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Inverse Boundary Problems for Biharmonic Operators and Nonlinear PDEs on Riemannian Manifolds

DISSERTATION

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by

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VITA

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ABSTRACT OF THE DISSERTATION

Inverse Boundary Problems for Biharmonic Operators and Nonlinear PDEs on Riemannian Manifolds

By

Lili Yan

Doctor of Philosophy in Mathematics

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Professor Katya Krupchyk, Chair

This thesis compiles my work on three projects.

In my first project, we proved a global uniqueness result for an inverse boundary problem for a first order perturbation of the biharmonic operator on a conformally transversally anisotropic (CTA) Riemannian manifold of dimension $n \geq 3$. Specifically, we established that a continuous first order perturbation can be determined uniquely from the knowledge of the Cauchy data set of solutions of the perturbed biharmonic operator on the boundary of the manifold provided that the geodesic X-ray transform on the transversal manifold is injective.

In my second project, we showed that a continuous potential can be constructively determined from the Cauchy data set of solutions to the perturbed biharmonic equation on a CTA Riemannian manifold of dimension ≥ 3 with boundary, assuming that the geodesic X-ray transform on the transversal manifold is constructively invertible. This is a constructive counterpart of our uniqueness result [119]. In particular, our result is applicable and new in the case of smooth bounded domains in the 3–dimensional Euclidean space as well as in the case of 3–dimensional CTA manifolds with simple transversal manifold.

In my third project joint with Katya Krupchyk and Gunther Uhlmann, we solved an inverse boundary problem for the nonlinear magnetic Schrödinger operator on a compact complex manifold, equipped with a Kähler metric and admitting sufficiently many global holomorphic functions.

Chapter 1

Introduction

In inverse problems, one aims to recover the internal properties of a medium by indirect measurements, say, measurements along the boundary of the medium or scattering measurements. Such problems arise in many important practical situations such as monitoring cardiac activity, lung function, and pulmonary perfusion in medical imaging, oil prospecting in exploration geophysics, and corrosion, cracks in non-destructive testing. For references see the survey [19].

In 1980, Calderón published a short paper entitled *On an inverse boundary value problem* [25], asking the following question:

Calderón's Problem: Is it possible to determine the electrical conductivity of a medium by making voltage and current measurements on its boundary?

To state this problem mathematically, let $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$, be a bounded open set with smooth boundary and let γ be a positive smooth function on $\overline{\Omega}$, representing the electrical conductivity of the domain. Under the assumption of no sources or sinks of current in Ω, a voltage f at the boundary $\partial\Omega$ induces a voltage potential u in Ω, which solves the Dirichlet problem for the conductivity equation,

$$\begin{cases} \operatorname{div}(\gamma \cdot \nabla u) = 0 \quad \text{in} \quad \Omega, \\ \quad u|_{\partial\Omega} = f. \end{cases} \tag{1.0.1}$$

There is a unique weak solution $u \in H^1(\Omega)$ for any boundary value $f \in H^{\frac{1}{2}}(\partial\Omega)$. One can define the Dirichlet–to–Neumann map associated to this problem as follows:

$$\Lambda_\gamma(f) = (\gamma \partial_\nu u)|_{\partial\Omega},$$

where ν is the unit outer normal to $\partial\Omega$. The Dirichlet–to–Neumann map Λ_γ encodes the voltage to current measurements performed along the boundary of the domain. That is, if the measured currents $\Lambda_\gamma(f)$ are known for all boundary voltages f, one would like to determine the conductivity γ. To ensure the possibility of unique recovery, one should have a global uniqueness result stating that if $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$ for two conductivities γ_1 and γ_2, then $\gamma_1 = \gamma_2$.

The inverse conductivity problem has been studied intensively starting with the work [25] of Calderón in 1980. The first global uniqueness result is obtained by Sylvester and Uhlmann [115] in their breakthrough work in 1987 for C^2 conductivities and $n \geq 3$. Haberman and Tataru [55] extended the uniqueness result to Lipschitz conductivities under a smallness condition, which has later been removed by Caro and Rogers [27]. The corresponding result in dimension 2 was given by Nachman [96] for conductivities of Sobolev class $W^{2,p}$ with some $p > 1$, and the regularity was later improved to L^∞ conductivities by Astala and Päivärinta [11]. The main contribution of Sylvester and Uhlmann [115] is the construction of *complex geometric optics* (CGO) solutions for the Schrödinger equation, which play an essential role in solving elliptic inverse problems. Among the three projects we are going to discuss, the first two rely heavily on the properties of appropriate CGO solutions.

Once uniqueness results for inverse boundary problems have been established, one is interested in upgrading them to a reconstruction procedure. The uniqueness result in [115] was extended to a reconstruction procedure by Nachman [95] and independently by Novikov [99] for $n \geq 3$. The reconstruction procedure for $n = 2$ is given by [96] along with the uniqueness result.

Another interesting inverse problem is to consider the stability result, i.e., does the closeness of Λ_{γ_1} and Λ_{γ_2} imply the closeness of γ_1 and γ_2? It is well-known that the Calderón problem is severely ill-posed. A log-type stability estimate was established by Alessandrini [1] for conductivities of Sobolev space H^s with $s > \frac{n}{2} + 2$, and it has been shown by Mandache [89] that this estimate is optimal up to the value of the exponent.

In the discussion above, we considered the case of full data, where one can do measurements on the whole boundary. However, making measurements on the entire boundary may not be possible in practice. For instance, one can only cover a tiny part of the Earth's surface with measurements devices in geophysical imaging. Inverse problems with such restrictions are more difficult. The first uniqueness result for partial data measurements is due to Bukhgeim and Uhlmann [24] for C^2 conductivities, where the Dirichlet–to–Neumann map is restricted to slightly more than half of the boundary. The result has been improved significantly by Kenig, Sjöstrand, and Uhlmann [63] where they show that the knowledge of the Dirichlet–to–Neumann map on a possibly very small open subset of the boundary determines the conductivity uniquely. The corresponding reconstruction procedure of [63] is obtained by Nachman and Street [97]. The approaches of [24, 63] are based on Carleman estimates with boundary terms. The reader is referred to the recent survey article [61] by Kenig and Salo for Calderón problems with partial data.

The results mentioned previously are concerned with isotropic materials with the conductivity γ being a scalar function. However, there are more complicated anisotropic materials with the conductivity γ being an $n \times n$ matrix, depending on directions. Muscle tissue in the human body is an important example of an anisotropic conductor, where cardiac muscle has a conductivity of 2.3 mho in the transversal direction and 6.3 mho in the longitudinal direction [13]. Unfortunately, in anisotropic case, the knowledge of the Dirichlet–to–Neumann map Λ_γ does not determine γ uniquely, an observation due to L. Tartar (see [65] for an account), and the best we can show is that the recovery is unique up to some diffeomorphism. It turned out that this is the only obstruction to uniqueness of the conductivity for $n = 2$; see [114], [96]. Lee and Uhlmann [83] conjectured that this is also true for $n \geq 3$. In the case $n \geq 3$, this is a problem of geometrical nature; see [83]. Thus it is natural to study inverse problems on more general Riemannian manifolds.

Calderón's problem can be reduced to the problem of determining an electric potential q from the Dirichlet–to–Neumann map Λ_q associated to the Schrödinger operator $-\Delta + q$ with $q = \gamma^{-\frac{1}{2}}\Delta\gamma^{\frac{1}{2}}$, a lower order perturbation the of Laplacian. It is of great interest in the study of inverse problems to consider more general elliptic PDEs. In spite of 40 years of intensive research and an impressive body of results in the field of inverse boundary problems, see [116], [117] for recent surveys, several fundamental questions remain unsolved. In this thesis, we shall proceed to discuss an inverse boundary problem for the biharmonic operator on a Riemannian manifold, which is a fundamental problem, arising in the Kirchhoff plate equation in the theory of elasticity, the Paneitz-Branson operator in conformal geometry, and the steady Stokes flows in viscous fluids; see [44, 33, 101]. We shall also discuss an inverse boundary problem for the nonlinear magnetic Schrödinger operator on a compact complex manifold, manifesting the phenomenon, discovered in [77], that the presence of nonlinearity may help to solve inverse problems.

The thesis is organized as follows. In Chapter 2, we proved a uniqueness result for inverse boundary problems for first-order perturbations of biharmonic operators on conformally transversally anisotropic manifolds with smooth boundaries, provided that the geodesic X-ray transform on the transversal manifold is injective. The corresponding reconstruction procedure for a potential perturbation of the biharmonic operator is established in Chapter 3. Chapter 4 is devoted to inverse boundary problems for nonlinear magnetic Schrödinger operators on a compact complex manifold, equipped with a Kähler metric and admitting sufficiently many global holomorphic functions.

Chapter 2

Inverse boundary problems for biharmonic operators in transversally anisotropic geometries

2.1 Introduction and statement of results

Let (M, g) be a smooth compact oriented Riemannian manifold of dimension $n \geq 3$ with smooth boundary ∂M. Let $-\Delta_g$ be the Laplace–Beltrami operator, and let $(-\Delta_g)^2$ be the biharmonic operator on M. Let $X \in C(M, TM)$ be a complex vector field and let $q \in C(M, \mathbb{C})$. In this paper we shall be concerned with an inverse boundary problem for the first order perturbation of the biharmonic operator,

$$L_{X,q} = (-\Delta_g)^2 + X + q.$$

Let us now introduce some notation and state the main result of the paper. Let $u \in H^3(M^{\text{int}})$ be a solution to

$$L_{X,q}u = 0 \quad \text{in} \quad M. \tag{2.1.1}$$

Here and in what follows $H^s(M^{\text{int}})$, $s \in \mathbb{R}$, is the standard Sobolev space on M^{int}, and $M^{\text{int}} = M \setminus \partial M$ stands for the interior of M. Let ν be the unit outer normal to ∂M. We shall define the trace of the normal derivative $\partial_\nu(\Delta_g u) \in H^{-1/2}(\partial M)$ as follows. Let $\varphi \in H^{1/2}(\partial M)$. Then letting $v \in H^1(M^{\text{int}})$ be a continuous extension of φ, we set

$$\langle \partial_\nu(-\Delta_g u), \varphi \rangle_{H^{-1/2}(\partial M)\times H^{1/2}(\partial M)} = \int_M \big(\langle \nabla_g(-\Delta_g u), \nabla_g v\rangle_g + X(u)v + quv\big) dV_g, \tag{2.1.2}$$

where dV_g is the Riemannian volume element on M. As u satisfies (2.1.1), the definition of the trace $\partial_\nu(\Delta_g u)$ on ∂M is independent of the choice of an extension v of φ. Associated to (2.1.1), we define the set of the Cauchy data,

$$\mathcal{C}_{X,q} = \{(u|_{\partial M}, (\Delta_g u)|_{\partial M}, \partial_\nu u|_{\partial M}, \partial_\nu(\Delta_g u)|_{\partial M}) : u \in H^3(M^{\text{int}}),\ L_{X,q}u = 0 \text{ in } M\}. \tag{2.1.3}$$

Note that the first two elements in the set of the Cauchy data $\mathcal{C}_{X,q}$ correspond to the Navier boundary conditions for the first order perturbation of the biharmonic operator. Physically, such operators arise when considering the equilibrium configuration of an elastic plate which is hinged along the boundary; see [44]. One can also define the set of the Cauchy data for the first order perturbation of the biharmonic operator, based on the Dirichlet boundary conditions $(u|_{\partial M}, \partial_\nu u|_{\partial M})$, which corresponds to the clamped plate equation,

$$\widetilde{\mathcal{C}}_{X,q} = \{(u|_{\partial M}, \partial_\nu u|_{\partial M}, \partial_\nu^2 u|_{\partial M}, \partial_\nu^3 u|_{\partial M}) : u \in H^3(M^{\text{int}}),\ L_{X,q}u = 0 \text{ in } M\}.$$

The explicit description for the Laplacian in the boundary normal coordinates shows that $\mathcal{C}_{X,q} = \widetilde{\mathcal{C}}_{X,q}$; see [83], [67].

The inverse boundary problem that we are interested in is to determine the vector field X and the potential q from the knowledge of the set of the Cauchy data $\mathcal{C}_{X,q}$.

This problem was studied extensively in the Euclidean setting; see [68], [67], [5], [6], [8], [56] [57] [17], [18], [45], [46], [120]. Specifically, it was shown in [68] that the set of the Cauchy data $\mathcal{C}_{X,q}$ determines the vector field X and the potential q uniquely. Let us note that the unique determination of a first order perturbation of the Laplacian is not possible due to the gauge invariance of boundary measurements and in this case the first order perturbation can be recovered only modulo a gauge transformation; see [98], [111].

Going beyond the Euclidean setting, inverse boundary problems for lower order perturbations of the Laplacian were only studied in the case when (M, g) is CTA (conformally transversally anisotropic; see Definition 2.1.1 below) and under the assumption that the geodesic X-ray transform on the transversal manifold is injective; see the fundamental works [36] and [38] which initiated this study, and see also [37], [35], [73], [72], [32].

Definition 2.1.1. *A compact Riemannian manifold (M, g) of dimension $n \geq 3$ with boundary ∂M is called conformally transversally anisotropic (CTA) if $M \subset\subset \mathbb{R} \times M_0^{int}$ where $g = c(e \oplus g_0)$, $(\mathbb{R}, e)$ is the Euclidean real line, (M_0, g_0) is a smooth compact $(n-1)$-dimensional manifold with smooth boundary, called the transversal manifold, and $c \in C^\infty(\mathbb{R} \times M_0)$ is a positive function.*

The injectivity of the geodesic X-ray transform is known when the manifold (M_0, g_0) is simple, in the sense that any two points in M_0 are connected by a unique geodesic depending smoothly on the endpoints and that ∂M_0 is strictly convex (see [4], [94]), when M_0 has strictly convex boundary and is foliated by strictly convex hypersurfaces [110], [118], and also when M_0 has a hyperbolic trapped set and no conjugate points [48], [49]. An example of the latter occurs when M_0 is a negatively curved manifold.

Turning our attention to the inverse boundary problem of determining the first order perturbation of the biharmonic operator, this problem was solved in [9] in the case when (M, g) is CTA and the transversal manifold (M_0, g_0) is simple, extending the result of [36] to the case of biharmonic operators. To be on par with the best results available for the perturbations of the Laplacian in the context of Riemannian manifolds, the goal of this paper is to solve the inverse problem for the first order perturbation of the biharmonic operator in the case when (M, g) is CTA and the geodesic X-ray transform is injective on the transversal manifold (M_0, g_0), generalizing the result of [38] to the case of biharmonic operators.

Let us recall some definitions related to the geodesic X-ray transform following [48], [36]. The geodesics on M_0 can be parametrized by points on the unit sphere bundle $SM_0 = \{(x, \xi) \in TM_0 : |\xi| = 1\}$. Let

$$\partial_{\pm} SM_0 = \{(x, \xi) \in SM_0 : x \in \partial M_0, \pm\langle \xi, \nu(x)\rangle > 0\}$$

be the incoming $(-)$ and outgoing $(+)$ boundaries of SM_0. Here ν is the unit outer normal vector field to ∂M_0. Here and in what follows $\langle \cdot, \cdot \rangle$ is the duality between T^*M_0 and TM_0.

Let $(x, \xi) \in \partial_- SM_0$ and $\gamma = \gamma_{x,\xi}(t)$ be the geodesic on M_0 such that $\gamma(0) = x$ and $\dot{\gamma}(0) = \xi$. Let us denote by $\tau(x, \xi)$ the first time when the geodesic γ exits M_0 with the convention that $\tau(x, \xi) = +\infty$ if the geodesic does not exit M_0. We define the incoming tail by

$$\Gamma_- = \{(x, \xi) \in \partial_- SM_0 : \tau(x, \xi) = +\infty\}.$$

When $f \in C(M_0, \mathbb{C})$ and $\alpha \in C(M_0, T^*M_0)$ is a complex valued 1-form, we define the geodesic X-ray transform on (M_0, g_0) as follows:

$$I(f,\alpha)(x,\xi) = \int_0^{\tau(x,\xi)} \big[f(\gamma_{x,\xi}(t)) + \langle \alpha(\gamma_{x,\xi}(t)), \dot{\gamma}_{x,\xi}(t)\rangle\big]dt, \quad (x,\xi) \in \partial_- SM_0 \setminus \Gamma_-.$$

A unit speed geodesic segment $\gamma = \gamma_{x,\xi} : [0, \tau(x,\xi)] \to M_0$, $\tau(x,\xi) > 0$, is called nontangential if $\gamma(0), \gamma(\tau(x,\xi)) \in \partial M_0$, $\dot{\gamma}(0), \dot{\gamma}(\tau(x,\xi))$ are nontangential vectors on ∂M_0, and $\gamma(t) \in M_0^{\text{int}}$ for all $0 < t < \tau(x,\xi)$.

Assumption 1. *We assume that the geodesic X-ray transform on (M_0, g_0) is injective in the sense that if $I(f,\alpha)(x,\xi) = 0$ for all $(x,\xi) \in \partial_- SM_0 \setminus \Gamma_-$ such that $\gamma_{x,\xi}$ is a nontangential geodesic, then $f = 0$ and $\alpha = dp$ in M_0 for some $p \in C^1(M_0, \mathbb{C})$ with $p|_{\partial M_0} = 0$.*

The main result of the paper is as follows.

Theorem 2.1.2. *Let (M, g) be a CTA manifold of dimension $n \geq 3$ such that Assumption 1 holds for the transversal manifold. Let $X^{(1)}, X^{(2)} \in C(M, TM)$ be complex vector fields, and let $q^{(1)}, q^{(2)} \in C(M, \mathbb{C})$. If $\mathcal{C}_{X^{(1)},q^{(1)}} = \mathcal{C}_{X^{(2)},q^{(2)}}$, then $X^{(1)} = X^{(2)}$ in M. Assuming furthermore that*

$$q^{(1)}|_{\partial M} = q^{(2)}|_{\partial M}, \tag{2.1.4}$$

we have $q^{(1)} = q^{(2)}$ in M.

Remark 2.1.3. *Examples of nonsimple manifolds M_0 satisfying Assumption 1 include in particular manifolds with a strictly convex boundary which are foliated by strictly convex hypersurfaces [110], [118], and manifolds with a hyperbolic trapped set and no conjugate points [48], [49].*

Remark 2.1.4. *To the best of our knowledge, Theorem 2.1.2 seems to be the first result where one recovers a vector field uniquely on general CTA manifolds.*

Remark 2.1.5. *The assumption* (2.1.4) *is made for simplicity only and can be removed by performing the boundary determination as done in Section 2.5 for the vector fields* $X^{(1)}$ *and* $X^{(2)}$. *This can be done by using the approach of [53] combined with its extensions in [74] and [41].*

Let us proceed to describe the main ideas in the proof of Theorem 2.1.2. The key step in the proof is a construction of complex geometric optics solutions for the equations $L_{X,q}u = 0$ and $L_{-\overline{X},-\operatorname{div}(\overline{X})+\overline{q}}u = 0$ in M. Here the operator $L_{-\overline{X},-\operatorname{div}(\overline{X})+\overline{q}}$ represents the formal L^2 adjoint of the operator $L_{X,q}$. In contrast to the work [9], where one deals with the same inverse problem in the case of a simple transversal manifold, here without a simplicity assumption, complex geometric optics solutions cannot be easily constructed by means of a global WKB method, and following [38], we shall construct complex geometric optics solutions based on Gaussian beam quasimodes for the biharmonic operator $(-\Delta_g)^2$ conjugated by an exponential weight corresponding to the limiting Carleman weight $\phi(x) = \pm x_1$ for $-h^2\Delta_g$ on the CTA manifold (M, g); see [36]. To convert the Gaussian beam quasimodes to exact solutions, we shall rely on the corresponding Carleman estimate with a gain of two derivatives established in [73]; see also [36].

Remark 2.1.6. *We would like to note that one can obtain Gaussian beam quasimodes for the biharmonic operator* $(-\Delta_g)^2$ *conjugated by an exponential weight as the Gaussian beam quasimodes for the Laplacian conjugated by an exponential weight. However, such quasimodes are not enough to prove Theorem 2.1.2 as in order to recover the vector field uniquely, one has to exploit a richer set of amplitudes which are not available for the Gaussian beam quasimodes for the Laplacian.*

Remark 2.1.7. *When constructing Gaussian beam quasimodes for the Laplacian conjugated by an exponential weight, one first reduces to the setting when the conformal factor* $c = 1$ *by* *using the following transformation:*

$$c^{\frac{n+2}{4}} \circ (-\Delta_g) \circ c^{-\frac{(n-2)}{4}} = -\Delta_{\widetilde{g}} + \widetilde{q},$$

where

$$\widetilde{g} = e \oplus g_0, \quad \widetilde{q} = -c^{\frac{n+2}{4}}(-\Delta_g)(c^{-\frac{(n-2)}{4}});$$

see [38]. However, it seems that no such useful reduction is available for the biharmonic operator and therefore, when constructing Gaussian beam quasimodes for the biharmonic operator $(-\Delta_g)^2$ conjugated by an exponential weight, we shall proceed directly accommodating the conformal factor in the construction which makes it somewhat more complicated.

Once complex geometric optics solutions are constructed, the next step is to substitute them into a suitable integral identity which is obtained as a consequence of the equality $\mathcal{C}_{X^{(1)},q^{(1)}} = \mathcal{C}_{X^{(2)},q^{(2)}}$ for the Cauchy data sets. Exploiting the concentration properties of the corresponding Gaussian beam together with Assumption 1, we first show that there exists $\psi \in C^1(\mathbb{R} \times M_0)$ with compact support in x_1 such that $\psi(x_1, \cdot)|_{\partial M_0} = 0$ and $X^{(1)} - X^{(2)} = \nabla_g \psi$. To show that $\psi = 0$, i.e., $X^{(1)} = X^{(2)}$, we use the concentration properties of the Gaussian beam for the biharmonic operator with a richer set of amplitudes which are not available for the Laplacian, combining with Assumption 1. Finally, we show that $q^{(1)} = q^{(2)}$ by using the concentration properties of the Gaussian beam together with Assumption 1 once again.

The plan of the paper is as follows. In Section 2.2 we construct Gaussian beam quasimodes for the biharmonic operator conjugated by an exponential weight corresponding to the limiting Carleman weight ϕ and establish some concentration properties of them. In Section 2.3 we convert the Gaussian beam quasimodes to the exact complex geometric optics solutions. Section 2.4 is devoted to the proof of Theorem 2.1.2. Finally, in Section 2.5 the boundary determination of a continuous vector field on a compact manifold with boundary, from the set of the Cauchy data, is presented.

2.2 Gaussian beam quasimodes for biharmonic operators on conformally anisotropic manifolds

Let (M,g) be a CTA manifold so that $(M,g) \subset\subset (\mathbb{R}\times M_0^{\text{int}}, c(e\oplus g_0))$. Here $(\mathbb{R},e)$ is the Euclidean real line, (M_0,g_0) is a smooth compact $(n-1)$–dimensional manifold with smooth boundary, and $c\in C^\infty(\mathbb{R}\times M_0)$ is a positive function. Let us write $x=(x_1,x')$ for local coordinates in $\mathbb{R}\times M_0$. Note that $\phi(x)=\pm x_1$ is a limiting Carleman weight for $-h^2\Delta_g$; see Definition 2.3.1 in Section 2.3, and see also [36].

In this section we shall construct Gaussian beam quasimodes for the biharmonic operator $(-\Delta_g)^2$ conjugated by an exponential weight corresponding to the limiting Carleman weight $\phi=\pm x_1$, i.e., suitable approximate solutions concentrated on a single curve; see [103], [104]. Due to the presence of the conformal factor c, our quasimodes will be constructed on the manifold M and will be localized to nontangential geodesics on the transversal manifold M_0.

The first main result of this section is as follows. In this result $H^1(M^{\text{int}})$ stands for the standard Sobolev space, equipped with the semiclassical norm,

$$\|u\|^2_{H^1_{\text{scl}}(M^{\text{int}})}=\|u\|^2_{L^2(M)}+\|h\nabla_g u\|^2_{L^2(M)}.$$

Proposition 2.2.1. *Let $s=\mu+i\lambda$ with $1\le\mu=1/h$ and $\lambda\in\mathbb{R}$ being fixed, and let $\gamma:[0,L]\to M_0$ be a unit speed nontangential geodesic on M_0. Then there exist families of*

Gaussian beam quasimodes $v_s, w_s \in C^\infty(M)$ *such that*

$$\|v_s\|_{H^1_{\mathrm{scl}}(M^{\mathrm{int}})} = \mathcal{O}(1), \quad \|e^{sx_1}(-h^2\Delta_g)^2 e^{-sx_1} v_s\|_{L^2(M)} = \mathcal{O}(h^{5/2}), \tag{2.2.1}$$

and

$$\|w_s\|_{H^1_{\mathrm{scl}}(M^{\mathrm{int}})} = \mathcal{O}(1), \quad \|e^{-sx_1}(-h^2\Delta_g)^2 e^{sx_1} w_s\|_{L^2(M)} = \mathcal{O}(h^{5/2}), \tag{2.2.2}$$

as $h \to 0$. *Moreover, in a sufficiently small neighborhood* U *of a point* $p \in \gamma([0, L])$, *the quasimode* v_s *is a finite sum,*

$$v_s|_U = v_s^{(1)} + \cdots + v_s^{(P)},$$

where $t_1 < \cdots < t_P$ *are the times in* $[0, L]$ *where* $\gamma(t_l) = p$. *Each* $v_s^{(l)}$ *has the form*

$$v_s^{(l)} = e^{is\varphi^{(l)}} a^{(l)}, \quad l = 1, \dots, P, \tag{2.2.3}$$

where $\varphi = \varphi^{(l)} \in C^\infty(\overline{U}; \mathbb{C})$ *satisfies for* t *close to* t_l,

$$\varphi(\gamma(t)) = t, \quad \nabla\varphi(\gamma(t)) = \dot{\gamma}(t), \quad \mathrm{Im}\,(\nabla^2\varphi(\gamma(t))) \geq 0, \quad \mathrm{Im}\,(\nabla^2\varphi)|_{\dot{\gamma}(t)^\perp} > 0,$$

and $a^{(l)} \in C^\infty(\mathbb{R} \times \overline{U})$ *is of the form*

$$a^{(l)}(x_1, t, y) = h^{-\frac{(n-2)}{4}} a_0^{(l)}(x_1, t)\chi\left(\frac{y}{\delta'}\right),$$

where for all $l = 1, \dots, P$, *either* $a_0^{(l)}$ *is given by*

$$a_0^{(l)} = e^{-\phi^{(l)}(x_1, t)}, \tag{2.2.4}$$

defining an amplitude of the first type, or $a_0^{(l)}$ *satisfies the equation*

$$\frac{1}{c(x_1,t,0)}(\partial_{x_1} - i\partial_t)(e^{\phi^{(l)}(x_1,t)}a_0^{(l)}) = 1, \tag{2.2.5}$$

defining an amplitude of the second type. Here

$$\phi^{(l)}(x_1,t) = \log c(x_1,t,0)^{\frac{n}{4}-\frac{1}{2}} + G^{(l)}(t), \quad \partial_t G^{(l)}(t) = \frac{1}{2}(\Delta_{g_0}\varphi^{(l)})(t,0), \tag{2.2.6}$$

(t,y) *are the Fermi coordinates for* γ *for* t *close to* t_l*,* $\chi \in C_0^\infty(\mathbb{R}^{n-2})$ *is such that* $0 \le \chi \le 1$*,* $\chi = 1$ *for* $|y| \le 1/4$ *and* $\chi = 0$ *for* $|y| \ge 1/2$*, and* $\delta' > 0$ *is a fixed number that can be taken arbitrarily small.*

In a sufficiently small neighborhood U *of a point* $p \in \gamma([0,L])$*, the quasimode* w_s *is a finite sum,*

$$w_s|_U = w_s^{(1)} + \cdots + w_s^{(P)},$$

where $t_1 < \cdots < t_P$ *are the times in* $[0,L]$ *where* $\gamma(t_l) = p$*. Each* $w_s^{(l)}$ *has the form*

$$w_s^{(l)} = e^{is\varphi^{(l)}}b^{(l)}, \quad l = 1,\dots,P, \tag{2.2.7}$$

where $\varphi^{(l)}$ *is the same as in* (2.2.3)*, and* $b^{(l)} \in C^\infty(\mathbb{R}\times\overline{U})$ *is of the form*

$$b^{(l)}(x_1,t,y) = h^{-\frac{(n-2)}{4}}b_0^{(l)}(x_1,t)\chi\left(\frac{y}{\delta'}\right),$$

where

$$b_0^{(l)} = e^{-\widetilde{\phi}^{(l)}(x_1,t)}. \tag{2.2.8}$$

Here

$$\widetilde{\phi}^{(l)}(x_1,t)=\log c(x_1,t,0)^{\frac{n}{4}-\frac{1}{2}}+F^{(l)}(t),\quad \partial_t F^{(l)}(t)=\frac{1}{2}(\Delta_{g_0}\varphi^{(l)})(t,0). \tag{2.2.9}$$

Remark 2.2.2. *Note that the first type of the amplitudes, i.e.,* $a_0^{(l)}$ *given by* (2.2.4)*, will be used to recover the potential* q *as well as the vector field* X *up to a suitable gauge transformation, while to recover* X *uniquely, we shall have to work with the second type of amplitudes, i.e.,* $a_0^{(l)}$ *solving* (2.2.5)*.*

Proof. To construct Gaussian beam quasimodes, we shall follow the standard approach; see [38], [73]. The novelty here is that when working with the biharmonic operator we have to accommodate the presence of the conformal factor c throughout the construction. We are also led to consider a richer class of amplitudes for the Gaussian beam quasimodes.

Step 1. Preparation. Let us isometrically embed the manifold (M_0, g_0) into a larger closed manifold $(\widehat{M_0}, g_0)$ of the same dimension. This is possible as we can form the manifold $\widehat{M_0} = M_0 \sqcup_{\partial M_0} M_0$, which is the disjoint union of two copies of M_0, glued along the boundary; see [38, Proof of Proposition 3.1]. We extend γ as a unit speed geodesic in $\widehat{M_0}$. Let $\varepsilon > 0$ be such that $\gamma(t) \in \widehat{M_0} \setminus M_0$ and $\gamma(t)$ has no self-intersection for $t \in [-2\varepsilon, 0) \cup (L, L+2\varepsilon]$. This choice of ε is possible since γ is nontangential.

Our aim is to construct Gaussian beam quasimodes near $\gamma([-\varepsilon, L+\varepsilon])$. We shall start by carrying out the quasimode construction locally near a given point $p_0 = \gamma(t_0)$ on $\gamma([-\varepsilon, L+\varepsilon])$. Let $(t,y) \in U = \{(t,y) \in \mathbb{R}\times\mathbb{R}^{n-2} : |t-t_0| < \delta, |y| < \delta'\}$, $\delta, \delta' > 0$, be Fermi coordinates near p_0; see [60]. We may assume that the coordinates (t,y) extend smoothly to a neighborhood of $\overline{U}$. The geodesic γ near p_0 is then given by $\Gamma = \{(t,y) : y = 0\}$, and

$$g_0^{jk}(t,0)=\delta^{jk},\quad \partial_{y_l} g_0^{jk}(t,0)=0.$$

Hence, near the geodesic

$$g_0^{jk}(t,y) = \delta^{jk} + \mathcal{O}(|y|^2). \tag{2.2.10}$$

Let us first construct the quasimode v_s in (2.2.1) for the operator $e^{sx_1}(-h^2\Delta_g)^2e^{-sx_1}$. In doing so, we consider the following Gaussian beam ansatz:

$$v_s(x_1,t,y) = e^{is\varphi(t,y)}a(x_1,t,y;s). \tag{2.2.11}$$

Here $\varphi \in C^\infty(U,\mathbb{C})$ is such that

$$\operatorname{Im}\varphi \geq 0, \quad \operatorname{Im}\varphi|_\Gamma = 0, \quad \operatorname{Im}\varphi(t,y) \sim |y|^2 = \operatorname{dist}((y,t),\Gamma)^2, \tag{2.2.12}$$

and $a \in C^\infty(\mathbb{R}\times U,\mathbb{C})$ is an amplitude such that $\operatorname{supp}(a(x_1,\cdot))$ is close to Γ; see [104], [59]. Notice that here we choose φ to depend on the transversal variables (t,y) only while a is a function of all the variables.

Let us first compute $e^{sx_1}(-h^2\Delta_g)^2e^{-sx_1}v_s$. To that end, letting

$$\widetilde{\varphi}(x_1,t,y) = x_1 - i\varphi(t,y), \quad \widehat{\varphi} = sh\widetilde{\varphi}, \tag{2.2.13}$$

we first get

$$e^{\frac{\widehat{\varphi}}{h}}(-h^2\Delta_g)e^{-\frac{\widehat{\varphi}}{h}} = -h^2\Delta_g + h(2\langle\nabla_g\widehat{\varphi},\nabla_g\cdot\rangle_g + \Delta_g\widehat{\varphi}) - \langle\nabla_g\widehat{\varphi},\nabla_g\widehat{\varphi}\rangle_g. \tag{2.2.14}$$

Here and in what follows we write $\langle\cdot,\cdot\rangle_g$ to denote the Riemannian scalar product on tangent and cotangent spaces. In view of (2.2.14), we see that

$$e^{s\widetilde{\varphi}}(-h^2\Delta_g)^2e^{-s\widetilde{\varphi}} = h^4\big(-\Delta_g + s(2\langle\nabla_g\widetilde{\varphi},\nabla_g\cdot\rangle_g + \Delta_g\widetilde{\varphi}) - s^2\langle\nabla_g\widetilde{\varphi},\nabla_g\widetilde{\varphi}\rangle_g\big)^2,$$

and therefore,

$$e^{sx_1}(-h^2\Delta_g)^2 e^{-sx_1} v_s = e^{is\varphi} h^4 \big(-\Delta_g + s(2\langle\nabla_g\widetilde{\varphi}, \nabla_g\cdot\rangle_g + \Delta_g\widetilde{\varphi}) - s^2\langle\nabla_g\widetilde{\varphi}, \nabla_g\widetilde{\varphi}\rangle_g\big)^2 a. \quad (2.2.15)$$

Step 2. Solving an eikonal equation to determine the phase function $\varphi(t, y)$. Following the WKB method, we start by considering the eikonal equation

$$\langle\nabla_g\widetilde{\varphi}, \nabla_g\widetilde{\varphi}\rangle_g = 0,$$

and we would like to find $\varphi = \varphi(t, y) \in C^\infty(U, \mathbb{C})$ such that

$$\langle\nabla_g\widetilde{\varphi}, \nabla_g\widetilde{\varphi}\rangle_g = \mathcal{O}(|y|^3), \quad y \to 0, \quad (2.2.16)$$

and

$$\mathrm{Im}\varphi \geq d|y|^2, \quad (2.2.17)$$

with some $d > 0$. Using that $g = c(e \otimes g_0)$ and (2.2.13), we see that

$$\langle\nabla_g\widetilde{\varphi}, \nabla_g\widetilde{\varphi}\rangle_g = c^{-1}(1 - \langle\nabla_{g_0}\varphi, \nabla_{g_0}\varphi\rangle_{g_0}),$$

and therefore, in view of (2.2.16), we have to find φ satisfying the standard eikonal equation,

$$1 - \langle\nabla_{g_0}\varphi, \nabla_{g_0}\varphi\rangle_{g_0} = \mathcal{O}(|y|^3), \quad y \to 0.$$

As in [38], [103], and [104], we can choose,

$$\varphi(t, y) = t + \frac{1}{2}H(t)y \cdot y, \quad (2.2.18)$$

where $H(t)$ is a unique smooth complex symmetric solution of the initial value problem for the matrix Riccati equation,

$$\dot{H}(t) + H(t)^2 = F(t), \quad H(t_0) = H_0, \tag{2.2.19}$$

with H_0 being a complex symmetric matrix with $\operatorname{Im}(H_0)$ positive definite and $F(t)$ being a suitable symmetric matrix, determined by the metric tensor; see [38, Proof of Proposition 3.1]. Hence, as explained in [38], [103], and [104], $\operatorname{Im}(H(t))$ is positive definite for all t.

Step 3. Solving a transport equation to find an amplitude a. We look for a smooth amplitude $a = a(x_1, x')$ satisfying the transport equation,

$$L^2 a = \mathcal{O}(|y|), \tag{2.2.20}$$

as $y \to 0$. Here

$$L := 2\langle \nabla_g \widetilde{\varphi}, \nabla_g \cdot \rangle_g + \Delta_g \widetilde{\varphi}. \tag{2.2.21}$$

To proceed let us first simplify the operator L. To that end, in view of (2.2.13), a direct computation shows that

$$\langle \nabla_g \widetilde{\varphi}, \nabla_g \cdot \rangle_g = \frac{1}{c}(\partial_{x_1} - i g_0^{-1}(x') \varphi'_{x'} \cdot \partial_{x'}), \tag{2.2.22}$$

$$\Delta_g \widetilde{\varphi} = \Delta_g x_1 - i \Delta_g \varphi(x'), \tag{2.2.23}$$

where

$$\Delta_g x_1 = \left(\frac{n}{2} - 1\right)\frac{1}{c^2}\partial_{x_1} c, \tag{2.2.24}$$

and

$$\Delta_g \varphi = \frac{1}{c}\Delta_{g_0}\varphi + \left(\frac{n}{2} - 1\right)\frac{1}{c^2}\langle \nabla_{g_0} c, \nabla_{g_0}\varphi\rangle_{g_0}. \tag{2.2.25}$$

In view of (2.2.22), (2.2.23), (2.2.24), (2.2.25), the operator L given by (2.2.21) becomes

$$L = \frac{2}{c}(\partial_{x_1} - i g_0^{-1}(x')\varphi'_{x'} \cdot \partial_{x'}) + \left(\frac{n}{2} - 1\right)\frac{1}{c^2}\partial_{x_1} c - \frac{i}{c}\Delta_{g_0}\varphi - \left(\frac{n}{2} - 1\right)\frac{i}{c^2}\langle \nabla_{g_0} c, \nabla_{g_0}\varphi\rangle_{g_0}. \tag{2.2.26}$$

Let us proceed to simplify the operator L further. Using (2.2.10) and (2.2.18), we see that

$$g_0^{-1}(x')\varphi'_{x'} \cdot \partial_{x'} = \partial_t + \mathcal{O}(|y|^2)\partial_t + H(t)y \cdot \partial_y + \mathcal{O}(|y|^2) \cdot \partial_y. \tag{2.2.27}$$

Using (2.2.10) and (2.2.18), we also have

$$\begin{aligned}(\Delta_{g_0}\varphi)(t, 0) &= |g_0|^{-1/2}\partial_{x'_j}(|g_0|^{1/2} g_0^{jk}\partial_{x'_k}\varphi)|_{y=0} = \delta^{jk}\partial_{x'_j}\partial_{x'_k}\varphi|_{y=0} \\ &= \delta^{jk} H_{jk} = \operatorname{tr} H(t),\end{aligned}$$

and therefore

$$(\Delta_{g_0}\varphi)(t, y) = (\Delta_{g_0}\varphi)(t, 0) + \mathcal{O}(|y|) = \operatorname{tr} H(t) + \mathcal{O}(|y|). \tag{2.2.28}$$

Finally, using (2.2.10) and (2.2.18), we get

$$\langle \nabla_{g_0} c, \nabla_{g_0}\varphi\rangle_{g_0} = \partial_t c + \mathcal{O}(|y|). \tag{2.2.29}$$

Using (2.2.27), (2.2.28), (2.2.29), the operator L in (2.2.26) becomes

$$
\begin{aligned}
L =& \frac{2}{c}\Bigg[\partial_{x_1} - i\partial_t - iH(t)y\cdot\partial_y + \left(\frac{n}{4}-\frac{1}{2}\right)(\partial_{x_1} - i\partial_t)\log c - \frac{i}{2}\operatorname{tr} H(t) \\
& + \mathcal{O}(|y|) + \mathcal{O}(|y|^2)\partial_t + \mathcal{O}(|y|^2)\partial_y\Bigg] \\
=& \frac{2}{c(x_1,t,0)}\Bigg[\partial_{x_1} - i\partial_t - iH(t)y\cdot\partial_y + (\partial_{x_1} - i\partial_t)\log c(x_1,t,0)^{\frac{n}{4}-\frac{1}{2}} \\
& - \frac{i}{2}\operatorname{tr} H(t) + \mathcal{O}(|y|) + \mathcal{O}(|y|)(\partial_{x_1},\partial_t) + \mathcal{O}(|y|^2)\partial_y\Bigg].
\end{aligned}
\tag{2.2.30}
$$

Let $\chi \in C_0^\infty(\mathbb{R}^{n-2})$ be such that $\chi = 1$ for $|y| \le 1/4$ and $\chi = 0$ for $|y| \ge 1/2$. We look for the amplitude a in the form

$$
a(x_1,t,y) = h^{-\frac{(n-2)}{4}} a_0(x_1,t)\chi\left(\frac{y}{\delta'}\right), \tag{2.2.31}
$$

where $a_0(\cdot,\cdot) \in C^\infty(\mathbb{R}\times\{t : |t-t_0| < \delta\})$ is independent of y. In view of (2.2.20), a_0 should satisfy the equation

$$
L^2 a_0 = \mathcal{O}(|y|), \tag{2.2.32}
$$

as $y \to 0$. In view of (2.2.30), we write

$$
L = \frac{2}{c(x_1,t,0)}(L_0 + R), \tag{2.2.33}
$$

where

$$
L_0 = (\partial_{x_1} - i\partial_t) + (\partial_{x_1} - i\partial_t)\log c(x_1,t,0)^{\frac{n}{4}-\frac{1}{2}} - \frac{i}{2}\operatorname{tr} H(t) \tag{2.2.34}
$$

and

$$
R = -iH(t)y\cdot\partial_y + \mathcal{O}(|y|) + \mathcal{O}(|y|)(\partial_{x_1},\partial_t) + \mathcal{O}(|y|^2)\partial_y. \tag{2.2.35}
$$

To solve our inverse problem, we need two types of amplitudes. Let us proceed to construct the first type of amplitudes. In doing so, first note that as a_0 is independent of y, if a_0 solves the equation

$$L_0 a_0 = 0, \tag{2.2.36}$$

then a_0 satisfies (2.2.32). Let us proceed to find a solution to (2.2.36). To that end, letting

$$\phi(x_1, t) = \log c(x_1, t, 0)^{\frac{n}{4}-\frac{1}{2}} + G(t), \quad \partial_t G(t) = \frac{1}{2} \operatorname{tr} H(t), \tag{2.2.37}$$

we see that

$$L_0 = e^{-\phi(x_1,t)}(\partial_{x_1} - i\partial_t)e^{\phi(x_1,t)}. \tag{2.2.38}$$

We solve (2.2.36) by taking

$$a_0 = e^{-\phi} = c(x_1, t, 0)^{\frac{1}{2}-\frac{n}{4}} e^{-G(t)}, \quad \partial_t G(t) = \frac{1}{2} \operatorname{tr} H(t). \tag{2.2.39}$$

Now we proceed to find the second type of amplitudes, which is given by more general solutions to (2.2.32). As a_0 is independent of y, using (2.2.33), (2.2.34), and (2.2.35), equation (2.2.32) becomes

$$\frac{2}{c(x_1, t, 0)}[L_0 + R]\left(\frac{2}{c(x_1, t, 0)} L_0 a_0(x_1, t) + \mathcal{O}(|y|)\right) = \mathcal{O}(|y|),$$

or simply

$$L_0\left(\frac{1}{c(x_1, t, 0)} L_0\right) a_0(x_1, t) = 0. \tag{2.2.40}$$

Using (2.2.38), we see that (2.2.40) becomes

$$(\partial_{x_1} - i\partial_t)\left(\frac{1}{c(x_1,t,0)}(\partial_{x_1} - i\partial_t)(e^{\phi(x_1,t)}a_0)\right) = 0. \tag{2.2.41}$$

To solve (2.2.41), we choose $a_0(x_1,t)$ to be a solution to

$$\frac{1}{c(x_1,t,0)}(\partial_{x_1} - i\partial_t)(e^{\phi(x_1,t)}a_0) = 1. \tag{2.2.42}$$

Note that (2.2.42) can be solved as it is a standard inhomogeneous $\overline{\partial}$ equation in the complex plane $z = x_1 - it$,

$$\overline{\partial}(e^{\phi(x_1,t)}a_0) = c/2. \tag{2.2.43}$$

Step 4. Establishing the estimates (2.2.1) *locally near the point* p_0*.* First it follows from (2.2.11) and (2.2.31) that

$$v_s(x_1,t,y) = e^{is\varphi(t,y)}h^{-\frac{(n-2)}{4}}a_0(x_1,t)\chi\left(\frac{y}{\delta'}\right). \tag{2.2.44}$$

Using (2.2.17), we have

$$|v_s(x_1,t,y)| \leq \mathcal{O}(1)h^{-\frac{(n-2)}{4}}e^{-\frac{1}{h}d|y|^2}\chi\left(\frac{y}{\delta'}\right), \quad (x_1,t,y) \in J \times U, \tag{2.2.45}$$

and therefore,

$$\|v_s\|_{L^2(J\times U)} \leq \mathcal{O}(1)\|h^{-\frac{(n-2)}{4}}e^{-\frac{1}{h}d|y|^2}\|_{L^2(|y|\leq\delta'/2)} = \mathcal{O}(1), \quad h \to 0, \tag{2.2.46}$$

where $J \subset \mathbb{R}$ is a large fixed bounded open interval. Similarly, it follows from (2.2.44) that

$$\|\nabla v_s\|_{L^2(J\times U)} = \mathcal{O}(h^{-1}). \tag{2.2.47}$$

Let us next estimate $\|e^{sx_1}(-h^2\Delta_g)^2 e^{-sx_1} v_s\|_{L^2(J\times U)}$. To that end, letting

$$f = \langle \nabla_g \widetilde{\varphi}, \nabla_g \widetilde{\varphi}\rangle_g = \mathcal{O}(|y|^3) \tag{2.2.48}$$

(cf. (2.2.16)), we obtain from (2.2.15) with the help of (2.2.21) that

$$\begin{aligned} e^{sx_1}(-h^2\Delta_g)^2 e^{-sx_1} v_s = e^{is\varphi} h^4 \big(&(-\Delta_g)^2 a - s\Delta_g(La) + s^2 \Delta_g(fa) \\ &+ sL(-\Delta_g a) + s^2 L^2 a - s^3 L(fa) + s^2 f(\Delta_g a) - s^3 fLa + s^4 f^2 a \big). \end{aligned} \tag{2.2.49}$$

We shall proceed to bound each term in (2.2.49) in $L^2(J\times U)$. First using (2.2.31) and (2.2.17), we get

$$\begin{aligned} \|e^{is\varphi} h^4 (-\Delta_g)^2 a\|_{L^2(J\times U)} &= h^4 \|e^{is\varphi} h^{-\frac{(n-2)}{4}} (-\Delta_g)^2 (a_0\chi)\|_{L^2(J\times U)} \\ &= \mathcal{O}(h^4) \|h^{-\frac{(n-2)}{4}} e^{-\frac{d}{h}|y|^2}\|_{L^2(|y|\le \delta'/2)} = \mathcal{O}(h^4), \end{aligned} \tag{2.2.50}$$

and similarly,

$$\|e^{is\varphi} h^4 s\Delta_g(La)\|_{L^2(J\times U)} = \mathcal{O}(h^3) \tag{2.2.51}$$

and

$$\|e^{is\varphi} h^4 sL(\Delta_g a)\|_{L^2(J\times U)} = \mathcal{O}(h^3). \tag{2.2.52}$$

Now to bound $e^{is\varphi} h^4 s^2 \Delta_g(fa)$ in $L^2(J\times U)$ we note that the worst case occurs when Δ_g falls on f, and in this case we have, using (2.2.48) and (2.2.31),

$$\|e^{is\varphi} h^4 s^2 \Delta_g(f) a\|_{L^2(J\times U)} \le \mathcal{O}(h^2) \|h^{-\frac{(n-2)}{4}} |y| e^{-\frac{d}{h}|y|^2}\|_{L^2(|y|\le\delta'/2)} = \mathcal{O}(h^{5/2}),$$

and therefore,

$$\|e^{is\varphi}h^4s^2\Delta_g(fa)\|_{L^2(J\times U)} = \mathcal{O}(h^{5/2}). \tag{2.2.53}$$

Here we have used the following bound:

$$\|h^{-\frac{(n-2)}{4}}|y|^k e^{-\frac{d}{h}|y|^2}\|_{L^2(|y|\le\delta'/2)} = \mathcal{O}(h^{k/2}), \quad k = 1, 2, \dots. \tag{2.2.54}$$

Similarly, using (2.2.32) and (2.2.54), we get

$$\|e^{is\varphi}h^4s^2L^2a\|_{L^2(J\times U)} \le \mathcal{O}(h^2)\|h^{-\frac{(n-2)}{4}}|y|e^{-\frac{d}{h}|y|^2}\|_{L^2(|y|\le\delta'/2)} = \mathcal{O}(h^{5/2}). \tag{2.2.55}$$

Using (2.2.48), (2.2.54), and the fact that $L(\mathcal{O}(|y|^3)) = \mathcal{O}(|y|^3)$, we obtain that

$$\begin{aligned}
&\|e^{is\varphi}h^4s^3L(fa)\|_{L^2(J\times U)} \le \mathcal{O}(h)\|h^{-\frac{(n-2)}{4}}|y|^3e^{-\frac{d}{h}|y|^2}\|_{L^2(|y|\le\delta'/2)} = \mathcal{O}(h^{5/2}),\\
&\|e^{is\varphi}h^4s^2f(\Delta_g a)\|_{L^2(J\times U)} \le \mathcal{O}(h^2)\|h^{-\frac{(n-2)}{4}}|y|^3e^{-\frac{d}{h}|y|^2}\|_{L^2(|y|\le\delta'/2)} = \mathcal{O}(h^{7/2}),\\
&\|e^{is\varphi}h^4s^3fLa\|_{L^2(J\times U)} \le \mathcal{O}(h)\|h^{-\frac{(n-2)}{4}}|y|^3e^{-\frac{d}{h}|y|^2}\|_{L^2(|y|\le\delta'/2)} = \mathcal{O}(h^{5/2}),\\
&\|e^{is\varphi}h^4s^4f^2a\|_{L^2(J\times U)} \le \mathcal{O}(1)\|h^{-\frac{(n-2)}{4}}|y|^6e^{-\frac{d}{h}|y|^2}\|_{L^2(|y|\le\delta'/2)} = \mathcal{O}(h^3).
\end{aligned} \tag{2.2.56}$$

Combining (2.2.49), (2.2.50), (2.2.51), (2.2.52), (2.2.53), (2.2.55), (2.2.56), we get

$$\|e^{sx_1}(-h^2\Delta_g)^2e^{-sx_1}v_s\|_{L^2(J\times U)} = \mathcal{O}(h^{5/2}). \tag{2.2.57}$$

This completes verification of (2.2.1) locally.

For later purposes we need estimates for $\|v_s(x_1,\cdot)\|_{L^2(\partial M_0)}$. If U contains a boundary point $x_0 = (t_0, 0) \in \partial M_0$, then $\partial_t|_{x_0}$ is transversal to ∂M_0. Let ρ be a boundary defining function for M_0 so that ∂M_0 is given by the zero set $\rho(t, y) = 0$ near x_0. Then $\nabla\rho(x_0)$ is normal to ∂M_0, and hence, $\partial_t\rho(x_0) \ne 0$. By the implicit function theorem, there is a smooth function

$y \mapsto t(y)$ near 0 such that ∂M_0 near x_0 is given by $\{(t(y), y) : |y| < r_0\}$ for some $r_0 > 0$ small; see also [60]. Then using (2.2.45), we get

$$\begin{aligned} \|v_s(x_1, \cdot)\|^2_{L^2(\partial M_0 \cap U)} &= \int_{|y|<r_0} |v_s(x_1, t(y), y)|^2 dS(y) \\ &\leq \mathcal{O}(1) \int_{\mathbb{R}^{n-2}} h^{-\frac{(n-2)}{2}} e^{-2\frac{d}{h}|y|^2} dy = \mathcal{O}(1). \end{aligned} \tag{2.2.58}$$

Step 5. Establishing estimates (2.2.1) *globally.* Now let us construct the quasimode v_s in M by gluing together quasimodes defined along small pieces of the geodesic. As $\gamma : (-2\varepsilon, L + 2\varepsilon) \to \widehat{M_0}$ is a unit speed non-tangential geodesic, an application of [60, Lemma 7.2] shows that $\gamma|_{[-\varepsilon, L+\varepsilon]}$ self-intersects only at finitely many times t_j with

$$0 \leq t_1 < \cdots < t_N \leq L.$$

We let $t_0 = -\varepsilon$ and $t_{N+1} = L + \varepsilon$. By [38, Lemma 3.5], there exists an open cover $\{(U_j, \kappa_j)\}_{j=0}^{N+1}$ of $\gamma([-\varepsilon, L + \varepsilon])$ consisting of coordinate neighborhoods having the following properties:

(i) $\kappa_j(U_j) = I_j \times B$, where I_j are open intervals and $B = B(0, \delta')$ is an open ball in $\mathbb{R}^{n-2}$. Here $\delta' > 0$ can be taken arbitrarily small and the same for each U_j,

(ii) $\kappa_j(\gamma(t)) = (t, 0)$ for each $t \in I_j$,

(iii) t_j only belongs to I_j and $\overline{I_j} \cap \overline{I_k} = \emptyset$ unless $|j - k| \leq 1$,

(iv) $\kappa_j = \kappa_k$ on $\kappa_j^{-1}((I_j \cap I_k) \times B)$.

To construct the quasimode v_s globally, we first find a function $v_s^{(0)} = e^{is\varphi^{(0)}} a^{(0)}$, $a^{(0)} = h^{-\frac{(n-2)}{4}} a_0^{(0)} \chi$, in U_0 as above. Choose some t_0' with $\gamma(t_0') \in U_0 \cap U_1$. To construct the phase $\varphi^{(1)}$ in U_1, we solve the Riccati equation (2.2.19) with the initial condition $H^{(1)}(t_0') = H^{(0)}(t_0')$. Continuing in this way, we obtain the phases $\varphi^{(0)}, \varphi^{(1)}, \ldots, \varphi^{(N+1)}$ such that $\varphi^{(j)} = \varphi^{(j+1)}$ on

$U_j \cap U_{j+1}$. In a similar way, by solving ODE in (2.2.37) with prescribed initial conditions we get $\phi^{(0)}, \dots, \phi^{(N+1)}$, and therefore, in view of (2.2.39) we obtain $a_0^{(0)}, a_0^{(1)}, \dots, a_0^{(N+1)}$, and hence, we construct the amplitude of the first type globally.

To construct the amplitude of the second type, we need to solve the inhomogeneous $\bar{\partial}$–type equations (2.2.43). To that end, we first find $a_0^{(0)}$ and $a_0^{(1)}$ which are solutions of (2.2.43) on $\widetilde{J} \times I_0$ and on $\widetilde{J} \times I_1$, respectively. Here $\widetilde{J} \subset \mathbb{R}$ is a bounded open interval. Then we see that $e^{\phi^{(1)}} a_0^{(1)} - e^{\phi^{(0)}} a_0^{(0)}$ is holomorphic on $\widetilde{J} \times (I_0 \cap I_1)$. By [16, Example 3.25], there are holomorphic functions g_1, g_0 on $\widetilde{J} \times I_1$ and $\widetilde{J} \times I_0$, respectively, such that $e^{\phi^{(1)}} a_0^{(1)} - e^{\phi^{(0)}} a_0^{(0)} = g_0 - g_1$ on $\widetilde{J} \times (I_0 \cap I_1)$. Thus, modifying $a_0^{(0)}$ and $a_0^{(1)}$, we can always arrange so that $a_0^{(0)} = a_0^{(1)}$ on $\widetilde{J} \times (I_0 \cap I_1)$. Proceeding in the same way, we can find $a_0^{(2)}, \dots, a_0^{(N+1)}$ so that $a_0^{(j)} = a_0^{(j+1)}$ on $\widetilde{J} \times (I_j \cap I_{j+1})$, and hence, we construct the amplitude of the second type globally.

Thus, we obtain the quasimodes $v_s^{(0)}, \dots, v_s^{(N+1)}$ such that

$$v_s^{(j)}(x_1, \cdot) = v_s^{(j+1)}(x_1, \cdot) \quad \text{in} \quad U_j \cap U_{j+1} \tag{2.2.59}$$

for all x_1. Let $\chi_j = \chi_j(t) \in C_0^\infty(I_j)$ be such that $\sum_{j=0}^{N+1} \chi_j = 1$ near $[-\varepsilon, L + \varepsilon]$, and define our quasimode v globally by

$$v_s = \sum_{j=0}^{N+1} \chi_j v_s^{(j)}.$$

Let us next give a local description of the quasimode v_s near self-intersecting points of the geodesic γ and near the other points of γ. To that end, let $p_1, \dots, p_R \in M_0$ be the distinct points where the geodesic self-intersects, and let $0 \le t_1 < \cdots < t_{R'}$ be the times of self-intersections. Let $V_1, \dots, V_R$ be small neighborhoods in $\widehat{M}_0$ around p_j, $j = 1, \dots, R$. Then choosing δ' small enough we obtain an open cover in $\widehat{M_0}$,

$$\text{supp }(v_s(x_1,\cdot))\cap M_0 \subset (\cup_{j=1}^R V_j)\cup(\cup_{k=1}^S W_k), \tag{2.2.60}$$

where in each V_j, the quasimode is a finite sum,

$$v_s(x_1,\cdot)|_{V_j} = \sum_{l:\gamma(t_l)=p_j} v_s^{(l)}(x_1,\cdot), \tag{2.2.61}$$

and in each W_k (where there are no self-intersecting points), in view of (2.2.59), there is some $l(k)$ so that the quasimode is given by

$$v_s(x_1,\cdot)|_{W_k} = v_s^{l(k)}(x_1,\cdot). \tag{2.2.62}$$

We also have

$$\text{supp }(v_s)\cap M \subset (\cup_{j=1}^R \widetilde{J}\times V_j)\cup(\cup_{k=1}^S \widetilde{J}\times W_k),$$

where $\widetilde{J}\subset\mathbb{R}$ is a bounded open interval.

Finally, the bounds in (2.2.1) follows from the bounds (2.2.46), (2.2.47), (2.2.57), and the representations (2.2.61) and (2.2.62) of v.

Step 6. Construction of the Gaussian beam quasimodes w_s. Now look for a Gaussian beam quasimode for the operator $e^{-sx_1}(-h^2\Delta_g)^2e^{sx_1}$ in the form

$$w_s(x_1,t,y) = e^{is\varphi(t,y)}b(x_1,t,y;s), \tag{2.2.63}$$

where $\varphi\in C^\infty(U)$ is the phase function given by (2.2.18), and $b\in C^\infty(\mathbb{R}\times U)$ is an amplitude, which we shall proceed to determine. To that end, first, similarly to (2.2.15), we get

$$e^{-sx_1}(-h^2\Delta_g)^2 e^{sx_1} w_s = e^{is\varphi} h^4 \big(-\Delta_g - s(2\langle\nabla_g\widetilde{\widetilde{\varphi}},\nabla_g\cdot\rangle_g + \Delta_g\widetilde{\widetilde{\varphi}}) - s^2\langle\nabla_g\widetilde{\widetilde{\varphi}},\nabla_g\widetilde{\widetilde{\varphi}}\rangle_g\big)^2 b, \quad (2.2.64)$$

where

$$\widetilde{\widetilde{\varphi}}(x_1,t,y) = x_1 + i\varphi(t,y). \quad (2.2.65)$$

With φ given by (2.2.18), we have

$$\langle\nabla_g\widetilde{\widetilde{\varphi}},\nabla_g\widetilde{\widetilde{\varphi}}\rangle_g = \mathcal{O}(|y|^3),$$

as $y \to 0$. We thus look for the smooth amplitude $b = b(x_1,x')$ satisfying the transport equation,

$$\widetilde{L}^2 b = \mathcal{O}(|y|), \quad (2.2.66)$$

where

$$\widetilde{L} = 2\langle\nabla_g\widetilde{\widetilde{\varphi}},\nabla_g\cdot\rangle_g + \Delta_g\widetilde{\widetilde{\varphi}}. \quad (2.2.67)$$

Let us simplify the operator $\widetilde{L}$. First using (2.2.65), we get

$$\langle\nabla_g\widetilde{\widetilde{\varphi}},\nabla_g\cdot\rangle_g = \frac{1}{c}(\partial_{x_1} + ig_0^{-1}(x')\varphi'_{x'}\cdot\partial_{x'}), \quad (2.2.68)$$

$$\Delta_g\widetilde{\varphi} = \Delta_g x_1 + i\Delta_g\varphi(x'). \quad (2.2.69)$$

Hence, using (2.2.68), (2.2.69), (2.2.24), and (2.2.25), the operator $\widetilde{L}$ given by (2.2.67) becomes

$$\widetilde{L} = \frac{2}{c}(\partial_{x_1} + ig_0^{-1}(x')\varphi'_{x'} \cdot \partial_{x'}) + \left(\frac{n}{2} - 1\right)\frac{1}{c^2}\partial_{x_1} c + \frac{i}{c}\Delta_{g_0}\varphi + \left(\frac{n}{2} - 1\right)\frac{i}{c^2}\langle \nabla_{g_0} c, \nabla_{g_0}\varphi\rangle_{g_0}. \quad (2.2.70)$$

Using (2.2.27), (2.2.28), (2.2.29), the operator $\widetilde{L}$ in (2.2.70) becomes

$$\begin{aligned} \widetilde{L} = &= \frac{2}{c(x_1, t, 0)}\Big[\partial_{x_1} + i\partial_t + iH(t)y \cdot \partial_y + (\partial_{x_1} + i\partial_t)\log c(x_1, t, 0)^{\frac{n}{4} - \frac{1}{2}} \\ &+ \frac{i}{2}\operatorname{tr} H(t) + \mathcal{O}(|y|) + \mathcal{O}(|y|)(\partial_{x_1}, \partial_t) + \mathcal{O}(|y|^2)\partial_y\Big]. \end{aligned} \quad (2.2.71)$$

We look for the amplitude b in the form

$$b(x_1, t, y) = h^{-\frac{(n-2)}{4}} b_0(x_1, t)\chi\left(\frac{y}{\delta'}\right), \quad (2.2.72)$$

where $b_0(\cdot, \cdot) \in C^\infty(\mathbb{R} \times \{t : |t - t_0| < \delta\})$ is independent of y, and in view of (2.2.66), b_0 should satisfy

$$\widetilde{L}^2 b_0 = \mathcal{O}(|y|), \quad y \to 0. \quad (2.2.73)$$

It follows from (2.2.70) that

$$\widetilde{L} = \frac{2}{c(x_1, t, 0)}(\widetilde{L}_0 + \widetilde{R}), \quad (2.2.74)$$

where

$$\widetilde{L}_0 = (\partial_{x_1} + i\partial_t) + (\partial_{x_1} + i\partial_t)\log c(x_1, t, 0)^{\frac{n}{4} - \frac{1}{2}} + \frac{i}{2}\operatorname{tr} H(t), \quad (2.2.75)$$

and

$$\widetilde{R} = iH(t)y \cdot \partial_y + \mathcal{O}(|y|) + \mathcal{O}(|y|)(\partial_{x_1}, \partial_t) + \mathcal{O}(|y|^2)\partial_y. \quad (2.2.76)$$

In contrast to the construction of the Gaussian beam quasimodes v_s, we shall only need amplitudes of the first type. To construct such amplitudes, we note that as b_0 is independent of y, if b_0 solves the equation

$$\widetilde{L}_0 b_0 = 0, \tag{2.2.77}$$

then b_0 satisfies (2.2.73). To find a solution to (2.2.77), we note that

$$\widetilde{L}_0 = e^{-\widetilde{\phi}(x_1,t)}(\partial_{x_1} + i\partial_t)e^{\widetilde{\phi}(x_1,t)}, \tag{2.2.78}$$

where $\widetilde{\phi}(x_1, t)$ is given by

$$\widetilde{\phi}(x_1, t) = \log c(x_1, t, 0)^{\frac{n}{4}-\frac{1}{2}} + F(t), \quad \partial_t F(t) = \frac{1}{2} \operatorname{tr} H(t). \tag{2.2.79}$$

We solve (2.2.77) by taking

$$b_0 = e^{-\widetilde{\phi}} = c(x_1, t, 0)^{\frac{1}{2}-\frac{n}{4}} e^{-F(t)}. \tag{2.2.80}$$

Proceeding further as in the construction of the quasimode v_s above, we obtain the quasimode $w_s \in C^\infty(M)$ such that (2.2.2) holds.

□

We shall need the following result.

Proposition 2.2.3. *Let $X \in C(M, TM)$ be a complex vector field, let $\psi \in C(M_0)$, and let $x_1' \in \mathbb{R}$. Then there exist the Gaussian beam quasimodes v_s and w_s given by Proposition*

2.2.1 such that v_s is obtained using amplidutes of the first type and we have

$$\lim_{h\to 0}\int_{\{x_1'\}\times M_0} v_s\overline{w_s}\psi dV_{g_0} = \int_0^L e^{-2\lambda t}c(x_1,\gamma(t))^{1-\frac{n}{2}}\psi(\gamma(t))dt \tag{2.2.81}$$

and

$$\lim_{h\to 0} h\int_{\{x_1'\}\times M_0} X(v_s)\overline{w_s}\psi dV_{g_0} = i\int_0^L X_t(x_1',\gamma(t))e^{-2\lambda t}c(x_1,\gamma(t))^{1-\frac{n}{2}}\psi(\gamma(t))dt. \tag{2.2.82}$$

Here $X_t(x_1',\gamma(t)) = \langle X(x_1',\gamma(t)),(0,\dot\gamma(t))\rangle_g$.

Proof. Step 1. Proof of (2.2.81)*.* Let $\psi \in C(M_0)$, $x_1' \in \mathbb{R}$. Using a partition of unity, in view of (2.2.60), it suffices to establish (2.2.81) for ψ having compact support in one of the sets V_j or W_k. First, assume that $\psi \in C_0(M_0)$, supp $(\psi) \subset W_k$. Thus, in view of (2.2.62), (2.2.44), (2.2.63), (2.2.72), on supp (ψ), we have

$$v_s = e^{is\varphi}h^{-\frac{(n-2)}{4}}a_0(x_1',t)\chi\left(\frac{y}{\delta'}\right), \quad w_s = e^{is\varphi}h^{-\frac{(n-2)}{4}}b_0(x_1',t)\chi\left(\frac{y}{\delta'}\right). \tag{2.2.83}$$

To proceed, we shall need the consequence of (2.2.10),

$$|g_0|^{1/2} = 1 + \mathcal{O}(|y|^2), \tag{2.2.84}$$

as well as

$$is\varphi - i\overline{s\varphi} = -2\frac{1}{h}\mathrm{Im}\varphi - 2\lambda\mathrm{Re}\varphi. \tag{2.2.85}$$

Using (2.2.83), (2.2.84), (2.2.85), (2.2.18), we get

$$\begin{aligned}
&\int_{\{x_1'\}\times M_0} v_s \overline{w_s}\psi dV_{g_0} \\
&= \int_0^L \int_{\mathbb{R}^{n-2}} e^{-2\frac{1}{h}\mathrm{Im}\varphi} e^{-2\lambda \mathrm{Re}\varphi} h^{-\frac{(n-2)}{2}} a_0(x_1',t)\overline{b_0(x_1',t)}\chi^2\left(\frac{y}{\delta'}\right)\psi(t,y)|g_0|^{\frac{1}{2}} dydt \\
&= \int_0^L \int_{\mathbb{R}^{n-2}} e^{-\frac{1}{h}\mathrm{Im}H(t)y\cdot y} e^{-2\lambda t} e^{\lambda \mathcal{O}(|y|^2)} h^{-\frac{(n-2)}{2}} a_0(x_1',t)\overline{b_0(x_1',t)}\chi^2\left(\frac{y}{\delta'}\right) \\
&\psi(t,y)(1+\mathcal{O}(|y|^2))dydt.
\end{aligned} \tag{2.2.86}$$

Making the change of variable $y = h^{1/2}\widetilde{y}$ in (2.2.86), we obtain that

$$\begin{aligned}
\int_{\{x_1'\}\times M_0} v_s \overline{w_s}\psi dV_{g_0} = \int_0^L \int_{\mathbb{R}^{n-2}} & e^{-\mathrm{Im}H(t)\widetilde{y}\cdot\widetilde{y}} e^{-2\lambda t} e^{\lambda h \mathcal{O}(|\widetilde{y}|^2)} a_0(x_1',t)\overline{b_0(x_1',t)} \\
& \chi^2\left(\frac{h^{1/2}\widetilde{y}}{\delta'}\right)\psi(t,h^{1/2}\widetilde{y})(1+h\mathcal{O}(|\widetilde{y}|^2))dtd\widetilde{y}.
\end{aligned} \tag{2.2.87}$$

Using that

$$\int_{\mathbb{R}^{n-2}} e^{-\mathrm{Im}H(t)y\cdot y} dy = \frac{\pi^{(n-2)/2}}{\sqrt{\det(\mathrm{Im}H(t))}}, \tag{2.2.88}$$

and the dominated covergence theorem, we get from (2.2.87) that

$$\begin{aligned}
&\lim_{h\to 0}\int_{\{x_1'\}\times M_0} v_s \overline{w_s}\psi dV_{g_0} \\
&= \int_0^L e^{-2\lambda t} a_0(x_1',t)\overline{b_0(x_1',t)}\psi(t,0)\int_{\mathbb{R}^{n-2}} e^{-\mathrm{Im}H(t)y\cdot y} dydt \\
&= \int_0^L e^{-2\lambda t} a_0(x_1',t)\overline{b_0(x_1',t)}\frac{\pi^{(n-2)/2}}{\sqrt{\det(\mathrm{Im}H(t))}}\psi(t,0)dt.
\end{aligned} \tag{2.2.89}$$

Let us proceed to simplify the expression in (2.2.89) in the case when a_0 is the amplitude of the first type, i.e., a_0 be given by (2.2.39), and let b_0 be given by (2.2.80). Then

$$a_0(x_1',t)\overline{b_0(x_1',t)}\frac{\pi^{(n-2)/2}}{\sqrt{\det(\mathrm{Im}H(t))}} = c(x_1,t,0)^{1-\frac{n}{2}} e^{-(G(t)+\overline{F(t)})}\frac{\pi^{(n-2)/2}}{\sqrt{\det(\mathrm{Im}H(t))}}. \tag{2.2.90}$$

Now it follows from (2.2.39) and (2.2.79) that

$$G(t) + \overline{F(t)} = G(t_0) + \overline{F(t_0)} + \int_{t_0}^{t} \operatorname{tr} \operatorname{Re}(H(s))ds. \tag{2.2.91}$$

Using (2.2.91) and the property of solutions of the matrix Riccati equation [59, Lemma 2.58],

$$\det(\operatorname{Im}H(t)) = \det(\operatorname{Im}H(t_0))e^{-2\int_{t_0}^{t} \operatorname{tr}\operatorname{Re}(H(s))ds},$$

we see that

$$e^{-(G(t)+\overline{F(t)})}\frac{\pi^{(n-2)/2}}{\sqrt{\det(\operatorname{Im}H(t))}} = e^{-(G(t_0)+\overline{F(t_0)})}\frac{\pi^{(n-2)/2}}{\sqrt{\det(\operatorname{Im}H(t_0))}} \tag{2.2.92}$$

is a constant in t. To fix this constant, when constructing the amplitude a_0 and b_0, specifically, when solving (2.2.39) and (2.2.79) in U_0, we choose initial conditions for G and F so that the constant in (2.2.92) is equal to 1. With this choice, it follows from (2.2.89), (2.2.90), (2.2.92) that

$$\lim_{h\to 0}\int_{\{x_1'\}\times M_0} v_s\overline{w_s}\psi dV_{g_0} = \int_0^L e^{-2\lambda t}c(x_1,t,0)^{1-\frac{n}{2}}\psi(t,0)dt. \tag{2.2.93}$$

This completes the proof of (2.2.81) in the case when supp $(\psi) \subset W_k$.

Let us now establish (2.2.81) when supp $(\psi) \subset V_j$. Here on supp (ψ) we have

$$v_s = \sum_{l:\gamma(t_l)=p_j} v_s^{(l)}, \quad w_s = \sum_{l:\gamma(t_l)=p_j} w_s^{(l)}, \tag{2.2.94}$$

and hence,

$$v_s\overline{w_s} = \sum_{l:\gamma(t_l)=p_j} v_s^{(l)}\overline{w_s^{(l)}} + \sum_{l\neq l',\gamma(t_l)=\gamma(t_{l'})=p_j} v_s^{(l)}\overline{w_s^{(l')}}. \tag{2.2.95}$$

We shall use a nonstationary phase argument as in [38, end of proof Proposition 3.1] to show that the contribution of the mixed terms vanishes in the limit $h \to 0$, i.e., if $l \neq l'$,

$$\lim_{h\to 0} \int_{\{x_1'\}\times M_0} v_s^{(l)} \overline{w_s^{(l')}} \psi dV_{g_0} = 0. \tag{2.2.96}$$

In doing so, write

$$v_s^{(l)} = e^{i\frac{1}{h}\operatorname{Re}\varphi^{(l)}} p^{(l)}, \quad p^{(l)} = e^{-\lambda \operatorname{Re}\varphi^{(l)}} e^{-s\operatorname{Im}\varphi^{(l)}} a^{(l)}$$

and

$$w_s^{(l')} = e^{i\frac{1}{h}\operatorname{Re}\varphi^{(l')}} q^{(l')}, \quad q^{(l')} = e^{-\lambda \operatorname{Re}\varphi^{(l')}} e^{-s\operatorname{Im}\varphi^{(l')}} b^{(l')},$$

and therefore,

$$v_s^{(l)} \overline{w_s^{(l')}} = e^{i\frac{1}{h}\phi} p^{(l)} \overline{q^{(l')}}, \tag{2.2.97}$$

where

$$\phi = \operatorname{Re}\varphi^{(l)} - \operatorname{Re}\varphi^{(l')}.$$

Thus, in view of (2.2.96) and (2.2.97) we shall show that for $l \neq l'$,

$$\lim_{h\to 0} \int_{\{x_1'\}\times M_0} e^{i\frac{1}{h}\phi} p^{(l)} \overline{q^{(l')}} \psi dV_{g_0} = 0. \tag{2.2.98}$$

Since $\partial_t \varphi^{(l)}(t,0) = \partial_t \varphi^{(l')}(t,0) = 1$ and the geodesic intersects itself transversally, as explained in [60, Lemma 7.2], we see that $d\phi(p_j) \neq 0$. By decreasing the set V_j if necessary, we may assume that $d\phi \neq 0$ in V_j.

To prove (2.2.98), we shall integrate by parts and in doing so, we let $\varepsilon > 0$ be fixed, and decompose $\psi = \psi_1 + \psi_2$, where $\psi_1 \in C^\infty(M_0)$, supp $(\psi_1) \subset V_j$ and and $\|\psi_2\|_{L^\infty(V_j \cap M_0)} \leq \varepsilon$. Notice that ψ may be nonzero on ∂M_0. We have

$$\left| \int_{\{x_1'\} \times M_0} e^{i\frac{1}{h}\phi} p^{(l)} \overline{q^{(l')}} \psi_2 dV_{g_0} \right| \leq \|v_s^{(l)}\|_{L^2} \|w_s^{(l)}\|_{L^2} \|\psi_2\|_{L^\infty} \leq \mathcal{O}(\varepsilon). \tag{2.2.99}$$

For the smooth part ψ_1, we integrate by parts using that

$$e^{i\frac{1}{h}\phi} = \frac{h}{i} L(e^{i\frac{1}{h}\phi}), \quad L = \frac{1}{|d\phi|^2} \langle d\phi, d\cdot \rangle_{g_0}.$$

We have

$$\begin{aligned} \int_{\{x_1'\} \times M_0} e^{i\frac{1}{h}\phi} p^{(l)} \overline{q^{(l')}} \psi_1 dV_{g_0} = & \int_{\{x_1'\} \times (V_j \cap \partial M_0)} h \frac{\partial_\nu \phi}{i|d\phi|^2} e^{i\frac{1}{h}\phi} p^{(l)} \overline{q^{(l')}} \psi_1 dS \\ & + h \frac{1}{i} \int_{\{x_1'\} \times M_0} e^{i\frac{1}{h}\phi} L^t (p^{(l)} \overline{q^{(l')}} \psi_1) dV_{g_0}, \end{aligned} \tag{2.2.100}$$

where $L^t = -L - \operatorname{div} L$ is the transpose of L.

In view of (2.2.58), the boundary term is of $\mathcal{O}(h)$ as $h \to 0$. To estimate the second term in the right-hand side of (2.2.100), we recall that

$$\begin{aligned} p^{(l)} \overline{q^{(l')}} = e^{-\lambda(\operatorname{Re}\varphi^{(l)} + \operatorname{Re}\varphi^{(l')})} e^{-i\lambda(\operatorname{Im}\varphi^{(l)} - \operatorname{Im}\varphi^{(l')})} e^{-\frac{1}{h}(\operatorname{Im}\varphi^{(l)} + \operatorname{Im}\varphi^{(l')})} h^{-\frac{(n-2)}{2}} \\ a_0^{(l)}(x_1', t) \overline{b_0^{(l')}(x_1', t)} \chi^2 \left(\frac{y}{\delta'} \right). \end{aligned}$$

This shows that to bound the second term in the right-hand side of (2.2.100), it is enough to analyze the contributions occurring when differentiating

$$e^{-\frac{1}{h}(\operatorname{Im}\varphi^{(l)} + \operatorname{Im}\varphi^{(l')})},$$

as all the other contributions are of $\mathcal{O}(h)$, as $h \to 0$.

As in [38], using (2.2.17), we have

$$|L(e^{-\frac{1}{h}(\operatorname{Im}\varphi^{(l)}+\operatorname{Im}\varphi^{(l')})})| \leq \mathcal{O}(h^{-1})|d(\operatorname{Im}\varphi^{(l)}+\operatorname{Im}\varphi^{(l')})|e^{-\frac{1}{h}d|y|^2} \leq \mathcal{O}(h^{-1}|y|)e^{-\frac{1}{h}d|y|^2},$$

which shows that the corresponding contribution to the second term in the right-hand side of (2.2.100) is of $\mathcal{O}(h^{1/2})$. This shows that the integral in the left-hand side of (2.2.100) goes to 0 as $h \to 0$, and this together with (2.2.99) establishes (2.2.96).

Using (2.2.93) for each of the factors $v_s^{(l)}\overline{w_s^{(l)}}$ in (2.2.95), we get

$$\lim_{h\to 0}\int_{\{x_1'\}\times M_0} v_s^{(l)}\overline{w_s^{(l)}}\psi dV_{g_0} = \int_{I_l} e^{-2\lambda t}c(x_1,t,0)^{1-\frac{n}{2}}\psi(t,0)dt.$$

Summing over I_l, appearing in the Fermi coordinates, such that $t_l \in I_l$ and $\gamma(t_l) = p_j$, we get (2.2.81) when supp $(\psi) \subset V_j$ and hence, in general.

Step 2. Establishing (2.2.82). Let $X \in C(M,TM)$ be a complex vector field, $\psi \in C(M_0)$, and $x_1' \in \mathbb{R}$. Using a partition of unity, it is enough to verify (2.2.82) in the following two cases: supp $(\psi) \subset W_k$ and supp $(\psi) \subset V_j$. Assume first that supp $(\psi) \subset W_k$. Using (2.2.83), we get

$$h\int_{\{x_1'\}\times M_0} X(v_s)\overline{w_s}\psi dV_{g_0} = I_{1,1} + I_{1,2} + I_2, \tag{2.2.101}$$

where

$$I_{1,1} = \int_{\{x_1'\}\times M_0} iX(\varphi)v_s\overline{w_s}\psi dV_{g_0}, \tag{2.2.102}$$

$$I_{1,2} = -h\int_{\{x_1'\}\times M_0} \lambda X(\varphi)v_s\overline{w_s}\psi dV_{g_0}, \tag{2.2.103}$$

$$I_2 = h\int_{\{x_1'\}\times M_0} h^{-\frac{(n-2)}{4}} e^{is\varphi} X(a_0\chi)\overline{w_s}\psi dV_{g_0}. \tag{2.2.104}$$

Using (2.2.1) and (2.2.2), we have

$$\begin{aligned} |I_{1,2}| &\le \mathcal{O}(h)\|v_s(x',\cdot)\|_{L^2(M_0)}\|w_s(x_1',\cdot)\|_{L^2(M_0)} = \mathcal{O}(h), \\ |I_2| &\le \mathcal{O}(h)\|e^{is\varphi}h^{-\frac{(n-2)}{4}}\|_{L^2(\{|y|\le\delta'/2\})}\|w_s(x_1',\cdot)\|_{L^2(M_0)} = \mathcal{O}(h). \end{aligned} \tag{2.2.105}$$

Let us now compute $\lim_{h\to 0} I_{1,1}$. To that end, we write

$$X = X_1\partial_{x_1} + X_t\partial_t + X_y\cdot\partial_y, \quad x = (x_1, t, y). \tag{2.2.106}$$

Using (2.2.18), we get

$$\partial_t\varphi = 1 + \mathcal{O}(|y|^2), \quad \partial_y\varphi = \mathcal{O}(|y|). \tag{2.2.107}$$

As X is continuous, it follows from (2.2.106) and (2.2.107) that

$$X(\varphi) = (X_t(x_1, t, 0) + o(1))(1 + \mathcal{O}(|y|^2)) + \mathcal{O}(|y|) = X_t(x_1, t, 0) + o(1), \tag{2.2.108}$$

as $y \to 0$, uniformly in x_1 and t. Using (2.2.108), as in (2.2.86), we obtain from (2.2.102) that

$$\begin{aligned} I_{1,1} = \int_0^L \int_{\mathbb{R}^{n-2}} & i(X_t(x_1', t, 0) + o(1))h^{-\frac{(n-2)}{2}} e^{-\frac{1}{h}\mathrm{Im}H(t)y\cdot y} e^{-2\lambda t} e^{\lambda\mathcal{O}(|y|^2)} \\ & a_0(x_1', t)\overline{b_0(x_1', t)}\chi^2\left(\frac{y}{\delta'}\right)\psi(t, y)(1 + \mathcal{O}(|y|^2))dydt. \end{aligned} \tag{2.2.109}$$

We first observe that

$$\lim_{h\to 0} I_{1,1,2} = 0, \tag{2.2.110}$$

uniformly in x_1' and t, where

$$I_{1,1,2}=\int_{\mathbb{R}^{n-2}} g(x_1',t,y)dy,\quad g(x_1',t,y)=o(1)h^{-\frac{(n-2)}{2}}e^{-\frac{1}{h}\mathrm{Im}H(t)y\cdot y}e^{-2\lambda t}$$
$$e^{\lambda\mathcal{O}(|y|^2)}a_0(x_1',t)\overline{b_0(x_1',t)}\chi^2\left(\frac{y}{\delta'}\right)\psi(t,y)(1+\mathcal{O}(|y|^2)).$$

Indeed, let $\varepsilon>0$ and let $\delta>0$ be such that $|o(1)|\le\varepsilon$ when $|y|\le\delta$. Then

$$\begin{aligned}|I_{1,1,2}|&\le\left|\int_{|y|\le\delta} g(x_1',t,y)dy\right|+\left|\int_{|y|\ge\delta} g(x_1',t,y)dy\right|\\&\le\varepsilon\mathcal{O}(1)\left|\int_{\mathbb{R}^{n-2}} h^{-\frac{(n-2)}{2}}e^{-\frac{1}{h}\mathrm{Im}H(t)y\cdot y}dy\right|+\mathcal{O}(e^{-d\delta^2/h})\le\varepsilon\mathcal{O}(1)+\mathcal{O}(e^{-d\delta^2/h}),\end{aligned}$$

showing (2.2.110).

Using (2.2.110), making the change of variables $y=h^{1/2}\widetilde{y}$ in (2.2.109), using the dominated convergence theorem, and (2.2.88), we get

$$\lim_{h\to0} I_{1,1}=i\int_0^L X_t(x_1',t,0)e^{-2\lambda t}a_0(x_1',t)\overline{b_0(x_1',t)}\psi(t,0)\frac{\pi^{(n-2)/2}}{\sqrt{\det(\mathrm{Im}H(t))}}dt. \tag{2.2.111}$$

It follows from (2.2.101) with the help of (2.2.105) and (2.2.111) that

$$\begin{aligned}&\lim_{h\to0} h\int_{\{x_1'\}\times M_0} X(v_s)\overline{w_s}\psi dV_{g_0}\\&\quad=i\int_0^L X_t(x_1',t,0)e^{-2\lambda t}a_0(x_1',t)\overline{b_0(x_1',t)}\psi(t,0)\frac{\pi^{(n-2)/2}}{\sqrt{\det(\mathrm{Im}H(t))}}dt.\end{aligned} \tag{2.2.112}$$

When a_0 is the amplitude of the first type, i.e. a_0 be given by (2.2.39), and b_0 be given by (2.2.80), using (2.2.90), (2.2.92), we get from (2.2.112) that

$$\lim_{h\to0} h\int_{\{x_1'\}\times M_0} X(v_s)\overline{w_s}\psi dV_{g_0}=i\int_0^L X_t(x_1',t,0)e^{-2\lambda t}c(x_1,t,0)^{1-\frac{n}{2}}\psi(t,0)dt. \tag{2.2.113}$$

This establishes (2.2.82) when supp $(\psi)\subset W_k$.

Assume now that supp $(\psi) \subset V_j$, and therefore, on supp (ψ), v_s and w_s are given by (2.2.94). Then

$$h \int_{\{x_1'\}\times M_0} X(v_s)\overline{w_s}\psi dV_{g_0} = h \sum_{l:\gamma(t_l)=p_j} \int_{\{x_1'\}\times M_0} X(v_s^{(l)})\overline{w_s^{(l)}}\psi dV_{g_0} + h \sum_{l\neq l':\gamma(t_l)=\gamma(t_{l'})=p_j} \int_{\{x_1'\}\times M_0} X(v_s^{(l)})\overline{w_s^{(l')}}\psi dV_{g_0}. \tag{2.2.114}$$

As before, we shall show that the mixed terms, i.e., $l \neq l'$, vanish in the limit as $h \to 0$,

$$\lim_{h\to 0} h \int_{\{x_1'\}\times M_0} X(v_s^{(l)})\overline{w_s^{(l')}}\psi dV_{g_0} = 0. \tag{2.2.115}$$

It follows from (2.2.101), (2.2.102), (2.2.103), (2.2.104), (2.2.105) that we only have to prove that

$$\lim_{h\to 0} \int_{\{x_1'\}\times M_0} iX(\varphi^{(l)})v_s^{(l)}\overline{w_s^{(l')}}\psi dV_{g_0} = 0. \tag{2.2.116}$$

Now (2.2.116) follows by repeating a nonstationary phase argument as in the proof of (2.2.96) replacing ψ by $X(\varphi^{(l)})\psi \in C(M_0)$. Thus, using (2.2.114) and (2.2.116), we see that

$$\lim_{h\to 0} h \int_{\{x_1'\}\times M_0} X(v_s)\overline{w_s}\psi dV_{g_0} = \sum_{l:\gamma(t_l)=p_j} i \int_{I_l} X_t(x_1', t, 0)e^{-2\lambda t}c(x_1, t, 0)^{1-\frac{n}{2}}\psi(t, 0)dt;$$

completing the proof of (2.2.82) when supp $(\psi) \subset V_j$. □

We shall also need the following result.

Proposition 2.2.4. *Let $\psi \in C^1(\mathbb{R} \times M_0)$ be such that $\psi(x_1, \cdot)|_{\partial M_0} = 0$ and with compact support in x_1. Then there exist Gaussian beam quasimodes v_s and w_s given by Proposition*

2.2.1 such that v_s is obtained using amplitudes of the second type and

$$\begin{aligned}
&\lim_{h\to 0}\left[h\int_{\mathbb{R}} e^{-2i\lambda x_1}\int_{M_0}(\nabla_g\psi)(v_s)\overline{w_s}c(x_1,x')^{\frac{n}{2}}dV_{g_0}dx_1\right.\\
&\quad\left.-\int_{\mathbb{R}} e^{-2i\lambda x_1}\int_{M_0}(\nabla_g\psi)_1 v_s\overline{w_s}c(x_1,x')^{\frac{n}{2}}dV_{g_0}dx_1\right] \qquad (2.2.117)\\
&\quad=\int_{\mathbb{R}}\int_0^L e^{-2i\lambda(x_1-it)}\psi(x_1,\gamma(t))c(x_1,\gamma(t))dtdx_1.
\end{aligned}$$

Proof. In view of (2.2.60), using a partition of unity, it suffices to check (2.2.117) for ψ such that supp $(\psi(x_1,\cdot))$ is in one of the sets V_j or W_k. Let us first consider the case when supp $(\psi(x_1,\cdot))\subset W_k$. Thus, on supp $(\psi(x_1,\cdot))$, v_s and w_s are given by (2.2.83) with a_0 being an amplitude of type two. To proceed, we note that

$$\nabla_g\psi=\frac{1}{c}(\partial_{x_1}\psi\partial_{x_1}+g_0^{-1}\partial_{x'}\psi\cdot\partial_{x'}), \qquad (2.2.118)$$

and therefore, using (2.2.10), we see that

$$(\nabla\psi)_t(x_1,t,0)=\frac{\partial_t\psi(x_1,t,0)}{c(x_1,t,0)}. \qquad (2.2.119)$$

Using (2.2.83), (2.2.118), and (2.2.119), a computation similar to that in the proof of Proposition 2.2.3 (cf. (2.2.89) and (2.2.112)) gives

$$\begin{aligned}
I=&\lim_{h\to 0}\left[h\int_{\mathbb{R}} e^{-2i\lambda x_1}\int_{M_0}(\nabla_g\psi)(v_s)\overline{w_s}c(x_1,x')^{\frac{n}{2}}dV_{g_0}dx_1\right.\\
&\quad\left.-\int_{\mathbb{R}} e^{-2i\lambda x_1}\int_{M_0}(\nabla_g\psi)_1 v_s\overline{w_s}c(x_1,x')^{\frac{n}{2}}dV_{g_0}dx_1\right]\\
&\quad=-\int_{\mathbb{R}}\int_0^L e^{-2i\lambda x_1}e^{-2\lambda t}((\partial_{x_1}-i\partial_t)\psi(x_1,t,0))a_0(x_1,t)\overline{b_0(x_1,t)} \qquad (2.2.120)\\
&\quad\frac{\pi^{(n-2)/2}}{\sqrt{\det(\mathrm{Im}H(t))}}c(x_1,t,0)^{\frac{n}{2}-1}dtdx_1.
\end{aligned}$$

When solving (2.2.37) and (2.2.79) for G and F, respectively, we choose the initial conditions

$G(t_0)$ and $F(t_0)$ so that the constant in (2.2.92) is equal to 1. Then using (2.2.80), (2.2.37), (2.2.92), we see that

$$
\begin{aligned}
&a_0(x_1,t)\overline{b_0(x_1,t)}\frac{\pi^{(n-2)/2}}{\sqrt{\det(\mathrm{Im}H(t))}}c(x_1,t,0)^{\frac{n}{2}-1}\\
&=a_0(x_1,t)c(x_1,t,0)^{\frac{n}{4}-\frac{1}{2}}e^{-\overline{F(t)}}\frac{\pi^{(n-2)/2}}{\sqrt{\det(\mathrm{Im}H(t))}}\\
&=a_0(x_1,t)c(x_1,t,0)^{\frac{n}{4}-\frac{1}{2}}e^{G(t)}=a_0(x_1,t)e^{\phi(x_1,t)}.
\end{aligned}
\tag{2.2.121}
$$

Combining (2.2.120) and (2.2.121), integrating by parts, using the fact that ψ compact support in x_1 and $\psi(x_1,\cdot)|_{\partial M_0}=0$, and using (2.2.42), we get

$$
\begin{aligned}
I&=-\int_{\mathbb{R}}\int_0^L e^{-2i\lambda(x_1-it)}((\partial_{x_1}-i\partial_t)\psi(x_1,t,0))a_0(x_1,t)e^{\phi(x_1,t)}dtdx_1\\
&=\int_{\mathbb{R}}\int_0^L e^{-2i\lambda(x_1-it)}\psi(x_1,t,0)(\partial_{x_1}-i\partial_t)(a_0(x_1,t)e^{\phi(x_1,t)})dtdx_1\\
&=\int_{\mathbb{R}}\int_0^L e^{-2i\lambda(x_1-it)}\psi(x_1,t,0)c(x_1,t,0)dtdx_1.
\end{aligned}
\tag{2.2.122}
$$

This completes the proof of (2.2.117) in the case when supp $(\psi(x_1,\cdot))\subset W_k$.

Let us now show (2.2.117) when supp $(\psi(x_1,\cdot))\subset V_j$. Then on supp (ψ), v_s and w_s are given by (2.2.94), and we have

$$
\begin{aligned}
&\int_{\mathbb{R}}e^{-2i\lambda x_1}\int_{M_0}(h(\nabla_g\psi)(v_s)-(\nabla_g\psi)_1v_s)\overline{w_s}c(x_1,x')^{\frac{n}{2}}dV_{g_0}dx_1\\
&=\sum_{l:\gamma(t_l)=p_j}\int_{\mathbb{R}}e^{-2i\lambda x_1}\int_{M_0}(h(\nabla_g\psi)(v_s^{(l)})-(\nabla_g\psi)_1v_s^{(l)})\overline{w_s^{(l)}}c(x_1,x')^{\frac{n}{2}}dV_{g_0}dx_1+\\
&\sum_{l\neq l':\gamma(t_l)=\gamma(t_{l'})=p_j}\int_{\mathbb{R}}e^{-2i\lambda x_1}\int_{M_0}(h(\nabla_g\psi)(v_s^{(l)})-(\nabla_g\psi)_1v_s^{(l)})\overline{w_s^{(l')}}c(x_1,x')^{\frac{n}{2}}dV_{g_0}dx_1.
\end{aligned}
\tag{2.2.123}
$$

Now when $l\neq l'$, as in (2.2.96) and (2.2.115), by a nonstationary phase argument we see that

$$\lim_{h\to 0}\int_{M_0}(h(\nabla_g\psi)(v_s^{(l)})-(\nabla_g\psi)_1 v_s^{(l)})\overline{w_s^{(l')}}c(x_1,x')^{\frac{n}{2}}dV_{g_0}=0,$$

uniformly in x_1, and therefore, the limit $h\to 0$ of the second sum in (2.2.123) is equal to 0. Hence,

$$\lim_{h\to 0}\int_{\mathbb{R}}e^{-2i\lambda x_1}\int_{M_0}(h(\nabla_g\psi)(v_s)-(\nabla_g\psi)_1 v_s)\overline{w_s}c(x_1,x')^{\frac{n}{2}}dV_{g_0}dx_1$$
$$=\sum_{l:\gamma(t_l)=p_j}\int_{\mathbb{R}}\int_{I_l}e^{-2i\lambda(x_1-it)}\psi(x_1,t,0)c(x_1,t,0)dtdx_1,$$

showing (2.2.117) when supp $(\psi(x_1,\cdot))\subset V_j$. $\square$

2.3 Construction of complex geometric optics solutions based on Gaussian beam quasimodes

Let (M,g) be a CTA manifold so that $(M,g)\subset\subset(\mathbb{R}\times M_0^{\text{int}},c(e\oplus g_0))$. Let $X,Y\in L^\infty(M,TM)$ be complex vector fields, and let $q\in L^\infty(M,\mathbb{C})$. Consider the following operator:

$$P_{X,Y,q}=(-\Delta_g)^2+X+\operatorname{div}(Y)+q. \tag{2.3.1}$$

Note that the operator $P_{X,Y,q}$ comprises both the operator $L_{X,q}$ as well as its formal adjoint $L_{X,q}^*=(-\Delta_g)^2-\overline{X}-\operatorname{div}(\overline{X})+\overline{q}$. Here $\operatorname{div}(Y)\in H^{-1}(M^{\text{int}})$ is given by

$$\langle\operatorname{div}(Y),\varphi\rangle_{M^{\text{int}}}:=-\int Y(\varphi)dV,\quad \varphi\in C_0^\infty(M^{\text{int}}), \tag{2.3.2}$$

where $\langle \cdot, \cdot \rangle_{M^{\text{int}}}$ is a distributional duality on M^{int}. We shall also view $\text{div}(Y)$ as multiplication operator,

$$\text{div}(Y) : C_0^\infty(M^{\text{int}}) \to H^{-1}(M^{\text{int}}). \tag{2.3.3}$$

Therefore, it follows from (2.3.1) that

$$P_{X,Y,q} : C_0^\infty(M^{\text{int}}) \to H^{-1}(M^{\text{int}}).$$

In this section, we will construct complex geometric optics solutions to the equation $P_{X,Y,q}u = 0$ in M based on the Gaussian beam quasimodes for the conjugated biharmonic operator, constructed in Section 2.2.

Assume, as we may, that (M, g) is embedded in a compact smooth manifold (N, g) without boundary of the same dimension, and let U be open in N such that $M \subset U$. Let $\varphi \in C^\infty(U, \mathbb{R})$ and let us consider the conjugated operator

$$P_\varphi = e^{\frac{\varphi}{h}}(-h^2\Delta_g)e^{-\frac{\varphi}{h}} = -h^2\Delta_g - |\nabla\varphi|_g^2 + 2\langle \nabla\varphi, h\nabla\rangle_g + h\Delta_g\varphi$$

with the semiclassical principal symbol

$$p_\varphi = |\xi|_g^2 - |d\varphi|_g^2 + 2i\langle \xi, d\varphi\rangle_g \in C^\infty(T^*U).$$

Following [63], [36], we have the following definition.

Definition 2.3.1. *We say that $\varphi \in C^\infty(U, \mathbb{R})$ is a limiting Carleman weight for $-h^2\Delta_g$ on (U, g) if $d\varphi \neq 0$ on U, and the Poisson bracket of $Re p_\varphi$ and $Im p_\varphi$ satisfies*

$$\{Re p_\varphi, Im p_\varphi\} = 0 \quad \textit{when} \quad p_\varphi = 0.$$

We refer to [36] for a characterization of Riemannian manifolds admitting limiting Carleman weights as well as for examples of limiting Carleman weights. In particular, note that $\phi(x) = \pm x_1$ is a limiting Carleman weight for $-h^2\Delta_g$ on a CTA manifold; see [36].

Our starting point is the following Carleman estimates for $-h^2\Delta_g$ with a gain of two derivatives, established in [73]; see also [36] and [106].

Proposition 2.3.2. *Let ϕ be a limiting Carleman weight for $-h^2\Delta_g$ on U. Then for all $0 < h \ll 1$ and $t \in \mathbb{R}$, we have*

$$h\|u\|_{H^{t+2}_{\mathrm{scl}}(N)} \le C\|e^{\frac{\phi}{h}}(-h^2\Delta_g)e^{-\frac{\phi}{h}}u\|_{H^t_{\mathrm{scl}}(N)}, \quad C > 0, \tag{2.3.4}$$

for all $u \in C_0^\infty(M^{int})$.

Here $H^t(N)$, $t \in \mathbb{R}$, is the standard Sobolev space, equipped with the natural semiclassical norm,

$$\|u\|_{H^t_{\mathrm{scl}}(N)} = \|(1 - h^2\Delta_g)^{\frac{t}{2}}u\|_{L^2(N)}.$$

Iterating (2.3.4), we get the following Carleman estimates for $(-h^2\Delta_g)^2$, for $0 < h \ll 1$ and $t \in \mathbb{R}$:

$$h^2\|u\|_{H^{t+4}_{scl}(N)} \le C\|e^{\frac{\phi}{h}}(-h^2\Delta_g)^2e^{-\frac{\phi}{h}}u\|_{H^t_{scl}(N)}, \quad C > 0, \tag{2.3.5}$$

for all $u \in C_0^\infty(M^{\mathrm{int}})$.

To construct complex geometric optics solutions for $P_{X,Y,q}u = 0$, we shall need the following Carleman estimates for the operator $P_{X,Y,q}$. In what follows we extend X, Y, and q to N by zero and we denote these extensions by the same letters so that $X, Y \in L^\infty(N, TN)$ and $q \in L^\infty(N, \mathbb{C})$.

Proposition 2.3.3. *Let ϕ be a limiting Carleman weight for $-h^2\Delta_g$ on U. Then for all $0 < h \ll 1$, we have*

$$h^2\|u\|_{H^1_{scl}(N)} \le C\|e^{\frac{\phi}{h}}(h^4 P_{X,Y,q})e^{-\frac{\phi}{h}}u\|_{H^{-3}_{scl}(N)}, \quad C > 0, \tag{2.3.6}$$

for all $u \in C_0^\infty(M^{int})$.

Proof. First letting $t = -3$ in (2.3.5), we get for all $0 < h \ll 1$,

$$h^2\|u\|_{H^1_{scl}(N)} \le C\|e^{\frac{\phi}{h}}(-h^2\Delta_g)^2 e^{-\frac{\phi}{h}}u\|_{H^{-3}_{scl}(N)} \tag{2.3.7}$$

for all $u \in C_0^\infty(M^{\text{int}})$. We also have

$$\|e^{\frac{\phi}{h}}h^4 X(e^{-\frac{\phi}{h}}u)\|_{H^{-3}_{scl}(N)} \le \|h^4X(u) - h^3X(\phi)u\|_{L^2(N)} = \mathcal{O}(h^3)\|u\|_{H^1_{scl}(N)}. \tag{2.3.8}$$

In order to estimate $\|h^4 \operatorname{div}(Y)u\|_{H^{-3}_{\text{scl}}(N)}$, we shall use the following characterization of the semiclassical norm in the Sobolev space $H^{-3}(N)$:

$$\|v\|_{H^{-3}_{\text{scl}}(N)} = \sup_{0\neq\psi\in C^\infty(N)} \frac{|\langle v,\psi\rangle_N|}{\|\psi\|_{H^3_{\text{scl}}(N)}}.$$

Using (2.3.2), for $0 \neq \psi \in C^\infty(N)$, we get

$$|\langle h^4 e^{\frac{\phi}{h}} \operatorname{div}(Y)e^{-\frac{\phi}{h}}u,\psi\rangle_N| \le \int_N h^4|Y(u\psi)|dV \le \mathcal{O}(h^3)\|u\|_{H^1_{\text{scl}}(N)}\|\psi\|_{H^3_{\text{scl}}(N)},$$

and therefore,

$$\|h^4 \operatorname{div}(Y)u\|_{H^{-3}_{\text{scl}}(N)} \le \mathcal{O}(h^3)\|u\|_{H^1_{\text{scl}}(N)}. \tag{2.3.9}$$

Finally, we have

$$\|h^4 qu\|_{H^{-3}_{\mathrm{scl}}(N)} \le \mathcal{O}(h^4)\|u\|_{H^1_{\mathrm{scl}}(N)}. \tag{2.3.10}$$

Combining (2.3.7), (2.3.8), (2.3.9), and (2.3.10), we obtain (2.3.6) for all $0 < h \ll 1$ and $u \in C_0^\infty(M^{\mathrm{int}})$. □

Note that the formal L^2 adjoint of $P_{X,Y,q}$ is given by $P_{-\overline{X},-\overline{X}+\overline{Y},\overline{q}}$. Using the fact that if ϕ is a limiting Carleman weight then so is $-\phi$, we obtain the following solvability result; see [36] and [70] for the details.

Proposition 2.3.4. *Let $X, Y \in L^\infty(M, TM)$ be complex vector fields, and let $q \in L^\infty(M, \mathbb{C})$. Let ϕ be a limiting Carleman weight for $-h^2\Delta_g$ on (U, g). If $h > 0$ is small enough, then for any $v \in H^{-1}(M^{int})$, there is a solution $u \in H^3(M^{int})$ of the equation*

$$e^{\frac{\phi}{h}}(h^4 P_{X,Y,q})e^{-\frac{\phi}{h}}u = v \quad \text{in} \quad M^{int},$$

which satisfies

$$\|u\|_{H^3_{\mathrm{scl}}(M^{int})} \le \frac{C}{h^2}\|v\|_{H^{-1}_{\mathrm{scl}}(M^{int})}.$$

Let

$$s = \mu + i\lambda, \quad 1 \le \mu = \frac{1}{h}, \quad \lambda \in \mathbb{R}, \quad \lambda \quad \text{fixed}.$$

We shall construct complex geometric optics solutions to the equation

$$P_{X,Y,q}u = 0 \quad \text{in} \quad M^{\mathrm{int}} \tag{2.3.11}$$

of the form

$$u = e^{-sx_1}(v_s + r_s), \tag{2.3.12}$$

where v_s is a Gaussian beam quasimode for $(-h^2\Delta_g)^2$, constructed in Proposition 2.2.1. Thus, u is a solution to (2.3.11) provided that

$$\begin{aligned} e^{sx_1}h^4 P_{X,Y,q} e^{-sx_1} r_s = &- e^{sx_1}h^4 P_{X,Y,q} e^{-sx_1} v_s = -e^{sx_1}(-h^2\Delta_g)^2 e^{-sx_1} v_s \\ &- e^{sx_1}h^4 X(e^{-sx_1}v_s) - e^{sx_1}h^4 \operatorname{div}(Y)(e^{-sx_1}v_s) - h^4 q v_s =: F. \end{aligned} \tag{2.3.13}$$

Let us estimates the terms in the right-hand side of (2.3.13) in $H^{-1}_{\mathrm{scl}}(M^{\mathrm{int}})$. First, it follows from (2.2.1) that

$$\|e^{sx_1}(-h^2\Delta_g)^2 e^{-sx_1} v_s\|_{H^{-1}_{\mathrm{scl}}(M^{\mathrm{int}})} \le \|e^{sx_1}(-h^2\Delta_g)^2 e^{-sx_1} v_s\|_{L^2(M)} = \mathcal{O}(h^{5/2}) \tag{2.3.14}$$

and

$$\|e^{sx_1}h^4 X(e^{-sx_1}v_s)\|_{H^{-1}_{\mathrm{scl}}(M^{\mathrm{int}})} \le \|h^4 X(v_s) - h^4 s X(x_1) v_s\|_{L^2(M)} = \mathcal{O}(h^3). \tag{2.3.15}$$

Letting $0 \ne \rho \in C_0^\infty(M^{\mathrm{int}})$ and using (2.3.2), we obtain that

$$\begin{aligned} |\langle e^{sx_1}h^4 \operatorname{div}(Y)(e^{-sx_1}v_s), \rho\rangle_{M^{\mathrm{int}}}| &\le h^4 \int |Y(v_s\rho)| dV \\ &= \mathcal{O}(h^3)\|v_s\|_{H^1_{\mathrm{scl}}(M^{\mathrm{int}})}\|\rho\|_{H^1_{\mathrm{scl}}(M^{\mathrm{int}})} = \mathcal{O}(h^3)\|\rho\|_{H^1_{\mathrm{scl}}(M^{\mathrm{int}})}, \end{aligned}$$

and therefore,

$$\|e^{sx_1}h^4 \operatorname{div}(Y)(e^{-sx_1}v_s)\|_{H^{-1}_{\mathrm{scl}}(M^{\mathrm{int}})} = \mathcal{O}(h^3). \tag{2.3.16}$$

We also have

$$\|h^4 q v_s\|_{H^{-1}_{\mathrm{scl}}(M^{\mathrm{int}})} = \mathcal{O}(h^4). \tag{2.3.17}$$

Using (2.3.14), (2.3.15), (2.3.16), (2.3.17), we get from (2.3.13) that $\|F\|_{H^{-1}_{\mathrm{scl}}(M^{\mathrm{int}})} = \mathcal{O}(h^{5/2})$. An application of Proposition 2.3.4 to (2.3.13) gives that for all $h > 0$ small enough, there exists $r_s \in H^3(M^{\mathrm{int}})$ such that $\|r_s\|_{H^3_{scl}(M^{\mathrm{int}})} = \mathcal{O}(h^{1/2})$. To summarize, we have proven the following result.

Proposition 2.3.5. *Let $X, Y \in L^\infty(M, TM)$ be complex vector fields, and let $q \in L^\infty(M, \mathbb{C})$. Let $s = \frac{1}{h} + i\lambda$ with $\lambda \in \mathbb{R}$ being fixed. For all $h > 0$ small enough, there is a solution $u_1 \in H^3(M^{int})$ of $P_{X,Y,q} u_1 = 0$ in M^{int} having the form*

$$u_1 = e^{-sx_1}(v_s + r_1),$$

where $v_s \in C^\infty(M)$ is the Gaussian beam quasimode given in Proposition 2.2.1 and $r_1 \in H^3(M^{int})$ such that $\|r_1\|_{H^3_{\mathrm{scl}}(M^{int})} = \mathcal{O}(h^{1/2})$ as $h \to 0$.

Similarly, for all $h > 0$ small enough, there is a solution $u_2 \in H^3(M^{int})$ of $P_{X,Y,q} u_2 = 0$ in M^{int} having the form

$$u_2 = e^{sx_1}(w_s + r_2),$$

where $w_s \in C^\infty(M)$ is the Gaussian beam quasimode given in Proposition 2.2.1 and $r_2 \in H^3(M^{int})$ such that $\|r_2\|_{H^3_{\mathrm{scl}}(M^{int})} = \mathcal{O}(h^{1/2})$ as $h \to 0$.

2.4 Proof of Theorem 2.1.2

Our starting point is the following integral identity.

Proposition 2.4.1. *Let $X^{(1)}, X^{(2)} \in C(M, TM)$ with complex valued coefficients, and $q^{(1)}, q^{(2)} \in C(M, \mathbb{C})$. If $\mathcal{C}_{X^{(1)},q^{(1)}} = \mathcal{C}_{X^{(2)},q^{(2)}}$, then*

$$\int_M \big((X^{(1)} - X^{(2)})(u_1)\overline{u_2} + (q^{(1)} - q^{(2)})u_1\overline{u_2}\big)dV_g = 0 \tag{2.4.1}$$

for $u_1, u_2 \in H^3(M^{int})$ satisfying

$$L_{X^{(1)},q^{(1)}}u_1 = 0 \quad \textit{and} \quad L_{-\overline{X^{(2)}},-\operatorname{div}(\overline{X^{(2)}})+\overline{q^{(2)}}}u_2 = 0. \tag{2.4.2}$$

Proof. First, using that $\overline{u_2}$ solves the equation

$$L_{-X^{(2)},-\operatorname{div}(X^{(2)})+q^{(2)}}\overline{u_2} = 0, \tag{2.4.3}$$

similar to (2.1.2), we define the boundary trace $\partial_\nu(\Delta_g\overline{u_2}) \in H^{-1/2}(\partial M)$ as follows. Letting $\varphi \in H^{1/2}(\partial M)$ and letting $v \in H^1(M^{\text{int}})$ be a continuous extension of φ, we set

$$\begin{aligned}\langle \partial_\nu(-\Delta_g\overline{u_2}), \varphi\rangle_{H^{-1/2}(\partial M)\times H^{1/2}(\partial M)} &= -\int_{\partial M}(X^{(2)}\cdot\nu)\overline{u_2}v\,dS_g \\ &+ \int_M \big(\langle\nabla_g(-\Delta_g\overline{u_2}), \nabla_g v\rangle_g + \overline{u_2}X^{(2)}(v) + q^{(2)}\overline{u_2}v\big)dV_g.\end{aligned} \tag{2.4.4}$$

It follows from (2.4.3) that the definition of the trace $\partial_\nu(\Delta_g\overline{u_2})$ is independent of the choice of extension v of φ.

As $\mathcal{C}_{X^{(1)},q^{(1)}} = \mathcal{C}_{X^{(2)},q^{(2)}}$, there exists $v_2 \in H^3(M^{\text{int}})$ such that

$$L_{X^{(2)},q^{(2)}}v_2 = 0 \quad \text{in} \quad M \tag{2.4.5}$$

and

$$\begin{aligned}u_1|_{\partial M} = v_2|_{\partial M}, \quad (\Delta_g u_1)|_{\partial M} = (\Delta_g v_2)|_{\partial M}, \quad \partial_\nu u_1|_{\partial M} = \partial_\nu v_2|_{\partial M}, \\ \partial_\nu(\Delta_g u_1)|_{\partial M} = \partial_\nu(\Delta_g v_2)|_{\partial M}.\end{aligned} \tag{2.4.6}$$

It follows from (2.4.6) in particular that

$$\langle \partial_\nu(\Delta_g u_1), \overline{u_2}\rangle_{H^{-1/2}(\partial M)\times H^{1/2}(\partial M)} = \langle \partial_\nu(\Delta_g v_2), \overline{u_2}\rangle_{H^{-1/2}(\partial M)\times H^{1/2}(\partial M)}. \tag{2.4.7}$$

Using that v_2 solves (2.4.5) and (2.1.2), we get

$$\begin{aligned}\langle \partial_\nu(-\Delta_g v_2), \overline{u_2}\rangle_{H^{-1/2}(\partial M)\times H^{1/2}(\partial M)}& \\ = \int_M \big(\langle \nabla_g(-\Delta_g v_2), \nabla_g \overline{u_2}\rangle_g + X^{(2)}(v_2)\overline{u_2} + q^{(2)} v_2 \overline{u_2}\big) dV_g.&\end{aligned} \tag{2.4.8}$$

Using (2.4.4) and integration by parts, we obtain that

$$\begin{aligned}\langle \partial_\nu(-\Delta_g \overline{u_2}), v_2\rangle_{H^{-1/2}(\partial M)\times H^{1/2}(\partial M)} &= -\int_{\partial M} (X^{(2)}\cdot\nu)\overline{u_2} v_2 dS_g \\ &+ \int_M \big(\langle \nabla_g \overline{u_2}, \nabla_g(-\Delta_g) v_2\rangle_g + \overline{u_2} X^{(2)}(v_2) + q^{(2)}\overline{u_2} v_2\big) dV_g \\ &+ \int_{\partial M} (\partial_\nu \overline{u_2})\Delta_g v_2 dS_g - \int_{\partial M} (\Delta_g \overline{u_2})\partial_\nu v_2 dS_g.\end{aligned} \tag{2.4.9}$$

Combining (2.4.8) and (2.4.9), using (2.4.6), we obtain that

$$\begin{aligned}\langle \partial_\nu(-\Delta_g v_2), \overline{u_2}\rangle_{H^{-1/2}(\partial M)\times H^{1/2}(\partial M)} &= \langle \partial_\nu(-\Delta_g \overline{u_2}), v_2\rangle_{H^{-1/2}(\partial M)\times H^{1/2}(\partial M)} \\ &+ \int_{\partial M} (X^{(2)}\cdot\nu)\overline{u_2} v_2 dS_g - \int_{\partial M} (\partial_\nu \overline{u_2})\Delta_g v_2 dS_g + \int_{\partial M} (\Delta_g \overline{u_2})\partial_\nu v_2 dS_g \\ &= \langle \partial_\nu(-\Delta_g \overline{u_2}), u_1\rangle_{H^{-1/2}(\partial M)\times H^{1/2}(\partial M)} + \int_{\partial M} (X^{(2)}\cdot\nu)\overline{u_2} u_1 dS_g \\ &- \int_{\partial M} (\partial_\nu \overline{u_2})\Delta_g u_1 dS_g + \int_{\partial M} (\Delta_g \overline{u_2})\partial_\nu u_1 dS_g \\ &= \int_M \big(\langle \nabla_g \overline{u_2}, \nabla_g(-\Delta_g) u_1\rangle_g + \overline{u_2} X^{(2)}(u_1) + q^{(2)}\overline{u_2} u_1\big) dV_g.\end{aligned} \tag{2.4.10}$$

On the other hand, using (2.4.2) for u_1 and (2.1.2), we get

$$\begin{aligned}\langle \partial_\nu(-\Delta_g u_1), \overline{u_2}\rangle_{H^{-1/2}(\partial M)\times H^{1/2}(\partial M)}& \\ = \int_M \big(\langle \nabla_g(-\Delta_g) u_1, \nabla_g \overline{u_2}\rangle_g + X^{(1)}(u_1)\overline{u_2} + q^{(1)} u_1 \overline{u_2}\big) dV_g.&\end{aligned} \tag{2.4.11}$$

The claim follows from (2.4.7), (2.4.10), and (2.4.11). $\square$

Now by Proposition 2.3.5, for $h > 0$ small enough, there are $u_1, u_2 \in H^3(M^{\text{int}})$ solutions to $L_{X^{(1)},q^{(1)}} u_1 = 0$ and $L_{-\overline{X^{(2)}},-\operatorname{div}(\overline{X^{(2)}})+\overline{q^{(2)}}} u_2 = 0$ in M^{int}, of the form

$$u_1 = e^{-sx_1}(v_s + r_1), \quad u_2 = e^{sx_1}(w_s + r_2), \tag{2.4.12}$$

where $v_s, w_s \in C^\infty(M)$ are the Gaussian beam quasimode given in Proposition 2.2.1 and

$$\|r_1\|_{H^1_{scl}(M^{\text{int}})} = \mathcal{O}(h^{1/2}), \quad \|r_2\|_{H^1_{scl}(M^{\text{int}})} = \mathcal{O}(h^{1/2}), \tag{2.4.13}$$

as $h \to 0$.

Let us denote $X = X^{(1)} - X^{(2)}$ and $q = q^{(1)} - q^{(2)}$. By the boundary determination of Proposition 2.5.1, we have that $X^{(1)}|_{\partial M} = X^{(2)}|_{\partial M}$, and therefore, we may extend X by zero to the complement of M in $\mathbb{R} \times M_0$ so that the extension $X \in C(\mathbb{R} \times M_0, T(\mathbb{R} \times M_0))$.

Step 1. Proving that there exists $\psi \in C^1(\mathbb{R} \times M_0)$ with compact support in x_1 such that $\psi(x_1, \cdot)|_{\partial M_0} = 0$ and $\nabla_g \psi = X$. In this step, we shall work with solutions u_1 and u_2 given by (2.4.12) with v_s and w_s being the Gaussian beam quasimode for which Proposition 2.2.3 holds. In particular, here v_s has an amplitude of the first type. Next, we would like to substitute u_1 and u_2 into the integral identity (2.4.1), multiply it by h, and let $h \to 0$. To that end, first using (2.4.13), (2.2.1), and (2.2.2), we get

$$\left| h \int_M q u_1 \overline{u_2} dV_g \right| = \left| h \int_M q e^{-2i\lambda x_1}(v_s + r_1)(\overline{w_s} + \overline{r_2}) dV_g \right| = \mathcal{O}(h). \tag{2.4.14}$$

Writing $x = (x_1, x')$, $x' \in M_0$, and $X = X_1 \partial_{x_1} + \widetilde{X} \cdot \partial_{x'}$, we obtain that

$$h \int_M X(u_1) \overline{u_2} dV_g = I_1 + I_2 + I_3 + I_4, \tag{2.4.15}$$

where

$$I_1 = h \int_M e^{-2i\lambda x_1} X(v_s) \overline{w_s} dV_g - \int_M X_1(x_1, x') e^{-2i\lambda x_1} v_s \overline{w_s} dV_g, \tag{2.4.16}$$

$$I_2 = -hi\lambda \int_M X_1(x_1, x') e^{-2i\lambda x_1} (v_s + r_1)(\overline{w_s} + \overline{r_2}) dV_g, \tag{2.4.17}$$

$$I_3 = -\int_M X_1(x_1, x') e^{-2i\lambda x_1} (v_s \overline{r_2} + \overline{w_s} r_1 + r_1 \overline{r_2}) dV_g, \tag{2.4.18}$$

$$I_4 = h \int_M e^{-2i\lambda x_1} (X(v_s)\overline{r_2} + X(r_1)\overline{w_s} + X(r_1)\overline{r_2}) dV_g. \tag{2.4.19}$$

Using (2.4.13), (2.2.1), and (2.2.2), we get

$$|I_2| = \mathcal{O}(h), \quad |I_3| = \mathcal{O}(h^{1/2}), \quad |I_4| = \mathcal{O}(h^{1/2}). \tag{2.4.20}$$

It follows from (2.4.1) with the help of (2.4.14), (2.4.15), and (2.4.20) that

$$\lim_{h \to 0} I_1 = 0. \tag{2.4.21}$$

Using that $X = 0$ outside of M, $dV_g = c^{\frac{n}{2}} dx_1 dV_{g_0}$, Fubini's theorem, and Proposition 2.2.3, we obtain from (2.4.21) that

$$
\begin{aligned}
0 &= \lim_{h\to 0} h \int_{\mathbb{R}} e^{-2i\lambda x_1} \int_{M_0} X(v_s)\overline{w_s} c(x_1,x')^{\frac{n}{2}} dV_{g_0} dx_1 \\
&\quad - \lim_{h\to 0} \int_{\mathbb{R}} e^{-2i\lambda x_1} \int_{M_0} X_1(x_1,x') v_s \overline{w_s} c(x_1,x')^{\frac{n}{2}} dV_{g_0} dx_1 \\
&= -\int_{\mathbb{R}} e^{-2i\lambda x_1} \int_0^L \big(X_1(x_1,\gamma(t)) - iX_t(x_1,\gamma(t))\big) c(x_1,\gamma(t)) e^{-2\lambda t} dt dx_1.
\end{aligned}
\tag{2.4.22}
$$

Now the Riemmanian metric g on M induces a natural isomorphism between the tangent and cotangent bundles given by

$$
TM \to T^*M, \quad (x,X) \mapsto (x,X^b), \tag{2.4.23}
$$

where $X^b(Y) = \langle X,Y\rangle$. In local coordinates, $X^b = \sum_{j,k=1}^n g_{jk} X_j dx_k$, and using that $g = c(e\oplus g_0)$, and (2.2.10), we get

$$
X_1^b(x_1,\gamma(t)) = c(x_1,\gamma(t)) X_1(x_1,\gamma(t)), \quad X_t^b(x_1,\gamma(t)) = c(x_1,\gamma(t)) X_t(x_1,\gamma(t)).
$$

Hence, it follows from (2.4.22), replacing 2λ by λ, that

$$
\int_{\mathbb{R}} \int_0^L e^{-i\lambda x_1 - \lambda t} (X_1^b(x_1,\gamma(t)) - iX_t^b(x_1,\gamma(t))) dt dx_1 = 0. \tag{2.4.24}
$$

Letting

$$
\begin{aligned}
f(\lambda,x') &= \int_{\mathbb{R}} e^{-i\lambda x_1} X_1^b(x_1,x') dx_1, \quad x' \in M_0, \\
\alpha(\lambda,x') &= \sum_{j=2}^n \Big(\int_{\mathbb{R}} e^{-i\lambda x_1} X_j^b(x_1,x') \Big) dx_j,
\end{aligned}
\tag{2.4.25}
$$

we have $f(\lambda,\cdot)\in C(M_0)$, $\alpha(\lambda,\cdot)\in C(M_0,T^*M)$, and (2.4.24) implies that

$$\int_0^L [f(\lambda,\gamma(t)) - i\alpha(\lambda,\dot\gamma(t))]e^{-\lambda t}dt = 0, \tag{2.4.26}$$

along any unit speed nontangential geodesic $\gamma:[0,L]\to M_0$ on M_0 and any $\lambda\in\mathbb{R}$. Arguing as in [73, Section 7], [32], using the injectivity of the geodesic X-ray transform on functions and 1-forms, we conclude from (2.4.26) that there exist $p_l\in C^1(M_0)$, $p_l|_{\partial M_0}=0$, such that

$$\partial_\lambda^l f(0,x') + lp_{l-1}(x') = 0, \quad \partial_\lambda^l \alpha(0,x') = idp_l(x'), \quad l = 0,1,2,\dots. \tag{2.4.27}$$

To proceed we shall follow [40, Section 5] and let

$$\psi(x_1,x') = \int_{-a}^{x_1} X_1^b(y_1,x')dy_1, \tag{2.4.28}$$

where supp $(X^b(\cdot,x'))\subset(-a,a)$. It follows from (2.4.27), (2.4.25) that

$$0 = f(0,x') = \int_{\mathbb{R}} X_1^b(y_1,x')dy_1,$$

and therefore, ψ has compact support in x_1. Thus, the Fourier transform of ψ with respect to x_1, which we denote by $\widehat{\psi}(\lambda,x')$, is real analytic with respect to λ, and therefore, we have

$$\widehat{\psi}(\lambda,x') = \sum_{k=0}^{\infty}\frac{\psi_k(x')}{k!}\lambda^k, \tag{2.4.29}$$

where $\psi_k(x') = (\partial_\lambda^k\widehat{\psi})(0,x')$. It follows from (2.4.28) that

$$\partial_{x_1}\psi(x_1,x') = X_1^b(x_1,x'), \tag{2.4.30}$$

and therefore, taking the Fourier transform with respect to x_1, and using (2.4.25)

$$i\lambda\psi(\lambda, x') = f(\lambda, x'). \tag{2.4.31}$$

Differentiating (2.4.31) $(l+1)$-times in λ, letting $\lambda = 0$, and using (2.4.27), we get

$$\partial_\lambda^l \widehat{\psi}(0, x') = ip_l(x'), \quad l = 0, 1, 2, \ldots. \tag{2.4.32}$$

Substituting (2.4.32) into (2.4.29), we obtain that

$$\widehat{\psi}(\lambda, x') = \sum_{k=0}^{\infty} \frac{ip_l(x')}{k!}\lambda^k,$$

and taking the differential in x' in the sense of distributions, and using (2.4.27), (2.4.25), we see that

$$d_{x'}\widehat{\psi}(\lambda, x') = \sum_{k=0}^{\infty} \frac{idp_l(x')}{k!}\lambda^k = \sum_{k=0}^{\infty} \frac{\partial_\lambda^k \alpha(0, x')}{k!}\lambda^k = \alpha(\lambda, x') = \sum_{j=2}^{n} \widehat{X}_j^b(\lambda, x')dx_j. \tag{2.4.33}$$

Taking the inverse Fourier transform $\lambda \mapsto x_1$ in (2.4.33), we get

$$d_{x'}\psi(x_1, x') = \sum_{j=2}^{n} X_j^b(x_1, x')dx_j. \tag{2.4.34}$$

We also have from (2.4.30) that

$$d_{x_1}\psi(x_1, x') = X_1^b(x_1, x')dx_1. \tag{2.4.35}$$

It follows from (2.4.34) and (2.4.35) that

$$d\psi = X^b. \tag{2.4.36}$$

Using the inverse of (2.4.23), we see from (2.4.36) that

$$\nabla_g \psi = X. \tag{2.4.37}$$

Recall that $\psi \in C(\mathbb{R} \times M_0)$ with compact support in x_1 and $\psi(x_1, \cdot)|_{\partial M_0} = 0$. It follows from (2.4.37) that $\psi \in C^1(\mathbb{R} \times M_0)$.

Step 2. Showing that $X = 0$. Returning to (2.4.1) and using (2.4.37), we get

$$\int_M \big((\nabla_g \psi)(u_1)\overline{u_2} + q u_1 \overline{u_2}\big) dV_g = 0, \tag{2.4.38}$$

for $u_1, u_2 \in H^3_{scl}(M^{\text{int}})$ satisfying $L_{X^{(1)},q^{(1)}} u_1 = 0$ and $L_{-\overline{X^{(2)}}, -\operatorname{div}(\overline{X^{(2)}}) + \overline{q^{(2)}}} u_2 = 0$. Let now u_1 and u_2 be given by (2.4.12) with v_s and w_s being the Gaussian beam quasimode for which Proposition 2.2.4 holds. In particular, here v_s has an amplitude of the second type. We would like to substitute u_1 and u_2 into the integral identity (2.4.38), multiply it by h, and let $h \to 0$. Similar to (2.4.21), using (2.4.14) and (2.4.20), we get

$$\lim_{h\to 0} h \int_M e^{-2i\lambda x_1} (\nabla_g \psi)(v_s) \overline{w_s} dV_g - \int_M (\nabla_g \psi)_1 e^{-2i\lambda x_1} v_s \overline{w_s} dV_g = 0. \tag{2.4.39}$$

It follows from (2.4.39) with the help of Proposition 2.2.4,

$$\int_{\mathbb{R}} \int_0^L e^{-2i\lambda(x_1 - it)} \psi(x_1, \gamma(t)) c(x_1, \gamma(t)) dt dx_1 = 0. \tag{2.4.40}$$

Now (2.4.40) can be written as

$$\int_\gamma \widehat{\psi c}(2\lambda, \gamma(t)) e^{-2\lambda t} dt = 0 \tag{2.4.41}$$

for any $\lambda \in \mathbb{R}$ and any nontangential geodesic γ in M_0, where

$$\widehat{\psi c}(2\lambda, x') = \int_{-\infty}^{\infty} e^{-2i\lambda x_1}(\psi c)(x_1, x')dx_1.$$

Equation (2.4.41) says that the attenuated geodesic ray transform of $\widehat{\psi c}$ with constant attenuation -2λ vanishes along all nontangential geodesics in M_0. Arguing as in [38, Proof of Theorem 1.2] and using the injectivity of the geodesic X-ray transform on functions, we conclude that $\psi c = 0$, and therefore $\psi = 0$, and hence $X = 0$.

Step 3. Proving that $q = 0$. Returning to (2.4.1) and substituting $X^{(1)} = X^{(2)}$, we get

$$\int_M q u_1 \overline{u_2} dV_g = 0 \tag{2.4.42}$$

for $u_1, u_2 \in H^3_{scl}(M^{\text{int}})$ satisfying $L_{X^{(1)},q^{(1)}}u_1 = 0$ and $L_{-\overline{X^{(2)}},-\operatorname{div}(\overline{X^{(2)}})+\overline{q^{(2)}}}u_2 = 0$. Let now u_1 and u_2 be given by (2.4.12) with v_s and w_s being the Gaussian beam quasimode for which Proposition 2.2.3 holds. In particular, here v_s has an amplitude of the first type. Substituting u_1 and u_2 into (2.4.42), we obtain that

$$0 = \int_M q u_1 \overline{u_2} dV_g = I_1 + I_2, \tag{2.4.43}$$

where

$$\begin{aligned} I_1 &= \int_M e^{-2i\lambda x_1} q v_s \overline{w_s} dV_g = \int_{\mathbb{R}} e^{-2i\lambda x_1} \int_{M_0} q v_s \overline{w_s} c^{\frac{n}{2}} dV_{g_0} dx_1, \\ I_2 &= \int_M e^{-2i\lambda x_1} q(v_s \overline{r_2} + r_1 \overline{w_s} + r_1 \overline{r_2}) dV_g. \end{aligned}$$

Here in view of the assumption (2.1.4), we extended q by zero to the complement of M in $\mathbb{R} \times M_0$ so that the extension $q \in C(\mathbb{R} \times M_0, \mathbb{C})$.

Using (2.4.13), (2.2.1), and (2.2.2), we see that

$$|I_2| = \mathcal{O}(h^{1/2}). \tag{2.4.44}$$

Letting $h \to 0$, we obtain from (2.4.43), (2.4.44) with the help of Proposition 2.2.3 that

$$\int_{\mathbb{R}} e^{-2i\lambda x_1} \int_0^L e^{-2\lambda t}(qc)(x_1, \gamma(t))dtdx_1 = 0.$$

Arguing as in [38, Proof of Theorem 1.2] and using the injectivity of the geodesic X-ray transform on functions, we conclude that $qc = 0$, and therefore $q = 0$. This complete the proof of Theorem 2.1.2.

2.5 Boundary determination of a first order perturbation of the biharmonic operator

When proving Theorem 2.1.2, an important step consists in determining the boundary values of the first order perturbation of the biharmonic operator. The purpose of this section is to carry out this step by adapting the method of [22], [73].

Proposition 2.5.1. *Let (M, g) be a CTA manifold of dimension $n \geq 3$. Let $X^{(1)}, X^{(2)} \in C(M, TM)$ with complex vector fields and $q^{(1)}, q^{(2)} \in L^\infty(M, \mathbb{C})$. If $\mathcal{C}_{g,X^{(1)},q^{(1)}} = \mathcal{C}_{g,X^{(2)},q^{(2)}}$, then $X^{(1)}|_{\partial M} = X^{(2)}|_{\partial M}$.*

Proof. We shall follow [22], [73] closely. We shall construct some special solutions to the equations $L_{X^{(1)},q^{(1)}}u_1 = 0$ and $L_{-\overline{X^{(2)}},-\operatorname{div}(\overline{X^{(2)}})+\overline{q^{(2)}}}u_2 = 0$, whose boundary values have an oscillatory behavior while becoming increasingly concentrated near a given point on the boundary of M. Substituting these solutions into the integral identity (2.4.1) will allow us to prove that $X^{(1)}|_{\partial M} = X^{(2)}|_{\partial M}$.

In doing so, let $x_0 \in \partial M$ and let $(x_1, \dots, x_n)$ be the boundary normal coordinates centered at x_0 so that in these coordinates, $x_0 = 0$, the boundary ∂M is given by $\{x_n = 0\}$, and M^{int} is given by $\{x_n > 0\}$. We shall assume, as we may, that

$$g^{\alpha\beta}(0) = \delta^{\alpha\beta}, \quad 1 \le \alpha, \beta \le n-1, \tag{2.5.1}$$

and therefore $T_0\partial M = \mathbb{R}^{n-1}$, equipped with the Euclidean metric. The unit tangent vector τ is then given by $\tau = (\tau', 0)$ where $\tau' \in \mathbb{R}^{n-1}$, $|\tau'| = 1$. Associated to the tangent vector τ' is the covector $\xi'_\alpha = \sum_{\beta=1}^{n-1} g_{\alpha\beta}(0)\tau'_\beta = \tau'_\alpha \in T^*_{x_0}\partial M$.

Let $\eta \in C_0^\infty(\mathbb{R}^n, \mathbb{R})$ be a function such that supp (η) is in a small neighborhood of 0, and

$$\int_{\mathbb{R}^{n-1}} \eta(x', 0)^2 dx' = 1. \tag{2.5.2}$$

Following [22], in the boundary normal coordinates, we set

$$v_0(x) = \eta\left(\frac{x}{\lambda^{1/2}}\right) e^{\frac{i}{\lambda}(\tau' \cdot x' + i x_n)}, \quad 0 < \lambda \ll 1, \tag{2.5.3}$$

so that $v_0 \in C^\infty(M)$ with supp (v_0) in $\mathcal{O}(\lambda^{1/2})$ neighborhood of $x_0 = 0$. Here τ' is viewed as a covector.

Let $v_1 \in H_0^1(M^{\mathrm{int}})$ be the solution to the following Dirichlet problem for the Laplacian:

$$\begin{aligned} -\Delta_g v_1 &= \Delta_g v_0 \quad \text{in} \quad M, \\ v_1|_{\partial M} &= 0. \end{aligned} \tag{2.5.4}$$

Let $\delta(x)$ be the distance from $x \in M$ to the boundary of M. As proved in the [73, Appendix], the following estimates hold:

$$\|v_0\|_{L^2(M)} \leq \mathcal{O}(\lambda^{\frac{n-1}{4}+\frac{1}{2}}), \tag{2.5.5}$$

$$\|v_1\|_{L^2(M)} \leq \mathcal{O}(\lambda^{\frac{n-1}{4}+\frac{1}{2}}), \tag{2.5.6}$$

$$\|dv_1\|_{L^2(M)} \leq \mathcal{O}(\lambda^{\frac{n-1}{4}}), \tag{2.5.7}$$

$$\|dv_0\|_{L^2(M)} \leq \mathcal{O}(\lambda^{\frac{n-1}{4}-\frac{1}{2}}), \tag{2.5.8}$$

$$\|\delta d(v_0+v_1)\|_{L^2(M)} \leq \mathcal{O}(\lambda^{\frac{n-1}{4}+\frac{1}{2}}), \tag{2.5.9}$$

$$\|v_0\|_{L^2(\partial M)} \leq \mathcal{O}(\lambda^{\frac{n-1}{4}}). \tag{2.5.10}$$

We shall also need Hardy's inequality,

$$\int_M |f(x)/\delta(x)|^2 dV_g \leq C \int_M |df(x)|^2 dV_g, \tag{2.5.11}$$

where $f \in H_0^1(M^{\text{int}})$; see [34].

Next we would like to show the existence of a solution $u_1 \in H^3(M^{\text{int}})$ to the equation

$$L_{X^{(1)},q^{(1)}} u_1 = 0 \quad \text{in} \quad M, \tag{2.5.12}$$

of the form

$$u_1 = v_0 + v_1 + r_1, \tag{2.5.13}$$

with

$$\|r_1\|_{H^3(M^{\text{int}})} \le \mathcal{O}(\lambda^{\frac{n-1}{4}+\frac{1}{2}}). \tag{2.5.14}$$

To that end, plugging (2.5.13) into (2.5.12), we obtain the following equation of r_1:

$$L_{X^{(1)},q^{(1)}} r_1 = -((-\Delta_g)^2 + X^{(1)} + q^{(1)})(v_0 + v_1) = -(X^{(1)} + q^{(1)})(v_0 + v_1) \quad \text{in} \quad M. \tag{2.5.15}$$

Applying Proposition 2.3.4 with $h > 0$ small but fixed, we conclude the existence of $r_1 \in H^3(M^{\text{int}})$ such that

$$\|r_1\|_{H^3(M^{\text{int}})} \le \mathcal{O}(1)\|(X^{(1)} + q^{(1)})(v_0 + v_1)\|_{H^{-1}(M^{\text{int}})}. \tag{2.5.16}$$

Let us now bound the norm in the right-hand side of (2.5.16). To that end, letting $\psi \in C_0^\infty(M^{\text{int}})$ and using (2.5.11), (2.5.9), we get

$$\begin{aligned} |\langle X^{(1)}(v_0 + v_1), \psi\rangle_{M^{\text{int}}}| &\le \mathcal{O}(1)\|X^{(1)}\|_{L^\infty(M)}\|\delta d(v_0 + v_1)\|_{L^2(M)}\|\psi\|_{H^1(M^{\text{int}})} \\ &\le \mathcal{O}(\lambda^{\frac{n-1}{4}+\frac{1}{2}})\|\psi\|_{H^1(M^{\text{int}})}. \end{aligned} \tag{2.5.17}$$

By (2.5.5) and (2.5.6), we have

$$\begin{aligned}|\langle q^{(1)}(v_0+v_1),\psi\rangle_{M^{\mathrm{int}}}| &\le \|q^{(1)}\|_{L^\infty(M^0)}\|v_0+v_1\|_{L^2(M)}\|\psi\|_{L^2(M)}\\ &\le \mathcal{O}(\lambda^{\frac{n-1}{4}+\frac{1}{2}})\|\psi\|_{H^1(M^{\mathrm{int}})}.\end{aligned} \tag{2.5.18}$$

The estimate (2.5.14) follows from (2.5.16), (2.5.17), and (2.5.18).

Let us show that there exists a solution $u_2\in H^3(M^{\mathrm{int}})$ of $L_{-\overline{X^{(2)}},-\operatorname{div}(\overline{X^{(2)}})+\overline{q^{(2)}}}u_2=0$ in M of the form

$$u_2=v_0+v_1+r_2, \tag{2.5.19}$$

where $r_2\in H^3(M^{\mathrm{int}})$ with

$$\|r_2\|_{H^3(M^{\mathrm{int}})}\le \mathcal{O}(\lambda^{\frac{n-1}{4}+\frac{1}{2}}). \tag{2.5.20}$$

Applying Proposition 2.3.4 with $h>0$ small but fixed to the equation,

$$L_{-\overline{X^{(2)}},-\operatorname{div}(\overline{X^{(2)}})+\overline{q^{(2)}}}r_2=(\overline{X^{(2)}}+\operatorname{div}(\overline{X^{(2)}})-\overline{q^{(2)}})(v_0+v_1)\quad\text{in}\quad M, \tag{2.5.21}$$

we conclude the existence of $r_2\in H^1(M^{\mathrm{int}})$ such that

$$\|r_2\|_{H^3(M^{\mathrm{int}})}\le \mathcal{O}(1)\|(\overline{X^{(2)}}+\operatorname{div}(\overline{X^{(2)}})-\overline{q^{(2)}})(v_0+v_1)\|_{H^{-1}(M^{\mathrm{int}})}. \tag{2.5.22}$$

To bound the norm in the right-hand side of (2.5.22), we let $\psi\in C_0^\infty(M^{\mathrm{int}})$, and using

(2.5.11), (2.3.2), (2.5.5), (2.5.6), (2.5.9), we get

$$
\begin{aligned}
|\langle \operatorname{div}(\overline{X^{(2)}})(v_0+v_1), \psi\rangle_{M^{\text{int}}}| &= \left| \int \overline{X^{(2)}}((v_0+v_1)\psi) dV_g \right| \\
&\leq \left| \int \psi \overline{X^{(2)}}(v_0+v_1) dV_g \right| + \left| \int (v_0+v_1)\overline{X^{(2)}}(\psi) dV_g \right| \\
&\leq \mathcal{O}(1)\|\delta d(v_0+v_1)\|_{L^2(M)}\|\psi\|_{H^1(M^{\text{int}})} + \mathcal{O}(1)\|v_0+v_1\|_{L^2(M)}\|\psi\|_{H^1(M^{\text{int}})} \\
&\leq \mathcal{O}(\lambda^{\frac{n-1}{4}+\frac{1}{2}})\|\psi\|_{H^1(M^{\text{int}})}.
\end{aligned}
\tag{2.5.23}
$$

The bound (2.5.20) follows from (2.5.22), (2.5.23), (2.5.17), (2.5.18).

The next step is to substitute the solution u_1 and u_2, given in (2.5.13) and (2.5.19), into the integral identity (2.4.1), multiply by $\lambda^{-\frac{(n-1)}{2}}$, and compute the limit as $\lambda \to 0$. In doing so, we write

$$
I := \lambda^{-\frac{(n-1)}{2}} \int_M X(u_1)\overline{u_2} + qu_1\overline{u_2} dV_g = I_1 + I_2 + I_3 + I_4 + I_5 + I_6, \tag{2.5.24}
$$

where

$$
\begin{aligned}
I_1 &= \lambda^{-\frac{(n-1)}{2}} \int_M X(v_0)\overline{v_0} dV_g, \quad I_2 = \lambda^{-\frac{(n-1)}{2}} \int_M X(v_0)\overline{v_1} dV_g, \\
I_3 &= \lambda^{-\frac{(n-1)}{2}} \int_M X(v_0)\overline{r_2} dV_g, \quad I_4 = \lambda^{-\frac{(n-1)}{2}} \int_M X(v_1)\overline{u_2} dV_g, \\
I_5 &= \lambda^{-\frac{(n-1)}{2}} \int_M X(r_1)\overline{u_2} dV_g, \quad I_6 = \lambda^{-\frac{(n-1)}{2}} \int_M qu_1\overline{u_2} dV_g.
\end{aligned}
$$

Let us compute $\lim_{\lambda\to 0} I_1$. To that end, writing $X = X_j\partial_{x_j}$, we have

$$
Xv_0 = e^{\frac{i}{\lambda}(\tau'\cdot x' + ix_n)}\Big[\lambda^{-\frac{1}{2}}(X\eta)\big(\frac{x}{\lambda^{\frac{1}{2}}}\big) + i\lambda^{-1}X(x)\cdot(\tau', i)\eta\big(\frac{x}{\lambda^{\frac{1}{2}}}\big)\Big] \tag{2.5.25}
$$

and

$$Xv_0\overline{v_0} = e^{-\frac{2x_n}{\lambda}}\Big[\lambda^{-\frac{1}{2}}(X\eta)\big(\frac{x}{\lambda^{\frac{1}{2}}}\big)\eta\big(\frac{x}{\lambda^{\frac{1}{2}}}\big) + i\lambda^{-1}X(x)\cdot(\tau', i)\eta^2\big(\frac{x}{\lambda^{\frac{1}{2}}}\big)\Big]. \tag{2.5.26}$$

Making the change of variable $y' = \frac{x'}{\lambda^{1/2}}$, $y_n = \frac{x_n}{\lambda}$, using that $X \in C(M, TM)$, η has compact support, (2.5.1) and (2.5.2), we get

$$\begin{aligned}\lim_{\lambda\to 0} I_1 &= \lim_{\lambda\to 0}\int_{\mathbb{R}^{n-1}}\int_0^\infty e^{-2y_n}\lambda^{\frac{1}{2}}(X\eta)(y', \lambda^{\frac{1}{2}}y_n)\eta(y', \lambda^{\frac{1}{2}}y_n)|g(\lambda^{\frac{1}{2}}y', \lambda y_n)|^{\frac{1}{2}}dy_n dy' \\ &+ \lim_{\lambda\to 0}\int_{\mathbb{R}^{n-1}}\int_0^\infty e^{-2y_n} iX(\lambda^{\frac{1}{2}}y', \lambda y_n)\cdot(\tau', i)\eta^2(y', \lambda^{\frac{1}{2}}y_n)|g(\lambda^{\frac{1}{2}}y', \lambda y_n)|^{\frac{1}{2}}dy_n dy' \\ &= \frac{i}{2}X(0)\cdot(\tau', i).\end{aligned} \tag{2.5.27}$$

The fact that $v_1 \in H_0^1(M^{\text{int}})$ together with the estimates (2.5.11), (2.5.9), (2.5.7) gives that

$$|I_2| \le \mathcal{O}(\lambda^{-\frac{(n-1)}{2}})\|X\|_{L^\infty(M)}\|\delta dv_0\|_{L^2(M)}\|\frac{v_1}{\delta}\|_{L^2(M)} = \mathcal{O}(\lambda^{\frac{1}{2}}). \tag{2.5.28}$$

To estimate I_3, first assume that (M, g) is embedded in a compact smooth manifold (N, g) without boundary of the same dimension. Let us extend $X \in C(M, TM)$ to a continuous vector field on N, and still write $X \in C(N, TN)$. Using a partition of unity argument together with a regularization in each coordinate patch, we see that there exists a family $X_\tau \in C^\infty(N, TN)$ such that

$$\|X - X_\tau\|_{L^\infty} = o(1), \quad \|X_\tau\|_{L^\infty} = \mathcal{O}(1), \quad \|\nabla X_\tau\|_{L^\infty} = \mathcal{O}(\tau^{-1}), \quad \tau \to 0. \tag{2.5.29}$$

We write

$$I_3 = I_{3,1} + I_{3,2}, \tag{2.5.30}$$

where

$$I_{3,1} = \lambda^{-\frac{(n-1)}{2}} \int_M (X - X_\tau)(v_0)\overline{r_2} dV_g, \quad I_{3,2} = \lambda^{-\frac{(n-1)}{2}} \int_M X_\tau(v_0)\overline{r_2} dV_g. \tag{2.5.31}$$

Using (2.5.29), (2.5.8), (2.5.20), we get

$$|I_{3,1}| \leq \mathcal{O}(\lambda^{-\frac{(n-1)}{2}})\|X - X_\tau\|_{L^\infty(M)}\|dv_0\|_{L^2(M)}\|r_2\|_{L^2(M)} = o(1), \tag{2.5.32}$$

as $\tau \to 0$. To estimate $I_{3,2}$, integrating by parts, we obtain that

$$I_{3,2} = J_1 + J_2 + J_3, \tag{2.5.33}$$

where

$$\begin{gathered} J_1 = -\lambda^{-\frac{(n-1)}{2}} \int_M v_0 X_\tau(\overline{r_2}) dV_g, \quad J_2 = -\lambda^{-\frac{(n-1)}{2}} \int_M \operatorname{div}(X_\tau) v_0 \overline{r_2} dV_g, \\ J_3 = \lambda^{-\frac{(n-1)}{2}} \int_{\partial M} (\nu \cdot X_\tau) v_0 \overline{r_2} dS_g. \end{gathered} \tag{2.5.34}$$

Using (2.5.29), (2.5.20), (2.5.5), we get

$$\begin{aligned} |J_1| &\leq \mathcal{O}(\lambda^{-\frac{(n-1)}{2}})\|X_\tau\|_{L^\infty(M)}\|v_0\|_{L^2(M)}\|dr_2\|_{L^2(M)} = \mathcal{O}(\lambda), \\ |J_2| &\leq \mathcal{O}(\lambda^{-\frac{(n-1)}{2}})\|\operatorname{div} X_\tau\|_{L^\infty(M)}\|v_0\|_{L^2(M)}\|r_2\|_{L^2(M)} = \mathcal{O}(\tau^{-1}\lambda). \end{aligned} \tag{2.5.35}$$

Using (2.5.10), (2.5.29), (2.5.20), and the trace theorem, we obtain that

$$|J_3| \leq \mathcal{O}(\lambda^{-\frac{(n-1)}{2}})\|\nu \cdot X_\tau\|_{L^\infty(M)}\|v_0\|_{L^2(\partial M)}\|r_2\|_{H^1(M)} = \mathcal{O}(\lambda^{1/2}). \tag{2.5.36}$$

Choosing $\tau = \lambda^{1/2}$, we conclude from (2.5.30), (2.5.31), (2.5.32), (2.5.33), (2.5.34), (2.5.35), (2.5.36) that

$$|I_3| = o(1), \quad \lambda \to 0. \tag{2.5.37}$$

Now (2.5.5), (2.5.6), (2.5.20) imply that

$$\|u_2\|_{L^2} = \mathcal{O}(\lambda^{\frac{n-1}{4}+\frac{1}{2}}). \tag{2.5.38}$$

Using (2.5.38) together with (2.5.7), we have

$$|I_4| \le \mathcal{O}(\lambda^{-\frac{(n-1)}{2}})\|dv_1\|_{L^2(M)}\|u_2\|_{L^2(M)} = \mathcal{O}(\lambda^{\frac{1}{2}}). \tag{2.5.39}$$

Using (2.5.38) together with (2.5.14), we get

$$|I_5| \le \mathcal{O}(\lambda^{-\frac{(n-1)}{2}})\|dr_1\|_{L^2(M)}\|u_2\|_{L^2(M)} = \mathcal{O}(\lambda). \tag{2.5.40}$$

Last let us estimate $|I_6|$. Using (2.5.38) and a similar bound for u_1, we see that

$$|I_6| \le \mathcal{O}(\lambda^{-\frac{(n-1)}{2}})\|q\|_{L^\infty(M)}\|u_1\|_{L^2(M)}\|u_2\|_{L^2(M)} = \mathcal{O}(\lambda). \tag{2.5.41}$$

Now it follows from (2.5.24), (2.5.27), (2.5.28), (2.5.37), (2.5.39), (2.5.40), and (2.5.41) that

$$\lim_{\lambda\to 0} I = \frac{i}{2}X(0)\cdot(\tau', i) = 0,$$

and therefore,

$$X^{(1)}(0)\cdot(\tau', i) = X^{(2)}(0)\cdot(\tau', i),$$

for all $\tau' \in \mathbb{R}^{n-1}$. This completes the proof of Proposition 2.5.1. □

Chapter 3

Reconstructing a potential perturbation of the biharmonic operator on transversally anisotropic manifolds

3.1 Introduction and statement of results

Let (M, g) be a smooth compact oriented Riemannian manifold of dimension $n \geq 3$ with smooth boundary ∂M. Let γ be the Dirichlet trace operator defined by

$$\gamma : H^2(M^{\text{int}}) \to H^{3/2}(\partial M) \times H^{1/2}(\partial M), \quad \gamma u = (u|_{\partial M}, \partial_\nu u|_{\partial M}), \tag{3.1.1}$$

which is bounded and surjective, see [47, Theorem 9.5]. Here and in what follows $M^{\text{int}} = M \setminus \partial M$, $H^s(M^{\text{int}})$ and $H^s(\partial M)$, $s \in \mathbb{R}$, are the standard L^2–based Sobolev spaces on M^{int} and its boundary ∂M, respectively, and ν is the exterior unit normal to ∂M. We also let

$H_0^2(M^{\text{int}}) = \{u \in H^2(M^{\text{int}}) : \gamma u = 0\}$. Let $-\Delta_g = -\Delta$ be the Laplace–Beltrami operator on M, and let Δ^2 be the biharmonic operator on M. Let $q \in C(M)$. By standard arguments, see for instance [71, Appendix A], the operator

$$\Delta^2 + q : H_0^2(M^{\text{int}}) \to H^{-2}(M^{\text{int}}) = (H_0^2(M^{\text{int}}))', \tag{3.1.2}$$

is Fredholm of index zero and has a discrete spectrum. We shall assume throughout the paper that

(A) 0 is not in the spectrum of the operator (3.1.2).

Thus, for any $f = (f_0, f_1) \in H^{3/2}(\partial M) \times H^{1/2}(\partial M)$, the Dirichlet problem

$$\begin{cases} (\Delta^2 + q)u = 0 & \text{in} \quad M^{\text{int}}, \\ \gamma u = f & \text{on} \quad \partial M, \end{cases} \tag{3.1.3}$$

has a unique solution $u \in H^2(M^{\text{int}})$, depending continuously on f. Physically, the Dirichlet boundary condition in (3.1.3) corresponds to the clamped plate equation, see [44]. We define the Dirichlet–to–Neumann map Λ_q by

$$\langle \Lambda_q f, g \rangle_{H^{-3/2}(\partial M)\times H^{-1/2}(\partial M), H^{3/2}(\partial M)\times H^{1/2}(\partial M)} = \int_M (\Delta u)(\Delta v) dV + \int_M quv dV, \tag{3.1.4}$$

where $g = (g_0, g_1) \in H^{3/2}(\partial M) \times H^{1/2}(\partial M)$, $v \in H^2(M^{\text{int}})$ is such that $\gamma v = g$, and u is the solution to (3.1.3). The linear map Λ_q is well defined and

$$\Lambda_q : H^{3/2}(\partial M) \times H^{1/2}(\partial M) \to H^{-3/2}(\partial M) \times H^{-1/2}(\partial M)$$

is continuous, see [71, Appendix A]. This corresponds to the fact that in the weak sense we have $\Lambda_q f = (-\partial_\nu(\Delta u)|_{\partial M}, \Delta u|_{\partial M})$.

Note that working with solutions $u \in H^4(M^{\text{int}})$ of the equation $(\Delta^2 + q)u = 0$, the explicit description for the Laplacian in the boundary normal coordinates, see (3.2.2) below, together with boundary elliptic regularity, see [47, Theorem 11.14], shows that the knowledge of the graph of the Dirichlet–to–Neumann map Λ_q, $\{(f, \Lambda_q f) : f \in H^{\frac{7}{2}}(\partial M) \times H^{\frac{5}{2}}(\partial M)\}$ is equivalent to the knowledge of the set of the Cauchy data,

$$\{(u|_{\partial M}, \partial_\nu u|_{\partial M}, \partial_\nu^2 u|_{\partial M}, \partial_\nu^3 u|_{\partial M}) : u \in H^4(M^{\text{int}}), (\Delta^2 + q)u = 0 \text{ in } M^{\text{int}}\}.$$

The areas of physics and geometry where biharmonic operators occur, include the study of the Kirchhoff plate equation in the theory of elasticity, and the study of the Paneitz-Branson operator in conformal geometry, see [44, 33]. In particular, in the elasticity theory, the biharmonic operator is used to model small transversal vibrations of a plate of negligible thickness, according to the Kirchhoff–Love model for elasticity. Furthermore, the biharmonic equation also arises in the theory of steady Stokes flows of viscous fluids, where it is the equation satisfied by the stream function, see [101].

The inverse boundary problem for a potential perturbation of the biharmonic operator is to determine the potential q in M from the knowledge of the Dirichlet–to–Neumann map Λ_q. In the case of domains in the Euclidean space $\mathbb{R}^n$ with $n \geq 3$, this problem was solved in [56], [57] showing that the bounded potential q can indeed be recovered from the knowledge of the Dirichlet–to–Neumann map Λ_q, see [71] for the case of unbounded potentials. We refer to [68], [67] where the inverse boundary problem of determination of a first order perturbation of the biharmonic operator was studied in the Euclidean case, see also [21], [6], [5], [8] for the case of non-smooth perturbations, and [18], [46] for the case of second order perturbations.

Going beyond the Euclidean setting, the global uniqueness in the inverse boundary problem for zero and first order perturbations of the biharmonic operator was only obtained in the case when the manifold (M, g) is admissible in [9], see Definition 3.1.2 below, and in the more general case when (M, g) is CTA (conformally transversally anisotropic, see Definitions 3.1.1) with the injective geodesic X-ray transform on the transversal manifold (M_0, g_0) in [119]. The works [9] and [119] are extensions of the fundamental works [36] and [38] which initiated this study in the case of perturbations of the Laplacian. We refer to the works [80], [43], [42], [74], for inverse boundary problems for nonlinear Schrödinger equations on CTA manifolds, and we remark that that there are no assumptions on the transversal manifold in these works.

Definition 3.1.1. *A compact Riemannian manifold (M, g) of dimension $n \geq 3$ with boundary ∂M is called conformally transversally anisotropic (CTA) if $M \subset\subset \mathbb{R} \times M_0^{int}$ where $g = c(e \oplus g_0)$, $(\mathbb{R}, e)$ is the Euclidean real line, (M_0, g_0) is a smooth compact $(n-1)$–dimensional manifold with smooth boundary, called the transversal manifold, and $c \in C^\infty(M)$ is a positive function.*

Definition 3.1.2. *A compact Riemannian manifold (M, g) of dimension $n \geq 3$ with boundary ∂M is called admissible if it is CTA and the transversal manifold (M_0, g_0) is simple, meaning that for any $p \in M_0$, the exponential map $\exp_p$ with its maximal domain of definition in T_pM_0 is a diffeomorphism onto M_0, and ∂M_0 is strictly convex.*

The proofs of the global uniqueness results in the works [36, 38, 9, 119] rely on construction of complex geometric optics solutions based on the techniques of Carleman estimates with limiting Carleman weights. Thanks to the work [36], we know that the property of being a CTA manifold guarantees the existence of limiting Carleman weights.

Once uniqueness results for inverse boundary problems have been established, one is interested in upgrading them to a reconstruction procedure. The reconstruction of a potential perturbation of the Laplacian from boundary measurements in the Euclidian space was obtained in the pioneering works [95] and [99], see also [100]. We refer to [97] for reconstruction in the case of partial data inverse boundary problems. In the case of admissible manifolds, a reconstruction procedure for a potential perturbation of the Laplacian was given in [62], complementing the uniqueness result of [36], see also [7]. In the case of more general CTA manifolds whose transversal manifolds enjoy the constructive invertibility of the geodesic ray transform, a reconstruction procedure for a potential perturbation of the Laplacian was established in [41], complementing the uniqueness result of [38]. We refer to [14], [15] for the reconstruction of a Riemannian manifold from the dynamical data.

Turning the attention to inverse boundary problems for a potential perturbation of the biharmonic operator, to the best of our knowledge, there is no reconstruction procedure available in the literature and the purpose of this paper is to provide such a reconstruction procedure. Our result will be stated in the most general setting possible, i.e. on a CTA manifold whose transversal manifold enjoys the constructive invertibility of the geodesic ray transform, but it is applicable and new already in the case of smooth bounded domains in the 3–dimensional Euclidean space and in the case of 3–dimensional admissible manifolds. To state our result, we shall need the following definition.

Definition 3.1.3. *We say that the geodesic ray transform on the transversal manifold* (M_0, g_0) *is constructively invertible if any function* $f \in C(M_0)$ *can be reconstructed from the knowledge of its integrals over all non-tangential geodesics in* M_0*. Here a unit speed geodesic* $\gamma : [0, L] \to M_0$ *is called non-tangential if* $\dot\gamma(0), \dot\gamma(L)$ *are non-tangential vectors on* ∂M_0 *and* $\gamma(t) \in M_0^{\text{int}}$ *for all* $0 < t < L$.

Our main result is as follows, and it gives a constructive counterpart of the uniqueness result of [119].

Theorem 3.1.4. *Let* (M, g) *be a given CTA manifold and assume that the geodesic ray transform on the transversal manifold* (M_0, g_0) *is constructively invertible. Let* $q \in C(M)$ *be such that assumption (A) is satisfied. Then the knowledge of* Λ_q *determines* q *in* M *constructively.*

Combining Theorem 3.1.4 with the constructive invertibility of the geodesic ray transform on a simple two-dimensional Riemannian manifold, see [102], [66], [107], see also [90], [91], we obtain the following unconditional result.

Corollary 3.1.5. *Let (M, g) be a given 3–dimensional admissible manifold, and let $q \in C(M)$ be such that assumption (A) is satisfied. Then the knowledge of Λ_q determines q in M constructively.*

Remark 3.1.6. *As explained in [36], bounded smooth domains in the Euclidean space are examples of admissible manifolds, and therefore, Corollary 3.1.5 is applicable and new in this case.*

Remark 3.1.7. *Beyond the case of a simple two-dimensional Riemannian manifold, the constructive invertibility of the geodesic ray transform is also known in particular in the following situations:*

- *(M_0, g_0) is a two-dimensional Riemannian manifold with strictly convex boundary, no conjugate points, and the hyperbolic trapped set (these conditions are satisfied in negative curvature, in particular), see [50].*
- *(M_0, g_0) is of dimension $n \geq 3$, has a strictly convex boundary and is globally foliated by strictly convex hypersurfaces, see [118].*

Remark 3.1.8. *The work [119] establishes that not only a continuous potential but an entire continuous first order perturbation can be determined uniquely from the knowledge of the set of the Cauchy data on the boundary of a CTA manifold provided that the geodesic ray transform on the transversal manifold is injective, and therefore, it would be interesting to propose a reconstruction procedure of the recovery of a full first order perturbation. We shall address this question in a future work. To the best of our knowledge, there are no reconstruction results even in the case of a first order perturbation of the Laplacian on admissible manifolds and the only available result is the work [26] in the case of compact domains contained in cylindrical manifolds of the form $\mathbb{R} \times \mathbb{T}^d$ with $\mathbb{T}^d$ being the d-dimensional torus, $d \geq 2$,* *see also [105] for the Euclidean case. Note that the problem of determining a first order perturbation of the biharmonic operator appears to be more challenging, as here one has to recover a first order perturbation uniquely while in the case of the Laplacian, one only needs to determine it up to a gauge transformation, which is only the first step in the corresponding program for the biharmonic operator, see [119].*

Let us proceed to discuss the main ideas in the proof of Theorem 3.1.4. The first step is the derivation of the integral identity,

$$\int_M q u_1 \overline{u_2} dV = \langle (\Lambda_q - \Lambda_0)\gamma u_1, \gamma \overline{u_2} \rangle_{H^{1/2}(\partial M)\times H^{3/2}(\partial M), H^{-1/2}(\partial M)\times H^{-3/2}(\partial M)}, \tag{3.1.5}$$

where $u_1, u_2 \in L^2(M)$ are solutions to $(\Delta^2 + q)u_1 = 0$ and $\Delta^2 u_2 = 0$ in M^{int}. The next step is to test the integral identity (3.1.5) agains suitable complex geometric optics solutions u_1 and u_2. Working on a general CTA manifold, we shall obtain such solutions based on Gaussian beam quasimodes for the conjugated biharmonic operator, constructed on M and localized to non-tangential geodesics on the transversal manifold M_0 times $\mathbb{R}_{x_1}$. Such solutions were constructed in [119] without any notion of uniqueness involved. In this paper, we propose an alternative construction to produce complex geometric optics solutions enjoying a uniqueness property. The key step in the proof is the constructive determination of the Dirichlet trace γu_1 on ∂M of the unique complex geometric optics solution u_1 from the knowledge of the Dirichlet–to–Neumann map Λ_q. Once this step is carried out, the quantity on the right hand side of (3.1.5) is reconstructed thanks to the knowledge of the manifold M and Λ_q. Another ingredient in the proof is the boundary reconstruction formula for $q|_{\partial M}$ from the knowledge of Λ_q. Using it together with the constructive invertibility of the geodesic ray transform and following the standard argument, see [38], [41], we reconstruct the potential q from the left hand side of (3.1.5), with u_1 and u_2 being the complex geometric optics solutions.

To the best of our knowledge there are two approaches to the reconstruction of the Dirichlet boundary traces of suitable complex geometric optics solutions to the Schrödinger equation in the Euclidean space in the literature. In the first one, suitable complex geometric optics solutions are constructed globally on all of $\mathbb{R}^n$, enjoying uniqueness properties characterized by decay at infinity, see [95], [99], while in the second one, complex geometric optics solutions are constructed by means of Carleman estimates on a bounded domain, and the notion of uniqueness is obtained by restricting the attention to solutions of minimal norm, see [97]. In both approaches, the boundary traces of the complex geometric optics solutions in question are determined as unique solutions of well posed integral equations on the boundary of the domain, involving the Dirichlet–to–Neumann map along with other known quantities. In the proof of Theorem 3.1.4 in order to reconstruct the Dirichlet trace $\gamma u_1 = (u_1|_{\partial M}, \partial_\nu u_1|_{\partial M})$ on ∂M of the unique complex geometric optics solution u_1 from the knowledge of the Dirichlet–to–Neumann map Λ_q, we follow the second approach, adapting the simplified version of it given in [41] to the case of perturbed biharmonic operators. Compared to [41], we not only need to reconstruct the boundary trace $u_1|_{\partial M}$ but also the boundary trace $\partial_\nu u_1|_{\partial M}$ of the normal derivative. In doing so, we introduced the single layer operator associated to the Green operator of the conjuagated semiclassical biharmonic operator.

Finally, let us mention that similarly to the reconstructions results of [62] and [41], we make no claims regarding practicality of the reconstruction procedure developed in this paper. Our purpose merely is to show that all the steps in the proof of the uniqueness result of [119] can be carried out constructively.

This article is organized as follows. In Section 3.2 we collect some essentially well known results related to the maximal domain of the biharmonic operator and boundary traces needed in the proof of Theorem 3.1.4. The derivation of the integral identify (3.1.5) is also given in Section 3.2. In Section 3.3 we present an extension of the Nachman–Street method [97] for the constructive determination of the boundary traces of suitable complex geometric optics solutions, developed for the Schrödinger equation, to the case of the perturbed biharmonic equation. In Section 3.4, we give a construction of complex geometric optics solutions to the perturbed biharmonic equations enjoying uniqueness property and complete the proof of Theorem 3.1.4. Finally, a reconstruction formula for the boundary traces of a continuous potential from the knowledge of Λ_q for the perturbed biharmonic operator is established in Section 3.5.

3.2 The Hilbert space $H_{\Delta^2}(M)$ and boundary traces

The purpose of this section is to collect some essentially well known results needed in the proof of Theorem 3.1.4, see also [47], [88]. Since we are dealing with the biharmonic operator Δ^2 rather than the Laplacian, some of the proofs are provided for the convenience of the reader.

Let (M,g) be a smooth compact oriented Riemannian manifold of dimension $n \geq 3$ with smooth boundary ∂M. We shall need the following Green formula for Δ^2, valid for $u, v \in H^4(M^{\text{int}})$,

$$\begin{aligned}\int_M (\Delta^2 u) v dV - \int_M u(\Delta^2 v) dV = \int_{\partial M} \partial_\nu u(\Delta v) dS - \int_{\partial M} u\partial_\nu(\Delta v) dS \\ + \int_{\partial M} \partial_\nu(\Delta u) v dS - \int_{\partial M} (\Delta u)\partial_\nu v dS,\end{aligned} \tag{3.2.1}$$

where ν is the unit exterior normal vector to ∂M, dV and dS are the Riemannian volume elements on M and ∂M, respectively, see [47].

We shall also need the following expressions for the operators Δ and $\partial_\nu\Delta$ on the boundary of M, valid for $v \in H^4(M^{\text{int}})$,

$$\begin{aligned}&\Delta v = \partial_\nu^2 v + H\partial_\nu v + \Delta_t v \quad \text{on} \quad \partial M, \\ &\partial_\nu \Delta v = \partial_\nu^3 v + \partial_\nu H \partial_\nu v + H\partial_\nu^2 v + \Delta_t \partial_\nu v \quad \text{on} \quad \partial M,\end{aligned} \tag{3.2.2}$$

where $H = \frac{1}{2}\partial_\nu \log|\det g| \in C^\infty(M)$ and $\Delta_t = \Delta_{g|_{\partial M}}$ is the tangential Laplacian on ∂M, see [83].

Consider the Hilbert space

$$H_{\Delta^2}(M) = \{u \in L^2(M) : \Delta^2 u \in L^2(M)\},$$

equipped with the norm

$$\|u\|^2_{H_{\Delta^2}(M)} = \|u\|^2_{L^2(M)} + \|\Delta^2 u\|^2_{L^2(M)}.$$

The space $H_{\Delta^2}(M)$ is the maximal domain of the bi-Laplacian Δ^2, acting on $L^2(M)$.

We shall need the following result concerning the existence of traces of functions in $H_{\Delta^2}(M)$.

Lemma 3.2.1. *(i) The trace map $\gamma_j : C^\infty(M) \to C^\infty(\partial M)$, $u \mapsto \partial_\nu^j u|_{\partial M}$, $j = 0, 1$, extends to a linear continuous map*

$$\gamma_j : H_{\Delta^2}(M) \to H^{-j-1/2}(\partial M). \tag{3.2.3}$$

(ii) The trace map $\widetilde{\gamma}_j : C^\infty(M) \to C^\infty(\partial M)$, $u \mapsto \partial_\nu^j(\Delta u)|_{\partial M}$, $j = 0, 1$, extends to a linear continuous map

$$\widetilde{\gamma}_j : H_{\Delta^2}(M) \to H^{-j-5/2}(\partial M).$$

Proof. We follow the arguments of [24, Section 1], carried out in the case of Δ.

(i). Let $j = 0$, $u \in C^\infty(M)$, and $w \in H^{1/2}(\partial M)$. By the Sobolev extension theorem, see [47, Theorem 9.5], there exists $v \in H^4(M^{\text{int}})$ such that

$$v|_{\partial M} = 0, \quad \partial_\nu v|_{\partial M} = 0, \quad \partial_\nu^2 v|_{\partial M} = 0, \quad \partial_\nu^3 v|_{\partial M} = w, \tag{3.2.4}$$

and

$$\|v\|_{H^4(M^{\text{int}})} \leq C\|w\|_{H^{1/2}(\partial M)}. \tag{3.2.5}$$

It follows from (3.2.1), (3.2.2), (3.2.4) that

$$-\int_{\partial M} uwdS = \int_M (\Delta^2 u)vdV - \int_M u(\Delta^2 v)dV,$$

and therefore, using (3.2.5), we get

$$\left|\int_{\partial M} uwdS\right| \leq C\|u\|_{H_{\Delta^2}(M)}\|v\|_{H^4(M^{\text{int}})} \leq C\|u\|_{H_{\Delta^2}(M)}\|w\|_{H^{1/2}(\partial M)}.$$

Hence,

$$\|\gamma_0 u\|_{H^{-1/2}(\partial M)} \leq C\|u\|_{H_{\Delta^2}(M)}. \tag{3.2.6}$$

By the density of the space $C^\infty(M)$ in $H_{\Delta^2}(M)$, see [88, Chapter 2, Section 8.1, page 192], and also [47, Theorem 9.8, and page 233], we conclude that the map γ_0 extends to a continuous linear map: $H_{\Delta^2}(M) \to H^{-1/2}(\partial M)$ and (3.2.6) holds for all $u \in H_{\Delta^2}(M)$. This shows (i) with $j = 0$.

Let next $j = 1$ in (i) and let us now prove that γ_1 extends to a continuous linear map: $H_{\Delta^2}(M) \to H^{-3/2}(\partial M)$. To that end, let $u \in C^\infty(M)$ and let $w \in H^{3/2}(\partial M)$. By the Sobolev extension theorem, there is $v \in H^4(M^{\text{int}})$ such that

$$v|_{\partial M} = 0, \quad \partial_\nu v|_{\partial M} = 0, \quad \partial_\nu^2 v|_{\partial M} = w, \quad \partial_\nu^3 v|_{\partial M} = -Hw, \tag{3.2.7}$$

where H is defined in (3.2.2), and

$$\|v\|_{H^4(M^{\text{int}})} \leq C\|w\|_{H^{3/2}(\partial M)}. \tag{3.2.8}$$

It follows from (3.2.2) and (3.2.7) that

$$\Delta v|_{\partial M} = w, \quad \partial_\nu(\Delta v)|_{\partial M} = 0. \tag{3.2.9}$$

Using (3.2.1), (3.2.7), (3.2.9), we get

$$\int_{\partial M} (\partial_\nu u) w dS = \int_M (\Delta^2 u) v dV - \int_M u(\Delta^2 v) dV,$$

and therefore, using (3.2.8), we see that

$$\left| \int_{\partial M} (\partial_\nu u) w dS \right| \leq C \|u\|_{H_{\Delta^2}(M)} \|w\|_{H^{3/2}(\partial M)}.$$

Thus,

$$\|\gamma_1 u\|_{H^{-3/2}(\partial M)} \leq C \|u\|_{H_{\Delta^2}(M)}. \tag{3.2.10}$$

By the density of the space $C^\infty(M)$ in $H_{\Delta^2}(M)$, we obtain that the map γ_1 extends to a continuous linear map: $H_{\Delta^2}(M) \to H^{-3/2}(\partial M)$ and (3.2.10) holds for all $u \in H_{\Delta^2}(M)$. This shows (i) with $j = 1$.

(ii). The proof here follows along the same lines as in the case (i). Let us only mention that when $j = 0$, we shall work with $w \in H^{5/2}(\partial M)$ and $v \in H^4(M^{\text{int}})$ such that

$$v|_{\partial M} = 0, \quad \partial_\nu v|_{\partial M} = w, \quad \partial_\nu^2 v = -Hw, \quad \partial_\nu^3 v = -(\partial_\nu H) w + H^2 w - \Delta_t w.$$

Therefore, this together with (3.2.2) implies that

$$\Delta v|_{\partial M} = 0, \quad \partial_\nu \Delta v|_{\partial M} = 0.$$

We also have $\|v\|_{H^4(M^{\text{int}})} \leq C \|w\|_{H^{5/2}(\partial M)}$.

When $j = 1$, we shall work with $w \in H^{7/2}(\partial M)$ and $v \in H^4(M^{\text{int}})$ such that

$$v|_{\partial M} = w, \quad \partial_\nu v|_{\partial M} = 0, \quad \partial_\nu^2 v = -\Delta_t w, \quad \partial_\nu^3 v = H\Delta_t w.$$

Therefore, by (3.2.2), we get

$$\Delta v|_{\partial M} = 0, \quad \partial_\nu \Delta v|_{\partial M} = 0.$$

We also have $\|v\|_{H^4(M^{\text{int}})} \leq C\|w\|_{H^{7/2}(\partial M)}$. This completes the proof of Lemma 3.2.1. $\square$

By Lemma 3.2.1, we have the following consequence of (3.2.1).

Corollary 3.2.2. *For any $u \in H_{\Delta^2}(M)$ and $v \in H^4(M^{int})$, we have the following generalized Green formula,*

$$\begin{aligned} \int_M (\Delta^2 u) v dV - \int_M u \Delta^2 v dV = &\int_{\partial M} \partial_\nu u (\Delta v) dS - \int_{\partial M} u \partial_\nu (\Delta v) dS \\ &+ \int_{\partial M} \partial_\nu (\Delta u) v dS - \int_{\partial \Omega} (\Delta u) \partial_\nu v dS, \end{aligned} \tag{3.2.11}$$

where

$$\begin{aligned} \int_{\partial M} \partial_\nu u (\Delta v) dS &:= \langle \gamma_1 u, \Delta v \rangle_{H^{-3/2}(\partial M), H^{3/2}(\partial M)}, \\ \int_{\partial M} u \partial_\nu (\Delta v) dS &:= \langle \gamma_0 u, \partial_\nu (\Delta v) \rangle_{H^{-1/2}(\partial M), H^{1/2}(\partial M)}, \\ \int_{\partial M} \partial_\nu (\Delta u) v dS &:= \langle \widetilde{\gamma}_1 u, v \rangle_{H^{-7/2}(\partial M), H^{7/2}(\partial M)}, \\ \int_{\partial \Omega} (\Delta u) \partial_\nu v dS &:= \langle \widetilde{\gamma}_0 u, \partial_\nu v \rangle_{H^{-5/2}(\partial M), H^{5/2}(\partial M)}. \end{aligned}$$

We shall need the following extension of [39, Theorem 26.3] to the case of the biharmonic operator Δ^2. Here for $u \in H_{\Delta^2}(M)$, we set

$$\gamma u = (\gamma_0 u, \gamma_1 u), \tag{3.2.12}$$

where γ_j, $j = 0, 1$, are given by (3.2.3). Note γ in (3.2.12) is an extension of the trace map in (3.1.1).

Theorem 3.2.3. *For each $g = (g_0, g_1) \in H^{-1/2}(\partial M) \times H^{-3/2}(\partial M)$, there exists a unique $u \in L^2(M)$ such that*

$$\begin{cases} \Delta^2 u = 0 & in \quad M^{int}, \\ \gamma u = g & on \quad \partial M, \end{cases} \tag{3.2.13}$$

and

$$\|u\|_{L^2(M)} \leq C \|g\|_{H^{-1/2}(\partial M) \times H^{-3/2}(\partial M)}. \tag{3.2.14}$$

Here $\|g\|^2_{H^{-1/2}(\partial M) \times H^{-3/2}(\partial M)} = \|g_0\|^2_{H^{-1/2}(\partial M)} + \|g_1\|^2_{H^{-3/2}(\partial M)}$.

Proof. We shall follow the proof of [39, Theorem 26.3]. Let $v \in H^4(M^{\text{int}})$ be such that $v|_{\partial M} = 0$, $\partial_\nu v|_{\partial M} = 0$. If there is $u \in L^2(M)$ satisfying (3.2.13) then by the generalized Green formula (3.2.11), we obtain

$$\int_M u \Delta^2 v dV = \langle g_0, \partial_\nu (\Delta v) \rangle_{H^{-1/2}(\partial M), H^{1/2}(\partial M)} - \langle g_1, \Delta v \rangle_{H^{-3/2}(\partial M), H^{3/2}(\partial M)}. \tag{3.2.15}$$

Consider the subspace

$$L := \{\Delta^2 v : v \in H^4(M^{\text{int}}), v|_{\partial M} = 0, \ \partial_\nu v|_{\partial M} = 0\} \subset L^2(M).$$

In view of (3.2.15), we define the linear functional F on L by

$$F(\Delta^2 v) := \langle g_0, \partial_\nu(\Delta v)\rangle_{H^{-1/2}(\partial M), H^{1/2}(\partial M)} - \langle g_1, \Delta v\rangle_{H^{-3/2}(\partial M), H^{3/2}(\partial M)}. \tag{3.2.16}$$

Using the Cauchy–Schwarz inequality, the following Sobolev trace theorem

$$\|(v, \partial_\nu v, \partial_\nu^2 v, \partial_\nu^3 v)\|_{(H^{7/2}\times H^{5/2}\times H^{3/2}\times H^{1/2})(\partial M)} \le C\|v\|_{H^4(M^{\text{int}})},$$

and (3.2.2), we obtain from (3.2.16) that

$$\begin{aligned} |F(\Delta^2 v)| &\le \|g_0\|_{H^{-1/2}(\partial M)}\|\partial_\nu(\Delta v)\|_{H^{1/2}(\partial M)} + \|g_1\|_{H^{-3/2}(\partial M)}\|\Delta v\|_{H^{3/2}(\partial M)} \\ &\le C\|g\|_{H^{-1/2}(\partial M)\times H^{-3/2}(\partial M)}\|v\|_{H^4(M^{\text{int}})}. \end{aligned} \tag{3.2.17}$$

Using the fact that $v|_{\partial M} = 0$, $\partial_\nu v|_{\partial M} = 0$, and boundary elliptic regularity, see [47, Theorem 11.14], we get

$$\|v\|_{H^4(M^{\text{int}})} \le C\|\Delta^2 v\|_{L^2(M)}. \tag{3.2.18}$$

Combining (3.2.17) and (3.2.18), we obtain that

$$|F(\Delta^2 v)| \le C\|g\|_{H^{-1/2}(\partial M)\times H^{-3/2}(\partial M)}\|\Delta^2 v\|_{L^2(M)},$$

which shows that F is bounded on L. Thus, by the Hahn-Banach theorem, F can be extended to a bounded linear functional on $L^2(M)$, and by Riesz representation theorem, there exists $u \in L^2(M)$ such that

$$F(\Delta^2 v) = \int_M (\Delta^2 v) u dV, \tag{3.2.19}$$

and (3.2.14) holds. Letting $v \in C_0^\infty(M^{\text{int}})$, we conclude from (3.2.19) and (3.2.16) that

$\Delta^2 u = 0$ in M^{int}.

Using (3.2.19), (3.2.16), and the generalized Green formula (3.2.11), we get

$$\begin{aligned} &\langle \gamma_0 u, \partial_\nu(\Delta v)\rangle_{H^{-1/2}(\partial M), H^{1/2}(\partial M)} - \langle \gamma_1 u, \Delta v\rangle_{H^{-3/2}(\partial M), H^{3/2}(\partial M)} \\ &= \langle g_0, \partial_\nu(\Delta v)\rangle_{H^{-1/2}(\partial M), H^{1/2}(\partial M)} - \langle g_1, \Delta v\rangle_{H^{-3/2}(\partial M), H^{3/2}(\partial M)}, \end{aligned} \tag{3.2.20}$$

for all $v \in H^4(M^{\text{int}})$ such that $v|_{\partial M} = 0$, $\partial_\nu v|_{\partial M} = 0$.

Letting $w \in H^{1/2}(\partial M)$, and taking $v \in H^4(M^{\text{int}})$ such that (3.2.4) holds, we see from (3.2.20) that $\gamma_0 u = g_0$. Furthermore, letting $w \in H^{3/2}(\partial M)$ and taking $v \in H^4(M^{\text{int}})$ such that (3.2.7) holds, in view of (3.2.9), we conclude from (3.2.20) that $\gamma_1 u = g_1$.

The uniqueness follows from the fact that if $u \in L^2(M)$ solves the Dirichlet problem (3.2.13) with $g = 0$ then by the boundary elliptic regularity, see [47, Theorem 11.14], $u \in H^4(M^{\text{int}})$, and therefore, $u = 0$. □

Corollary 3.2.4. *Let $q \in C(M)$ be such that assumption (A) is satisfied, and let*

$$H_q := \{u \in L^2(M) : (\Delta^2 + q)u = 0\} \subset H_{\Delta^2}(M).$$

Then the trace map

$$\gamma : H_q \to H^{-1/2}(\partial M) \times H^{-3/2}(\partial M) \tag{3.2.21}$$

is bijective.

Proof. We begin by showing that the map γ in (3.2.21) is surjective. To that end, letting $g \in H^{-1/2}(\partial M) \times H^{-3/2}(\partial M)$, by Theorem 3.2.3, we get a unique $u \in L^2(M)$ satisfying

(3.2.13). Assumption (A) implies that there is a unique $v \in H_0^2(M^{\text{int}})$ such that

$$\begin{cases} (\Delta^2 + q)v = qu & \text{in} \quad M^{\text{int}}, \\ \gamma v = 0 & \text{on} \quad \partial M. \end{cases} \tag{3.2.22}$$

Now letting $w = u - v \in L^2(M)$, in view of (3.2.13) and (3.2.22), we see that $w \in H_q$ and $\gamma w = g$. This shows the surjectivity of γ in (3.2.21).

The injectivity of γ in (3.2.21) follows from the fact that if $u \in H_q$ is such that $\gamma u = 0$ then the boundary elliptic regularity, see [47, Theorem 11.14], shows that $u \in (H^4 \cap H_0^2)(M^{\text{int}})$, and by assumption (A), $u = 0$. □

In view of Corollary 3.2.4, we can define the Poisson operator as follows,

$$\mathcal{P}_q = \gamma^{-1} : H^{-1/2}(\partial M) \times H^{-3/2}(\partial M) \to H_q. \tag{3.2.23}$$

We have

$$\|\mathcal{P}_q f\|_{L^2(M)} \le C \|f\|_{H^{-1/2}(\partial M) \times H^{-3/2}(\partial M)}, \tag{3.2.24}$$

for all $f \in H^{-1/2}(\partial M) \times H^{-3/2}(\partial M)$.

Finally, let us derive the integral identity which will be used to reconstruct the potential. To that end, let $f, g \in H^{3/2}(\partial M) \times H^{1/2}(\partial M)$, let $u = u^f \in H^2(M^{\text{int}})$ be the unique solution to the Dirichlet problem

$$\begin{cases} (\Delta^2 + q)u = 0 & \text{in} \quad M^{\text{int}}, \\ \gamma u = f & \text{on} \quad \partial M, \end{cases} \tag{3.2.25}$$

and let $v = v^g \in H^2(M^{\text{int}})$ be the unique solution to the Dirichlet problem

$$\begin{cases} \Delta^2 v = 0 & \text{in} \quad M^{\text{int}}, \\ \gamma v = g & \text{on} \quad \partial M. \end{cases} \tag{3.2.26}$$

By the definition of the Dirichlet–to–Neumann map (3.1.4), we get

$$\langle \Lambda_q f, g \rangle_{H^{-3/2}(\partial M)\times H^{-1/2}(\partial M), H^{3/2}(\partial M)\times H^{1/2}(\partial M)} = \int_M (\Delta u^f)(\Delta v^g) dV + \int_M q u^f v^g dV, \tag{3.2.27}$$

and

$$\begin{aligned} \langle \Lambda_0 g, f \rangle_{H^{-3/2}(\partial M)\times H^{-1/2}(\partial M), H^{3/2}(\partial M)\times H^{1/2}(\partial M)} &= \int_M (\Delta v^g)(\Delta u^f) dV \\ = \int_M (\Delta v^g)(\Delta v^f) dV &= \langle \Lambda_0 f, g \rangle_{H^{-3/2}(\partial M)\times H^{-1/2}(\partial M), H^{3/2}(\partial M)\times H^{1/2}(\partial M)}. \end{aligned} \tag{3.2.28}$$

In the penultimate equality of (3.2.28) we used the fact that the definition of the Dirichlet–to–Neumann map Λ_0 is independent of the choice of extension of $f \in H^{3/2}(\partial M) \times H^{1/2}(\partial M)$ to an $H^2(M^{\text{int}})$ element whose trace is equal to f. Considering the difference of (3.2.27) and (3.2.28), we obtain the following integral identity,

$$\langle (\Lambda_q - \Lambda_0) f, g \rangle_{H^{-3/2}(\partial M)\times H^{-1/2}(\partial M), H^{3/2}(\partial M)\times H^{1/2}(\partial M)} = \int_M q u v dV, \tag{3.2.29}$$

where $u = u^f, v = v^g \in H^2(M^{\text{int}})$ are solutions to (3.2.25) and (3.2.26), respectively.

We would like to extend the Nachman–Street argument [97] to reconstruct the potential q from the knowledge of the Dirichlet–to–Neumann map for the biharmonic operator and therefore, as in [97], we shall work with $L^2(M)$ solutions rather than $H^2(M^{\text{int}})$ solutions to the Dirichlet problems (3.2.25), (3.2.26). Thus, we shall need to extend the integral identity (3.2.29) to such solutions. In doing so, we first claim that $\Lambda_q - \Lambda_0$ extends to a linear continuous map

$$\Lambda_q - \Lambda_0 : H^{-1/2}(\partial M) \times H^{-3/2}(\partial M) \to H^{1/2}(\partial M) \times H^{3/2}(\partial M). \tag{3.2.30}$$

To that end, letting $f, g \in C^\infty(\partial M) \times C^\infty(\partial M)$, we conclude from (3.2.29), (3.2.14), and (3.2.24) that

$$\begin{aligned} |\langle (\Lambda_q - \Lambda_0) f, g \rangle_{L^2(\partial M)\times L^2(\partial M), L^2(\partial M)\times L^2(\partial M)}| &\le C \|u\|_{L^2(M)} \|v\|_{L^2(M)} \\ &\le C \|f\|_{H^{-1/2}(\partial M)\times H^{-3/2}(\partial M)} \|g\|_{H^{-1/2}(\partial M)\times H^{-3/2}(\partial M)}. \end{aligned}$$

Hence,

$$\|(\Lambda_q - \Lambda_0) f\|_{H^{1/2}(\partial M)\times H^{3/2}(\partial M)} \le C \|f\|_{H^{-1/2}(\partial M)\times H^{-3/2}(\partial M)},$$

which together with the density of $C^\infty(\partial M) \times C^\infty(\partial M)$ in the space $H^{-1/2}(\partial M) \times H^{-3/2}(\partial M)$ gives the claim (3.2.30).

Now letting $f, g \in H^{-1/2}(\partial M) \times H^{-3/2}(\partial M)$, approximating them by $C^\infty(\partial M) \times C^\infty(\partial M)$–functions, using (3.2.30), (3.2.14), and (3.2.24), we obtain from (3.2.29) that

$$\langle (\Lambda_q - \Lambda_0) f, g \rangle_{H^{1/2}(\partial M)\times H^{3/2}(\partial M), H^{-1/2}(\partial M)\times H^{-3/2}(\partial M)} = \int_M quv dV, \tag{3.2.31}$$

where $u = u^f, v = v^g \in L^2(M)$ are solutions to (3.2.25) and (3.2.26), respectively.

3.3 The Nachman–Street argument for biharmonic operators

The goal of this section is to extend the Nachman–Street argument [97] for constructive determination of the boundary traces of suitable complex geometric optics solutions, developed for the Schrödinger equation, to the case of the perturbed biharmonic equation. Specifically, we shall extend to the case of the perturbed biharmonic equation the simplified version of the Nachman–Street argument, presented in [41] in the full data case in the setting of compact Riemannian manifolds with boundary admitting a limiting Carleman weight.

Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 3$ with smooth boundary ∂M, and let $-h^2\Delta_g = -h^2\Delta$ be the semiclassical Laplace–Beltrami operator on M, where $h > 0$ is a small semiclassical parameter. Assume, as we may, that (M, g) is embedded in a compact smooth Riemannian manifold (N, g) without boundary of the same dimension, and let U be open in N such that $M \subset U$. When $\varphi \in C^\infty(U; \mathbb{R})$, we let

$$P_\varphi = e^{\frac{\varphi}{h}}(-h^2\Delta)e^{-\frac{\varphi}{h}}$$

be the conjugated operator, and let p_φ be its semiclassical principal symbol. Following [63], [36], we say that $\varphi \in C^\infty(U; \mathbb{R})$ is a limiting Carleman weight for $-h^2\Delta$ on (U, g) if $d\varphi \neq 0$ on U, and the Poisson bracket of $\operatorname{Re} p_\varphi$ and $\operatorname{Im} p_\varphi$ satisfies,

$$\{\operatorname{Re} p_\varphi, \operatorname{Im} p_\varphi\} = 0 \quad \text{when} \quad p_\varphi = 0.$$

Using Carleman estimates for $-h^2\Delta$, established in [36], it was shown in [97], see also [41, Proposition 2.2], that for all $0 < h \ll 1$ and any $v \in L^2(M)$, there exists a unique solution

$u \in (\mathrm{Ker}(P_\varphi))^\perp$ of the equation

$$P_\varphi u = v \quad \text{in} \quad M^{\mathrm{int}}.$$

Here

$$\mathrm{Ker}(P_\varphi) = \{u \in L^2(M) : P_\varphi u = 0\}.$$

Based on this unique solution, the Green operator G_φ for P_φ was constructed in [97], see also [41, Theorem 2.3], enjoying the following properties: for all $0 < h \ll 1$, there exists a linear continuous operator $G_\varphi : L^2(M) \to L^2(M)$ such that

$$\begin{aligned} &P_\varphi G_\varphi = I \text{ on } L^2(M), \quad \|G_\varphi\|_{\mathcal{L}(L^2(M),L^2(M))} = \mathcal{O}(h^{-1}), \\ &G_\varphi^* = G_{-\varphi}, \quad G_\varphi P_\varphi = I \text{ on } C_0^\infty(M^{\mathrm{int}}). \end{aligned} \tag{3.3.1}$$

Here G_φ^* denotes the $L^2(M)$–adjoint of G_φ. Letting P_φ^* be the formal $L^2(M)$–adjoint of P_φ, we see that $P_\varphi^* = P_{-\varphi}$. Note also that if φ is a limiting Carleman weight for $-h^2\Delta$ then so is $-\varphi$.

In this paper we shall work with the semiclassical biharmonic operator $(-h^2\Delta)^2$. We have

$$P_\varphi^2 = e^{\frac{\varphi}{h}}(-h^2\Delta)^2 e^{-\frac{\varphi}{h}}.$$

We shall use $G_\varphi^2 : L^2(M) \to L^2(M)$ as Green's operator for P_φ^2. It follows from (3.3.1) that G_φ^2 enjoys the following properties,

$$\begin{aligned} &P_\varphi^2 G_\varphi^2 = I \text{ on } L^2(M), \quad \|G_\varphi^2\|_{\mathcal{L}(L^2(M),L^2(M))} = \mathcal{O}(h^{-2}), \\ &(G_\varphi^2)^* = G_{-\varphi}^2, \quad G_\varphi^2 P_\varphi^2 = I \text{ on } C_0^\infty(M^{\mathrm{int}}). \end{aligned} \tag{3.3.2}$$

Furthermore, the first identity in (3.3.2) implies that

$$G_\varphi^2 : L^2(M) \to e^{\varphi/h} H_{\Delta^2}(M). \tag{3.3.3}$$

Next we shall proceed to introduce single layer operators associated to the Green operator G_φ^2. First note that the trace map γ given by (3.2.12) has the following mapping properties,

$$\gamma : e^{\pm\varphi/h} H_{\Delta^2(M)} \to e^{\pm\varphi/h}(H^{-1/2}(\partial M) \times H^{-3/2}(\partial M)) = H^{-1/2}(\partial M) \times H^{-3/2}(\partial M), \tag{3.3.4}$$

and therefore, using (3.3.3), we get

$$\gamma \circ G_\varphi^2 : L^2(M) \to H^{-1/2}(\partial M) \times H^{-3/2}(\partial M)$$

is continuous. Here and below the operator norms for the various continuous maps depend on the semiclassical parameter h, and we only indicate explicitly this dependence when needed. This implies that the L^2–adjoint

$$(\gamma \circ G_\varphi^2)^* : H^{1/2}(\partial M) \times H^{3/2}(\partial M) \to L^2(M) \tag{3.3.5}$$

is also continuous. For any $g \in H^{1/2}(\partial M) \times H^{3/2}(\partial M)$, we have

$$P_{-\varphi}^2((\gamma \circ G_\varphi^2)^* g) = 0 \quad \text{in} \quad \mathcal{D}'(M^{\text{int}}). \tag{3.3.6}$$

The proof is based on the following observation. Letting $f \in C_0^\infty(M^{\text{int}})$, using the fourth property in (3.3.2), we get

$$
\begin{aligned}
(P^2_{-\varphi}((\gamma\circ G^2_\varphi)^*g), f)_{L^2(M)} &= ((\gamma\circ G^2_\varphi)^*g, P^2_\varphi f)_{L^2(M)}\\
&= (g, (\gamma\circ G^2_\varphi)P^2_\varphi f)_{H^{1/2}(\partial M)\times H^{3/2}(\partial M), H^{-1/2}(\partial M)\times H^{-3/2}(\partial M)} = 0.
\end{aligned}
$$

Now (3.3.5) and (3.3.6) imply that $e^{\varphi/h}(\gamma\circ G^2_\varphi)^*g \in H_{\Delta^2}(M)$, and therefore, we have the following mapping properties for the operator $(\gamma\circ G^2_\varphi)^*$,

$$
(\gamma\circ G^2_\varphi)^* : H^{1/2}(\partial M)\times H^{3/2}(\partial M) \to e^{-\varphi/h}H_{\Delta^2}(M),
$$

which improves (3.3.5). Thus, in view of (3.3.4), we have that the map

$$
\gamma\circ(\gamma\circ G^2_\varphi)^* : H^{1/2}(\partial M)\times H^{3/2}(\partial M) \to H^{-1/2}(\partial M)\times H^{-3/2}(\partial M)
$$

is well defined and continuous, and therefore, its L^2–adjoint

$$
(\gamma\circ(\gamma\circ G^2_\varphi)^*)^* : H^{1/2}(\partial M)\times H^{3/2}(\partial M) \to H^{-1/2}(\partial M)\times H^{-3/2}(\partial M)
$$

is also continuous. We introduce the single layer operator associated to the Green operator G^2_φ as follows:

$$
\begin{aligned}
S_\varphi =& e^{-\varphi/h}(\gamma\circ(\gamma\circ G^2_\varphi)^*)^* e^{\varphi/h}\\
&\in \mathcal{L}(H^{1/2}(\partial M)\times H^{3/2}(\partial M), H^{-1/2}(\partial M)\times H^{-3/2}(\partial M)).
\end{aligned}
\tag{3.3.7}
$$

Note that definition (3.3.7) looks similar to the corresponding single layer operator in the case of the Laplacian in [97], see also [41], with the only difference that here the Green operator is G^2_φ instead of G_φ and the trace γ has two components.

Now in view of (3.2.30) and (3.3.7), we have

$$S_\varphi(\Lambda_q - \Lambda_0) : H^{-1/2}(\partial M) \times H^{-3/2}(\partial M) \to H^{-1/2}(\partial M) \times H^{-3/2}(\partial M).$$

is continuous. We claim that

$$S_\varphi(\Lambda_q - \Lambda_0) = \gamma \circ e^{-\varphi/h} \circ G_\varphi^2 \circ e^{\varphi/h} \circ q \circ \mathcal{P}_q \tag{3.3.8}$$

in the sense of linear continuous operators on the space $H^{-1/2}(\partial M) \times H^{-3/2}(\partial M)$. Here $\mathcal{P}_q$ is the Poisson operator given by (3.2.23). To see (3.3.8), letting $f, g \in C^\infty(\partial M) \times C^\infty(\partial M)$, we get

$$\begin{aligned}
&\langle \gamma \circ e^{-\varphi/h} \circ G_\varphi^2 \circ e^{\varphi/h} \circ q \circ \mathcal{P}_q f, g \rangle_{H^{-1/2}(\partial M)\times H^{-3/2}(\partial M), H^{1/2}(\partial M)\times H^{3/2}(\partial M)} \\
&\quad = \langle q \circ \mathcal{P}_q f, e^{\varphi/h}(\gamma \circ G_\varphi^2)^* e^{-\varphi/h} g \rangle_{L^2(M), L^2(M)} \\
&\quad = \langle (\Lambda_q - \Lambda_0) f, \gamma \circ e^{\varphi/h}(\gamma \circ G_\varphi^2)^* e^{-\varphi/h} g \rangle_{H^{-1/2}(\partial M)\times H^{-3/2}(\partial M), H^{1/2}(\partial M)\times H^{3/2}(\partial M)} \\
&\quad = \langle S_\varphi(\Lambda_q - \Lambda_0) f, g \rangle_{H^{-1/2}(\partial M)\times H^{-3/2}(\partial M), H^{1/2}(\partial M)\times H^{3/2}(\partial M)},
\end{aligned}$$

showing (3.3.8). Here in the penultimate equality, we used the fact that $\Delta^2(e^{\varphi/h}(\gamma \circ G_\varphi^2)^* e^{-\varphi/h} g) = 0$ in M^{int} in view of (3.3.6) and the integral identity (3.2.31), and in the last equality we used (3.3.7).

Similar to [41, Proposition 2.4], we have the following result.

Proposition 3.3.1. *Let $f, g \in H^{-1/2}(\partial M) \times H^{-3/2}(\partial M)$. Then*

$$(1 + h^4 S_\varphi(\Lambda_q - \Lambda_0)) f = g \tag{3.3.9}$$

if and only if

$$(1 + e^{-\varphi/h} \circ G_\varphi^2 \circ e^{\varphi/h} h^4 q)\mathcal{P}_q f = \mathcal{P}_0 g. \tag{3.3.10}$$

Proof. Assume first that (3.3.9) holds. To show that (3.3.10) holds, we first observe that $(h^2\Delta)^2\mathcal{P}_q f = -h^4 q\mathcal{P}_q f$. Using the first property in (3.3.2), we also obtain that

$$(h^2\Delta)^2(1 + e^{-\varphi/h} \circ G_\varphi^2 \circ e^{\varphi/h}h^4 q)\mathcal{P}_q f = 0 \quad \text{in} \quad M^{\text{int}}. \tag{3.3.11}$$

Furthermore, (3.3.8) and (3.3.9) imply that

$$\gamma(1 + e^{-\varphi/h} \circ G_\varphi^2 \circ e^{\varphi/h}h^4 q)\mathcal{P}_q f = f + h^4 S_\varphi(\Lambda_q - \Lambda_0))f = g. \tag{3.3.12}$$

By the uniqueness result of Theorem 3.2.3 applied to (3.3.11) and (3.3.12), we obtain (3.3.10).

Now if (3.3.10) holds then (3.3.9) can be obtained by taking the trace γ on both sides of (3.3.10). □

The recovery of the boundary traces of suitable complex geometric optics solutions to the equation $(\Delta^2 + q)u = 0$ will be based on the following result, which is similar to [41, Proposition 2.5].

Proposition 3.3.2. *The operator* $1+h^4 S_\varphi(\Lambda_q-\Lambda_0) : H^{-1/2}(\partial M)\times H^{-3/2}(\partial M) \to H^{-1/2}(\partial M)\times H^{-3/2}(\partial M)$ *is a linear homemorphism for all* $0 < h \ll 1$.

Proof. First using that $\|G_\varphi^2\|_{L^2(M)\to L^2(M)} = \mathcal{O}(h^{-2})$, see (3.3.2), we observe that the operator $1 + e^{-\varphi/h} \circ G_\varphi^2 \circ e^{\varphi/h}h^4 q$ in (3.3.10) is a linear homemorphism on $L^2(M)$ for all $0 < h \ll 1$. Thus, for all $0 < h \ll 1$ and for all $v \in L^2(M)$, the equation

$$(1 + e^{-\varphi/h} \circ G_\varphi^2 \circ e^{\varphi/h}h^4 q)u = v \quad \text{in} \quad M^{\text{int}}$$

has a unique solution $u \in L^2(M)$. Furthermore, if $v \in H_0$ then $u \in H_q$ by the first property of (3.3.2). Hence, for all $0 < h \ll 1$, the operator $1 + e^{-\varphi/h} \circ G_\varphi^2 \circ e^{\varphi/h}h^4 q : H_q \to H_0$ is an isomorphism. It follows from (3.2.23) that the operator $(1 + e^{-\varphi/h} \circ G_\varphi^2 \circ e^{\varphi/h}h^4 q) \circ \mathcal{P}_q :$

$H^{-1/2}(\partial M) \times H^{-3/2}(\partial M) \to H_0$ is an isomorphism for all $0 < h \ll 1$. This together with Proposition 3.3.1 implies the claim. □

3.4 Proof of Theorem 3.1.4

Let (M, g) be a CTA manifold so that $(M, g) \subset\subset (\mathbb{R} \times M_0^{\text{int}}, c(e \oplus g_0))$. Since (M, g) is known, the transversal manifold (M_0, g_0) as well as the conformal factor c are also known. Therefore, the Dirichlet–to–Neumann map Λ_0 is also known. Furthermore, we assume the knowledge of the Dirichlet–to–Neumann map Λ_q. Using the integral identity (3.2.31), we would like to reconstruct the potential q from this data.

Let $x = (x_1, x')$ be the local coordinates in $\mathbb{R} \times M_0$. We know from [36] that the function $\varphi(x) = x_1$ is a limiting Carleman weight for the semiclassical Laplacian $-h^2\Delta$. Our starting point is the following result about the existence of Gaussian beam quasimodes for the biharmonic operator, constructed on M and localized to non-tangential geodesics on the transversal manifold M_0 times $\mathbb{R}_{x_1}$, established in [119, Propositions 2.1, 2.2]. See also [12], [103], [104], [38], [69] for related constructions of Gaussian beam quasimodes for second order operators and applications to inverse boundary problems.

Theorem 3.4.1. *[119, Propositions 2.1, 2.2] Let $s = \frac{1}{h} + i\lambda$, $0 < h < 1$, $\lambda \in \mathbb{R}$ and let $\gamma : [0, L] \to M_0$ be a unit speed non-tangential geodesic on M_0. Then there are families of Gaussian beam quasimodes $v_s, w_s \in C^\infty(M)$ such that*

$$\|v_s\|_{H^1_{\text{scl}}(M^{\text{int}})} = \mathcal{O}(1), \quad \|e^{sx_1}(h^2\Delta)^2 e^{-sx_1} v_s\|_{L^2(M)} = \mathcal{O}(h^{5/2}), \tag{3.4.1}$$

$$\|w_s\|_{H^1_{\text{scl}}(M^{\text{int}})} = \mathcal{O}(1), \quad \|e^{-sx_1}(h^2\Delta)^2 e^{sx_1} w_s\|_{L^2(M)} = \mathcal{O}(h^{5/2}), \tag{3.4.2}$$

as $h \to 0$. Furthermore, letting $\psi \in C(M_0)$, and letting $x_1 \in \mathbb{R}$, we have

$$\lim_{h\to 0}\int_{\{x_1\}\times M_0} v_s\overline{w_s}\psi dV_{g_0} = \int_0^L e^{-2\lambda t}c(x_1,\gamma(t))^{1-\frac{n}{2}}\psi(\gamma(t))dt. \tag{3.4.3}$$

We shall use the Gaussian beam quasimodes of Theorem 3.4.1 to construct solutions $u_2, u_1 \in L^2(M)$ to the biharmonic equation $\Delta^2 u_2 = 0$ and the perturbed biharmonic equation $(\Delta^2 + q)u_1 = 0$ in M, which will be used to test the integral identity (3.2.31). Note that some solutions of the perturbed biharmonic equations based on the Gaussian beam quasimodes of Theorem 3.4.1 were constructed in [119] with the help of Carleman estimates. Here our construction will be different as we need to be able to reconstruct their traces $\gamma u_1 = (u_1|_{\partial M}, \partial_\nu u_1|_{\partial M})$. Specifically, we construct complex geometric optics solutions enjoying a uniqueness property based on the Green operator G^2_φ for the conjugated biharmonic operator P^2_φ.

First, let us define $u_2 \in L^2(M)$ by

$$u_2 = e^{sx_1}(w_s + \widetilde{r}_2), \tag{3.4.4}$$

where w_s is the Gaussian beam quasimode given by Theorem 3.4.1 and $\widetilde{r}_2 \in L^2(M)$ is the remainder term. Now u_2 solves $\Delta^2 u_2 = 0$ if $\widetilde{r}_2$ satisfies

$$P^2_{-\varphi}e^{i\lambda x_1}\widetilde{r}_2 = -e^{i\lambda x_1}e^{-sx_1}h^4\Delta^2 e^{sx_1}w_s. \tag{3.4.5}$$

Looking for $\widetilde{r}_2$ in the form $\widetilde{r}_2 = e^{-i\lambda x_1}G^2_{-\varphi}r_2$ with $r_2 \in L^2(M)$, we see from (3.4.5) and (3.3.2) that $r_2 = -e^{i\lambda x_1}e^{-sx_1}h^4\Delta^2 e^{sx_1}w_s$. It follows from (3.4.2) that $\|r_2\|_{L^2(M)} = \mathcal{O}(h^{5/2})$, and therefore, using (3.3.2), we get

$$\|\widetilde{r}_2\|_{L^2(M)} = \mathcal{O}(h^{1/2}), \tag{3.4.6}$$

as $h \to 0$.

Next we look for $u_1 \in L^2(M)$ solving

$$(\Delta^2 + q)u_1 = 0 \quad \text{in} \quad M^{\text{int}} \tag{3.4.7}$$

in the form,

$$u_1 = u_0 + e^{-sx_1}\widetilde{r}_1. \tag{3.4.8}$$

Here $u_0 \in L^2(M)$ is such that

$$\Delta^2 u_0 = 0 \quad \text{in} \quad M^{\text{int}}, \tag{3.4.9}$$

and u_0 has the form,

$$u_0 = e^{-sx_1}(v_s + \widetilde{r}_0), \tag{3.4.10}$$

where v_s is the Gaussian beam quasimode given by Theorem 3.4.1, and $\widetilde{r}_0, \widetilde{r}_1 \in L^2(M)$ are the remainder terms. First in view of (3.4.9), $\widetilde{r}_0$ should satisfy

$$P_\varphi^2 e^{-i\lambda x_1}\widetilde{r}_0 = -e^{-i\lambda x_1}e^{sx_1}h^4\Delta^2 e^{-sx_1}v_s. \tag{3.4.11}$$

Looking for $\widetilde{r}_0$ in the form $\widetilde{r}_0 = e^{i\lambda x_1}G_\varphi^2 r_0$, we conclude from (3.4.11) that

$$r_0 = -e^{-i\lambda x_1}e^{sx_1}h^4\Delta^2 e^{-sx_1}v_s.$$

Thus, it follows from (3.4.1) that $\|r_0\|_{L^2(M)} = \mathcal{O}(h^{5/2})$, and therefore, using (3.3.2), we obtain that

$$\|\widetilde{r}_0\|_{L^2(M)} = \mathcal{O}(h^{1/2}), \tag{3.4.12}$$

as $h \to 0$. Now u_1 given by (3.4.8) is a solution to (3.4.7) provided that

$$(P_\varphi^2 + h^4 q)e^{-i\lambda x_1}\widetilde{r}_1 = -h^4 e^{\varphi/h} q u_0 \quad \text{in} \quad M^{\text{int}}. \tag{3.4.13}$$

Looking for $\widetilde{r}_1$ in the form $\widetilde{r}_1 = e^{i\lambda x_1} G_\varphi^2 r_1$ with $r_1 \in L^2(M)$, we see from (3.4.13) that

$$(1 + h^4 q G_\varphi^2) r_1 = -h^4 e^{\varphi/h} q u_0 \quad \text{in} \quad M^{\text{int}}. \tag{3.4.14}$$

In view of (3.3.2), (3.4.10), (3.4.1), and (3.4.12), for all $0 < h \ll 1$, there exists a unique solution $r_1 \in L^2(M)$ to (3.4.14) such that

$$\|r_1\|_{L^2(M)} = \mathcal{O}(h^4)\|e^{\varphi/h} u_0\|_{L^2(M)} = \mathcal{O}(h^4),$$

and therefore,

$$\|\widetilde{r}_1\|_{L^2(M)} = \mathcal{O}(h^2). \tag{3.4.15}$$

Next we would like to reconstruct the boundary traces $\gamma u_1 = (u_1|_{\partial M}, \partial_\nu u_1|_{\partial M})$, where the complex geometric optics solution u_1 to (3.4.7) is given by (3.4.8), from the knowledge of the Dirichlet–to–Neumann map Λ_q. First we claim that u_1 satisfies the equation

$$(1 + h^4 e^{-\varphi/h} G_\varphi^2 q e^{\varphi/h}) u_1 = u_0. \tag{3.4.16}$$

Indeed, applying the operator G_φ^2 to (3.4.14) and then multiplying it by $e^{-\varphi/h}$, we get

$$e^{-sx_1}\widetilde{r}_1 + h^4 e^{-\varphi/h} G_\varphi^2 q e^{\varphi/h} u_1 = 0. \tag{3.4.17}$$

Adding u_0 to both sides of (3.4.17) gives us (3.4.16).

Using Proposition 3.3.1, we obtain from (3.4.16) that $f = \gamma u_1 \in H^{-1/2}(\partial M) \times H^{-3/2}(\partial M)$ satisfies the boundary integral equation

$$(1 + h^4 S_\varphi(\Lambda_q - \Lambda_0))f = \gamma u_0. \tag{3.4.18}$$

Since (M, g) is known, u_0 and therefore, γu_0 are also known as well as the single layer operator S_φ, and the Dirichlet–to–Neumann map Λ_0. Furthermore, Dirichlet–to–Neumann map Λ_q is known as well. By Proposition 3.3.2, for all $0 < h \ll 1$, the boundary trace $f = \gamma u_1$ can be reconstructed as the unique solution to (3.4.18).

Now substituting u_1 and u_2, given by (3.4.8) and (3.4.4), respectively, into the integral identity (3.2.31), we get

$$\int_M q u_1 \overline{u_2} dV = \langle (\Lambda_q - \Lambda_0)\gamma u_1, \gamma \overline{u_2} \rangle_{H^{1/2}(\partial M)\times H^{3/2}(\partial M), H^{-1/2}(\partial M)\times H^{-3/2}(\partial M)}. \tag{3.4.19}$$

Now as u_2 solves $\Delta^2 u_2 = 0$ in M^{int}, it is a known function. This together with the reconstruction of γu_1 shows that the expression in the right hand side of (3.4.19) can be reconstructed from our data. Thus, we can reconstruct the integral

$$\begin{aligned}\int_M q u_1 \overline{u_2} dV = \int_M q e^{-2i\lambda x_1}(\overline{w_s} v_s + \overline{\widetilde{r}_2}(v_s + \widetilde{r}_0 + \widetilde{r}_1) + \overline{w_s}(\widetilde{r}_0 + \widetilde{r}_1))dV \\ = \int_M q e^{-2i\lambda x_1} \overline{w_s} v_s dV + \mathcal{O}(h^{1/2}).\end{aligned} \tag{3.4.20}$$

Here we have used (3.4.8), (3.4.10), (3.4.4), (3.4.1), (3.4.2), (3.4.6), (3.4.12), and (3.4.15).

By Theorem 3.5.1, we can determine $q|_{\partial M}$ from the knowledge of Λ_q and (M, g) in a constructive way. Thus, we extend q to a function in $C_0(\mathbb{R}\times M_0^{\text{int}})$ in such a way that $q|_{(\mathbb{R}\times M_0)\setminus M}$ is known. This together with (3.4.20) and $dV = c^{\frac{n}{2}}dx_1 dV_{g_0}$ allows us to reconstruct

$$\int_{\mathbb{R}} e^{-2i\lambda x_1}\int_{M_0} q(x_1, x')\overline{w_s(x_1, x')}v_s(x_1, x')c(x_1, x')^{n/2}dV_{g_0}dx_1 + \mathcal{O}(h^{1/2}). \tag{3.4.21}$$

Letting $h \to 0$ in (3.4.21), and using (3.4.3), we obtain from (3.4.21) that

$$\int_{\mathbb{R}} e^{-2i\lambda x_1}\int_0^L e^{-2\lambda t}q(x_1, \gamma(t))c(x_1, \gamma(t))dtdx_1 = \int_0^L \widehat{\widetilde{q}}(2\lambda, \gamma(t))e^{-2\lambda t}dt, \tag{3.4.22}$$

for any $\lambda \in \mathbb{R}$ and any non-tangential geodesic γ in M_0. Here $\widetilde{q} = qc$ and

$$\widehat{\widetilde{q}}(\lambda, x') = \int_{\mathbb{R}} e^{-i\lambda x_1}\widetilde{q}(x_1, x')dx_1.$$

The integral in the right hand side of (3.4.22) is the attenuated geodesic ray transform of $\widehat{\widetilde{q}}(2\lambda, \cdot)$ with constant attenuation -2λ. Note that if M_0 is simple then it was shown in [107] that the attenuated ray transform is constructively invertible for any attenuation, and using the inversion procedure in [107], we reconstruct the potential q.

In general, proceeding similarly to the end of the proof of [41, Theorem 1.4], using the constructive invertibility assumption of the geodesic ray transform on M_0, we reconstruct the potential q in M. This completes the proof of Theorem 3.1.4.

3.5 Boundary reconstruction of a continuous potential for the perturbed biharmonic operator

The goal of this section is to give a reconstruction formula for the boundary values of a continuous potential q from the knowledge of the Dirichlet–to–Neumann map for the perturbed biharmonic operator Δ^2+q on a smooth compact Riemannian manifold of dimension $n\ge 2$ with smooth boundary. In the case of Schrödinger operator, the constructive determination of the boundary values of a continuous potential from boundary measurements is given in [41, Appendix A], and our reconstruction here will rely crucially on this work. For the non-constructive boundary determination of a continuous potential in the case of the Schrödinger operator, we refer to the works [53], [74], [87]. For the boundary determination of smooth perturbations based on pseudodifferential techniques, see [83] and [68]. Our result is as follows.

Theorem 3.5.1. *Let (M,g) be a given compact smooth Riemannian manifold of dimension $n\ge 2$ with smooth boundary, and let $q\in C(M)$ be such that assumption (A) is satisfied. For each point $x_0\in\partial M$, there exists an explicit family of functions $f_\lambda\in C^\infty(\partial M)\times C^\infty(\partial M)$, $0<\lambda\ll 1$, depending only on (M,g), such that*

$$q(x_0)=2\lim_{\lambda\to 0}\langle(\Lambda_q-\Lambda_0)f_\lambda,\overline{f_\lambda}\rangle_{H^{-3/2}(\partial M)\times H^{-1/2}(\partial M),H^{3/2}(\partial M)\times H^{1/2}(\partial M)}.$$

Proof. Let $f\in H^{3/2}(\partial M)\times H^{1/2}(\partial M)$ and let us start by considering the special case of the integral identity (3.2.29),

$$\langle(\Lambda_q-\Lambda_0)f,\overline{f}\rangle_{H^{-3/2}(\partial M)\times H^{-1/2}(\partial M),H^{3/2}(\partial M)\times H^{1/2}(\partial M)}=\int_M qu\overline{v}dV. \tag{3.5.1}$$

Here $u, v \in H^2(M^{\rm int})$ are solutions to

$$\begin{cases} (\Delta^2 + q)u = 0 & \text{in} \quad M^{\rm int}, \\ \gamma u = f & \text{on} \quad \partial M, \end{cases} \tag{3.5.2}$$

and

$$\begin{cases} \Delta^2 v = 0 & \text{in} \quad M^{\rm int}, \\ \gamma v = f & \text{on} \quad \partial M, \end{cases} \tag{3.5.3}$$

respectively.

We would like to construct suitable solutions to (3.5.2) and (3.5.3) to test the integral identity (3.5.1). The construction of these solutions will be based on an explicit family of functions v_λ, whose boundary values have a highly oscillatory behavior as $\lambda \to 0$, while becoming increasingly concentrated near a given point on the boundary of M. Such a family of functions v_λ was introduced in [20], [22], see also [41], [73], [74], [68].

To define v_λ, we let $x_0 \in \partial M$ and let $(x_1, \dots, x_n)$ be the boundary normal coordinates centered at x_0 so that in these coordinates, $x_0 = 0$, the boundary ∂M is given by $\{x_n = 0\}$, and $M^{\rm int}$ is given by $\{x_n > 0\}$. In these local coordinates, we have $T_{x_0}\partial M = \mathbb{R}^{n-1}$, equipped with the Euclidean metric. The unit tangent vector τ is then given by $\tau = (\tau', 0)$ where $\tau' \in \mathbb{R}^{n-1}$, $|\tau'| = 1$. Associated to the tangent vector τ' is the covector $\xi'_\alpha = \sum_{\beta=1}^{n-1} g_{\alpha\beta}(0)\tau'_\beta = \tau'_\alpha \in T^*_{x_0}\partial M$.

Let $\eta \in C_0^\infty(\mathbb{R}^n; \mathbb{R})$ be such that supp (η) is in a small neighborhood of 0, and

$$\int_{\mathbb{R}^{n-1}} \eta(x', 0)^2 dx' = 1. \tag{3.5.4}$$

Let $\frac{1}{3} \le \alpha \le \frac{1}{2}$. Following [22], [74, Appendix C], [41, Appendix A] in the boundary normal coordinates, we set

$$v_\lambda(x) = \lambda^{-\frac{\alpha(n-1)}{2}-\frac{1}{2}}\eta\left(\frac{x}{\lambda^\alpha}\right)e^{\frac{i}{\lambda}(\tau'\cdot x'+ix_n)}, \quad 0<\lambda\ll 1, \tag{3.5.5}$$

so that $v_\lambda \in C^\infty(M)$, with supp (v_λ) in $\mathcal{O}(\lambda^\alpha)$ neighborhood of $x_0 = 0$. Here τ' is viewed as a covector. A direct computation shows that

$$\|v_\lambda\|_{L^2(M)} = \mathcal{O}(1), \tag{3.5.6}$$

as $\lambda \to 0$, see also [74, Appendix C]. Following [41, Appendix A], we let

$$v = v_\lambda + r_1, \tag{3.5.7}$$

where $r_1 \in H_0^1(M^{\text{int}})$ is the solution to the Dirichlet problem,

$$\begin{cases} -\Delta r_1 = \Delta v_\lambda \quad \text{in} \quad M^{\text{int}}, \\ r_1|_{\partial M} = 0. \end{cases} \tag{3.5.8}$$

By boundary elliptic regularity, we have $r_1 \in C^\infty(M)$, and therefore, $v \in C^\infty(M)$. It was established in [41, Appendix A] that when $\alpha = 1/3$,

$$\|r_1\|_{L^2(M)} = \mathcal{O}(\lambda^{1/12}), \tag{3.5.9}$$

as $\lambda \to 0$. In what follows, we fix $\alpha = 1/3$.

Note that $v \in C^\infty(M)$ solves the Dirichlet problem (3.5.3) with

$$f = f_\lambda := (v_\lambda|_{\partial M}, \partial_\nu(v_\lambda + r_1)|_{\partial M}). \tag{3.5.10}$$

Now since the manifold (M, g) is known, the harmonic function v as well as the trace f_λ are known.

Next we look for a solution u to (3.5.2) with the Dirichlet data $f = f_\lambda$ given by (3.5.10) in the form

$$u = v_\lambda + r_1 + r_2. \tag{3.5.11}$$

Thus, $r_2 \in H^2(M^{\text{int}})$ is the solution to the following Dirichlet problem,

$$\begin{cases} (\Delta^2 + q)r_2 = -q(v_\lambda + r_1) & \text{in} \quad M^{\text{int}}, \\ \gamma r_2 = 0 & \text{on} \quad \partial M. \end{cases} \tag{3.5.12}$$

It follows from [47, Section 11, p. 325, 326] that for all $s > 3/2$,

$$\|r_2\|_{H^s(M^{\text{int}})} \leq C\|q(v_\lambda + r_1)\|_{H^{s-4}(M^{\text{int}})}. \tag{3.5.13}$$

In particular, letting $s = 3$ in (3.5.13), we get

$$\begin{aligned} \|r_2\|_{L^2(M)} \leq C\|q(v_\lambda + r_1)\|_{H^{-1}(M^{\text{int}})} &\leq C(\|qv_\lambda\|_{H^{-1}(M^{\text{int}})} + \|r_1\|_{L^2(M)}) \\ &= o(1) + \mathcal{O}(\lambda^{1/12}) = o(1), \end{aligned} \tag{3.5.14}$$

as $\lambda \to 0$. Note that here we used the following bound

$$\|qv_\lambda\|_{H^{-1}(M^{\text{int}})} = o(1),$$

as $\lambda \to 0$, cf. [41, Appendix A, (A.20)], together with (3.5.9).

Substituting v and u given by (3.5.7) and (3.5.11), respectively, into (3.5.1) and taking the limit $\lambda \to 0$, we obtain that

$$\lim_{\lambda\to 0}\langle(\Lambda_q - \Lambda_0)f_\lambda, \overline{f_\lambda}\rangle_{H^{-3/2}(\partial M)\times H^{-1/2}(\partial M), H^{3/2}(\partial M)\times H^{1/2}(\partial M)} = \lim_{\lambda\to 0}(I_1 + I_2), \tag{3.5.15}$$

where

$$I_1 = \int_M q|v_\lambda|^2 dV, \quad I_2 = \int_M q(v_\lambda \overline{r_1} + (r_1 + r_2)(\overline{v_\lambda} + \overline{r_1}))dV.$$

Using (3.5.9) and (3.5.14), we get

$$\lim_{\lambda\to 0} I_2 = 0. \tag{3.5.16}$$

A direct computation shows that

$$\lim_{\lambda\to 0} I_1 = \frac{1}{2}q(0), \tag{3.5.17}$$

cf. [41, Appendix A, (A.24)]. Combining (3.5.15), (3.5.16), and (3.5.17), we see that

$$q(0) = 2\lim_{\lambda\to 0}\langle(\Lambda_q - \Lambda_0)f_\lambda, \overline{f_\lambda}\rangle_{H^{-3/2}(\partial M)\times H^{-1/2}(\partial M), H^{3/2}(\partial M)\times H^{1/2}(\partial M)}.$$

This completes the proof of Theorem 3.5.1. □

Chapter 4

A remark on inverse problems for nonlinear magnetic Schrödinger equations on complex manifolds

4.1 Introduction

Let M be an n–dimensional compact complex manifold with C^∞ boundary, equipped with a Kähler metric g. Consider the nonlinear magnetic Schrödinger operator

$$L_{A,V}u = d^*_{\overline{A(\cdot,u)}} d_{A(\cdot,u)}u + V(\cdot,u),$$

acting on $u \in C^\infty(M)$. Here the nonlinear magnetic $A : M \times \mathbb{C} \to T^*M \otimes \mathbb{C}$ and electric $V : M \times \mathbb{C} \to \mathbb{C}$ potentials are assumed to satisfy the following conditions:

(i) the map $\mathbb{C} \ni w \mapsto A(\cdot, w)$ is holomorphic with values in $C^\infty(M, T^*M \otimes \mathbb{C})$,

(ii) $A(z, 0) = 0$ for all $z \in M$,

(iii) the map $\mathbb{C} \ni w \mapsto V(\cdot, w)$ is holomorphic with values in $C^\infty(M)$,

(iv) $V(z,0) = \partial_w V(z,0) = 0$ for all $z \in M$.

Thus, A and V can be expanded into power series

$$A(z,w) = \sum_{k=1}^{\infty} A_k(z)\frac{w^k}{k!}, \quad V(z,w) = \sum_{k=2}^{\infty} V_k(z)\frac{w^k}{k!}, \tag{4.1.1}$$

converging in $C^\infty(M, T^*M \otimes \mathbb{C})$ and $C^\infty(M)$ topologies, respectively. Here

$$A_k(z) := \partial_w^k A(z,0) \in C^\infty(M, T^*M \otimes \mathbb{C}), \quad V_k(z) := \partial_w^k V(z,0) \in C^\infty(M).$$

We write $T^*M \otimes \mathbb{C}$ for the complexified cotangent bundle of M,

$$d_{A(\cdot,w)} = d + iA(\cdot, w) : C^\infty(M) \to C^\infty(M, T^*M \otimes \mathbb{C}), \quad w \in \mathbb{C}, \tag{4.1.2}$$

where $d : C^\infty(M) \to C^\infty(M, T^*M \otimes \mathbb{C})$ is the de Rham differential, and $d^*_{A(\cdot,w)} : C^\infty(M, T^*M \otimes \mathbb{C}) \to C^\infty(M)$ is the formal L^2–adjoint of $d_{A(\cdot,w)}$ taken with respect to the Kähler metric g.

It is established in [74, Appendix B] that under the assumptions (i)-(iv), there exist $\delta > 0$ and $C > 0$ such that for any $f \in B_\delta(\partial M) := \{f \in C^{2,\alpha}(\partial M) : \|f\|_{C^{2,\alpha}(\partial M)} < \delta\}$, $0 < \alpha < 1$, the Dirichlet problem for the nonlinear magnetic Schrödinger operator

$$\begin{cases} L_{A,V} u = 0 \quad \text{in} \quad M^{\text{int}}, \\ u|_{\partial M} = f, \end{cases} \tag{4.1.3}$$

has a unique solution $u = u_f \in C^{2,\alpha}(M)$ satisfying $\|u\|_{C^{2,\alpha}(M)} < C\delta$. Here $C^{2,\alpha}(M)$ and $C^{2,\alpha}(\partial M)$ stand for the standard Hölder spaces of functions on M and ∂M, respectively, and $M^{\text{int}} = M \setminus \partial M$ stands for the interior of M. Associated to (4.1.3), we introduce the

Dirichlet–to–Neumann map

$$\Lambda_{A,V} f = \partial_\nu u_f|_{\partial M}, \quad f \in B_\delta(\partial M), \tag{4.1.4}$$

where ν is the unit outer normal to the boundary of M.

The inverse boundary problem for the nonlinear magnetic Schrödinger operator that we are interested in asks whether the knowledge of the Dirichlet–to–Neumann map $\Lambda_{A,V}$ determines the nonlinear magnetic A and electric V potentials in M. Such inverse problems have been recently studied in [74] in the case of conformally transversally anisotropic manifolds and in [78] and [86] in the case of partial data in the Euclidean space and on Riemann surfaces, respectively.

To state our result, following [51], we assume that the manifold M satisfies the following additional assumptions:

(a) M is holomorphically separable in the sense that if $x, y \in M$ with $x \neq y$, there is some $f \in \mathcal{O}(M) := \{f \in C^\infty(M) : f \text{ is holomorphic in } M^{\text{int}}\}$ such that $f(x) \neq f(y)$,

(b) M has local charts given by global holomorphic functions in the sense that for every $p \in M$ there exist $f_1, \dots, f_n \in \mathcal{O}(M)$ which form a complex coordinate system near p.

As explained in [51], examples of complex manifolds satisfying all of the assumptions above including (a) and (b) are as follows:

- any compact C^∞ subdomain of a Stein manifold, equipped with a Kähler metric,
- any compact C^∞ subdomain of a complex submanifold of $\mathbb{C}^N$, equipped with a Kähler metric,

- any compact C^∞ subdomain of a complex coordinate neighborhood on a Kähler manifold.

The main result of this note is as follows.

Theorem 4.1.1. *Let M be an n–dimensional compact complex manifold with C^∞ boundary, equipped with a Kähler metric g, satisfying assumptions (a) and (b). Let $A^{(1)}, A^{(2)} : M\times\mathbb{C} \to T^*M \otimes \mathbb{C}$ and $V^{(1)}, V^{(2)} : M \times \mathbb{C} \to \mathbb{C}$ be such that the assumptions* (i)–(iv) *hold. If $\Lambda_{A^{(1)},V^{(1)}} = \Lambda_{A^{(2)},V^{(2)}}$ then $A^{(1)} = A^{(2)}$ and $V^{(1)} = V^{(2)}$ in $M \times \mathbb{C}$.*

Remark 4.1.2. *Theorem 4.1.1 in the case of a semilinear Schrödinger operator, i.e. when $A = 0$, was obtained in [87].*

Remark 4.1.3. *The corresponding inverse problems for the linear Schrödinger operator $-\Delta_g+V_0$, $V_0 \in C^\infty(M)$, as well as for the linear magnetic Schrödinger operator $d^*_{\overline{A_0}}d_{A_0}+V_0$, $A_0 \in C^\infty(M, T^*M \otimes \mathbb{C})$, in the geometric setting of Theorem 4.1.1 are open. Theorem 4.1.1 can be viewed as a manifestation of the phenomenon, discovered in [77], that the presence of nonlinearity may help to solve inverse problems. We refer to [51] for the solution of the linearized inverse problem for the linear Schrödinger operator in this geometric setting, and would like to emphasize that our proof of Theorem 4.1.1 is based crucially on this result. We also refer to [52], [53], [54] for solutions to inverse boundary problems for the linear Schrödinger and magnetic Schrödinger operators on Riemann surfaces.*

Remark 4.1.4. *The known results for the inverse boundary problem for the linear Schrödinger and magnetic Schrödinger operators on Riemannian manifolds of dimension ≥ 3 with boundary beyond the Euclidean ones, see [115], [98], [70], and real analytic ones, see [82], [81], [83], all require a certain conformal symmetry of the manifold as well as some additional assumptions about the injectivity of geodesic ray transforms, see [36], [38], [32], [73]. The known results for inverse problems for the nonlinear Schrödinger operators $L_{0,V}$ [43], [80], and nonlinear magnetic Schrödinger operators $L_{A,V}$ [74] still require the same conformal symmetry of the manifold, while the injectivity of the geodesic transform is no longer needed.*

Note that the need to require a certain conformal symmetry of the manifold in all of the known results in dimensions $n \geq 3$ is to due to the existence of limiting Carleman weights on such manifolds, see [36], which are crucial for the construction of complex geometric optics solutions used for solving inverse problems for elliptic PDE since the fundamental work [115]. However, it is shown in [85], [2] that a generic manifold of dimension $n \geq 3$ does not admit limiting Carleman weights.

Remark 4.1.5. *As in [51, Theorem 1.1], manifolds considered in Theorem 4.1.1 need not admit limiting Carleman weights. For example, it was established in [3] that* $\mathbb{C}P^2$ *with the Fubini-Study metric* g *does not admit a limiting Carleman weight near any point. However,* $(\mathbb{C}P^2, g)$ *is a Kähler manifold, and as explained in [51], compact* C^∞ *subdomains of it provide examples of manifolds where Theorem 4.1.1 applies.*

Remark 4.1.6. *In contrast to the inverse boundary problem for the linear magnetic Schrödinger equation, where one can determine the magnetic potential up to a gauge transformation only, see for example [98], [70], in Theorem 4.1.1 the unique determination of both nonlinear magnetic and electric potentials is achieved. This is due to our assumptions (ii) and (iv) which lead to the first order linearization of the nonlinear magnetic Schrödinger equation given by* $-\Delta_g u = 0$ *rather than by the linear magnetic Schrödinger equation, see also [74] for a similar unique determination in the case of conformally transversally anisotropic manifolds.*

Let us finally mention that inverse problems for the semilinear Schrödinger operators and for nonlinear conductivity equations have been investigated intensively recently, see for example [43], [79], [80], [84], [76], [75], and [30], [64], [29], [28], [93], [109], respectively.

Theorem 4.1.1 is a direct consequence of the main result of [51], combined with some boundary determination results of [87] and of Section 4.3, as well as the higher order linearization procedure introduced in [77] in the hyperbolic case, and in [43], [80] in the elliptic case. We refer to [58] where the method of a first order linearization was pioneered in the study of inverse problems for nonlinear PDE, and to [10], [31], [112], and [113] where a second order linearization was successfully exploited. The crucial fact used in the proof of the main result of [51], indispensable for our Theorem 4.1.1, is that both holomorphic and antiholomorphic functions are harmonic on Kähler manifolds. The assumptions (a) and (b) in Theorem 4.1.1 are needed as they are used in [51] to construct suitable holomorphic and antiholomorphic functions by extending the two dimensional arguments of [23] and [53] to the case of higher dimensional complex manifolds.

The plan of the note is as follows. The proof of Theorem 4.1.1 is given in Section 4.2. Section 4.3 contains the boundary determination result needed in the proof of Theorem 4.1.1.

4.2 Proof of Theorem 4.1.1

First using that $d_A^* = d^* - i\langle \overline{A}, \cdot\rangle_g$ and (4.1.2), we write the nonlinear magnetic Schödinger operator $L_{A,V}$ as follows,

$$
\begin{aligned}
L_{A,V}u &= d^*_{\overline{A(\cdot,u)}} d_{A(\cdot,u)}u + V(\cdot,u)\\
&= -\Delta_g u + d^*(iA(\cdot,u)u) - i\langle A(\cdot,u), du\rangle_g + \langle A(\cdot,u), A(\cdot,u)\rangle_g u + V(\cdot,u),
\end{aligned}
$$

for $u \in C^\infty(M)$. Here $\langle\cdot,\cdot\rangle_g$ is the pointwise scalar product in the space of 1-forms induced by the Riemannian metric g, compatible with the Kähler structure.

Using the mth order linearization of the Dirichlet–to–Neumann map $\Lambda_{A,V}$ and induction on $m = 2, 3, \dots$, we shall show that the coefficients A_{m-1} and V_m in (4.1.1) can all be recovered from $\Lambda_{A,V}$.

First, let $m = 2$ and let us proceed to carry out a second order linearization of the Dirichlet–to–Neumann map. To that end, let $f_1, f_2 \in C^\infty(\partial M)$ and let $u_j = u_j(x,\varepsilon) \in C^{2,\alpha}(M)$ be the unique small solution of the following Dirichlet problem,

$$\begin{cases} -\Delta_g u_j + id^*(\sum_{k=1}^\infty A_k^{(j)}(x)\frac{u_j^k}{k!}u_j) - i\langle\sum_{k=1}^\infty A_k^{(j)}(x)\frac{u_j^k}{k!}, du_j\rangle_g \\ \quad +\langle\sum_{k=1}^\infty A_k^{(j)}(x)\frac{u_j^k}{k!}, \sum_{k=1}^\infty A_k^{(j)}(x)\frac{u_j^k}{k!}\rangle_g u_j + \sum_{k=2}^\infty V_k^{(j)}(x)\frac{u_j^k}{k!} = 0 \text{ in } M^{\text{int}}, \\ u_j = \varepsilon_1 f_1 + \varepsilon_2 f_2 \text{ on } \partial M, \end{cases} \tag{4.2.1}$$

for $j = 1, 2$. It was established in [74, Appendix B] that for all $|\varepsilon|$ sufficiently small, the solution $u_j(\cdot, \varepsilon)$ depends holomorphically on $\varepsilon = (\varepsilon_1, \varepsilon_2) \in \text{neigh}(0, \mathbb{C}^2)$. Applying the operator $\partial_{\varepsilon_l}|_{\varepsilon=0}$, $l = 1, 2$, to (4.2.1) and using that $u_j(x, 0) = 0$, we get

$$\begin{cases} -\Delta_g v_j^{(l)} = 0 & \text{in } M^{\text{int}}, \\ v_j^{(l)} = f_l & \text{on } \partial M, \end{cases}$$

where $v_j^{(l)} = \partial_{\varepsilon_l} u_j|_{\varepsilon=0}$. By the uniqueness and the elliptic regularity, it follows that $v^{(l)} := v_1^{(l)} = v_2^{(l)} \in C^\infty(M)$, $l = 1, 2$. Applying $\partial_{\varepsilon_1}\partial_{\varepsilon_2}|_{\varepsilon=0}$ to (4.2.1), we obtain the second order linearization,

$$\begin{cases} -\Delta_g w_j + 2id^*(A_1^{(j)} v^{(1)} v^2) - i\langle A_1^{(j)}, d(v^{(1)}v^{(2)})\rangle_g + V_2^{(j)} v^{(1)} v^{(2)} = 0 \text{ in } M^{\text{int}}, \\ w_j = 0 \text{ on } \partial M, \end{cases} \tag{4.2.2}$$

where $w_j = \partial_{\varepsilon_1}\partial_{\varepsilon_2} u_j|_{\varepsilon=0}$, $j = 1, 2$. Using that

$$d^*(Bv) = (d^*B)v - \langle B, dv\rangle_g, \tag{4.2.3}$$

for any $B \in C^\infty(M, T^*M \otimes \mathbb{C})$ and $v \in C^\infty(M)$, (4.2.2) implies that

$$\begin{cases} -\Delta_g w_j - 3i\langle A_1^{(j)}, d(v^{(1)}v^{(2)})\rangle_g + (2id^*(A_1^{(j)}) + V_2^{(j)}) v^{(1)} v^{(2)} = 0 \text{ in } M^{\text{int}}, \\ w_j = 0 \text{ on } \partial M, \end{cases} \tag{4.2.4}$$

$j = 1, 2$. The equality $\Lambda_{A^{(1)},V^{(1)}}(\varepsilon_1 f_1 + \varepsilon_2 f_2) = \Lambda_{A^{(2)},V^{(2)}}(\varepsilon_1 f_1 + \varepsilon_2 f_2)$ yields that $\partial_\nu u_1|_{\partial M} = \partial_\nu u_2|_{\partial M}$, and hence, $\partial_\nu w_1|_{\partial M} = \partial_\nu w_2|_{\partial M}$. Multiplying the difference of two equations in (4.2.4) by a harmonic function $v^{(3)} \in C^\infty(M)$, integrating over M and using Green's formula, we obtain that

$$\int_M \big(3i\langle A, d(v^{(1)}v^{(2)})\rangle_g v^{(3)} - (2id^*(A) + V)v^{(1)}v^{(2)}v^{(3)}\big) dV_g = 0, \tag{4.2.5}$$

valid for all harmonic functions $v^{(l)} \in C^\infty(M)$, $l = 1, 2, 3$. Here $A = A_1^{(1)} - A_1^{(2)}$ and $V = V_2^{(1)} - V_2^{(2)}$. Interchanging $v^{(3)}$ and $v^{(1)}$ in (4.2.5), we also have

$$\int_M \big(3i\langle A, d(v^{(3)}v^{(2)})\rangle_g v^{(1)} - (2id^*(A) + V)v^{(1)}v^{(2)}v^{(3)}\big) dV_g = 0. \tag{4.2.6}$$

Subtracting (4.2.6) from (4.2.5) and letting $v^{(3)} = 1$, we get

$$\int_M \langle A, dv^{(1)}\rangle_g v^{(2)} dV_g = 0, \tag{4.2.7}$$

for all harmonic functions $v^{(1)}, v^{(2)} \in C^\infty(M)$. Applying Proposition 4.3.1 to (4.2.7), we conclude that $A|_{\partial M} = 0$. Using this together with Stokes' formula,

$$\int_M \langle dw, \eta\rangle_g dV_g = \int_M w d^*\eta dV_g + \int_{\partial M} \omega(\mathbf{n}\eta),\ \omega \in C^\infty(M),\ \eta \in C^\infty(M, T^*M \otimes \mathbb{C}),$$

where the $(2n-1)$-form $\mathbf{n}\eta$ on the boundary is the normal trace of η, see [108, Proposition 2.1.2], we obtain from (4.2.5) that

$$\int_M \big(3id^*(Av^{(3)}) - (2id^*(A) + V)v^{(3)}\big)v^{(1)}v^{(2)} dV_g = 0, \tag{4.2.8}$$

for all harmonic functions $v^{(l)} \in C^\infty(M)$, $l = 1, 2, 3$. Applying [51, Theorem 1.1] together with the boundary determination result of [87, Proposition 3.1] to (4.2.8), we get

$$3id^*(Av^{(3)}) - (2id^*(A) + V)v^{(3)} = 0, \tag{4.2.9}$$

for every harmonic function $v^{(3)} \in C^\infty(M)$. Using (4.2.3), we obtain from (4.2.9) that

$$(id^*(A) - V)v^{(3)} - 3i\langle A, dv^{(3)}\rangle_g = 0, \tag{4.2.10}$$

for every harmonic function $v^{(3)} \in C^\infty(M)$. Letting $v^{(3)} = 1$ in (4.2.10), we get

$$id^*(A) - V = 0, \tag{4.2.11}$$

and therefore,

$$\langle A, dv^{(3)}\rangle_g = 0, \tag{4.2.12}$$

for every harmonic function $v^{(3)} \in C^\infty(M)$. Let $p \in M^{\text{int}}$ and by assumption (b), there exist $f_1, \dots, f_n \in \mathcal{O}(M)$ which form a complex coordinate system near p. Hence, $df_j(p)$, $d\overline{f_j}(p)$ is a basis for $T_p^*M \otimes \mathbb{C}$. Since on a Kähler manifold the Laplacian on functions satisfies

$$\Delta_g = d^*d = 2\partial^*\partial = 2\overline{\partial}^*\overline{\partial},$$

see [51, Lemma 2.1], [92, Theorem 8.6, p. 45], we have that all functions $f_1, \dots, f_n$ as well as $\overline{f_1}, \dots, \overline{f_n}$ are harmonic, and therefore, it follows from (4.2.12) that

$$\langle A, df_j\rangle_g(p) = 0, \quad \langle A, d\overline{f_j}\rangle_g(p) = 0.$$

Hence, $A = 0$, and therefore, $A_1^{(1)} = A_1^{(2)}$ in M. It follows from (4.2.11) that $V = 0$, and therefore, $V_2^{(1)} = V_2^{(2)}$ in M.

Let $m \geq 3$ and let us assume that

$$A_k^{(1)} = A_k^{(2)}, \quad k = 1, \dots, m-2, \quad V_k^{(1)} = V_k^{(2)}, \quad k = 2, \dots, m-1. \tag{4.2.13}$$

To prove that $A_{m-1}^{(1)} = A_{m-1}^{(2)}$ and $V_m^{(1)} = V_m^{(2)}$, we shall use the mth order linearization of the Dirichlet–to–Neumann map. Such an mth order linearization with $m \geq 3$ is performed in [74], and combining with (4.2.13), it leads to the following integral identity,

$$\int_M \big((m+1)i\langle A, d(v^{(1)} \cdots v^{(m)})\rangle_g v^{(m+1)} - (mid^*(A) + V)v^{(1)} \cdots v^{(m+1)}\big) dV_g = 0, \tag{4.2.14}$$

for all harmonic functions $v^{(l)} \in C^\infty(M)$, $l = 1, \dots, m+1$, see [74, Section 5]. Here $A = A_{m-1}^{(1)} - A_{m-1}^{(2)}$ and $V = V_m^{(1)} - V_m^{(2)}$. Letting $v^{(1)} = \dots = v^{(m-2)} = 1$ in (4.2.14) and arguing as in the case $m = 2$, we complete the proof of Theorem 4.1.1.

Remark 4.2.1. *Thanks to the density of products of two harmonic functions in the geometric setting of Theorem 4.1.1 established in [51], we recover the nonlinear magnetic and electric potentials of the general form* (4.1.1) *here. On the other hand, in the case of conformally transversally anisotropic manifolds of real dimension* ≥ 3*, only the density of products of four harmonic functions is available, see [43], [80], [74], and therefore, the nonlinear magnetic and electric potentials of the form* (4.1.1) *with* $k \geq 2$ *and* $k \geq 3$*, respectively, were determined from the knowledge of the Dirichlet–to–Neumann map in [74].*

4.3 Boundary determination of a 1-form on a Riemannian manifold

When proving Theorem 4.1.1, we need the following essentially known boundary determination result on a general compact Riemannian manifold with boundary, see [22], [72,

Appendix A], [74, Appendix C], [86] for similar results. We present a proof for completeness and convenience of the reader.

Proposition 4.3.1. *Let (M, g) be a compact smooth Riemannian manifold of dimension $n \geq 2$ with smooth boundary. If $A \in C(M, T^*M \otimes \mathbb{C})$ satisfies*

$$\int_M \langle A, du\rangle_g \overline{u} dV_g = 0, \tag{4.3.1}$$

for every harmonic function $u \in C^\infty(M)$, then $A|_{\partial M} = 0$.

Proof. In order to show that $A|_{\partial M} = 0$, we shall construct a suitable harmonic function $u \in C^\infty(M)$ to be used in the integral identity (4.3.1). When doing so, we shall use an explicit family of functions v_λ, constructed in [20], [22], whose boundary values have a highly oscillatory behavior as $\lambda \to 0$, while becoming increasingly concentrated near a given point on the boundary of M. We let $x_0 \in \partial M$ and we shall work in the boundary normal coordinates centered at x_0 so that in these coordinates, $x_0 = 0$, the boundary ∂M is given by $\{x_n = 0\}$, and M^{int} is given by $\{x_n > 0\}$. We have $T_{x_0}\partial M = \mathbb{R}^{n-1}$, equipped with the Euclidean metric. The unit tangent vector τ is then given by $\tau = (\tau', 0)$ where $\tau' \in \mathbb{R}^{n-1}$, $|\tau'| = 1$. Associated to the tangent vector τ' is the covector $\sum_{\beta=1}^{n-1} g_{\alpha\beta}(0)\tau'_\beta = \tau'_\alpha \in T^*_{x_0}\partial M$.

Letting $\frac{1}{3} \leq \alpha \leq \frac{1}{2}$ and following [22], see also [41, Appendix A], we set

$$v_\lambda(x) = \lambda^{-\frac{\alpha(n-1)}{2} - \frac{1}{2}} \eta\left(\frac{x}{\lambda^\alpha}\right) e^{\frac{i}{\lambda}(\tau' \cdot x' + i x_n)}, \quad 0 < \lambda \ll 1,$$

where $\eta \in C_0^\infty(\mathbb{R}^n; \mathbb{R})$ is such that supp (η) is in a small neighborhood of 0, and

$$\int_{\mathbb{R}^{n-1}} \eta(x', 0)^2 dx' = 1.$$

Here τ' is viewed as a covector. Thus, we have $v_\lambda \in C^\infty(M)$ with supp (v_λ) in $\mathcal{O}(\lambda^\alpha)$

neighborhood of $x_0 = 0$. A direct computation shows that

$$\|v_\lambda\|_{L^2(M)} = \mathcal{O}(1), \tag{4.3.2}$$

as $\lambda \to 0$, see also [41, Appendix A, (A.8)]. Furthermore, we have

$$\|dv_\lambda\|_{L^2(M)} = \mathcal{O}(\lambda^{-1}), \tag{4.3.3}$$

as $\lambda \to 0$, see [74, Appendix C, bound (C.42)].

Following [22], we set

$$u = v_\lambda + r, \tag{4.3.4}$$

where $r \in H^1_0(M^{\text{int}})$ is the unique solution to the Dirichlet problem,

$$\begin{cases} -\Delta_g r = \Delta_g v_\lambda \quad \text{in} \quad M^{\text{int}}, \\ r|_{\partial M} = 0. \end{cases} \tag{4.3.5}$$

Boundary elliptic regularity implies $r \in C^\infty(M)$, and hence, $u \in C^\infty(M)$. Following [41, Appendix A], we fix $\alpha = 1/3$. The following bound, proved in [41, Appendix A, bound (A.15)], will be needed here,

$$\|r\|_{L^2(M)} = \mathcal{O}(\lambda^{1/12}), \tag{4.3.6}$$

as $\lambda \to 0$. The proof of (4.3.6) relies on elliptic estimates for the Dirichlet problem for the Laplacian in Sobolev spaces of low regularity. We shall also need the following rough bound

$$\|r\|_{H^1(M^{\text{int}})} = \mathcal{O}(\lambda^{-1/3}), \tag{4.3.7}$$

as $\lambda \to 0$, established in [74, Appendix C, bound (C.41)].

Substituting u into (4.3.1), and multiplying (4.3.1) by λ, we get

$$0 = \lambda \int_M \langle A, dv_\lambda + dr\rangle_g (\overline{v_\lambda} + \overline{r}) dV_g = \lambda(I_1 + I_2 + I_3), \tag{4.3.8}$$

where

$$I_1 = \int_M \langle A, dv_\lambda\rangle_g \overline{v_\lambda} dV_g, \quad I_2 = \int_M \langle A, dr\rangle_g (\overline{v_\lambda} + \overline{r}) dV_g, \quad I_3 = \int_M \langle A, dv_\lambda\rangle_g \overline{r} dV_g.$$

It was computed in [74, Appendix C], see bounds (C.44) and (C.45) there, that

$$\lim_{\lambda \to 0} \lambda I_1 = \frac{i}{2}\langle A(0), (\tau', i)\rangle. \tag{4.3.9}$$

It follows from (4.3.7), (4.3.2), and (4.3.6) that

$$\lambda |I_2| \leq \mathcal{O}(\lambda)\|dr\|_{L^2(M)}\|v_\lambda + r\|_{L^2(M)} = \mathcal{O}(\lambda^{2/3}). \tag{4.3.10}$$

Using (4.3.3) and (4.3.6), we get

$$\lambda |I_3| \leq \mathcal{O}(\lambda)\|dv_\lambda\|_{L^2(M)}\|r\|_{L^2(M)} = \mathcal{O}(\lambda^{1/12}). \tag{4.3.11}$$

Passing to the limit $\lambda \to 0$ in (4.3.8) and using (4.3.9), (4.3.10), (4.3.11), we obtain that $\langle A(0), (\tau', i)\rangle = 0$, and arguing as in [74, Appendix C], we get $A|_{\partial M} = 0$. This completes the proof of Proposition 4.3.1. □

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