Projections of a Learning Space*

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Abstract

Any proper subset Q' of the domain Q of a learning space $\mathcal{K}$ defines a projection of $\mathcal{K}$ on Q' which is itself a learning space consistent with $\mathcal{K}$. Such a construction defines a partition of Q having each of its classes either equal to $\{\varnothing\}$, or preserving some key properties of the learning space $\mathcal{K}$, namely closure under union and wellgradedness. If the set Q' satisfies certain conditions, then each of the equivalence classes is essentially, via a trivial transformation, a learning space. We give a direct proof of these and related facts which are instrumental in parsing large learning spaces.

This paper is dedicated to George Sperling whose curious, incisive mind rarely fails to produce the unexpected creative idea. George and I have been colleagues for the longest time in both of our careers. The benefit has been mine.

Learning spaces, which are special cases of knowledge spaces (cf. Doignon and Falmagne, 1999), are mathematical structures designed to model the cognitive organization of a scholarly topic, such as Beginning Algebra or Chemistry 101. The definition of 'learning space' is recalled in our Definition 1. Essentially, a learning space is a family of sets, called knowledge states, satisfying a couple of conditions. The elements of the sets are 'atoms' of knowledge, such as facts or problems to be solved. A knowledge state is a set gathering some of these atoms. Each of the knowledge states in a learning space is intended as a possible representation of some individual's competence in the topic. Embedded in a suitable stochastic framework, the concept of a learning space provides a mechanism for the assessment of knowledge in the sense that efficient questioning of a subject on a well chosen subset of atoms leads to gauge his or her knowledge state[1]. Many aspects of these structures have been investigated and the results were reported in various publications; for a sample, see

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[1]As such, it offers an alternative to standardized testing, the theoretical basis of which is fundamentally different and based on the measurement of aptitudes (see Nunnally and Bernstein, 1994, for example).

Doignon and Falmagne (1985); Falmagne and Doignon (1988a); Albert and Lukas (1999); Falmagne et al. (2006). The monograph by Doignon and Falmagne (1999) contains most of the results up to that date[2].

In practice, in an educational context for example, a learning space can be quite large, sometimes numbering millions of states. The concept of a 'projection' at the core of this paper provides a way of parsing such a large structure into meaningful components. Moreover, when the learning space concerns a scholarly curriculum such as high school algebra, a projection may provide a convenient instrument for the programming of a placement test. For the complete algebra curriculum comprising several hundred types of problems, a placement test of a few dozens problems can be manufactured automatically via a well chosen projection.

The key idea is that if $\mathcal{K}$ is a learning space on a domain Q, then any subset Q' of Q defines a learning space $\mathcal{K}_{|Q'}$ on Q' which is consistent with $\mathcal{K}$. We call $\mathcal{K}_{|Q'}$ a 'projection' of $\mathcal{K}$ on Q', a terminology consistent with that used by Cavagnaro (2008) and Eppstein et al. (2007) for media. Moreover, this construction defines a partition of $\mathcal{K}$ such that each equivalence class is a subfamily of $\mathcal{K}$ satisfying some of the key properties of a learning space. In fact, Q' can be chosen so that each of these equivalence classes is essentially (via a trivial transformation) either a learning space consistent with $\mathcal{K}$ or the singleton $\{\varnothing\}$.

These results, entitled 'Projection Theorems' (13 and 16), are formulated in this paper. They could be derived from corresponding results for the projections of media (Cavagnaro, 2008; Falmagne and Ovchinnikov, 2002; Eppstein et al., 2007). Direct proof are given here. This paper extends previous results from Doignon and Falmagne (1999, Theorem 1.16 and Definition 1.17) and Cosyn (2002).

Basic Concepts

1 Definition. We denote by $K \triangle L = (K \setminus L) \cup (L \setminus K)$ the symmetric difference between two sets K, L, and by $d(K, L) = |K \triangle L|$ the symmetric difference distance between these sets. (All the sets considered in this chapter are finite.) The symbols "+ and $\subset$ stand for the disjoint union and the proper inclusion of sets respectively. A *(knowledge) structure* is a pair $(Q, \mathcal{K})$ where Q is a non empty set and $\mathcal{K}$ is a family of subsets of Q containing $\varnothing$

[2]An extensive database on knowledge spaces, with hundreds of titles, is maintained by Cord Hockemeyer at the University of Graz: http://wundt.uni-graz.at/kst.php (see also Hockemeyer, 2001).

and $Q = \cup\mathcal{K}$. The latter is called the *domain* of $(Q, \mathcal{K})$. The elements of Q are called *items* and the sets in $\mathcal{K}$ are *(knowledge) states*. Since $Q = \cup\mathcal{K}$, the set Q is implicitly defined by $\mathcal{K}$ and we can without ambiguity call $\mathcal{K}$ a knowledge structure. A knowledge structure $\mathcal{K}$ is *well–graded* if for any two distinct states K and L with $|K \triangle L| = n$ there exists a sequence $K_0 = K, K_1, \ldots, K_n = L$ such that $d(K, L) = n$ and $d(K_i, K_{i+1}) = 1$ for $0 \leq i \leq n-1$. We call such a sequence $K_0 = K, K_1, \ldots, K_n = L$ a *tight path* from K to L. We say that $\mathcal{K}$ is a *knowledge space* if it is closed under union, or $\cup$-*closed*.

A knowledge structure $(Q, \mathcal{K})$ is a *learning space* (cf. Cosyn and Uzun, 2008) if it satisfies the following two conditions:

[L1] Learning smoothness. For any two $K, L \in \mathcal{K}$, with $K \subset L$ and $|L \setminus K| = n$, there is a chain $K_0 = K \subset K_1 \subset \cdots \subset K_n = L$ such that, for $0 \leq i \leq n-1$, we have $K_{i+1} = K_i + \{q_i\} \in \mathcal{K}$ for some $q_i \in Q$.

[L2] Learning consistency. If $K \subset L$ are two sets in $\mathcal{K}$ such that $K + \{q\} \in \mathcal{K}$ for some $q \in Q$, then $L \cup \{q\} \in \mathcal{K}$.

A learning space is also known in the combinatorics literature as an 'antimatroid', a structure introduced independently by Edelman and Jamison (1985) with slightly different, but equivalent axioms (cf. also Welsh, 1995; Björner et al., 1999). Another name is 'well–graded knowledge space' (Falmagne and Doignon, 1988b); see our Lemma 10.

A family $\mathcal{F}$ of subsets of a set Q is a *partial knowledge structure* if it contains the set $Q = \cup\mathcal{F}$. We do not assume that $|\mathcal{F}| \geq 2$. We also call 'states' the sets in $\mathcal{F}$. A partial knowledge structure $\mathcal{F}$ is a *partial learning space* if it satisfies Axioms [L1] and [L2]. Note that $\{\varnothing\}$ is vacuously well-graded and vacuously $\cup$-closed, with $\cup\{\varnothing\} = \varnothing$. Thus, it a partial knowledge structure and a partial learning space (cf. Lemma 11).

The following preparatory result will be helpful in shortening some proofs.

2 Lemma. *A $\cup$-closed family of set $\mathcal{K}$ is well–graded if, for any two sets $K \subset L$, there is a tight path from K to L.*

Proof. Suppose that the condition holds. For any two distinct sets K and L, there exists a tight path $K_0 = K \subset K_1 \subset \ldots \subset K_n = K \cup L$ and another tight path $L_0 = L \subset L_1 \subset \ldots \subset L_m = K \cup L$. These two tight paths can be concatenated. Reversing the order of the sets in the latter tight path and redefining $K_{n+1} = L_{m-1}, K_{n+2} = L_{m-2}, \ldots, K_{n+m} = L_0 = L$ we get the tight path $K_0 = K, K_1, \ldots, K_{n+m} = L_0 = L$, with $|K \triangle L| = n + m$. □

Projections

As mentioned in our introduction, some knowledge structures may be so large that a splitting is required, for convenient storage in a computer's memory for example. Also, in some situations, only a representative part of a large knowledge structure may be needed. The concept of a projection is of critical importance in this respect. We introduce a tool for its construction.

3 Definition. Let $(Q, \mathcal{K})$ be a partial knowledge structure with $|Q| \geq 2$ and let Q' be any proper non empty subset of Q. Define a relation $\sim_{Q'}$ on $\mathcal{K}$ by

$$K \sim_{Q'} L \iff K \cap Q' = L \cap Q' \tag{1}$$

$$\iff K \triangle L \subseteq Q \setminus Q'. \tag{2}$$

Thus, $\sim_{Q'}$ is an equivalence relation on $\mathcal{K}$. When the context specifies the subset Q', we sometimes use the shorthand $\sim$ for $\sim_{Q'}$ in the sequel. The equivalence between the right hand sides of (1) and (2) is easily verified. We denote by $[K]$ the equivalence class of $\sim$ containing K, and by $\mathcal{K}_\sim = \{[K] \,|\, K \in \mathcal{K}\}$ the partition of $\mathcal{K}$ induced by $\sim$. We may say for short that such a partition is induced by the set Q'. In the sequel we always assume that $|Q| \geq 2$, so that $|Q'| \geq 1$.

4 Definition. Let $(Q, \mathcal{K})$ be a partial knowledge structure and take any non empty proper subset Q' of Q. The family

$$\mathcal{K}_{|Q'} = \{W \subseteq Q \,|\, W = K \cap Q' \text{ for some } K \in \mathcal{K}\} \tag{3}$$

is called the *projection* of $\mathcal{K}$ on Q'. We have thus $\mathcal{K}_{|Q'} \subseteq 2^{Q'}$. As shown by Example 5, the sets in $\mathcal{K}_{|Q'}$ may not be states of $\mathcal{K}$. For any state K in $\mathcal{K}$ and with $[K]$ as in Definition 3, we define the family

$$\mathcal{K}_{[K]} = \{M \,|\, M = L \setminus \cap[K] \text{ for some } L \sim K\}. \tag{4}$$

(If $\varnothing \in \mathcal{K}$, we have thus $\mathcal{K}_{[\varnothing]} = [\varnothing]$.) The family $\mathcal{K}_{[K]}$ is called a *Q'-child*, or simply a *child* of $\mathcal{K}$ (*induced by Q'*). As shown by the example below, a child of $\mathcal{K}$ may take the form of the singleton $\{\varnothing\}$ and we may have $\mathcal{K}_{[K]} = \mathcal{K}_{[L]}$ even when $K \not\sim L$. The set $\{\varnothing\}$ is called the *trivial* child.

5 Example. Equation (5) defines a learning space $\mathcal{F}$ on the domain $Q = \{a, b, c, d, e, f\}$:

$$\mathcal{F} = \{\varnothing, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}, \{b, c, e\}, \{b, d, f\},$$
$$\{a, b, c, d\}, \{a, b, c, e\}, \{b, c, d, e\}, \{b, c, d, f\}, \{b, c, e, f\}, \{a, b, d, f\}, \{a, b, c, d, e\},$$
$$\{a, b, c, d, f\}, \{a, b, c, e, f\}, \{b, c, d, e, f\}, \{a, b, c, d, e, f\}, \{a, b, c, d, e, f, g\}\}. \tag{5}$$

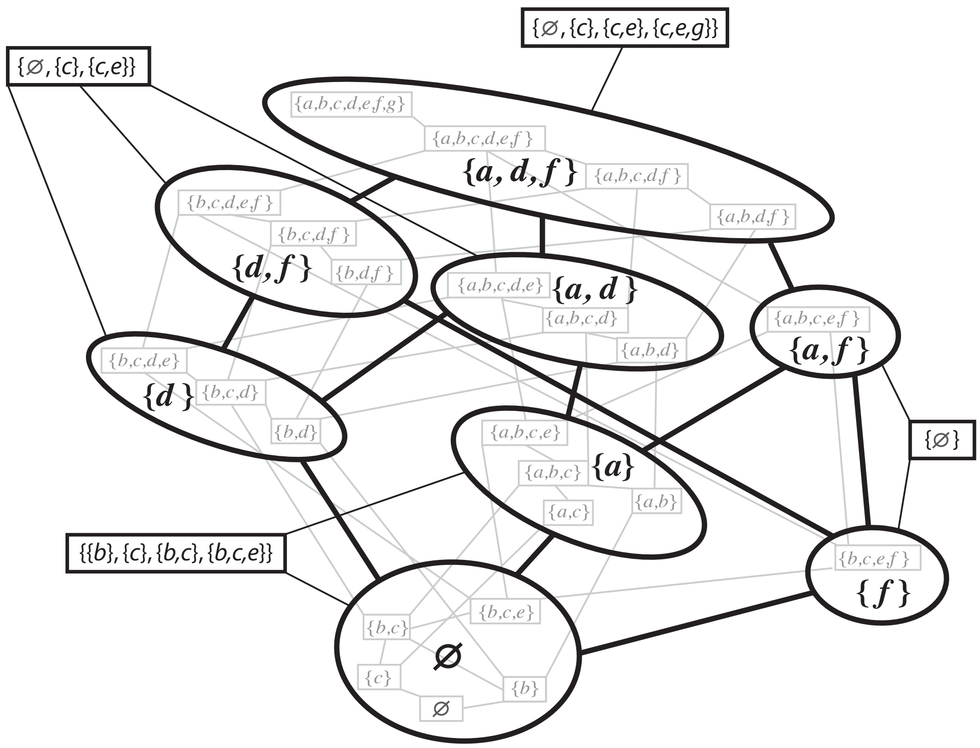

Figure 1: In grey, the inclusion graph of the learning space $\mathcal{F}$ of Equation (5). Each oval surrounds an equivalence class $[K]$ (in grey) and a particular state (in black) of the projection $\mathcal{F}_{|\{a,d,f\}}$ of Q on $Q' = \{a, d, f\}$, signaling a 1-1 mapping $\mathcal{F}_{\sim} \to \mathcal{F}_{|\{a,d,f\}}$ (cf. Lemma 7(ii)). Via the defining equation (4), the eight equivalence classes produce four children of $\mathcal{F}$, which are represented in the black rectangles of the figure. One of these children is the singleton set $\{\varnothing\}$ (thus, a trivial child), and the others are learning spaces or partial learning spaces (cf. Projection Theorems 13 and 16).

The inclusion graph of this learning space is pictured by the grey parts of the diagram of Figure 1. The sets marked in black in the eight ovals of the figure represents the states of the projection $\mathcal{F}_{|\{a,d,f\}}$ of $\mathcal{F}$ on the set $\{a, d, f\}$. It is clear that $\mathcal{F}_{|\{a,d,f\}}$ is a learning space[3]. Each of these ovals also surrounds the inclusion subgraph corresponding to an equivalence class of the partition $\mathcal{F}_{\sim}$. This is consistent with Lemma 7 (ii) according to which there is a

[3]In fact, $\mathcal{F}_{|\{a,d,f\}} = 2^{\{a,d,f\}}$ in this particular case. This property does hold in general.

1-1 correspondence between $\mathcal{F}_\sim$ and $\mathcal{F}_{|\{a,d,f\}}$. In this example, the 'learning space' property is hereditary in sense that not only is $\mathcal{F}_{|\{a,d,f\}}$ a learning space, but also any child of $\mathcal{F}$ is a learning space or a partial learning space. Indeed, we have

$$\mathcal{F}_{[\{b,c,e\}]} = \mathcal{F}_{[\{a,b,c,e\}]} = \{\{b\},\{c\},\{b,c\},\{b,c,e\}\},$$
$$\mathcal{F}_{[\{b,c,d,e\}]} = \mathcal{F}_{[\{b,c,d,e,f\}]} = \mathcal{F}_{[\{a,b,c,d,e\}]} = \{\varnothing,\{c\},\{c,e\}\},$$
$$\mathcal{F}_{[\{a,b,c,d,e,f,g\}]} = \{\varnothing,\{c\},\{c,e\},\{c,e,g\}\}$$
$$\mathcal{F}_{[\{b,c,e,f\}]} = \mathcal{F}_{[\{a,b,c,e,f\}]} = \{\varnothing\}.$$

These four children are represented in the four black rectangles in Figure 1.

Theorem 13 shows that this hereditary property is general in the sense that the children of a partial learning space are always partial learning spaces. In the particular case of this example, just adding the set $\{\varnothing\}$ to the child not containing it already, that is, to the child $\mathcal{F}_{[\{b,c,e\}]} = \mathcal{F}_{[\{a,b,c,e\}]}$, would result in having all the children being learning spaces or trivial. This is **not** generally true. The situation is clarified by Theorem 16.

6 Remark. The concept of projection for learning spaces is closely related to the concept bearing the same name for media introduced by Cavagnaro (2008). The Projection Theorems 13 and 16, the main results of this chapter, could be derived via similar results concerning the projections of media (cf. Theorem 2.11.6 in Eppstein et al., 2007). This would be a detour, however. The route followed here is direct.

In the next two lemmas, we derive a few consequences of Definition 4.

7 Lemma. *The following two statements are true for any partial knowledge structure* $(Q,\mathcal{K})$.

(i) *The projection* $\mathcal{K}_{|Q'}$, *with* $Q' \subset Q$, *is a partial knowledge structure. If* $(Q,\mathcal{K})$ *is a knowledge structure, then so is* $\mathcal{K}_{|Q'}$.

(ii) *The function* $h : [K] \mapsto K \cap Q'$ *is a well defined bijection of* $\mathcal{K}_\sim$ *onto* $\mathcal{K}_{|Q'}$.

Proof. (i) Both statements stem from the observations that $\varnothing \cap Q' = \varnothing$ and $Q \cap Q' = Q'$.

(ii) That h is a well defined function is due to (1). It is clear that $h(\mathcal{K}_\sim) = \mathcal{K}_{|Q'}$ by the definitions of h and $\mathcal{K}_{|Q'}$. Suppose that, for some $[K],[L] \in \mathcal{K}_\sim$, we have $h([K]) = K \cap Q' = h([L]) = L \cap Q' = X$. Whether or not $X = \varnothing$, this entails $K \sim L$ and so $[K] = [L]$. □

8 Lemma. *If $\mathcal{K}$ is a $\cup$-closed family, then the following three statements are true.*

(i) $K \sim \cup[K]$ *for any* $K \in \mathcal{K}$.

(ii) $\mathcal{K}_{|Q'}$ *is a* $\cup$*-closed family. If* $\mathcal{K}$ *is a knowledge space, so is* $\mathcal{K}_{|Q'}$.

(iii) *The children of* $\mathcal{K}$ *are also* $\cup$*-closed.*

In (ii) *and* (iii)*, the converse implications are not true.*

For knowledge spaces, Lemma 8 (ii) was obtained by Doignon and Falmagne (1999, Theorem 1.16 on p. 25) where the concept of a projection was referred to as a 'substructure.' Their proof applies here. We include it for completeness.

PROOF. (i) As $\cup[K]$ is the union of states of $\mathcal{K}$, we get $\cup[K] \in \mathcal{K}$. We must have $K \cap Q' = (\cup[K]) \cap Q'$ because $K \cap Q' = L \cap Q'$ for all $L \in [K]$; so $K \sim \cup[K]$.

(ii) Since $\cup\mathcal{K} \in \mathcal{K}$ by hypothesis, $\mathcal{K}$ is a knowledge structure. Lemma 7(i), implies that $\mathcal{K}_{|Q'}$ is a partial knowledge structure. Any subfamily $\mathcal{H} \subseteq \mathcal{K}_{|Q'}$ is associated to the family $\mathcal{H}' = \{H' \in \mathcal{K} \,|\, H = H' \cap Q' \text{ for some } H \in \mathcal{H}\}$. As $\mathcal{K}$ is a partial knowledge space, we get $\cup\mathcal{H}' \in \mathcal{K}$, yielding $Q' \cap (\cup\mathcal{H}') \in \mathcal{K}_{|Q'}$, with

$$Q' \cap \big(\cup\mathcal{H}'\big) = \cup_{H' \in \mathcal{H}'}(H' \cap Q') = \cup\mathcal{H}.$$

Thus $\mathcal{K}_{|Q'}$ is a partial knowledge space. This argument is valid for knowledge spaces.

(iii) Take $K \in \mathcal{K}$ arbitrarily. We must show that $\mathcal{K}_{[K]}$ is $\cup$-closed. If $\mathcal{K}_{[K]} = \{\varnothing\}$, this is vacuously true. Otherwise, for any $\mathcal{H} \subseteq \mathcal{K}_{[K]}$ we define the associated family

$$\mathcal{H}' = \{H' \in \mathcal{K} \,|\, H' \sim K,\, H' \setminus \cap[K] \in \mathcal{H}\}.$$

So, $\mathcal{H}' \subseteq [K]$, which gives $L \cap Q' = K \cap Q'$ for any $L \in \mathcal{H}'$. We get thus $\cup\mathcal{H}' \sim K$.

Since $\mathcal{K}$ is $\cup$-closed, we have $\cup\mathcal{H}' \in \mathcal{K}$. The $\cup$-closure of $\mathcal{K}_{[K]}$ follows from the string of equalities

$$\cup\mathcal{H} = \cup_{H' \in \mathcal{H}'}(H' \setminus \cap[K]) = \cup_{H' \in \mathcal{H}'}(H' \cap (\overline{\cap[K]}) = (\cup_{H' \in \mathcal{H}'} H') \setminus \cap[K]$$

which gives $\cup\mathcal{H} \in \mathcal{K}_{[K]}$ because $K \sim \cup\mathcal{H}' \in \mathcal{K}$.

Example 9 shows that the reverse implications in (ii) and (iii) do not hold. □

9 Example. Consider the projection of the knowledge structure

$$\mathcal{G} = \big\{\, \varnothing,\ \{a\},\ \{b\},\ \{c\},\ \{a,b\},\ \{a,c\},\ \{a,b,c\}\big\},$$

on the subset $\{c\}$. We have thus the two equivalence classes $[\{a,b\}]$ and $[\{a,b,c\}]$, with the projection $\mathcal{G}_{|\{c\}} = \{\varnothing, \{c\}\}$. The two $\{c\}$-children are $\mathcal{G}_{[\varnothing]} = \big\{\varnothing, \{a\}, \{b\}, \{a,b\}\big\}$ and $\mathcal{G}_{[\{c\}]} = \big\{\, \varnothing,\ \{a\}, \{a,b\}\,\big\}$. Both $\mathcal{G}_{[\varnothing]}$ and $\mathcal{G}_{[\{c\}]}$ are well-graded and $\cup$-closed, and so is $\mathcal{G}_{|\{c\}}$, but $\mathcal{G}$ is not since $\{b,c\}$ is not a state.

We omit the proof of the next result which is due to Cosyn and Uzun (2008) .

10 Lemma. *A knowledge structure $(Q, \mathcal{K})$ is a learning space if and only if it a well–graded knowledge space.*

As indicated by the next lemma, the equivalence ceases to hold in the case of partial spaces.

11 Lemma. *Any well-graded $\cup$-closed family is a partial learning space. The converse implication is false.*

Proof. Let $\mathcal{K}$ be a well-graded $\cup$-closed family. Axiom [L1] is a special case of the well-gradedness condition. If $K \subset L$ for two sets K and L in $\mathcal{K}$ and $K + \{q\}$ is in $\mathcal{K}$, then the set $(K + \{q\}) \cup L = L \cup \{q\}$ is in $\mathcal{K}$ by $\cup$–closure, and so [L2] holds. The example below disproves the converse. □

12 Example. The family of sets

$$\mathcal{L} = \{\{a,b,c\}, \{c,d,e\}, \{a,b,c,f\}, \{c,d,e,g\}, \{a,b,c,f,d\}, \{c,d,e,g,b\},$$
$$\{a,b,c,f,d,e\}, \{c,d,e,g,b,a\}, \{a,b,c,d,e,f,g\}\}$$

is a partial learning space since it is the union of the two chains

$$\{a,b,c\} \subset \{a,b,c,f\} \subset \{a,b,c,f,d\} \subset \{a,b,c,f,d,e\} \subset \{a,b,c,d,e,f,g\},$$
$$\{c,d,e\} \subset \{c,d,e,g\} \subset \{c,d,e,g,b\} \subset \{c,d,e,g,b,a\} \subset \{a,b,c,d,e,f,g\}$$

with the only common set $\cup\mathcal{L}$. However, $\mathcal{L}$ is neither $\cup$-closed nor well-graded.

We state the first of our two projection theorems.

13 Projection Theorem. *Let $\mathcal{K}$ be a learning space (resp. a well-graded $\cup$-closed family) on a domain Q with $|Q| = |\cup\mathcal{K}| \geq 2$. The following two properties hold for any proper non empty subset Q' of Q:*

(i) *The projection $\mathcal{K}_{|Q'}$ of $\mathcal{K}$ on Q' is a learning space (resp. a well-graded $\cup$-closed family);*

(ii) *In either case, the children of $\mathcal{K}$ are well-graded and $\cup$-closed families.*

Note that we may have $\mathcal{K}_{[K]} = \{\varnothing\}$ in (ii) (cf. Example 5).

PROOF. (i) If $\mathcal{K}$ is a learning space, then $\mathcal{K}_{|Q'}$ is a knowledge structure by Lemma 7(i). Since $\mathcal{K}$ is $\cup$-closed by Lemma 10, so is $\mathcal{K}_{[K]}$ by Lemma 8 (ii). It remains to show that $\mathcal{K}_{[K]}$ is well-graded. (By Lemma 10 again, this will imply that $\mathcal{K}_{|Q'}$ is a learning space.) We use use Lemma 2 for this purpose. Take any two states $K' \subset L'$ in $\mathcal{K}_{|Q'}$ with $d(K', L') = n$ for some positive integer n. By Lemma 7 (ii), we have thus $K' = K \cap Q'$ and $L' = L \cap Q'$ for some $K, L \in \mathcal{K}$. As $\mathcal{K}$ well-graded by Lemma 10, there exists a tight path $K_0 = K, K_1, \ldots, K_m = L$ from K to L, with either $K_j = K_{j-1} + \{p_j\}$ or $K_{j-1} = K_j + \{p_j\}$ for some $p_j \in Q$ and $1 \leq i \leq m$. Let $j \in \{1, \ldots, m\}$ be the first index such that $p_j \in Q'$. We have then necessarily $p_j \in Q' \cap L = L'$ and $K_j = K_{j-1} + \{p_j\}$. This yields

$$K' = K_0 \cap Q' = K_1 \cap Q' = \ldots = K_{j-1} \cap Q',$$

and for $1 \leq j \leq n$,

$$K_j \cap Q' = (K_{j-1} \cap Q') + \{p\},$$

with $p \in L' \setminus K'$. Defining $K'_0 = K'$ and $K'_1 = K_j \cap Q'$, we have $d(K'_1, L') = n - 1$, with K'_1 a state of $\mathcal{K}_{|Q'}$. By induction, we conclude that $\mathcal{K}_{|Q'}$ a wellgraded $\cup$ closed knowledge structure. Since $\mathcal{K}_{|Q'}$ is $\cup$-closed, it must be a learning space by Lemma 10.

Suppose now that $\mathcal{K}$ is a well-graded $\cup$-closed family (rather than learning space). In such a case, there is no need to invoke Lemma 10 and the above argument can be used to prove that $\mathcal{K}_{|Q'}$ a wellgraded $\cup$ closed family.

(ii) Take any child $\mathcal{K}_{[K]}$ of $\mathcal{K}$. By Lemma 8(iii), $\mathcal{K}_{[K]}$ is a $\cup$-closed family. We use Lemma 2 to prove that $\mathcal{K}_{[K]}$ is also well-graded. Take any two states $M \subset L$ in $\mathcal{K}_{[K]}$. We have thus $L = L' \setminus (\cap[K])$ and $M = M' \setminus (\cap[K])$ for some L' and M' in $[K]$, with

$$\cap\,[K] \subseteq L' \subset M'. \tag{6}$$

Since $\mathcal{K}$ is well-graded, there is a tight path

$$L'_0 = L' \subset L'_1 \subset \ldots \subset L'_n = M' \tag{7}$$

with all its states in $[K]$. Indeed, $L' \subset L'_j \subset M'$ and $L' \cap Q' = M' \cap Q'$ imply $L' \cap Q' = L'_j \cap Q' = M' \cap Q'$ for any index $1 \leq j \leq n-1$. We now define the sequence $L_j = L'_j \setminus \cap[K]$, $0 \leq j \leq n$. It is clear that (6) and (7) imply

$$L_0 = L \subset L_1 \subset \ldots \subset L_n = M,$$

and it is easily verified that $L_0 = L, L_1, \ldots, L_n = M$ is a tight path from L to M. Applying Lemma 2, we conclude that $\mathcal{K}_{[K]}$ is well-graded. □

14 Remark. In Example 5, we had a situation in which the non trivial children of a learning space were either themselves learning spaces, or would become so by the addition of the set $\{\varnothing\}$. This can happen if and only if the subset Q' of the domain defining the projection satisfies the condition spelled out in the next definition.

15 Definition. Suppose that $(Q, \mathcal{K})$ is a partial knowledge structure, with $|Q| \geq 2$. A subset $Q' \subset Q$ is *yielding* if for any state L of $\mathcal{K}$ that is minimal for inclusion in some equivalence class $[K]$, we have $|L \setminus \cap[K]| \leq 1$. We recall that $[K]$ is the equivalence class containing K in the partition of $\mathcal{K}$ induced by Q' (cf. Definition 3). For any non trivial child $\mathcal{K}_{[K]}$ of $\mathcal{K}$, we call $\mathcal{K}^+_{[K]} = \mathcal{K}_{[K]} \cup \{\varnothing\}$ a *plus* child of $\mathcal{K}$.

16 Projection Theorem. *Suppose that $(Q, \mathcal{K})$ is a learning space with $|Q| \geq 2$, and let Q' be a proper non empty subset of Q. The two following conditions are then equivalent.*

(i) *The set Q' is yielding.*

(ii) *All the plus children of $\mathcal{K}$ are learning space*[4].

(It is easily shown that any learning space has always at least on non trivial child.)

PROOF. (i) $\Rightarrow$ (ii). By Lemma 8(iii), we know that any non trivial child $\mathcal{K}_{[K]}$ is $\cup$-closed. This implies that the associated plus child $\mathcal{K}^+_{[K]}$ is a knowledge space. We use Lemma 2 to prove that $\mathcal{K}^+_{[K]}$ is also well–graded. Suppose that L and M are states of $\mathcal{K}^+_{[K]}$, with $\varnothing \subseteq L \subset M$ and, say $d(L, M) = n$.

CASE 1. Suppose that $L \neq \varnothing$. By the definition of $\mathcal{K}^+_{[K]}$, we have $L = L' \setminus \cap[K]$ and $M = M' \setminus \cap[K]$ for some $L', M' \in [K]$. Since $L \subset M$, we must have $L' \subset M'$. Because L'

[4]Note that we may have $\{\varnothing\} \in \mathcal{K}_{[K]}$, in which case $\mathcal{K}^+_{[K]} = \mathcal{K}_{[K]}$ (cf. Example 5).

and M' are sets of $[K]$, there exists in $[K]$ (by the Projection Theorem 13 (ii)) a tight path $L'_0 = L' \subset L'_1 \subset \ldots \subset L'_m = M'$. We show that this tight path defines a tight path

$$L_0 = L, L_1, \ldots, L_n = M \tag{8}$$

lying entirely in $\mathcal{K}^+_{[K]}$ (actually, in $\mathcal{K}_{[K]}$). By definition of a tight path, we have $L'_0 + \{p_1\} = L'_1$ for some $p_1 \in Q \setminus Q'$. Defining $L_1 = L'_1 \setminus \cap[K]$, we get $L_0 + \{p_1\} = L_1$ and thus $d(L_1, M) = n - 1$. Note that L_1 is in $\mathcal{K}^+_{[K]}$ because L'_1 is is $[K]$. The existence of the tight path (8) follows by induction.

Case 1. Suppose now that $L = \varnothing$. In view of what we just proved, we only have to show that, for any non empty $M \in \mathcal{K}^+_{[K]}$, there is a singleton set $\{q\} \in \mathcal{K}^+_{[K]}$ with $q \in M$. By definition of $\mathcal{K}^+_{[K]}$, we have $M = M' \setminus \cap[K]$ for some $M' \in [K]$. Take a minimal state N in $[K]$ such that $N \subseteq M'$ and so $N \setminus \cap[K] \subseteq M$. Since Q' is yielding, we get $|N \setminus \cap[K]| \leq 1$. If $|N \setminus \cap[K]| = 1$, then $N \setminus \cap[K] = \{q\} \subseteq M$ for some $q \in Q$ with $\{q\} \in \mathcal{K}^+_{[K]}$. Suppose that $|N \setminus \cap[K]| = 0$. Thus $N \setminus \cap[K] = \varnothing$ and N must be the only minimal set in $[K]$, which implies that $\cap[K] = N$. By the wellgradedness of $[K]$ established in the Projection Theorem 13(ii), there exists some $p \in M$ such that $N + \{p\} \subseteq M$. We get thus

$$(N + \{p\}) \setminus \cap[K] = (N + \{p\}) \setminus N = \{p\} \subseteq M \quad \text{with} \quad \{p\} \in \mathcal{K}^+_{[K]}.$$

The tight path (8) from L to M exists thus in both cases. Applying Lemma 2, we can assert that $\mathcal{K}^+_{[K]}$ is well-graded. We have shown earlier that $\mathcal{K}^+_{[K]}$ is a knowledge space. Accordingly, the plus child $\mathcal{K}^+_{[K]}$ is a learning space.

(ii) $\Rightarrow$ (i). If some equivalence class $[K]$ is a chain or is a single set[5] $\{K\} = [K]$, then $|K \setminus \cap[K]| = 0$. Otherwise $[K]$ contains more than one minimal state. Let L be one of these minimal states. Thus $L \setminus \cap[K]$ is a minimal non empty set of $\mathcal{K}^+_{[K]}$, which by hypothesis is a learning space. By the wellgradedness of $\mathcal{K}^+_{[K]}$, there is a tight path from $\varnothing$ to $L \setminus \cap[K]$. Because $L \setminus \cap[K]$ is non empty and minimal in $\mathcal{K}_{[K]}$, it must be a singleton. We get thus $|L \setminus \cap[K] = 1$. □

Summary

Performing an assessment in a large learning space $(Q, \mathcal{K})$ may be impractical in view of memory limitation or for other reasons. In such a case, a two-step or an n-step procedure

[5] We saw in Example 5 that this is possible.

may be applied. On Step 1, a representative subset Q' of items from the domain Q is selected, and an assessment is performed on the projection learning space $\mathcal{K}_{|Q'}$ induced by Q' (cf. Projection Theorem 13(i)). The outcome of this assessment is some knowledge state $K \cap Q'$ of $\mathcal{K}_{|Q'}$ which corresponds (1-1) to equivalence class $[K]$ of the partition of $\mathcal{K}$ induced by Q' (cf. Lemma 7 (ii)). On Step 2, the child $\mathcal{K}_{[K]}$ is formed by removing all the common items in the states of $[K]$. The assessment can then be pursued on $\mathcal{K}_{[K]}$ of $\mathcal{K}$ which is a partial learning space (cf. Projection Theorem 13(ii)). The outcome of Step 2 is a set $L \setminus \cap [K]$, where L is a state in the learning space $\mathcal{K}$. This 2-step procedure can be expanded into a n-step recursive algorithm if necessary.

If the set Q' is yielding, then any equivalence class $[K]$containing more than one set can be made into a learning space by a trivial transformation. This property is not critical for the 2-step procedure outlined above.

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