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ABSTRACT

In some current theories of elementary particle interactions, the elastic two-particle scattering amplitude on the second Riemann sheet of $\sqrt{s-4m^2}$, $\phi^{(2)}(s, \cos\theta)$, possesses singularities dense everywhere on the real negative s-axis for arbitrary complex $\cos\theta$. The real negative s-axis is therefore a natural boundary of $\phi^{(2)}(s,\cos\theta)$ for arbitrary $\cos\theta$, whereas the partial-wave amplitude $\phi_\ell^{(2)}(s)$ is known not to possess this natural boundary. Inspection of the integral defining $\phi_\ell^{(2)}(s)$ in terms of $\phi^{(2)}(s, \cos\theta)$ might lead one to expect $\phi_\ell^{(2)}(s)$ to possess the same singularities as $\phi^{(2)}(s, \cos\theta=\pm1)$. In the present work the integral defining $\phi_\ell^{(2)}(s)$ is examined carefully, and it is demonstrated here why $\phi_\ell^{(2)}(s)$ does not possess the "expected" singularities.

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I. INTRODUCTION

The two-particle-into-two-particle scattering amplitude $T(s,t)$ is generally written as a function of the two Lorentz invariants s and t, which are formed from the two initial and the two final 4-momentum vectors. In 1958, Mandelstam[1] proposed that the scattering amplitude $T(s,t)$ should be considered as the boundary value of an analytic function of two complex variables simultaneously regular over specified domains. He furthermore conjectured a representation for $T(s,t)$ in the form of a double Cauchy integral over the so-called double spectral functions. A physical theory, when described in the S-matrix language, is determined by the specific choice of the double spectral functions. The failure of a proof of the Mandelstam conjecture within the framework of local field theory has led Chew[2] and others to propose the adoption of the Mandelstam representation and the unitarity condition of the S Matrix as <u>the</u> theory of strong interaction physics. It should be remarked at this point that the Mandelstam representation is a conjecture of the analyticity properties of the scattering amplitude on the so-called physical Riemann sheet only; nothing is explicitly said about the complete structure of the scattering amplitude. By analytically continuing the unitarity condition of the partial-wave amplitude $\phi_\ell(s)$ across the elastic cut, it has recently become possible[3-6] to define a partial-wave scattering amplitude on the second Riemann sheet of $\sqrt{s - 4m^2}$ as $\phi_\ell^{(2)}(s)$, which is expressible entirely in terms of $\phi_\ell(s)$. Furthermore, the unitarity condition of the full amplitude relates the scattering amplitude $\phi^{(2)}(s, \cos\theta)$ on the second sheet to $\phi(s, \cos\theta)$ on the first through an integral equation. Zimmermann[6] utilized this integral equation to demonstrate that $\phi^{(2)}(s, \cos\theta)$ possesses infinitely many more singularities than $\phi(s, \cos\theta)$ does. Freund and Karplus[7] investigated the distribution of these infinitely many singularities and showed that $\phi^{(2)}(s, \cos\theta)$ has singularities that are dense everywhere along the real negative s-axis for arbitrary fixed $\cos\theta$; hence the real negative s-axis is a natural boundary of $\phi^{(2)}(s, \cos\theta)$. It is known by the crossing symmetry[4,8] that the partial wave $\phi_\ell^{(2)}(s)$ of $\phi^{(2)}(s, \cos\theta)$ does not possess a natural boundary on the real negative axis, because $\phi_\ell(s)$ does not; therefore the logical conclusion to be drawn is that the partial-wave integration somehow "erases" the natural boundary. The present study is devoted to the delineation of this phenomenon. It is demonstrated in this study that the reason is simply the following: the partial-wave amplitude $\phi_\ell^{(2)}(s)$ does not possess those endpoint singularities[9] [see eq.(2.23) below] that would otherwise constitute the natural boundary because a special property of the kernel function $H(\xi, \eta, z)$ of the above-mentioned integral equation [see eqs.(2.16) and (2.17)] enables the partial-wave projection integration to integrate out these singularities <u>point by point</u>. In Section II, a rather detailed description of the background material is provided for the sake of continuity and for proper perspective of this study. Section III is devoted to a careful demonstration of the reason why the partial-wave amplitude $\phi_{\ell}^{(2)}(s)$ is not singular at the "expected" endpoint singularities that would otherwise constitute a natural boundary along $-\infty \leq s \leq 0$. In Section IV the partial-wave integration of $\phi^{(2)}(s,z)$ on $P_{\ell}(z)$ is exactly carried out. Then some general remarks conclude this study. A few necessary mathematical formulae and identities are recorded in the Appendix A.

II. BACKGROUND MATERIAL[1-9]

This section, which closely follows Zimmermann's paper,[6] Eden's and others,[9] is devoted to a recapitulation of the background materials necessary for the present study and at the same time serves to establish some of the more important notation and formulae that will be employed in the report. The background materials fall roughly into five groups: (A) the definition of the scattering amplitude T, (B) the unitarity condition, (C) the analyticity property, ~~and~~ (D) the combined results of (A), (B), and, (C), and (E) the notion of endpoint singularities of definite integrals.

(A) The definition of the scattering amplitude T. We choose to deal with a pair theory in which two particle states are only coupled to states with an even number of particles. The scattering amplitude which is a function of the two Lorentz invariants s and t is designated by T(s,t) and is defined by

$$\delta(k_1+k_2-k_1'-k_2')\ T(s,\ t) = T(k_1k_2|k_1'k_2') = -i(\phi_{k_1k_2}{}^{in},\ (S-1)\phi_{k_1'k_2'}{}^{in}) \tag{2.1}$$

where

$$\begin{cases} k_1,\ k_2 \text{ are the incident particle 4-momenta,} \\ k_1',\ k_2' \text{ are the outgoing particle 4-momenta,} \end{cases}$$

and

$$\begin{cases} s = (k_1 + k_2)^2 = (k_1 + k_2)_o^2 - (\underline{k}_1 + \underline{k}_2)^2 \\ t = (k_1 - k_1')^2 \end{cases} \tag{2.2}$$

The symbol S is the Heisenberg S matrix. The scattering amplitude $T(s,t)$, which will occasionally be written as $T(s, \cos\theta)$, is defined over the physical interval

$$4m^2 \leq s, \quad 0 \leq -t \leq s-4m^2 .$$

(B) The unitarity condition. The conservation of probability requires the Heisenberg S matrix to be unitary. Hence the scattering amplitude T (2.1), in the elastic scattering region, satisfies

$$\mathrm{Im}\, T(k_1 k_2 | k_1' k_2') = \frac{1}{2} \int D\ell_1 \, D\ell_2 \, T(k_1 k_2 | \ell_1 \ell_2) \, T^*(\ell_1 \ell_2 | k_1' \, k_2'),$$

$$4m^2 \leq s < 16m^2$$

$$D\ell = d^4\ell \, \theta(\ell) \, \delta(\ell^2 - m^2) . \qquad (2.3)$$

By using the definition of $T(k_1 k_2 | k_1' \, k_2')$ in (2.1), six of the eight integrations can be immediately carried out on account of the δ-functions, and (2.3) simplifies to

$$\mathrm{Im}\, T(s, \cos\theta) = \frac{1}{2i} \left[T(s, \cos\theta) - T^*(s, \cos\theta) \right]$$

$$= \frac{1}{8} \frac{\sqrt{s-4m^2}}{\sqrt{s}} \int_{-1}^{1} d\xi \int_{-1}^{1} d\eta \; T(s, \xi) \, \frac{\theta(-K(\xi, \eta, \cos\theta))}{\sqrt{-K(\xi, \eta, \cos\theta)}} \times$$

$$\times \, T^*(s, \eta) , \qquad 4m^2 \leq s < 16m^2 , \qquad (2.4)$$

where

$$K(\xi, \eta, z) = \xi^2 + \eta^2 + z^2 - 2\xi\eta z - 1, \quad -1 \leq z \leq +1.$$

$T(s, \cos\theta)$ is said to be partial-wave decomposed into $T_\ell(s)$ if

$$T_\ell(s) = \frac{1}{2}\int_{-1}^{1} T(s, z)\, P_\ell(z)\, dz.$$

Partial-wave decomposition of the unitarity condition for the full amplitude $T(s, \cos\theta)$ (2.4) results in a unitarity condition for the partial-wave amplitude $T_\ell(s)$, which can be written, with the help of Lemma 2, of the Appendix A,

$$\text{Im}\, T_\ell(s) = \frac{\pi}{4}\,\frac{\sqrt{s-4m^2}}{\sqrt{s}}\, T_\ell(s)\, T_\ell^*(s), \quad 4m^2 \leq s < 16m^2 . \qquad (2.5)$$

It should be remarked here that the unitarity condition for the full amplitude $T(s, t)$ as in (2.4) is an integral equation relating T and T^*, whereas for the partial-wave amplitude $T_\ell(s)$ as in (2.5) it is an algebraic relation between $T_\ell(s)$ and $T_\ell^*(s)$. These two facts, in conjunction with the analyticity of $T(s, t)$ and $T_\ell(s)$ in postulate (C) below, are of paramount importance. The latter permits the unique determination of the analytic structure of $T_\ell(s)$ at $s = 4m^2$ to be $F_\ell(s) + i\sqrt{s-4m^2}\, G_\ell(s)$, where $F_\ell(s)$ and $G_\ell(s)$ are analytic for $0 \lessapprox s < 16m^2$; hence the possibility of analytic continuation from the first sheet into the second sheet, across the elastic region. The former permits the determination of the singularities of $T^{(2)}(s, \cos\theta)$ when suitably defined on the second Riemann sheet of $\sqrt{s-4m^2}$.

(C) Analyticity. The analyticity of the scattering amplitude $T(s, t)$ is provided by the Mandelstam conjecture, which asserts that $T(s, t)$ is the boundary value

$$T(s,t) = \lim_{\epsilon,\epsilon'\to 0^+} \phi(s+i\epsilon,\ t+i\epsilon'),\quad s \geq 4m^2,\quad 0 \leq -t \leq s-4m^2$$

of an analytic function of both complex variables s and t regular everywhere except for the cuts

$$s \geq 4m^2, \qquad t \geq 4m^2, \qquad u = 4m^2 - s - t \geq 4m^2.$$

The full amplitude $\phi(s, \cos\theta)$ is even in $\cos\theta$ because we are dealing with a pair theory of one kind of particles only. In terms of the partial wave $\phi_\ell(s)$, Mandelstam's conjecture asserts that $T_\ell(s)$ is the boundary value

$$T_\ell(s) = \lim_{\epsilon\to 0^+} \phi_\ell(s + i\epsilon)$$

of an analytic function of the complex variable s regular everywhere except for the cuts

$$s \leq 0, \qquad s \geq 4m^2.$$

The partial-wave amplitude $\phi_\ell(s)$ is a real analytic function, satisfying

$$\lim_{\epsilon\to 0^+} \phi_\ell(s + i\epsilon) = \left\{\lim_{\epsilon\to 0^+} \phi_\ell(s - i\epsilon)\right\}^*, \quad 4m^2 < s < 16m^2. \qquad (2.6)$$

Or in words, $\phi_\ell(s)$ is real on the segment $0 < s < 4m^2$ of the real axis, and by Schwartz' reflection principle takes on complex conjugate values at complex conjugate s points. The discontinuity across any point on the real axis is given by twice the imaginary part of $\phi_\ell(s)$ at that point. For a fixed complex s, not on the cut $s \leq 0$, $\phi(s, \cos\theta)$ can be represented by the Legendre series

$$\phi(s, \cos\theta) = \sum_{\ell=0}^{\infty} (2\ell + 1)\, \phi_\ell(s)\, P_\ell(\cos\theta)$$

which converges in the $\cos\theta$-plane inside an ellipse through the points

$$z_1 = \pm\left(1 + \frac{8m^2}{s - 4m^2}\right) = \pm\,\frac{s + 4m^2}{s - 4m^2} ,$$

and with foci at ± 1. This is quite easily seen from the denominators in the following form of the Mandelstam representation, (the pair theory)

$$\phi(s,z) = \frac{1}{\pi}\,\frac{2}{s-4m^2}\int_{4m^2}^{\infty}\frac{dt'A_2(s,t')}{\left(1+\frac{2t'}{s-4m^2}\right) - z} + \frac{1}{\pi}\,\frac{2}{s-4m^2}\int_{4m^2}^{\infty}\frac{du'A_3(s,u')}{\left(1+\frac{2u'}{s-4m^2}\right) + z}$$

$$\equiv \phi^{(+)}(s,z) + \phi^{(-)}(s,z) ,$$

where

$$\phi^{(+)}(s,z) = \phi^{(-)}(s,-z). \qquad (2.7)$$

The last equality follows from the fact that the physical t-channel and the physical u-channel are equivalent, $A_2(s,t) = A_3(s,t)$.

(D) The combined results of (A), (B), and (C). The combined results of (A), (B), and (C) are the possibility of the analytic continuation of $\phi_\ell(s)$ in s across the boundary of the physical sheet through the elastic interval $4m^2 \leq s < 16m^2$ and the determination of the singularities of $\phi(s, \cos\theta)$ in the second Riemann sheet of $\sqrt{s-4m^2}$. We start out by arbitrarily defining a function

$$\phi^{irr}(s) = \frac{\phi_\ell(s)}{1 + i\rho(s)\phi_\ell(s)} \tag{2.8}$$

where

$$\rho(s) = \frac{\pi}{4}\frac{\sqrt{s - 4m^2}}{\sqrt{s}}$$

is defined by the following two cuts in the complex s plane:

$$-\infty \leq s \leq 0\ , \quad 4m^2 \leq s \leq \infty\ , \quad \mathrm{Im}\ s = 0,$$

and

$$\rho(s + i0) = \text{real positive for } s \geq 4m^2\ .$$

Then $\rho(s + i0)^* = -\rho(s - i0)$ and (2.6) require $\phi_\ell{}^{irr}(s)$, Eq.(2.8), to satisfy the condition,

$$\left\{\lim_{\epsilon\to 0^+} \phi_\ell{}^{irr}(s - i\epsilon)\right\}^* = \lim_{\epsilon\to 0^+} \phi_\ell{}^{irr}(s + i\epsilon).$$

Furthermore, by the partial-wave unitarity condition (2.5), the imaginary part of $\phi_\ell^{irr}(s)$ in the elastic scattering region vanishes identically,

$$\mathrm{Im}\, \phi_\ell^{irr}(s) = 0, \quad 4m^2 \leq s < 16m^2 ;$$

hence $\phi_\ell^{irr}(s)$ is regular at $s = 4m^2$.

By solving Eq. (2.8), $\phi_\ell(s)$ is found to be

$$\phi_\ell(s) = \frac{\phi_\ell^{irr}(s)}{1-i\rho(s)\phi_\ell^{irr}(s)} = F_\ell(s) + i\rho(s)\, G_\ell(s); \; 4m^2 \leq s < 16m^2. \tag{2.9}$$

where

$$F_\ell(s) = \frac{\phi_\ell^{irr}(s)}{1 + \rho^2(s)\, \phi_\ell^{irr}(s)^2} \tag{2.10}$$

$$G_\ell(s) = \frac{\phi_\ell^{irr}(s)^2}{1 + \rho^2(s)\, \phi_\ell^{irr}(s)^2} \tag{2.11}$$

Equation (2.9) exhibits clearly the fact that $\phi_\ell(s)$ has only a $\sqrt{s-4m^2}$ singularity at $s = 4m^2$, that connects the two Riemann sheets of $\phi_\ell(s)$. The analytic continuation of the partial-wave $\phi_\ell(s)$ across the elastic cut $4m^2 \leq s < 16m^2$ onto the second Riemann sheet of $\sqrt{s-4m^2}$ is designated by $\phi_\ell^{(2)}(s)$, and is given by (2.9) as

$$\phi_\ell^{(2)}(s) = \phi_\ell^*(s) = F_\ell(s) - i\rho(s)\, G_\ell(s)$$

$$= \phi_\ell(s) - 2i\rho(s)\, G_\ell(s), \quad 4m^2 \le s < 16m^2 . \qquad (2.12)$$

Multiplying (2.12) by $(2\ell + 1)\, P_\ell(\cos\theta)$ and summing over all ℓ, we obtain

$$\phi^{(2)}(s, \cos\theta) = \phi^*(s, \cos\theta) = F(s, \cos\theta) - i\rho(s)\, G(s, \cos\theta)$$

$$= \phi(s, \cos\theta) - 2i\rho(s)\, G(s, \cos\theta), \qquad (1.13)$$

where

$$G(s,\cos\theta) = \sum_{\ell=0}^{\infty} (2\ell + 1)\, G_\ell(s)\, P_\ell(\cos\theta).$$

The Legendre series of $G(s, \cos\theta)$ converges in general in a larger ellipse than that of $\phi(s,\cos\theta)$ because it is the imaginary part of $\phi(s, \cos\theta)$. Hence the Legendre series $\phi^{(2)}(s, \cos\theta)$[6] (2.13) has the same ellipse of convergence in $\cos\theta$ as that of $\phi(s, \cos\theta)$ for arbitrary fixed s, provided that s does not assume any of the isolated points at which $G_\ell(s)$ has a pole for some ℓ. The importance of Eqs. (2.12) and (2.13) lie in the fact that $\phi_\ell^*(s)$ and $\phi^*(s,\cos\theta)$ $(4m^2 \le s < 16m^2,\ -1 \le \cos\theta \le +1)$ have been identified as the boundary values of $\phi_\ell^{(2)}(s)$ and $\phi^{(2)}(s, \cos\theta)$ respectively as s approaches the real axis from above on the <u>second</u> Riemann sheet of $\sqrt{s-4m^2}$. When these identifications are fed back into the unitarity condition (2.4) and (2.5), an integral equation and an algebraic equation relating ϕ and $\phi^{(2)}$ emerge:

$$\phi(s,\cos\theta) - \phi^{(2)}(s,\cos\theta)$$

$$= \frac{i}{\pi}\rho(s)\int_{-1}^{1} d\xi \int_{-1}^{1} d\eta\, \phi(s,\xi)\, \phi^{(2)}(s,\eta)\, \frac{\theta(-K(\xi,\eta,\cos\theta))}{\sqrt{-K(\xi,\eta,\cos\theta)}} \;;$$

$$4m^2 \leq s < 16m^2 . \qquad (2.14)$$

$$\phi_\ell^{(2)}(s) = \frac{\phi_\ell(s)}{1 + 2i\rho(s)\,\phi_\ell(s)} \;; \quad 4m^2 \leq s < 16m^2. \qquad (2.15)$$

The analytic continuation of $\phi^{(2)}(s,\cos\theta)$ and $\phi_\ell^{(2)}(s)$ are obtained by the analytic continuation of $\phi(s,\cos\theta)$ and $\phi_\ell(s)$ away from the physical region. For a given $\phi(s,\cos\theta)$, the integral equation (2.14) in principle yields $\phi^{(2)}(s,\cos\theta)$. In summary, it is the analytic continuation of the unitarity condition that enables one to obtain information of the scattering amplitude on the second Riemann sheet of $\sqrt{s-4m^2}$.

Zimmermann[6] recasts Eq. (2.14) into a slightly more convenient form for the investigation of the singularities of $\phi^{(2)}(s,\cos\theta)$,

$$\phi^{(2)}(s,z) = \phi(s,z) - \frac{i}{4\pi^3}\rho(s) \oint_C d\xi \oint_C d\eta\, \phi(s,\xi)\, \phi^{(2)}(s,\eta) H(\xi,\eta,z) \qquad (2.16)$$

where C is an ellipse encircling ± 1, and

$$H(\xi,\eta,z) \equiv \int_{-1}^{1} \frac{d\xi'}{\xi'-\xi} \int_{-1}^{1} \frac{d\eta'}{\eta'-\eta} \frac{\theta\left[-K(\xi',\eta',z)\right]}{\sqrt{-K(\xi',\eta',z\ )}}$$

$$= \frac{-\pi}{\sqrt{K(\xi,\eta,z)}} \ \ell n \left[\frac{z-\xi\eta + \sqrt{K(\xi,\eta,z)}}{z-\xi\eta - \sqrt{K(\xi,\eta,z)}} \right], \quad -1 \leq z \leq +1,$$

(2.17)

with the condition

$$H(\xi,\eta,z) = \text{real positive at } z=1,\ \xi,\eta > 1. \qquad (2.18)$$

He deduces the following set of properties of $\phi^{(2)}(s, \cos\theta)$: $\phi^{(2)}(s, \cos\theta)$ is analytic in s and $\cos\theta$ except for

(a) the normal cuts

(1) $s \geq 4m^2$

(2) $\cos\theta = \pm \left(1 + \dfrac{2t}{s - 4m^2}\right)$, $t \geq 4m^2$,

for s lying on the cut (1) the boundary values of $\phi^{(2)}(s, \cos\theta)$ are still analytic in $\cos\theta$ except for the cuts (2)

(b) the cut $s \leq 0$

(c) poles

(d) the domains $D^{\pm}_n$, defined by

$$D^{\pm}_n : \cos\theta = \pm \cosh\left[\gamma(s,t_1) + \dots + \gamma(s,t_n)\right]; \quad t_i \geq 4m^2, \; n = 2, 3 \dots$$

$$\cosh\gamma(s, t_i) = 1 + \frac{2t_i}{s - 4m^2} \quad .$$

In this paper we are only concerned with the cut (b) $s \leq 0$ and the apexes of $D^{\pm}_n$ (i.e. $D^{\pm}_n$ evaluated at $t_i = 4m^2$),

$$\cos\theta = \pm\cosh\left[n\gamma(s)\right], \quad n = 1, 2, 3, \dots$$

$$\gamma(s) = \cosh^{-1}\left(1 + \frac{8m^2}{s-4m^2}\right) \quad . \tag{2.19}$$

For a fixed complex s, not on the cut $s \leq 0$, (2.19) gives a partial distribution of the singularities of $\phi^{(2)}(s, \cos\theta)$ at the $\cos\theta$-plane.

Freund and Karplus[7] invert (2.19) and obtain, for an arbitrary but fixed complex θ, the singularities in s (or equivalently in $\sigma \equiv 1 + \frac{8m^2}{s-4m^2}$):

$$\sigma_{n,k} \equiv 1 + \frac{8m^2}{s_{n,k} - 4m^2} = \cos\left[\frac{\theta}{n} + k\frac{\pi}{n}\right], \begin{cases} n=1, 2, 3, \dots \\ k=0, 1, \dots, 2n-1. \end{cases} \tag{2.20}$$

Now it is quite obvious that the interval $-1 \leq \sigma \leq +1$ (or equivalently the interval $-\infty \leq s \leq 0$) is dense with singularities everywhere; hence it is a natural boundary of $\phi^{(2)}(s, \cos\theta)$.

From crossing symmetry, it has been shown[8] [see formula (IV.7) of Ref. 8] that the imaginary part of $\phi_\ell(s)$, $\mathrm{Im}\phi_\ell(s)$, for $-32m^2 < s < 0$ can be expressed in terms of Legendre polynomials P_ℓ's and $\mathrm{Im}\,\phi_\ell(s)$ for s <u>physical</u> ,

$$\mathrm{Im}\,\phi_\ell(\nu) \simeq \frac{2}{\nu}\int_0^{-(m^2+\nu)} d\nu'\, P_\ell\left(1 + 2\frac{m^2+\nu'}{\nu}\right) \sum_{\ell'=0}^{\infty} (2\ell'+1) \times$$

$$\times\, \mathrm{Im}\,\phi_\ell(\nu')\, P_\ell\left[1 + 2\,\frac{m^2+\nu}{\nu'}\right]$$

where ν is restricted by

$$-9m^2 < \nu < -m^2 , \qquad s \equiv 4(\nu + m^2) .$$

Since $\mathrm{Im}\,\phi_\ell(s)$ is analytic in $4m^2 \leq s < 16m^2$ (or $0 \leq \nu < 3m^2$) [see Eq. (2.9)], ~~therefore~~ the above equation implies that $\mathrm{Im}\,\phi_\ell(s)$ is analytic in $-12m^2 < s < 0$ (or $-4m^2 < \nu < -m^2$) . By writing a Cauchy integral representation for $\phi_\ell(s)$,

$$\phi_\ell(s) \simeq \int_0^{-12m^2} ds'\, \frac{\mathrm{Im}\,\phi_\ell(s')}{s'-s} + R(s), \qquad -12m^2 < s < 0,$$

where $R(s)$ is analytic at $-12m^2 < s < 0$, one easily deduces by analytically deforming the contour of integration that $\phi_\ell(s)$ is analytic in $-12m^2 < s < 0$ because $\text{Im}\,\phi_\ell(s)$ is analytic there. Being a rational function of $\phi_\ell(s)$, $\phi_\ell^{(2)}(s)$ cannot have a natural boundary <u>at least</u> along $-12m^2 < s < 0$ (or $-1 < \sigma < \frac{1}{2}$). Although it has not been demonstrated here that $\phi_\ell(\sigma)$ doesnot have a natural boundary along $\frac{1}{2} < \sigma < 1$. But the indications are that it doesnot; whether it does or not we already have a phenomenon to explain. So from now on, for simplicity, we talk as if $\phi_\ell^{(2)}(s)$ doesnot possess a natural boundary along $-1 \leq \sigma \leq 1$. Therefore, it is of interest to ascertain the reason for the absence of the natural boundary when $\phi^{(2)}(s, \cos\theta)$ is projected on $P_\ell(\cos\theta)$. The presentation of this reason has to be temporarily postponed until after the notion of endpoint singularities of integral transform is introduced in the next paragraph.

(E) The notion of endpoint singularities of definite integrals.[9] This notion can be simply illustrated by the following definite integral:

$$w(c) = \int_{-1}^{1} f(c,z)\,dz \equiv -2\int_{-1}^{1} \frac{dz}{\sqrt{c-z}} = \sqrt{c-1} - \sqrt{c+1}\,. \qquad (2.21)$$

The integrand f(c,z) as a function of two complex variables is singular at $c = z$ for an arbitrary fixed z. Since the integration of z is over the closed interval $[-1, +1]$, it is conceivable that w(c) be singular at every point of the closed interval $[-1, +1]$. But Eq. (2.21) demonstrates that this is not the case, w(c) is singular only at $c = \pm 1$. These two points of singularities in c coincide with the singularities of the integrand $f(c, z = \pm 1)$ <u>evaluated</u> at the endpoints of integration. This phenomenon is summed up in Lemma 1A of Tarski paper[9] as follows :

Lemma 1A. Let an arc A be given in the complex z plane as a contour of integration, let N denote a neighborhood of the contour A, and let D be a domain in the complex c plane. Let f(z,c) be regular in either variable, except for a finite number of isolated singularities or branch points, for any value of the other variable, when $z \epsilon N$, $c \epsilon D$. (We have to include the possibility that the domains D and N extend over more than one Riemann sheet of f.) Then

$$w(c) = \int_A f(z,c)dz$$

can be singular at $c = c_0 \epsilon D$ only if one of the following two conditions holds:

(1) $f(z,c_0)$ as a function of z has a singularity at an endpoint of the contour, or

(2) for c_1 in a neighborhood of c_0, $f(z,c_1)$ is singular at $z = z_0 + \eta_1$ and at $z = z_0 - \eta_2$,

where $z_0 + \eta_1$ and $z_0 - \eta_2$ lie on opposite sides of the contour A. z_0 is a point of the contour, and $\eta_1, \eta_2 \to 0$ as $c_1 \to c_0$.

Since the condition (2) of the Lemma 1A is not encountered in our study, we do not wish to discuss it here.

Applying the notion of endpoint singularities of the preceding paragraph ($\mathcal{E}$) to the singularities of $\phi^{(2)}(\sigma, \cos\theta)$ as enumerated in Eq. (2.20), we expect $\phi_\ell^{(2)}(\sigma)$

$$\phi_{\ell}^{(2)}(\sigma) = \frac{1}{2}\int_{-1}^{1} \phi^{(2)}(\sigma, z)\, P_{\ell}(z)dz \tag{2.22}$$

to have endpoint singularities at (i.e. the singularities of $\phi^{(2)}(\sigma,\ z = \pm 1)$)

$$\sigma_{n,k}(\theta = 0) = \sigma_{n,k}(\theta = \pi) = \cos\left(k\,\frac{\pi}{n} \right),\ \begin{cases} n=1,2,3,\ldots \\ k=0,1,\ldots,2n-1. \end{cases}, \tag{2.23}$$

which is still dense everywhere in σ, $-1 \le \sigma \le 1$ (or $-\infty \le s \le 0$). Therefore $\phi_{\ell}^{(2)}(\sigma)$ is expected to have a natural boundary there. It is to be shown in the (next) Section III, that due to special properties of the function $H(\xi, \eta, z)$, (2.17) the "expected" endpoint singularities of (2.23) in σ along $-1 \le \sigma \le +1$ disappear during the partial-wave integration. Hence the partial amplitude $\phi_{\ell}^{(2)}(\sigma)$ does not possess a natural boundary there.

III. THE ABSENCE OF THE NATURAL BOUNDARY OF $\phi_\ell^{(2)}(\sigma)$

We have seen that the partial-wave amplitude $\phi_\ell^{(2)}(\sigma)$ does not possess a natural boundary for $-1 \leq \sigma \leq +1$; nevertheless, its defining integral does seem to have sufficiently many endpoint singularities (2.23) along $-1 \leq \sigma \leq +1$ so as to constitute a natural boundary there. The aim of this section is to work out in detail the resolution of this seemingly incompatible situation. The explanation turns out to be as follows: the kernel function $H(\xi,\eta,z)$ [see (3.9) and (2.16)], for suitable values of ξ and η, has certain special properties at $z = \pm 1$ such that when it is integrated in the neighborhoods of $z = \pm 1$ and the resulting integral is evaluated exactly at $z = \pm 1$, the resulting integral is an analytic function of ξ and η. This remarkable property of $H(\xi,\eta,z)$ enables us to demonstrate inductively how the "expected" endpoint singularities (2.23) except for $\sigma = \pm 1$, are integrated out point by point; hence the absence of the natural boundary for $\phi_\ell^{(2)}(\sigma)$ along $-1 \leq \sigma \leq +1$ as concluded from crossing symmetry. Since we are going to treat the infinite set of singularities (2.23) point by point, it is cogent to introduce first a classification scheme that pinpoints the origin of the singularities of the full amplitude $\phi^{(2)}(\sigma,z)$. Probably the most natural classification scheme for the singularities (2.19) and (2.20) of $\phi^{(2)}(\sigma,z)$ is the Liouville series, and this scheme is introduced in Subsection (III.A). The n^{th} term of the Liouville series $\phi^{(2)}(s,z)$ is singular only at the 2n (a finite number) points of (2.20) corresponding to the integer n. Consequently, the partial-wave projection of the n^{th} term of the Liouville series contributes to the partial-wave $\phi_\ell^{(2)}(\sigma)$ (2.22) a finite number of the "expected" endpoint singularities of (2.23) corresponding to the integer n. Also discussed in Subsection(III.A), are some detailed properties of the kernel function $H(\xi,\eta,z)$. In subsection (III.B), we employ the material of (III.A) to show that the partial-wave projection of the second Liouville term is not singular at two of the four "expected" endpoint singularities. In (III.C) the argument of (III.B) is repeated on the third term of the Liouville series to bring out its generalization to the arbitrary n^{th} term. In (III.D), we demonstrate that the partial-wave projection of the n^{th} term of the Liouville series is singular only at $\sigma = \pm 1$, but regular at the "expected" endpoint singularities of (2.23) corresponding to the integer n, $\sigma = \cos(\frac{\pi}{n}k)$, $k \neq 0,n$. Hence the partial-wave amplitude $\phi_\ell^{(2)}(\sigma)$ (2.22) does not possess the "expected" natural boundary in $-1 \leq \sigma \leq +1$. [$\phi_\ell^{(2)}(\sigma)$ is singular at $\sigma = \pm 1$.]

III.A. The "Expected" Endpoint Singularities of $\phi_\ell^{(2)}(\sigma)$ and the Kernel Function $H(\xi,\eta,z)$

A classification scheme for the singularities of the amplitudes $\phi^{(2)}(\sigma,z)$ will be given presently. As remarked before this report is only concerned with the following set of singularities of $\phi^{(2)}(s,z)$: [see (2.19) and (2.20)]

$$\begin{cases} \text{for a fixed } \sigma(s) \equiv 1 + \dfrac{8m^2}{s - 4m^2}, & z \equiv \cos\theta = \pm\cos n \cos^{-1}\sigma, \\ & n = 1,2,\ldots \quad \text{or} \\ \text{for a fixed (complex) } \theta, & \sigma_{n,k}(\theta) = \cos\left[\dfrac{\theta}{n} + \dfrac{\pi}{n}k\right] \\ & n = 1,2,\ldots \\ & k = 0,1,\ldots 2n-1. \end{cases} \tag{3.1}$$

Zimmermann derives[6] this set (3.1) of singularities by an iterative procedure, therefore it is most natural to classify this same set by the Liouville series of $\phi^{(2)}(\sigma, z)$ which is obtained by iterating the integral equation (2.16):

$$\phi^{(2)}(\sigma,z) = \phi(\sigma,z) + \rho_0(s) \oint_{C_\xi} d\xi \oint_{C_\eta} d\eta \; \phi(\sigma,\xi) \; H(\xi,\eta,z) \; \phi(\sigma,\eta) + \cdots$$

$$\equiv \sum_{n=1}^{\infty} \left[\rho_0(s)\right]^{n-1} M_n(\sigma,z), \left(M_1(\sigma,z) \equiv \phi(\sigma,z), \rho_0(s) \equiv \frac{2i\rho(s)}{(2\pi)^3}\right)$$

$$= \phi(\sigma,z) + \ldots + (\rho_0)^n \oint_{C_\xi} d\xi \oint_{C_\eta} d\eta \, \phi(\sigma,\xi) H(\xi,\eta,z) M_{n-1}(\sigma,\eta) + \ldots \quad (3.2)$$

The n^{th} term of the Liouville series (3.2) is singular only at the finite set of singular points of (3.1) corresponding to the integer n. The validity of this statement is simply illustrated by the second term $M_2(\sigma,z)$ of the Liouville series. For a fixed σ , $\phi(\sigma,\xi)$ and $\phi(\sigma,\eta)$ are singular at $\xi = \pm\sigma$ and $\eta = \pm\sigma$ respectively, therefore the contours of integration C_ξ and C_η can not be analytically deformed about these points respectively. Next the kernel function $H(\xi,\eta,z)$ [see detailed properties of $H(\xi,\eta,z)$ below] combines these four pairs of points, $(\xi,\eta) = (\pm\sigma, \pm\sigma)$ into two distinct points in $z = \pm\cos\left(2\cos^{-1}\sigma\right)$ which is (3.1) for n = 2. The singularities of all other terms of the Liouville series (3.2) can be similarly built up. At this point it is advisable to make a momentary detour and enumerate some relevant properties of the kernel function $H(\xi,\eta,z)$

so that their repeated use from now on may be made easier.

For computations carried out in this study, we recast the function $H(\xi,\eta,z)$ (2.17) into a more suitable form, whose multi-valuedness is enumerated by the <u>arbitrary</u> negative as well as positive integer n,

$$H(\xi,\eta,z) = \frac{2\pi}{\sqrt{K_+}\sqrt{K_-}}\left[\ln\frac{\sqrt{K_+}+\sqrt{K_-}}{\sqrt{K_+}-\sqrt{K_-}} + 2\pi i(n)\right];$$

(n=0 branch for (2.16))

where

$$\begin{cases} K_+ \equiv z - \left[\xi\eta - \sqrt{1-\xi^2}\sqrt{1-\eta^2}\right] \equiv z - \cos(\alpha+\beta) \\ K_- \equiv z - \left[\xi\eta + \sqrt{1-\xi^2}\sqrt{1-\eta^2}\right] \equiv z - \cos(\alpha-\beta) \end{cases}$$

$$\begin{cases} \xi \equiv \cos\alpha \\ \eta \equiv \cos\beta, \end{cases} \quad \alpha,\beta \text{ may be } \underline{\text{complex}} \tag{3.3}$$

The choice of cosine instead of the hyperbolic-cosine for the variables ξ and η is motivated by the fact that we are later mainly concerned with ξ- or η-values in neighborhoods of the real line segment $[-1,+1]$ because the natural boundary $-\infty \leq s \leq 0$ is mapped onto $-1 \leq \sigma(s) \leq +1$. All branches (except one) of the function $H(\xi,\eta,z)$ (3.3) associated with all arbitrary integers n (except $n = 0$) possess double-valued (square-root type) branch points at $z = \cos(\alpha+\beta)$, $\cos(\alpha-\beta)$, and an infinitely many-valued (logarithmic type) branch point at $z = \infty$. The appropriate branch of the function $H(\xi,\eta,z)$ (3.3) for the unitarity relation or the integral equation (2.16) is the one corresponding to $n = 0$, which is <u>regular</u> at $z = \cos(\alpha - \beta)$.

One important property of the Kernel function $H(\xi, \eta, z)$ to be recorded here is that the kernel function $H(\xi, \eta, z)$ is singular at $z = \cos(\alpha + \beta)$ for a given pair of values, $\xi \equiv \cos\alpha$, and $\eta \equiv \cos\beta$ (this is called the "addition" property). Or stated differently, if we designate an ellipse of semi-major axis cosh b, $b > 1$, by $E(b)$, then for fixed values of ξ and η such that

$$\begin{cases} \xi \equiv \cos\alpha \text{ is on or outside of } E(a) \quad a = |\mathrm{Im}\,\alpha| > 1 \\ \eta \equiv \cos\beta \text{ is on or outside of } E(b) \quad b = |\mathrm{Im}\,\beta| > 1, \end{cases}$$

then the function $H(\cos\alpha, \cos\beta, z)$ of (3.3) is regular in z where

$$z \text{ is inside of } E(a + b), \tag{3.4}$$

because K_+ of (3.3) cannot vanish. This "addition" property of $H(\xi, \eta, z)$ is of great importance to (this) Section III.

We record another important property of the function $H(\xi, \eta, z)$ in this paragraph. Let a neighborhood $N_z(z_0)$ of a point z_0 be defined as the following open set;

$$N_z(z_0) \equiv \left\{ z \;\middle|\; |z - z_0| < \epsilon, \epsilon \text{ a small positive number} \right\}. \tag{3.5}$$

Then if the variables z, ξ and η are restricted to the following set of neighborhoods;

$$\begin{cases} z \equiv \cos\theta & \text{in} \quad N_z(\cos(\alpha_0+\beta_0)) \\ \xi \equiv \cos\alpha & \text{in} \quad N_\xi(\cos\alpha_0) \\ \eta \equiv \cos\beta & \text{in} \quad N_\eta(\cos\beta_0) \end{cases},$$

such that

$$\begin{cases} \cos(\alpha_0+\beta_0) \approx 1 \\ \cos(\alpha_0-\beta_0) \neq 1 \end{cases}, \tag{3.6}$$

the kernel function $H(\xi,\eta,z)$ of (3.3) can be approximated as,

$$H(\xi,\eta,z) \approx \frac{1}{\sqrt{z-\cos(\alpha+\beta)}} . \tag{3.7}$$

<u>In particular</u>, if $\alpha_0+\beta_0 = 2\pi$ and $\alpha_0-\beta_0 \neq 0$ or 2π, then the indefinite integral of the approximation (3.7), evaluated at $z = 1$, is a <u>remarkable</u> function of the variables ξ and η ;

$$G(\alpha,\beta) \equiv \left[\int dz\; H(\xi,\eta,z)\right]_{z=1} \approx \int \frac{dz}{\sqrt{z-\cos(\alpha+\beta)}}\Bigg|_{z=1} \approx \sqrt{1-\cos(\alpha+\beta)}$$

$$\approx \sin\left(\frac{\alpha+\beta}{2}\right), \text{ an } \underline{\text{entire}} \text{ function of } \alpha \text{ and } \beta .. \tag{3.8}$$

Or in terms of the variables ξ and η, G is analytic in ξ and η

$(\alpha \equiv \cos^{-1}\xi\,,\quad \beta \equiv \cos^{-1}\eta\,)$,

$$G(\alpha,\beta) \equiv G_2(\xi,\eta), \begin{cases} \text{analytic in } \xi \text{ and } \eta \\ \text{except for } \xi = \pm 1 \ \underline{\text{or}}\ \eta = \pm 1, \end{cases} \qquad (3.9)$$

and this is the other crucial property of the function $H(\xi,\eta,z)$ used in explaining the absence of the natural boundary of the partial-wave amplitude $\phi_\ell^{(2)}(\sigma)$ [(2.22) and (2.23)] . We have recorded the two important properties of the kernel function $H(\xi,\eta,z)$, hereby end our little detour, and return to the continuation of the main text.

The partial-wave projection (2.22) of the full amplitude $\phi^{(2)}(\sigma,z)$ on $P_\ell(z)$ [or in brief, the partial wave or the partial amplitude of $\phi^{(2)}(\sigma,z)$] in terms of the Liouville series (3.2) results in the partial wave $M_{n,\ell}(\sigma)$ of the individual term of the series;

$$M_{n,\ell}(\sigma) \equiv \tfrac{1}{2}\int_{-1}^{1} M_n(\sigma,z)\,P_\ell(z)\,dz = \int_{0}^{1} M_n(\sigma,z)\,P_\ell(Z)\,dz\,, \qquad (3.10)$$

where $M_n(\sigma,z)$ and $P_\ell(z)$ are both even in z in the "pair" theory of this study. Since the n^{th} term $M_n(\sigma,z)$ of the Liouville series is singular only at the singular points of (3.1) corresponding to the integer n , the partial wave $M_{n,\ell}(\sigma)$ of (3.10) is "expected" to be singular only at the endpoint singularities of (2.23) also corresponding to the integer n . It is the aim of this section to show that $M_{n,\ell}(\sigma)$ of (3.10) is regular at the "expected" endpoint singularities of (3.23) (except for $\sigma = \pm 1$),

$$\sigma = \cos\left(\frac{\pi}{n} k\right), \qquad k \neq 0, n. \tag{3.11}$$

Now we proceed to examine the functional behavior of the partial wave $M_{n,\ell}(\sigma)$ of (3.10) in neighborhoods of the σ-points of (3.11). For the variable σ restricted to <u>any</u> of the neighborhoods of the points of (3.11),

$$\sigma \text{ in } N_\sigma\left(\cos k \frac{\pi}{n}\right), \; k \neq 0, \; n , \tag{3.12}$$

the integrand $M_n(\sigma, z)$ of the integral (3.10) has two singular z-points in the neighborhoods of $z = \pm 1$,

$$z = \pm \cos n \cos^{-1}\left(\cos k \frac{\pi}{n}\right) = \pm \cos k\pi = \pm 1 . \tag{3.13}$$

Hence for σ as restricted in (3.12), the integral (3.10) can be conveniently split up into two parts,

$$M_{n,\ell}(\sigma) = \left[\int_0^{1-\epsilon} + \int_{1-\epsilon}^{1} \right] M_n(\sigma, z)\, P_\ell(z)\, dz, \quad (\epsilon \text{ small}) \tag{3.14}$$

such that the first integral is <u>regular</u> at σ . Since the ϵ is quite arbitrary, the function that really determines whether or not

$M_{n,\ell}(\sigma)$ is singular at σ of (3.12) is precisely the following integral,

$$I_{n,\ell}(\sigma) \equiv \left[\int M_n(\sigma,z)\; P_\ell(z)dz\right]_{z\,=\,1} . \qquad (3.15)$$

Furthermore, the polynomial $P_\ell(z)$ can be dropped in so far as the ascertainment of whether or not $M_{n,\ell}(\sigma)$ is singular at σ of (3.12) is concerned,

$$I_n(\sigma) \equiv \left[\int dz M_n(\sigma,z)\right]_{z\,=\,1} . \qquad (3.16)$$

In the integral (3.16) the z-dependence of $M_n(\sigma,z)$ resides entirely in the kernel function $H(\xi,\eta,z)$ which was discussed between Eqs.(3.5) to (3.9).

The expression (3.16) and the two important properties of the kernel function $H(\xi,\eta,z)$ will be used repeatedly in the remainder of this section to demonstrate the absence of the "expected" natural boundary (2.23) of the partial wave $\phi_\ell^{(2)}(\sigma)$ along $-1 \le \sigma \le 1$.

III. B. $M_{2,\ell}(\sigma)$ is Regular at $\sigma = \cos\frac{\pi}{2}k$, $k \neq 0, 2$

We are going to demonstrate in this subsection that the partial-wave projection of the second Liouville term $M_2(\sigma,z)$ doesnot contribute two of the four endpoint singularities to $\phi_\ell^{(2)}(\sigma)$. We start by rewriting from Eq. (3.2) the second term of the Liouville series,

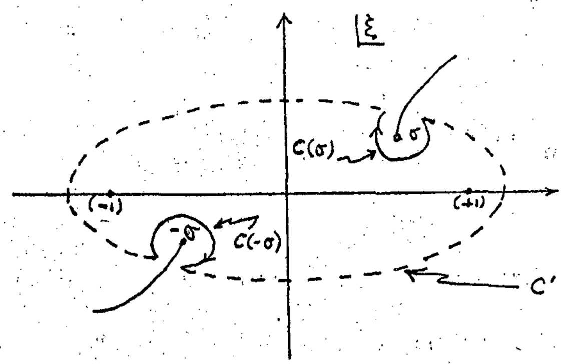
$$M_2(\sigma,z) = \oint_{C_\xi} d\xi \oint_{C_\eta} d\eta\; \phi(\sigma,\xi)\; H(\xi,\eta,z)\; \phi(\sigma,\eta). \qquad (3.17)$$

Its associated singularities are

$$\begin{cases} \text{for a fixed } \sigma \ , \ z = \pm \cos 2 \cos^{-1}\sigma \quad \text{or} \\ \text{for a fixed } \theta \ , \ \sigma_{2,k}(\theta) = \cos\left(\frac{\theta}{2} + \frac{\pi}{2} k\right), \quad k=0,1,2,3. \end{cases} \quad (3.18)$$

The contour C_ξ (and similarly the contour C_η) is an ellipse encircling the closed interval $[-1, +1]$ with two indentations at $\xi = \pm \sigma$,

Figure 1.



$$\begin{cases} \text{the little loops encircling } \xi = \pm\sigma \text{ are designated as } C(\pm\sigma)\text{, and} \\ \text{the dotted } \underline{\text{portions}} \text{ of } C_\xi \text{ are designated as } C'. \end{cases} \quad (3.19)$$

Now if the contour C_ξ (and C_η) is divided into three parts according to (3.19), or symbollically,

$$\begin{Bmatrix} C_\xi = C(\sigma) + C(-\sigma) + C' \\ C_\eta = C(\sigma) + C(-\sigma) + C' \end{Bmatrix}, \ (C' \text{ consists of two disjoint arcs}), \quad (3.20)$$

the expression $M_2(\sigma, z)$ of (3.17) can be written as a sum of nine integrals associated with the nine combinations in (ξ, η) contours of (3.20). The nine combinations of integration in (ξ, η) can be compactly written with the symbolic multiplication sign (x);

$$C_{\xi} \times C_{\eta} = C(\sigma) \times C(\sigma) + C(\sigma) \times C(-\sigma) + C(\sigma) \times C' + \ldots \quad . \qquad (3.21)$$

Furthermore, for ease of reference, the associated nine integrals of $M_2(\sigma,z)$ are designated by the following rather self-explanatory notation :

$$M_2(\sigma,z) \equiv M_2\left[\sigma,z \middle| C(\sigma) \times C(\sigma)\right] + \ldots$$
$$\equiv M\left[C(\sigma) \times C(\sigma)\right] + \ldots\ldots$$

where for example

$$M\left[C(\sigma) \times C(\sigma)\right] \equiv \int_{C(\sigma)} d\xi \int_{C(\sigma)} d\eta \; \phi(\sigma,\xi)\, H(\xi,\eta,z)\, \phi(\sigma,\eta). \qquad (3.22)$$

The ϕ's in Eq. (3.22) can be replaced by $\phi^{(+)}$ or $\phi^{(-)}$ according to whether the assocated contour of integration is $C(\sigma)$ or $C(-\sigma)$ because the Mandelstam representation can be written as a sum of two terms (2.7). For example, if the integration is carried over $C(\sigma)$ then $\phi^{(-)}$ is integrated to zero. Hence (3.22) can be rewritten as

$$M\left[C(\sigma) \times C(\sigma)\right] \equiv M[+,+] \equiv \int_{C(\sigma)} d\xi \int_{C(\sigma)} d\eta \; \phi^{(+)}(\sigma,\xi) H(\xi,\eta,z) \phi^{(+)}(\sigma,\eta). \qquad (3.23)$$

The singularities as enumerated in (3.18) can not arise from any one of the <u>five</u> integrals of (3.22) where at least one of the integration in either ξ or η is carried over C' because of the "addition" property of $H(\xi,\eta,z)$, as discussed in the paragraph containing (3.4). Also due to a symmetry property of the function $H(\xi,\eta,z)$,

$$H(\xi,\eta,z) = H(-\xi,-\eta,z). \tag{3.24}$$

Of the remaining four integrals of (3.22), only two are distinct,

$$\begin{cases} M[+,+] = M[-,-] \\ M[+,-] = M[-,+] \,. \end{cases} \tag{3.25}$$

Hence for a fixed σ ,

$$\begin{cases} M\,[+,+] \text{ gives rise to the singularity } z = +\cos 2\cos^{-1}\sigma\,, \\ M\,[+,-] \text{ gives rise to the singularity } z = -\cos 2\cos^{-1}\sigma\,. \end{cases} \tag{3.26}$$

Now the origin of the singularities of $M_2(\sigma,z)$ is isolated and identified in (3.26). We proceed to examine the expressions (3.26) in terms of the partial-wave integration (3.16). It is the goal of this subsection to show that two of the four "expected" endpoint singularities

$$\begin{cases} \sigma = \cos k\,\frac{\pi}{2}\,,\ k \neq 0,2 \quad \text{or} \\ \sigma = \pm\, i0 \end{cases} \tag{3.27}$$

are absent in the partial-wave amplitude $M_{2,\ell}(\sigma)$. Therefore, we will work in small neighborhoods (3.5) of these points (3.27). For values z and σ restricted to the following neighborhoods:

$$\begin{cases} z \text{ in } N_z(1) \\ \sigma \text{ in } N_\sigma(+i0), \end{cases} \tag{3.28}$$

the expression M [+,-] of (3.26) is the appropriate one to use in the partial wave projection,

$$I_2(\sigma) = \int_{z=1} dz M[+,-] = \int_{C(\sigma)} d\xi \int_{C(-\sigma)} d\eta\, \phi^{(+)}(\sigma,\xi)\phi^{(-)}(\sigma,\eta) \int_{z=1} dz H(\xi,\eta,z).$$

$$(\sigma \approx +i0). \tag{3.29}$$

It should be pointed out here that the order of integrations between dz and $d\xi d\eta$ has been <u>interchanged</u> ; and this interchange needs justification. We emphasize here that as long as $\sigma = +i\epsilon$, $\epsilon > 0$, the integrand of Eq. (3.29) is <u>analytic</u> in the product region in the variables $(z,\xi,\eta,)$

$$\left[z \in N_z(1)\right] \otimes \left[\xi \in C(\sigma)\right] \otimes \left[\eta \in C(-\sigma)\right]$$

so that the interchange of the order of integration in (3.29) is justified. The function $I_2(\sigma)$ at $\sigma = i0$ is <u>defined</u> by the <u>analytic continuation of $I_2(\sigma)$ at $\sigma = i\epsilon$</u> as <u>calculated</u> from (3.29). And this is the only meaningful definition of $I_2(\sigma)$ at $\sigma = i0$ (or equivalently the only meaningful definition of the partial-wave $\phi_\ell(s)$ on the <u>negative</u> s-axis) since the integral defining $I_2(\sigma)$ of Eq. (3.29) is <u>meaningless</u> at $\sigma = i0$. We illustrate the above seemingly obscure but important remarks by the following integral,

$$\phi(s) \equiv \int_0^\infty dz s e^{-zs} \qquad \text{Re } s > 0.$$

The integral $\phi(s)$ is meaningless for Re s = 0. (for that matter Re s ≤ 0) But $\phi(s) \equiv 1$ identically, hence $\phi(s)$ is <u>defined</u> for s, Re s ≤ 0 by the <u>analytic continuation</u> of $\phi(s)$ at s, Re s > 0. Now we return to the subject matter proper. For z as restricted in (3.28), and for ξ and η restricted to following neighborhoods (because of the contours $C(\sigma)$ and $C(-\sigma)$):

$$\begin{Bmatrix} \xi \text{ in} & N_\xi(+i0) \\ \eta \text{ in} & N_\eta(-i0) \\ z \text{ in} & N_z(\ 1\ ) \end{Bmatrix} \text{ or } \begin{Bmatrix} \xi & = \cos\alpha, \alpha \approx \frac{\pi}{2} \\ \eta & = \cos\beta, \beta \approx \frac{\pi}{2}\cdot 3 \\ z & = \cos\theta, \theta \approx 2\pi \end{Bmatrix} ; \qquad (3.30)$$

the function $H(\xi,\eta,z)$ in (3.29) satisfies the conditions of (3.6). Hence all the results of that paragraph are applicable here. Thus Eq.(3.29) can be rewritten as;

$$I_2(\sigma) = \int_{C(\sigma)} d\xi \int_{C(-\sigma)} d\eta\ \phi^{(+)}(\sigma,\xi)\ \phi^{(-)}(\sigma,\eta)\ G_2(\xi,\eta), \qquad (3.31)$$

where $G_2(\xi,\eta)$ is defined in (3.9) and is <u>analytic</u> in ξ and η, restricted to the neighborhoods of (3.30). The analytic property of $G_2(\xi,\eta)$ in ξ and η implies that $I_2(\sigma)$ is regular at $\sigma = +i0$ because the integration contours $C(\sigma)$ and $C(-\sigma)$ can now be analytically deformed to avoid the advancing singularities $\xi = +i0$ and $\eta = -i0$ respectively as σ approaches $+i0$.

By entirely analogous argument, one can show that $I_2(\sigma)$ is also regular at $\sigma = -i0$. Hence the partial-wave projection $M_{2,\ell}(\sigma)$ of the second Liouville term $M_2(\sigma,z)$ is regular at $\sigma = \cos\frac{\pi}{2}k$, $k \neq 0, 2$.

In summary, the "expected" endpoint singularity of $M_{2,\ell}(\sigma)$ at $\sigma = +i0$ is absent because of the analytic form of the kernel function $H(\xi,\eta,z)$. In particular, the factor $\left[z - \cos(\alpha+\beta)\right]^{-\frac{1}{2}}$ of $H(\xi,\eta,z)$, when integrated and evaluated at $z = \pm 1$, becomes an <u>entire</u> function in α and β, [see Eq.(3.8)]

$$\sqrt{1 - \cos(\alpha+\beta)} = \sqrt{2}\,\sin\frac{\alpha+\beta}{2},\ \sqrt{1 + \cos(\alpha+\beta)} = \sqrt{2}\,\cos\frac{\alpha+\beta}{2}\,.$$

Or in terms of ξ and η, it is an analytic function of (ξ,η) except for $\xi = \pm 1$ <u>or</u> $\eta = \pm 1$. Consequently the two contours of integration $C(\pm\sigma)$ can be analytically deformed to avoid the two advancing singular points in ξ and η as σ approaches $+i0$. This property persists as a characteristic feature for all $M_{n,\ell}(\sigma)$ terms to be discussed later. We now go on to treat the $M_3(\sigma,z)$ term in analogous fashion, so as to arrive at the generalization for the arbitrary n^{th} term $M_n(\sigma,z)$ of the Liouville series.

III. C. $M_{3,\ell}(\sigma)$ is Regular $\sigma = \cos\frac{\pi}{3}k$, $k \neq 0, 3$

The treatment of the term $M_3(\sigma,z)$ follows closely that of $M_2(\sigma,z)$. We start out by writing $M_3(\sigma,z)$ in full;

$$M_3(\sigma,z) = \oint_{C_{\xi_1}} d\xi_1 \oint_{C_\eta} d\eta\,\phi(\sigma,\xi_1)H(\xi_1,\eta,z) \oint_{C_{\xi_2}} d\xi_2 \oint_{C_{\xi_3}} d\xi_3\,\phi(\sigma,\xi_2)\times$$
$$\times H(\xi_2,\xi_3,\eta)\phi(\sigma,\xi_3). \tag{3.32}$$

The singularities of the term $M_3(\sigma,z)$ are:

$$\begin{cases} \text{for a fixed } \sigma, & z = \pm \cos 3 \cos^{-1}\sigma \quad \text{or} \\ \text{for a fixed } \theta, & \sigma_{3,k}(\theta) = \cos\left[\frac{\theta}{3} + \frac{\pi}{3} k\right], \quad k = 0,1,\ldots 5. \end{cases} \tag{3.33}$$

Again the contours C_{ξ_i} $(i = 1,2,3)$ are chosen as in (3.19) and (3.20),

$$C_{\xi_i} = C(\sigma) + C(-\sigma) + C' \qquad (i = 1,2,3) . \tag{3.34}$$

The expression (3.32) for $M_3(\sigma, z)$ can now be written as a sum of 27 integrals corresponding to the 27 combinations of integration in ξ_i($i = 1,2,3$) of (3.34),

$$C_{\xi_1} \times C_{\xi_2} \times C_{\xi_3} = C(\sigma) \times C(\sigma) \times C(\sigma) + \ldots \tag{3.35}$$

However, <u>any</u> integral involving an integration in <u>any</u> of the three variables $\xi_i (i = 1,2,3)$ over C' is analytic at the points (σ, z) of (3.33) again because of the "addition" property of $H(\xi, \eta, z)$ [see the discussion that leads to (3.4)] . Therefore, as far as the study of the singularities of $M_3(\sigma, z)$ (3.33) is concerned, it suffices to discuss only 8 of the total 27 integrals,

$$\begin{aligned} M_3(\sigma, z) \equiv\ & M_3\left[\sigma, z \,\middle|\, C(\sigma) \times C(\sigma) \times C(\sigma)\right] \\ & + M_3\left[\sigma, z \middle| C(\sigma) \times C(\sigma) \times C(-\sigma)\right] + \ldots\ldots \\ \equiv\ & M_3\,[+,+,+] + M_3[+,+,-] + \ldots \quad . \end{aligned} \tag{3.36}$$

Furthemore, due to the symmetry of the function $H(\xi, \eta, z)$ (3.24), only four of the eight integrals are distinct,

$$\begin{cases} M_3\,[+,+,+] = M_3\,[+,-,-] \\ M_3\,[+,+,-] = M_3\,[+,-,+] \\ M_3\,[-,+,+] = M_3\,[-,-,-] \\ M_3\,[-,+,-] = M_3\,[-,-,+] \quad . \end{cases} \tag{3.37}$$

In any of the M_3 - integrals of (3.37), the integration over η is still over a closed contour C_η [see (3.32)] . For a fixed pair of (ξ_2, ξ_3) -values of (say) $C(\sigma) \times C(\sigma)$;

$$\begin{cases} \xi_2 \equiv \cos\alpha_2 & \text{on} \quad C(\sigma) \\ \xi_3 \equiv \cos\alpha_3 & \text{on} \quad C(\sigma), \end{cases} \tag{3.38}$$

then the closed contour C_η can be broken up into two pieces, [see definition (3.19)]

$$C_\eta = C(\cos[\alpha_2 + \alpha_3]) + \left\{C_\eta - C(\cos[\alpha_2 + \alpha_3])\right\} . \tag{3.39}$$

Again by the "addition" property of the function $H(\xi_2, \xi_3, \eta)$ (3.4) only the integral involving the integration in η over the contour $C(\cos[\alpha_2 + \alpha_3])$ contributes to the singularity of (3.33). Hence keeping this in mind and remembering the replacement of ϕ by $\phi^{(+)}$ or $\phi^{(-)}$ as done in (3.22) to (3.23), we write a typical integral of (3.37) in detail as :

$$M_3[+,+,+] \equiv \int_{C(\sigma)} d\xi_1 \int_{C(\sigma)} d\xi_2 \int_{C(\sigma)} d\xi_3 \int_{C(\cos[\alpha_2+\alpha_3])} d\eta\, \phi^{(+)}(\sigma,\xi_1)\phi^{(+)}(\sigma,\xi_2)\phi^{(+)}(\sigma,\xi_3) \cdot$$

$$\cdot H(\xi_1,\eta,z)H(\xi_2,\xi_3,\eta) \quad . \tag{3.40}$$

According to the expressions (2.23) and (3.33), we wish to show that the partial-wave amplitude $M_{3,\ell}(\sigma)$ of the third Liouville term is regular at,

$$\sigma_{3,k} = \cos \frac{\pi}{3} k \ , \quad k = 1,2,4,5 \ (k \neq 0,3). \tag{3.41}$$

For definiteness, we choose to approach the point

$$\sigma = \sigma_{3,2} = \cos \left(\frac{\pi}{3}, 2\right) = -\tfrac{1}{2} + i0 \tag{3.42}$$

in the expression (3.10), then both $M_3[+,+,+]$ and $M_3[-,+,-]$ of (3.37) contribute to the integral (3.16) because of the "addition" property of the function $H(\xi,\eta,z)$. But we will illustrate the procedure by working out the consequence due to the integral $M[+,+,+]$ only.

For ξ_i $(i = 1,2,3)$, z and η restricted to the following neighborhoods:

$$\begin{cases} \xi_i \equiv \cos \alpha_i & \text{in } N_{\xi_i}(-\tfrac{1}{2} + i0) & \text{or } \alpha_i \approx \frac{\pi}{3} \cdot 2 \\ z \equiv \cos \theta & \text{in } N_z(1) & \text{or } \theta \approx 2\pi \\ \eta \equiv \cos \beta & \text{in } N_\eta(-\tfrac{1}{2} - i0) & \text{or } \beta \approx \frac{\pi}{3} \cdot 4 \end{cases} \tag{3.43}$$

the following approximations for the two H's in (3.40) are valid;

$$\begin{cases} H(\xi_1,\eta,z) \approx \dfrac{1}{\sqrt{z - \cos(\alpha_1 + \beta)}} \\ H(\xi_2,\xi_3,\eta) \approx \dfrac{1}{\sqrt{\eta - \cos(\alpha_2 + \alpha_3)}} \end{cases} \tag{3.44}$$

Again the discussion from (3.5) to (3.9) is applicable, the integral $I_3(\sigma)$ of (3.16) with $M_3\,[+,+,+]$ of (3.40) substituted can be rewritten as

$$I_3(\sigma) \equiv \int_{z=1} dz\, M\,[+,+,+]$$

$$= \int_{C(\sigma)} d\xi_1 \int_{C(\sigma)} d\xi_2 \int_{C(\sigma)} d\xi_3 \phi^{(+)}(\sigma,\xi_1)\phi^{(+)}(\sigma,\xi_2)\phi^{(+)}(\sigma,\xi_3) G_3(\xi_1,\xi_2,\xi_3),$$

where

$$G_3(\xi_1,\xi_2,\xi_3) \equiv \int_{C(\cos[\alpha_2+\alpha_3])} d\eta \frac{G_2(\xi_1,\eta)}{\sqrt{\eta - \cos(\alpha_2+\alpha_3)}}$$

$$= \int_{C(0)} \frac{dx}{\sqrt{x}}\, G_2(\xi_1, \cos(\alpha_2+\alpha_3) + x\,). \qquad (3.45)$$

The interchange of the order of integration is again justified by entirely analogous arguments that justifies Eq. (3.29) previously. Because of the analytic property of $G_2(\xi_1,\eta)$ (3.9), the integral of (3.45) integrates to an <u>analytic</u> function $G_3(\xi_1,\xi_2,\xi_3)$,

$$\left.\begin{array}{l} G_3(\xi_1,\xi_2,\xi_3) \text{ is analytic in } \xi_i (i = 1,2,3) \\ \text{provided} \\ \xi_i \neq \pm 1\,,\ (i = 1,2,3), \quad \text{or } \cos(\alpha_2+\alpha_3) \neq \pm 1 \end{array}\right\} \qquad (3.46)$$

since the loop contour C(0) can be made arbitrarily small. For

ξ_i (i = 1,2,3) as restricted in the neighborhoods of (3.43), the conditions of (3.46) are satisfied; hence $I_3(\sigma)$ of (3.45) is <u>regular</u> at $\sigma = -\frac{1}{2} + i0$ (3.42) because the contours $C(\sigma)$ can now be analytically deformed to avoid the advancing singularities $\xi_i = -\frac{1}{2} + i0$ (i = 1,2,3) as σ approaches $(-\frac{1}{2} + i0)$.

The other three points of (3.41) can be similarly treated so that $M_{3,\ell}(\sigma)$ is regular at $\sigma = \cos\frac{\pi}{3} k$, $k \neq 0, 3$.

Now the essential element for the generalization to the arbitrary n^{th} term of the Liouville series is present in the function $G_3(\xi_i)$ (i = 1, 2, 3) of (3.46) or (3.45). For the general n^{th} term $M_n(\sigma, z)$, analogous treatment will lead to an analytic function $G_n(\xi_i)$ (i = 1, 2, ... , n), instead of $G_3(\xi_i)$ (i = 1, 2, 3) as here. The function will be analytic in ξ_i(i = 1, . . . , n) provided that $\xi_i \neq \pm 1$ or $\cos(\alpha_2 + \ldots + \alpha_n) \neq \pm 1$, etc.

III. D. $M_{n,\ell}(\sigma)$ is Regular at $\sigma = \cos\frac{\pi}{n} k$, $k \neq 0, n$

The general n^{th} term of the Liouville series of $\phi^{(2)}(\sigma, z)$ is given as : (3.2)

$$M_n(\sigma,z) = \oint_{C_{\xi_1}} d\xi_1 \oint_{C_{\eta_1}} d\eta_1 \, \phi(\sigma,\xi_1) \, H(\xi_1,\eta_1,z) \, M_{n-1}(\sigma,\eta_1)$$

$$= \oint_{C_{\xi_1}} d\xi_1 \ldots \oint_{C_{\xi_n}} d\xi_n \oint_{C_{\eta_1}} d\eta_1 \ldots \oint_{C_{\eta_{n-2}}} d\eta_{n-2} \, \phi(\sigma,\xi_1)\ldots\phi(\sigma,\xi_n)$$

$$H(\xi_1,\eta_2,z) \, H(\xi_2,\eta_2,\eta_1)\ldots H(\xi_{n-1},\xi_n,\eta_{n-2}). \qquad (3.47)$$

The singularities of $M_n(\sigma,z)$ are,

$$\begin{cases} \text{for a fixed } \sigma, & z = \pm \cos n \cos^{-1} \sigma \quad \text{or} \\ \text{for a fixed } \theta, & \sigma_{n,k}(\theta) = \cos\left[\frac{\theta}{n} + \frac{\pi}{n} k\right], \quad k = 0,\dots,2n-1. \end{cases} \tag{3.48}$$

We want to show as before that the partial-wave $M_{n,\ell}(\sigma)$ of $M_n(\sigma,z)$ is regular at

$$\sigma = \sigma_{n,k}(\theta) = \cos\left[\frac{\pi}{n} k\right], \quad k \neq 0,n. \tag{3.49}$$

Therefore the steps taken from (3.34) to (3.45) for the $M_3(\sigma,z)$ case can be exactly repeated here for the $M_n(\sigma,z)$ case except for the complication in retaining the appropriate portions of the contours C_η. For illustration, we examine the second H factor of the expression (3.47), $H(\xi_2,\eta_2,\eta_1)$. For fixed $\xi_2 \equiv \cos\alpha_2$ and $\eta_2 \equiv \cos\beta_2$, only the contours $C(\pm \cos[\alpha_2 + \beta_2])$ [see (3.19) for definition] contribute to the singularities (3.48). Furthermore, for a chosen pair of (σ,z) values, only one of the two contours $C(\pm \cos[\alpha_2 + \beta_2])$ contributes. Hence by repeating the steps from (3.34) to (3.45), we start from the expression (3.47) for $M_n(\sigma,z)$, and arrive at the following expression for the determination of whether or not the partial wave $M_{n,\ell}(\sigma)$ is singular at the points of (3.49) :

$$I_n(\sigma) = \int_{z=1} M_n[+,+,\dots,+]\, dz + \dots, \quad (\sigma \simeq \cos\left[\frac{\pi}{n} k\right],\ k \neq 0,n)$$

$$\approx \int_{C(\sigma)} d\xi_1 \dots \int_{C(\sigma)} d\xi_n \phi^{(+)}(\sigma,\xi_1)\dots\phi^{(+)}(\sigma,\xi_n)\; G_n(\xi_1,\dots,\xi_n) + \dots, \tag{3.50}$$

where

$$G_n(\xi_1,\ldots,\xi_n)\equiv \int\limits_{C(\cos[\alpha_{n-1}+\alpha_n])} d\eta_{n-2}\cdots \int\limits_{C(\cos[\alpha_2+\beta_2])} d\eta_1 G_2(\xi_1,\eta_1)\frac{1}{\sqrt{\eta_1-\cos(\alpha_2+\beta_2)}}\times$$

$$\times\frac{1}{\sqrt{\eta_2-\cos(\alpha_3+\beta_3)}}\cdots\frac{1}{\sqrt{\eta_{n-2}-\cos(\alpha_{n-1}+\alpha_n)}}. \tag{3.51}$$

The integral in (3.51) involving only the integration in η_1 is just the function $G_3(\xi_1,\xi_2,\eta_2)$ of (3.45) and (3.46), whereas the integral involving the integrations in η_1 and η_2 would be $G_4(\xi_1,\xi_2,\xi_3,\eta_3)$. In general the integral involving the integrations of η_1 up to η_{j-2} inclusively will be a function G_j such that,

$G_j(\xi_1,\xi_2,\ldots,\xi_{j-1},\ \eta_{j-1})$ is analytic in all the variables

except for

$$\begin{cases}\xi_i=\pm 1, \quad (i=1,2,\ldots,j-1.) \text{ and } \eta_{j-1}=\pm 1\\ \cos(\alpha_2+\ldots+\alpha_{j-1}+\beta_{j-1}=\pm 1,\end{cases}$$

$$\begin{cases}\xi_i=\cos\alpha_i\\ \eta_j=\cos\beta_j\end{cases}. \tag{3.52}$$

By such inductive arguments, the analytic property of G_n, Eq. (3.51) is the following :

$G_n(\xi_1, \ldots, \xi_n)$ is analytic in all ξ_i $(i = 1, \ldots, n)$

except for

$$\begin{cases} \xi_i = \pm 1, \quad (i = 1, \ldots, n) \quad \text{and} \\ \cos(\alpha_2 + \ldots + \alpha_n) = \pm 1 \end{cases} \tag{3.53}$$

But for all the σ- points of (3.49), the conditions (3.53) is satisfied by the integration domains of the integral $I_n(\sigma)$ of (3.50) :

$$\begin{cases} \xi_i \equiv \cos\left(\frac{\pi}{n} k\right), \quad k \neq 0, n \Rightarrow \xi_i \neq \pm 1 \\ \cos(\alpha_2 + \ldots + \alpha_n) = \cos(n-1)\frac{\pi}{n}k, \quad k \neq 0, n \Rightarrow \cos\left(\frac{n-1}{n} k\right)\pi \neq \pm 1 \quad . \end{cases} \tag{3.54}$$

Therefore, $I_n(\sigma)$ of (3.50) is again regular at $\sigma = \cos\left(\frac{\pi}{n} k\right)$, $k \neq 0, n$. Since the integer n is arbitrary, we have demonstrated that the partial-wave projection of all terms of the Liouville series $\phi^{(2)}(\sigma, z)$ is regular at the "expected" endpoint singularities of (2.23) <u>except</u> for the two points $\sigma = \pm 1$. Hence formally (or rigorously if the Liouville series converges) the partial-wave $\phi_\ell^{(2)}(\sigma)$ of $\phi^{(2)}(\sigma, z)$ does not possess the "expected" natural boundary in $-1 \leq \sigma \leq +1$ because the "expected" endpoint singularities are integrated out <u>point by point</u> as demonstrated in this section.

IV. The Partial-wave Projection of the Liouville Series of $\phi^{(2)}(\sigma,z)$ and Conclusion

The explanation for the absence of the natural boundary of $\phi_{\ell}^{(2)}(\sigma)$ in Section III is very explicitly given. There the singularities of $\phi^{(2)}(\sigma,z)$ are isolated and then it is shown that the "expected" end-point singularities are integrated out point by point except for the points $\sigma = \pm 1$. Actually, the partial-wave projection $M_{n,\ell}(\sigma)$ of the arbitrary n^{th} term $M_n(\sigma,z)$ of the Liouville series can be carried out exactly with the help of the Lemmas in the Appendix A,

$$M_{n,\ell}(\sigma) = \frac{1}{2}\int_{-1}^{+1} dz\, M_n(\sigma,z)P_\ell(z) \approx \left[\phi_\ell(\sigma)\right]^n \quad . \tag{4.1}$$

Then it is obvious that $M_{n,\ell}(\sigma)$ is singular only at $\sigma = \pm 1$ because $\phi_\ell(\sigma)$ is . This is in agreement with the results of Section III as briefly mentioned above in this paragraph.

The z - dependence of the individual term of the Liouville series for $\phi^{(2)}(\sigma,z)$ (3.2) resides entirely in the kernel function $H(\xi,\eta,z)$. The partial-wave projection of $H(\xi,\eta,z)$ is a product of two Legendre functions [see Lemma 3 of the Appendix A]

$$H_\ell(\xi,\eta) \equiv \frac{1}{2}\int_{-1}^{1} dz H(\xi,\eta,z)P_\ell(z) = 2\pi Q_\ell(\xi)\, Q_\ell(\eta) \quad , \tag{4.2}$$

which agrees with the analytic properties of the function $G_2(\xi,\eta)$ of (3.9). By the repeated use of Lemma 1 and Lemma 3 of the Appendix A , $M_{2,\ell}(\sigma)$, $M_{3,\ell}(\sigma)$ and $M_{n,\ell}(\sigma)$ are integrated to;

$$\begin{cases} M_{2,\ell}(\sigma) = (-)\ (2\pi)^3 \left[\phi_\ell(\sigma)\right]^2 \\ M_{3,\ell}(\sigma) = (+)\ (2\pi)^6 \left[\phi_\ell(\sigma)\right]^3 \\ \vdots \\ M_{n,\ell}(\sigma) = (-)^{n-1}(2\pi)^{3(n-1)} \left[\phi_\ell(\sigma)\right]^n \\ \vdots \end{cases} \qquad (4.3)$$

which is the assertion of Eq. (4.1).

By putting the expressions (4.3) back into (3.2), the partial-wave $\phi_\ell^{(2)}(\sigma)$ of the full amplitude $\phi^{(2)}(\sigma, z)$ formally sums back to the partial-wave unitarity relation (2.15);

$$\phi_\ell^{(2)}(\sigma) = \frac{1}{2}\int_{-1}^{1} dz \phi^{(2)}(\sigma, z) P_\ell(z) = \sum_{n=1}^{\infty} \left[\rho_0(s)\right]^{n-1} \frac{1}{2}\int_{-1}^{1} dz M_n(\sigma, z) P_\ell(z)$$

$$= \sum_{n=1}^{\infty} \left[\rho_0(s)\right]^{n-1} (-)^{n-1} \left[\phi_\ell(\sigma)\right]^n (2\pi^3)^{n-1}$$

$$= \phi_\ell(\sigma) \left\{ \sum_{n=0}^{\infty} \left[-2i\rho(s)\ \phi_\ell(\sigma)\right]^n \right\}$$

$$= \frac{\phi_\ell(\sigma)}{1 + 2i\rho(s)\ \phi_\ell(\sigma)} \qquad (4.4)$$

which is the starting point of our study. Hence our study comes to an end.

We have carried the present study through a complete cycle: we start out with the Mandelstam representation and the partial-wave unitarity relation (2.15) between $\phi_\ell^{(2)}(s)$ and $\phi_\ell(s)$, then go on to define the full amplitude $\phi^{(2)}(s,z)$. From the integral equation (2.16) [the unitarity relation between $\phi^{(2)}(s,z)$ and $\phi\ (s,z)$], aninfinite set (3.1) of singular points of $\phi^{(2)}\ (s,z)$ is determined. This infinite set of singular points is shown to constitute a natural boundary along the real negative s - axis for arbitrary complex z . In an attempt to understand the absence of the "expected" natural boundary for the partial-wave amplitude $\phi_\ell^{(2)}(s)$, the set (3.1) of singularities is classified by the Liouville series whose n^{th} term is designated by $M_n(s,z)$. It is then shown that the partial-wave projection $M_{n,\ell}(s)$ of $M_n(s,z)$ on $P_\ell(z)$ is singular only at $s = 0, \infty$ ($\sigma = \pm 1$) whereas all other "expected" endpoint singularities (2.23) are integrated out <u>point by point</u> due to a remarkable property of the kernel function $H(\xi,\eta,z)$. Since each term $M_{n,\ell}(s)$ in the infinite Liouville series is singular only at $s = 0,\infty$, the formal sum $\phi_\ell^{(2)}(s)$ is then singular only at $s = 0,\infty$; hence the absence of the "expected" natural boundary along $-\infty \le s \le 0$ for the partial-wave amplitude $\phi_\ell^{(2)}(s)$ is understood. Furthermore, the partial-wave projection of the Liouville series of $\phi_\ell^{(2)}(s)$ sums exactly back to the partial-wave unitarity relation (2.15), which is the starting point of the cycle.

We conclude by making a few general remarks :

(1) The demonstration that the "expected" endpoint singularities of $\phi_\ell^{(2)}(s)$ is integrated out point by point ,is carried out via the Liouville series of $\phi^{(2)}(s,z)$. If the series converges, the demonstration is rigorous; if the series diverges, the demonstration is then formal. Even in the latter case, we still believe that the proposed explanation in this study for the absence of the "expected" natural boundary of $\phi_\ell^{(2)}(s)$ is correct.

(2) A careful study of Zimmermann's derivation [6] of this infinite set (3.1) of singularities shows that it has its origin in the kernel function $H(\xi,\eta,z)$. Our proposed explanation is also based on a remarkable property of $H(\xi,\eta,z)$ at $z = \pm 1$; namely, the factor $[z - \cos(\alpha+\beta)]^{+\frac{1}{2}}$ which when evaluated at the endpoints of integration $z = \pm 1$, $\sqrt{1 \pm \cos(\alpha+\beta)}$, is an <u>entire</u> function of α and β or an analytic function of ξ and η except for $\xi = \pm 1$ <u>or</u> $\eta = \pm 1$. It should be noticed that the endpoint singularities $s = 0, -\infty$ (or $\sigma = \pm 1$) for $M_{n,\ell}(\sigma)$ are not integrated out by our proposed mechanism because now the function $H(\xi,\eta,z)$ (3.3) can no longer be approximated by $[z - \cos(\alpha+\beta)]^{-\frac{1}{2}}$ at $z \approx 1$ because $K_- \approx 0$ also.

(3) Zimmermann's work [6] can be carried out within the frame-work of the rigorously proved analyticity properties of the physical amplitude $\phi(s,z)$ [i.e. his work does not need all the analyticity properties of the Mandelstam conjecture]. Since this study is based on Zimmermann's work, therefore, the result of this study has greater validity than that implied by the Mandelstam conjecture.

(4) One possible implication of the result of this study is the following : the amplitude $\phi^{(2)}_{AB}(s,z)$ given by the unitarity condition in Freund and Karplus' [10] paper, formula (6),

$$\phi^{(2)}_{AB}(s,z) = \phi_{AB}(s,z) - \frac{2i\rho(s)}{2\pi}\int_{-1}^{1}dx\int_{-1}^{1}dy\; \phi_{AB}(s,x)\frac{\theta(-K(x,y,z))}{\sqrt{-K(x,y,z)}} \times \phi^{(2)}_{AA}(s,y) ,$$

is ~~also~~ expected to have a natural boundary because $\phi^{(2)}_{AA}(s,y)$ does;

there the angular integrations do not have the simplifying properties that cause the singularities to disappear.

(5) It is mathematically interesting to continue the individual partial-wave amplitude $\phi_\ell^{(2)}(s)$ across the real negative s - axis and then to write a Legendre series,

$$\phi^{(2)}_{\text{pseudo}}(s,z) = \sum_{n=0}^{\infty} \phi_\ell^{(2)}(s)\, P_\ell(z) .$$

We call this series a "pseudo - continuation" of $\phi^{(2)}(s,z)$ if it converges. How is the "pseudo - continuation" $\phi^{(2)}_{\text{pseudo}}(s,z)$ related to $\phi^{(2)}(s,z)$ <u>across</u> the natural boundary ?

(6) Lemma 5 of the Appendix A is an interesting identity among the same Legendre function Q 's of different arguments.

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APPENDIX A MATHEMATICAL FORMULAS AND IDENTITIES[11]

Lemma 1. Let $f(z)$ be an analytic function of z in E, where E is a neighborhood of the interval $-1 \leq z \leq +1$. Further, let $P_\ell(z)$ and $Q_\ell(z)$ denote the legendre functions of the first and second kind. Then

$$\frac{1}{2} \int_{-1}^{1} f(z)\, P_\ell(z)\, dz = \frac{1}{2\pi i} \oint_C f(z)\, Q_\ell(z)\, dz$$

where C is wholly inside E and encircles the interval $-1 \leq z \leq +1$.

Proof. By substituting the Neumann formula

$$Q_\ell(z) = \frac{1}{2} \int_{-1}^{1} P_\ell(z') \frac{dz'}{z-z'} \qquad (A.1)$$

into the right-hand side and interchanging the order of integration, an identity results.

Lemma 2.

$$\frac{\theta(-K(x,y,z))}{\sqrt{-K(x,y,z)}} = \pi \sum_{\ell=0}^{\infty} \left(\frac{2\ell+1}{2}\right) P_\ell(x)\, P_\ell(y)\, P_\ell(z)$$

where

$$K(x,y,z) = x^2 + y^2 + z^2 - 2xyz - 1.$$

Proof. The proof of this identity starts with the addition formula of Legendre polynomials,

$$P_\ell(\cos\alpha) = P_\ell(\cos\theta)\, P_\ell(\cos\theta') + 2\sum_{m=1}^{\ell} \frac{(\ell-m)!}{(\ell+m)!} \cdot$$

$$\cdot\; P_\ell^{\,m}(\cos\theta)\, P_\ell^{\,m}(\cos\theta') \cos m\phi$$

where $\cos\alpha = \cos\theta \cos\theta' + \sin\theta \sin\theta' \cos\phi$.

We integrate in the variable ϕ from 0 to π, and get

$$\int_0^\pi P_\ell\left(xy + \sqrt{1-x^2}\,\sqrt{1-y^2}\,\cos\phi\right) d\phi = \pi P_\ell(x)\, P_\ell(y). \qquad (A.2)$$

Now we multiply (A.2) by $P_\ell(z)\left(\frac{2\ell+1}{2}\right)$ and sum, we get

$$\int_{-1}^{1} \delta\left(xy + \sqrt{1-x^2}\,\sqrt{1-y^2}\,\cos\phi - z\right) \frac{d(\cos\phi)}{\sin\phi} =$$

$$= \pi \sum_{\ell=0}^{\infty} \left(\frac{2\ell+1}{2}\right) P_\ell(x)\, P_\ell(y)\, P_\ell(z) \quad . \qquad (A.3)$$

The left-hand side of (A.3) simplifies to

$$\frac{\theta(-K(x,y,z))}{\sqrt{-K(x,y,z)}} = \pi \sum_{\ell=0}^{\infty} \left(\frac{2\ell+1}{2}\right) P_\ell(x)\, P_\ell(y)\, P_\ell(z) \; . \qquad (A.4)$$

<u>Lemma 3.</u> Lemma 2 leads immediately to another identity,

$$2\pi\, Q_\ell(x)\, Q_\ell(y) = \frac{1}{2} \int_{-1}^{1} P_\ell(z)\, H(x,y,z)\, dz$$

Proof. From Neumann's formula

$$Q_\ell(z) = \frac{1}{2}\int_{-1}^{1} P_\ell(z')\,\frac{dz'}{z-z'}$$

We are led to multiply (A.4) by $\frac{1}{2}\,\frac{1}{x-x'}\quad\frac{1}{y-y'}\,P_\ell(z)\;dx'\;dy'\;dz$ and integrate from -1 to +1. This step results in the following expression

$$2\pi Q_\ell(x)\;Q_\ell(y) = \frac{1}{2}\int_{-1}^{1} P_\ell(z)\;dz\left[\int_{-1}^{1}\frac{dx'}{x-x'}\int_{-1}^{1}\frac{dy'}{y-y'}\,\frac{\theta(-K(x',y',z))}{\sqrt{-K(x',y',z)}}\right]. \tag{A.5}$$

But the expression inside the bracket is nothing other than the function H(x,y,z) [6] [ref. (2.17)], hence

$$2\pi\;Q_\ell(x)\;Q_\ell(y) = \frac{1}{2}\int_{-1}^{1} P_\ell(z)\;dz\,H(x,y,z). \tag{A.6}$$

Lemma 4.[12]

$$Q_\ell(x)\;Q_\ell(y) = \int_{1}^{\infty}\frac{dw}{\sqrt{w^2-1}}\;Q_\ell\left(xy + w\sqrt{x^2-1}\;\sqrt{y^2-1}\right) .$$

Proof. By substituting[6]

$$H(x,y,z) = 2\pi\int_{xy+\sqrt{x^2-1}\sqrt{y^2-1}}^{\infty}\frac{dz'}{z'-z}\;\frac{1}{\sqrt{K(x,y,z')}}$$

into (A.6) of Lemma 3, we get

$$2\pi Q_\ell(x)Q_\ell(y) = \frac{1}{2}\int_{-1}^{1} P_\ell(z)\,dz\; 2\pi \int_{xy+\sqrt{x^2-1}\sqrt{y^2-1}}^{\infty} \frac{dz'}{z'-z}\,\frac{1}{\sqrt{K(x,y,z')}}$$

$$= 2\pi \int_{xy+\sqrt{x^2-1}\sqrt{y^2-1}}^{\infty} \frac{dz'\; Q_\ell(z')}{\sqrt{K(x,y,z')}}\;. \qquad (A.7)$$

By the following change of variable,

$$z' = xy + w\sqrt{x^2 - 1}\sqrt{y^2 - 1}$$

(A.7) is transformed into the desired result.

Lemma 5.

$$\prod_{i=0}^{n} Q_\ell(x_i) = \int_1^{\infty} \frac{dw_1}{\sqrt{w_1^{\,2}-1}} \;\cdots\; \int_1^{\infty} \frac{dw_n}{\sqrt{w_n^{\,2}-1}}\; Q_\ell(x_n')$$

where x_n' is defined by a recursion relation,

$$x_n' = x_n x'_{n-1} + w_n\sqrt{x_n^{\,2}-1}\;\sqrt{x'^{\,2}_{n-1} - 1}, \qquad x_0' \equiv x_0 \;.$$

Proof. Repeated application of Lemma 4 yields Lemma 5.

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12. We are inspired by Ref. 5 in writing down Lemmas 4 and 5.