fixed by any nontrivial element of Γ. We define the *$\bar\theta$-Dirichlet domains*

$$D := D_o^{\bar\theta} := \{x \mid d^{\bar\theta}(o,x) = \min d^{\bar\theta}(\cdot,x)|_{\Gamma o}\} \subset X$$

for the Γ-action on X. Clearly,

$$\Gamma \cdot D = X.$$

For each sequence $x_n \to \infty$ in D which converges at infinity, $x_n \to [b] \in \partial_\infty^{\bar\theta} X$, we obtain:

$$b(\gamma o) - b(o) = \lim_{n\to+\infty} (d_{x_n}(\gamma o) - d_{x_n}(o)) \geqslant 0.$$

Hence, $b_\xi(\gamma o) \geqslant b_\xi(o)$ and thus:

$$\partial_\infty^{\bar\theta} D \subseteq \{[b] \mid b(o) = \min b|_{\Gamma o}\} \subset \partial_\infty^{\bar\theta} X. \tag{7.1}$$

Notation 7.2. *As before, throughout this section we let $\bar\theta$ be an almost root type.*

7.2 Ideal boundaries of Dirichlet domains

Suppose now that D is the Dirichlet domain of Γ defined in the previous section.

In [KLP1] and [KLP2] we defined and analyzed the notion of *conical limit chambers* of regular subgroups of G: This notion is a higher rank generalization of the one of *conical limit* *point* for discrete subgroups of rank 1 Lie groups. We let $\Lambda^{con}_{ch}(\Gamma)$ denote the set of conical limit chambers in $\Lambda_{ch}(\Gamma)$. We will now prove that in the case when all limit chambers of Γ are conical, the Finsler ideal boundary of the Dirichlet domain D is disjoint from the thickened limit set; this is analogous to [KLP1, §8.1.3], where similar result was established in the context of the visual ideal boundary.

Lemma 7.3. *$\partial^{\bar\theta}_{\infty} D \cap Th_{\bar\theta}(\sigma) = \emptyset$ for all conical limit chambers $\sigma \in \Lambda^{con}_{ch}(\Gamma)$.*

Proof. Suppose that $[b] \in \mathrm{Th}_{\bar\theta}(\sigma) \subset \partial^{\bar\theta}_{\infty} X$. Then

$$\mathrm{slope}(b, \cdot) < 0$$

on $\sigma \cap \theta^{-1}(\Theta)$.

Since the limit chamber σ is conical, there exists a sequence $\gamma_n \to \infty$ in Γ such that the sequence $\gamma_n o \to \infty$ in X converges to some $\lambda \in \sigma \cap \Lambda(\Gamma) \subset \sigma \cap \theta^{-1}(\Theta)$ *conically* with respect to the Weyl chamber $V(o, \sigma)$, i.e.

$$d(\gamma_n o, V(o, \sigma)) \leqslant const.$$

Let $x_n \in V(o, \sigma)$ denote the nearest point projection of $\gamma_n o$ to $V(o, \sigma)$. Then x_n lies on a Riemannian geodesic ray $o\eta_n$ with $\eta_n \in \sigma \subset \partial_{\infty} X$, and

$$\eta_n \to \lambda.$$

Since $\mathrm{slope}(b, \cdot)$ is a continuous function on $\partial_{\infty} X$, we have that

$$\mathrm{slope}(b, \eta_n) \leqslant s < 0$$

for large n. It follows that

$$\lim_{n \to +\infty} b(x_n) = -\infty,$$

and, since $d(\gamma_n o, x_n) \leqslant const$, also

$$\lim_{n \to +\infty} b(\gamma_n o) = -\infty.$$

Thus,

$$\inf b|_{\Gamma o} = -\infty,$$

which implies that $[b] \notin \partial^{\bar\theta}_{\infty} D$, cf. (7.1). ☐

Corollary 7.4. *If $\Lambda_{ch}(\Gamma)$ is conical, i.e., $\Lambda_{ch}(\Gamma) = \Lambda^{con}_{ch}(\Gamma)$, then*

$$\overline{D}^{\bar\theta} \subset X \cup \Omega_{Th_{\bar\theta}}(\Gamma).$$

7.3 Cocompactness

We now use compactified Dirichlet domains $D = D_o^{\bar\theta}$ in order to prove cocompactness of discrete group actions on bordified symmetric spaces: The domains Ω below are Γ-invariant open subsets of the Finsler boundary $\partial_\infty^{\bar\theta} X$.

Lemma 7.5. *Suppose that $X \cup \Omega \subset \overline{X}^{\bar\theta}$ is a domain of proper discontinuity for the Γ-action and that $\partial_\infty^{\bar\theta} D_o^{\bar\theta} \subset \Omega$ for some base point $o \in X$. Then $\overline{D}_o^{\bar\theta}$ has nonempty intersection with each orbit of the action $\Gamma \curvearrowright X \cup \Omega$. In particular, the action $\Gamma \curvearrowright X \cup \Omega$ is cocompact.*

Proof. Let $[b] \in \Omega$, and let $x_n \to \infty$ be a sequence in X such that $x_n \to [b]$.

Suppose that the sequence hits infinitely many Dirichlet domains $\gamma D = D_{\gamma o}^{\bar\theta}$, i.e., $x_n \in D_{\gamma_n o}^{\bar\theta}$ with $\gamma_n \to \infty$ in Γ. Then

$$C := \overline{D}_o^{\bar\theta} \cup \{x_n : n \in \mathbb{N}\} \cup \{[b]\} \subset X \cup \Omega$$

is compact and it holds that

$$\gamma_n C \cap C \neq \emptyset$$

for all n, contradicting the proper discontinuity of the action $\Gamma \curvearrowright X \cup \Omega$.

It follows that the sequence (x_n) is contained in a finite union of Dirichlet domains. After passing to a subsequence, we may assume that it is contained in a single one, $x_n \in D_{\gamma o}^{\bar\theta}$ for some $\gamma \in \Gamma$ and all n. Then $[b] \in \partial_\infty^{\bar\theta} D_{\gamma o}^{\bar\theta}$, i.e.

$$[b] \in \Gamma \cdot \partial_\infty^{\bar\theta} D_o^{\bar\theta}.$$

This shows that also every Γ-orbit in Ω hits $\overline{D}_o^{\bar\theta}$. ☐

We now apply this lemma to the domain $X \cup \Omega_{Th_{\bar\theta}}(\Gamma)$.

Theorem 7.6 (Cocompactness). *Let $\Gamma < G$ be a Θ-regular discrete subgroup and suppose that $\bar\theta \in \mathrm{int}(\sigma_{mod})$ is a $(\bar\rho, \Theta)$-almost root type. Suppose in addition that $\Lambda_{ch}(\Gamma) \subset \partial_{F\ddot{u}} X$ is conical. Then the properly discontinuous action*

$$\Gamma \curvearrowright X \cup \Omega_{Th_{\bar\theta}}(\Gamma) \subset \overline{X}^{\bar\theta} \tag{7.7}$$

is cocompact.

Proof. That the action is properly discontinuous, we know from Theorem 6.21. According to Corollary 7.4, the compactified Dirichlet domains avoid the thickening of the chamber limit set,

$$\overline{D}_o^{\bar\theta} \subset X \cup \Omega_{Th_{\bar\theta}}(\Gamma).$$

Lemma 7.5 then yields the assertion. ☐

Corollary 7.8. *If $\Gamma < G$ is an RCA subgroup and $\bar{\theta} \in \mathrm{int}(\sigma_{mod})$ is an almost root type (as in the previous theorem), then the action*

$$\Gamma \curvearrowright X \cup \Omega_{Th_{\bar{\theta}}}(\Gamma)$$

is properly discontinuous and cocompact. The quotient

$$\left(X \cup \Omega_{Th_{\bar{\theta}}}(\Gamma)\right)/\Gamma$$

has a structure as a real-analytic orbifold with corners induced from that one of $\overline{X}^{\bar{\theta}}$, and

$$\left(\Omega_{Th_{\bar{\theta}}}(\Gamma)\right)/\Gamma$$

is the boundary of this orbifold.

Proof. It is proven in [KLP1] as well as in [KLP2] that each RCA subgroup $\Gamma < G$ is uniformly regular. The RCA property includes the *conicality* assumption, i.e., that each limit chamber of Γ is conical. Now, the statement follows from Theorem 7.6. Note that the real-analytic structure on the quotient comes from the G-equivariant homeomorphism between $\overline{X}^{\bar{\theta}}$ and the maximal Satake compactification of X: The latter is a real-analytic manifold with corners. The last the statement of the corollary is a special case of Corollary 6.22. ☐

8 Interlude

In order to extend our main results on proper discontinuity and cocompactness for discrete group actions from the regular to the weakly regular case, we need some additional background material, primarily concerning the asymptotic geometry of symmetric spaces.

8.1 Weak regularity

8.1.1 Basic definitions

We recall the concept of τ_{mod}*-regularity* from [KLP2, §2.4.2 and §5.1]:

The *open star*

$$\mathrm{ost}(\tau_{mod}) \subset \sigma_{mod}$$

is the union of all open faces of σ_{mod} whose closure contains τ_{mod}. Its complement

$$\partial\,\mathrm{st}(\tau_{mod}) := \sigma_{mod} - \mathrm{ost}(\tau_{mod})$$

is the union of all (closed) faces of σ_{mod} which do not contain τ_{mod}.

An ideal point $\xi \in \partial_\infty X$ is τ_{mod}*-regular* if $\theta(\xi) \in \mathrm{ost}(\tau_{mod})$, and τ_{mod}*-singular* if $\theta(\xi) \in \partial\,\mathrm{st}(\tau_{mod})$.

For a simplex $\tau \subset \partial_\infty X$, the *open star*

$$\mathrm{ost}(\tau) \subset \mathrm{st}(\tau)$$

is the union of all open simplices in $\mathrm{st}(\tau)$ whose closure contains τ, equivalently, the subset of τ_{mod}-regular points in $\mathrm{st}(\tau)$. Furthermore,

$$\partial\,\mathrm{st}(\tau) := \mathrm{st}(\tau) - \mathrm{ost}(\tau)$$

is the union of all (closed) simplices in $\mathrm{st}(\tau)$ which do not contain τ, equivalently, the subset of τ_{mod}-singular points in $\mathrm{st}(\tau)$.

A sequence (δ_n) in Δ is τ_{mod}*-regular* if

$$d(\delta_n, V(0, \partial\,\mathrm{st}(\tau_{mod}))) \to +\infty.$$

A sequence (x_n) in X is τ_{mod}*-regular* if for some, equivalently, any base point $o \in X$ the sequence of Δ-distances $d_\Delta(o, x_n)$ is τ_{mod}-regular.

Remark 8.1. A sequence (x_n) is τ_{mod}-regular if and only if every subsequence has a τ_{mod}-regular subsequence. (Simply, because a sequence of positive numbers is unbounded if and only if every subsequence has an unbounded subsequence.)

We call a sequence (g_n) in G τ_{mod}*-regular* if some (any) orbit $(g_n x)$ in X has this property.

We call a sequence (δ_n) in Δ *uniformly* τ_{mod}*-regular*, if it diverges from $V(0, \partial\,\mathrm{st}(\tau_{mod}))$ at a linear rate, i.e.

$$\liminf_{n\to+\infty} d(\delta_n, V(0, \partial\,\mathrm{st}(\tau_{mod})))/\|\delta_n\| > 0,$$

equivalently if it accumulates $\overline{\Delta}$ at a compact subset of $\mathrm{ost}(\tau_{mod})$.

A sequence (x_n) in X is *uniformly* τ_{mod}*-regular* if for some (any) base point $o \in X$ the sequence of Δ-distances $d_\Delta(o, x_n)$ is uniformly τ_{mod}-regular, equivalently, if (x_n) accumulates at a compact subset of $\theta^{-1}(\mathrm{ost}(\tau_{mod}))$. Lastly, we call a sequence (g_n) in G *uniformly* τ_{mod}*-regular* if some (any) orbit $(g_n x)$ in X has this property.

A subgroup $\Gamma < G$ is *(uniformly)* τ_{mod}*-regular* if every sequence of distinct elements in Γ has this property.

8.1.2 Relation of weakly regular convergence and stratification at infinity

We consider sequences $\delta_n \to \infty$ in Δ and their accumulation sets at infinity.

Lemma 8.2. *A sequence $\delta_n \to \infty$ in Δ accumulates in $\overline{\Delta}^{\bar\theta}$ at a compact subset of $S^{\Delta}_{\tau_{mod}}$ if and only if it is contained in a tubular neighborhood of $V(0, \tau_{mod})$ and $d(\delta_n, V(0, \partial\tau_{mod})) \to +\infty$.*

Proof. If (δ_n) has the property that it is contained in a tubular neighborhood of $V(0, \tau_{mod})$ and $d(\delta_n, V(0, \partial\tau_{mod})) \to +\infty$, then Lemma 3.19 implies that it accumulates at a compact subset of $S^{\Delta}_{\tau_{mod}}$. If (δ_n) does not have this property, then, after passing to a subsequence, it has this property for a different face type $\tau'_{mod} \neq \tau_{mod}$, and it follows that (δ_n) has accumulation points in the different stratum $S^{\Delta}_{\tau'_{mod}}$. ☐

Lemma 8.3. *A sequence $\delta_n \to \infty$ in Δ accumulates at $\overline{S}^{\Delta}_{\tau_{mod}}$ if and only if it is τ_{mod}-regular, i.e. $d(\delta_n, V(0, \partial\,\mathrm{st}(\tau_{mod}))) \to +\infty$.*

Proof. Suppose that the sequence (δ_n) is not τ_{mod}-regular. Then a subsequence is contained in a tubular neighborhood of $V(0, \partial\,\mathrm{st}(\tau_{mod}))$. Hence there exists a smallest face $\nu_{mod} \not\supseteq \tau_{mod}$ with the property that a subsequence (δ_{n_k}) is contained in a tubular neighborhood of $V(0, \nu_{mod})$. By the previous lemma, (δ_{n_k}) accumulates at a compact subset of $S^{\Delta}_{\nu_{mod}} \not\subset \overline{S}^{\Delta}_{\tau_{mod}}$.

Vice versa, suppose that (δ_n) is τ_{mod}-regular and has an accumulation point in $S^{\Delta}_{\nu_{mod}}$. By the previous lemma, a subsequence of (δ_n) is contained in a tubular neighborhood of $V(0, \nu_{mod})$. If $\nu_{mod} \not\supseteq \tau_{mod}$, then $\nu_{mod} \subset \partial\,\mathrm{st}(\tau_{mod})$ and we run into a contradiction with τ_{mod}-regularity. Therefore $\nu_{mod} \supseteq \tau_{mod}$ and consequently $S^{\Delta}_{\nu_{mod}} \subset \overline{S}^{\Delta}_{\tau_{mod}}$. ☐

We now state corresponding facts for sequences $x_n \to \infty$ in X.

Fix a base point $o \in X$ and a face type τ_{mod}. There exist simplices $\tau_n \subset \partial_\infty X$ of type τ_{mod} such that $x_n \in V(o, \mathrm{st}(\tau_n))$.

Lemma 8.4. *The sequence (x_n) accumulates at $\overline{X}_\tau$ if and only if it is τ_{mod}-regular and $\tau_n \to \tau$.*

Proof. The sequence (x_n) accumulates at $\overline{X}_\tau$ if and only if $\tau_n \to \tau$ and the sequence of Δ-lengths $d_\Delta(o, x_n)$ accumulates in $\overline{\Delta}^{\bar\theta}$ at $\overline{S}^{\Delta}_{\tau_{mod}}$. By the previous lemma, the second condition is equivalent to the τ_{mod}-regularity of the sequence (x_n). ☐

Corollary 8.5. *The sequence (x_n) accumulates at $\overline{S}_{\tau_{mod}}$ if and only if it is τ_{mod}-regular.*

Proof. Suppose that (x_n) accumulates at $\overline{S}_{\tau_{mod}}$. In view of Remark 8.1, we may assume that (x_n) accumulates at a small stratum closure $\overline{X}_\tau$ with a simplex τ of type τ_{mod}. The previous lemma shows that (x_n) is τ_{mod}-regular.

Conversely, suppose that (x_n) is τ_{mod}-regular and converges at infinity. After passing to a subsequence, we may assume that $x_n \in V(o, \mathrm{st}(\tau_n))$ with a convergent sequence $\tau_n \to \tau$ of simplices of type τ_{mod}. Again by the previous lemma, it follows that the limit point belongs to $\overline{X}_\tau \subset \overline{S}_{\tau_{mod}}$. ☐

8.2 Relative position at infinity and folding order

We recall some concepts from [KLP1] and refer to the discussion there for more details.

Let $\sigma_0, \sigma \subset \partial_\infty X$ be chambers. We view them also as points $\sigma_0, \sigma \in \partial_{Fü} X$. There exists an (in general non-unique) apartment $a \subset \partial_\infty X$ containing these chambers, $\sigma_0, \sigma \subset a$, and a unique apartment chart $\alpha : a_{mod} \to a$ such that $\sigma_0 = \alpha(\sigma_{mod})$. We define the *position of* σ *relative to* σ_0 as the chamber

$$\operatorname{pos}(\sigma, \sigma_0) := \alpha^{-1}(\sigma) \subset a_{mod}.$$

Abusing notation, it can be regarded algebraically as the unique element

$$\operatorname{pos}(\sigma, \sigma_0) \in W$$

such that

$$\sigma = \alpha\big(\operatorname{pos}(\sigma, \sigma_0)\sigma_{mod}\big),$$

cf. [KLP1, Def 4.8]. It does not depend on the choice of the apartment a. To see this, choose regular points $\xi_0 \in \operatorname{int}(\sigma_0)$ and $\xi \in \operatorname{int}(\sigma)$ which are not antipodal, $\angle_{Tits}(\xi, \xi_0) < \pi$. Then the segment $\xi_0\xi$ is contained in a by convexity, and its image $\alpha^{-1}(\xi_0\xi)$ in a_{mod} is independent of the chart α because its initial portion $\alpha^{-1}(\xi_0\xi \cap \sigma_0)$ in σ_{mod} is.

The level sets of $\operatorname{pos}(\cdot, \sigma_0)$ in $\partial_{Fü} X$ are the *Schubert cells* relative σ_0, i.e. the orbits of the Borel subgroup $B_{\sigma_0} \subset G$ fixing σ_0.

More generally, we define the position relative σ_0 of an arbitrary simplex $\tau \subset \partial_\infty X$ as follows. Let again $a \subset \partial_\infty X$ be an apartment containing σ_0 and τ, and let $\alpha : a_{mod} \to a$ be a chart such that $\sigma_0 = \alpha(\sigma_{mod})$. We define the *position of* τ *relative to* σ_0 as the simplex

$$\operatorname{pos}(\tau, \sigma_0) := \alpha^{-1}(\tau) \subset a_{mod}.$$

It can be interpreted algebraically as a coset in $W/W_{\tau_{mod}}$ where $\tau_{mod} = \theta(\tau)$.

Even more generally, we define the position of a simplex $\nu \subset \partial_\infty X$ relative to a simplex $\tau_0 \subset \partial_\infty X$ of type τ_{mod}. Let $a \subset \partial_\infty X$ be an apartment containing τ_0 and ν, and let $\alpha : a_{mod} \to a$ be a chart such that $\tau_0 = \alpha(\tau_{mod})$. We define the *position of* ν *relative to* τ_0 as the $W_{\tau_{mod}}$-orbit of the simplex $\alpha^{-1}(\nu) \subset a_{mod}$. It can be interpreted algebraically as a double coset

$$\operatorname{pos}(\nu, \tau_0) \in W_{\tau_{mod}} \backslash W / W_{\nu_{mod}}$$

where $\nu_{mod} = \theta(\nu)$. In particular, for chambers σ we have that

$$\operatorname{pos}(\sigma, \tau_0) \in W_{\tau_{mod}} \backslash W.$$

The *Bruhat order* "$\prec$" on the Weyl group W has the following geometric interpretation as *folding order*, cf. [KLP1, §4.2]. For distinct elements $w_1, w_2 \in W$, it holds that

$$w_1 \prec w_2$$

if and only if there exists a folding map $a_{mod} \to a_{mod}$ fixing σ_{mod} and mapping $w_2\sigma_{mod} \mapsto w_1\sigma_{mod}$, cf. [KLP1, Def 4.3]. Here, by a *folding map* $a_{mod} \to a_{mod}$ we mean a type preserving continuous map which sends chambers isometrically onto chambers.

The folding order on relative positions coincides with the inclusion order on Schubert cycles, i.e. $w_1 \leq w_2$ if and only if the Schubert cell $\{\text{pos}(\cdot, \sigma_0) = w_1\}$ is contained in the closure of the Schubert cell $\{\text{pos}(\cdot, \sigma_0) = w_2\}$, and the *Schubert cycles* relative σ_0 are the sublevel sets of $\text{pos}(\cdot, \sigma_0)$, cf. [KLP1, Prop 4.9].

More generally, we have a folding order on the simplices of type τ_{mod} in a_{mod}: For distinct simplices $\bar\tau_1, \bar\tau_2 \subset a_{mod}$, it holds that

$$\bar\tau_1 \prec \bar\tau_2$$

iff there exists a folding map $a_{mod} \to a_{mod}$ fixing σ_{mod} and mapping $\bar\tau_2 \mapsto \bar\tau_1$, cf. [KLP1, §4.3]. Again, the folding order coincides with the inclusion order on Schubert cycles. Note that the Schubert cycles in $\text{Flag}_{\tau_{\text{mod}}}(\text{X})$ are projective subvarieties; in particular, they admit finite triangulations.

We also need to define the folding order on positions of chambers relative to simplices τ_0 of an arbitrary face type τ_{mod}. We say that

$$W_{\tau_{mod}}\bar\sigma_1 \leq_{\tau_{mod}} W_{\tau_{mod}}\bar\sigma_2$$

for chambers $\bar\sigma_1, \bar\sigma_2 \subset a_{mod}$ iff there exist $\bar\sigma_i' \in W_{\tau_{mod}}\bar\sigma_i$ such that

$$\bar\sigma_1' \leq \bar\sigma_2',$$

equivalently, geometrically, if for some (any) chambers $\bar\sigma_i' \in W_{\tau_{mod}}\bar\sigma_i$ there exists a folding map $a_{mod} \to a_{mod}$ fixing τ_{mod} and mapping $\bar\sigma_2'$ to $\bar\sigma_1'$. (Note that the elements in W_{mod} are such folding maps.)

Lemma 8.6. *$\prec_{\tau_{mod}}$ is a partial order.*

Proof. Transitivity holds, because the composition of folding maps is again a folding map.

To verify reflexivity, pick points $\xi_{mod} \in \text{int}(\tau_{mod})$ and $\eta_{mod} \in \text{int}(\sigma_{mod})$.

Let $\bar\sigma = w\sigma_{mod} \subset a_{mod}$ be a chamber and $f : a_{mod} \to a_{mod}$ a folding map fixing τ_{mod}. We denote $\bar\eta = w\eta_{mod}$. If the f-image of the segment $\xi_{mod}\bar\eta$ is again an unbroken geodesic segment, then the two geodesic segments are congruent by an element of $W_{\tau_{mod}}$, because their initial directions at ξ_{mod} are. On the other hand, if the f-image of $\xi_{mod}\bar\eta$ is a broken geodesic segment, then the distance of its endpoints is strictly smaller than its length, and consequently $f\bar\sigma \not\geq \bar\sigma$. This shows that

$$W_{\tau_{mod}}\bar\sigma_1 \leq_{\tau_{mod}} W_{\tau_{mod}}\bar\sigma_2 \leq_{\tau_{mod}} W_{\tau_{mod}}\bar\sigma_1 \quad \Rightarrow \quad W_{\tau_{mod}}\bar\sigma_1 = W_{\tau_{mod}}\bar\sigma_2$$

and hence reflexivity. ☐

We will use the following notation.

For a simplex $\tau \subset \partial_\infty X$ of type $\theta(\tau) = \hat{\tau}_{mod} := \iota\tau_{mod}$, we denote by

$$C_{\tau_{mod}}(\tau) := \{\hat{\tau} : \hat{\tau} \text{ opposite to } \tau\big\} \subset \mathrm{Flag}_{\tau_{\mathrm{mod}}}(\mathrm{X})$$

the open Schubert cell associated with τ in $\mathrm{Flag}_{\tau_{\mathrm{mod}}}(\mathrm{X})$, and by

$$C_{F\ddot{u}}(\tau) := \bigcup\big\{\mathrm{st}(\hat{\tau}) : \hat{\tau} \text{ opposite to } \tau\big\} \subset \partial_{F\ddot{u}}X$$

the set of chambers which have a face opposite to τ, equivalently, the set of chambers which are opposite to a chamber in $\mathrm{st}(\tau)$. It equals the union of open Schubert cells $C_{F\ddot{u}}(\sigma)$ over all chambers $\sigma \subset \mathrm{st}(\tau)$. Here, and later, we will abuse notation and regard stars as sets of chambers. Note that $C_{F\ddot{u}}(\tau)$ is the preimage of $C_{\tau_{mod}}(\tau)$ under the natural fibration $\partial_{F\ddot{u}}X \to \mathrm{Flag}_{\tau_{\mathrm{mod}}}(\mathrm{X})$.

The following result will be useful to compare relative positions.

Lemma 8.7. *(i) Let $\sigma_0, \sigma_1, \sigma_2 \subset \partial_\infty X$ be chambers, and suppose that there exists a segment $\xi_0\xi_2$ with $\xi_0 \in \mathrm{int}(\sigma_0)$ and $\xi_2 \in \mathrm{int}(\sigma_2)$ containing a point $\xi_1 \in \mathrm{int}(\sigma_1)$. Then*

$$pos(\sigma_1, \sigma_0) \preceq pos(\sigma_2, \sigma_0)$$

with equality iff $\sigma_1 = \sigma_2$.

(ii) More generally, let $\sigma_1, \sigma_2 \subset \partial_\infty X$ be chambers and let $\tau_0 \subset \partial_\infty X$ be a simplex of type τ_{mod}. Suppose that there exists a segment $\xi_0\xi_2$ with $\xi_0 \in \mathrm{int}(\tau_0)$ and $\xi_2 \in \mathrm{int}(\sigma_2)$ containing a point $\xi_1 \in \mathrm{int}(\sigma_1)$. Then

$$pos(\sigma_1, \tau_0) \preceq_{\tau_{mod}} pos(\sigma_2, \tau_0)$$

with equality iff $\sigma_1 = \sigma_2$.

Proof. We prove the more general assertion (ii). After perturbing ξ_2, we can arrange that the subsegment $\xi_1\xi_2$ avoids codimension two faces. Along this subsegment we find a gallery of chambers connecting σ_1 to σ_2. We may therefore proceed by induction and assume that the chambers σ_1 and σ_2 are adjacent, i.e. share a panel π which is intersected transversally by $\xi_1\xi_2$. Working in an apartment containing $\tau_0, \sigma_1, \sigma_2$, the wall through π does not contain τ_0 and separates $\mathrm{st}(\tau_0) \cup \sigma_1$ from σ_2. Folding at this wall yields the desired inequality. ☐

8.3 Further properties of the folding order

This is a technical section whose results are used in the proof of Proposition 9.3 which is the key to proving proper discontinuity of actions of τ_{mod}-regular subgroups.

We work with the spherical building structure on the visual boundary $\partial_\infty X$. We fix a reference chamber $\sigma_0 \subset \partial_\infty X$.

Let $\tau \subset \partial_\infty X$ be a simplex. For any interior points $\eta \in \mathrm{int}(\tau)$ and $\xi_0 \in \mathrm{int}(\sigma_0)$, the segment $\eta\xi_0$ enters the *interior* of a chamber $\sigma_- \supset \tau$, i.e.

$$\eta\xi_0 \cap \mathrm{int}(\sigma_-) \neq \emptyset.$$

Note that the chamber σ_- does not depend on the interior points η, ξ_0. Moreover, it is contained in any apartment containing σ_0 and τ. We call σ_- the chamber in $\mathrm{st}(\tau)$ *pointing towards* σ_0.

Similarly, if $\xi_0\xi_+ \supsetneq \xi_0\eta$ is an extension of the segment $\xi_0\eta$ beyond η, then there exists a chamber $\sigma_+ \supset \tau$ such that $\eta\xi_+ \cap \mathrm{int}(\sigma_+) \neq \emptyset$, and we call σ_+ *a* chamber in $\mathrm{st}(\tau)$ *pointing away from* σ_0.

Let $a \subset \partial_\infty X$ be an apartment containing σ_0 and τ.

Then $\sigma_- \subset a$. Moreover, since geodesic segments inside a extend uniquely, there exists a *unique* chamber $\sigma_+ \subset \mathrm{st}(\tau) \cap a$ pointing away from σ_0. The chambers $\sigma_\pm \subset a$ can be characterized as follows in terms of separation from σ_0 by walls:

Lemma 8.8. *Let $\sigma \subset \mathrm{st}(\tau) \cap a$ be a chamber. Then*

(i) $\sigma = \sigma_+$ iff σ is separated from σ_0 by every wall $s \subset a$ containing τ.

(ii) $\sigma = \sigma_-$ iff σ is not separated from σ_0 by any wall $s \subset a$ containing τ.

Proof. (i) Clearly, σ_+ is separated from σ_0 by every wall $s \supset \tau$ because, using the above notation, $\xi_0\xi_+ \cap s = \eta$. Vice versa, if σ is separated from σ_0 by all such walls s, then σ and σ_+ lie in the same hemispheres bounded by the walls $s \supset \tau$ in a, and therefore must coincide.

(ii) Similarly, σ_- is not separated from σ_0 by any wall $s \supset \tau$ because $\xi_0\eta \cap s = \eta$, and vice versa, if σ is not separated from σ_0 by any wall $s \supset \tau$, then σ and σ_- lie in the same hemispheres bounded by the walls $s \supset \tau$ in a, and therefore must coincide. ☐

Remark 8.9. The assertion of the lemma remains valid if one only admits the walls $s \subset a$ such that $s \cap \sigma$ is a panel containing τ.

The chambers pointing towards and away from σ_0 in $\partial_\infty X$ can also be characterized in terms of the folding order:

Lemma 8.10. *The restriction of the function $pos(\sigma_0, \cdot)$ to the set of chambers contained in* $\mathrm{st}(\tau)$ *attains a unique global minimum in σ_- and global maxima precisely in the chambers pointing away from σ_0.*

Proof. Let $\sigma \supset \tau$ be a chamber and let $a \subset \partial_\infty X$ be an apartment containing σ_0 and σ. Then $\sigma_- \subset a$. Let $\sigma_+ \subset \mathrm{st}(\tau) \cap a$ be the unique chamber pointing away from σ_0.

Still using the above notation, let $\xi_0\xi_+ \supset \xi_0\eta$ be an extension of the segment $\xi_0\eta$ with endpoint $\xi_+ \in \mathrm{int}(\sigma_+)$. Let $\xi_- \in \xi_0\eta \cap \mathrm{int}(\sigma_-)$. The points ξ_- and η appear in this order on the (oriented) segment $\xi_0\xi_+$.

We now perturb the segment $\xi_0\xi_+$ to a segment $\xi_0\xi'_+$ which intersects $\mathrm{int}(\sigma)$ in a point η' close to η and $\mathrm{int}(\sigma_-)$ in a point ξ'_- close to ξ_-. The perturbation is possible because $\sigma \supset \tau$. Again, the points ξ'_- and η' appear in this order on the perturbed segment $\xi_0\xi'_+$. Lemma 8.7 implies that

$$\mathrm{pos}(\sigma_-, \sigma_0) \preceq \mathrm{pos}(\sigma, \sigma_0) \preceq \mathrm{pos}(\sigma_+, \sigma_0)$$

with equality in the first (second) inequality if and only if $\sigma = \sigma_-$ ($\sigma = \sigma_+$). The assertion of the lemma follows because $\mathrm{pos}(\sigma_+, \sigma_0)$ does not depend on the choice of a. ☐

Let now $\sigma \subset \partial_\infty X$ be an arbitrary chamber.

We say that a face $\tau \subset \sigma$ *faces towards* σ_0 if there exist points $\xi_0 \in \mathrm{int}(\sigma_0)$ and $\xi \in \mathrm{int}(\sigma)$ such that $\xi_0\xi \cap \mathrm{int}(\tau) \neq \emptyset$. Equivalently, σ is a chamber in $\mathrm{st}(\tau)$ pointing away from σ_0. Again equivalently, with an apartment $a \supset \sigma_0 \cup \sigma$, all walls $s \subset a$ through τ separate σ from σ_0, see Lemma 8.8. The last characterization remains valid, if one only admits the walls which intersect σ in a panel, cf. Remark 8.9.

The last characterization implies that σ has a unique smallest face

$$\mathrm{front}_{\sigma_0}(\sigma) \subseteq \sigma$$

facing towards σ_0, namely the intersection of all panels facing towards σ_0. Note that $\mathrm{front}_{\sigma_0}(\sigma) = \sigma$ iff $\sigma = \sigma_0$, and $\mathrm{front}_{\sigma_0}(\sigma) = \emptyset$ iff σ is antipodal to σ_0.

Similarly, we say that $\tau \subset \sigma$ *faces away from* σ_0 if there exist points $\xi_0 \in \mathrm{int}(\sigma_0)$ and $\eta \in \mathrm{int}(\tau)$ such that $\xi_0\eta \cap \mathrm{int}(\sigma) \neq \emptyset$. Equivalently, σ is the unique chamber in $\mathrm{st}(\tau)$ pointing towards σ_0, equivalently, no wall $s \subset a$ through τ separates σ from σ_0. Again, σ has a unique smallest face

$$\mathrm{back}_{\sigma_0}(\sigma) \subseteq \sigma$$

facing away from σ_0, namely the intersection of all panels facing away from σ_0. Moreover, $\mathrm{back}_{\sigma_0}(\sigma) = \emptyset$ iff $\sigma = \sigma_0$, and $\mathrm{back}_{\sigma_0}(\sigma) = \sigma$ iff σ is antipodal to σ_0.

The front and back faces of σ are complementary, i.e. each vertex of σ belongs to exactly one of them.

Let $\sigma \subset \partial_\infty X$ be a chamber and let $\tau \subset \sigma$ be a face.

Lemma 8.11. *The restriction of the function* $\mathrm{pos}(\sigma_0, \cdot)$ *to the set of chambers contained in* $\mathrm{st}(\tau)$ *attains a maximum in* σ *iff* $\mathrm{front}_{\sigma_0}(\sigma) \subseteq \tau$.

Proof. By definition, $\mathrm{front}_{\sigma_0}(\sigma) \subseteq \tau$ iff τ faces towards σ_0 iff σ is a chamber in $\mathrm{st}(\tau)$ pointing away from σ_0. By Lemma 8.10, the latter holds iff the restriction of $\mathrm{pos}(\sigma_0, \cdot)$ to the set of chambers contained in $\mathrm{st}(\tau)$ is maximal in σ. ☐

8.4 Thickenings

A *thickening* (of the neutral element) in W is a subset

$$\mathrm{Th} \subset W$$

which is a union of sublevels for the folding order, i.e. which contains with every element w also every element w' satisfying $w' \prec w$, cf. [KLP1, Def 4.16]. In the theory of posets, such subsets are called *ideals.*

Note that

$$\mathrm{Th}^c := w_0(W - \mathrm{Th})$$

is again a thickening. Here, $w_0 \in W$ denotes the longest element of the Weyl group, that is, the element of order two mapping σ_{mod} to the opposite chamber in a_{mod}. It holds that

$$W = \mathrm{Th} \sqcup w_0 \mathrm{Th}^c$$

and we call Th^c the thickening *complementary* to Th.

The thickening $\mathrm{Th} \subset W$ is called *fat* if $\mathrm{Th} \cup w_0 \mathrm{Th} = W$, equivalently, $\mathrm{Th} \supseteq \mathrm{Th}^c$. It is called *slim* if $\mathrm{Th} \cap w_0 \mathrm{Th} = \emptyset$, equivalently, $\mathrm{Th} \subseteq \mathrm{Th}^c$. It is called *balanced* if it is both fat and slim, equivalently, $\mathrm{Th} = \mathrm{Th}^c$, cf. [KLP1, Def 4.17].

For types $\tilde{\vartheta}_0, \tilde{\vartheta} \in \sigma_{mod}$ and a radius $r \in [0, \pi]$ we define the *metric thickening*

$$\mathrm{Th}_{\tilde{\vartheta}_0, \tilde{\vartheta}, r} := \{w \in W : d(w\tilde{\vartheta}, \tilde{\vartheta}_0) \leqslant r\},$$

using the natural W-invariant spherical metric d on a_{mod}, cf. [KLP1, §4.4].

For a face type $\tau_{mod} \subseteq \sigma_{mod}$, we denote by $W_{\tau_{mod}}$ its stabilizer in W. Furthermore, $\iota = -w_0 : \sigma_{mod} \to \sigma_{mod}$ denotes the canonical involution of the model spherical Weyl chamber.

Lemma 8.12. *(i) If $\tilde{\vartheta}_0 \in \tau_{mod}$, then $W_{\tau_{mod}} Th_{\tilde{\vartheta}_0, \tilde{\vartheta}, r} = Th_{\tilde{\vartheta}_0, \tilde{\vartheta}, r}$.*

(ii) If $\iota\tilde{\vartheta}_0 = \tilde{\vartheta}_0$, then $Th_{\tilde{\vartheta}_0, \tilde{\vartheta}, r}$ is fat for $r \geqslant \frac{\pi}{2}$ and slim for $r < \frac{\pi}{2}$.

Proof. (i) For $w' \in W_{\tau_{mod}}$, we have that $w'\tilde{\vartheta}_0 = \tilde{\vartheta}_0$ and hence

$$d(w'w\tilde{\vartheta}, \tilde{\vartheta}_0) = d(w\tilde{\vartheta}, \underbrace{w'^{-1}\tilde{\vartheta}_0}_{\tilde{\vartheta}_0}).$$

(ii) Since $w_0\tilde{\vartheta}_0 = -\iota\tilde{\vartheta}_0 = -\tilde{\vartheta}_0$, we have

$$d(w_0 w\tilde{\vartheta}, -\tilde{\vartheta}_0) = d(w\tilde{\vartheta}, \underbrace{-w_0\tilde{\vartheta}_0}_{\tilde{\vartheta}_0}),$$

whence the assertion. ∎

Corollary 8.13 (Existence of balanced thickenings). *If the face type τ_{mod} is ι-invariant, $\iota\tau_{mod} = \tau_{mod}$, then there exists a $W_{\tau_{mod}}$-invariant balanced thickening $Th \subset W$.*

Proof. Since $\iota\tau_{mod} = \tau_{mod}$, there exists $\tilde{\vartheta}_0 \in \tau_{mod}$ such that $\iota\tilde{\vartheta}_0 = \tilde{\vartheta}_0$. Moreover, there exists $\tilde{\vartheta} \in \sigma_{mod}$ such that $d(\cdot\tilde{\vartheta}, \tilde{\vartheta}_0) \neq \frac{\pi}{2}$ on W. (This holds for an open dense subset of types $\tilde{\vartheta} \in \sigma_{mod}$.) According to the lemma, the metric thickening $\mathrm{Th}_{\tilde{\vartheta}_0, \tilde{\vartheta}, \frac{\pi}{2}}$ is balanced and $W_{\tau_{mod}}$-invariant. ∎

Given a thickening $\mathrm{Th} \subset W$, we obtain *thickenings at infinity* as follows.

First, we define the thickening in $\partial_{F\ddot{u}} X$ of a chamber $\sigma \in \partial_{F\ddot{u}} X$ as

$$\mathrm{Th}_{F\ddot{u}}(\sigma) := \{\mathrm{pos}(\cdot, \sigma) \in \mathrm{Th}\} \subset \partial_{F\ddot{u}} X.$$

It is a finite union of Schubert cycles relative σ. We then define the thickening of σ inside the Finsler ideal boundary as the "suspension" of its thickening inside the Fürstenberg boundary,

$$\mathrm{Th}^{\bar\theta}(\sigma) := \{[b^{\bar\theta}_{\nu,p}] : \mathrm{st}(\nu) \subset \mathrm{Th}_{F\ddot u}(\sigma)\} = \bigcup\{S_\nu : \mathrm{st}(\nu) \subset \mathrm{Th}_{F\ddot u}(\sigma)\} \subset \partial^{\bar\theta}_\infty X$$

where we view $\mathrm{st}(\nu)$ as a subset of $\partial_{F\ddot u}X$, namely as the set of chambers containing ν as a face. Note that $\mathrm{Th}_{F\ddot u}(\sigma) = \mathrm{Th}^{\bar\theta}(\sigma) \cap \partial_{F\ddot u}X$.

Lemma 8.14. *$Th^{\bar\theta}(\sigma)$ is compact.*

Proof. Consider a sequence of points $[b^{\bar\theta}_{\nu_n,p_n}] \in \mathrm{Th}^{\bar\theta}(\sigma)$, and suppose that it converges in $\partial^{\bar\theta}_\infty X$,

$$[b^{\bar\theta}_{\nu_n,p_n}] \to [b^{\bar\theta}_{\mu,q}].$$

We must show that also $[b^{\bar\theta}_{\mu,q}] \in \mathrm{Th}^{\bar\theta}(\sigma)$.

After passing to a subsequence, we may assume that all simplices ν_n have the same type $\theta(\nu_n) = \nu_{mod}$. According to Lemma 3.30, $\nu_n \to \nu \subseteq \mu$. By assumption, $\mathrm{st}(\nu_n) \subset \mathrm{Th}_{F\ddot u}(\sigma)$, and we must show that $\mathrm{st}(\mu) \subset \mathrm{Th}_{F\ddot u}(\sigma)$. Since $\mathrm{st}(\nu) \supseteq \mathrm{st}(\mu)$, this follows from $\mathrm{st}(\nu) \subset \mathrm{Th}_{F\ddot u}(\sigma)$.

The latter follows from the closedness of $\mathrm{Th}_{F\ddot u}(\sigma)$ in $\partial_{F\ddot u}X$, because every chamber $\sigma' \subset \mathrm{st}(\nu)$ is a limit of a sequence of chambers $\sigma'_n \subset \mathrm{st}(\nu_n)$. ☐

Remark 8.15. One can show that $\mathrm{Th}^{\bar\theta}(\sigma) \subset \partial^{\bar\theta}_\infty X$ is a contractible CW-complex. In the second version of this preprint, we proved that it is Čech acyclic, see [KL2, Thm 8.21 in §8.5].

Example 8.16. Suppose that the Weyl group W of X is of type A_2, i.e. is isomorphic to the permutation group on 3 letters. Let $s_1, s_2 \in W$ denote the generators which are the reflections in the walls of the positive chamber σ_{mod}. There is the unique balanced thickening $\mathrm{Th} = \{e, s_1, s_2\} \subset W$. The thickening $\mathrm{Th}^{\bar\theta}(\sigma) \subset \partial^{\bar\theta}_\infty X$ is the wedge of two closed disks connected at the point σ: These disks are the visual compactifications $\overline{X}_{\tau_i}, i = 1, 2$, of two rank 1 symmetric spaces X_{τ_i}. Here τ_1, τ_2 are the two vertices of the edge σ.

More generally, we define the thickening in $\partial^{\bar\theta}_\infty X$ of a set of chambers $A \subset \partial_{F\ddot u}X$ as

$$\mathrm{Th}^{\bar\theta}(A) := \bigcup_{\sigma\in A} \mathrm{Th}^{\bar\theta}(\sigma) \subset \partial^{\bar\theta}_\infty X.$$

Lemma 8.17. *If A is compact, then $Th^{\bar\theta}(A)$ is compact.*

Proof. Since $\partial_{F\ddot u}X$ is a homogeneous space also for the maximal compact subgroup K, there exists a chamber $\sigma_0 \in A$ and a compact subset $C \subset K$ such that $A = C\sigma_0$. Then

$$\mathrm{Th}^{\bar\theta}(A) = C \cdot \mathrm{Th}^{\bar\theta}(\sigma_0)$$

and is hence compact as a consequence of the previous lemma. ☐

If the thickening $\mathrm{Th} \subset W$ is $W_{\tau_{mod}}$-invariant, then we can define the thickening in $\partial^{\bar\theta}_\infty X$ of a simplex $\tau \subset \partial_\infty X$ of type τ_{mod} as

$$\mathrm{Th}^{\bar\theta}(\tau) := \mathrm{Th}^{\bar\theta}(\sigma) \subset \partial^{\bar\theta}_\infty X$$

for a chamber $\sigma \supseteq \tau$. It does not depend on σ. For a set $A \subset \mathrm{Flag}_{\tau_{\mathrm{mod}}}(\mathrm{X})$ of simplices of type τ_{mod}, we define its thickening in $\partial_\infty^{\bar\theta} X$ as

$$\mathrm{Th}^{\bar\theta}(A) := \bigcup_{\tau \in A} \mathrm{Th}^{\bar\theta}(\tau) \subset \partial_\infty^{\bar\theta} X.$$

Again, $\mathrm{Th}^{\bar\theta}(A)$ is compact if A is.

Lemma 8.18 (Fibration of thickenings). *Let $A \subset \mathrm{Flag}_{\tau_{\mathrm{mod}}}(\mathrm{X})$ be compact, and suppose that the thickenings $Th^{\bar\theta}(\tau)$ of the simplices $\tau \in A$ are pairwise disjoint. Then the natural map*

$$Th^{\bar\theta}(A) \xrightarrow{\pi} A$$

is a fiber bundle.

Proof. Regarding continuity of π, suppose that $\xi_n \to \xi$ in $\mathrm{Th}^{\bar\theta}(A)$ and $\tau_n \to \tau$ in A with $\xi_n \in \mathrm{Th}^{\bar\theta}(\tau_n)$. Then $\xi \in \mathrm{Th}^{\bar\theta}(\tau)$ by semicontinuity of relative position, and hence $\pi(\xi) = \tau$.

In order to show that π is a fibration, we need to construct local trivializations. Fix $\tau \in A$ and an opposite simplex $\hat\tau$. Let U denote the unipotent radical of the stabilizer of $\hat\tau$ in G. Then U acts simply transitively on an open neighborhood of τ in $\mathrm{Flag}_{\tau_{\mathrm{mod}}}(\mathrm{X})$. Now, let $S \subset U$ denote the closed subset consisting of all $u \in U$ which send τ to elements of A. Then $S \cdot \tau$ is a neighborhood of τ in A. Restricting the action of U to the subset S, we obtain a topological embedding

$$S \times \mathrm{Th}^{\bar\theta}(\tau) \to \mathrm{Th}^{\bar\theta}(A)$$

and a local trivialization of π over a neighborhood of τ in A. ☐

9 Proper discontinuity: general case

9.1 Accumulation of individual orbits for divergent sequences of isometries

Let $\tau_\pm \subset \partial_\infty X$ be a pair of opposite simplices, $\theta(\tau_+) = \tau_{mod}$, and let $H = H(\tau_-, \tau_+) < G$ be the subgroup of isometries preserving the parallel set $P = P(\tau_-, \tau_+)$, compare (2.1). We study now the dynamics of divergent sequences of isometries in H on the Finsler compactification $\overline{X}^{\bar\theta}$.

We begin by relating the dynamics on the symmetric space to the dynamics at infinity on the flag manifolds. We let $o \in X$ denote a base point. In the sequel, we study the asymptotic dynamics of sequences of isometries $h_n \to \infty$ in H.

Lemma 9.1. *The following are equivalent:*

(i) For any point $x \in X$, the orbit $(h_n x)$ is contained in a tubular neighborhood of the Weyl cone $V(o, \mathrm{st}(\tau_+))$ and $d(h_n x, V(o, \partial\,\mathrm{st}(\tau_+))) \to +\infty$.

(i') The (h_n)-orbits in X accumulate in $\overline{X}^{\bar\theta}$ at $\overline{X}_{\tau_+}$.

(ii) All (h_n)-orbits in $C_{F\ddot{u}}(\tau_-) \subset \partial_{F\ddot{u}}X$ accumulate at $\mathrm{st}(\tau_+)$.

(ii') All (h_n)-orbits in $C_{\tau_{mod}}(\tau_-) \subset \mathrm{Flag}_{\tau_{\mathrm{mod}}}(\mathrm{X})$ converge to τ_+.

Proof. Note that if property (i) or (i') is satisfied for some point $x \in X$ then also for every other point, cf. Lemma 3.27.

Suppose that (i) holds. Then $h_n o \in V(o, \mathrm{st}(\tau_+))$ for large n, and Lemma 8.4 with $\tau_n = \tau_+$ implies (i').

Vice versa, suppose that (i') holds. Let $\tau_n \subset \partial_\infty P(\tau_-, \tau_+)$ be simplices of type τ_{mod} such that $h_n o \in V(o, \mathrm{st}(\tau_n))$. Lemma 8.4 yields that the sequence $(h_n o)$ is τ_{mod}-regular and $\tau_n \to \tau_+$. Since τ_+ is an isolated point of $\{\tau \in \mathrm{Flag}_{\tau_{\mathrm{mod}}}(\mathrm{X}) : \tau \subset \partial_\infty P(\tau_-, \tau_+)\}$, it follows that $\tau_n = \tau_+$ for large n, and thus property (i) is satisfied.

To see that conditions (ii) and (ii') are equivalent, consider the natural fibration

$$\partial_{F\ddot{u}}X \longrightarrow \mathrm{Flag}_{\tau_{\mathrm{mod}}}(\mathrm{X})$$

whose fibers $\mathrm{st}(\tau)$ for $\tau \in \mathrm{Flag}_{\tau_{\mathrm{mod}}}(\mathrm{X})$ are compact. The equivalence follows because $C_{F\ddot{u}}(\tau_-)$ is the preimage of $C_{\tau_{mod}}(\tau_-)$.

Our next goal is to show that (ii') implies (i).

We first observe that the pointwise convergence $h_n \to \tau_+$ on the Schubert cell $C = C_{\tau_{mod}}(\tau_-)$ implies locally uniform convergence. Indeed, the unipotent radical $U = U_{\tau_-}$ of the parabolic subgroup P_{τ_-} acts simply transitively on C. It is normalized by H, and the action $H \curvearrowright C$ corresponds to the action $H \curvearrowright U$ by conjugation. We realize G as a matrix group. Then U becomes a subset of a space of matrices. We note that the action

$$M \overset{g}{\mapsto} gMg^{-1}$$

of G by conjugation on the space of matrices is linear. Therefore, the pointwise convergence $h_n \to$ *const* of the sequence of transformations h_n on U implies locally uniform convergence (on the linear span of the subset U).

We now prove that (ii') implies (i). We deduce this implication from results in our earlier paper [KLP2]. The locally uniform convergence $h_n \to \tau_+$ on C implies that the sequence (h_n) acting on $\mathrm{Flag}_{\tau_{\mathrm{mod}}}(\mathrm{X})$ is contracting in the sense of [KLP2, Def. 5.9]. Therefore, according to [KLP2, Thm. 5.23], the sequence (h_n) is τ_{mod}-regular.

Let $\tau_n \subset \partial_\infty P(\tau_-, \tau_+)$ be simplices of type τ_{mod} such that $h_n o \in V(o, \mathrm{st}(\tau_n))$. Then (τ_n) is a shadow sequence for $(h_n o)$ in the sense of [KLP2, Def. 5.13]. By [KLP2, Lem. 5.16], $\tau_n \to \tau_+$. Since τ_+ is an isolated point of $\{\tau \in \mathrm{Flag}_{\tau_{\mathrm{mod}}}(\mathrm{X}) : \tau \subset \partial_\infty P(\tau_-, \tau_+)\}$, it follows that $\tau_n = \tau_+$ for large n, and thus property (i) is satisfied.

It remains to prove that (i)$\Rightarrow$(ii'). Condition (i) implies that $h_n^{-1} o \in V(o, \mathrm{st}(\tau_-))$ and $d(h_n^{-1} o, V(o, \partial\, \mathrm{st}(\tau_-))) \to +\infty$. Therefore, for a simplex $\hat\tau_-$ opposite to τ_-, it follows that

$$d(o, P(\tau_-, h_n\hat\tau_-)) = d(h_n^{-1}o, P(\tau_-, \hat\tau_-)) \to 0$$

as $n \to +\infty$. This implies that $h_n \hat\tau_- \to \tau_+$. ☐

Remark 9.2. Property (ii') is equivalent to the condition that $d(h_n^{-1}o, P(\tau_-, \hat\tau_-)) \to 0$ for all $\hat\tau_- \in C$. This observation can be used to give a more geometric proof for (ii')⇒(i).

The next result describes the accumulation of individual (h_n)-orbits in $\overline{X}^{\bar\theta}$.

Let $\emptyset \neq \mathrm{Th} \subsetneq W$ be a $W_{\tau_{mod}}$-invariant thickening. Then the thickenings $(\mathrm{Th}^c)^{\bar\theta}(\tau_-)$ and $\mathrm{Th}^{\bar\theta}(\tau_+)$ are H-invariant disjoint compact subsets of $\overline{X}^{\bar\theta}$.

Proposition 9.3. *Suppose that all (h_n)-orbits in $C_{F\ddot u}(\tau_-) \subset \partial_{F\ddot u}X$ accumulate at* $\mathrm{st}(\tau_+)$. *Then, for any $W_{\tau_{mod}}$-invariant thickening $\emptyset \neq Th \subsetneq W$, all (h_n)-orbits in the (P_{τ_-}-invariant) subset $\overline{X}^{\bar\theta} - (Th^c)^{\bar\theta}(\tau_-) \subset \overline{X}^{\bar\theta}$ accumulate at $Th^{\bar\theta}(\tau_+)$.*

Proof. According to the implication (ii)⇒(i') in Lemma 9.1, the orbits in X accumulate at $\overline{S}_{\tau_+} \subseteq \mathrm{Th}^{\bar\theta}(\tau_+)$. The latter inclusion holds, because $\mathrm{Th} \neq \emptyset$ and hence $\mathrm{st}(\tau_+) \subseteq \mathrm{Th}_{F\ddot u}(\tau_+)$.

The issue is therefore to prove the assertion for the orbits at infinity. We first verify it for the orbits in the (P_{τ_-}-invariant) subset

$$\mathrm{Th}^{\bar\theta}(C_{\tau_{mod}}(\tau_-)) = \bigcup \big\{\mathrm{Th}^{\bar\theta}(\hat\tau_-) : \hat\tau_- \text{ opposite to } \tau_-\big\} \subset \partial_\infty^{\bar\theta} X.$$

Due to the implication (ii)⇒(ii') in Lemma 9.1, our assumption implies that

$$h_n\hat\tau_- \to \tau_+ \tag{9.4}$$

for every simplex $\hat\tau_-$ opposite to τ_-. This in turn implies the (Hausdorff) convergence of compact subsets

$$h_n \,\mathrm{Th}^{\bar\theta}(\hat\tau_-) = \mathrm{Th}^{\bar\theta}(h_n\hat\tau_-) \to \mathrm{Th}^{\bar\theta}(\tau_+).$$

Indeed, we may write $h_n\hat\tau_- = g_n\tau_+$ with a sequence $g_n \to e$ in G, and then see that $h_n \,\mathrm{Th}^{\bar\theta}(\hat\tau_-) = g_n \,\mathrm{Th}^{\bar\theta}(\tau_+) \to \mathrm{Th}^{\bar\theta}(\tau_+)$. It follows that the (h_n)-orbits in $\mathrm{Th}^{\bar\theta}(C_{\tau_{mod}}(\tau_-))$ accumulate at $\mathrm{Th}^{\bar\theta}(\tau_+)$.

We are left with the orbits in the "annulus"

$$\mathrm{Ann} := \partial_\infty^{\bar\theta} X - \big((\mathrm{Th}^c)^{\bar\theta}(\tau_-) \cup \mathrm{Th}^{\bar\theta}(C_{\tau_{mod}}(\tau_-))\big).$$

Here, a finer argument is needed.

Note that Ann is a union of small strata. Let $\nu \subset \partial_\infty X$ be a simplex such that $X_\nu \subset \mathrm{Ann}$. We must show that the orbits $(h_n[b^{\bar\theta}_{\nu,x}])$ in $\partial_\infty^{\bar\theta} X$ accumulate at $\mathrm{Th}^{\bar\theta}(\tau_+)$ for all $x \in X$. We will deduce this from the dynamics of the orbits of points in the boundary ∂X_ν of the small stratum, more precisely, in $\partial X_\nu \cap \partial_{F\ddot u}X$.

We need to carefully analyze the position of ν relative to τ_-.

There exists a maximal flat $F' \subset X$ such that the apartment $a' = \partial_\infty F' \subset \partial_\infty X$ contains τ_- and ν. Let $\hat\tau_- \subset a'$ denote the simplex opposite to τ_-. Furthermore, choose interior points $\xi_- \in \mathrm{int}(\tau_-)$ and $\eta \in \mathrm{int}(\nu)$, and let $\hat\xi_- \in \mathrm{int}(\hat\tau_-)$ be the antipode of ξ_-. (Note that $\nu \neq \tau_-, \hat\tau_-$.) Then the geodesic segment $\xi_-\eta$ extends inside a' to the segment $\xi_-\eta\hat\xi_-$ of length π. Let $\mu_\pm \supset \nu$ be the simplices such that $\eta\xi_- \cap \mathrm{int}(\mu_-) \neq \emptyset$ and $\eta\hat\xi_- \cap \mathrm{int}(\mu_+) \neq \emptyset$.

The last property implies that $W_{\mu_-} \leqslant W_{\tau_-}$, where e.g. $W_{\tau_-} < \mathrm{Isom}(a')$ denotes the group generated by reflections at the walls containing τ_-, because the walls containing μ_- also contain τ_-. Thus, the chambers in $\mathrm{st}(\mu_-)$ are contained in an orbit of $W_{\tau_-} = W_{\hat\tau_-}$, and in particular they have the same position relative τ_- and $\hat\tau_-$. Similarly, the chambers in $\mathrm{st}(\mu_+)$ have the same relative position.

Lemma 9.5. *The chambers in* $\mathrm{st}(\mu_-)$ *have strictly smaller position relative* τ_- *than the other chambers in* $\mathrm{st}(\nu)$*. Similarly, the chambers in* $\mathrm{st}(\mu_+)$ *have strictly smaller position relative* $\hat\tau_-$ *than the other chambers in* $\mathrm{st}(\nu)$*.*

Proof. Let $\sigma \subset \mathrm{st}(\nu)$ be a chamber. We perturb $\eta \in \mathrm{int}(\nu)$ generically to a point $\eta' \in \mathrm{int}(\sigma)$. Then the perturbed segment $\xi_-\eta'$ intersects $\mathrm{ost}(\mu_-)$ and contains a point $\zeta_- \in \mathrm{int}(\sigma_-)$ for a chamber $\sigma_- \subset \mathrm{st}(\mu_-)$. By Lemma 8.7, $\mathrm{pos}(\sigma, \tau_-) \succeq_{\tau_{mod}} \mathrm{pos}(\sigma_-, \tau_-)$ with equality iff $\sigma \subset \mathrm{st}(\mu_-)$. The same argument applies to the chambers in $\mathrm{st}(\mu_+)$. ☐

We continue with the proof of the proposition.

Since $X_\nu \subset \mathrm{Ann}$, it holds that

$$X_\nu \not\subset (\mathrm{Th}^c)^{\bar\theta}(\tau_-) \quad \text{and} \quad X_\nu \not\subset \mathrm{Th}^{\bar\theta}(\hat\tau_-).$$

This is equivalent to the conditions

$$\mathrm{st}(\nu) \not\subset \mathrm{Th}^c_{F\ddot{u}}(\tau_-) \quad \text{and} \quad \mathrm{st}(\nu) \not\subset \mathrm{Th}_{F\ddot{u}}(\hat\tau_-). \tag{9.6}$$

Since the function $\mathrm{pos}(\cdot, \tau_-)$ has the same range on $\mathrm{st}(\nu)$ and $\mathrm{st}(\nu) \cap \partial_{F\ddot{u}} F'$, (9.6) is in turn equivalent to the conditions $\mathrm{st}(\nu) \cap \partial_{F\ddot{u}} F' \not\subset \mathrm{Th}^c_{F\ddot{u}}(\tau_-)$ and $\mathrm{st}(\nu) \cap \partial_{F\ddot{u}} F' \not\subset \mathrm{Th}_{F\ddot{u}}(\hat\tau_-)$. The latter are, in view of the disjoint decomposition

$$\partial_{F\ddot{u}} F' = (\partial_{F\ddot{u}} F' \cap \mathrm{Th}^c_{F\ddot{u}}(\tau_-)) \sqcup (\partial_{F\ddot{u}} F' \cap \mathrm{Th}_{F\ddot{u}}(\hat\tau_-)),$$

in turn equivalent to the conditions

$$\mathrm{st}(\nu) \cap \mathrm{Th}^c_{F\ddot{u}}(\tau_-) \neq \varnothing \quad \text{and} \quad \mathrm{st}(\nu) \cap \mathrm{Th}_{F\ddot{u}}(\hat\tau_-) \neq \varnothing.$$

In view of the lemma, these conditions imply that $\mathrm{st}(\mu_-) \subset \mathrm{Th}^c_{F\ddot{u}}(\tau_-)$ and $\mathrm{st}(\mu_+) \subset \mathrm{Th}_{F\ddot{u}}(\hat\tau_-)$, equivalently, that

$$\overline{X}_{\mu_-} \subset (\mathrm{Th}^c)^{\bar\theta}(\tau_-) \quad \text{and} \quad \overline{X}_{\mu_+} \subset \mathrm{Th}^{\bar\theta}(\hat\tau_-). \tag{9.7}$$

The second condition is most important.

Now we return to dynamics and apply the sequence (h_n). Recall that our goal is to show that the (h_n)-orbits of points in X_ν accumulate only at $\mathrm{Th}^{\bar\theta}(\tau_+)$. Therefore we are free to pass to subsequences.

Since $h_n\hat\tau_- \to \tau_+$, compare (9.4), we have that $h_n\hat\xi_- \to \xi_+$ with the antipode $\xi_+ \in \mathrm{int}(\tau_+)$ of ξ_-. After passing to a subsequence, we also have convergence $h_n\nu \to \nu_\infty$ and $h_n\mu_\pm \to \mu_\infty^\pm$. Then also $h_n\eta \to \eta_\infty \in \mathrm{int}(\nu_\infty)$ and the sequence of geodesics $h_n \cdot \xi_-\eta\hat\xi_-$ of length π converges to an a priori broken geodesic path γ_∞ of the same length π connecting the points $\xi_\pm$ and passing through η_∞. Since $\xi_\pm$ are antipodal, the path γ_∞ is also a geodesic segment.

It follows that $\text{pos}(\cdot, \tau_-)$ takes the same values on $\text{st}(\mu_\pm)$ and $\text{st}(\mu_\infty^\pm)$ and, accordingly, the range of $\text{pos}(\cdot, \hat{\tau}_-)$ on $\text{st}(\mu_\pm)$ is the same as the range of $\text{pos}(\cdot, \tau_+)$ on $\text{st}(\mu_\infty^\pm)$. Thus, (9.7) implies

$$\overline{X}_{\mu_\infty^+} \subset \text{Th}^{\bar{\theta}}(\tau_+).$$

It therefore suffices to show that the orbits $(h_n[b_{\nu,x}^{\bar{\theta}}])$ accumulate at $\overline{X}_{\mu_\infty^+}$.

As mentioned before, we will deduce this from the dynamics of the orbits of the points in $\partial X_\nu \cap \partial_{F\ddot{u}} X$. The control on their dynamics comes from the following observation.

Let $\hat{\mu}_- \supset \nu$ be an arbitrary simplex which is ν-opposite to μ_- inside $\text{st}(\nu)$ in the sense that the geodesic segment $\xi_-\eta$ can be extended beyond η into the interior of $\hat{\mu}_-$. It then extends further to a segment $\xi_-\eta\hat{\xi}'_-$ of length π, with endpoint $\hat{\xi}'_- \in \text{int}(\hat{\tau}'_-)$ and $\hat{\tau}'_-$ a simplex opposite to τ_-. Since also $h_n\hat{\tau}'_- \to \tau_+$ by (9.4), it follows that also

$$h_n\hat{\mu}_- \to \mu_\infty^+ \tag{9.8}$$

We now identify the strata $X_{h_n\nu}$ with the limit stratum X_{ν_∞} and translate the assertion on the dynamics of (h_n) into an assertion on the dynamics of a related divergent sequence of isometries preserving X_{ν_∞}. This is done as follows.

In view of the convergence $h_n\mu_\pm \to \mu_\infty^\pm$, there exists a sequence $g_n \to e$ in G so that $h_n\mu_\pm = g_n\mu_\infty^\pm$ and, consequently, $h_n\nu = g_n\nu_\infty$. Moreover, let $g \in G$ so that $\mu_\pm = g\mu_\infty^\pm$ and $\nu = g\nu_\infty$. The isometries $h'_n := g_n^{-1}h_ng \to \infty$ in G then fix $\mu_\infty^\pm$ and ν_∞, and in particular preserve the stratum X_{ν_∞}. Since (h_n)-orbits accumulate at the same compact subsets as the corresponding $(g_n^{-1}h_n)$-orbits, we are reduced to showing that the (h'_n)-orbits in X_{ν_∞} accumulate at $\overline{X}_{\mu_\infty^+} \subset \partial X_{\nu_\infty}$.

Property (9.8) translates into

$$h'_n\hat{\mu}_\infty^- \to \mu_\infty^+ \tag{9.9}$$

for all simplices $\hat{\mu}_\infty^- \supset \nu_\infty$ which are ν_∞-opposite to μ_∞^- inside $\text{st}(\nu_\infty)$.

Recall that, as all small strata of $\partial_\infty^{\bar{\theta}} X$, the stratum X_{ν_∞} carries a natural structure as a symmetric space of noncompact type, being canonically identified with the space of strong asymptote classes of Weyl sectors $V(\cdot, \nu_\infty)$. Moreover, there is a natural identification of spherical buildings

$$\partial_{Tits} X_{\nu_\infty} \cong \Sigma_{\nu_\infty} \partial_{Tits} X.$$

The simplices $\mu_\infty^\pm$ correspond to a pair of opposite simplices $\bar{\mu}_\pm := \Sigma_{\nu_\infty}\mu_\infty^\pm \subset \partial_\infty X_{\nu_\infty}$.

Condition (9.9) means that all (h'_n)-orbits in $C_{F\ddot{u}}(\bar{\mu}_-) \subset \partial_{F\ddot{u}} X_{\nu_\infty}$ accumulate at $\text{st}(\bar{\mu}_+)$. The simplices $\bar{\mu}_\pm$ are fixed by the h'_n, and invoking implication (ii)⇒(i) of Lemma 9.1, it follows that every (h'_n)-orbit $([b_{\nu_\infty,h'_nx}^{\bar{\theta}}])$ in X_{ν_∞} remains in a tubular neighborhood of the Weyl cone $V(\bar{o}, \text{st}(\bar{\mu}_+))$ and

$$\bar{d}([b_{\nu_\infty,h'_nx}^{\bar{\theta}}], V(\bar{o}, \partial\,\text{st}(\bar{\mu}_+))) \to +\infty. \tag{9.10}$$

Here, $\bar{o} \in X_{\nu_\infty}$ denotes a base point, say $\bar{o} = [b_{\nu_\infty,o}^{\bar{\theta}}]$ for a base point $o \in X$, and we measure distances in X_{ν_∞} using a P_{ν_∞}-invariant Riemannian or Finsler metric $\bar{d}$.

The relation between euclidean Weyl chambers and Weyl cones in X and X_{ν_∞} is as follows. A simplex $\tau \supsetneq \nu_\infty$ corresponds to the simplex $\bar\tau := \Sigma_{\nu_\infty}\tau \subset \partial_\infty X_{\nu_\infty}$, the star $\mathrm{st}(\tau)$ to the star $\mathrm{st}(\bar\tau)$, and the open star $\mathrm{ost}(\tau)$ to the open star $\mathrm{ost}(\bar\tau)$. The euclidean Weyl sector $V(\bar o, \bar\tau) \subset X_{\nu_\infty}$ consists of the points $[b^{\bar\theta}_{\nu_\infty,x}]$ with $x \in V(o,\tau) \subset X$. The faces of the sector $V(\bar o,\bar\tau)$ correspond to the faces of the sector $V(o,\tau)$ which properly contain the face $V(o,\nu_\infty)$. The Weyl cone $V(\bar o, \mathrm{st}(\bar\tau))$ consists of the points $[b^{\bar\theta}_{\nu_\infty,x}]$ with $x \in V(o,\mathrm{st}(\tau))$.

The last condition on the (h'_n)-orbit $([b^{\bar\theta}_{\nu_\infty,h'_n x}])$ in X_{ν_∞}, see formula (9.10) and the sentence before, therefore means that we can replace the orbit points $h'_n x$ by points x_n in a tubular neighborhood of the Weyl cone $V(o,\mathrm{st}(\mu^+_\infty)) \subset X$ such that $[b^{\bar\theta}_{\nu_\infty,h'_n x}] = [b^{\bar\theta}_{\nu_\infty,x_n}]$ and $d(x_n, V(o,\partial\,\mathrm{st}(\mu^+_\infty))) \to +\infty$. Then the sequence (x_n) accumulates at $\overline{X}_{\mu^+_\infty}$, cf. Lemma 8.4. The same applies to every sequence (y_n) of points $y_n \in V(x_n,\nu_\infty)$. It follows by approximation that the (h'_n)-orbit $([b^{\bar\theta}_{\nu_\infty,x_n}])$ in X_{ν_∞} accumulates at $\overline{X}_{\mu^+_\infty}$. This finishes the proof. ∎

Invoking the implication (i')⇒(ii) of Lemma 9.1, we conclude:

Corollary 9.11. *If the (h_n)-orbits in X accumulate in $\overline{X}^{\bar\theta}$ at $\overline{X}_{\tau_+}$, then for any $W_{\tau_{mod}}$-invariant thickening $\emptyset \neq Th \subsetneq W$, all (h_n)-orbits in $\overline{X}^{\bar\theta} - (Th^c)^{\bar\theta}(\tau_-)$ accumulate at $Th^{\bar\theta}(\tau_+)$.*

9.2 Locally uniform accumulation of orbits

We continue the discussion of the previous section. Our next goal is to show that the accumulation of (h_n)-orbits as in Corollary 9.11 is locally uniform. Here we need to assume *uniform* τ_{mod}-regularity for the sequence of isometries (h_n). This can be expressed by replacing the assumption that the (h_n)-orbits in X accumulate in the Finsler compactification $\overline{X}^{\bar\theta}$ at the stratum closure $\overline{X}_{\tau_+}$ with the stronger assumption that they accumulate in the visual compactification $\overline{X}$ at a compact subset of the open star $\mathrm{ost}(\tau_+)$.

Proposition 9.12. *If the (h_n)-orbits in X accumulate in $\overline{X}$ at a compact subset of* $\mathrm{ost}(\tau_+)$*, and if $\emptyset \neq Th \subsetneq W$ is a $W_{\tau_{mod}}$-invariant thickening, then for every compact subset $C \subset \overline{X}^{\bar\theta} - (Th^c)^{\bar\theta}(\tau_-)$, the sequence of subsets $h_n C$ accumulates at $Th^{\bar\theta}(\tau_+)$.*

Proof. We argue by contradiction. Suppose that the assertion fails for some compact subset C. Recall that the thickenings $(\mathrm{Th}^c)^{\bar\theta}(\tau_-)$ and $\mathrm{Th}^{\bar\theta}(\tau_+)$ are disjoint compact subsets. Hence, after passing to a suitable subsequence, there exists an open neighborhood U_+ of $\mathrm{Th}^{\bar\theta}(\tau_+)$ such that $\bar U_+ \cap (\mathrm{Th}^c)^{\bar\theta}(\tau_-) = \emptyset$ and

$$h_n C \not\subset U_+ \quad \forall\, n. \tag{9.13}$$

Moreover, due to the uniform τ_{mod}-regularity of the sequence (h_n), we may assume that for a base point $o \in P$ and a compact τ_{mod}-Weyl convex subset $\Theta \subset \mathrm{ost}(\tau_{mod})$, it holds that

$$h_n o \in V(o, \mathrm{st}_\Theta(\tau_+)) \quad \forall\, n,$$

cf. [KLP2, Defs. 2.15 and 2.16 in §2.4.2]. The last condition is equivalent to $o \in V(h_n o, \mathrm{st}_{\iota\Theta}(\tau_-))$ and, since $h_n\tau_\pm = \tau_\pm$, to $h_n^{-1} o \in V(o, \mathrm{st}_{\iota\Theta}(\tau_-))$. Thus, the (h_n^{-1})-orbits in X accumulate in $\overline{X}$ at a compact subset of $\mathrm{ost}(\tau_-)$.

By passing to a subsequence again, we can arrange that

$$h_{n+1}^{-1}o \in V(h_n^{-1}o, \mathrm{st}_{\iota\Theta}(\tau_-)) \quad \forall\, n, \tag{9.14}$$

compare [KLP2, Prop. 2.18 and Cor. 2.19]. After (reindexing and) filling in the sequence (h_n) in H, we may assume that in addition to (9.14) it holds that the coarse path

$$n \mapsto h_n^{-1}o$$

in P is a quasigeodesic ray, compare [KLP2, Lemma 7.12]. Instead of (9.13), we then only have that

$$h_{n_k}C \not\subset U_+ \quad \forall\, k$$

for a subsequence of indices $n_k \to +\infty$. Since C is compact, we may assume after passing to a subsequence of (n_k) that there exists a convergent sequence of points $\bar{x}_k \to \bar{x}$ in $\overline{X}^{\bar{\theta}} - (\mathrm{Th}^c)^{\bar{\theta}}(\tau_-)$ such that

$$h_{n_k}\bar{x}_k \to \bar{x}' \notin U_+.$$

According to Corollary 9.11, the orbit $(h_n\bar{x})$ accumulates at $\mathrm{Th}^{\bar{\theta}}(\tau_+)$. We may therefore assume, after shrinking U_+ and replacing the sequence $\bar{x}_k \to \bar{x}$ by its h_m-image for sufficiently large m, that the entire orbit $(h_n\bar{x})$ is contained in U_+, and that $\bar{x}' \notin \bar{U}_+$.

Then $h_{n_k}\bar{x}_k \notin \bar{U}_+$ for large k, and we replace n_k by the minimal index ("exit time") such that $h_{n_k}\bar{x}_k \notin U_+$. In view of the continuity of the action, the orbits $(h_n\bar{x}_k)_{n\in\mathbb{N}}$ converge to the orbit $(h_n\bar{x})$ as $k \to +\infty$. It follows that still $n_k \to +\infty$ and, after passing to a subsequence, that $h_{n_k}\bar{x}_k \to \bar{x}' \notin U_+$ with a different limit point $\bar{x}'$.

By shifting the orbits $(h_n\bar{x}_k)_{n\in\mathbb{N}}$ and taking a limit, we now find a backward orbit inside $\bar{U}_+$ for a modified divergent sequence of isometries in H as follows. For every $m \geqslant 1$, the points $h_{n_k-m}\bar{x}_k \in U_+$ are defined for large k. By a diagonal argument, after passing to a subsequence of (n_k), we may assume simultaneous convergence

$$h_{n_k-m}\bar{x}_k \to \bar{x}'_{-m} \in \bar{U}_+$$

as $k \to +\infty$ for all m. Since the coarse paths

$$n \mapsto h_{n_k}h_{n+n_k}^{-1}o$$

defined for $n \geqslant -n_k$ are uniform coarse quasigeodesic rays passing through o at time $n = 0$, we may assume in addition that, as $k \to +\infty$, they subconverge to a coarse quasiline

$$n \mapsto {h'_n}^{-1}o$$

defined on $\mathbb{Z}$, and that we have simultaneous convergence

$$h_{n+n_k}h_{n_k}^{-1} \to h'_n$$

in H as $k \to +\infty$ for all $n \in \mathbb{Z}$. Using the continuity of the H-action on $\overline{X}^{\bar{\theta}}$, we obtain

$$\underbrace{h_{n_k-m}\bar{x}_k}_{\to \bar{x}'_{-m}} = h_{n_k-m}h_{n_k}^{-1} \cdot h_{n_k}\bar{x}_k \to h'_{-m}\bar{x}'$$

and hence

$$\bar{x}'_{-m} = h'_{-m}\bar{x}'$$

for all $m \geqslant 1$. Moreover,

$$h'_{-m} \to \infty$$

in H as $m \to +\infty$ because $m \mapsto {h'_m}^{-1}o$ is a quasiline.

By (9.14), we have that $h_n^{-1}o \in V(h_{n-m}^{-1}o, \mathrm{st}_{\iota\Theta}(\tau_-))$, equivalently, $h_{n-m}h_n^{-1}o \in V(o, \mathrm{st}_{\iota\Theta}(\tau_-))$, and thus, by taking a limit,

$$h'_{-m}o \in V(o, \mathrm{st}_{\iota\Theta}(\tau_-)).$$

In particular, the $(h'_{-m})_{m\in\mathbb{N}}$-orbits in X accumulate in $\overline{X}$ at $\mathrm{st}_{\iota\Theta}(\tau_-) \subset \mathrm{ost}(\tau_-)$, and consequently in $\overline{X}^{\bar{\theta}}$ at $\overline{X}_{\tau_-}$.

We now apply Corollary 9.11 to the sequence (h'_{-m}) in H by reversing the roles of the simplices $\tau_\pm$ and their thickenings: Since $\bar{x}' \notin \mathrm{Th}^{\bar{\theta}}(\tau_+)$, the orbit $(h'_{-m}\bar{x}')$ must accumulate in $\overline{X}^{\bar{\theta}}$ at $(\mathrm{Th}^c)^{\bar{\theta}}(\tau_-)$. On the other hand, by construction, the orbit is contained in the closed neighborhood $\bar{U}_+$, which is disjoint from $(\mathrm{Th}^c)^{\bar{\theta}}(\tau_-)$, a contradition. ☐

The proposition has the following implication for dynamical relations:

Corollary 9.15 (Dynamical relations with respect to sequences in H). *If $\bar{x}_\pm \in \overline{X}^{\bar{\theta}}$ are points such that $\bar{x}_-$ is dynamically related to $\bar{x}_+$ with respect to the sequence (h_n),*

$$\bar{x}_- \overset{(h_n)}{\sim} \bar{x}_+,$$

then $\bar{x}_- \in (Th^c)^{\bar{\theta}}(\tau_-)$ or $\bar{x}_+ \in Th^{\bar{\theta}}(\tau_+)$.

9.3 Dynamical relation

We extend the last corollary to arbitrary uniformly τ_{mod}-regular sequences (g_n) in G.

Uniform τ_{mod}-regularity means that the (g_n)-orbits in X accumulate in $\overline{X}$ at a compact subset of the τ_{mod}-regular part $\theta^{-1}(\mathrm{ost}(\tau_{mod})) \subset \partial_\infty X$ of the visual boundary. After passing to a subsequence of (g_n), we may suppose that there exists a pair of (in general not antipodal) simplices $\tau_\pm \subset \partial_\infty X$ of types $\theta(\tau_+) = \tau_{mod}$ and $\theta(\tau_-) = \iota\tau_{mod}$ such that the $(g_n^{\pm 1})$-orbits in X accumulate at compact subsets $C_\pm$ of the open stars $\mathrm{ost}(\tau_\pm)$,

$$g_n^{\pm 1}x \to C_\pm \subset \mathrm{ost}(\tau_\pm)$$

for $x \in X$.

Generalizing our earlier result Corollary 6.3 in the regular case, we obtain:

Corollary 9.16 (Dynamical relation with respect to τ_{mod}-regular sequences of isometries). *Let $\emptyset \neq Th \subsetneq W$ be a $W_{\tau_{mod}}$-invariant thickening. If $\bar{x}_\pm \in \overline{X}^{\bar{\theta}}$ are points such that $\bar{x}_-$ is dynamically related to $\bar{x}_+$ with respect to a uniformly τ_{mod}-regular sequence (g_n) in G,*

$$\bar{x}_- \overset{(g_n)}{\sim} \bar{x}_+,$$

then $\bar{x}_- \in (Th^c)^{\bar{\theta}}(\tau_-)$ or $\bar{x}_+ \in Th^{\bar{\theta}}(\tau_+)$.

Proof. We deduce this version of the result using the KA_+K-decomposition of G.

Let $\tau'_\pm \subset \partial_\infty X$ be a pair of antipodal simplices of the same types $\theta(\tau'_\pm) = \theta(\tau_\pm)$, and let $o \in P(\tau'_-, \tau'_+)$ be a base point and $K < G$ the maximal compact subgroup fixing it. We may write

$$g_n = k_n h_n k'_n$$

with isometries $k_n, k'_n \in K$ and $h_n \to \infty$ in $H(\tau'_-, \tau'_+)$ such that, in view of the uniform τ_{mod}-regularity of (g_n), also (h_n) is uniformly τ_{mod}-regular and the $(h_n^{\pm 1})$-orbits in X accumulate at compact subsets $C'_\pm \subset \mathrm{ost}(\tau'_\pm)$,

$$h_n^{\pm 1} x \to C'_\pm \subset \mathrm{ost}(\tau'_\pm).$$

After passing to a subsequence, we may assume convergence $k_n \to k_\infty$ and $k'_n \to k'_\infty$. Then $\tau_+ = k_\infty \tau'_+$ and $\tau_- = k'^{-1}_\infty \tau'_-$, and our assumption on dynamical relation translates into

$$k'_\infty \bar{x}_- \overset{(h_n)}{\sim} k_\infty^{-1} \bar{x}_+.$$

Corollary 9.15 therefore yields that $k'_\infty \bar{x}_- \in (\mathrm{Th}^c)^{\bar{\theta}}(\tau'_-)$ or $k_\infty^{-1} \bar{x}_+ \in \mathrm{Th}^{\bar{\theta}}(\tau'_+)$, equivalently, that $\bar{x}_- \in (\mathrm{Th}^c)^{\bar{\theta}}(k'^{-1}_\infty \tau'_-) = (\mathrm{Th}^c)^{\bar{\theta}}(\tau_-)$ or $\bar{x}_+ \in \mathrm{Th}^{\bar{\theta}}(k_\infty \tau'_+) = \mathrm{Th}^{\bar{\theta}}(\tau_+)$, as claimed. □

9.4 Proper discontinuity

We deduce a version of the last result for discrete group actions. Generalizing our earlier result for regular subgroups, see Proposition 6.12, we obtain:

Corollary 9.17 (Dynamical relation with respect to uniformly τ_{mod}-regular subgroups). *Let τ_{mod} be a ι-invariant face type, and let $Th \subset W$ be a $W_{\tau_{mod}}$-invariant balanced thickening. Suppose that $\Gamma < G$ is a uniformly τ_{mod}-regular discrete subgroup. If two points $\bar{x}_\pm \in \overline{X}^{\bar{\theta}}$ are dynamically related with respect to the Γ-action,*

$$\bar{x}_- \overset{\Gamma}{\sim} \bar{x}_+,$$

then at least one of them is contained in $Th^{\bar{\theta}}(\Lambda_{\tau_{mod}}(\Gamma))$.

Proof. By assumption, there exists a sequence $\gamma_n \to \infty$ in Γ such that

$$\bar{x}_- \overset{(\gamma_n)}{\sim} \bar{x}_+.$$

According to the definition of the τ_{mod}-limit set, after passing to a subsequence, there exist limit simplices $\lambda_\pm \in \Lambda_{\tau_{mod}}(\Gamma)$ such that

$$\gamma_n^{\pm 1} x \to C_\pm \subset \mathrm{ost}(\lambda_\pm)$$

in $\overline{X}$ for suitable compact subsets $C_\pm \subset \mathrm{ost}(\lambda_\pm)$ and all points $x \in X$. Since the thickening Th is balanced, $\mathrm{Th}^c = \mathrm{Th}$, Corollary 9.16 yields at least one of the containments $\bar{x}_\pm \in \mathrm{Th}^{\bar{\theta}}(\lambda_\pm)$. □

We can now extend Theorem 6.21 to the weakly regular case. The last result translates into:

Theorem 9.18 (Domain of proper discontinuity). *Suppose that $\Gamma < G$ is a uniformly τ_{mod}-regular discrete subgroup and that $Th \subset W$ is a $W_{\tau_{mod}}$-invariant balanced thickening. Then the action*

$$\Gamma \curvearrowright \overline{X}^{\bar\theta} - Th^{\bar\theta}(\Lambda_{\tau_{mod}}(\Gamma))$$

is properly discontinuous.

Proof. According to the last corollary, there are no dynamical relations between points outside $\mathrm{Th}^{\bar\theta}(\Lambda_{\tau_{mod}}(\Gamma))$. Therefore, the action is properly discontinuous, see [F] and [KL1]. ☐

9.5 Nonemptyness of domains of proper discontinuity at infinity

Let Γ and Th be as in Theorem 9.18, and suppose in addition that Γ is τ_{mod}-antipodal, i.e. any two limit simplices in $\Lambda_{\tau_{mod}}(\Gamma)$ are antipodal.

Lemma 9.19. *If $rank(X) \geqslant 2$, then $Th^{\bar\theta}(\Lambda_{\tau_{mod}}(\Gamma)) \neq \partial_\infty^{\bar\theta} X$.*

Proof. The set of chambers in the model apartment a_{mod} decomposes as

$$\mathrm{Th}_{F\ddot{u}}(\sigma_{mod}) \sqcup \mathrm{Th}_{F\ddot{u}}(w_0\sigma_{mod}).$$

We regard both thickenings as subcomplexes of a_{mod}. Their union covers a_{mod} and they have disjoint interiors. Since $\mathrm{rank}(X) \geqslant 2$, a_{mod} is connected and the two subcomplexes have a common face $\bar\nu$. Then $\mathrm{st}(\bar\nu) \cap \mathrm{Th}_{F\ddot{u}}(\sigma_{mod}) \neq \emptyset$ and $\mathrm{st}(\bar\nu) \cap \mathrm{Th}_{F\ddot{u}}(w_0\sigma_{mod}) \neq \emptyset$.

Fix a limit simplex $\tau \in \Lambda_{\tau_{mod}}(\Gamma)$. Let $a \supset \tau$ be an apartment, let $\hat\tau \subset a$ be the simplex opposite to τ, and let $f : \partial_\infty X \to a$ be a folding map which restricts to an isometry on every apartment $a' \supset \tau$. Then for every simplex $\hat\tau' \subset \partial_\infty X$ opposite to τ it holds that $f(\mathrm{Th}_{F\ddot{u}}(\hat\tau')) \subset \mathrm{Th}_{F\ddot{u}}(\hat\tau)$. Let $\nu \subset \partial_\infty X$ be a simplex with $\mathrm{pos}(\nu, \tau) = \bar\nu$. It follows that $\mathrm{st}(\nu) \not\subset \mathrm{Th}_{F\ddot{u}}(\lambda)$ for every limit simplex $\lambda \in \Lambda_{\tau_{mod}}(\Gamma)$. Consequently, $X_\nu \not\subset \mathrm{Th}^{\bar\theta}(\Lambda_{\tau_{mod}}(\Gamma))$. ☐

Remark 9.20. Note that the nonemptyness of domains of proper discontinuity is much harder to prove for the Γ-actions on flag manifolds and requires additional assumptions. For the case of actions on the Fürstenberg boundary, see [KLP1, Thm. 8.39 in §8.3].

10 Cocompactness: general case

Suppose that $\Gamma < G$ is a τ_{mod}-RCA subgroup and that $\mathrm{Th} \subset W$ is a $W_{\tau_{mod}}$-invariant balanced thickening. In this section, we will use the following notation:

$$\hat\Sigma := \overline{X}^{\bar\theta}, \quad \hat\Lambda := \mathrm{Th}^{\bar\theta}(\Lambda_{\tau_{mod}}(\Gamma)), \quad \hat\Omega := \hat\Sigma - \hat\Lambda$$

By Theorem 9.18, the action

$$\Gamma \curvearrowright \hat\Omega$$

is properly discontinuous. The main goal of this section is to show that this action is also cocompact.

10.1 Decompositions and collapses

A *decomposition* $\mathcal{R}$ of a set Z is an equivalence relation on Z. We let $\mathcal{D} = \mathcal{D}_{\mathcal{R}}$ denote the subset of the power set 2^Z consisting of the equivalence classes of $\mathcal{R}$.

A decomposition of a Hausdorff topological space Z is *closed* if the elements of $\mathcal{D}$ are closed subsets of Z; a decomposition is *compact* if its elements are compact subsets. Given a decomposition $\mathcal{R}$ of Z, one defines the quotient space $Z/\mathcal{R}$. Quotient spaces of closed decompositions are T_1 but in general not Hausdorff.

Definition 10.1. A decomposition of Z is *upper semicontinuous* (usc) if it is closed and for each $D \in \mathcal{D}$ and each open subset $U \subset Z$ containing D, there exists another open subset $V \subset Z$ containing D such that every $D' \in \mathcal{D}$ intersecting V nontrivially is already contained in U.

Lemma 10.2 ([D, Proposition 1, page 8]). *The following are equivalent for a closed decomposition $\mathcal{R}$ of Z:*

(i) $\mathcal{R}$ is usc.

(ii) For every open subset $U \subset Z$, the saturated subset

$$U^* = \bigcup \{D \in \mathcal{D} : D \subset U\}$$

is open.

(iii) The quotient projection

$$Z \xrightarrow{\kappa} Z/\mathcal{R}$$

is closed.

Proof. (i)⇒(ii): Let $x \in U$, and let $D \in \mathcal{D}$ be the decomposition subset through x. The usc property implies that U^* contains a neighborhood of x.

(ii)⇒(i): Take $V = U^*$.

(ii)⇒(iii): Let $C \subset Z$ be closed, and let U be the complement. Then $U^* = \kappa^{-1}\kappa(Z - C)$, and it follows that $\kappa(C)$ is closed.

(iii)⇒(ii): Let $U \subset Z$ be open. Then $U^* = \kappa^{-1}(Z/\mathcal{R} - \kappa(Z - U))$ is open. □

Let $Z' \subset Z$ be the union of all elements of $\mathcal{D}$ which are not singletons, and denote by $\mathcal{R}'$ the equivalence relation on Z' induced by $\mathcal{R}$.

Lemma 10.3. *Suppose that Z' is closed. Then $\mathcal{R}$ is usc iff $\mathcal{R}'$ is usc.*

Proof. Suppose that $\mathcal{R}'$ is usc. Let $D \in \mathcal{D}$. If D is a singleton, then $Z - Z'$ is a saturated open neighborhood of D. On the other hand, if $D \subset Z'$ then D has a saturated open neighborhood V' in Z'. It is an intersection $V' = V \cap Z'$ with an open subset $V \subset Z$ which is necessarily again saturated. This verifies that $\mathcal{R}$ is usc.

Vice versa, suppose that $\mathcal{R}$ is usc. Then the intersection of a saturated open subset in Z with Z' is open and saturated in Z'. Hence $\mathcal{R}'$ is usc. □

We will use the following result:

Proposition 10.4 ([D, Proposition 2, page 13]). *If Z is metrizable and $\mathcal{R}$ is a compact usc decomposition of Z, then $Z/\mathcal{R}$ is again metrizable.*

We now apply the notion of usc decompositions in the context of the Finsler thickening of $\Lambda_{\tau_{mod}}(\Gamma) \subset \mathrm{Flag}_{\tau_{\mathrm{mod}}}(X)$. Since Γ is τ_{mod}-antipodal and the thickening Th is slim, we obtain a compact decomposition $\mathcal{R}$ of $\hat{\Sigma}$, whose elements are singletons, namely the points in $\hat{\Omega}$, and the thickenings $\mathrm{Th}^{\bar{\theta}}(\tau)$ of the simplices $\tau \in \Lambda_{\tau_{mod}}(\Gamma)$. (One can show that the latter are contractible, cf. Remark 8.15.) We let

$$\kappa : \hat{\Sigma} \to \Sigma$$

denote the quotient projection, and

$$\Lambda := \kappa(\hat{\Lambda}) \cong \Lambda_{\tau_{mod}}(\Gamma), \quad \Omega := \kappa(\hat{\Omega}) \cong \hat{\Omega}.$$

Lemma 10.5. *The decomposition $\mathcal{R}$ of $\hat{\Sigma}$ is compact usc.*

Proof. The restriction $\hat{\Lambda} \to \Lambda$ of κ is a map of compact Hausdorff spaces and hence closed. Thus the restriction of the decomposition $\mathcal{R}$ to $\hat{\Lambda}$ is usc, cf. Lemma 10.2. Hence, by Lemma 10.3, the decomposition $\mathcal{R}$ is usc as well. It is also compact. ☐

Corollary 10.6. *$\Sigma = \hat{\Sigma}/\mathcal{R}$ is metrizable.*

This corollary is relevant to us in order to do computations with Čech cohomology.

Remark 10.7. We showed in [KL2, Lemma 10.7] that Σ is Čech acyclic.

10.2 Convergence actions

Since the τ_{mod}-RCA property of Γ is equivalent to τ_{mod}-asymptotic embeddedness, we have that $\Lambda_{\tau_{mod}}(\Gamma) \cong \partial_\infty \Gamma$ equivariantly, see [KLP2]. We continue using the notation from the previous section. The action of Γ on $\hat{\Sigma}$ descends to a continuous action

$$\Gamma \curvearrowright \Sigma. \tag{10.8}$$

The results of section 9.2 imply that this action is a convergence action.[1] (We will not use this fact and hence will omit the proof.) We note that $\Sigma = \Omega \sqcup \Lambda$, where Ω is the domain of discontinuity of $\Gamma \curvearrowright \Sigma$ and Λ is the limit set of the action.

There is the following natural question from the general theory of convergence groups, due to P. Haïssinsky [H][2]:

[1] Cf. [PS] for a similar constructions of convergence actions starting with isometric actions on CAT(0) spaces.

[2] An equivalent question was asked by P. Tukia in [T2, p. 77], we owe the observation of equivalence of the questions to V. Gerasimov.

Question 10.9. Let $\Gamma \curvearrowright \Sigma$ be a convergence group action of a hyperbolic group on a metrizable compact space Σ, and suppose that $\Lambda \subset \Sigma$ is an invariant compact subset which is equivariantly homeomorphic to $\partial_\infty \Gamma$. Then the action $\Gamma \curvearrowright \Omega = \Sigma - \Lambda$ is properly discontinuous. Is it always cocompact?

Remark 10.10. This is true for actions which are *expanding* at the limit set Λ, cf. [KLP1].

In our situation of actions derived from τ_{mod}-RCA actions on symmetric spaces, we have the following additional information of which we will make crucial use:

Whenever $\Gamma \curvearrowright \widetilde{R}$ is a properly discontinuous cocompact isometric action on a locally compact geodesic metric space, there exists a continuous Γ-equivariant map of triads

$$(\overline{\widetilde{R}}, \widetilde{R}, \partial_\infty \widetilde{R}) \xrightarrow{\tilde{f}} (\Sigma, \Omega, \Lambda) \tag{10.11}$$

which comes from extending an orbit map $\Gamma \to \Gamma x \subset X$, see [KLP3, Thm. 1.4], [KLP2, Thm. 7.35 and the discussion before] and Lemma 3.38. Here, $\overline{\widetilde{R}}$ denotes the Gromov compactification of $\widetilde{R}$ and $\partial_\infty \widetilde{R}$ its Gromov boundary,

$$\overline{\widetilde{R}} = \widetilde{R} \sqcup \partial_\infty \widetilde{R}.$$

We observe that the map of triads satisfies:

(i) $\tilde{f}|_{\widetilde{R}} : \widetilde{R} \to \Omega$ is proper.

(ii) $\tilde{f}|_{\partial_\infty \widetilde{R}} : \partial_\infty \widetilde{R} \to \Lambda$ is a homeomorphism.

The first property comes from the proper discontinuity of the Γ-action on Ω.

In section 10.4 we will give a positive answer to Question 10.9 for torsion-free hyperbolic groups under the assumption that Ω has finitely many path-connected components.

10.3 Cocompactness theorems

We consider now an action $\Gamma \curvearrowright \Sigma$ of a hyperbolic group as in (10.8), i.e. which is derived from a τ_{mod}-RCA action $\Gamma \curvearrowright X$ on the symmetric space X. Note that the domain $\Omega \subset \Sigma$ is path connected. We recall that we work with Alexander-Spanier-Čech cohomology.

Theorem 10.12. *Ω/Γ is compact.*

Proof. Since $\Gamma < G$ is a finitely-generated linear group, it is virtually torsion-free by Selberg's Lemma. Therefore, from now on we shall assume that Γ is torsion-free.

Let $\widetilde{R}$ be a contractible finite-dimensional locally compact simplicial complex on which Γ acts properly discontinuously and cocompactly, e.g. a suitable Rips complex of Γ. We set $R = \widetilde{R}/\Gamma$. Then $\pi_1(R) \cong \Gamma$. Furthermore, the Gromov compactification $\overline{\widetilde{R}}$ of $\widetilde{R}$ is contractible and metrizable, cf. [BM].

We need to "thicken" R to a closed manifold without changing the fundamental group. To do so, we first embed R as a subcomplex into the (suitably triangulated) euclidean space E^{2n+1}, where $n = \dim(R)$. We denote by N the regular neighborhood of R in E^{2n+1}, and let $D = \partial N$.

Lemma 10.13. *D is connected and $\pi_1(D) \to \pi_1(N) \cong \pi_1(R)$ is surjective.*

Proof. Let $N' := N - R$. We claim that the map $D \hookrightarrow N'$ is a homotopy equivalence. The proof is the same as the one for the homotopy equivalence $R \to N$: Each simplex $c \subset N$ is the join $c_1 \star c_2$ of a simplex c_1 disjoint from R (and, hence, contained in D) and a simplex $c_2 \subset R$ (in the extreme cases, c_1 or c_2 could be empty). Now, use the straight line segments given by these join decompositions to homotop each $c - R$ to $c_1 \subset D$.

Since R has codimension $\geqslant 2$ in N, it does not separate N' and each loop in N is homotopic to a loop in N'. Hence, N' is connected and

$$\pi_1(D) \xrightarrow{\cong} \pi_1(N') \longrightarrow \pi_1(N)$$

is surjective. ☐

Lemma 10.14. *There exists a closed connected manifold M which admits a map $h : R \to M$ inducing an isomorphism of fundamental groups $\pi_1(R) \to \pi_1(M)$.*

Proof. We start with N (the regular neighborhood of $R \subset E^{2n+1}$) as above. As noted in the proof of the previous lemma, the inclusion $R \to N$ is a homotopy equivalence, and N is a compact manifold with boundary. Consider the closed manifold M obtained by doubling N along its boundary D,

$$M = N_1 \cup_D N_2,$$

where N_1, N_2 are the two copies of N. We let $i : D \to M, i_k : N_k \to M$ denote the inclusion maps. Since M is the double of N, we have the canonical retraction $r : M \to N_1$ (whose restriction to N_2 is a homeomorphism). Define the map $h = i_1 \circ g$,

$$h : R \xrightarrow{g} N_1 \xrightarrow{i_1} M,$$

where g corresponds to the inclusion $R \to N$ and hence is a homotopy equivalence. We claim that h induces an isomorphism h_* of fundamental groups.

The existence of the retraction r implies the injectivity of i_{1*} and hence of h_*.

By Lemma 10.13, D is connected. Hence, the Seifert–van Kampen theorem implies that $\pi_1(M)$ is generated by the two subgroups $i_{k*}(\pi_1(N_k)), k = 1, 2$. Since the homomorphisms

$$\pi_1(D) \to \pi_1(N_k)$$

are surjective (Lemma 10.13), we obtain

$$i_{1*}(\pi_1(N_1)) = i_*(\pi_1(D)) = i_{2*}(\pi_1(N_2)).$$

Hence, both homomorphisms $i_{k*} : \pi_1(N_k) \to \pi_1(M)$ are surjective. The surjectivity of h_* follows. ☐

We let $m = 2n + 1$ denote the dimension of the manifold M and its universal cover $\widetilde{M}$.

We now consider the triads (10.11) and the diagonal Γ-action on their products with $\widetilde{M}$. Dividing out the action, we obtain bundles over M and $\tilde{f}$ induces the map of triads of bundles

$$(\underbrace{(\overline{\widetilde{R}} \times \widetilde{M})/\Gamma}_{\bar{E}_{mod}}, \underbrace{(\widetilde{R} \times \widetilde{M})/\Gamma}_{E_{mod}}, \underbrace{(\partial_\infty \widetilde{R} \times \widetilde{M})/\Gamma}_{L_{mod}}) \xrightarrow{F} (\underbrace{(\Sigma \times \widetilde{M})/\Gamma}_{\bar{E}}, \underbrace{(\Omega \times \widetilde{M})/\Gamma}_{E}, \underbrace{(\Lambda \times \widetilde{M})/\Gamma}_{L}) \quad (10.15)$$

Note that E also fibers over Ω/Γ with fiber $\widetilde{M}$.

The map F of triads of bundles satisfies:

(i) $F|_{E_{mod}} : E_{mod} \to E$ is proper.

(ii) $F|_{L_{mod}} : L_{mod} \to L$ is a homeomorphism of bundles.

Lemma 10.16. *Both spaces $\bar{E}, \bar{E}_{mod}$ are metrizable.*

Proof. These spaces are fiber bundles with compact metrizable bases and fibers. Therefore, $\bar{E}, \bar{E}_{mod}$ are both compact and Hausdorff. Hence, they are metrizable, for instance, by Smirnov's metrization theorem, because they are paracompact, Hausdorff and locally metrizable. □

Our approach to proving Theorem 10.12 is based on the following observation.

In a noncompact connected manifold, the point represents the zero class in H_0^{lf}. Similarly, let $\iota : F \to E \xrightarrow{\pi} B$ be a fiber bundle over a noncompact connected manifold, where $\iota : F \to E_b$ is the homeomorphism of F to the fiber $E_b = \pi^{-1}(b)$. If the base B is path-connected, then the induced map

$$\iota_* : H_*^{lf}(F) \to H_*^{lf}(E)$$

on locally finite homology is independent of the choice of b. In order to show triviality of this map provided that B is noncompact, note that for each class $[\eta] \in Z_c^i(E)$ and each locally finite class $[\xi] \in H_m^{lf}(F)$, if b is chosen so that E_b is disjoint from the support of η, then $\langle [\eta], [\xi] \rangle = 0$. Here and in the sequel we use (co)homology with $\mathbb{Z}_2$-coefficients. Hence, $\iota_* = 0$.

The compactness of Ω/Γ therefore follows from showing that the fiber of the bundle

$$\widetilde{M}^m \to E \to \Omega/\Gamma$$

represents a nontrivial class in $H_m^{lf}(E)$, i.e. that the locally finite fundamental class $[\widetilde{M}] \in H_m^{lf}(\widetilde{M})$ has a non-zero image under the inclusion induced map $H_m^{lf}(\widetilde{M}) \to H_m^{lf}(E)$.

The proper map $F : E_{mod} \to E$ induces the map $F_* : H_m^{lf}(E_{mod}) \to H_m^{lf}(E)$ which carries the class represented by the $\widetilde{M}$-fiber in the model E_{mod} to the corresponding class in E. It therefore suffices to show that

$$\underbrace{H_m^{lf}(\widetilde{M})}_{\cong \mathbb{Z}_2} \xrightarrow{\iota_*} H_m^{lf}(E_{mod}) \xrightarrow{F_*} H_m^{lf}(E) \quad (10.17)$$

is a composition of injective maps.

Step 1: Injectivity of F_*. We pass to compactly supported cohomology. We recall that locally finite homology (with field coefficients) is dual to compactly supported cohomology in the same degree via Kronecker duality. We therefore must show that the dual map

$$H_c^m(E) \xrightarrow{F^*} H_c^m(E_{mod})$$

is surjective.

We now switch the fiber direction and regard E and E_{mod} as bundles over M. We use their compactifications $\bar{E}$ and $\bar{E}_{mod}$ mentioned earlier which allow us to replace compactly supported cohomology by relative cohomology. Since E is compact and metrizable, while L is compact, we have a natural isomorphism of Alexander-Spanier cohomology groups (cf. [Sp, Lemma 11, p. 321]):

$$H_c^m(E) \cong H^m(\bar{E}, L)$$

Similarly, we have a natural isomorphism

$$H_c^m(E_{mod}) \cong H^m(\bar{E}_{mod}, L_{mod}).$$

Thus, the surjectivity of the previous map F^* is equivalent to the surjectivity of the map

$$H^m(\bar{E}, L) \xrightarrow{F^*_{rel}} H_c^m(\bar{E}_{mod}, L_{mod})$$

induced by the map of pairs

$$(\bar{E}_{mod}, L_{mod}) \xrightarrow{F} (\bar{E}, L). \tag{10.18}$$

To verify the surjectivity of F^*_{rel}, we use the long exact cohomology sequence of F:

$$\begin{array}{ccccccccc}
\dots\ H^{m-1}(\bar{E}) & \longrightarrow & H^{m-1}(L) & \longrightarrow & H^m(\bar{E}, L) & \longrightarrow & H^m(\bar{E}) & \longrightarrow & H^m(L)\ \dots \\
\downarrow & & \downarrow \cong & & \downarrow F^*_{rel} & & \downarrow F^*_{abs} & & \downarrow \cong \\
\dots H^{m-1}(\bar{E}_{mod}) & \longrightarrow & H^{m-1}(L_{mod}) & \longrightarrow & H^m(\bar{E}_{mod}, L_{mod}) & \xrightarrow{j} & H^m(\bar{E}_{mod}) & \longrightarrow & H^m(L_{mod}) \dots
\end{array}$$

A diagram chase (as in the proof of the 5-lemma) shows that the surjectivity of F^*_{rel} follows from the surjectivity of F^*_{abs}. Indeed, one first checks that $\ker j \subset \operatorname{im} F^*_{rel}$, and uses this to verify the inclusion

$$j^{-1}(\operatorname{im} F^*_{abs}) \subset \operatorname{im}(F^*_{rel}).$$

To see that F^*_{abs} is surjective, we consider the map of bundles:

$$\begin{array}{ccc}
\bar{E}_{mod} & \xrightarrow{F} & \bar{E} \\
 & {\scriptstyle \pi_{\bar{E}_{mod}}} \searrow \quad \swarrow {\scriptstyle \pi_{\bar{E}}} & \\
 & M &
\end{array}$$

The fibration $\pi_{\bar{E}_{mod}}$ is a homotopy equivalence because its fibers $\overline{\widetilde{R}}$ are contractible. Let

$$s : M \to \bar{E}_{mod}$$

denote a section. It follows that $s \circ \pi_{\bar{E}}$ is a left homotopy inverse for F, i.e. $s \circ \pi_{\bar{E}} \circ F \simeq \mathrm{id}_{\bar{E}_{mod}}$. Thus, the induced map on cohomology F^*_{abs} is surjective.

Step 2: Injectivity of ι_*. We consider the fiber bundle

$$\widetilde{M} \to E_{mod} \to R.$$

The map $h : R \to M$ in Lemma 10.14 yields a section of this bundle. Since the base R of the bundle is a finite CW complex and its fiber $\widetilde{M}$ is a connected m-manifold, Lemma 2.6 implies that the induced map

$$H^{lf}_m(\widetilde{M}) \xrightarrow{\iota_*} H^{lf}_m(E_{mod}),$$

is injective.

This concludes the proof of Theorem 10.12. □

We note that our proof required no assumptions on algebro–topological properties of Σ. We only used that Σ is compact metrizable, that Ω is path-connected, the existence of the map of triads (10.11) and that Γ is virtually torsion-free. Our proof, thus, shows

Theorem 10.19. *Let $\Sigma = \Omega \sqcup \Lambda$ be a compact metrizable space, where $\Lambda \subset \Sigma$ is closed. Let Γ be a virtually torsion-free hyperbolic group and let $\tilde{R}$ be a contractible Rips complex for Γ. Assume that $\Gamma \curvearrowright \Sigma$ is a continuous action, such that Λ is Γ-invariant and Γ-equivariantly homeomorphic to $\partial_\infty\Gamma$ and such that the action of Γ on Ω is properly discontinuous. Assume also that Ω is path-connected and that there exists a continuous equivariant map of triads*

$$\tilde{f} : (\overline{\tilde{R}}, \tilde{R}, \partial_\infty \tilde{R}) \to (\Sigma, \Omega, \Lambda).$$

Then Ω/Γ is compact.

This theorem provides strong evidence for a positive answer to Question 10.9 in the case of convergence group actions with path-connected discontinuity domains, see Theorem 10.22 in the next section.

By combining Theorems 9.18 and 10.12 with Theorem 1.1, we obtain:

Theorem 10.20. *Let $\Gamma < G$ be a τ_{mod}-RCA subgroup, let $Th \subset W$ be a $W_{\tau_{mod}}$-invariant balanced thickening (which always exists), and let $\bar{\theta} \in \mathrm{int}(\sigma_{mod})$. Then the action*

$$\Gamma \curvearrowright \overline{X}^{\bar{\theta}} - Th^{\bar{\theta}}(\Lambda_{\tau_{mod}}(\Gamma))$$

is properly discontinuous and cocompact. The quotient

$$(\overline{X}^{\bar{\theta}} - Th^{\bar{\theta}}(\Lambda_{\tau_{mod}}(\Gamma))/\Gamma$$

has a natural structure as a compact real-analytic orbifold with corners.

Remark 10.21. The starting point of our proof of Theorem 10.12, namely, the usage of the bundles E and E_{mod}, is similar to the one in [GW, Prop. 8.10 on pages 40-41]. However, we avoid the use of Poincaré duality and do not need homological assumptions on the space Σ. Unlike [GW], an essential ingredient in our proof is the map of triads (10.11), i.e. the existence of a continuous extension of the equivariant proper map $\tilde{f} : \tilde{R} \to \Omega$ to a map of compactifications.

10.4 Haïssinsky's conjecture for nonelementary torsion-free convergence groups

Theorem 10.22. *Let $\Gamma \curvearrowright \Sigma$ be a convergence group action of a torsion-free hyperbolic group on a metrizable compact space Σ, and suppose that $\Lambda \subset \Sigma$ is an invariant compact subset which is equivariantly homeomorphic to $\partial_\infty\Gamma$. Then the action $\Gamma \curvearrowright \Omega = \Sigma - \Lambda$ is cocompact provided that Ω has finitely many path connected components.*

Proof. After passing to a finite index subgroup of Γ preserving each connected component of Ω, it suffices to consider the case when Ω is path connected (and nonempty). If we had an equivariant map of triads (10.11), we would be done by Theorem 10.19. Below we modify the space Σ so that such a map of triads exists.

Pick a point $x \in \Omega$ and define the orbit map

$$f : \Gamma \to \Omega, \quad \gamma \mapsto \gamma x.$$

This map is injective since Γ is torsion-free and, hence, acts freely on Ω. Let $f_\infty : \partial_\infty\Gamma \to \Lambda$ be an (the) equivariant homeomorphism. We further let $\overline{\Gamma} = \Gamma \cup \partial_\infty\Gamma$ denote the Gromov compactification of Γ.

Lemma 10.23. *The extension of the map f by the map $f_\infty : \partial_\infty\Gamma \to \Lambda$,*

$$f' : \Gamma \cup \partial_\infty\Gamma \to \Sigma,$$

is an equivariant homeomorphism to $\Gamma x \cup \Lambda$.

Proof. We first note that the natural action $\Gamma \curvearrowright \overline{\Gamma}$ is a convergence action.

Suppose that (γ_n) is a sequence in Γ converging to $\xi \in \partial_\infty\Gamma$; $\lambda = f_\infty(\xi)$. We claim that

$$\lim_{n\to\infty} f(\gamma_n) = \lambda.$$

Case 1: Γ is nonelementary. Without loss of generality (in view of compactness of Σ and the convergence property of the action $\Gamma \curvearrowright \Sigma$), there exists $\lambda_- \in \Lambda$ such that the sequence $\gamma_n|_{\Sigma-\{\lambda_-\}}$ converges to some $\lambda_+ \in \Lambda$ uniformly on compacts. Since f_∞ is a homeomorphism, γ_n converges to $f_\infty^{-1}(\lambda_+)$ uniformly on compacts in $\partial_\infty\Gamma - f_\infty^{-1}(\lambda_-)$. The assumption that Γ is nonelementary implies that $\partial_\infty\Gamma - f_\infty^{-1}(\lambda_-)$ consists of more than one point. Therefore, in view of the convergence property for the action $\Gamma \curvearrowright \overline{\Gamma}$, it follows that γ_n converges to $f_\infty^{-1}(\lambda_+)$ on Γ (here we again pass to a subsequence if necessary). Hence, $\xi = f_\infty^{-1}(\lambda_+)$, $\lambda_+ = \lambda$ and the continuity of f' follows (cf. Lemma 2.4).

Case 2: Γ is elementary, i.e, $\Gamma \cong \mathbb{Z}$. Then Γ is generated by a single *loxodromic* homeomorphism $\gamma : \Sigma \to \Sigma$, i.e., $\Lambda = \{\lambda_+, \lambda_-\}$. Tukia proved [T1, Lemma 2D] that the sequence (γ^n) converges uniformly on compacts in $\Sigma - \{\lambda_-\}$ to λ_+, while the sequence (γ^{-n}) converges uniformly on compacts in $\Sigma - \{\lambda_+\}$ to λ_-. This implies continuity of the map f. □

We now amalgamate the spaces $\overline{\widetilde{R}}$ and Σ using the homeomorphism

$$f' : \overline{\Gamma} = \Gamma \cup \partial_\infty \Gamma \to \overline{\Gamma x} = \Gamma x \cup \Lambda,$$

where we identify Γ with the vertex set of the Rips complex $\widetilde{R}$. We denote by Σ' the result of the amalgamation. This space is metrizable, for instance, by Proposition 10.4. (This can be also easily proven more directly, as Σ' is Hausdorff, compact and 1st countable; hence, it is metrizable by Urysohn's metrization theorem.)

Since f' is Γ-equivariant, the topological action of Γ on $\overline{\widetilde{R}} \sqcup \Sigma$ descends to a topological action $\Gamma \curvearrowright \Sigma'$. This action is properly discontinuous on $\Omega' := \widetilde{R} \cup \Omega \subset \Sigma'$ as for each compact $C \subset \Omega'$, its intersections with $\widetilde{R}$ and Ω are both compact and the actions $\Gamma \curvearrowright \widetilde{R}, \Gamma \curvearrowright \Omega$ are properly discontinuous. Lastly, we note that, in view of connectivity of $\widetilde{R}$, since Ω is path connected, so is Ω'. We set $\Lambda' := \Lambda$.

The inclusion map $i : \overline{\widetilde{R}} \hookrightarrow \Sigma'$ is Γ-equivariant and its (co)restriction $\widetilde{R} \to \Omega'$ is clearly proper. Therefore, i yields an equivariant map of triads

$$\widetilde{f} : (\overline{\widetilde{R}}, \widetilde{R}, \partial_\infty \Gamma) \to (\Sigma', \Omega', \Lambda')$$

which is proper on $\widetilde{R}$ and restricts to an equivariant homeomorphism $\partial_\infty \Gamma \to \Lambda'$. Since the embedding $\Omega \to \Omega'$ is proper, Ω/Γ is compact if and only if Ω'/Γ is compact.

Remark 10.24. It is not hard to check that $\Gamma \curvearrowright \Sigma'$ is a convergence action, however, this is not needed for our proof.

With this modification, Theorem 10.19 implies that Ω'/Γ' is compact, hence, Ω/Γ is compact as well. ☐

11 Equivalent characterizations of τ_{mod}-RCA actions

We call an open subset $S \subset \partial_\infty^{\bar{\theta}} X$ *saturated* if it is a union of small strata X_ν.

We start with the following simple observation about Finsler convergence at infinity: If (x_n) and (y_n) are sequenes in X which are bounded distance apart (i.e. $d(x_n, y_n)$ is uniformly bounded) and $x_n \to [b], y_n \to [b'] \in \partial_\infty^{\bar{\theta}} X$, then the limit points $[b]$ and $[b']$ lie in the same small stratum X_ν, see Lemma 3.27. In particular, for each saturated open subset $S \subset \partial_\infty^{\bar{\theta}} X$,

$$[b] \in S \iff [b'] \in S$$

It follows that if $[b] \in S$, then the entire accumulation set

$$\mathrm{Acc}((B(x_n, R))) \subset \partial_\infty^{\bar{\theta}} X$$

is a compact subset of S.

Lemma 11.1. *Let $\Gamma < G$ be a discrete subgroup. If $S \subset \partial_\infty^{\bar\theta} X$ is a Γ-invariant saturated open subset such that Γ acts properly discontinuously on $X \sqcup S$, then each compact subset $C \subset X \sqcup S$ satisfies the following* uniform finiteness property*: There exists a function $f_C(R)$ such that for each ball $B(x,R) \subset X$ it holds that*

$$\mathrm{card}(\{\gamma \in \Gamma : \gamma C \cap B(x,R) \neq \varnothing\}) \leqslant f_C(R).$$

Proof. Suppose the contrary. Then there is a sequence of balls $B(x_i, R)$ intersecting C and a sequence $\gamma_i \to \infty$ in Γ such that also the balls $B(\gamma_i x_i, R)$ intersect C. We may assume that $x_i \to \bar x$ and $\gamma_i x_i \to \bar x'$ in $\overline{X}^{\bar\theta}$. By the observation preceeding the lemma, it holds that $\bar x, \bar x' \in X \cup S$. Since these points are dynamically related with respect to the Γ-action, we obtain a contradiction with proper discontinuity. ☐

The lemma leads to the following definition.

Definition 11.2. A discrete subgroup $\Gamma < G$ is *S-cocompact* if there exists a Γ-invariant saturated open subset $\Omega \subset \partial_\infty^{\bar\theta} X$ such that the action $\Gamma \curvearrowright X \sqcup \Omega$ is properly discontinuous and cocompact.

Note that each S-cocompact subgroup is necessarily finitely generated because it acts properly discontinuously and cocompactly on a connected manifold with boundary.

Theorem 11.3. *Each S-cocompact subgroup $\Gamma < G$ admits a Γ-equivariant coarse Lipschitz retraction $r : X \to \Gamma$. In particular, Γ is undistorted in G.*

Proof. Let $\Omega \subset \partial_\infty^{\bar\theta} X$ be as in the definition. Let $C \subset X \cup \Omega$ be a compact subset whose Γ-orbit covers the entire $X \cup \Omega$. We define the coarse retraction r first by sending each point $x \in X$ to the subset

$$r(x) := \{\gamma \in \Gamma : x \in \gamma C\} \subset \Gamma.$$

This subset is clearly finite because of the proper discontinuity of the Γ-action, and the assignment $x \mapsto r(x)$ is equivariant. According to Lemma 11.1, the cardinality of the subset

$$\{\gamma \in \Gamma : \gamma \in r(B(x,1))\} = \{\gamma \in \Gamma : B(x,1) \cap \gamma C \neq \varnothing\}$$

is bounded by $f_C(1)$, independently of x. It follows that r is coarse Lipschitz. ☐

We now apply the previous theorem to the cocompact domains of proper discontinuity obtained earlier by removing Finsler thickenings of limit sets.

In the regular case, we make the following observation regarding the antipodality condition:

Corollary 11.4. *Every uniformly regular conical subgroup $\Gamma < G$ is RCA. In other words, uniform regularity and conicality imply antipodality in the regular case.*

Proof. We choose $\bar\theta$ to be an almost root type as in Theorems 6.21 and 7.6 and conclude that the subgroup Γ is S-cocompact. By Theorem 11.3, Γ is undistorted in G. Hence, Γ is an URU subgroup of G. By [KLP3, Theorem 1.5], Γ is RCA. ☐

It seems unclear whether, without assuming uniformity, regularity and conicality still imply antipodality. Note that RCA implies uniform regularity, see [KLP2].

We now proceed to the general weakly regular case. Here, we need to assume antipodality. The next result relates conicality and S-cocompactness:

Theorem 11.5. *Suppose that $\Gamma < G$ is τ_{mod}-uniformly regular and antipodal. Then Γ is τ_{mod}-conical if and only if it is S-cocompact.*

Proof. The direction τ_{mod}-RCA $\Rightarrow$ S-cocompact is proven in Theorem 10.20. To prove the converse, note that each S-cocompact subgroup Γ is undistorted in G by Theorem 11.3. Hence, it is τ_{mod}-URU and therefore τ_{mod}-RCA by [KLP3, Thm. 1.5]. ☐

Remark 11.6. The proof shows that, without assuming antipodality, uniform τ_{mod}-regularity and S-cocompactness imply τ_{mod}-RCA. One may wonder whether the antipodality condition can be dropped altogether, as in the regular case. This would yield the implication uniformly τ_{mod}-RC$\Rightarrow$ τ_{mod}-RCA. Furthermore, since S-cocompactness is strictly stronger than undistortedness, one may ask whether each τ_{mod}-regular S-cocompact subgroup is τ_{mod}-uniformly regular.

We now can prove a converse to Theorem 10.20:

Corollary 11.7. *Suppose that $\Gamma < G$ is a uniformly τ_{mod}-regular discrete subgroup and that $Th \subset W$ is a $W_{\tau_{mod}}$-invariant balanced thickening. Then the following are equivalent:*

(i) The properly discontinuous action (see Theorem 9.18)

$$\Gamma \curvearrowright \overline{X}^{\bar\theta} - Th^{\bar\theta}(\Lambda_{\tau_{mod}}(\Gamma))$$

is cocompact.

(ii) There exists a Γ-invariant saturated open subset $\Omega \subset \partial^{\bar\theta}_{\infty} X$ such that the action

$$\Gamma \curvearrowright X \cup \Omega$$

is properly discontinuous and cocompact.

(iii) Γ is τ_{mod}-RCA.

Proof. (i)$\Rightarrow$(ii) is obvious.

(ii)$\Rightarrow$(iii): Γ is S-cocompact, hence τ_{mod}-URU, hence τ_{mod}-RCA.

(iii)$\Rightarrow$(i) is the content of Theorem 10.20. ☐

We are now ready to state the equivalence of a variety of conditions on discrete subgroups:

Theorem 11.8. *The following are equivalent for τ_{mod}-uniformly regular subgroups $\Gamma < G$:*

1. *Γ is a coarse equivariant retract.*
2. *Γ is a coarse retract.*
3. *Γ is undistorted in G, i.e. τ_{mod}-URU.*

4. Γ *is* τ_{mod}*-RCA.*

5. Γ *is* τ_{mod}*-asymptotically embedded.*

6. Γ *is* τ_{mod}*-Anosov.*

7. Γ *is S-cocompact.*

Proof. The implications 1⇒2⇒3 are immediate. The equivalence 3⇔4 is one of the main results of [KLP3], see Corollary 1.6 of that paper. The equivalences 4⇔5⇔6 are established in [KLP2]. The implication 5⇒7 is Theorem 10.20 of this paper, while the implication 7⇒1 is established in Theorem 11.3. ☐

In the regular case, this result can be strengthened to:

Theorem 11.9. *The following are equivalent for uniformly regular subgroups* $\Gamma < G$*:*

1. Γ *is RCA.*

2. Γ *is conical.*

3. The Finsler ideal boundary in $\partial_{\infty}^{\bar\theta} X$ *of each* $\bar\theta$*-Dirichlet domain* $D_o^{\bar\theta}$ *of the group* Γ *in* X *is contained in* $\Omega_{Th_{\bar\theta}}(\Gamma)$*, compare (6.20).*

4. Γ *is S-cocompact.*

Proof. The implication 1⇒2 is trivial. The converse is Corollary 11.4. The equivalence 4⇔1 holds in general in the weakly regular case. The implication 1⇒3 has been proven in Corollary 7.4. The implication 3⇒4 was how we proved cocompactness in Theorem 7.6. ☐

We note that this list of equivalences is nearly a perfect match to the list of equivalent definitions of convex cocompact subgroups of rank 1 Lie groups, except that convex-cocompactness is (by necessity) missing, see [KlL].

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