Geometric Interpretation of Donkin's Tensor Product Theorem

by

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Abstract

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This thesis gives a geometric interpretation of the Donkin's Tensor Product Theorem, whose original proof is purely algebraic.

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Chapter 1

Quantum Group at Roots of Unity

1.1 Notations

- Let G be a connected complex simply connected semisimple algebraic group, $G^\vee$ be its Langlands dual group which is of adjoint type.
- Let $\mathfrak{g}$ be the Lie algebra of G, $\mathfrak{t} \subseteq \mathfrak{g}$ be the Cartan subalgebra. Let $R \subset \mathfrak{t}^*$ be the root system of G, with $\{\alpha_i\}_{i\in I}$ being the simple roots. Let $\rho = \frac{1}{2}\sum_{i\in I}\alpha_i$. Let $R^\vee \subset \mathfrak{t}$ be the coroot system and $\{\alpha_i^\vee\}_{i\in I}$ be the simple coroots. Hence $a_{ij} = \langle \alpha_i^\vee, \alpha_j\rangle$ gives the Cartan matrix.
- Let $X = \{\mu \in \mathfrak{t}^* | \ \langle \alpha_i^\vee, \mu\rangle \in \mathbb{Z}, \forall i \in I\}$ be the weight lattice of G. Let $X^+ = \{\mu \in X | \ \langle \alpha_i^\vee, \mu\rangle \geq 0, \forall i \in I\}$ be the dominant integral weights of G. Let $Y = \Sigma_{i\in I}\mathbb{Z}\alpha_i \subseteq X$ be the root lattice of G. Let $Y^+ = X^+ \cap Y$ be the positive root lattice of G. Let $Y^\vee = \mathrm{Hom}(Y, \mathbb{Z})$ be the coweight lattice of G.
- Choose a Killing form $(\cdot,\cdot)$ such that $(\alpha_i, \alpha_i) = 2d_i$, $d_i \in \{1, 2, 3\}$. Note that we'll always have the Cartan matrix $a_{ij} = \frac{(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)}$ back.
- Choose a borel subalgebra $\mathfrak{b} \subseteq \mathfrak{g}$. Let $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{t} \oplus \mathfrak{n}_-$ be the induced triangular decomposition.
- Let $\mathbb{C}(q)$ be the field of rational functions in q. Let $U_q(\mathfrak{g})$ be the Drinfeld-Jimbo quantum group generated by the Chevalley generators E_i, F_i and $K_{\mu^\vee}$, where $i \in I$ and $\mu \in Y^\vee$ is a coweight. They satisfy the following relations:

$$K_{\mu_1^\vee}K_{\mu_2^\vee} = K_{\mu_1^\vee+\mu_2^\vee}, \quad K_{\mu^\vee}E_iK_{\mu^\vee}^{-1} = q^{\langle\alpha_i,\mu^\vee\rangle}E_i, \quad K_{\mu^\vee}F_iK_{\mu^\vee}^{-1} = q^{-\langle\alpha_i,\mu^\vee\rangle}F_i,$$

$$E_iF_j - F_jE_i = \delta_{ij}\frac{K_{d_i\alpha_i^\vee} - K_{d_i\alpha_i^\vee}^{-1}}{q^{d_i} - q^{-d_i}},$$

They also satisfy the quantum Serre relations.

- Let $U_q^{\mathbb{Z}}(\mathfrak{g}) \subset U_q(\mathfrak{g})$ be Lusztig's integral form[10], which is a subalgebra generated over $\mathbb{C}[q,q^{-1}]$ by the quantum divided powers $E_i^{(n)} = \frac{E_i^n}{[n]_{d_i}!}$, $F_i^{(n)} = \frac{F_i^n}{[n]_{d_i}!}$ and $\left[\begin{array}{c} K_{\mu^\vee}, m \\ n \end{array}\right]_{d_i}$, where $[n]_d! = \prod_{j=1}^n \frac{q^{dj}-q^{-dj}}{q^d-q^{-d}}$.
- Fix an odd positive integer l which is greater than the Coxeter number of R, l should also be coprime to 3 if G contains a factor of type G_2. Fix a primitive l-th root of unity ξ_l. Define the quantum group at l-th root of unity as $U_{\xi_l}(\mathfrak{g}) = U_q^{\mathbb{Z}}(\mathfrak{g}) \otimes_{\mathbb{C}[q,q^{-1}]} \frac{\mathbb{C}[q,q^{-1}]}{(q-\xi_l)}$.
- Once the choice $\mathfrak{b}$ of Borel is fixed, there are the following induced triangular decompositions:
$$U_q^{\mathbb{Z}}(\mathfrak{g}) = U_q^{\mathbb{Z},+}(\mathfrak{g}) \otimes_{\mathbb{C}[q,q^{-1}]} U_q^{\mathbb{Z},0}(\mathfrak{g}) \otimes_{\mathbb{C}[q,q^{-1}]} U_q^{\mathbb{Z},-}(\mathfrak{g}),$$
$$U_{\xi_l}(\mathfrak{g}) = U_{\xi_l}^+(\mathfrak{g}) \otimes_{\mathbb{C}} U_{\xi_l}^0(\mathfrak{g}) \otimes_{\mathbb{C}} U_{\xi_l}^-(\mathfrak{g})$$

1.2 Review of the Representation Theory of $U_{\xi_l}(\mathfrak{g})$

The category of finite dimensional complex representations of $U_{\xi_l}(\mathfrak{g})$ is not semisimple. However, simple objects are still labelled by the dominant integral weights $\mu \in X^+$ of G. We denote the simple object of highest weight μ as $L(\mu)$. This category also has the structure of a highest weight category, with standard objects $\Delta(\mu)$, costandard objects $\nabla(\mu)$ and tilting objects $T(\mu)$.

Let $Fr : Rep(G) \to U_{\xi_l}(\mathfrak{g})$-mod be the quantum Frobenius functor sending the irreducible complex representation V_λ with highest weight $\lambda \in X^+$ to the irreducible representation $L(l\lambda)$ of highest weight $l\lambda$.

Any highest weight can be uniquely written as $\mu = \lambda_0 + l\lambda_1$, where λ_1 is a dominant integral weight and λ_0 belongs to the set of l-restricted dominant integral weights $X_1^+ = \{\lambda \in X | 0 \leq \langle \lambda, \alpha_i^\vee \rangle < p \text{ for all simple roots } \alpha_i\}$.

Now the Steinberg tensor product theorem [9] states that in the category of finite dimensional modules of $U_{\xi_l}(\mathfrak{g})$, if $\mu = \lambda_0 + l\lambda_1$ as above, we have

$$L(\mu) = L(\lambda_0 + l\lambda_1) = L(\lambda_0) \otimes Fr(V_{\lambda_1})$$

Since the category is not semisimple, we're interested in the block decomposition such that representations belonging to different blocks do not have any non-zero derived homomorphisms between each other. We're primarily interested in the principal block $\text{Block}_0(U_{\xi_l}(\mathfrak{g}))$ containing highest weight 0, which is one of the biggest blocks.

To label the he highest weights in this block, we need more notations. Let $W_{aff} = W \ltimes X$ be the affine Weyl group. A group element in this semi-direct product is denoted by wt_λ, where $w \in W$ and $\lambda \in X$. Define an action of W_{aff} on X centered at $-\rho$ given by

$$(wt_\lambda) \cdot \mu = w(l\lambda + \mu + \rho) - \rho$$

The highest weights that belong to the principal block containing highest weight 0 are exactly in the set [9] $X^+ \cap W_{aff} \cdot 0$. They are in bijection with the set of right W-minimal elements in W_{aff}. Let's denote a right W-minimal element by $w_{aff,R}$. Therefore the simple objects in the principle block $\mathrm{Block}_0(U_{\xi_l}(\mathfrak{g}))$ are precisely $L(w_{aff,R}^{-1} \cdot 0)$. Each right W-minimal element can be uniquely written as xt_λ, where x is a restricted right W-minimal element and $\lambda \in -X^+$ is a weight of G in the opposite of the dominant integral cone X^+. Each restricted element x can be uniquely factored as $wt_{\lambda'}$, where $w \in W$ and $\langle \lambda', \alpha_i^\vee \rangle = 0$ for a simple coroot $\alpha_i^\vee$ of G if $w(\alpha_i^\vee)$ is still a positive coroot, $\langle \lambda', \alpha_i^\vee \rangle = -1$ if not. As a result there are exactly $|W|$-many restricted elements. Therefore with the notations above, the highest weights in $\mathrm{Block}_0(U_{\xi_l}(\mathfrak{g}))$ are in the form $(xt_\lambda)^{-1} \cdot 0$.

For quantum group at roots of unity there's a Donkin's tilting tensor product theorem for tilting modules that resembles the Steinberg tensor product theorem:

Theorem 1. *[2] Suppose λ_0 is l-restricted (in X_1^+) and μ is dominant integral, we have $T((l-1)\rho + \lambda_0 + l\mu) \cong T((l-1)\rho + \lambda_0) \otimes Fr(V_\mu)$.*

If we restrict this theorem to the case where $(l-1)\rho + \lambda_0$ belongs to the principal block, we can rewrite it using the notations introduced above as:

$$T((xt_{-\rho+w_0\mu})^{-1} \cdot 0) \cong T((xt_{-\rho})^{-1} \cdot 0) \otimes Fr(V_\mu)$$

Here μ is dominant integral and w_0 is the longest element in W.

In the next chapter we'll give a geometric interpretation of this formula with the help of the Geometric Langlands Correspondence, which provides a different way to understand this theorem apart from the existing algebraic proof.

Chapter 2

Perverse Sheaves on the Affine Grassmannian and the Principal Block

In this chapter we use Langlands Duality and the geometry of the affine grassmannian of $G^\vee$ to study tilting representations of the principal block.

2.1 Sheaves on the Affine Grassmannian of $G^\vee$ and Representations of G

Let $K = \mathbb{C}((t))$ be the field of Laurent power series in t and $O = \mathbb{C}[[t]]$ be the subring of power series. The affine grassmannian of $G^\vee$ is defined as $Gr_{G^\vee} = G^\vee(K)/G^\vee(O)$. Let $D^b_c(G^\vee(O)\backslash G^\vee(K)/G^\vee(O))$ be the equivariant derived category of constructible sheaves with complex coefficients. There is a perverse t-structure [4] on this dg category and the heart $\mathrm{Perv}(G^\vee(O)\backslash G^\vee(K)/G^\vee(O)) \subset D^b_c(G^\vee(O)\backslash G^\vee(K)/G^\vee(O))$ is the Abelian category of equivariant perverse sheaves, with simple objects being equivariant intersection cohomology sheaves on the closure of each stratum.

To work with perverse sheaves, we will frequently use the geometric notion of a semi-small map and the decomposition theorem.

Recall that a stratified map $\pi : (X, \{X_\alpha\}) \to (Y, \{Y_\beta\})$ is a map between varieties with Whitney stratifications such that $\pi(X_\alpha)$ is a union of stata Y_β on Y and that π is a smooth fibration over each Y_β with fiber F_β. π is called stratified semi-small if

$$2\ \mathrm{dim} F_\beta \leq \mathrm{dim} X_\alpha - \mathrm{dim} Y_\beta$$

holds for all strata X_α and Y_β such that $\pi(X_\alpha)$ contains Y_β.

If π is proper and stratified semi-small, then the pushforward functor $\pi_* = \pi_!$ preserves the perverse t-structure on the derived category of constructible sheaves [4]. Moreover if the inequality is strict whenever $\dim F_\beta \neq 0$, the map is called stratifed small and

$$\pi_*(IC_X) = IC_Y$$

Simple objects in $\mathrm{Perv}(G^\vee(O)\backslash G^\vee(K)/G^\vee(O))$ are intersection cohomology sheaves IC_λ labeled by the cosets in $G^\vee(O)\backslash G^\vee(K)/G^\vee(O)$, i.e. the dominant cocharacters of $G^\vee$ or the dominant integral weights $\lambda \in X^+$ of G. Moreover the convolution on $G^\vee(O)\backslash G^\vee(K)/G^\vee(O)$ induces a monoidal structure on $D^b_c(G^\vee(O)\backslash G^\vee(K)/G^\vee(O))$. This monoidal structure preserves the perverse t-structure because the convolution map is stratified semi-small [8] [11]. Therefore we have a monoidal abelian category

$$(\mathrm{Perv}(G^\vee(O)\backslash G^\vee(K)/G^\vee(O)), *)$$

whose simple objects are labeled by the dominant integral weights $\lambda \in X^+$ of G. We then have the following Geometric Satake equivalence:

Theorem 2. *[8] [11] There exists an equivalence of monoidal Abelian categories*

$$Sat\colon (Perv(G^\vee(O)\backslash G^\vee(K)/G^\vee(O)), *) \to (Rep(G), \otimes)$$

sending IC_λ to V_λ.

Besides constructible sheaves on the complex affine grassmannian, one can also consider etale constructible sheaves on the affine grassmannian defined over $\mathbb{F}_p$.

We shall also make use of a variant of the Geometric Satake equivalence called the Iwahori-Whittaker model. For convenience we will consider etale constructible sheaves. Let $N^\vee_- \subset G^\vee$ be the unipotent subgroup corresponding to the negative roots. Let $I^\vee_{u,-}$ denote the preimage of $N^\vee_-$ in $G^\vee(O)$. Consider a generic additive character

$$\psi : I^\vee_{u,-} \to N^\vee_- \to N^\vee_-/[N^\vee_-, N^\vee_-] \to \mathbb{G}_a$$

The convenience of using etale sheaves is such that the category $D^b(G^\vee(O)\backslash G^\vee(K)/(I^\vee_{u,-}, \psi))$ of sheaves on $G^\vee(O)\backslash G^\vee(K)$ which are right twisted equivariant with respect to the character $(I^\vee_{u,-}, \psi)$ is a subcategory of etale constructible sheaves on $G^\vee(O)\backslash G^\vee(K)$, thanks to the existence of Artin-Schreier local systems on $\mathbb{G}_a$ in etale topology.

The cosets in $G^\vee(O)\backslash G^\vee(K)/(I^\vee_{u,-})$ that can support an equivariant local system twisted by ψ are labelled by the subset of elements in W_{aff} that are left W-maximal and right W-minimal. Such kind of elements are precisely $t_{-\rho-\lambda}$ for $\lambda \in X^+$. Therefore the simple perverse sheaves are again in one to one correspondence with irreducible representations of G. Let $w_0 \in W$ be the longest element. Then the simple perverse sheaves are denoted by

$$IC_{G^\vee(O),IW}(t_{-\rho+w_0(\mu)}) \in \mathrm{Perv}(G^\vee(O)\backslash G^\vee(K)/(I^\vee_{u,-}, \psi))$$

where μ runs through all dominant integral weights of G.

Note that $\mathrm{Perv}(G^\vee(O)\backslash G^\vee(K)/G^\vee(O))$ acts on $\mathrm{Perv}(G^\vee(O)\backslash G^\vee(K)/(I^\vee_{u,-}, \psi))$ via convolution on the left. The following theorem, called the Iwahori-Whittaker model of Geometric Satake, states that $IC_{G^\vee(O),IW}(t_{-\rho})$ is the generator of $\mathrm{Perv}(G^\vee(O)\backslash G^\vee(K)/(I^\vee_{u,-}, \psi))$ as a rank one free left module category of $\mathrm{Perv}(G^\vee(O)\backslash G^\vee(K)/G^\vee(O))$.

Theorem 3. *[6] There exists an equivalence of Abelian categories*

$$Perv(G^\vee(O)\backslash G^\vee(K)/G^\vee(O)) \to Perv(G^\vee(O)\backslash G^\vee(K)/(I^\vee_{u,-},\psi))$$

given by convolution with $IC_{G^\vee(O),IW}(t_{-\rho})$.

From this theorem we get the following useful formula:

$$IC_\mu *_{G^\vee(O)} IC_{G^\vee(O),IW}(t_{-\rho}) \cong IC_{G^\vee(O),IW}(t_{-\rho+w_0(\mu)}).$$

2.2 The ABG Equivalence

In the seminal work [3] the authors identified the Abelian category $\mathrm{Perv}(I^\vee_u\backslash G^\vee(K)/G^\vee(O))$ with $\mathrm{Block}_0(U_{\xi_l}(\mathfrak{g}))$. First note that both categories are highest weight categories with simple objects indexed by right W-minimal elements in W_{aff}. The simple objects in $\mathrm{Block}_0(U_{\xi_l}(\mathfrak{g}))$ are $L((xt_\lambda)^{-1}\cdot 0)$ and the simple objects in $\mathrm{Perv}(I^\vee_u\backslash G^\vee(K)/G^\vee(O))$ are denoted $IC_{I^\vee_u,G^\vee(O)}(xt_\lambda)$. The indecomposable tilting representations and tilting perverse sheaves are denoted respectively by $T((xt_\lambda)^{-1}\cdot 0)$ and $T_{I^\vee_u,G^\vee(O)}(xt_\lambda)$.

Moreover, both categories are module categories of $\mathrm{Rep}(G)$. $V\in\mathrm{Rep}(G)$ acts on $\mathrm{Block}_0(U_{\xi_l}(\mathfrak{g}))$ by sending a module M to $M\otimes Fr(V)$. $\mathrm{Rep}(G)\cong\mathrm{Perv}(G^\vee(O)\backslash G^\vee(K)/G^\vee(O))$ acts on $\mathrm{Perv}(I^\vee_u\backslash G^\vee(K)/G^\vee(O))$ by convolution on the right.

Theorem 4. *[3] There exists an equivalence of highest weight Abelian categories between* $Perv(I^\vee_u\backslash G^\vee(K)/G^\vee(O))$ *and* $Block_0(U_{\xi_l}(\mathfrak{g}))$. *In particular the equivalence matches* $T((xt_\lambda)^{-1}\cdot 0)$ *and* $T_{I^\vee_u,G^\vee(O)}(xt_\lambda)$. *This equivalence also commutes with the Frobenius twisted action of* $Rep(G)$ *on both sides as described above.*

Hence after the ABG equivalence, we can translate the Donkin's tensor product theorem in Chapter 2 into a statement of tilting perverse sheaves:

Corollary 1. *The Donkin's tensor product formula is equivalent to the following isomorphism of perverse sheaves:*

$$T_{I^\vee_u,G^\vee(O)}(xt_{-\rho}) *_{G^\vee(O)} IC_\mu \cong T_{I^\vee_u,G^\vee(O)}(xt_{-\rho+w_0(\mu)})$$

in $Perv(I^\vee_u\backslash G^\vee(K)/G^\vee(O))$

In the next section we'll recall the notion of monoidal Koszul Duality which exchanges indecomposable tilting sheaves and IC sheaves, to help us justify the isomorphism above.

2.3 Koszul Duality of Loop Groups and Gaitsgory's Central Sheaves

Studying tilting representations or tilting perverse sheaves directly is not easy, luckily for sheaves with characteristic zero coefficients we have the Koszul duality exchanging tilting sheaves and intersection cohomology sheaves, which are easier to work with.

Bezrukavnikov and Yun [5] proved a Koszul Duality between the free monodromic completion of the $I^\vee$-bimonodromic mixed derived category $\widehat{D^b_{mix}}^{bim}(I^\vee_u\backslash G^\vee(K)/I^\vee_u)$ and the $I^\vee$ bivariant derived category $D^b_{mix}(I^\vee\backslash G^\vee(K)/I^\vee)$. Under this equivalence, free monodromic indecomposable mixed tilting perverse sheaves on the left hand side corresponds to indecomposable mixed IC sheaves on the right. This Koszul duality is an equivalence of monoidal categories. Moreover, they also showed that there's a Koszul duality between $D^b_{mix}(I^\vee_u\backslash G^\vee(K)/G^\vee(O))$ and $D^b_{mix}(I^\vee\backslash G^\vee(K)/(I^\vee_{u,-},\psi))$ which sends mixed tilting sheaves to mixed IC sheaves and that Koszul duality preserves the left module structures induced by convolution on these two categories. Here $(I^\vee_{u,-},\psi)$ is again a generic Iwahori-Whittaker equivariant condition as discussed in section 1.

In [7] Gaitsgory constructed certain objects called central sheaves in $D^b_{mix}(I^\vee\backslash G^\vee(K)/I^\vee)$. Let $Z^{mix}(IC_\mu)$ be the mixed central sheaf in $D^b_{mix}(I^\vee\backslash G^\vee(K)/I^\vee)$ obtained from applying the nearby cycle functor of the central degeneration to the mixed perverse sheaf $\delta_{B^\vee\backslash B^\vee/B^\vee}\boxtimes IC_\mu$ in $Perv^{mix}(B^\vee\backslash G^\vee/B^\vee\times G^\vee(O)\backslash G^\vee(K)/G^\vee(O))$. Let $\widehat{Z^{mix}}(IC_\mu)$ be the mixed central sheaf in $\widehat{D^b_{mix}}^{bim}(I^\vee_u\backslash G^\vee(K)/I^\vee_u)$ obtained from applying the nearby cycle functor of the central degeneration to the mixed free monodromic perverse sheaf $\delta_{N^\vee\backslash \tilde{B}^\vee/N^\vee}\boxtimes IC^\mu$ in $Perv^{mix,bim}(N^\vee\backslash G^\vee/N^\vee\times G^\vee(O)\backslash G^\vee(K)/G^\vee(O))$.

We have the following conjecture, which is the key to the geometric interpretation of Donkin's tensor product theorem.

Conjecture 1. *Under the Koszul duality of Bezrukavnikov and Yun [5],*

$$\widehat{Z^{mix}}(IC_\mu) \text{ corresponds to } Z^{mix}(IC_\mu)$$

.

Before proceeding to the last section where we deduce Donkin's theorem assuming this conjecture, let's recall some properties [7] of these central sheaves that are useful.

The first property is that for any sheaf $\mathcal{F}$ in $D^b_{mix}(I^\vee\backslash G^\vee(K)/I^\vee)$,

$$\mathcal{F}*_{I^\vee} Z^{mix}(IC_\mu)\cong Z^{mix}(IC_\mu)*_{I^\vee}\mathcal{F}$$

The second property is that

$$\pi_*(Z^{mix}(IC_\mu))\cong IC_\mu$$

where π is the left projection $\pi: I^\vee\backslash G^\vee(K)\to G^\vee(O)\backslash G^\vee(K)$ or the right projection.

The third prperty is that for any sheaf $\mathcal{F}$ in $D^b_{mix}(I_u^\vee \backslash G^\vee(K)/I^\vee)$,

$$\mathcal{F} *_{I^\vee} Z^{mix}(IC_\mu) \cong \widehat{Z^{mix}}(IC_\mu) *_{I_u^\vee} \mathcal{F}$$

2.4 Geometric Interpretation of Donkin's formula

In this section let's finish the proof of the main theorem:

Theorem 5. *Assuming Conjecture 1, the isomorphism from section 3.2*

$$T_{I_u^\vee, G^\vee(O)}(xt_{-\rho}) *_{G^\vee(O)} IC_\mu \cong T_{I_u^\vee, G^\vee(O)}(xt_{-\rho+w_0(\mu)})$$

holds in $Perv(I_u^\vee \backslash G^\vee(K)/G^\vee(O))$

To prove this isomorphism, we take the preferred lift of both sides in the mixed category and apply Koszul duality. By the centrality of central sheaves (the first property) and the fact that central sheaves averages to IC sheaves on the affine grassmannian(the second property),

$$\widehat{Z^{mix}}(IC_\mu) *_{I_u^\vee} T^{mix}_{I_u^\vee, G^\vee(O)}(xt_{-\rho}) \cong T^{mix}_{I_u^\vee, G^\vee(O)}(xt_{-\rho}) *_{G^\vee(O)} IC_\mu$$

. Therefore, after applying Koszul duality and our first theorem, it suffices to show that

$$Z^{mix}(IC_\mu) *_{I^\vee} IC^{mix}_{I^\vee, IW}(xt_{-\rho}) \cong IC^{mix}_{I^\vee, IW}(xt_{-\rho+w_0(\mu)})$$

in $D^b_{mix}(I^\vee \backslash G^\vee(K)/(I^\vee_{u,-}, \psi))$ Since the preferred lift is possible and unique, it's enough to show the non-mixed version.

Thus we want to show that

$$Z(IC_\mu) *_{I^\vee} IC_{I^\vee, IW}(xt_{-\rho}) \cong IC_{I^\vee, IW}(xt_{-\rho+w_0(\mu)})$$

in $D^b(I^\vee \backslash G^\vee(K)/(I^\vee_{u,-}, \psi))$.

Since $l(xt_{-\rho}) = l(x) + l(t_{-\rho})$ and $t_{-\rho}$ is left W-maximal, by the decomposition theorem the left hand side is a direct summand of

$$Z(IC_\mu) *_{I^\vee} IC_{I^\vee, G^\vee(O)}(x) *_{G^\vee(O)} IC_{G^\vee(O), IW}(t_{-\rho})$$

. Here $IC_{I^\vee, G^\vee(O)}(x)$ is the IC sheaf in $D^b(I^\vee \backslash G^\vee(K)/G^\vee(O))$ lebelled by the right W- minimal element x and $IC_{G^\vee(O), IW}(t_{-\rho})$ is the IC sheaf in $D^b(G^\vee(O) \backslash G^\vee(K)/(I^\vee_{u,-}, \psi))$ labelled by the element $t_{-\rho}$ which is both left W-maximal and right W-minimal.

We plan to show that this triple convolution product is an indecomposable IC sheaf, so that its summand will also be an indecomposable IC sheaf.

By centrality and the fact that central sheaves averages to IC sheaves, we can move the central sheaf to the middle:

$$Z(IC_\mu) *_{I^\vee} IC_{I^\vee, G^\vee(O)}(x) *_{G^\vee(O)} IC_{G^\vee(O), IW}(t_{-\rho}) \cong IC_{I^\vee, G^\vee(O)}(x) *_{G^\vee(O)} IC_\mu *_{G^\vee(O)} IC_{G^\vee(O), IW}(t_{-\rho}).$$

Then, applying the Iwahori-Whittaker model of the Satake category, $IC_{G^\vee(O),IW}(t_{-\rho})$ is the generator of $Perv(G^\vee(O)\backslash G^\vee(K)/(I^\vee_{u,-},\psi))$ as a rank one free left module of $Perv(G^\vee(O)\backslash G^\vee(K)/G^\vee(O))$. We have

$$IC_\mu *_{G^\vee(O)} IC_{G^\vee(O),IW}(t_{-\rho}) \cong IC_{G^\vee(O),IW}(t_{-\rho+w_0(\mu)}).$$

Therefore, it suffices to show that $IC_{I^\vee,G^\vee(O)}(x)*_{G^\vee(O)}IC_{G^\vee(O),IW}(t_{-\rho+w_0(\mu)})$ is an indecomposable IC sheaf, which is a corollary of the following theorem.

Theorem 6. *[1] The convolution functor*

$$* : Perv^{res}(I^\vee_u\backslash G^\vee(K)/G^\vee(O))\times Perv(G^\vee(O)\backslash G^\vee(K)/(I^\vee_{u,-},\psi)) \to Perv(I^\vee_u\backslash G^\vee(K)/(I^\vee_{u,-},\psi))$$

is fully faithful on both factors, with coefficients in any field. The notation res in the first factor means the subcategory of sheaves supported on restricted strata. Therefore when we convolve two indecomposable objects, the endomorphism algebra of the result is the tensor product of the endomorphisms of the two objects, which is again a local algebra.

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