Title
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Publication Date
2011

Peer reviewed|Thesis/dissertation

Modeling the co-development of strategic and conceptual knowledge in mathematical problem solving

By

Mariana Elaine Campbell

A dissertation submitted in partial satisfaction of the

requirements for the degree of

Doctor of Philosophy

in

Science and Mathematics Education

in the

Graduate Division

of the

University of California, Berkeley

Committee in charge:

Professor Alan H. Schoenfeld, Chair
Professor Andrea A. diSessa
Professor Geoffrey B. Saxe

Fall 2011

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Abstract

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This dissertation explores the question of how strategic and conceptual knowledge co-develop over the course of several episodes of mathematical problem solving. The core analytic work involves an in-depth microgenetic case study of a single pre-algebra student, Liam, who over six hours of videotaped interaction with a tutor/researcher constructs a deterministic and essentially algebraic algorithm for solving algebra word problems that have an underlying linear structure. Over six hours of videotaped interaction with a tutor/researcher, Liam's later and conceptually more sophisticated strategy is seen to emerge as a gradual refinement of his initial strategy. This focal case study is used to develop a theoretical model of how strategic and conceptual knowledge co-evolve. A novel aspect of the present analysis is that both strategies and the knowledge needed to implement them in problem solving are modeled as complex knowledge systems. The analytic methodology employed in developing the theoretical model is a coordination of *Knowledge Analysis* (diSessa, 1993; Sherin, 2001) and *Microgenetic Learning Analysis* (Parnafes & diSessa, submitted; Schoenfeld, Smith, & Arcavi, 1993). The model of co-development of strategic and conceptual knowledge that is developed through the analysis is one of mutual bootstrapping: (1) Within a given strategic frame, a solver activates a particular projection of conceptual knowledge and (2) As the solver creates new conceptual schemes in the context of working within a given particular strategic frame, novel refinements to existing strategies can emerge.

This dissertation is dedicated to my husband, Aaron

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Acknowledgments

This dissertation is the product of my interactions with many people, and though it is impossible to adequately express in words the depth of gratitude I feel, I would like to at least try to communicate my appreciation.

Firstly, I thank Liam for his enthusiasm and willingness to participate in this research that proved to be such a productive point of departure for an empirical investigation into the nature of knowing and learning mathematics.

I sincerely thank my committee members, Alan Schoenfeld, Andy diSessa, and Geoff Saxe, for the wonderful learning opportunities they provided me throughout my training. The mentorship I received from each of them has been so perfectly tuned to my needs and abilities. They were each at once challenging and constructive in their feedback and relentless in their support, trust, and patience. For this individualized and personal treatment, I am very grateful. I am humbled to have such amazing role models to emulate both in scholarship and mentorship. I would like to say just a few words in thanks about each of them.

Alan fully embodies the ideal of a scholar who makes important theoretical contributions while also substantially contributing to improving the learning and teaching of mathematics at local, national, and international levels. Alan has been an exemplary advisor for me – helping me to make a comfortable transition from the mathematics department to the world of education research and pushing my thinking in many ways through our joint work in the Diversity in Mathematics Education Project and the Algebra Teaching Study. I appreciate his dedication and diligence as a mentor and how steadfastly he has encouraged me in developing my own identity as a researcher.

Geoff has played many different roles throughout my graduate career. His theoretical ideas about form-function shifts, emergent goals, and coordinated genetic analyses have inspired me from the very beginning of graduate school and though not always fore-grounded in the analysis, have continued to shape my thinking throughout the analytic process. I greatly appreciate the support I received through the IES Research in Cognition and Mathematics Education (RCME) Research Training Grant program at Berkeley, headed by Na'ilah Nasir and Geoff and I would also like to thank both Geoff and Na'ilah for the wonderful professional opportunity to design and co-teach the introductory doctoral course "Survey of current research and issues in mathematics education" at UC Berkeley.

Finally, I wish to warmly thank Andy diSessa, whose mentorship has played a strong role in shaping me personally and professionally. I owe a large intellectual debt to his theoretical ideas that have been so influential in my own thinking. I was in dialogue with Andy throughout the entire period in which I was developing the analysis presented in this dissertation and our conversations strongly shaped the final version of the dissertation, both conceptually and rhetorically. In addition to the work on my dissertation, working together on organizing conferences and symposia sessions all around the world on the topics that are near to my heart – developing theory and methods for studying knowledge and learning – has truly been one of the great pleasures of graduate school and some of the most formative professional experiences for me.

I wish to also thank Randi Engle, who was very involved in supporting my development as a researcher, especially in the early phases of this research. Randi's tremendous enthusiasm and keen attention to detail have been so inspirational to me throughout graduate school. I admire her intensity and absolute dedication as a scholar.

When I first started out graduate school, I had no idea how central the theme of "community" would come to be in bringing this dissertation to fruition. Being part of multiple, intergenerational communities has always been a part of my identity, and perhaps the experience of trying to work on a Ph.D. while moving all over the world, made the need for communities of all sorts more salient and poignant. I'm so grateful for all of the wonderful people in my life whose part in this journey I treasure.

Penny Bartlett – Thank you for welcoming me into your home and into your life. You have got to be one of the most gifted listeners who ever lived! Many well-timed cups of tea and long chats in your living room helped me to make my way through this process.

Janet Casperson – We truly did all of the ups and downs of graduate school together, sharing many delightful memories along the way including morning walks along the Ohlone trail, costume parties, and innumerable craft nights. Your relentless efforts to keep me connected with Berkeley even when I was in Providence or Pisa or East Lansing made it possible for me to finish this dissertation.

Ann Edwards – Did you know that you were the first person I talked with about SESAME? If it hadn't been for that conversation, perhaps I wouldn't have done this program! Thanks, also, for putting a violin back in my hands after several years away from serious practicing. It truly changed my life for the better to have violin back in my life. Thanks for many happy hours of playing as co-concertmasters of Prometheus and for the chamber music jam sessions. I still can't believe that your neighbors never complained about our Shostakovich quartet sight-reading sessions.

Katie Lewis – Our late-night writing and editing sessions of 210 papers into the wee hours were some of my first memories of grad school and our writing boot camp and daily check-ins were some of my last memories of grad school. Thanks for teaching me the discipline of apples and hours.

Marianne Noland – There is so much to thank you for! Thanks for including me in your fabulous summertime BBQ's, your Hanukkah parties, for caring for me when I was sick and when Aaron was out of town, and for all of our impromptu excursions from MoMA to the opera. So much of what Berkeley is to me was shaped through my early experiences in Berkeley here with you.

Orit Parnafes – How could we have predicted back in Geoff's Piaget & Vygotsky class or Andy's scientific writing seminar that we would become such close friends and colleagues? I've learned so much from you and from our collaborations. One of the things I admire most is your uncompromising commitment to both quality and depth in your work. Among many, many pleasant shared memories, my mind often wanders to your kitchen table in Ra'anana, thinking about planning the KAIA conference together.

Sally Picciotto – It's uncanny how we've been following each other around for the last decade: from San Diego to Italy to Boston/Providence and now back to Berkeley. So when are you moving to East Lansing? Being more serious, many of my adult values have been shaped through our friendship and I'm glad that we've stayed in touch as our lives have taken us around the globe.

Kim Seashore – Your effervescence, warmth and endlessly giving character are a true inspiration. Our chats over community dinners and morning walks to campus will stay with me always. Thank you, especially, for introducing me to the Berkeley co-housing community. The opportunity to live amongst such a wonderful group of people has been a transformative experience.

Many musical friendships sustained me on this journey. Thanks to the members of the Prometheus and Kensington Symphonies, and especially to Ann Edwards, Kristina Forester, Alexia Schulz, Jane Leber Herr, Jessica Cande, Tim McKnight, and Katherine Minakov. I owe special thanks to my violin teachers, Michael and Irina Tseitlin, for nurturing in me from a young age, many of the same qualities that I would need to draw upon throughout the process of completing this dissertation.

One of the most amazing experiences during "the graduate school years" was the two years Aaron and I spent at the Centro di Ricerca Matematica "Ennio De Giorgi" in Pisa, Italy. All the carnivals, weekend hikes, festivals, delicious food, summer days in Rosignano, weekend adventures through the Tuscan countryside, and celebrations small and large, will be imprinted on our memories forever. Thank you to Anna Baccaglini-Frank, Ana Bakovic, Tanja Begenesic, Pablo Claudevilla & Erika Anson, Margherita d'Ayala Valva and Alex Auf der Heyde (and of course Johannes/Nanni), Giovanni De Matteis, Andrea de Meo, Roberto De Pasquale, Laure Falk, Carla Frontini, Ilaria Gabbani, Antonella Gregorace, Myrthe Juliane, Daniel Kiss, Antoine Lemenant & Stephanie, Olivia Levrini, Maria Alessandra Mariotti, Lisa Mignone, Adoyo Owour, Bojan Simic, Claudia Tardelli Terry, Rachel van Dusen, and Aya Yamasaki, for sharing this remarkable and dreamlike chapter of our lives with us. When I returned to Berkeley after three years away, it really did seem as if I had just awoken from one beautiful dream and returned to another.

Several professional communities have played important roles during my graduate career. My first community in the GSE was related to my work with the DiME center. I learned so much from this work and I'd particularly like to thank Indigo Esmonde, Ann Edwards, Vanessa Pitts Bannister, Mara Landers, Meghan Shaughnessy, Tracy Craig, Vicki Hand, Anita Wager, Mary Foote, and Tonya Gau Bartell for many productive and formative conversations. The most constant professional community in the GSE has been the "Functions" research group, led by Alan Schoenfeld. I'd particularly like to thank Adi Adiredja, Danielle Champney, Darrell Earnest, Jose Gutiérrez, Lina Chopra Haldar, Hee-Jeong Kim, Angie Little, Nicole Louie, Sarah Nix, Daniel Reinholz, Kim Seashore, Niral Shah, Yasmin Sitabkan, and Dan Zimmerlin. I would also like to thank the members of the Knowledge Analysis seminar at UC Berkeley including Adi Adiredja, Lauren Barth-Cohen, Janet Casperson, Shuly Kapon, Jennifer King Chen, Colleen Lewis, Angie Little, Hillary Swanson, and of course, Andy diSessa.

A very important touchstone for me has been the formation of the KiP community and the workshops and conferences that we have organized around KiP themes. I'd especially like to thank Orit Parnafes, Rosemary Russ, Victor Lee, Ellie Sayre, Ayush Gupta, and Nathaniel Brown for wonderful collaborations around organizing these events. I'm very much looking forward to the continued development of this community!

Making the transition in moving to East Lansing was made easier by my connections with Michiganders working on the Algebra Teaching Study project. I'd also like to thank Glenda Lappan, Betty Phillips and all of the folks at the Center for the Study of Mathematics Curriculum at MSU for welcoming me so wholeheartedly to East Lansing and providing a wonderful place to write and work as I completed this dissertation.

Finally, I would like to thank my family. I am fortunate to feel so supported by my entire family, including my in-laws, Jerry and Victoria. Naturally, I find it hard to even know where to begin in thanking my parents, Kaye and Bruce, for all they have done to help me become the person I am. My parents have never been anything but completely encouraging of me in pursuing my passions and I thank them for instilling in me a strong sense of both independence and idealism. I would also like to thank my brothers, Bob and Greg. I have always admired Greg's focus and determination and Bob's deep thoughtfulness and compassion for those around him. (They also know when silliness and a good laugh are absolutely essential, and I thank them for that, too!)

Lastly, I thank my husband, Aaron – my toughest critic and closest ally. It would just not be possible for me to list all the ways that he has sustained me throughout this work. Our life, including the completion of this dissertation amongst everything else, has truly been a remarkable odyssey and there is no one in the world I'd rather be sharing this journey with.

Chapter 1: Introduction and Overview

> *Attaining a deep understanding of change mechanisms is the holy grail of developmental psychology: a profound goal, not fully attainable but worth pursuing nonetheless. Among the obstacles that prevent full realization of the goal are the impossibility of directly observing change mechanisms, and the sheer complexity of the developmental process. Nonetheless, the pursuit of this goal is worth the effort because of what we learn along the way.*
>
> *- Siegler, 2006*

Liam is a seventh grade student who has just finished a course in pre-algebra. During a series of sessions with a university tutor/researcher, he constructs a deterministic and essentially algebraic algorithm that he uses to for solving algebra word problems that have an underlying linear structure. This strategy seemed to emerge continuously over several episodes of problem solving in which he gradually refined earlier approaches. The question is, how does this happen? What mechanisms result in the development of his new conceptions?

While this is a particular case, the pursuit of an adequate scientific explanation of such learning events immediately implicates a number of foundational and general issues. Did this student learn only a new way to accomplish the goal of solving problems that he recognized as having a similar structure? Or was there some deeper conceptual residue of this experience? What kind of theoretical and analytical tools would allow us to identify the knowledge that students like Liam draw upon when they are solving problems? How can we model the processes by which new knowledge is constructed?

This dissertation explores the question of how strategic and conceptual knowledge co-develop. Solving problems, both in cases where individuals have a ready algorithm and in cases where they need to generate one on the spot is an extremely knowledge intensive process. Thus, the research is situated squarely at the intersection of fundamental research in problem solving and research on knowledge and learning.

The analytic focus of this dissertation is on addressing the following research questions:

1. What conceptual knowledge do individuals draw upon in implementing strategies?
2. How do new strategies and new conceptual knowledge co-develop during mathematical problem solving?

Overview of the dissertation

The core analytic work in this dissertation revolves around an in-depth microgenetic analysis of strategy emergence in the case study of Liam. The case study is used to develop a theoretical model that traces the evolution of the novel strategy at a fine-grained level of detail. A novel aspect of the present analysis is the consideration and the elaboration of the nature of the knowledge needed to implement observed strategies.

In particular, both strategies and the conceptual knowledge that undergirds their implementation are modeled as complex knowledge systems. This epistemologically grounded approach gives a new window into the study of strategy construction by characterizing not only the behaviorally observable actions that characterize the implementation of the strategy, but the forms of knowledge behind those actions.

The organization of this document is set up to prepare for and explain the evolution of the model of strategy-conceptual co-development from an ongoing dialogue with theory and with the empirical data.

Following this introductory chapter, I present the literature in mathematics education on the nature of mathematical knowledge and knowledge construction in Chapter Two. In the literature review, I make a particular effort to evaluate the contribution of the literature and its potential applicability in developing the kind of moment-to-moment account of learning processes that I aim to develop.

Chapter Three presents a brief overview of the background and design of the study from which the data for the focal case study of Liam came from.

In Chapter Four, I discuss the epistemological framework that guides the analysis of the data – Knowledge in Pieces or KiP (diSessa, 1993). I discuss the central principles of the framework as well as give some exemplar theoretical work that will serve as reference models in the analysis presented in this dissertation.

In Chapter Five, I discuss the general empirical and analytical strategies that accompany this choice of epistemological perspective. The analytical perspective I chose and developed in this dissertation is Knowledge Analysis (KA). First, I discuss the general strategies and KA and how it departs from other perspectives on cognitive modeling. I then mention a number of methodological problems that are inherent in the practice of KA and how they are negotiated. The second half of chapter five is devoted to a discussion of a complementary research method, microgenetic learning analysis (MLA) and neighboring perspectives on studying learning processes. This material prepares for the central analysis of the dissertation, which involves coordinating techniques from both KA and MLA.

Chapter Six is the analytic core of the dissertation. The presentation of the analysis roughly follows the process of developing the model. Successive stages in the research process are presented in strands so that the reader can understand how the final theoretical model was developed in dialogue with both the data and the orienting theoretical perspective. I begin with a discussion of Liam's initial and final strategies. This provides an opportunity to frame the change that was observed in the sessions and to think about what might be helpful to model in order to understand it. To help the reader understand what the epistemological analysis I will perform offers in light of competing or contrasting perspectives, I describe contrasting framings of the same data. Following this, I move into a discussion of the main analysis. The first strand of the main phase of the analysis is to formulate the analytic enterprise in epistemological terms, and in particular, in terms that make contact with the general orienting theoretical perspective, Knowledge in Pieces (diSessa, 1993). Here I indicate how one can think about strategies as complex knowledge systems, an idea that will be central to developing the analysis. I then give a detailed discussion of how the analytical vocabulary of co-variation schemes and knowledge of controlling the variation of linear functions was schematized “bottom-up” from an analysis of the data. At this point, the reader will be familiar with how both the central conceptual territory (controlling linear variation) and strategic territory are viewed through a complex systems perspective. With this preparation established, it will be possible to go through a series of focal episodes in order to trace the co-evolution of Liam’s strategic and conceptual knowledge. For each focal episode, micro-developments and changes to the strategy system and growing conceptual system are noted. This discussion prepares the reader to schematize general features of the change processes, sketching mechanisms that are potentially involved in this particular learning process. The final strand of analysis builds on this work and extracts general features of the interplay between conceptual and strategic knowledge and how they developed in this case to propose a theoretical model that describes the process of co-evolution of strategies and concepts.

The final chapter, Chapter Seven, summarizes the findings and limitations of the case study and maps out directions for future work along substantive, methodological, and theoretical dimensions.

Chapter 2: Literature Review

As discussed in the introductory chapter, the heart of this dissertation involves developing a fine-grained account of how particular mathematical understandings were constructed by an individual student. Developing such an account poses significant analytical challenges and the main thrust of the dissertation is to develop theoretical and analytical tools that can help make sense of the complexities inherent in a close analysis of individual learning processes. However, before embarking on this analytical work, I first survey the literature, both to take stock of the state of the art and to set the stage for how the account that will be developed in this dissertation differs in significant and important ways from those already existing in the literature.

This chapter provides an overview of the way that researchers in mathematics and science education have conceptualized knowing and understanding in mathematics and science for the purpose of providing accounts of learning processes (at various time scales). Throughout the chapter, the ways in which the literature does or does not make contact with the aims of the current study will be explicitly noted. This discussion will be used to build the argument that an alternative perspective on knowledge and learning will be required in order to engage in the analytic work of this dissertation.

Conceptions, misconceptions, and intuitions about mathematics and science

I begin with a review of the literature on conceptions, misconceptions, and intuitions in mathematics and science education. Central questions that guide this body of work include:

1. What kind of intuitions or ideas do students have about mathematical and scientific phenomena? What is the source of such intuitions?
2. What is the status of such intuitions or ideas in learners' conceptual systems? Are they only obstacles or can they also be resources?
3. What makes "conceptual change" a difficult process?
4. What does it mean to "have" or to understand a concept?

Many early studies of mathematics thinking in the constructivist tradition focused on the nature of students' conceptions and preconceptions about particular conceptual domains (e.g., Confrey & Smith, 1995; Steffe & Cobb, 1998, Thompson, 1994; and Vergnaud, 1994). Like the current work, this research sought to analyze episodes of student thinking for the purpose of uncovering mental schemes that undergird children's performance in these domains. An impetus of this line of work was to build domain-specific theories of instruction (diSessa & Cobb, 2004).

A subset of the early work on examining students' conceptual structures focused on examining the nature of the systematic patterns of errors that students made on tasks across multiple domains. A sample of the domains studied includes functions and graphs (Bell & Janvier, 1981), subtraction (Brown & Burton, 1978; Resnick, 1982; VanLehn, 1982; 1990) and algebra (Clement, 1982; Matz, 1982; Sleeman, 1984). Confrey (1990) provides a comprehensive review. Though this work does focus on the nature of students' mental schemes and reasoning processes, it mainly targets systematic difficulties students have with learning procedures. Because it neither focuses on students' conceptual resources nor on their role in learning, this literature does not offer tools for engaging with the focus of this dissertation: constructing an account of strategy emergence.

In contrast to the literature on systematic errors, an emerging literature on embodied cognition (Barsalou, 1999; Johnson, 1987; Lakoff & Nuñez, 2000; Varela, Thompson, & Rosch, 1991) works to identify conceptual schemata such as image and spatial schemata that are generated from our experience in the world in human bodies. Such schemata can be understood as both resources and obstacles in developing understanding of mathematics through mechanisms such as conceptual metaphor and blending. They are potential resources because they resonate with our everyday experiences, but they can also help explain why mathematical topics that are not possible to directly experience through our everyday interactions with the world are more difficult to learn (e.g., reasoning about infinite processes, visualizing concepts in projective, hyperbolic, or higher-dimensional geometry, etc.)

A number of mathematics education researchers (e.g., Abrahamson, 2009; Edwards, 2009) are working on programs of research that develop and empirically ground these theoretical ideas in ways that make contact with the interests of mathematics and science educators. However, while this literature draws attention to what is potentially an important class of conceptual resources, in the data under study in this dissertation, there was not extensive opportunity to observe the function of such schemata. Thus, embodied frameworks were not chosen as a primary means or theoretical frames for interpreting this data.

Other researchers have studied the nature and role of individuals' intuitions about mathematics for learning (Ben-Zeev & Star, 2001; Fischbein, Tirosh, & Melamed, 1981; Fischbein, 1987; Tversky & Kahneman, 1974). From this perspective, the source of students' intuitions about mathematics is both experiential and curricular. Within the field of mathematics education, Fischbein's work was instrumental in establishing that students do have intuitions about mathematics and that effective instruction should make contact with and address students' ideas. Much of Fischbein's work is developed in the domain of the development of probabilistic thinking and concerns the "intuitive biases" that individuals have about chance and probability. While the general commitment to investigating mathematical intuitions and their role in thinking and learning is a shared concern, in both content and methodology (many of the designs are cross-sectional and instructionally-oriented), this line of work does not support the construction of the particular account being developed in this dissertation.

In a similar vein, Stavy & Tirosh have proposed a theory of "intuitive rules" that they use to account for error patterns in assessments of individual reasoning (Stavy & Tirosh, 1996; 2000; Tirosh & Stavy, 1999). Though some of the intuitive rules they propose (e.g., *More A corresponds to More B*) bear some similarity to intuitive schemata that are drawn upon in the account I construct, their methodology (that relies upon post-hoc attribution of the use of the scheme based on paper-pencil assessment items) does not allow them to make real-time attributions in the reasoning processes of individuals nor to know what in an individuals' knowledge system would cue such patterns of reasoning. Both issues are critical in the account I construct.

The accounts above have all focused on the nature of students' pre-instructional knowledge. However, what is of interest in this dissertation study is how to coordinate representations of students' knowledge structures with representations of their real-time reasoning processes. In this direction, I turn now to accounts within the broad literature on conceptual change that addresses issues of change or lack of change in knowledge organization.

Theories of conceptual change in science and mathematics

Theories of conceptual change are characteristically focused around deep, difficult, and problematic learning. One family of explanations for the difficulty of conceptual change is that nature of students' intuitive or pre-instructional knowledge is the source of difficulty. Intuitive ideas are understood as coherent or theory-like and seen to stand in contrast to normative knowledge. Further, in order to achieve a normative understanding of a domain, deeply entrenched ideas may need to be substantially restructured or entirely replaced (Carey, 1999; Ionnides & Vosniadou, 2001). However, the view that students' ideas are coherent and theory-like is called into question by numerous empirical studies demonstrating the context-sensitivity and fragmentation in naïve knowledge systems. (See diSessa, Gillespie & Esterly, 2004 for a quasi-replication study that challenges the findings of Ionnides & Vosniadou.) Clark, D'Angelo & Schleigh (2011) conducted a multi-national comparison study that confirmed the results of diSessa, Gillespie & Esterly.

The debate on the nature and form of students' pre-instructional knowledge has historically received more prominence in the science education community than in the mathematics education community. However, recent work drawing upon the framework theories approach (Christou, Vosniadou & Vamvakoussi, 2007; De Bock, Van Dooren, Janssens, & Verschaffel, 2002; Tirosh & Tsamir, 2004; Vamvakoussi & Vosniadou, 2007; Vosniadou & Greer, 2004) has brought the debate into mathematics education. Such work typically focused on identifying students' non-normative ways of thinking about literal symbols, proportionality, and rational number. The work in mathematics from this perspective has not focused on the issues of context-sensitivity and fragmentation or on processes of change. A contrasting body of work on conceptual change in mathematics informed by complex systems perspectives on thinking and learning has focused more analytic attention on understanding process data of real-time reasoning and has uncovered the same kinds of contextuality and fragmentation as was found in the science education literature (e.g., Chiu, Kessel, Moschkovich, & Muñoz-Nuñez, 2001; Schoenfeld, Smith, & Arcavi, 1993; Izsák, 2000; 2005; Pratt & Noss, 2002; Wagner, 2006; 2010).

Knowledge and learning: Perspectives within mathematics education

Given the kind of data studied (real-time processes of individual reasoning) in this dissertation, the microgenetic methods used in these analyses is of particular importance to developing the analysis in this dissertation study. Especially important features of such analyses, along with a discussion of several illustrative exemplars of microgenetic analyses of learning processes, will be reviewed separately in the methodological review section in the chapter on analytic methodology (Chapter Five).

In contrast to the work on conceptual change, which has been primarily developed thus far in science education, I now turn to three accounts of development of knowledge structures that have been primarily developed within mathematics education. The three families of developmental accounts that are discussed are (1) Radical constructivism, (2) Process-Object theories of learning, and (3) Abstraction in Context. Following this, as a perspective intimately concerned with the development of mathematical competence, but contrasting from the previously discussed perspectives in that it concerns a *discursive* approach to the development of expertise in mathematics, Sfard's "commognitive" perspective is briefly discussed.

Radical constructivism, as a theoretical perspective for studying the structure and processes of mathematical thinking and learning, was introduced into mathematics education by the work of von Glasersfeld and Steffe (Steffe, von Glasersfeld, Richards, & Cobb, 1983; Steffe, Cobb, & von Glasersfeld, 1988, Steffe, 1994). The primary focus of attention is on mental schemes and how they are constructed and coordinated: the theory draws heavily on the learning mechanisms and structures posited by Piaget. For example, mechanisms of development that are of particular interest from this perspective have included accommodation, interiorization, and reflective abstraction. Mental schemes are described as having a very specific structure including (1) an assimilatory structure that is activated when contexts of use are recognized (2) a set of mental and/or physical operations associated with the context, and (3) an anticipated outcome of the result of the operations. Inspired by Piagetian trajectories of development, the expected developmental accomplishment is that individuals will come to be able to anticipate the results of operations without needing to perform them. The original context for work in mathematics education guided by the perspective of radical constructivism concerned studies of children constructing meaning for whole number and whole number operations through counting activities. However, this focus has been extended in further work (e.g., Olive, 1999; Thompson, 1993)

Though the focus on mental schemes and their coordination is a shared focus between work on radical constructivism and the kind of account I seek to develop in this dissertation analysis, the *a priori* presumptions about the form and structure of schemes and the expected developmental mechanisms for their construction and coordination differ from the kind of account that I seek to develop. The critical difference is that I seek to uncover the particular structure, content, form, and mechanism through the analysis itself.

I now turn to discussing process-object theories, a second family of accounts within the mathematics education literature that concern the development of concepts as mental schemes. This includes work by Dubinsky, 1991; Gray & Tall, 1994; Sfard, 1991; Sfard & Linchevski, 1994, among others. In process-object theories, "process" understandings (e.g. thinking of $3x+5$ in terms of specific numeric calculations for each possible value of x) are posited to precede "object" understandings (e.g. being able to operate on $3x+5$ as an object, such as what allows one to operate on the graph of $y=3x$ by shifting it up by 5 units to obtain the graph of $y=3x+5$.) Process-object theories give a way to describe the compression that happens as earlier understandings become more routine and automatized. In the account of Sfard & Linchevksi (1994), there are three stages offered in the transition from process understandings to object understandings (1) learning procedures (interiorization), (2) thinking more in terms of actions and results as opposed to focusing on all processes as equally important (condensation), and (3) processes become objects that can then become acted upon (reification). Interiorization and condensation are thought to be slow, hard to observe processes and reification is thought to be a fast, hard to observe process.

Besides the issue of how much of mathematical learning can be characterized in terms of the trajectory from processes to objects, process-object theories also suffer from being difficult to trace closely with empirical data. Because the nature of the development in the account I seek to construct is not clearly on the process-object spectrum and also the time-scale of observations in my case is much finer than the learning mechanisms of interiorization, condensation, and reification, this family of accounts remains of limited use in my empirical work on tracking the knowledge construction processes that accompany strategy emergence.

We now turn to discussing a third perspective: Abstraction in Context (AiC). Hershkowitz, Schwarz & Dreyfus (2001) present a model for the growth and change of knowledge structures in which *abstraction* is a key mechanism. The model for the process of abstraction that they posit consists of three nested *epistemic actions:* R-actions ("recognizing with"), B-actions ("building with"), and C-actions ("constructing with).” The reason for identifying these observable actions in the empirical data of students reasoning is that though Hershkowitz, Schwarz, and Dreyfus study real-time reasoning processes of subjects, they do not hypothesize extensively about the structure and coordination of mental schemes underlying knowledge construction processes. Instead, they focus on identifying the nested epistemic actions described above which are observable. The actions they describe are intended to be general and apply equally well across different content and tasks, and social and material settings. Furthermore, the activities under study involve target strategies, concepts, and methods that can be described through an *a priori* task analysis.

The focus on observable student actions and the comparison with hypothesized partial states of construction of target concepts are two features of the studies of Hershkowitz, Schwarz, and Dreyfus that distinguish them from the kind of approach needed for developing the analysis in this dissertation. For the authors, the study of partial learning in terms of RBC actions helps inform the design of task sequences that are better tuned to be able to move students from initial conceptions to target conceptions (Ron, Dreyfus, & Hershkowitz, 2010). This represents a fundamentally different goal than that of my study, which is to uncover empirically grounded knowledge structures and learning mechanisms that take into account what the learner perceives and infers. As the work of Lobato (2003) and others has shown, what the student learns from an instructional activity may not align with the expected trajectory of learning. Indeed, the focal case study developed in this dissertation involved an unexpected learning event that would not have been predicted by any *a priori* task analysis of the curriculum materials.

The linear interpolation strategy was not an *a priori* instructional goal, but rather emerged as through interactions around the activity.

I now turn to our discussion of Sfard's commognitive perspective. Thus far in this review, we have focused our attention on theoretical perspectives about knowledge and knowledge construction involving mental structures and processes. In contrast, Sfard's perspective aims to conceptualize mathematics thinking and learning in terms communicative processes.

Informed by Wittgenstein and Vygotsky, Sfard (2008) reconceptualizes learning as a process of "individualizing genres of interpersonal communication." Empirically, this means that phenomena that have been traditionally approached from the perspective of conceptual change and cognitive structures are approached instead in terms of communicative structures and discourse change. Of particular interest in Sfard's personal empirical work include how new mathematical objects are formed and the development of discourses around particular mathematical topics (e.g., algebra, negative number, etc.).

One aim of Sfard's commognitive program is to create a foundation for studying thinking and learning that does not involve the observation of mental entities. She makes the argument for the necessity of her perspective in her discussion of five controversies that research on cognition has struggled to resolve. These include (1) contextuality and situativity of knowing, (2) what "understanding" means, (3) number, (4) misconceptions, and (5) learning disabilities. Drawing upon Blumer (1969), Sfard claims that current terms in the psychological literature such as "abstraction" and "understanding" fail several criteria: (1) instances of the underlying phenomenology of interest cannot be clearly identified based on the specification of these terms, (2) the theoretical boundary around the constructs and related theoretical constructs is not sharply drawn, and (3) because of these foundational referential problems, knowledge about the constructs cannot be accumulated.

It is of interest to note that researchers from the Knowledge in Pieces community (e.g., diSessa & Sherin, 1998) have made several of these same kinds of arguments about the inadequacy of much existing theoretical machinery for studying concepts and conceptual development. However, though the commognitive and KiP perspectives are similarly broad in scope, they diverge on the foundational issue of the sensibility of studying mental structures and processes.

Chapter 3: Research Design and Data Collection

In this chapter, the broad context of the study and the methodology guiding data collection are introduced. Because the most detailed analyses presented in this dissertation focuses on only a very small part of the larger corpus of data collected, I first provide some details about the broad context for my study. In particular, I first discuss at some length pilot data that I collected in middle school algebra classrooms and how it informed the later design of the tutorial/interview study. The core analytic work in this dissertation involves a case study of one student, Liam, developed from the tutorial study data.

The broader context framing the study

The focus of this dissertation grew out of data collection and work I was doing as part of the Diversity in Mathematics Education (DiME) project, led at UC Berkeley by Alan Schoenfeld.[1] I will start with some broad framing of the local context.

Developing a way to address a district mandate to teach algebra to all eighth graders was an issue of critical importance to the teachers in the local middle schools. One of the main difficulties in implementing the mandate was the issue of whether all students in the district had been sufficiently prepared by their previous mathematical experiences in order to take the standard yearlong algebra course.[2] To address this concern, the decision was made to offer two versions of algebra: the standard course and a slowed version of the same course (called "Algebra 1A/1B") using the same curricular materials, offered over two years instead of the usual pacing of one year.

In line with the concerns of the DiME project, my initial interest was to compare the tracked algebra and algebra 1A classes taught by the same teachers in order to examine the way that mathematics was presented, discussed, and taken up by students in the "standard" and "transitional" versions of algebra. Through my work with DiME, I was paired with an eighth-grade algebra teacher who was simultaneously teaching both the algebra and algebra 1A classes. As part of this project, I planned to follow the enactment of a particular strand of the curriculum, common to both the algebra and algebra 1A classes.

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[1] The DiME project was a NSF-funded Center for Learning and Teaching (also including UCLA and UW-Madison) that focused on issues of equity and diversity in mathematics education. Each university campus had a local school district as a partner. As a graduate student fellow on this project, over the course of three years on the project, one of my roles was that of participant observer and classroom support for a middle school algebra teacher.

[2] The course used the reform "College Preparatory Mathematics (CPM) Algebra Year One" curriculum.

In consultation with the classroom teacher, the decision was made to collect data on how students understand and model word problems. Based on my observations of the classroom instruction in the focal classes, I chose to focus on how students make the transition from "informal" solution processes of "guessing and checking" organized in a chart form to a standard algebraic solution technique of setting up and solving equations.

This strand of curriculum was of interest from a comparative perspective because in my classroom observations, I had observed that while both classes started from guessing and checking approaches, the Algebra class successfully and rather rapidly made the intended transition to variable modeling approaches, while the Algebra 1a class never really progressed beyond guessing and checking to solve problems.

A particular sketch of steps that this transition was expected to follow was suggested by the curriculum used in these classes: College Preparatory Mathematics (Sallee, Kysh, Hoey & Kasmatis, 2000). The transition model has five instructional phases: (1) exploring students' pre-instructional approaches, including trial and error (2) introducing a chart form (a "Guess and Check chart," see below) to record sequences of trial values (3) making a connection to future symbolic expression of the relationships in the problem (4) symbolizing the relationships in the problem in the chart, and (5) purely symbolic expression without the chart.

It was of interest to me to examine the continuities and discontinuities between this heuristic stage model for how students' competence with solving algebra word problems develops and how students' thinking actually develops as they engage with instruction attempting to follow this model. This was of interest both in the classroom context and in a later follow-up tutorial study (described later in this chapter). The purpose of the follow-up tutorial study was to get higher resolution data on processes of student thinking than would be possible with sampling students across several weeks or months of instruction in the classroom context.

Below I discuss an example that illustrates the stages of the expected transition model.[3] I use the following word problem as an example:

> *The length of a rectangle is six more than three times the width. If the perimeter is 148 ft, find the length and width.*

Phase One (Pre-instruction): Students who engage in solving the word problem might make use of informal strategies (guess and check, diagrams, unwinding). These strategies might be rather idiosyncratic and suited to the problem at hand (i.e. context dependent), but not represent a general approach to solving word problems. Another possibility is that students at this stage may not attempt to engage in solving the problem. One objective at this stage is to uncover students' pre-instructional approaches.

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[3] This is based both on material in the CPM guide for teachers and also observations of how the teacher understood and set up for the "steps" in this transition model in her classroom instruction.

Phase Two: The Guess and Check chart is introduced by the instructor and/or co-constructed with students. Instruction with Guess and Check charts initially focuses on constructing the representation with the explicitly communicated goal of solving the word problems via a sequence of trial cases recorded in the charts. The sequence of trial values is meant to develop students' conception of variable quantities.

Width	Length	Perimeter	Check
10	3(10)+6=36	10+36+10+36=92	Too low
20	3(20)+6=66	20+66+20+66=172	Too high
17	3(17)+6=57	17+57+17+57=148	Check

Figure 1. A "Guess and Check" chart and its use in solving a word problem.

Phase Three: At this point in the trajectory, students will have used the charts to solve a variety of problems (albeit of a rather narrowly defined semantic type). After having been encouraged to write out how the variables in the problem are related, students are prompted to express these patterns symbolically in the last row of the chart. The instructional goal is to connect the activity of solving the problems using guess and check to future symbolic methods of solving word problems.[4]

Width	Length	Perimeter	Check
10	3(10)+6=36	10+36+10+36=92	Too low
20	3(20)+6=66	20+66+20+66=172	Too high
17	3(17)+6=57	17+57+17+57=148	Check
W	3(*w*)+6	*W*+3*w*+6+*w*+3*w*+6	=148

Figure 2. Symbolic expression of relationships in the last row of the chart.

Phase Four: Students are to make the charts, but only record their expressions for variables and relationships between variables in the problems symbolically.

Width	Length	Perimeter	Check
W	3(*w*)+6	*W*+3*w*+6+*w*+3*w*+6	=148

Figure 3. Recoding only the last row of the chart in solving the problem.

Phase Five: Students are asked to not make the charts anymore, but just to write down the assignment of variables and solve the problem symbolically: From the problem, we are given that $l=6+3w$. So, $P=2(6+3w)+2w$. Solving for w gives $w=17$ feet. Substituting for l gives 57 feet.

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[4] We should note that understanding how, exactly, the critical step: bootstrapping from specific example calculations to "algebra" was precisely one of the potential discontinuities we wished to explore and get process data around in the follow-up tutorial study.

From the perspective of the development of algebraic reasoning, this strand was of interest because the transition model proposed by the curriculum and appropriated into the instruction of Ms. S. appeared to be consistent with mathematics education literature on the use of spreadsheets for supporting students in setting up and solving word problems using variable expressions (Abramovich & Nabors, 1998; Bills, Ainley, & Wilson, 2006; Haspekian, 2003; Rojano, 1996; Sutherland & Rojano, 1993). This literature claims to provide some evidence that spreadsheets are a valuable tool to mediate the transition between solution approaches (making variable modeling approaches more meaningful by making algebraic symbolism more meaningful). However, none of the research studies in the literature provided data of a very high resolution on what students were actually understanding about spreadsheets, about variables, or about how exactly the spreadsheets mediated this “arithmetic” to “algebra” transition.

The follow-up tutorial study I conducted was organized around the same trajectory as is found in the curriculum and discussed above. The objective of the tutorial sessions was exploratory—to see what aspects of the “transition” using this approach seemed sensible to students and whether and how the chart approach seemed to be supporting the intended transition.[5]

Prior to this study, very little literature (and, as noted above, mainly only in the context of computer spreadsheets) existed on any aspect of this relatively recent curricular proposal (Abramovich & Nabors, 1998; Bills, Ainley, & Wilson, 2006; Haspekian, 2003; Rojano, 1996; Sutherland & Rojano, 1993). Related perspectives come from the much wider literature on algebra learning and pattern generalization (e.g. Ellis, 2007; Lannin, 2005; Noss, Hoyles, Mavrikis, Geraniou, Gutierrez-Santos, & Pearce, 2009). The sole research study found that made direct contact with the transition as specified by the CPM curriculum was a recent study of Izsák, Caglayan & Olive (2009) exploring classroom episodes around word problem solving from a meta-representational (diSessa, 2004) perspective.

From the perspective of learning theory, this transition was of interest because it could provide an example of how students build on previous understandings (solving problems through a series of trial calculations) in building more “advanced” understandings (solving problems through solving for an unknown value). Previous literature (Johanning, 2004; 2007) had identified guessing and checking to solve word problems as an activity that was of some value. This was in contrast to previous literature that positioned “guessing and checking” as a problem solving approach that was

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[5] One difficulty that was evident both in the classroom observations and in the tutorial sessions is that the function of writing out calculations and recording them in the chart so that they can be available to reflect upon in the future is not necessarily well motivated and sensible to students. From the perspective of students, the current objective is simply to find the answer to the particular problem at hand. Asking them to do something that will be helpful in the future but that is not connected with the activity they are currently engaging in is a hard to justify pedagogical move.

an obstacle to learning the algebraic approach to solving problems (Stacey & MacGregor, 2000).[6]

Main phase of research

Six students participated in the main phase tutorial study. These students were recruited from the seventh grade pre-algebra class at the same school as the classroom observations were conducted. The interviews took place over the summer between the students' seventh and eighth grade years.

The subjects were:

Table 1. Tutorial Study Subjects

Student	Gender	Prior performance	Class placement in 8th grade	Group
IP (Liam)	Male	Average	Algebra	A
AV	Male	High	Honors Algebra	A
SD	Male	Average	Algebra	B
YA	Female	Average	Algebra	B
MD	Female	Average	Algebra	B
EC	Male	Low	Algebra 1a	B
DT[7]	Female	Low	Algebra 1a	A

Prior performance in mathematics was based on self-report of the student and the assessment of the students' seventh-grade teacher. The sessions were done over the course of two, two-week periods. "Group" in the above chart describing the subjects refers to whether the student participated in the sessions conducted in the first two-week period (Group "A") or second two-week period (Group "B").

Tutorial Sessions

Each subject participated in a series of six sessions. Each of these sessions was designed to last one hour. My role in the sessions was as a tutor-researcher (described more below).

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[6] One of the major themes that emerged in the pilot work (and that was followed up on in the tutorial study) was that the way that students make their choices about "next guesses" gives information about the state of their conceptual understanding. The specific kind of knowledge that gets used is knowledge about how to *control the variation of functions*. I will expand upon this type of knowledge later in the dissertation.

[7] DT had a great deal of difficulty engaging with the first problem in the session and asked to remove herself from the study. Though she withdrew from the study, her mother agreed to let me continue to tutor her on topics she identified (e.g., fraction operations) as problematic to her to help her prepare for eighth grade.

The sessions were designed to allow me insight into how a particular curricular proposal (as implemented in a local classroom) functioned. The main difference was that instead of studying the trajectory over the course of the year in the classroom, I would attempt to simulate conditions on a shorter timescale (over the course of six session over two weeks). Because there appeared to be a "singular point" in the curricular trajectory (and certainly the classroom implementation of the curriculum) between when students engaged in guessing and checking and when they later engaged in variable modeling, I wanted to "turn up the microscope" on this proposed trajectory.

Table 2. Overview of the Objectives of Each Session.

Session	Description
One	The first session covered three major themes: - "Start-unknown" versus "start-known" problems – Could students articulate what was different about paired problems where one used a given value to compute the desired answer versus one where one "worked backwards" to solve the problem? - Introducing students to organizing sequences of guesses organized in charts – with no particular focus on strategy for choosing values - Introducing students to actually solving problems using the charts. The first session functioned partially as a pre-assessment to ascertain what approaches students would use to solving simple algebra word problems if none were instructed.
Two	The second session continued with a warm-up pair of start-known and start-unknown problems. The student was asked to organize the solution in the chart. By design, all of the rest of the problems were to be solved using the charts. Practices around the charts (e.g. writing out operations on inputs and intermediate calculations, noting and marking whether a particular guess was too high or too low, etc.) were discussed.
Three	The third session involved students solving problems using guessing and checking and then creating a final row where they wrote out the general form of the trial calculations they had been recording. Setting up equations from this information was discussed. Students were asked about what the answer to the equation would be (to see if they understood that the answer they had just found using guessing and checking would also be the same answer as when solved by equation).
Four	Session four involved additional problems where students initially found a solution by using "guessing and checking" and then solved the problem a second time by setting up equations and solving them.
Five	Session five involved students setting up problems without reference to the chart form.
Six	Session six was a "post-assessment" where students solving problems using whichever method (guessing and checking or equation solving) they chose.

Above, I outlined the broad objectives of each of the six sessions that were conducted with students. Sessions one and six were intended to "bookend" the sessions with an opportunity to observe how students approached problems at the beginning and ending of the sessions. The interior sessions were each focused around a particular phase in the previously mentioned transition model.

All problems worked in the sessions appear in the appendix outlining the interview session protocol (Appendix 1).

Selection of problem pool

A number of features of the problems in the session are worth noting. Some example features include: (1) virtually all problems involved the underlying structure of a linear equation in one variable, (2) had integer-valued solutions, (3) involved largely discrete quantities (integer relations, integer age relations, relations between mystery numbers, relations between number of objects like candy bars, etc.), though some problems involved familiar continuous quantities coming from measurement (e.g. yards in a football game, perimeters of rectangles, etc.). There was also some attempt to make the numbers in the problems not so small that the problems could be solved easily by inspection.

The problems mirrored problems that were solved in the eighth grade algebra class that this study was meant to inform (In some cases they were taken directly from the curriculum, or slightly modified). The problem contexts were chosen so as to be simple and accessible, relying upon conceptual content that it could be assumed students would find familiar.

Implementation of the protocol

Though the sessions followed the instructional arc described above, during the time for the student to work on the tasks, the tutor/researcher usually avoided intervening in directive ways. Example interactions included:

1. Clarifying and reminding the student of conventions of the agreed upon approach (e.g. reminding the student to write out calculations)
2. Asking the student to justify choices for guesses (getting the student to verbally articulate their reasons for choosing particular guesses)
3. Encouraging students to continue/finish their calculations so that the trial could become a basis for making reasonable choices for next guesses.

On occasion, some verbal feedback or encouragement was given concerning the specifics of a students' approach. In retrospect, it is not clear that this was strictly necessary to the sessions and should probably be avoided. However, the affective dimension of "maintaining the interaction" is important and cannot be discounted.

Though there was a specified pool of problems and the sessions were globally structured around transitioning from guessing and checking approaches to solving problems to variable modeling approaches, there was some variation in the implementation of the protocol.

Some variation in the sessions was due to the fact that students moved through problems more quickly or slowly. The same basic pool of questions was used for all students, but some students needed to do another problem of the same type or others could move through problems more quickly. Thus, the sessions were thematically organized and the specific problems (within the initial pool) were decided upon during the sessions. Another constraint of the sessions was that they were to last exactly one hour. Sometimes problems would be added to fill out time in the sessions or on occasion, work on problems would be cut short due to time constraints. This happened rarely.

Because the core analysis of this dissertation will focus on an in-depth analysis of the protocol from the sessions with one of the students, the variation in the implementation of the protocol across subjects is not relevant to our present concerns.

Data collection procedures

Problems were given to the student on 8.5 x 11 pieces of paper with ample room to write. Students were given pens to work with so that all of their calculations would be recorded. Students were also allowed to use a calculator. A calculator with particularly large keys was selected so that any calculations that were done would be able to be later discerned in viewing the video.

Each of the six sessions was conducted individually with each of the six students over a two-week period. (So, there were 36 total hours of interviews). The data corpus includes video and all written work for all of the sessions. A basic, but novel, methodological advantage of the current study over previous studies documenting students' "informal" approaches to word problem solving (Bednarz & Janvier, 1996; Johanning 2004; 2007) was that, because of the video record, the current study could examine the choices students were making as they were solving the problems. As we will see in the focal case study, the richness of the video record is crucial in interpreting students' work.

Each of the six sessions with the six subjects was both video-recorded and audio recorded. Because the main focus was being able to coordinate the students' written productions with their verbal justifications and explanations, as well as any calculations they did with the calculator that was provided to them, the video camera was set up to focus on the subjects' papers. The camera was set-up behind the desk (over the shoulders) where the tutor and student were working. This also allowed for the analysis of any gestures that students made as they were constructing charts and solving problems. This allowed for the disambiguation of verbal descriptions (e.g. when students would say things like "for every one of *these*" such a statement could then be interpreted by referring to the video to see where the student was pointing). The sessions were also audio-recorded as a back-up measure in case collection of video data failed in a particular session. This did not occur and the audiotapes of the sessions were not used for analytic purposes. To aid in the collection of sufficiently good audio to capture students' utterances, an external flat microphone was used.

A re-design of data collection procedures could include capturing video that allowed a view of the entire interaction between student and tutor—allowing the tracking of eye gaze and facial expressions, as well as more interactional detail. However, for the purposes of the particular analysis undertaken here, such detail did not present itself as crucial. Further, the single camera data collection set-up of this study was chosen because of logistical considerations.

Narrowing of the analytical focus

In this section, I briefly describe the process of selecting a focal phenomenon of interest from the larger data corpus. The actual procedures of analysis and the arguments and principles that the analysis is subject to are discussed in the chapter on creating my analytical frame and then exemplified in my actual analysis.

The design and implementation of the tutorial study was meant to track a particular arc of learning: students moving from one approach to solving problems (guessing and checking) to another approach (setting up and solving equations). As noted above, there were many issues related to the implementation of this trajectory that were already noticed in the pilot classroom data. The tutorial studies, in the attempt to re-create the same learning trajectory as in the classroom, allowed for a cleaner documentation of some of these same issues. The higher resolution was meant to provide a baseline for re-design of instruction in the classroom the following year and/or in future iterations of a tutorial study. Along these lines, an important aspect of studying the process of moving from one approach to another was to find out what was more or less sensible to students about this activity.

Within the framework of *this* focus, an extended sub-arc of episodes of considerable interest occurred within one of the student protocols. A completely unexpected, but very interesting, development happened with one of the students, Liam. Over the course of the sessions, Liam refined his strategy for solving the problems in the session from a simple guessing and checking solution strategy to a more sophisticated and algorithmic approach based on the idea of linear interpolation. This development was of great interest to me, partly because this new conceptual and strategic direction seemed to be so different, yet emerge so naturally from interactions around the "intended" trajectory of the sessions. Finding a scientifically adequate way to describe the emergence of this novel strategy became the focal pursuit of the research reported in this dissertation.

In the next chapter, the theoretical perspective that guides the analysis will be introduced and discussed in more depth.

Chapter 4: Theoretical Perspective

The previous chapter discussed the broad framing for the study from which the focal case study data analyzed in this dissertation was drawn. It is now time to re-focus attention on what will be my central concern for the remainder of the dissertation: the process by which novel strategies emerge during mathematical problem solving. To prepare for the case study analysis that will be the analytical core of the dissertation, I now turn attention to a discussion of the epistemological framework that will play a central role in developing the analysis.

Empirical approaches to epistemological questions

Before discussing the particular epistemological framework that informs the current study, I first make explicit some of my orientations and aesthetics with respect to the enterprise of modeling conceptual understanding and learning.

First, one of the fundamental commitments taken in this dissertation is that analyses of learning processes *should* be epistemologically grounded, with the aim of extending our understanding of learning processes at a principled and theoretical level. An analysis being "epistemologically grounded" means that the assumptions being made about the nature of knowing and constraints on knowledge representation and change are made explicit by the researcher. Furthermore, an explicit goal of the analysis is to make an epistemological contribution – that is, to further our understanding of what it means to know and learn mathematics. The orientation to theory-building taken in this analysis could thus be identified as a type of "empirical epistemology" in that rather than introspective or philosophical arguments about the nature of mathematical knowledge and learning, we build up a theory of mathematical knowing by studying episodes of individuals and groups thinking about mathematics and engaging in a wide variety of mathematical activities. The theory produced should apply across all of these contexts because it should say something fundamental about what it means to know and do mathematics.

An empirical approach to studying epistemological issues was pioneered by Piaget, who established *genetic epistemology* as a field of study (Piaget, 1970). DiSessa (1994) discusses an approach to "epistemological micro-modeling" in which the constraints of cognitive modeling and information processing concerns are joined with a focus on the perspective of developmental psychology in creating accounts of intelligent activity its development. The current program of work is broadly informed by diSessa's epistemological micro-modeling approach.

I now discuss the epistemological framework that guides the analysis in this dissertation. The rationale for choosing KiP as a theoretical frame to guide the analysis is based on its potential to give a unified approach to theorizing about a variety of important phenomena about the nature of knowing and learning mathematics that have been documented in the cognitive and learning sciences literatures as well as in the mathematics education literature.

Knowledge in pieces (KiP)

The Knowledge in Pieces (KiP) epistemological perspective (diSessa, 1993) is a powerful and evolving heuristic framework for considering the nature of knowledge, its deployment, and how knowledge organization changes as individuals learn. Though many of the principles can be (and have been) stated quite generally, KiP remains most elaborated in the domain of physics (Notable and more recent exceptions in the domain of mathematics that have strongly guided the current analysis include Izsák, 2000; 2005; Pratt & Noss, 2002; and Wagner, 2006; 2010).

Historically, Knowledge in Pieces as a perspective on knowing and learning was informed by two main traditions: developmental psychology (e.g., Piaget's genetic epistemology) and artificial intelligence. From developmental psychology came an interest in and ultimate accountability to be able to explain long-term and central learning and developmental processes. From artificial intelligence came a commitment to the fine-grained accountability that is required by attempts to model moment-to-moment cognitive activity, ultimately computationally. Thus, one can see that the commitment to understanding change across very long time scales is paired with the commitment to understand the details of learning at very small, moment-to-moment timescales.

According to KiP (diSessa, 1993), individuals' intuitive knowledge about the physical world (their "sense of mechanism") can be productively modeled as a complex system of a diverse set of elements. Naïve knowledge systems are typically comprised of many, loosely coupled knowledge elements whose activation and use is highly context-sensitive. The development of expertise largely involves the progressive systematization and re-organization of the naïve knowledge system.

In choosing KiP as an orienting frame for the purpose of studying thinking and learning in other domains (such as mathematics), one is implicitly asking the question of to what degree that hypothesis about the nature of knowledge and how it changes as people learn is shared across domains. In addition to the studies of mathematical thinking and learning cited above, the potential for KiP to provide a unified way to think about a wide variety of issues that emerge in learning mathematics, recommends it as a powerful heuristic frame for guiding analyses. Though the specific elements (especially their source) may be different across domains, the idea of a "systems perspective" on conceptual development offers a lot (See Smith, diSessa, & Roschelle, 1993) to re-framing many and diverse learning issues in mathematics (e.g. gradual construction of understanding of multiple and diverse facets of concepts, coordination of understanding across representational forms, the difference between "having" knowledge, recognizing that it is applicable, and using it effectively to construct proofs and solve problems, etc.)

Other orienting assumptions of KiP include an attention to knowledge-in-use, especially with respect to potentially productive role of prior knowledge resources; a commitment to understanding the inner workings of concepts as they function for individuals at a sub-conceptual grain-size; and an accountability to the details of learning *processes* at multiple time-scales, especially including very short time-scales. This accountability gives an opportunity to coordinate our accounts of how novice knowledge structures develop over time into expert ones with the important nuances involved in local learning events.

To most important principles of KiP (as organized in diSessa, in preparation) are summarized below:

1. **The complexity of the naïve state.** The assumption that students come to school with many intuitive ideas that should be leveraged in instruction is a fundamental assumption of constructivism and of KiP in particular. The key phrase here is "complexity" – in the sense that students' knowledge structures and systems are diverse and complex to understand and we need theoretical tools that help us to make sense of that complexity.
2. **Potential productivity of intuitive knowledge.** This principle stems from the constructivist roots of KiP. In contrast to work on misconceptions, KiP studies of knowledge and learning aim to document knowledge structures that maintain continuity with both pre-cursor and successor forms.
3. **Grain size and structure.** The structure of knowledge is an empirical question and adopting a small grain size helps us to capture fine nuances in change.
4. **Focus on micro-details of learning processes.** In contrast to before/after studies of mathematics learning, KiP studies are accountable to the details of learning events and thinking processes within them.

Knowledge systems and their dynamics

We now discuss some of the ways that KiP models of knowledge structures, organization, and dynamics capture the situativity and reactivity of knowledge. One of the key questions one would ask about any knowledge system is how its elements are cued to an active state. Activation of elements depends very particularly on aspects of the mental context. Thus, what is salient to a person and they are attending to in a context is very important for determining what knowledge is activated. Other important influences on what knowledge elements are cued to an active state are the knowledge that is already in play in the situation. The constructs of cueing priority and reliability priority are measures of how likely knowledge elements are to be active at any given time.

1. **Cueing priority** is a measure of how likely, given a particular set of conditions, it is for a certain element to be activated. When this priority is high, little is required (in terms of the activation of other elements or the focusing of attention on particular aspects of the situation) for the element to be activated.
2. **Reliability priority** is a measure of how likely the element will stay active once it is activated. If the reliability priority is high, the reasoning supported by the element is likely to be more confidently expressed. The source of a high reliability priority could be previous empirical feedback that indicated that the activation of that element was useful or explanatory in the particular context.

There are several simple ways that knowledge systems are tuned toward expertise through experience.

1. New experiences generate **new elements** that help individuals with the work of the knowledge system (e.g., explain their physical experience, figure out how to control quantities, etc.)

2. **Changing of priorities** (e.g., the weights of the elements). Priority changes result in certain elements becoming more important and active in the system and other elements becoming less important. To induce this mechanism of learning through instruction, one must have a very good sense of the relevant elements of the conceptual terrain. Further, one must be aware of how the elements might be related since it is possible that several elements can be promoted/demoted in priority at the same time.
3. Elements can take on **new functions** within the system.

These general possibilities for how knowledge systems can increase in complexity will become important in the predictions we make about how the strategy and conceptual systems of interest in this dissertation grow and change.

Reference models

We close this chapter by introducing two reference models of knowledge structures and systems developed from the KiP perspective. The analysis developed in this dissertation does not directly make use of the constructs of p-prims and coordination classes. However, it is useful to have a sense for one, well theorized example of a more elemental class of knowledge structures (p-prims) and another example of knowledge systems (coordination classes). To preview, the analysis in this dissertation will involve both elements (co-variation schemes) and systems (MEA and linear interpolation/extrapolation strategy systems and the linear control of variation system).

Phenomenological primitives

The first and most familiar theory from the Knowledge in Pieces tradition is the theory of phenomenological primitives (p-prims). Phenomenological primitives are *phenomenological* in the sense that they are mental structures that are abstracted from experience and help individuals to form interpretations of their experiences. They are *primitive* in the sense that they are self-explanatory to individuals and that they are encoded and evoked as a single unit (e.g., in contrast to concepts that may be very complexly encoded and evoked).

As a knowledge system, the system of phenomenological primitives is quite large and is dense in the areas in which people have a lot of experience with the physical world. In diSessa, 1993, several dozen p-prims were documented and discussed as part of the argument for showing the wide breadth of contexts in which p-prims function and also to demonstrate through example how this theoretical construct helps us to understand how people reason about the physical world.

The hypothesis is that p-prims are abstracted from everyday experiences in the world and that the proper contextual recognizing takes years to establish through a process of "tuning towards expertise." In terms of encoding, p-prims are not well aligned with linguistic expression. Researchers give labels to different p-prims, but the actual mental structure is not necessarily connected with language at all. The prototypical p-prims is Ohm's p-prim *"More effort begets more result."* This p-prim is what accounts for what someone knows when they "know" that throwing a ball harder will make it go farther or when they know that they will need to work harder in order to finish their dissertation in time!

Coordination classes

A coordination class (diSessa, 1994; diSessa & Sherin, 1998; diSessa & Wagner, 2005) is a model of a particular kind of concept. Unlike p-prims, coordination classes are a *system* of elements rather than just a unitary element. The motivation for the idea of a coordination class is how individuals come to "see" theoretical kinds in a world that is perceptually diverse and complex. For example, one determines "force" in situations differently in situations where one sees a book on a table, a ball tossed in the air, or an object pushed across a table. Typical issues in determining theoretical kinds in the world are that coordination classes (e.g., individuals' knowledge systems) for determining particular quantities lack *span* (e.g., means of determining in different contexts) and lack *alignment* (e.g., the different strategies that individuals do have for making determinations do not give the same result across contexts). As has been documented in several cases, the process of "generalization" of knowledge (e.g., increasing span) occurs through the accumulation of specifics – new strategies of determination as opposed to through processes like abstraction.

In terms of actually "seeing" coordination classes in data, diSessa (2006; 2011) describes two key heuristics:

1. Follow the "eye" (perceptual component – "readout strategies"). Track what individuals are attending to, why, and how.
2. Follow the "mind" (inferential component – "causal net"). Track what individuals are inferring from what they actually see.

Coordination classes and p-prims are discussed here because they will be used as reference models in the work of the dissertation.

Chapter 5: Analytical Perspectives

While Knowledge in Pieces (KiP) describes the theoretical (epistemological) framework that informs the analysis in this dissertation, the family of empirical and analytical strategies that are well suited for investigations aligned with the central principles of KiP (e.g., that knowledge is diverse in form, complex in organization, and usage is highly contextual), is called *Knowledge analysis* (diSessa, 1993, 2004; Sherin, 2001; Parnafes & diSessa, submitted). A characteristic feature of this family of analytic strategies is that the nature and form of knowledge and how it changes as individuals learn is not assumed *a priori*, but rather the structure and dynamics of particular classes of knowledge elements and systems are typically *results* of analyses.[8] Further, the descriptions obtained through the analysis are accountable not only to the empirical data at hand in a given study, but other constraints also come into play (e.g., existing theory and literature, plausible developmental history of knowledge elements and systems, etc.).

In this sense, the analysis in this dissertation is a prototypical example of developing a knowledge analysis – the structure and function of new (candidate) knowledge elements and systems are schematized and their dynamics studied in a way that is constrained both by theory and empirical data.

Because this is a nascent genre of research and because the KiP/KA community investigates processes of thinking and learning across a wide variety of content domains (e.g., mechanics, algebra, basic probability and statistics, special relativity, etc.) using a range of empirical set-ups (e.g., clinical interviews and tutoring interactions, classroom discussions, peer work in pairs or groups), understanding and schematizing this range of analytic strategies is an ongoing and community effort.[9] In an attempt to contribute to this effort, the presentation of the analysis in the following chapter (Chapter Six) has been

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[8] As we become more knowledgeable about how individual knowledge is structured, how it functions in activity, and how the organization of knowledge systems changes as individuals learn, we also need tools and techniques to reliably identify the already discovered knowledge types and their properties in data.

[9] Charting characteristic empirical strategies and analytic practices associated with doing a wide spectrum of work related to "Knowledge analysis" has been the focus of an ongoing research seminar at Berkeley (A. diSessa and M. Levin), conference symposia at AERA, ICLS, and EARLI (organized by M. Levin, O. Parnafes, L. Barth-Cohen, and S. Kapon), as well as the theme of recent workshops of the KiP community ("KiPshop 2009: What is knowledge analysis?" and "KiPshop 2010: Knowledge analysis and Interaction analysis" and an AERA funded Educational Research Conference Program in 2011 on synthesizing Knowledge analysis and Interaction analysis perspectives to studying learning and conceptual change). Much of the material in this chapter is informed by discussions, presentations, and papers in these workshops, seminars, and conferences.

purposefully arranged so as to make visible the strategies and techniques of knowledge analysis that were particularly relevant to developing the final model of strategic and conceptual co-development presented in this dissertation. It is important to understand the programmatic nature of this work. Thus, one function of the current chapter is to prepare the reader to understand both the way the analysis in this dissertation was conducted, but also where we can expect such analyses to fit into larger programs of work.

The first part of this chapter explores some of the general heuristics and strategies that guide some of the prototypical work and questions of interest to this community. The second part of the chapter discusses some related analytic perspectives that are related to, but differ in important ways from Knowledge Analysis. The contrast perspectives we briefly survey include microgenetic analysis (Siegler, 2006), grounded theory (Glaser & Strauss, 1967; Shkedi, 2005), and phenomenography (Marton, 1981; 1986; Marton & Booth, 1996).

Knowledge Analysis (KA)

One indicator of the robustness of our scientific understanding of a phenomenon is our ability to build computational models of it. A central focus of attention in knowledge analysis research is to work *towards* the ambitious program of creating computational models that capture the richness of real-time dynamics of thinking and learning processes. The issue of "modeling at the knowledge level" and how such a program is distinguished from contrasting approaches to building computational models of cognition (e.g., symbolic modeling and approaches to unified models of cognition in the work of Newell & Simon, 1972 and John Anderson, 1996) is discussed in diSessa, 1994 and diSessa, in preparation. However, while an ultimate goal of this program of work involves computational explicitness, this is certainly not a proximal goal. At present, most research is focused on the very question of characterizing the nature and form of students' knowledge structures and systems (and, at the cutting edge, sketching mechanisms of change in form and function of elements and systems).

To give an indication of the shape of this general program using the most prototypical and well-studied knowledge type in this line of research as an example, the empirical core of diSessa (1993) involved giving extended and explicit descriptions of the form and function of a large number of knowledge structures involved in individuals' reasoning about the physical world (p-prims). Modeling the behavior of these structures in terms of connectionist networks and structured priorities was discussed in general terms with the intent of sketching out a way to operationalize the program of modeling thinking and learning processes at the knowledge level. However, while modeling knowledge systems in terms of networks of elements with weightings that encode the likelihood of activation and suppression of elements and subsystems has been productive at a heuristic level, most current KA research does not yet devote significant attention to explicitly operationalizing models.[10] As mentioned above, there is still such substantial

[10] Sherin, in preparation, is an exception in his efforts to computationally realize some aspects of the methodology through the use of Latent Semantic Analysis (LSA), among other techniques. However, Sherin's suite of computational coding techniques serves a

work to be done at the level of mapping out the terrain of individual cognition in terms of *types* of knowledge that capture characteristic intellectual functions. From this perspective, the program of computational realization is still premature.

An area where KA research is starting to break new ground is in the coordination of descriptions of the content, form, and organization of knowledge with descriptions of processes by which knowledge systems grow and change. In the section of this chapter on *microgenetic learning analysis* (Parnafes & diSessa, submitted), we will discuss several studies that exemplify the coordination between a focus on the content, form, and organization of knowledge and processes of change (e.g., diSessa, in preparation; Izsák, 2000; Kapon & diSessa, in preparation; Parnafes, 2007, submitted; Schoenfeld, Smith & Arcavi, 1993 and Wagner, 2006). The discussion of microgenetic learning analysis will be especially important in laying the groundwork for the analysis in this dissertation, that builds on and extends this line of work.

Heuristics and strategies for KA

I now turn to a discussion of some of the heuristics and strategies that KA research uses in working with empirical data. As KA research entails schematizing features of knowledge that are explanatory for making sense of the moment-to-moment details of individual thinking and learning processes, capturing and modeling conceptual nuances is an unavoidable complexity in this program of work. Though making models of mental structures involves making hypotheses about entities that are not directly observable, video-based data of real-time thinking and learning processes can reveal quite a lot that constrains the models that one *can* build. The table below summarizes just a few examples of the kind of nuance in real-time data that is potentially useful to track and the relevant indications of this nuance that are typically available in video records.

Table 3. Features of real-time data useful for tracking of the content and form of knowledge.

Aspect	Data Indications/Guiding Questions for the Analyst
Content	- Individuals' utterances – What did subjects *actually* literally say about the topic across *all* of the data? - Tracking the possibly many different ways that similar ideas are expressed (e.g., important differences in linguistic expression, encoding of understanding in gesture, etc.)
Form	- What do subjects seem to be focusing their attention on? What seems salient to them about a situation? (e.g., eye gaze, gestures, etc. can be indicators, as well as what they say and do). - What do subjects infer about the aspects of the situation that they find salient? - What do they *do* with the information that take in and reason about?

very different purpose in the broader landscape of KA work than work that centers around building models of knowledge structures and systems and how they function in activity.

Table 4. Features of real-time data useful for tracking the organization and dynamics of knowledge systems.

Organization	- Evidence that a subject takes a statement to be obvious and with no further explanation necessary can indicate a unitary encoding of structure (e.g., Explicit statement of obviousness, taking the idea as an unquestioned basis for reasoning, etc.) - Similar ideas expressed at different levels of abstraction can indicate a hierarchical structure with a common root element - What ideas has the subject expressed before a given point in time? Tracking the ideas before and after a focal point can give information about how individuals' knowledge systems are organized - Are some ideas generated on the spot out of other concepts or ideas that are at high cueing or does the subject mainly seem to be reasoning from stable structures that are recognized in the situation? Evidence can be expressions like "Oh!" or wait time that accompanies the assembly of a particular chain of reasoning. - Source of the knowledge element can give clues about what other knowledge or competences might be "in the vicinity" because of history of past usage.
Stability and Dynamics	- What ideas does the subject express first? - Do they remain consistent across instances where the idea should apply? What if the subject's attention shifts slightly within the context? Is the inferential process still consistent? - Does the expression of the idea remain consistent across contexts? - Confidence or conviction in the expression of ideas - Dynamic of the interaction: How does the subject respond to correct or incorrect ideas that are "available" over the course of the interaction. Do they find the "correct" ideas sensible and draw upon them? - In moments where it appears that there is a shift in conceptualization or thinking, are there candidate sources that could have supported the shift? (e.g., interactional dynamics, interaction with a representation or physical model, etc.)

The above chart is certainly not exhaustive with respect to the kinds of indications and evidence one can draw out of real-time data of reasoning processes. However, it does give some sense for the kinds of things that knowledge analysts would try to track or follow in the video of real-time data they are analyzing.[11] Further, these indications in

[11] As I was creating the analytic narratives that were instrumental in the process of performing the analysis discussed in chapter six, I used a similar set of guiding questions. Because I did not begin the analysis with such a list of heuristics, in the case of my analysis such a list of what I was looking for in the data was one of the results that came from iterative analysis of the tapes and through comparison with reference models and contrasting methods.

the data are exactly the kinds of indications of interest that should be available in transcriptions that are prepared for the purpose of knowledge analyses.

Note that the listed dimensions and sources of evidence can also be turned around to generate heuristics for the attributes of data that would be useful to *collect* about content, structure, organization, and dynamics.

I note that looking for indications *in the data* about how knowledge is used and how it changes is a *very* important feature of knowledge analysis. Subjects will not typically be able to access or easily describe the cognitive structures that guide their thinking in a situation (e.g., especially if the knowledge was encoded kinesthetically, in which case verbal descriptions are only a pale approximation).

The heuristics and guiding questions listed above for "open coding" of the data were especially useful in the case under study in this dissertation because the research was not conducted in a content domain in which extensive prior work in charting the conceptual territory had already been done. Thus, heuristics for interpreting and interacting with empirical data like those given above were useful in the process of defining a relevant analytic vocabulary for the analysis. This process is described in more detail in Chapter Six.

In addition to the general kinds of indications about knowledge structure and organization that one can read out of data, principles and heuristics need to be developed for reliably identifying previously studied knowledge structures according to their characteristic features. In addition to a list of some three-dozen p-prims that include typical structural and content features of each of the particular p-prims and their contexts of typical use, diSessa (1993) provides a description of fifteen heuristics for the identification of p-prims in general (as opposed to other kinds of knowledge elements). Many of these principles are meant to be specific to p-prims (e.g., principle of the body). However, some of them can be adapted and used as much more general principles in the analysis of knowledge structures and systems (e.g., diversity, coverage, functionality, continuity). See diSessa, 1993 for more details on these heuristics and their role in guiding analyses.

Contrast methodologies

We will now briefly discuss two prominent perspectives on qualitative research that guide many current studies: grounded theory and phenomenography. The purpose of highlighting the similarities and differences between these two methodologies and KA is to better understand which research practices are particular to KA (because of the nature of the data and intended scope of theorizing) and which practices are shared with other standard qualitative research methods.

Grounded theory

Grounded theory (Glaser & Strauss, 1967; Shkedi, 2005) is a qualitative research methodology in which theories emerge from iterative analyses of data – prototypically narrative accounts of individuals about some aspect of their experience. The methods of grounded theory are particularly well suited in cases where either theoretical or researcher bias is a concern. For example, grounded theory is a foundational methodology in the study of culture for the reason that there is a strong danger of researchers overlaying their own cultural categories on the experiences of others. "Member checking" is one means of validating the accounts that are generated by grounded theory research.

The analytic process begins with an "open coding" of the data in which initial line-by-line passes through the data corpus result in covering the entire corpus with a set of codes that capture important features of the data. Through subsequent analysis, synthesis, and refinement, a coding scheme and relationships between coding categories emerge, from which the grounded theory is developed. The refinement of the coding scheme happens through a method of "constant comparison" of coded instances for the purpose of determining how to merge and separate categories. The process of constant comparison continues until the categories are "saturated." Saturation happens when all of the instances of interest to the theory are covered by codes in the scheme. In terms of workflow, grounded theory researchers maintain records of the iterative process of generating transcripts, coding, and refining the coding in the transcripts.

Shkedi (2005) describes a version of grounded theory that he uses in qualitative studies of multiple case narratives. The first step he discusses in the analysis involves the categorization of transcripts and subsequent mapping of the categories that are present in the transcripts. After this stage, the analysis proceeds through a reduction and focusing phase in which the researcher decides what is more core to the data and what the focus of theorizing will be about. After having located focal categories and relations, the researcher attempts to created a focused narrative report. These stages are all interpretive stages that take place before the stages in which the theory-building stages. The development of theoretical categories is a negotiation anew with the data and in dialogue with the literature. The process continues in dialogue with the data to make sure that the categories that have emerged/been refined still cover the focal phenomena in the data. In the conceptual/theoretical stage, relations between categories are searched for, especially those that bring together many topics.

Grounded theory has been rarely used in studies of cognition. Taber & García-Franco (2010) is one exception. In this study, grounded theory is positioned as being involved in earlier stage work around identifying core constructs (e.g., like specific p-prims in relevant to chemistry education) and then one moves out to larger N studies to validate the findings of the grounded studies. Note, however, that Taber's focus in these studies is on finding categories that capture students' intuitions about physical and chemical phenomena, not on studying learning *processes.*

Phenomenography

The focus of phenomenographic research concerns the qualitatively different ways that people experience the world. Phenomenographic studies are by definition "second order" – that is, the object of study is how *people perceive* their own experiences. Phenomenographic studies can be of groups or individuals and can span many different aspects of experience. The aim is to find categories of experience that apply across individuals. Phenomenographic research typically focuses on participants' narratives about their experience as opposed to inferring what subjects are noticing or attending to based on their actions and words in an interaction. As such, a typical output of phenomenographic studies is the generation of a set of conceptual categories shared across several individuals. Though many researchers who are informed by phenomenography choose to use the techniques and research processes of grounded theory, the perspective itself has no highly schematized and shared research process.

Comparison with Knowledge Analysis

Grounded Theory (Glaser & Strauss, 1967) and Phenomenography (Marton, 1981; Marton & Booth, 1996) provide interesting points of similarity and contrast to KA. An important shared feature among all of the approaches is that they are focused around theory generation (as opposed to theory testing). Further, all three perspectives are concerned with the "bottom-up" generation of categories that capture the perceptions and perspectives of the subjects in the study (as opposed to data reduction techniques that use an *a priori* defined coding scheme).

Both grounded theory and phenomenography rely upon narrative accounts as data and member checking as means of validation. However, participant narratives would not be the most useful data for analyzing the structure, organization, and dynamics of learners' knowledge systems. For one thing, learners do not necessarily have access to language to describe their perceptions (especially for meaning that is encoded through other modalities). The data that is of most interest for KA research is that of knowledge systems-in-use. KA research also requires different means of validation of the accounts generated than grounded theory methods given that whether or not the technical analytic vocabulary generated through analysis is sensible to learners is not consequential in determining its validity. Means of validating the models and analyses generated through KA are still being discussed and developed. There is a large space of issues to consider. Among others, these issues range from the *replicability of analyses* (e.g., Given a precise enough description of the final constructs and models developed, can other researchers reliably "see" the same constructs and models in data?) to the *theoretical cogency* of constructs or models resulting from analyses.[12]

In terms of theoretical scope, while KA methodologies have been designed to develop theory concerning the structure, organization, and dynamics of knowledge systems, both grounded theory and phenomenography have typically been used more broadly to study the experiences of individuals in different cultural and situational contexts (e.g., how managers in companies interpret organizational changes, how individuals interpret poetry, the experiences of dying individuals in a health care system, etc.). In particular, grounded theory and phenomenography methods are severely limited

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[12] Certainly, inter-rater reliability would be a way to address the standard issue common to all qualitative analyses about whether analytical constructs are adequately operationalized. However, the issue of theoretical cogency of the epistemological constructs generated through the analysis and requires argumentation that is likely to be quite specific to KA research. Competitive argumentation is one general technique that could inform this pursuit. See vanLehn, Brown, & Greeno (1984) and Schoenfeld, Smith, & Arcavi (1993) for discussions of competitive argumentation.

in their ability to capture micro-processes of change. Change in studies employing grounded theory tends to be restricted in scope to rather coarse-grained stage models (e.g., Isabella, 1990) and not well adapted for capturing the moment-to-moment shifts in understanding that KA research is accountable to.

The specific prescribed research steps characteristic of grounded theory research are not a workflow processes shared uniformly by KA researchers, though some researchers implicitly do a version of this. One critical difference is that KA research requires a coordination of grounded "bottom-up" methods (open analysis) and "top-down" methods (negotiation with epistemological principles that guide, but do not dictate the course of analyses). There are many cycles of refinement of the constructs and models generated through KA[13], but it is not easy to describe in advance the exact process that a researcher will need to go through in order to develop the specific models relevant to a particular research study. Trying to characterize regularities of the research practice through methodological case studies is a current objective of the KiP/KA community.

Microgenetic methods for studying learning

This final section of the chapter discusses two qualitative research methods that are specifically designed for the study of processes of learning and change*: microgenetic analysis* (e.g., Siegler, 2006) and its KA counterpart, *microgenetic learning analysis* (Parnafes & diSessa, submitted).

This discussion of these two methodologies is especially important in laying the groundwork for the analysis in this dissertation. Children's strategy use and construction has been extensively studied by Siegler and his colleagues and in fact, microgenetic analytic techniques were invented for exactly this purpose. However, the approach to studying strategy change that is developed in this dissertation entails characterizing both strategies and the knowledge involved in implementing them as complex knowledge systems and therefore requires a somewhat different methodology, building more on the techniques of knowledge analysis and microgenetic learning analysis.

I start with a discussion of the way strategy use and change has been researched in the psychological literature using the techniques of microgenetic analysis and then turn to a review of a sample of exemplar microgenetic learning analyses and their characteristic features.

Siegler's approach to microgenetic analysis

The use of microgenetic methods in cognitive psychology was developed substantially by Siegler in his work on children's arithmetical strategies. One of the earliest and most detailed investigations of strategy use and construction was conducted by Siegler & Jenkins (1989) in which they studied kindergarten students solving arithmetic problems and inventing new ways to accomplish these tasks. In this study,

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[13] See diSessa, 1991 for a discussion of Observe, Schematize, Systematize (OSS) cycles and Parnafes & diSessa (submitted) for a methodological case study and a discussion of theory-building phases.

the strategic transition in question concerned how children constructed the idea that they could "count on" from either of the two addends as opposed to a common earlier strategy of "counting all." The experimental set-up involved giving the children a large number of problems to solve and watching how their approaches changed over time. Within the design of the sequence of tasks was the manipulation later on to give children problems that *could* be solved by "counting all," but that would be easier to solve if one counted on from one of the addends instead. In addition, the interviewer would ask the students how they solved the problem in between each problem. A summary of the findings of Siegler & Jenkins' study includes:

1. Children used multiple strategies including
    a. the "count all" strategy, sometimes called the "sum strategy,"
    b. the "shortcut sum" strategy – where to compute 2+3, children would represent the addends with their fingers and count 1, 2 on one hand and then continue 3, 4, 5 on the other as opposed to a less efficient version of "count all" where children would count 1, 2 on one hand and then 1, 2, 3 on the other hand, and then count all of fingers on both hands 1, 2, 3, 4, 5.
    c. the "count on" strategy – where children would find the sum by counting on from either one of the addends instead of both counting up to the first addend and then counting on from there to find the sum),
    d. the "count on from larger" optimization of "count on,"
    e. retrieval from memory, and
    f. recognizing a given problem such as 7+3 as equivalent to another that they knew such as 5+5.
2. Students made articulate and adaptive choices between the multiple strategies that they have available to them (e.g., the strategies listed above).
3. Children were able to generate the new strategies simply through engaging in the task sequence. It was also noted that specific features of the task sequence, such as the presentation of "challenge" problems that included one addend much larger than the other, seemed to support change and adaptation.
4. The "shortcut sum" strategy seemed to be a transitional strategy to "count on."
5. New strategies appeared to re-use and re-purpose parts of previous, successful strategies.
6. New strategies or adaptations seemed to appear after previously used strategies failed, but that impetus for new strategy development did not need to be the failure of previously used strategies.
7. The adaptation of strategies for use across a broader range of problems seemed to be a very slow process.

Siegler's work on children's strategy usage spans many related phenomena. His earlier work focused more on understanding *why* strategy shifts occurred for individuals (aligned with the focus of this dissertation) whereas later work became more focused around understanding patterns in children's strategy choices and the reasons for particular strategies becoming more or less prominent in students' repertoires. See Siegler (1996) for an extended discussion of Overlapping Waves Theory.

In Siegler's early writing on strategy discovery, the default assumption was that the discovery of entirely new strategies involved sudden breakthroughs, whereas the extension of existing strategies to wider classes of problems was a more gradual and incremental process. However, further evidence (including evidence from computational modeling studies that built on the corpus of data collected in the studies of children's use and discovery of arithmetic strategies) indicated that both processes were gradual and incremental (Siegler & Araya, 2005).

In conducting this line of research on strategy discovery and change, Siegler called the methodology he employed in order to study strategy discovery and change "microgenetic analysis" (Siegler & Crowley, 1991; Siegler, 2006). General features of the approach include:

1. Observations span the entire period of change
2. Observations must be dense compared to the timescale of expected change
3. Intensive trial by trial analysis is used

In Siegler's formulation of microgenetic analysis, change is tracked at the level of performance on a single trial (e.g., where a "trial" is synonymous with one problem). The trials are then coded by a summary judgment of the key features of the procedure used by students to solve the problem. In this way, trends and patterns in the approaches taken by students to solving a large battery of problems can be tracked at a problem-by-problem grainsize. At this level of granularity, one can identify the exact problem where a transition in strategy occurred. One can then look before this point for clues concerning what preceded the change and one can also look at the reaction of the child beyond the trial in question to see whether they recognized and drew upon certain features of their discovery in future trials. Such a design stands in contrast to designs that would sample students' performance every couple of months (as in a longitudinal design), before or after an intervention (as in a pre-post design), or sample performance across groups of differing ages (as in a cross-sectional design).

The strategy discovery and change literature places emphasis on tracking observable changes in problem solving procedures over several instances of solving similar problems. The nature and form of the mental representations and processes that undergird the implementation of such procedures are not typically the focus of the majority of these studies. Notable exceptions include Opfer & Siegler (2007), Siegler, Thompson, & Opfer (2009), and Alibali (2005) who were concerned with tracking change at both the level of internal knowledge/representations and external problem solving actions. However, in these cases, the relationship between internal representation and problem solving procedures is rather different than the model of the relationship that I formulated through the process of doing the analysis in this dissertation.

Microgenetic Learning Analysis (MLA)

The following studies illustrate microgenetic studies of emerging conceptual competence in a variety of educational contexts (tutorial sessions, pair problem session work between peers, and classroom discussions), over a variety of topics in science and mathematics (elementary statistics, oscillation, modeling/algebra, and equilibration) and across a range of student populations (middle school, high school, and university students).

These studies all share the property that they are focused on giving moment-by-moment accounts of learning processes. In that sense, the temporal resolution and density of observations in the following studies is similar to that of Siegler. However, because the process of change in question in these studies concern the development of conceptual competence, the nature of the relevant observations can look quite different than the types of observations that Siegler used to track strategy discovery and change (e.g. repeated trials).

In the studies of open problem solving and explanation of how physical or mathematical phenomena work, there is no longer the forced "unit" of analysis that is provided by repeated trials. However, analogues could be proposed for situations involving open problem solving and explanation generation. One possibility is that "threads of inquiry" (e.g., Why does X happen?) serve an analogous function to the performance units (e.g., work on simple, individual problems) in these analyses. The process of reasoning is more open to inspection in inquiry threads and the changes that are observed within and across inquiry threads may be of a different sort than the kind of changes that are observed across repeated trials. One constraint of many microgenetic studies with repeated trials is that the trials or observations are simple enough that there is not much opportunity to observe the process *within* a given trial.

Further, a key difference between the following microgenetic studies and the microgenetic studies conducted by Siegler is that a driving goal of the following studies is to develop explanatory models of learning processes in epistemological terms. That is, to use qualitative case studies of learning processes in order to theories of learning that are accountable to the empirical details of how knowledge functions and changes organization in the episodes under study. These studies all share an important and highly consequential intellectual commitment to developing accounts that are attentive to what the *learner* attends to, perceives, infers, etc as opposed to characterizing emerging competencies in terms of a pre-determined inventory of performance types.

These two features, (1) the nature of the observations under study as described above and (2) an eye toward theoretical innovation through the close analysis of episodes of conceptual learning, led Parnafes & diSessa (submitted) to distinguish studies of the form reviewed in this section to be exemplars of what they have termed "microgenetic learning analysis."

I organize the discussion of studies of microgenetic learning analysis according to contexts. I start by discussing microgenetic learning analyses that take place in the context of clinical studies of teaching and tutoring (Schoenfeld, Smith, & Arcavi, 1993; Wagner, 2006; Kapon & diSessa, in preparation). I then turn to the context of pairs of peers working together on open-ended tasks (Parnafes, 2007; submitted; Izsák, 2000). The final category includes a microgenetic learning analysis of a classroom discussion (diSessa, in preparation).

Following the discussion of the exemplars of microgenetic learning analysis, I will describe how the analytic approach I employ in my dissertation shares features with both versions of microgenetic analysis ("Siegler school" and "Microgenetic learning analysis"). I begin with a description of three microgenetic studies of clinical teaching and tutoring.

Schoenfeld, Smith & Arcavi (1993) provides an important and foundational reference point for the microanalytic work in this dissertation. Like in the analysis in this dissertation, the analysis in Schoenfeld, Smith & Arcavi is concerned with giving a rich and detailed account of the learning process of an individual student over a medium length of time (7 hours) in terms of knowledge structures and how they are organized. The context of the study was a series of tutorial sessions in which a 16 year-old student, IN, and a graduate student tutor jointly explored equations, functions, and graphs of linear functions in a computer microworld.

The data and analysis in Schoenfeld et al. are illustrative of some of the particular aspects of contextuality and fragmentation in learners' knowledge systems. Their analysis explains, in terms of the organization of the student's knowledge structures and system, why they did not build lasting and stable knowledge of symbolic and graphical representations of functions. For example, at the local level of knowledge structures, the "3-slot schema" (that three pieces of information are necessary to determine a linear function: slope, y-intercept *and* extraneously, the x-intercept) is an example of a knowledge structure that Schoenfeld, et al. found that the subject used.[14] At a more systemic level, some fundamental misunderstandings about the *Cartesian connection* – that a point satisfies a functional relation if and only if the point is on the graph of the function in the plane – explained observed difficulties the student had in interpreting the meaning of the y-intercept across contexts.

In conjunction with the analysis in the paper, the authors develop a description of a knowledge architecture that is consistent with the line-by-line analysis of IN's performance. In the model they propose, knowledge systems have four levels that are characterized by grain size. The topmost level includes *schemata* that organize expectations based on prior experiences. The example given of a schemata is that "A line has equation $y=mx+b$ if and only if its slope is m and its y-intercept is b." The next level down includes *concepts and conceptual entities* (e.g., Greeno, 1983). These are the objects that fill slots in the schemata in level one. The third level is the level of *fine-grained structure* that determine properties of objects. The example given here is the "Cartesian connection" (discussed above). The fourth level is that of *conceptual elements*. The example given of "conceptual elements" are the particular and non-normative meanings that IN had for "y-intercept" across contexts.

Schoenfeld et al. hypothesized that the procedures that the subject was using (that were abstracted from experience with the microworld and previous learning experiences) primarily involved "conceptual elements" from level four and because they were not connected to knowledge in the higher levels (e.g., fine-grained structures, conceptual

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[14] Moschkovich, 1999 found this same structure in use by other students in a separate and independent study. This indicated that the 3-slot schema was not an idiosyncratic structure of one student but rather an understanding that could be identified in a broader population.

entities, and schemata), they were more easily used but also that new, correct knowledge that was introduced and used in the sessions was more easily forgotten.

The microgenetic analytic methodology as developed in Schoenfeld et al. is a foundational reference point for the work in this dissertation. Like other microgenetic studies (e.g., Siegler, 2006) it involves a high density of observations of knowledge function and use in a particular domain over a short period of change. The particular focus of the analysis on elaborating relevant knowledge structures and making models of how knowledge is organized distinguishes it from microgenetic studies reviewed in Siegler.

The program of elaborating and validating the four level model proposed in the paper would involve elucidating through the analysis of data in other specific learning events the form of conceptual elements, fine-grained structures, conceptual entities, and schemata and how they interconnect. Because there were few learning events observed in the data, this analysis contributed more to elaborating reasonable constraints on knowledge organization and learning as opposed to describing mechanisms and processes of learning. Thus, to complement this work, a natural frontier is to consider data that is rich in learning events for the purpose of seeing knowledge construction in action.

I turn now to a microgenetic case study of an undergraduate student, Maria, working with a tutor/researcher in the domain of elementary statistics (Wagner, 2006). Over the course of several sessions, Maria worked on a series of problems all of which concern the expected value of a random variable and the law of large numbers (e.g., that the average of a set of values randomly sampled from a population is more likely to be near the expected value – the population average – when the sample size is large). The phenomenology of interest to Wagner's study is how the student eventually came to reason about a diverse set of contexts in terms of a single principle, the law of large numbers, instead of offering solutions that involved different rationales and reasoning across cases.

Through this extended analysis, Wagner develops a theoretical model for understanding knowledge transfer, a "transfer-in-pieces" perspective that builds on Knowledge in Pieces (diSessa, 1993) and coordination class theory (diSessa & Sherin, 1998). Wagner's analysis thus gives a knowledge-based alternative account for transfer to that given in the psychological literature. Instead of Maria developing an abstract mental representation that allow her to solve of the problems with the same principle, the focus of Wagner's account largely concerns when and how Maria perceives productive knowledge that she already has and uses in local contexts to be more widely relevant and useful. In coordination class theory terms, this observed incremental knowledge transfer is described as the span of her coordination class for expected value being extended through the addition of situation specific knowledge using various means such as sampling activities, interaction with a BOXER microworld, and direct comparison of problems. Wagner's analysis is squarely in the regime of microgenetic learning analysis. Though no new knowledge ontologies are uncovered through the analysis, the study extends coordination class theory in a fundamental way to develop an alternate model of knowledge transfer.

I now discuss a third example of a microgenetic learning analysis that took place in a clinical teaching/tutoring context. Kapon & diSessa (submitted) conducted a microgenetic cross-case comparison that analyzed the striking variability in how students made sense of an instructional analogical sequence about the existence of the normal force in physics (Kapon, 2010; Kapon & diSessa, submitted). An output of the analysis was the "explanatory primitive" model that examined students' reasoning in terms of classes of knowledge elements that students took to be explanatory as they reasoned about forces. The instruction in the study was based on an augmented version of a "bridging analogies" instructional sequence designed by Brown & Clement, 1989. The explanatory primitive model is a functional model (as opposed to a structural model) that can be thought of as a generalization of the p-prim model. This model was useful for explaining the differences in individual students' reactions to and shifts in judgment about the clinical instruction. Additionally, the model developed from this microgenetic analysis accounts for patterns in students' reasoning in the target domain that were not explained by the literature on analogical reasoning (e.g., Structure Mapping Theory of Gentner, 1983).

I now discuss microgenetic studies that take place in the context of pairs of students working together to solve problems and reason about physical phenomena.

Parnafes, 2007 investigates the role that computer-based simulations play in the development of students' conceptual understanding of harmonic oscillation. Through a moment-by-moment microgenetic analysis of observations from eight pairs of middle and high school students engaging in explorations involving physical oscillators and computer-based simulations, Parnafes constructed a model to explain how representations mediated the aspects of the students' growing understanding. The fine conceptual and temporal resolution of the knowledge-in-pieces perspective (diSessa, 1993) and coordination class theory, in particular (diSessa & Sherin, 1998) was instrumental in tracing how the students' knowledge systems interacted with representational features in the simulations. The resulting model consists of four mechanisms: (1) detecting a pattern in the simulation (2) mapping the pattern detected in the representation onto corresponding physical phenomena (3) identifying a conceptual challenge in explaining the observed pattern, and (4) extension of existing inferential relations to new contexts. The methodological approach employed by Parnafes to elucidating the content, form, and mechanisms of change evident in the data was instrumental in devising the analytical framework and evolving stages of analysis (e.g., knowledge schematization and then mechanism schematization) in the case study developed in this dissertation.

A second study by Parnafes (submitted) investigates the explanations that pairs of students jointly constructed as they generated drawings of scientific phenomena that they were trying to explain. Parnafes examines the process by which pairs of students work together to generate shared explanations and shared representations that explain how the phenomenon of the moon phases works. A single pair of students worked together over ninety minutes to produce an explanation. The study shows how the students, in interaction, guided occasionally by a researcher/teacher, came to activate and coordinate many pieces of knowledge of diverse types (ranging from mental models and general schemata to declarative facts) into a shared explanation. Increasing the *resolution* and *range* of the explanation were shown to be mechanisms that drove this learning process. As with Parnafes (2007), the focus on both the content and form of knowledge, in addition to learning mechanisms was a notable feature of this analysis. However, it is worth noting that one aspect of this study that differs from previous studies reviewed thus far is that in this study, the activation and use of many, different forms of knowledge is traced through the analysis (e.g., as opposed to analyses in which only one knowledge system or knowledge type is of focal interest to the analysis, such as coordination classes). This feature of accounting for and tracing diversity in forms of knowledge also informs the analysis in the current dissertation study.

A third exemplar of a microgenetic learning analysis taking place in the context of pairs of peers working together is Izsák (2000). Izsák studied students working together to model the behavior of a physical device called a winch. The relevant knowledge structures that came into play in Izsák's analysis were *symbolic forms* (Sherin, 1996; 2001), a knowledge structure that binds a symbol template with patterns of experience with how the world works. Izsak's analysis revealed the construction of a particular symbolic form: "base-plus-change" and the use of two mechanisms "mapping variation" (e.g., adjusting correspondences between the physical and algebraic representation) and "notation variation" (e.g., adjusting the number of terms and their compositions) in constructing a valid equation that would model the behavior of the winch. Izsák's study contributes to the program of ongoing theory development and elaboration through tracing the construction and use of a particular knowledge type that had been identified in previous clinical work (Sherin, 1996; 2001) and then uncovering mechanisms of learning in his analysis.

The final context in which I look at microgenetic studies of learning includes a case drawn from a study of classroom interaction.

DiSessa (in preparation) builds off extensive earlier work on characterizing elements of students' intuitive conceptual ecology (diSessa, 1993) in order to understand an episode of learning taking place between students in a classroom. The topic the students in the class are discussing is Newton's law of heating and cooling (that the rate of temperature change between two objects in contact or between an object and its ambiance is proportional to the difference between the two temperatures). The central episode under study in the paper concerns the students in the class observing a graph of the temperature curve obtained from experimental data and trying to explain its behavior.

The explanation that is constructed by the students included a number of previously documented intuitive schemata (diSessa, 1993). A notable feature of the analysis is the coordination of careful schematizations and descriptions of knowledge (building off a large corpus of previous work in the context of interviews with individual students) that can capture the nuances in meaning with their use in the social construction of an explanation. Another advance of the analysis is the development of empirically-grounded learning mechanisms. Some examples of relevant learning mechanisms introduced in the analysis and described in detail in the paper include: "context shifting" (i.e., In this case, the perception of agency in a situation where none is usually perceived.), "composition by causal chaining," "causal interpolation," "binding" (i.e., the mapping between features of intuitive schemes and attributes of the physical situation), and "emergence" (i.e., generation of an explanation and judgment of satisfaction with the emergent explanation).

Discussion

As I described in the introduction to this section, although my methodology involves a version of "repeated trials" (and is thus similar to Siegler's methodology), my analytic approach is strongly informed by the focus on theory development shared in the microgenetic studies described in the previous section. One essential similarity between my study and the MLA studies is that in my analysis, I develop an explanatory account of strategy emergence through modeling both strategies and the conceptual knowledge that implements them as complex knowledge systems. The content and form of the systems is schematized from the empirical data in the case, as are hypothetical mechanisms of learning. This approach contrasts both with the approach of tracing students' emerging conceptual understanding in terms of a pre-determined inventory of concepts that a researcher assumes students either have or lack and also with the approach of tracing emergent strategies in terms of performance types that roughly characterize the strategic actions that students took in solving a sequence of problems.

Given the limited scope and span of the particular problems in any given problem sequence, the slice of conceptual knowledge activated certainly cannot allow us to make a representation of a very wide breadth of an individual's knowledge about relevant concepts that might be identified in an a priori task analysis of the problems in the problem sequence. It may not even be the case that the "pieces" of knowledge that the individual displays (e.g. what they base their reasoning upon) correspond neatly to "pieces" of the understanding that would be identified by an expert or by a textbook task analysis as component "pieces" of understanding.[15]

The approach taken in this dissertation is that a major component of the analytic work lies in trying to uncover what the relevant pieces of knowledge in the context of solving particular problems are through studying the students' thinking processes in action. Thus, the microgenetic study of problem solving over time is not intended to be diagnostic of the *breadth* of conceptual understanding of a particular topic. However, the variety of microgenetic analysis I pursue does have the potential to allow a window into the quality and functioning of the knowledge required to solve the problems in the sequence.

Now that we have introduced the research focus, the context of the study, the theoretical framework that will guide the analysis and relevant analytical perspectives

[15] This is certainly the case in the analysis presented in this dissertation. The progression of understanding that is tracked is an emergent phenomenon of the interaction between Liam and the tutor/researcher that is mediated by his prior knowledge, the way the activity is structured, the artifacts that are available to him or that he constructs. Through the interaction, the student has the opportunity to learn much more (and different things) than an *a priori* task analysis would suggest. See Saxe, 1988 and 2002 for a theoretical framing of the issue of *emergent goals* in the study of mathematics thinking and learning.

that frame the work of the dissertation, the next chapter will move into a discussion of the core analysis in the dissertation and how it was developed.

Chapter 6: Analysis

The aim of this chapter is to develop an in-depth microgenetic case study of a single pre-algebra student, Liam, constructing a deterministic algorithm for solving algebra word problems that have an underlying linear structure (i.e., problems that one familiar with the domain would recognize as being of the form: "For a specified y, find the x value that solves the problem $y=ax+b$"). At the time of the study, Liam had not yet learned about linear functions in his pre-algebra class or about how to solve problems by formulating equations that model the situations described in problems.

The deterministic strategy Liam constructs will be seen to emerge as a gradual refinement of his initial strategy: a simple, yet purposeful guessing and checking strategy based around a version of *means end analysis* (MEA) or the idea of "getting closer and closer to the solution." The strategy that Liam comes to construct over time can be recognized as *linear interpolation/extrapolation*. This later, conceptually more developed strategy is based around the idea of using the results of two trial values in order to find how much a unit change in input would affect the output. One can then extrapolate from a base value using this information of the worth of a unit-increment in order to determine the solution to the problem. To elaborate on these two problem solving approaches

- **Means end analysis** is a problem solving approach used to control searches over solution spaces. Briefly, in this context, the idea behind the strategy is that one aims to choose a sequence of actions (e.g., a sequence of trial input values) that will result in successively reducing the difference between the results of the current trial and the target value specified in the text of the problem.
- **Linear interpolation** (See Meijering, 2002 for a historical overview of interpolation techniques) is a method for approximating a value of a function using known values at other points. In *linear* interpolation, approximations are obtained for function values between two given points based on assuming a model of linear variation between those points. [16]

To an observer with no knowledge of the intervening episodes, this shift in strategies would look quite dramatic and discontinuous. The analysis in this chapter closely examines the content and form of Liam's performance on several intermediate problems (and notably, the conceptual structures that supported the implementation of his strategies), for the purpose of tracing the process by which the novel strategy mentioned above emerged. With the analysis at this grain size and with this analytic focus, one sees significant continuity. The analysis developed in this chapter is organized around the goal of understanding, at a level of mechanism, how a novel strategy is constructed.

[16] In the case we study here, Liam uses interpolation to find the worth of a unit increment and then determines out how he needs to adjust the input value through extrapolation to solve the problem.

Microgenetic analyses of strategy change have focused on developing techniques for tracking shifts in strategy usage at a fine-grained level of detail. The analysis that follows also involves tracking strategy change at a fine-grained level of detail (both temporally and conceptually). However, the focus of interest in the current analysis is on processes by which novel strategies can be constructed out of conceptual resources as opposed to the processes by which individuals come to reliably activate and use one strategy over a competitor strategy. In addition to an interest in the identifying and tracking changes in the content and form of the knowledge needed to implement strategies, a key analytic move in developing an understanding of how a novel strategy is constructed was to give a specification of a problem solving strategy *itself* as a type of knowledge (a complex knowledge system, in fact).

The analysis presented in this chapter offers an explicit look at strategic performance and mechanisms of strategy change, while also attending to the structure and organization of knowledge that undergirds strategic performance and how it changes. Such explicit and fine-grained focus on the interrelations between strategic and conceptual knowledge as they co-develop elaborates the kind of description of the construction of strategies that is found in the current psychological literature on strategy construction and strategy change (See Siegler & Jenkins, 1989 and Rittle-Johnson, Siegler, & Alibali, 2001).

Outline of the analysis

The presentation of the analysis will proceed in several phases, mirroring the process by which the analysis evolved in response to the analytic challenges involved in constructing a model of the process of strategy and knowledge co-development. This rhetorical strategy was chosen to help the reader understand the particular choices that were made at each juncture of the analysis. In the presentation of the analysis there is a deliberate and constant dialogue between (1) the model that is being developed in the context of explaining particular data (e.g. describing the process by which a particular strategy was constructed out of particular conceptual knowledge) and (2) the theoretical framework that both informs and is developed through the process of the analysis.

The six strands of analysis that will be discussed are:

1. **Analytic strand one:** Documenting, describing, and framing change in strategies
2. **Analytic strand two:** Selecting an epistemological framework and interpreting the data to be analyzed with respect to the principles of the framework
3. **Analytic strand three:** A "bottom-up" approach – Generating a base vocabulary of strategic and conceptual knowledge structures implicated in the process of change by schematizing knowledge-in-use
4. **Analytic strand four:** Putting the new vocabulary to work – Re-describing snapshots of the strategy construction process in terms of underlying knowledge structures
5. **Analytic strand five:** Locating instances of development across the entire learning process under study and positing sketches of learning mechanisms that result in the micro-developments identified

6. **Analytic strand six:** Putting it all together – Creating a model of the process of the co-development of strategic and conceptual knowledge

To preview, and to help the reader orient toward the ultimate goal of this chapter I present here a top-level view of the model resulting from the analysis. The model shows the co-development of strategic and conceptual knowledge in this case to be a process of mutual bootstrapping. That is, there is a dialectic relationship between the performance of a strategy in the course of solving a problem and the conceptual underpinning of the performance.

- *Approaching problems within a particular "strategic frame" involves the activation of particular conceptual knowledge:* Engaging in the activity of solving the particular problems activates a certain "pool" or slice of conceptual knowledge that is perceived (by the learner) to be necessary for solving the problems at hand. In the case analyzed in this chapter, an important class of knowledge that was identified as being relevant to solving problems is how to control the variation of a linear function. The structure and function of this form of knowledge in this activity will be elaborated through the analysis.
- *Conceptual developments (in relation to the activity of solving these problems in this context) facilitate the construction of a new strategy:* Over time and through engagement, the knowledge that is recruited in solving these problems forms an increasingly organized conceptual scheme.[17] The development of this conceptual scheme then facilitates the development at other levels of the strategy as a knowledge system (e.g., the solver's conceptual categories and relations). Developments at these levels then lead to the construction of a new strategy.

Through the strands of analysis outline above, the specific details (e.g., the form and function of the strategy system in question, the nature of the conceptual scheme and component parts that are activated and refined, etc.) of this model will be elaborated. In terms of mechanisms or mediators that drive the co-development process forward, in the particular case studied here, a concern for both *efficiency* and *accuracy* (both identified as mechanisms of development in the previous literature on strategy change and construction – see Siegler & Araya, 2005) was seen to be instrumental. Thus, the model presented here gives a fine-grained view of how efficiency and accuracy drive the process of strategy construction forward, in addition to demonstrating the dialectic between strategic and conceptual growth.[18]

The amount of both strategic and conceptual development observed was influenced by the pragmatics of the activity (e.g., continued development was contingent

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[17] Note, also, that in general, not all of the knowledge used in the service of solving a particular problem need come from the same conceptual scheme. The pool of knowledge activated over the course of solving a given problem may involve many pieces of knowledge of many forms (e.g., remembered facts or theorems, intuitive "rules," embodied schemes, etc.)

[18] Liam was exceptional in his concern for efficiency and accuracy. In Chapter Seven, I discuss this issue further.

upon what Liam considered "good enough" for the purposes of solving the particular problems he was given to solve). It would be a mistake to consider the development of either Liam's strategy or the ongoing tuning and articulation of his conceptual scheme without reference to the specifics of the context (including interactions) and the activity in which he was engaged. However, the end goal of the analysis is *not* only to explain what happened with this one particular student in this case. The goal is to develop this understanding in a way that motivates future work. As we will see, negotiating with a theoretical framework throughout the analysis establishes grounds for and puts us in a position to conclude this analysis with a set of conjectures that will guide future work.

In presenting the final model up-front, it is important to note that this is a top-level summary of a model that was developed through the analysis presented in the rest of the chapter. The analytic process involved iterative and successive attempts at capturing the moment-to-moment dynamics of the co-development of strategic and conceptual knowledge. The model we ultimately present was judged to be more explanatory and have a more adequate fit with the data than alternative models generated in earlier stages of this research.[19] This brief aside is meant merely to attune the reader to a constant theme throughout the analysis: the critical re-description and refinement of working models of processes of knowledge construction.

Strand 1: Documenting, describing, and framing change

To prepare for the discussion of *how* the conceptually more advanced strategy in this case emerged, data excerpts illustrating the approach taken by Liam at the beginning and ending of the sessions are given below. One can see that while Liam's initial approach was based on purposeful guessing and checking, his later approach is deterministic, building of the linear structure underlying the problem contexts, and in fact no longer involves "guessing" at all. As the descriptions in this section are meant to highlight broad differences in Liam's approach at beginning and ending points in the sessions, the presentation of the data here suppresses many details of the context and interaction. The same two episodes will be revisited in fuller detail in strand four as the process of change is traced in close detail across the sessions.

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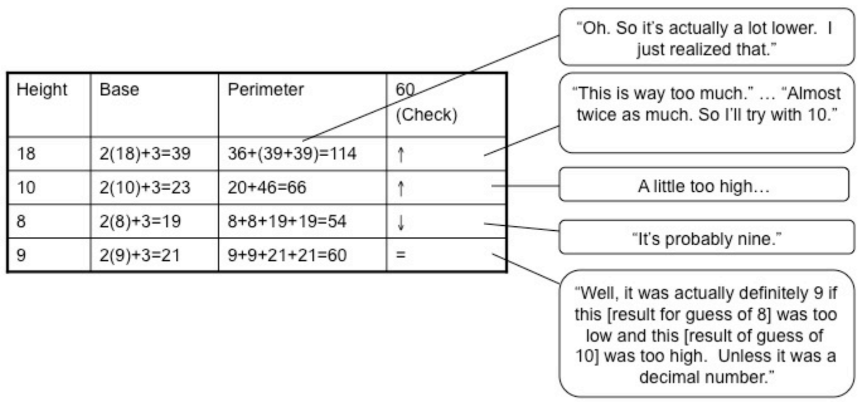
[19] An example of an earlier, related, but incomplete, working model involved an asymmetric conceptualization of the relationship between conceptual development and strategic development with shifts in conceptual understanding allowing a new strategy to emerge or be constructed. The earlier model was revised because it did not attend sufficiently to the structure of the conceptual scheme being developed, nor did it capture the structure and function of the strategy as a system of knowledge that selects for particular conceptual knowledge in its implementation. However, one thing that made the earlier model plausible as a working model was that a key conceptual refinement at a particular moment along the trajectory *is* what eventually allowed the strategy construction to take place. So, strategy implementation invoked a conceptual scheme, the conceptual scheme got enriched and refined (not only adding in the unit iteration idea but also the idea of iteration of co-varying change). This "new idea" then allowed for the construction of a new strategy.

Initial strategy: Systematic and purposeful guessing and checking

In the episode discussed below, one can observe that Liam is using a systematic and purposeful guessing and checking approach to solve the given word problem. This problem was the first in the sessions where the tutor had suggested Liam organize his guessing and checking strategy in a Guess and Check chart. Previously, Liam had used the invented strategy of "guessing and checking" though not arranged in a chart.

> *The base of a rectangle is three centimeters more than twice the height. The perimeter is 60 cm. Find the base and the height of the rectangle.*



Height	Base	Perimeter	60 (Check)
18	2(18)+3=39	36+(39+39)=114	↑
10	2(10)+3=23	20+46=66	↑
8	2(8)+3=19	8+8+19+19=54	↓
9	2(9)+3=21	9+9+21+21=60	=

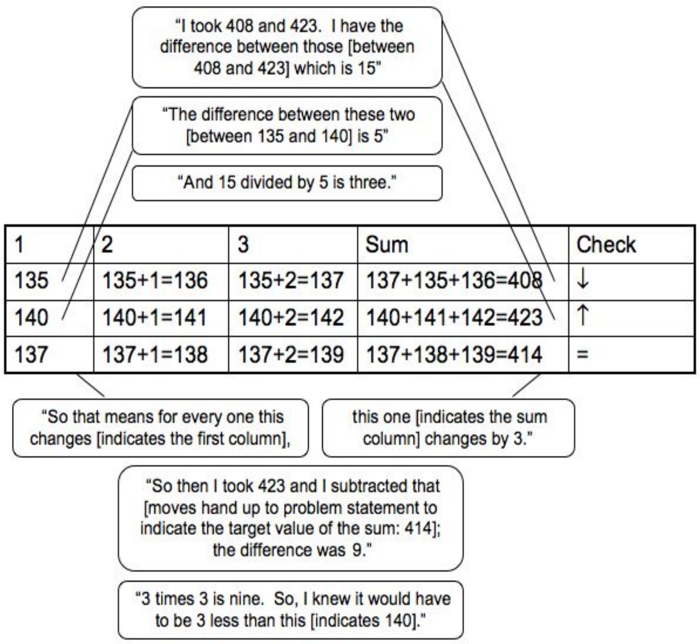
Figure 4. A schematized version of transcript coordinated with a typed reproduction of Liam's written work.

Already in this episode, Liam is making very purposeful choices about the sequence of trial values he chooses. Inferences and utterances that Liam made on earlier problems give weight to the idea that Liam has a sense for the nature of the variation. There does seem to be some intuition (however inexplicit and inarticulate at this point) to back up the judgments he makes. Certainly, his choices for guesses are far from random. In fact, he already appears to have at least an approximate sense for how the input/output pairs he generates co-vary linearly. One can also note that he is making inferences in terms of both scalar judgments ("a lot lower") and also approximately proportional judgments ("it's a little less than twice" the target value).

Later strategy: Leveraging linear structure to determine solutions

Later in the sessions (session 5 of 6), Liam had refined his strategy from "purposeful guessing and checking" to "linear interpolation/extrapolation." In the following data excerpt, one can see Liam deploy his newly constructed linear interpolation strategy to solve a problem of a similar underlying (linear) form as in the problem above. The problem he was working on was:

Three consecutive integers sum to 414. Find the three integers.



1	2	3	Sum	Check
135	135+1=136	135+2=137	137+135+136=408	↓
140	140+1=141	140+2=142	140+141+142=423	↑
137	137+1=138	137+2=139	137+138+139=414	=

Figure 5. A schematized version of transcript coordinated with Liam's written work on a contrasting problem in which he used "linear interpolation/extrapolation" to solve the problem.

In solving this problem, Liam continues to organize his work in a "guess and check" chart (as shown above). After having solved the problem and when asked to explain his solution strategy and Liam says

> *"I took 408 and 423 [see chart above]. I have the difference between those [between 408 and 423] which is 15. The difference between these two [between 135 and 140] is 5. And 15 divided by 5 is three. So that means that **for every one this changes** [indicates the first column], **this one** [indicates the sum column] **changes by 3.** So, then I took 423 and I subtracted that [moves hand up to problem statement to indicate the target value of the sum: 414]; the difference was 9. 3 times 3 is nine. So, I knew it [the value of the input that solves the problem] would have to be three less than this [indicates 140]."*

To give a quick recap of Liam's activity with the second episode, we see that after Liam has finished the computations with two trial input values, he uses this information to determine the unit worth of one guess: the amount the output will change corresponding to a change of one in the input. Liam then takes the output corresponding to a particular trial input (he chooses the input of 140) as a reference and figures out how far that output is away from the target output. He then uses the unit worth of one guess to figure out how much he should change the input by in order to produce the change in output he just computed would be required in order to achieve the solution.

In this episode, we see that Liam has refined his sense for how inputs and outputs co-vary. He has now found a way to quantify and explicitly leverage his intuitions about the underlying linear relationship that all the input-output pairs in these problems satisfy. Note that the idea that a given input is "worth" a fixed amount in terms of its effect on the output is a refinement of the earlier approximate, qualitative versions of proportionality Liam used in the previous episode. This is the sense in which Liam is learning to control the variation of a *linear* function—he uses the worth of one guess to extrapolate and move the input up or down by an amount that he has determined in advance will solve the problem. The extrapolation that he performs (e.g. the scaling of unit worth) depends on the underlying linear structure of the problems.

Discussion of the two contrasting episodes

An important point of contrast between the two episodes described above is that in the first, Liam's method is highly dependent on the inferences he draws about each *particular* guess. Each one is chosen purposefully, and the resulting output is compared to the target output given in the problem. However, in the second episode, Liam has realized that his solution method is *general*, and depends only on determining the unit worth of one guess. Furthermore, he purposefully uses two trial values, not for the purpose of converging to the solution to the problem, but for the purpose of determining an invariant (the rate of change between any two input/output pairs) of the underlying functional relation. Once he has determined this invariant, he uses it to deduce the unknown value that will solve the problem.

In the following sections, we will discuss the development of a way of seeing Liam's activity that allows us to understand the co-development of Liam's strategic and conceptual knowledge. We will return to these two particular episodes and put them in context later in strand four when we map out the process by which the second strategy emerged over the course of the activities of the sessions.

Alternative framings of what changed and how

One of the major motivations involved in analyzing the arc of strategy-knowledge development further than the bookend treatment given in the previous section is the desire to explore and characterize what Liam might have learned and accomplished in addition to developing a new (and more efficient) strategy for solving similar problems over the course of a small number of sessions.

Describing what, exactly, he learned that supported his understanding of and development of this strategy is more of a challenge (and, indeed, responding to this challenge is the focus of the remaining five strands of analysis). Through the analysis that unfolds over the next several sections, we will ultimately be developing a grounded way to describe what Liam learned. Before we describe the characterization of strategy and conceptual knowledge systems resulting from this analysis, we first present some of the initial formulations of answers to the question "What changed?" These are presented by way of exposing alternative perspectives on the same data and also by way of discussing the trajectory of the analysis. The very question of how one frames the issue of what Liam learned or what changed is part of the analysis. The pros and cons of the different framings will be discussed for the purpose of motivating the need for the kind of analysis and the theoretical framework that we ultimately chose as the basis for our theorizing.

The contrasting framings include:

1. *Development of algebraic thinking and reasoning* (e.g., Carraher & Schliemann, 2007; Kaput, 2007; Kieran, 2007)
2. *Development of particular "concepts" such as function, rate of change, co-variation* (e.g., Harel & Dubinsky, 1992; Sfard & Linchevski, 1991; co-variational reasoning of Blanton & Kaput, 2008; Carlson & Oehrtman, 2005; Confrey & Smith, 1995; Saldahna & Thompson, 1998).

Development of algebraic thinking and reasoning

One might think of the arc of learning that we described as one pathway bridging "arithmetic" and "algebraic" problem solving approaches. There are reasons to describe aspects of his second strategy as "algebraic" whereas his first strategy was "arithmetic" in nature. Liam's second strategy involved an algorithm to *determine* the unknown solution to the problem. In contrast, Liam's first strategy involved converging to the answer through a sequence of specific numerical calculations. Furthermore (as we will see in the more elaborated discussion of the episodes in strand four), Liam came to view the "cells" of the Guess and Check chart as standing for general quantities that could take on any value (e.g., generalized numbers). In his algorithm for determining the solutions to problems, Liam was also describing general operations on cells in the chart that had specific properties (e.g., being the outputs of values that had been operated on via the functional relation given in the problem statement). As he was constructing the linear interpolation/extrapolation strategy previewed in the previous section, he anticipated the actions he planned to take to solve the problem before any values and outputs he been specified.

Researchers in mathematics education have hotly debated what exactly characterizes "algebra" and "arithmetic" and "algebraic" thinking. Kaput (2007) describes algebraic thinking in terms of symbolization and syntactically guided manipulations on the symbolizations individuals create. Driscoll (2000) describes the habits of mind associated with algebra, especially those concerned with thinking about functions and how the impact a system's structure has on calculations. Greenes and Findell (1998) identify the big ideas of algebraic thinking as representation, proportional reasoning, balance, meaning of variable, patterns and function, inductive reasoning, and deductive reasoning. Standards documents (Common Core Standards for School Mathematics, 2010; National Council of Teachers of Mathematics, 2000) identify these among others in a broad pool of concepts and competences associated with algebra and algebraic thinking.

Certainly, the move from "inductive" (e.g., guessing-based approaches) to "deductive" approaches (e.g., *determining* the solution), Liam's recognition of the cells as generalized numbers that he can operate on, and his awareness of functional co-variation indicate that some of the hallmarks of algebraic thinking are in the process of being developed. However, while the descriptors "arithmetic" and "algebraic" might suffice for some purposes, they do not provide enough resolution to track moment-by-moment shifts in understanding. A more "tailored" description of how Liam understood and solved each problem is going to be necessary for the purpose of tracking moment-by-moment conceptual dynamics. Thus, it is the coarse temporal and conceptual resolution of frameworks for studying algebraic thinking, as well as the lack of definition around the core terms that make such frameworks unsuitable for a microanalytic study of knowledge growth and change.

Development of specific concepts

As noted above, Liam's second strategy involved setting up a proportionality and determining the solution to the problem while in contrast, Liam's first strategy involved converging to the answer through a sequence of specific numerical calculations. Thus, another possible way to frame the change in the sessions would be to zoom in from the general framing of Liam's reasoning processes becoming more "algebraic" and instead focus on the development of particular key ideas such as proportionality, function, and co-variation. There has been extensive literature on students' thinking about these topics including

1. proportional reasoning and its development (e.g., Fuson & Abrahamson, 2005; Karplus, Pulos & Stage, 1981; Noelting, 1980a; 1980b; Post, Behr, & Lesh, 1988; Tourniare & Pulos, 1985).
2. function (e.g., Dubinsky & Harel, 1992; Leinhart, Zazlavsky & Stein, 1990; Sfard & Linchevski, 1991; Thompson, 1994)
3. co-variation (e.g., Carlson & Oehrtman, 2005; Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Confrey & Smith, 1995; Saldanha & Thompson, 1998).

However, none of the studies named above focus on the micro-development of these concepts for individual learners. Furthermore, the original learning goals of the activity were framed around developing symbol sense (e.g., Arcavi, 1994; Schoenfeld & Arcavi, 1988; Usiskin, 1988) and "algebraic" means of solving the problems (e.g., Stacey & MacGregor, 2000). The task sequence itself was not particularly intended to develop ideas of proportional reasoning, function, co-variation, and rate of change. Further, only very specific projections of these larger concepts show themselves in the learning trajectory that we are interested in modeling in this case study. Thus, the data trace for the development of the many facets of any of these larger ideas is rather thin. Though maintaining a focus on the content of what Liam learned is a central commitment of the analysis, developing a more direct approach of determining what Liam was specifically drawing upon and using in his reasoning processes was determined productive.

Although any of the above framings would be suitable for illuminating or highlighting certain features of the process of change that we examine in detail in this case study, they all leave unexamined the crucial question of the nature and form of Liam's knowledge elaborated in a way that could be insightful for understanding the micro-processes of change underlying the shift in strategies that we observe. For this reason, the heart of the analysis in this dissertation focuses on developing an analytic vocabulary for describing the nature and form of Liam's knowledge-in-use and the processes by which it changes.

Strand 2: Negotiating with the epistemological frame

The first strand of the analysis involved recognizing that the approaches in these two episodes discussed were of a qualitatively different nature. We then discussed alternative perspectives on how to think about what changed in Liam's understanding that could be used to help us make sense of the data. However, to understand how Liam moved from one strategy to the other, we will have to go deeper than a top-level description of contrasting features of his initial and later strategies. The task before us is to find a way to describe the relevant shifts in conceptual understanding that allowed this strategy "shift" to take place. The question before us concerns making a representation of the organization of Liam's knowledge-in-use in the sessions and tracking how that organization changes, and thus can be framed productively as an epistemological question.

As previewed in the theory chapter, the analysis in this dissertation is informed by the Knowledge in Pieces (KiP) epistemological perspective. In the literature review chapter, I described alternative theoretical perspectives on knowledge and knowledge construction (e.g., Radical constructivism, Process-Object theories, Sfard's Commognitive perspective, Abstraction in Context, etc.) and also the pros/cons of each for an analysis like the one I pursue here. This review, combined with the growing body of evidence within mathematics and science education that show systems perspectives on thinking and learning to well-capture key aspects of developing competence, is used to support the reasonableness of the decision to use and adapt a theoretical perspective from "outside" the mathematics education.

As was discussed in Chapter Five, a generic attitude of KiP/KA research is that this theoretical frame is not simply "applied" in analyses, but rather the process is one of deliberate negotiation and adaptation of general epistemological principles so as to be tuned for use in the context under study. See diSessa & Cobb (2004) for a discussion of this approach of generating and refining "humble theories" through particular analyses. Here, we remind the reader of some of the central features of the epistemological framework we will be adopting/adapting through the upcoming analytic strands.

Initial contact between epistemological framework and data

The Knowledge in Pieces (KiP) epistemological perspective (diSessa, 1993) proposes that individual knowledge can be productively modeled as a complex system of knowledge elements that are diverse in form and contextually sensitive in their activation and use. In particular, naive knowledge systems are weakly organized (in contrast to being highly integrated, coherent systems). Learning and conceptual change involve refining when and how particular knowledge elements are activated and used, the coordination of the use of multiple knowledge elements, and the construction of new knowledge elements. Thus, this perspective brings into focus the structure and form of intuitive knowledge and its positive role in the subsequent refinement, integration, and systematization of students' knowledge systems.

In my initial attempts to negotiate between the empirical data and the theoretical frame, I made the conjecture that it could be possible to present Liam's problem solving strategies as compositions of smaller, sub-strategic and sub-conceptual knowledge elements. Further, I hypothesized that the smaller pieces might potentially "belong" to many concepts and could be appropriately or inappropriately activated according to context and that the development across the sessions could be traced in terms of these sub-strategic and sub-conceptual pieces becoming increasingly coordinated. As the case study progresses, I will show how this initial gloss for how the framework informed the analysis becomes increasingly specified and elaborated.

Considering existing constructs of the theory

In tackling the question of trying to exhibit the putative sub-strategic and sub-conceptual elements that get activated and coordinated in the process of strategy construction, I first asked whether it made sense to adapt and/or modify existing constructs within the framework in relation to my data and context (e.g. p-prims, diSessa, 1993; coordination classes, diSessa & Sherin, 1998 as described in more detail in chapter four). In this initial phase, I found that the principles of KiP as an orienting framework were useful, but these specific existing constructs elaborated in the theory were not recognizable to me in a way that immediately helped me in thinking about the question of strategy emergence in my particular data. For example, though the case of Liam did have to do with tracking increasing competence in a domain, it did not obviously have anything to do with the problem of establishing invariance of determinations across multiple contexts (the prototypical function for coordination classes) and the specific topic Liam was reasoning about did not seem obviously related to his comfort or surprise with aspects of his physical experience (the prototypical function for p-prims).

However, as examples of extensive knowledge systems and how they function and develop, existing "primitive" knowledge structures like p-prims, and more extensive knowledge systems like coordination classes, served as reference models in the analysis. As the analysis progressed, as opposed to p-prims and coordination classes themselves being the central constructs of interest, elements and systems that played (in some sense) analogous roles did emerge (e.g., co-variation schemes, control of variation systems, the two contrasting strategy systems, etc.) These constructs will be elaborated in the later phases of discussion of the analysis. The important idea to stress at this stage is that though these existing constructs of the theory were themselves not directly importable to the current analysis, studying the literature that documented their relationship in other contexts provided useful reference models for how to interpret my data in terms of knowledge systems that were more adept at capturing the relevant features of the context under study.

Reformulation: Strategies as complex systems

Informed by the Knowledge in Pieces (KiP) perspective, I (eventually came to) consider a strategy to be a particular kind of complex knowledge system. In attempting to reformulate strategic knowledge from a complex systems perspective, one can take a functional perspective – asking what kind of role does the element or system play in a learner's conceptual system. From this perspective, the function of strategies, as complex knowledge systems, is to coordinate the many, diverse kinds of knowledge necessary in order to solve problems in a way that leads to a solution.

There are many important aspects of understanding and using strategies effectively. In this section, will first give an indication of the range of kinds of knowledge that *could* comprise a strategy system and then I will turn to explaining those facets that will be most useful for the analysis developed in this dissertation.

To give an example of the breadth of kinds of knowledge that could be involved in the implementation of a strategy, consider the: (1) knowledge that would allow an individual to recognize when an already constructed strategy in their repertoire would apply, (2) knowledge about what it would mean to solve the problem (e.g., stopping criteria), (3) knowledge of the information one needs to determine or read out in order to reach the goal of solving the problem, (4) conceptual knowledge that enables the operation on this information in order to solve the problem, (5) control knowledge about how to determine how well the strategy is working and whether one should try adapting the strategy or choosing a new strategy mid-solution attempt, (6) knowledge about the alternative strategies and criteria for selecting other potentially viable strategies.

Mapping out the diversity of kinds of knowledge involved in strategy systems and how this knowledge functions is a program of research in and of itself. This analysis takes some initial steps in this direction of modeling strategic knowledge as a complex system. However, not all of the aspects listed above will be particularly accessible or relevant in the particular episodes we will be analyzing. The purpose in listing them is merely to give some indication of the breadth of types of knowledge involved in implementing strategies – briefly, to make the case that strategy implementation is a highly knowledge intensive enterprise.

For current purposes, the components of a strategy that we will trace through several focal episodes from Liam's problem solving sessions include (1) strategic path, (2) conceptual categories, and (3) knowledge structures that encode relations between conceptual categories. Below, each of these components of the strategy system is described in more detail and selected illustrative examples that are relevant to the case study in question are given.

1. **Strategic path.** Implementing strategies involves a strategic path – the solver's image of the goal state (e.g., what it means to have solved the problem at hand, what information needs to be determined, and an image of how it should be operated on in order to achieve the goal of solving the problem).

In this case, the information that Liam perceives as necessary to collect shifts over the course of the sessions from a sequences of trial values that get closer and closer to the solution *to* a two trial values from which the answer to the problem can be determined.

2. **Conceptual categories:** Conceptual categories are the categories of attention that the solver either *looks to* in order to *read out* information and/or *set* in order to solve the problem.

For example, In the case in question, the solver has to "set" the input variable and then "read out" information about the resulting output, and the target. Here, "setting" could be understood as "selecting a possible value for a conceptual category," such as the input. Output and target are related in that one can read out the error involved for each guess and work to reduce it. By determining the output and comparing it to the target, error is a conceptual category that a solver can know to look for and to read out information about. Briefly, input is a conceptual category that is *set* and output, target, and error are conceptual categories that are *read out*. Looking ahead to future developments over the sessions, in the context of the activity in question, Liam eventually comes to recognize "unit increment" as a relevant conceptual category for determining solutions.[20]

3. **Knowledge structures that encode relations between conceptual categories:** These are the inferential relations the solver forms between conceptual categories.

We will later discuss the elements of the solver's knowledge system that are particularly relevant for the case study here – those that capture expectations about the effect of varying the input. We call these relational elements *co-variation schemes* – relations between inputs and resulting outputs that concern the effect on the output of varying the input (e.g., Input increases → Output increases, etc.) The control of variation system is a conceptual system that gets build up and elaborated over the course of solving several problems. The relations between conceptual categories form the "inferential net" (cf. diSessa & Sherin, 1998) of the strategy system, providing a conceptual justification for the choices of operations the solver takes in order to implement the strategy.

In addition to the latter two components of the strategy system (that one could call "attentional" and "relational" components) listed above, there are other aspects of the strategy system that we will trace in our analysis. For instance, there is possibly quite diverse pool of knowledge that is activated in the course of solving problems. Though knowledge like arithmetic facts with simple units may not be a site for conceptual *development* (e.g., Liam may not learn anything new about multiplication for instance), these pieces of knowledge still support the implementation of the strategy. In the line-by-line analysis in strand four, I will group such knowledge under the category "activated auxiliary knowledge."

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[20] An interesting methodological issue concerns the process by which the analyst determines how individuals construe what we identify as their conceptual categories and relations. In this regard, it is helpful to recall the observation heuristics for coordination classes that were discussed in Chapter Five.

The reader will have ample opportunity to see these ideas concretized in the analysis when they are used to trace the development of the strategy over several episodes of problem solving (e.g., analytic strand four).

Reconsidering the connection with coordination class theory

In the above, the first class of knowledge in the strategy system (the attentional component: conceptual categories the solver looks to and/or sets) is more related to how the individual "sees" the problem context and what is required to solve the problem. The second class of knowledge (the relational component: relations between conceptual categories) concerns more what the solver knows about how the categories of attention in the problem are related that will enable them to make progress toward a solution. One could loosely make an analogy with *coordination class theory* (diSessa & Sherin, 1998) and say that the first class above is related to "readouts/setting" (perceptual component of a coordination class) and the latter class is related to the "inferential net" (inferential component of a coordination class). Though this is merely a rough analogy between two classes of knowledge systems, one function of pointing out parallels between the definition of the strategy system given above and the definition of coordination classes is that it can be suggestive of phenomena that it could perhaps be helpful to be attentive to. For example, is there empirical evidence that other prototypical issues associated with coordination class theory, such as *span* and *alignment* across determinations, are relevant in the case of strategy systems? For example, is having multiple and well-aligned means of determining particular conceptual categories (say, unit increment) a relevant issue?

The next section describes the process by which the base vocabulary for knowledge-in-use that we use in modeling the process of strategy construction was schematized from transcripts.

Strand 3: Developing an analytic vocabulary

As indicated in the previous section, at the most general level, Knowledge in Pieces (KiP) posits that individual knowledge can be productively modeled as a complex system, both strategies and relevant conceptual knowledge are modeled as complex systems of elements that are diverse in form and function. The overall goal of our analysis is to uncover the elements of these systems, how they are organized, and also how the systems relate to each other.

While KiP suggested making the conjecture that it would be possible to re-frame the process of strategy emergence in terms the dynamics of complex systems, the task was still to think about how it made sense to deconstruct this process of knowledge construction into relevant sub-systems and elements. Having temporarily eschewed the "top-down" approach of looking for specific types of knowledge elements such as p-prims and coordination classes in my data, I instead settled on a "bottom-up" approach that involved trying to characterize Liam's knowledge-in-use.[21]

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[21] In terms of chronology, this phase of bottom-up schematization of knowledge-in-use preceded the reformulation discussed in the last section of strategies as complex knowledge systems. Some of the relevant pieces of knowledge activated in context (co-

In this section, I will present the analytical framework for tracing this co-development process. This analytical framework is one of the outputs of the analysis in that it was developed in a grounded fashion through observations of the empirical data and in negotiation with the theoretical perspective that guides this analysis. Thus, while it is presented in advance of the analysis so that the reader can see the framework in action as we track some of the dynamics of change from initial to final strategy, this particular framework was the result of iterative passes through analyzing the empirical data and ongoing negotiation with the theoretical perspective that informed the analysis.

A bottom-up schematization of knowledge-in-use

In the initial phase of the analysis, the videos and transcripts were openly studied in order to identify and schematize the nature and form of the kinds of knowledge that Liam drew upon as he worked on problems.

There were some aspects of the strategy implementation that remained constant. For example, unlike many students[22], Liam had no difficulty "seeing" the paths of determination from input to resulting output that were specified by the conditions given in each problem. From this open analysis, it was observed that the fundamental shift of interest to the case study involved a change in organization of knowledge related to the *kind of justifications Liam gave for his next guesses*. Thus, an analytic decision was made to first focus analytic attention on characterizing the nature of Liam's knowledge system in this area and how it changed.

Liam adopted an "input/output" strategic frame in which he knew that at the very most general level, the organization of the solution strategy would follow the pattern of choosing a value for the input, operating on it as specified by the relation given in the problem, reading out the corresponding output, and checking that against the constraint of the "target output" given in the problem. The solution would proceed by him adjusting the input in a way that would result in the corresponding output move closer to the target and ultimately achieve it.

What is of interest is the way his awareness of how to adjust the input value so that the corresponding output would approach the target value became articulated over the course of solving several problems. What started out as a general and inarticulate

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variation schemes) were observed first and then it became necessary to try to model the knowledge systems involved in which we could observe these elements functioning.

[22] Seeing "paths of determination" is actually highly nontrivial for many of the other students observed in these sessions and in classroom sessions. For many students, flexibility in seeing which number to guess values for and being able to determine other values from their choice was difficult. Some of the exercise involved in working with the charts is actually to give students experience in recording and reflecting on how the independent variable gets operated on and one can use one value to determine the others. This is completely unproblematic for Liam. It remains a difficulty in instruction to figure out how to support this kind of "intuition" about which number is "easiest" to guess values for. It seems obvious to some students and is a completely nontrivial accomplishment for others.

intuition about how the inputs and outputs in each of the problems co-varied was eventually refined into a means to explicitly determine solutions based on the results of two guesses.

Methodological notes concerning the observation of elements in data

In general, it can be hard or impossible for an analyst to know *a priori* when they can expect to see particular knowledge invoked in an episode of reasoning. However, the regular structure of the activity: (1) making a guess (2) reflecting on its effect and then (3) choosing another guess that will improve on the result of the previous guess, naturally implicates thought elements of an expected form and grain size. That is, the chain of reasoning that occurs is for the individual to *read out* the output, compare it with the target value, determine (roughly or precisely) how big the error or discrepancy is and then make an adjustment to the input value. Thus, the regularity of the activity allows the analyst to predict what to expect in terms of the nature and type of thought elements invoked to support the reasoning process.

Another assumption is that each time Liam chooses a next guess, this choice is, in fact, guided by his understanding of the underlying variation of the function. A rival hypothesis to Liam drawing upon some knowledge or understanding of how the relationships in the problem contexts work would be that Liam is merely invoking a "More X corresponds to More Y" scheme because this has worked in the past for him. Throughout the sessions Liam has an explicit orientation to focus on "timesaving" and efficiency and the sequence of guesses he chooses reflect his version of the task which appears to be to solve the problem in the fewest number of guesses. That is, as we will see when we go through the set of focal episodes in sequence, there is a cumulative focus across the sessions to build on the problem solving approach used in previous trials and to refine it. Liam appears to be attuned to the direct effect of his choice of input on the resulting output via the relation given in problem contexts. That is, the assumptions that he is thinking about co-variation and change appears warranted. (Of course, I only flag this issue here. Readers will have the chance to judge for themselves in analytic strand four).

Creating a schematization of thought elements from transcript

In the data excerpt below (that has been discussed in the previous section describing the before/after view of strategy change), I give an illustrative example of how I schematized the decisions Liam was making as he was choosing next guesses. Recall that the problem Liam was working on was:

> *The base of a rectangle is three more than twice the height. The perimeter of a rectangle is 60 inches. Find the dimensions of the rectangle.*

In line with the instructional expectations of the sessions, Liam organized his solution to this problem in a guess and check chart (a typed reproduction of which is given below).

Height	Base	Perimeter	60 (Check)
18	2(18)+3=39	36+(39+39)=114	Too high
10	2(10)+3=23	20+46=66	Too high
8	2(8)+3=19	8+8+19+19=54	Too low
9	2(9)+3=21	9+9+21+21=60	Check

Figure 6. A typed reproduction of Liam's work.

For nearly all choices of input value, Liam either spontaneously offered or responded to a request with a justification of his choice. This provided a sense as the problem was unfolding of what was salient to him.

In the table below, I show the literal transcript that accompanied Liam's choice of each guess. One should note that since we are currently focused on schematizing the nature and form of Liam's knowledge-in-use, for now, I show only *Liam's* utterances. I then provide a schematization of the thought process behind what Liam says as he is making his next guesses and finally, I show how these are interpreted in terms of (1) what Liam notices about each guess *("readouts")* and (2) the knowledge structures that support his choice of next guess *("co-variation schemes")*. In the section on dynamics (strand four), more of the interactional details between the tutor and Liam will be reintroduced.

Table 5. Schematization of knowledge elements from transcript.

Input	**Transcript**	**Schematization of thought process**	**Schematization of element activation in the episode**
18	"So it's actually a lot lower. I just realized that." … "This is way too much."	If the result of my guess is *way too high* with respect to the target, then I should adjust by making the next guess *much lower*.	**Readout:** Too high (with respect to the target). **Co-variation scheme:** *Decreasing X → Decreasing Y.*
18	"It's almost twice as much. So I'll try with 10."	If the result my guess is about twice as high as the target, then I should adjust by making the next guess about half as much.	**Readout:** Too high, in fact, about twice too high. **Co-variation scheme:** *Decreasing X by about a factor of 2 → Decreasing Y by about a factor of 2.*
10	"OK. So now I'll try an 8. That'll be about right."	If the result of my guess is a little bit too high with respect to the target, then I should adjust by making the next guess a little lower.	**Readout:** Too high, in fact, a little bit too high. **Co-variation scheme:** *Decreasing X by a little bit → Decreasing Y by a little bit.*
8	"It's probably nine." … "Well it was actually definitely 9 if this *[result of guess of 8]* was too low and this *[result of guess of 10]* was too high. Unless it was a decimal number."	If the result of one guess is too low and the result of another is too high with respect to the target, then I should choose my next guess to be *in between* the guesses that resulted in the outputs being compared.	**Readout:** Too low. Combine this with the observation that the input of 10 resulted in an output that was too high. **Composition of co-variation schemes:** *If a target Y value lies between two specified Y values, then the X value that achieves the target must lie between the corresponding X values.*

In the above, Liam is attending to whether a particular guess is too high or too low, amending to that various attributions that qualify the initial assessment of too high or too low (such as "way too much" or "twice as much"). Another important thing to note about the general form of the inferences that Liam is making to guide his next guess is that, at this point, they refer to the variation of the function as if it were assumed that the underlying linear function passed through the origin, combined with a rough sense of proportionality (e.g., that is, he initially makes inferences that would not be true for all linear functions of the form $y=mx+b$, but that are consistent with $y=mx$). That is, it is not always true that reducing the input by a factor of two will result in reducing the output by a factor of two. This is true for functions of the form $y=mx$, but not $y=mx+b$.

As we see from the above, patterns of related variation that Liam notices and uses to guide his choices include:

- If I change the input (X) by A LOT, this will result in the output (Y) changing by A LOT.
- If I increase/decrease X by a little bit, this will result in Y increasing/decreasing by a little bit
- If I increase X by approximately a factor of 2, Y will increase by approximately a factor of 2

An interesting issue is the degree to which any of these "elements" are distinct from each other or whether they all have a common root element "Change in X → (like) Change in Y." The model that I will develop in this case assumes that the elements are *not independent* from each other and they *do* share a common root element. Another theme of the (upcoming) analysis will be showing how this approximate version of a linear model of the variation that initially guides Liam's choices of next guess gets refined into a quantitative and "incremental" linear model that Liam can use to solve problems involving general linear functions of the form $y=mx+b$. It is not *a priori* clear that judgments that are consistent with a linear model (e.g., the initial, "global" judgments that Liam makes) imply (for Liam) the quantitative and incremental linear model that he later uses. However, based on Liam's patterns of reasoning and the extended time it takes him to "put together" certain inferences on the spot later, we will argue that we are directly observing the construction and refinement process of Liam's model.

Iterative description of knowledge elements

Using this method of schematizing knowledge-in-use across the episodes allows us to get a first-pass idea of the nature and form of the relevant knowledge. However, this is just a first iteration of giving descriptions of the knowledge in the sessions. There is much more we want to know about the knowledge in the sessions in addition to just its rough form. As we indicated in the theoretical framework and methods section, we will want to describe the structure and form of knowledge elements, how the relevant knowledge systems in which these elements function is organized, how the elements are activated and used, and how the system organization changes over time.

Now that we have seen examples of them in data, below, I give definitions for co-variation and correspondence schemes that explain some of their more general features.

The definitions below include specification of function, structure, dynamics, and hypothetical source of the knowledge structures. The particular dimensions of specification are informed by the principles of KiP (in particular, see "Theorizing about any knowledge system, diSessa, 1993, p. 8).

This discussion is followed by a complete list of the schemes that appear in the data analyzed. Again, in strand four the reader will have the opportunity to see these schemes used in the context of Liam's activity.

Co-variation schemes
Below, we give a definition of co-variation schemes, a description of potential conditions of activation, and a sketch of potential genetic roots.

1. **Definition.** A co-variation scheme is a knowledge structure which functions to help an individual predict the effect of controlling a "cause" (or an independent variable). The relevant structural features of the knowledge structure are (1) input, (2) cause or "path of determination" from input to output, (3) resulting output, (4) hypothetical change in input, (5) resulting change in output given the hypothetical change in input.
2. **Activation.** In the context of the problem solving sessions we study in this analysis (a student solving algebra word problems by "guessing and checking,") appropriate co-variation schemes are recognized based on (1) perception of distance and direction away from the target value and (2) awareness of the role of the functional relation given in the problem in determining the output and in determining how the output changes when the input is varied. Activation of appropriate co-variation schemes is also reinforced through interactions that make the distance between results and the target salient. A further factor that influences the activation of appropriate co-variation schemes is the history of the activity and the effectiveness of previously activated co-variation schemes (e.g., the activation of the co-variation schemes coinciding with expectations increases its cueing priority). Increased quantification of the amount of error between the target and the output is also driven by the activity and the goal of solving problems that are recognized to be similar in more efficient ways.
3. **Source.** I now give some examples of contexts outside of the one under study and in which individuals likely use co-variation schemes to structure their thinking. Firstly, note that in real life settings, time is adjustable and consider the situation of one asking oneself "How long should I pour in order to fill a glass up with water?" In this example, time is the input that one has control over, the relationship between time and the height of the liquid is given by a linear function (assuming one is pouring at a constant rate). The water level rises as one continues to pour. One can either use qualitative descriptions like "One would need to pour a lot longer because the glass has been filling up slowly" or one could alternatively determine the exact amount the liquid level goes up per unit time.

Correspondence schemes

Here, we mention a special class of co-variation schemes – correspondence schemes (Y changes by the same amount every time that X changes by some fixed amount).

1. **Definition.** *Correspondence schemes* are knowledge structures that are involved in proportional reasoning. The structural components of this knowledge structure are (1) number of units of one quantity (e.g., the input quantity) (2) number of units of another quantity (3) correspondence between these quantities. Correspondence schemes are used in extrapolating relationships and in conducting unit conversions.
2. **Activation.** In these sessions, worth schemes are activated by the need to find how to *precisely* adjust the input variable (e.g., by the need to determine how much an increment in X in worth in terms of Y).
3. **Source.** One of the genetic roots for *correspondence schemes* are proposed to be "worth" schemes (diSessa, 1993). In everyday life and from early ages, we assign correspondences between a certain number of discrete objects (e.g., movie tickets) and say prices. We assume that if we know the price for one of these units, then we can predict the price for any number of them. (In real life, the practice of giving discounts for purchasing more units competes with this scheme).

Both co-variation and correspondence are recognized as two different facets of functional thinking (Blanton & Kaput, 2008; Carlson & Oehrtman, 2005; Chazan, 2000; Confrey & Smith, 1995; Saldanha &Thompson, 1998). The activity under study in this analysis requires the coordination of these two perspectives. That is, a student needs to think both in terms of the process that acts on inputs leading to resulting outputs and also think about the effect on the output of varying the input. Though correspondence remains the "traditional" perspective on functional reasoning, a growing number of studies recognize the value of co-variational reasoning.

In the above, the distinction between co-variation schemes and correspondence schemes is a slightly different one. Correspondence schemes establish the "worth" of a number of increments in input to increments in output. In this sense, they are more related to the formation of composite units (e.g., Steffe, 1994).

In the tables that follow, I list the co-variation schemes (qualitative and quantitative co-variation schemes are listed in separate tables) observed in all the sessions with Liam. Although it is easy to imagine that there could be many other co-variation schemes, apart from the ones listed here (just as a simple example, the family of primitives associated with inverse variation "If I increase X, Y decreases" is not listed here, the reader should note that this list contains only co-variation schemes that came up during the tutorial sessions (and because of the nature of the problems in the sessions, all the schemes involved at their root, direct, linear variation).

Notation: In the following description, X stands for "input" values and Y stands for "output" values. The "→" notation should be read as "results in." So, "Change X →

Change Y" is read "A change in X *results in* a change in Y." More X → More Y should be read "More change in X results in more change in Y" and so on.

Table 6. Qualitative co-variation schemes from across the sessions

Descriptor	**Co-variation scheme**	**Brief Description**	**Transcript segments**
Direct variation	More X →More Y	If I increase X, then Y increases.	"I would know that I should go a little bit higher." [His initial guess of 15 was too low and so Liam chose a next guess of 16 because *increasing X → increasing Y*] *(Episode 1)*
	Less X→Less Y	If I decrease X, then Y decreases	**"**Well, maybe 18 and I'll see what happens there. So for 18 – twice the height so it's 36 then plus the … Ohhh, so it's actually a lot lower." [So, I need to decrease the input since *decreasing X → decreasing Y] (Episode 2)*
Qualitative proportionality	Small change X → Small change Y	If I increase/decrease X by a small amount, then Y increases/decreases by a small amount	"Yeah. so 15 plus 51 is 66, so I would know that I should go a little bit higher." [Because 66 is just a little bit lower than 70, so you only need to *increase X by a little bit* to achieve a *corresponding little increase in Y*] *(Episode 1)*
	Moderate change X → Moderate change Y	If I increase/decrease X by a moderate amount, then Y increases/decreases by a moderate amount	*N/A. It is possible that only "big" and "little" adjustments are really seen in this data In any case, these are relative judgments to the other judgments subjects made in reasoning about the same problem.*
	Large change X→Large change Y	If I increase/decrease X by a large amount, then Y increases/decreases by a large amount	**"**Well, maybe 18 and I'll see what happens there. So for 18 – twice the height so it's 36 then plus the … Ohhh, so it's actually *a lot lower." (Episode 2)*
	Increasing/decreasing X by factor 2 → Increasing/decreasing in Y by a factor of 2 (approximately)	If I increase/decrease X by approx a factor of two, then Y also increases/decreases by approximately a factor of two.	"This is way too much." … "Uh, it's almost twice as much." *(Episode 2)*

Table 7. Quantitative or incremental co-variation schemes across the sessions.

Descriptor	Co-variation scheme	Brief Description	Transcript
Incremental worth	Increase or Decrease in X by d → Increase or Decrease in Y by D	If I increase or decrease X by an amount of d, then Y increases or decreases by an amount of D	"That moved up 3 [input of 17]. Then this [corresponding output] moved up 12." (Episode four, post-problem reflection) "So you could tell how many of these [indicating input column] equaled how many of these [indicating output column]" *(Episode 5)*
Unit worth	Increase or Decrease in X by 1 → Increase or Decrease in Y by D/d	If I increase or decrease X by 1, then Y increases or decreases by D/d	"So, that means for every one that this changes [indicating first column], the answer changes 3." *(Episode 5)*
Iterated unit worth	Increase or Decrease in X by a → Increase or Decrease in Y by a(D/d)	If I increase or decrease X by a, then Y increases or decreases by a(D/d)	"So, then I took 423 and I subtracted that [target value of 414]; the difference was 9 and 3 times 3 is 9. Yeah, 3 times 3 is nine, ***so I knew it would have to be 3 less, for this [input of 140] and yeah."*** *[For every one the input changes, the output changes 3. So, for every three the input changes, the output changes 9.] (Episode 5)*

Table 8. Compositions of co-variation schemes across the sessions.

Descriptor	Composition	Brief Description	Transcript
In between	Qualitative Composite of direct variation (More → More and Less → Less)	If the result of one guess undershoots the target output and another overshoots, then the target input is in between these two guesses.	"Well it was actually definitely 9 if this *[result for a guess of 8]* was too low and this *[result of a guess of 10]* was too high. Unless it was a decimal number." (Episode one)
Exactly in between	In-betweenness with the halfway point as a reference.	If the result of two guesses overshoots the target by the same amount, then the target input should be exactly in between the first two inputs.	"I just thought that it *[the answer]* was in between these two *[between 12 and 16]* because this *[points to result for 12]* was 20 too low and this *[points to result for 20]* was 20 too high. And this *[16]* was exactly between those *[12 and 20]*. (Episode three)
Closer to one than the other	In-betweenness (making instrumental use of half as a reference point)	If guess one overshoots the target by less than guess two undershoots, then the target input should be closer to guess one.	"I don't know how you'll put it in words, but this was lower *[result from a guess of 15]* and this *[result from guess of 20]* was too high. So, it would go – it would be further up [sweeping gesture] because I know that it'll be in between these two [15 and 20] already. So, it would be higher up instead of being right in the middle. It'll be higher up."

As discussed in the theory chapter, Knowledge in Pieces, as a deeply constructivist program of work is concerned with constraints placed by developmental perspectives on theorizing about knowledge systems. In the course of one analysis in which knowledge elements are schematized from the data under study, it is not possible (unless a substantial corroborative literature base is already in existence) to have extensive information about the genetic roots of the elements of the systems under study. In large measure, this constraint is a point towards future work in situating the current analysis. As a proxy for developmental study of some of the main ideas involved in the knowledge systems under current study, we engage in the brief thought experiment below that attempts to make the case that an idea like "in-betweenness" is a developmentally plausible idea that occurs in many contexts (not just the one under study). Recall that we engaged in a similar though exercise in discussing possible sources for co-variation and correspondence schemes.

In-betweenness is an intuitive idea that is substantially supported by the continuity of experience. In-betweenness is plausibly an intuitive precursor of ideas that students will encounter later in their studies in the form of the Intermediate Value Theorem. For example, the following scenarios are something that students will feel a great deal of intuition for, despite not being able to prove the mathematical theorem that justifies this intuition. Continuity of experience seems to strongly undergird intuitions about "intermediate values."

- **Example:** Imagine that one is traveling 40 miles per hour at one point in time and then at a later point in time one is traveling 60 miles per hour. One knows that there existed a point in time in between these two when they must have been going 50 miles per hour. Similarly, one would expect to be able to estimate when this point in time occurred if they knew a time when they were going slightly less than 50 miles per hour and time when they were traveling slightly more than 50 miles per hour.
- **Example:** Benjamin's parents measure his height at 52 inches in March, but they forget to measure him again for another couple of months. They find that he's 54 inches by August. Thus, they know that sometime between March and August he was 53 inches (and, actually, given the continuity of human growth, there exist points in time at which he achieved all values in between 52 and 54 inches).

The two examples above have slightly different "extra" assumptions in that people know that in order to go from 40 to 60 miles an hour, one has to speed up through all intermediate values. However, it is still possible for the speed of the car to vary over an interval in ways that are not monotonic. The growth example also appeals to the individuals' intuition rooted in experience that generally people do not shrink and then get taller over some interval of time – their height only increases as time progresses until it plateaus when they reach their adult height.

- **Example:** Tuning a radio dial to find a particular frequency is a process that involves the continuous control of an input variable over a range of frequencies. Furthermore, one likely engages in a successive under/overshoot process in order to zero in on the frequency. A similar example comes from the continuous tuning of string instruments above and below the pitch in order to zero in on the pitch.[23] One difference between the examples of tuning a radio dial and tuning an instrument and the examples above (growth and car speed) is that in the second two examples, the focus is on a personal *agent* that is doing the tuning.

Coordinating the top-down reformulation and the bottom-up schematization

I have now discussed given an indication for how the theoretical framework guides the analysis (1) top-down by "suggesting" the reformulation of our problem in terms of complex knowledge systems and (2) bottom-up by looking directly at what Liam attends to and infers in activity. The outline of the analytical framework that we are constructing is now starting to take shape. In this section, we will take another pass through thinking about what we should expect about both relevant strategies and concepts as complex systems.

Any strategy system relies on (possibly several) conceptual schemes to implement the strategy (in general). In the case studied in this dissertation, there are two main strategy systems *(means-end strategy system and the linear interpolation/extrapolation strategy system)* and one main conceptual system *(control of variation system)*. We are interested in the interrelations and co-development between these two knowledge systems.

Given that what is of interest is the conceptual schemes that are used in implementing the strategy, it is important to trace what particular aspects of Liam's understanding of linear control of variation are called upon in particular instances of problem solving. Furthermore, while we would contend that knowledge of how to control the variation of functions eventually becomes a more general conceptual system, we only get a chance to see this knowledge system engaged and activated in the context of this one activity. So, for the purposes of the analysis, we show the conceptual pool of knowledge that is engaged in the problem solving activity (the control of variation system) as a "subsystem" of the *strategy* system. This subsystem is refined over the course of the episodes of problem solving that we analyze.

System dynamics and generic mechanisms of change

Continuing with the elaboration of our model of strategies and concepts as system of knowledge, the elements in the knowledge systems described above ("means-end" strategy system and the "control of linear variation" conceptual system) can be assigned various priorities that allow us to determine which elements will be activated when. One of the qualities of the elements to note for an individual is how to tell which ideas are more or less likely to be activated. The way we would say this is to ask whether elements are at high *cueing priority* (e.g. a measure of how likely is an element to be turned on or recognized in a context). Another kind of priority involves *reliability priority* (e.g., a

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[23] While this may appear to be an incidental example, the student who is the subject of the case study in this dissertation happened to be a violin student and so this particular example of overshooting and undershooting pitch would certainly be sensible to him.

measure of how likely the element is to be turned off once it is turned on). Priorities can be increased/decreased by various mechanisms including interactions with others, with representations, etc. The language of priorities is introduced here so that in the analysis it will be sensible to talk about system dynamics in terms of priorities.

Strand 4: Using the analytical framework to trace the learning trajectory

This section describes how the analytic vocabulary I discussed in the previous section makes contact with my data and is instrumental in studying the learning process under study. I have selected six episodes along the trajectory of strategy construction on which to illustrate the fit between the data and the analytic vocabulary. The six episodes span (1) spontaneous guessing and checking (2) purposeful guessing and checking organized in a chart (3) exactly in between (4) closer to one than to the other (5) re-constructing the linear interpolation/extrapolation strategy and (6) re-applying the linear interpolation/extrapolation strategy.

For each episode, I give the statement of the problem and describe the work on that problem in line-by-line detail. This level of detail is important for showing how each episode gives a snapshot view of the strategy system in action as it is implemented. The conceptual categories that Liam looks to and sets, the inferences that support the actions he takes toward solving problems the conditions under which these elements are activated, as well as changes to the strategy system, will all be noted after each episode.

The six selected episodes are presented in chronological order and span the six hours of interaction between Liam and the tutor/researcher. All episodes that either prepared for or involved Liam solving problems using and refining a trial and error strategy were analyzed. The selection criterion for presentation in this section was that the episodes needed to illustrate the introduction of new ideas or developments related to the focal arc of strategy construction. For each episode, an indication of what happened in between focal episodes is given as the data is presented.

The following section can be expected to be quite dense to read, especially as we move through the events in the selected episodes line by line, and discuss after each one the features that are relevant to the model of strategy and conceptual co-development that we are constructing. After each episode is a description of (1) the strategy system in terms of strategic frame, conceptual categories and relations, (2) dynamics of knowledge activation noted in the episode, (3) a "snapshot" of the current organization of the control of variation scheme, and (4) micro-developmental events of note during the episode.

Analytic narratives like the six presented here were prepared for all seventeen episodes that were initially identified as potentially relevant to the trajectory of strategy construction. Thus, this section provides a sample of the kind of analytic work that was done in the construction of the final model. Following the extended analytic narratives of selected focal episodes provided in this section, there are two sections that attempt to synthesize important features noted in these episodes and move the analysis toward the creation of the model of strategic and conceptual co-development.

Focal episode one: Spontaneous guessing and checking

The first problem in all six sessions was the following:

> *Ben and Jerry each have a mystery number. Ben's mystery number is six more than three times Jerry's. If the sum of Ben's and Jerry's numbers is 70, find the two numbers.*

In this episode, we will see Liam generating a means, on the spot, for solving this problem. The interest in this episode is mainly as a baseline for the further developments over the course of the sessions.

Though the primary mode of the sessions was tutorial in nature, Liam was asked to work on this first problem independently for the purpose of establishing what his previous experience solving similar problems was (e.g., whether he had a ready algorithm and whether this had been something he had explicit previous instruction about). Incidentally, Liam was the only student in the study willing to engage working on a problem that he had not previously learned (or could readily remember) a process for solving.

As Liam begins work on this problem, he began by doing some calculations in his head. Since he is not writing down and calculations, I ask him to tell me what he's thinking about doing.

> ML: Can you just sort of tell me what approach you're thinking of right now? What are you thinking about doing?
>
> Liam: I'm thinking of … like I'm thinking of a number, then multiplying it by three, and then …I'm not exactly sure.

In the above, Liam has started on the path of choosing a specific number *("I'm thinking of a number"*) and he then multiplies it by three. It is interesting and of note also that he remarks that he's "not exactly sure." This may indicate that even though Liam is guessing numbers for Jerry (he confirms this in the next turn) that he does not have a fully conceived of plan before beginning work on the problem for how he will solve the problem. He may be trying to understand the relationships in the problem using a specific example and is constructing a strategy on the spot.[24]

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[24] Liam used an "unwinding" approach to solve a problem later in this session. If the problem had been "Ben is thinking of a number, 70, and it is six more than three times the number Jerry is thinking of. What number is Jerry thinking of?" then Liam may have started from Ben's number, 70, subtracted six and then divided by three to find Jerry's number. However, the problem as stated gives the condition that both Ben and Jerry's numbers are unknown and that the sum of them is 70. This additional complexity in the task may be part of explaining why Liam decided to start from a particular choice for Jerry, but then was not immediately sure how to proceed. However, if it is the case that

It is of interest that Liam tells me the approach in general terms *"I'm thinking of a number and then multiplying it by three"* as opposed to telling me the specific calculation he was working on. This could be because I have asked him to describe his "approach" (implying that I am interested in a general description). It does signal that he is thinking in terms of operating on a general category (e.g., he would do the same thing to any number) as opposed to only operating on specific numbers. This is pertinent as a landmark in the development of his conception of variable (See Schoenfeld & Arcavi, 1988, Usiskin, 1998) in that operating on a conceptual category (e.g., adjustable input) is different than operating on individual numbers (e.g., Sfard & Linchevski, 1991).

After asking him if he wants to give me an example of one of the calculations he just did, he says:

Liam: Well, basically like 15 … I guess I would do 15 times three is 45. Then, like it's six more than three times.

Note that here that Liam is starting with Jerry's number and using the relations given in the problem to operate on Jerry's number in order to get Ben's number. So far, he has found what Ben's number must be if Jerry's number is 15.

Recall that we call the ways one can go from the value of a quantity one *"sets"* to determining the value of another as following *"paths of determination."* Here Liam uses a determination path that involves identifying Jerry's number as the "input" and Ben's number as an "output." The path to get from Jerry to Ben is that one multiples Jerry by three and then adds six.

Liam: Yeah, so 15+51 *[the sum of Ben and Jerry's numbers]* is 66, so I would know that I should go a little bit higher.

ML: OK.

Liam: So I'll try 16 times 2 equals … plus six is 54. *[Liam stops working at this point because 16 and 54 sum to 70 and thus the problem is solved.]*

Several things are of note thus far. Firstly, at this point, Liam has (or has constructed on the spot) *conceptual categories* for *input, the resulting output, the target value, and the error between the resulting output and the target*. That is, these are categories that Liam sees the problem in terms of and uses to organize his work.

Secondly, Liam concludes that an output of 66 means that he "should go a little bit higher" (with the input). In making this determination, he has implicitly referenced another conceptual category of the *target [70, the sum of Ben and Jerry's numbers in this case]* in his comparison of 66 to 70, noting that it is too low. Further, this episode shows that Liam thinks about changing inputs as *leading to or controlling* results in changing

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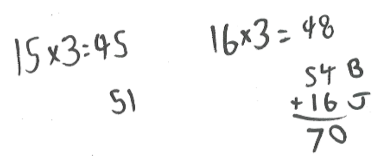
he is initially hesitant because he first "saw" the problem in terms of different conceptual categories, he constructs the strategy of guessing and adjusting fairly readily in the next turn.

outputs (since the output of 66 was too low with respect to the target, he decided needed to go a little bit higher with the input). I argue that his recognition of how to control the variation (e.g., increasing the input by a little bit will result in increasing the output by a little bit) is based upon a growing conceptual scheme (that will become increasingly organized over the sessions) about how the underlying functional relation given in the problem statement behaves when *x* is varied.

It is important to recognize this aspect of *control of variation* that is present from the beginning of the sessions for Liam. Note that Liam's focus on the *path of determination* and *co-variation* between input-output pairs stands in contrast to the conceptualization of a student who only recognizes that a given output corresponds to a particular input. One would expect that a student who was less attuned to the variation *between* input and output pairs would treat each trial input as an independent event, rather than using the results from previous trials to get closer and closer to the solution with each subsequent trial.

After he has finished his calculations that solve the problem, I ask him about his choices. Liam explains as follows:



ML: Okay, good. So yeah, I'm very interested in what you did over here *[pointing to Liam's work on the page, shown above]*. So you were telling me that you were guessing numbers, right?

Liam: ***Yeah, just like something that I figure in the middle range – 15 is a good even – you know. It's a rounded number sort of – 15 – and it turned out to be relatively – well, almost correct. So. And then from there, I would have thought, "Do I need more, a higher number or a lower number?" And then I need to go up or down with this*** *[points to 15 in his written calculations].*

Before we move to working on another task, I ask him whether this is a way that his seventh grade (pre-algebra) math teacher had taught him to solve "these problems" or if this was something that he came up with himself. He says

Liam: I really forget, but I think that I came up with this myself *[laughs a little bit, sounding proud]*. ***It's probably not as efficient***, but I forgot what Mr. R taught us about this problem in particular.

What is of particular interest here is that Liam remarks on the efficiency of this method in contrast to other methods that he may or may not have been taught for solving such problems. In any case, this shows that, from the very beginning of the sessions, *efficiency* is a criterion that Liam considers pertinent in evaluating and contrasting approaches to solving problems.

Our current goal is to be able to track the micro-shifts in strategy implementation and knowledge organization over the course of six focal episodes. To this end, I now present a thematically organized summary discussion of the first episode. This will be a model for summary discussions that will follow the future five focal episodes, as well.

The strands of the summary include: (1) snapshot of the strategy system in the episode, (2) organization of the growing control of variation scheme, (3) dynamics of knowledge activation, including the role of artifacts, interactions, and the activity, and (4) micro-developments that occurred during the episode.

Snapshot of the strategy system

Recall that the definition we gave of a strategy as a complex system included three components that we intended to track in the analysis: strategic path, conceptual categories, and relations between conceptual categories.

1. **Strategic path.** In this episode, Liam uses a strategic path based on the idea of trying to get "closer and closer" with subsequent trials. As discussed in the discussion of the analytic framework, such a strategic path is based on a version of "means end analysis" (MEA).
2. **Conceptual categories.** The conceptual categories that Liam recognizes or constructs in this episode include: (1) adjustable input value, (2) resulting output, (3) target value, (4) error between each output and the target. Liam sets the input value and then uses a path of determination so that the resulting output can be read out, and the target and output can be compared. In this case he notes that there is a difference between the target and output, but the resulting output in this case was just a little bit too low.
3. **Relations between conceptual categories.** In terms of relations between conceptual categories, with Liam's first guess of 15, he explicitly notes that "15 + 51 is 66, so I would know that I should go a little bit higher." One can schematize his thought process as "If the result of my guess is *a little bit too low* with respect to the target, then I should adjust by making the next guess *a little bit higher* than the previous one. This relation is an example of a *co-variation scheme*, a unitary judgment about the effect of controlling an input value.

Organization of the control of variation scheme

Recall that in addition to changes that occur at the level of Liam's strategy system, we are modeling changes in Liam's conceptual understanding of how to control the variation of linear functions. One of the claims of the analysis is that Liam's knowledge about how to control the variation of linear functions gets increasingly organized over the course of solving several problems. To trace this increasing organization, after each episode, we will provide a "snapshot" of what we know about the organization of Liam's knowledge of how to control the variation of a linear function.

Throughout the sessions, one has to keep in mind that a *control of variation scheme* is being built up and called upon by Liam each time he solves a problem is just a projection of what Liam may know about co-variation and control of variation. In studying how Liam solves the problems in these sessions and how that changes, we are observing Liam exercising his knowledge about how to control variation across a short period of time and in a particular context. There may be several elements that are related to the ones that are cued in solving a problem that Liam also has, but we do not have the opportunity to observe in action. By observing Liam's actions and utterances (e.g., wait time, confidence, etc.) we can make arguments about which elements may be constructed or recognized on the spot and which are more stable elements of his conceptual system. Despite these challenges, it is still of interest to track over several episodes what we observe and when about Liam's knowledge of how to control the variation of linear functions.

In this episode, Liam used (1) Changes in input result in Changes in output and (2) "Increasing X (by a little bit) → Increasing Y (by a little bit)." The following template will be used to track elements that Liam is observed to have used (along with the sessions in which they were used). Also, note that the organization of the template below has the "meta-element" *Change in X → Like Change in Y* at the top, since many of the individual inferences that Liam makes based on the information he reads out about the effect of his choice of guesses can be interpreted as "special cases" of this more general element.

Change in X → Like Change in Y

Monotonicity

- Increasing X → Increasing Y *[Episode 1]*

  Proportionality

  - Increasing X (by a little bit) → Increasing Y (by a little bit) *[Episodes 1 & 2]*

Figure 7. Active elements in Liam's control of variation scheme in episode one.

We now turn to a summary of what we observed in episode one about how and why particular knowledge elements were activated in Liam's process of reasoning.

In episode one, there were two general features of Liam's activity that give evidence about where his attention was focused and what elements were likely to be cued as a result.

1. **Awareness that the path of determination mediates the effect of modulating the input value.** Liam's statement about how to choose his first guess (e.g., *"just something that I figure in the middle range – 15 is a good even – you know – it's a rounded number sort of..."*) indicates that Liam is aware that the path of

determination between input and resulting output would be useful to attend to in making a determination about reasonable trial values.

Across the focal episodes, we will be interested in evidence in the data that Liam is aware that the effect of modulating the input is mediated by the functional relation given in the problem statement. It will be interesting to track whether this awareness of the relation connecting input to output is tacit or more articulate.

2. **Readout of error**. Readout of error plays a role in activating the thought element "Increasing X (by a little bit) results in increasing Y (by a little bit)." In this episode, we have that Liam is attentive to both the *direction* and the relative *magnitude* of the error between the resulting output and the target value.

One of the aspects that we will trace throughout the sessions is the changing nature of Liam's attention to the direction and magnitude of errors. To preview, we will see that Liam's attention to the error becomes *increasingly quantified* and this has consequences for the development of Liam's conceptual system.

Based on the above discussion, two kinds of micro-development in the strategy system are worth noting and tracing in the analysis of subsequent episodes.

1. **Creation and stabilization of conceptual categories.** In the episode described above, the conceptual categories that Liam used to organize his work included adjustable input, resulting output, error, and target. However, an alternate way to solve the same problem would involve starting from the target value (e.g., 70) and "unwinding" to find Ben and Jerry's numbers. Thus, what was previously the target could be "seen" as the input or the starting place for a different computation leading to the solution. That is, though the conceptual categories listed here may seem like the only possibility, the example just given shows that problems can be seen and parsed in many different ways and in terms of alternate sets of conceptual categories. Thus, recognizing that the role of creation and stabilization of conceptual categories in the construction of strategies is important.
2. **New relations between categories incorporated into the existing (and growing) conceptual scheme** (the control of variation scheme). This point is related to our discussion above concerning the tracking of the structure of the control of variation scheme. This will be important to track because adding relations between categories may enable different strategic actions and hence be a critical component of the development of a new strategy.

Overview of what happened in between episode one and episode two

Episode one was the first problem in the first session and episode two (discussed below) was the first problem of the second session (after a warm-up). During the balance of the first session, Liam worked with problem stems and representing them in t-charts and then we revisited solving word problems, attempting to introduce the chart organization. The final episode of session one (analyzed, but not discussed in length here) was interesting for the reason that it revealed evidence that at least under certain conditions Liam found the "unwinding" approach to solving problems to be natural (and hence the conceptual categories discussed above were not completely "stable"). However, the increased negotiation around the use of the charts in the next episode served to stabilize the conceptual categories that are used in organizing a solution using guessing and checking. Since we are interested primarily in the trajectory of refinement of Liam's guessing and checking approach to his linear interpolation/extrapolation approach, this intermediate stage is not included as a focal episode.

Focal episode two: Purposeful guessing and checking organized in a chart

We will now discuss the second of six focal episodes. In this episode, Liam is working on the following problem:

> *The base of a rectangle is three centimeters more than twice the height. The perimeter is 60 cm. Find the base and the height of the rectangle.*

This is the first problem of the second one-hour session and Liam is specifically asked to solve a word problem using guessing and checking organized in a chart. In discussing this episode, I will break the discussion into several phases of activity: introduction and framing of the task, Liam's work on the task, and a brief post-solution reflection. This pattern is fairly typical of work on the tasks in the sessions.

As mentioned above, the task is introduced as one that we'll try to solve by organizing our work in a Guess and Check chart. Before working on the problem, Liam and I talk briefly about how Liam processes the information given to him in problems. Liam says that he typically doesn't need to take notes or write anything down to process the information in word problems. Typically he can just *"write down the equation quickly"* or *"just do it in my head if they're really easy."* He decides that the problem we're working on (above) would be one that he'd have to think through a bit, remarking *"This is pretty confusing to me."* Liam's awareness of his own level of comfort or confusion as he is thinking about problems is notable.[25]

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[25] Liam's confidence in his own methods as opposed to the instructionally sanctioned methods is interesting, an orientation that *[as observed in his classroom the following year after this interview session was conducted]* often got him in trouble with his classroom teacher. For this reason, my requests in the session for him to write down his work in the chart and to show all calculations and operations on variables were onerous to Liam and he needed to be reminded several times to continue to do this. Sometimes I motivated these requests by asking him to write out his work in order to be able to "see the pattern" or finish writing out his calculations so that maybe knowing the result of the final calculation could be of help with choosing the next guess. As we will remark in the discussion of the analysis, the reliance upon such reasons (e.g., seeing particular patterns and preparing to express the relationships in variable notation) is not desirable because

The first part of the interaction around this problem involves us discussing the problem and the conventions for the construction of the chart (e.g., input, determination path to second variable, relation between variables given by the conditions in the problem and resulting output, and check are all included in the chart). These terms were not used in discussing the problem, but rather we talked through the problem statement, identifying information that was given and that we might choose to record in a chart that we would construct while solving the problem that would serve as a record of the problem solving process.

I ask him which of the height or base "helps us find the other" and Liam replies

Height	Base	Perimeter	60 (Check)

Liam: It could be either way. It just depends which one you find first. Like, yeah, it could be either way.

This is interesting because it signals Liam's flexibility with paths of determination between variables – he could either find the base by adding three to twice the height or he could find the height by subtracting three from the base and dividing the result by two. Liam is notable in that throughout the sessions he did not have difficulty managing his choice of determination path and he usually chose the computationally easier path to manage.[26]

In this problem, Liam has chosen the height to be the input. The resulting output in this problem is the perimeter. The path of determination between input and output involves a relation between the base and the height (the base is three more than twice the height) and another relation: using the base and height to find the perimeter. The constraint in this problem is that the perimeter has to be 60 cm. In terms of a "conceptual category" that is more consistent with Liam's eventual construal, this becomes "target"

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these are not meaningful reasons to students for why they should express their work in one mode or another. The reasons and function are clear to the instructor and to the designer, but not to the student.

[26] Most people would find the computations easier to manage if they let the height be the independent variable. However, this issue of which variable is more natural to choose values for (e.g., to let be the "input" or the independent variable) was not always clear to students. Some students, both in the tutorial sessions and in the classroom pilot work, seemed to "intuitively" make the computationally simpler choice based upon a reading of the conditions in the problem and other students would not. Often, the students who didn't choose the computationally simpler path of determination also had difficulty managing the path that they chose.

(as opposed to other descriptors for the category, like "constraint"). The check column is intended as a place to record whether the resulting output for a given input was above or below the "target" output (in this case, 60 *cm*).

We begin below as he is beginning to work on the problem in the chart. Even though we have just discussed the construction of charts, Liam begins work in this episode by wanting to guess the answer to the problem exactly (either through unwinding or other means – this is not clear as he is whispering to himself). In line with the objectives of the sessions, I encourage him to solve the problem using guessing and checking in the chart form.

Liam: Okay, so if I guessed ***about 18*** *[for the height]* it wouldn't work. OK, I'll just say 20 for now. ***I know it's a bit too high.***

ML: OK.

Liam: ***Well, maybe 18 and I'll see what happens there.*** So for 18 – twice the height so it's 36 then plus the – *[starts writing down the calculations without writing how the variables are operated on]*

Height	Base	Perimeter	60 (Check)
18			

Throughout the episode above, it is of interest that Liam is explaining to me that he knows already that it will be too high and that his plan is to just see what happens with one value. This seems to suggest that perhaps Liam believes that I am expecting that it is possible to solve the problem on the next try and he is explicitly letting me know that this next trial is just part of a process toward the solution.

At this point, I interject and ask him to write out how the variable he has chosen to guess values for is being operated on. Initially he had kept condensing the calculations in his head (e.g., writing down 36 instead of 2 times 18). In the middle of this negotiation around the conventions for recording calculations, while beginning to write down the calculation for the perimeter, Liam notices that his first guess of 18 is going to lead to a result that greatly overshoots the target.

Liam: Yeah. **Ohh::: so it's actually *a lot lower.***
ML: Oh.
Liam: *[Laughing]* I just realized that.

Liam's laughter and the comment that he just realized that his initial guess was way too high may indicate that, to him, something about the result of that choice should have been obvious to him before that point. This monitoring of the effect of his choices during the calculation is noticeable. He has not yet finished the calculation, yet has noticed that it will result in an outcome that is too high. Before having finished the calculation, he spontaneously leaves the calculation uncompleted after having ascertained that it is way too high and begins trying to make a next guess. I suggest that we go through this example and then adjust. At the same time as my suggestion, Liam guesses that the correct answer is probably six. Without having completed the calculation at that point, his guess is notable. It gives a sense for about how much he feels it would be necessary to adjust, but we have very little to access in terms of how he arrived at a next guess of six in particular. Because purposeful guessing was one of the aspects of the activity that I wanted to encourage generally (and because I wanted all of the calculations to be recorded in the chart for later review and reflection), I asked him to go through with and complete the calculations for his initial trial value of 18.

ML: ML: Well, why don't we go through with this?//
Liam: //I think it's six.
ML: Why don't we go through this example and then we can adjust, OK?

At this point (see below), Liam gives some insight into how he sees the enterprise and the relations in this problem context by explicitly explaining this to me. It appears as though he may be summarizing this information out loud for the purpose of displaying that this is now how he interprets the enterprise of solving problems in this way.

Liam: OK. So basically, you try to double this *[base]* then double that *[height]* then add them to try to get 60.

This narration of the general plan for solving the problems indicates that he *now* sees an *adjustable* input, path of determination, *resulting* output, and target clearly**.** Again, this may indicate that, at least in his mind, when he chose the initial guess of 18, the full structure of this problem solving strategy may not have been fully evident to him. That is, the categories above may not have been stable for him at the beginning of the session. This interpretation is consistent with the negotiation around solving problems that happened at the end of the first session: Liam found "unwinding" to be a natural approach to solving problems, and because he had parsed the problems and solution path in terms of different conceptual categories, it was difficult for him to reconcile the two approaches.

After some negotiation and discussion as the remaining calculations for the first guess were entered into the chart, Liam remarks again that the first guess and its result are way too much. This is consistent in quality with his earlier assessment. I push on that assessment and ask for some attention to degree of error, suggesting that this could be useful information for guiding the solution process. Liam does not specifically acknowledge this and instead finishes writing the calculations for the first guess out.

Liam: This is way too much.
ML: How much too much is it? Because even if it's way too high, maybe we can use it to help us.

Liam: //36 plus 36 – wait, no.
ML: Okay, so let's mark that as too high.
Liam: Yeah.

Height	Base	Perimeter	60 (Check)
18	2(18)+3=39	36+(39+39)=114	Too high

After he finishes this, I return to the point of how much too high the first guess was.

ML: Now, about how much too high was it?

Liam: *[Laughing]* ***Uh, it's almost twice as much***. So I'll try it with 10.

Here Liam laughs again, perhaps surprised by how off his first guess was. As will become more and more evident through the discussion of the subsequent episodes, Liam has a very strong number sense and ability to estimate. So, it could be that Liam is personally surprised by how far off he was since this is not typical for him. Notice that after Liam has written out the rest of the calculation for the guess of 18, he now has chosen to amend his initial choice for next guess from 6 to 10. We don't have access to why he thought 6 would be a good choice before, but that estimate came before he had articulated his image of the strategic plan. Further, the estimate of 10 comes directly after I have asked him again about how much too much his first guess was. He gives an approximate reply of "About twice too much" which is notable for being a multiplicative comparison. Relationships of half as much as twice as much are more intuitively grounded and likely to be more salient than other multiplicative comparisons

After having worked through the calculation for an input of 10 (and recording them exactly has been negotiated), Liam remarks

Liam: Okay, 20 plus 46 equals … huh … so that's a little bit too high.

Height	Base	Perimeter	60 (Check)
18	2(18)+3=39	36+(39+39)=114	Too high
10	2(10)+3=23	20+46=66	Too high

Note that Liam now spontaneously reflects about the result of the guess – both its direction and magnitude *("a little bit too high").* What is of particular interest to us will be what Liam does next *as a consequence* of that noticing. We see that Liam adjusts the input only a little, trying a next guess of 8. Liam explicitly notes that he believes that this choice would be about right (because he's only a little too high and he is only moving the input by two units.) Note that he only says that this guess would be about right, he does not make a claim about whether it is right. Note that the trial to solve the problem exactly model *does* show us more information about how he thinks about the relationship in the problem. This orientation to problem solving means he is more attuned to attending to the nuance in the variation than a student who is just trying to make an incremental improvement over the results of the previous trials.

Liam: Okay. Okay, ***so now I'll try an 8***. *[writing]* ***That'll be about right.*** So, 16. *[writing]*

He goes on to reflect that this might not be right either. Notice this as a shift towards working out the entire guess to see not only if it is right, but he also attends to what kind of adjustment is needed.

Liam: I think that might not be right either. 8 plus 8 plus 2 plus 2 equals. So that's 16 plus. Hold on I'll just use this *[picks up the calculator]*.

Liam determines that the result of his computation is 54, that it is too low and that therefore it is probably nine. It is at this point in the transcript that we see the composition of inferences to deduce that the input value that will achieve the target is probably nine. In the current model of his learning process, this is attributed as either an on the spot composition of two previously unconnected inferences or as the invocation of a previously constructed knowledge element in his knowledge system. However, it is plausible, for example, that a version of a "Too high, too low, just right" scheme is something that could be independently recognized and invoked in reasoning about the procedures to solve such problems.

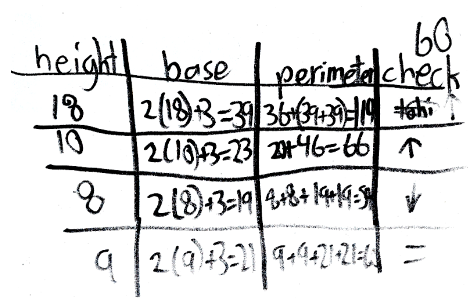
Liam: 54. ***54, too low. So it's probably nine.***
ML: It's okay. *[In relation to him running out of room on his paper]*
Liam: It's probably nine.

Height	Base	Perimeter	60 (Check)
18	2(18)+3=39	36+(39+39)=114	Too high
10	2(10)+3=23	20+46=66	Too high
8	2(8)+3=19	8+8+19+19=54	Too low

Liam does the computations for a height of nine, resulting in the following final chart:

Height	Base	Perimeter	60 (Check)
18	2(18)+3=39	36+(39+39)=114	Too high
10	2(10)+3=23	20+46=66	Too high
8	2(8)+3=19	8+8+19+19=54	Too low
9	2(9)+3=21	9+9+21+21=60	Check

After Liam has finished solving the problem, I ask him about his noticing that the answer was probably nine. Before Liam addresses that question, he annotates his chart with the express purpose of making the result of the calculation clear to others. This is an interesting positioning of the use of the chart – at this point it functions as a tool to communicate and display his thinking to other people. This is in contrast to its potential use as an organizer or record of his thinking and problem solving processes. And, yet again, that function of the chart in the process of solving the problem is different than the function of the chart to hold information in a particular format for later reflection. I continue with asking about how he knew that the input that would solve the problem was definitely nine.



ML: Yeah, okay. So here *[pointing to input column]* when we were going between 10 and then you went to 8 and then you decided to go to 9 and you said that it was *probably* 9?

Liam: ***Well, it was actually definitely 9 if this*** *[guess of 8]* ***was too low and this*** *[guess of 10]* ***was too high. Unless it was a decimal number.***

Liam's care in noting that the answer could potentially be a decimal number is interesting. Also of interest is his conviction that the answer was between 8 and 10.

Based on his utterance and his conviction, that the answer lies between 8 and 10 is not in doubt to him.

Snapshot of strategy system

As in the discussion of the previous focal episode, we begin by giving a snapshot of the current state of the Liam's strategy in terms of our reformulation of strategies as complex system involving (1) strategic paths, (2) conceptual categories, and (3) relations between conceptual categories.

1. **Strategic path.** The strategic path that Liam uses in this problem becomes Means End (MEA), but this is not established until the end of the first calculation. Liam's initially "parsing" of the problem may have included a different set of conceptual categories (see below).
2. **Conceptual categories** In this episode, there was some initial negotiation around the conceptual categories that Liam used to organize the solution to the problem. The categories of adjustable input, resulting output, error, and target were eventually established and these categories stabilized some after he had worked through the calculation with one trial value (e.g., when he narrates the actions he is taking *"OK. So basically, you try to double this [base] then double that [height] then add them to try to get 60."*)
3. **Relations between conceptual categories** The ***co-variation schemes*** that are activated in this episode include:
   a. Decreasing X → Decreasing Y and Decreasing X (by a lot) → Decreasing Y (by a lot). These elements are activated in response to Liam's assessment that his initial guess of 18 was almost twice as much and that as a response he'd try a next guess of 10. Decreasing X (by about a factor of 2) → Decreasing Y (by about a factor of 2) was activated in response to a question to Liam about how much too high his guess of 18 was.
   b. Decreasing X (by a little bit) → Decreasing Y (by a little bit). This is activated in response to Liam's second guess of 10 being a bit too high.
   c. Increasing X (by a little bit) → Increasing Y (by a little bit) is activated once Liam sees (e.g., reads out) that his third guess of 8 is a bit too low.
   d. In addition, *inbetweenness* a composite of two co-variation schemes is another form of element that is activated in this episode. This is in response to noticing that one trial value resulted in an output that was too high with respect to the target and another resulted in an output that was too low.

Organization of the control of variation scheme

Overall, the elements that we have seen activated so far (over both sessions one and two) in the control of variation scheme include:

Change in X → Like Change in Y

Monotonicity
Increasing X → Increasing Y *[Episode 1 & 2]*
Decreasing X → Decreasing Y *[Episode 2]*

Proportionality
- Decreasing X (by a lot) → Decreasing Y (by a lot) *[Episode 2]*
- Increasing X (by a little bit) → Increasing Y (by a little bit) *[Episodes 1 & 2]*
- Decreasing X (by a little bit) → Decreasing Y (by a little bit) *[Episode 2]*
- Decreasing X (by about a factor of 2) → Decreasing Y (by about a factor of 2) *[Episode 2]*

In-betweenness
Somewhere in between *[Episode 2]*

Figure 8. Active elements of Liam's control of variation scheme in focal episode two.

As noted in the previous focal episode, there appears to be a *meta knowledge element* that underlies all of these inferences that Liam is drawing: *Changing X in some way* → *Changing Y* in a "similar" way.

As in the first episode, the strategy system incrementally changed with the formation of new conceptual categories and relations.

1. **New conceptual category formed**
   a. The idea to focus attention on an *interval in X and a corresponding interval in Y* was seeded in this episode with the activation of the idea of "inbetweenness." This noticing is important for the future development of the strategy and underlying conceptual scheme.
2. **New relations activated or formed** (through *progressive quantification*)
   a. Liam's assessment that his first trial value was too high was in response to my question about how much too high his guess was. He spontaneously had noticed it was too high mid-calculation. I encouraged him to finish the calculation so that he could see the actual result of the calculation, he made the assessment that it was too high, and then I asked how much too high.
   b. Liam's use of a *multiplicative comparison* (noting that his result was about twice too high). His assessment "It's almost twice as much" now encodes not only direction of error, but a more quantitative assessment of error than just too high or a lot too high. Thus, Decreasing X (by about a factor of 2) → Decreasing Y (by about a factor of 2) is now in the conceptual scheme.

Overview of what happened in between episode two and episode three

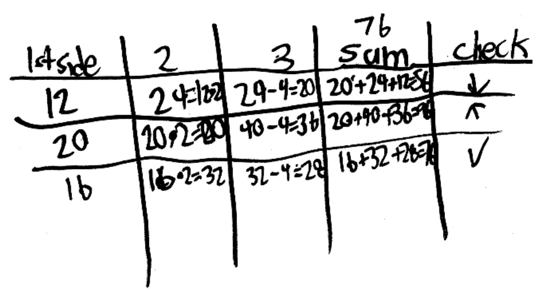
In between episode two and three, Liam worked on two related problems about sums of consecutive integers and sums of consecutive odd integers. In both cases, he used estimation to determine the solutions directly. The first problem he solved directly in one trial and the second involved two trials. Because his work on these problems was not related to the arc of refining his guessing and checking strategy, they were not selected as focal episodes. These were the only two problems in between the focal episode concerning the perimeter of the rectangle and the next focal episode that concerned the perimeter of a triangle.

Focal episode three: Exactly in between

The next episode highlights a serendipitous, but important move toward progressive quantification of the error between outputs and the target that will be consequential in the construction of Liam's linear interpolation/extrapolation strategy. Liam was working on the following problem:

> *The perimeter of a triangle is 76 cm. The second side is twice as long as the first side. The third side is four cm shorter than the second side. How long is each side?*

In this problem, Liam constructs a chart and begins right away guessing values for the first side. Before moving on, I decide to mark this and check what Liam would say about how he knew which value was the independent variable.



Liam: Okay. So I'll guess the first side is –

ML: Okay, so have you decided that the first side is going to help you get the other ones?

Liam: Yeah.

I ask him if he's sure about this choice of input and he provides a justification for what he was attending to – the given relation in the problem (that the second side is twice as long as the first). Liam takes as given that it is the relation in the problem that determines how the output results from the input. The relation determines that the first side is what allows you to get the second side. In a way, this seems self-evident to him – of course it is the relation in the problem that indicates to him which value to guess values for.

ML: Are you sure of that?
Liam: Because it says the second side is twice as long as the first side. ***So that determines that the first side is the first number, which is the only way that you can get the second side.***

I respond with an "okay", and Liam continues to explain that the relation determines how the first side allows you to find all the other sides.

Liam: ***By doubling the first side and then, from the second side, that's the way that you get the third side because you subtract four from the second side.***

I validate this response, while Liam is already off to solving the problem. He chooses (without verbalizing a justification) that his first guess will be 12.

ML: ML: Okay, excellent. So it looks like the first side is a good//
Liam: I'll guess 12.
ML: 12. Okay.
Liam: Okay, so I'll make this 24//

Liam finishes the calculation for the guess of 12 and notices that a guess of 12 resulted in 56. Without giving an explicit reason for his next guess, he decides to try 20 next. Note that this is consistent with the usual constraints on guesses (the previous guess resulted in an output that was too low) and so he should increase it. He judges that this is more than a mere fine-tuning required – 56 and 76 aren't that close and so a more substantial adjustment is necessary in order to achieve the target value. Though this is not stated, his prior and current actions are consistent with this attribution.

Liam: 20 plus 24 plus 12, okay? So that's 44 *[calculating]* 56.
ML: OK.
Liam: I'll try 20 now.
ML: You're going to try what?
Liam: 20.
ML: 20, OK.

Liam then works through his computation for 20 and immediately assesses that it is "way too much."

Liam: 40 minus 4 equals 36. 20 plus 40 – oh, that's way too much. *[writing]* It's too high.

In response, I ask for a quantification of the error. This time, I just ask how much too high it is, not for an approximation of how much too high it is.[27]

ML: So you got 96? Okay. How much too high is it?
Liam: 20.
ML: 20 too high.

It is curious that Liam asks if it is OK if he tries to guess again. It may be that Liam has an idea for how to use the previous two guesses and he is eager to see if it works (and he wants to provide space for himself to make that guess as opposed to being asked further questions about this particular guess).

Liam: I'll just keep – is it okay if I try to guess again?
ML: Yeah, of course.
Liam: OK, I'll say 16.
ML: OK.

Liam's choice of 16 for the length of the first side of the triangle does indeed solve the given problem. I ask him how he decided upon this guess. That he starts out by saying that "he doesn't know" is interesting. His following explanation is cogent and articulate and showing the purposeful choice of 16 for the input based upon the previous two choices. He first states his reason qualitatively – that he knows that the target output lies in between the outputs the first two guesses generated and so therefore the target input should be in between 12 and 20, the corresponding inputs. After running through the qualitative reasoning for where the target should lie, he adds the rationale for his specific choice based upon the quantitative information about exactly how much too high and too low the previous outputs were relative to the target. He invokes the idea that the two initial guesses are an equal amount too high and too low (stated in terms of the specific amounts too high and too low, in this case 20). Noticing that the target lies *exactly halfway* in between the two previous outputs invokes the expectation (or conjecture) that perhaps the target input that will solve the problem will also be exactly in between the input values. This is an invocation of "like changes in Y *correspond* to like changes in X." It is not immediately clear whether this invocation is based on the idea of *correspondence* (e.g., changes in Y of a particular sort *correspond to* changes in X of a similar sort) or whether it is based on the idea of the input (continuously and linearly) *controlling* the result.[28] (e.g., If I change X by a little, then *this will result in* a change in

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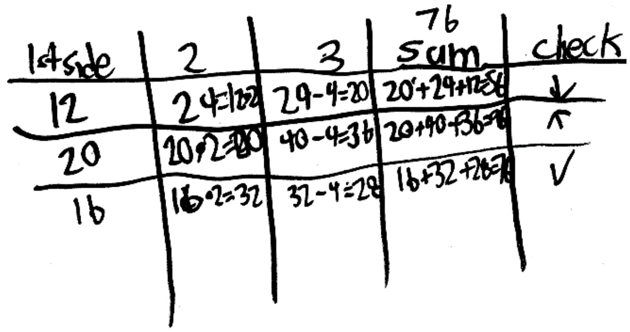
[27] My intention in asking such questions is merely to encourage students to choose next guesses that build on previous guesses. We will see evidence that the transition from answering such a question in an approximate form to answering in terms of an exact numerical value is important for Liam's later conceptual development.
[28] This conceptual distinction between correspondence and co-variation is important. The literature on co-variation and function has typically emphasized a pointwise "correspondence" view and what this activity encourages instead is a co-variational

Y by a little). In any case, the idea of "like change" here is instantiated as "halfway." That is, if the target output value lies halfway between two previous output values, then, according to this expectation, the target input value should also lie halfway between the corresponding two previous input values.



Liam: *[Writing]* Okay, so that, yeah. 76, wait. Okay, so check.

ML: Okay, so how did you decide a guess of 16 here?

Liam**: I don't know.** ***I just thought it was in between these two*** *[12 and 20]* ***because this*** *[56]* ***was 20 too low and this*** *[96]* ***was 20 too high. And this*** *[16]* ***was exactly between those*** *[12 and 20].*

One question that remains could be whether Liam would notice that one guess was 20 too high and another was 20 too low if this information was merely summarized in the chart, not in the context of solving problems. The particular sequence of guesses in this example (first guess 20 too low, second guess 20 too high) makes noticing "halfway" more salient than if this information needed to be discerned from a longer sequence of trials and outputs.

Snapshot of the strategy system

Below I give a snapshot of the strategy system in episode three:

1. **Strategic path.** The strategic path that Liam starts this episode with is MEA.
2. **Conceptual categories.** The *conceptual categories* that Liam starts out with are the same as before: adjustable input, resulting output, target, and error.
3. **Relations between conceptual categories.** The following co-variation schemes were activated in this episode:
    a. Increasing X → Increasing Y
    b. Increasing X (by a lot) → Increasing Y (by a lot).
    c. Another scheme that is activated (or constructed) is the halfway in between or "midpoint" scheme. This is a composite that builds upon the

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perspective on function. For this reason, it is of interest to track the usage of correspondence and co-variational ways of thinking about function over the course of the episodes.

> simple composite from episode two that uses two trial values and the assumption of monotonicity to bound the search space (e.g., if one guess is too low and another is too high, then the solution must lie in between the first two).

With respect to where Liam's attention is focused and the role that played in activating the categories and relations above, I note:

1. **Attention to amount and direction of error.** In this episode, Liam's co-variation schemes are activated by him reading out that his first guess was too low and that he would need to increase it substantially in order to achieve the target. After he notices that his first trial value resulted in an output that was 20 too low and his second resulted in an output that was an equal amount too high, he decided to try the midpoint between his previous two inputs with the expectation that this would achieve the midpoint between the previous two outputs.
2. **Awareness of mediating role of function relation on effect of modulating input.** In this episode, it is clear that Liam "sees" clearly the path of determination from input to output – he describes the way you get from one to the other using the relation. However, it is not clear if this plays a role in his choice of next guesses – it appears that, in this case, the equal amount above and below the target plays more of a role in the construction of the midpoint scheme than the actual relation.

Organization of the control of variation scheme

As in the previous episode, we will keep track of all of the co-variation schemes that Liam has activated in order to guide his choices of next guesses. This is for the purpose of getting a cumulative sense of the control of variation scheme that Liam is building up.

Change in X → Like Change in Y

Monotonicity

- Increasing X → Increasing Y
- Decreasing X → Decreasing Y

Proportionality

- Increasing X (by a lot) → Increasing Y (by a lot) *[Episode 3]*
- Decreasing X (by a lot) → Decreasing Y (by a lot) *[Episode 2]*
- Increasing X (by a little bit) → Increasing Y (by a little bit) *[Episodes 1 & 2]*
- Decreasing X (by a little bit) → Decreasing Y (by a little bit) *[Episode 2]*
- Decreasing X (by about a factor of 2) → Decreasing Y (by about a factor of 2) *[Episode 2]*

In-betweenness

- *Somewhere* in between *[Episode 2]*
- *Halfway* in between *[Episode 3]*

Figure 9. Liam's active elements in his control of variation scheme in focal episode three.

Here we discuss micro-developments and mechanisms of change that are indicated in this episode:

1. **Development of new relation (composite co-variation scheme).** The "in-betweenness" element from episode two has now been refined to be *exactly* in between. This episode was selected because of its clear role in the trajectory of *"progressive quantification."* By Liam quantifying the errors of his first two guesses and seeing that they were equal, this opened up the possibility for Liam to construct the new element or relation of "exactly in between." This is an advance over "in-betweenness." This development is possible *because* the errors were quantified and it was possible to recognize them as the same.

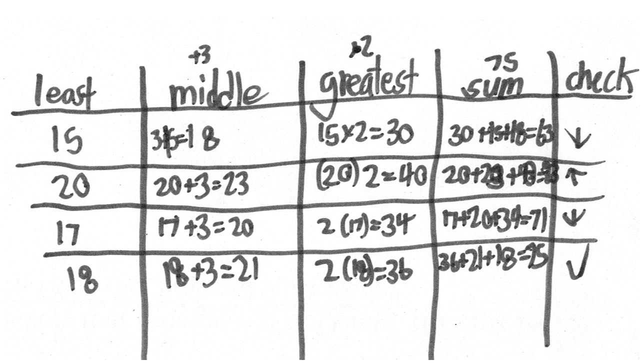
Overview of what happened in between episode three and episode four

The next two problems in between focal episode three and four were a pair of problems that concerned guessing and checking strategies for problems involving systems of two linear equations. With relative ease, Liam succeeded in solving the first problem, but struggled with the second problem and ultimately did not solve it. Partly because these problems pertain to his work with systems of equations and partly because after quite some time he didn't finish working on the second problem (we ran out of time in the session and Liam needed to leave), Liam's work on these problems was not selected as focal episodes. The very next problem in the sessions (at the beginning of session three, end of the first week of sessions) was focal episode four.

Focal episode four: Closer to one than the other

This episode involves both Liam's independent work on the problem and a discussion with the researcher-tutor afterwards in which they reflect on solving this problem.

> *Jabari is thinking of three numbers. The greatest is twice the least. The middle is three more than the least. The numbers total 75. Find the three numbers.*



least	middle (+3)	greatest (×2)	sum (75)	check
15	3+15=18	15×2=30	30+15+18=63	↓
20	20+3=23	(20)2=40	20+23+40=83	↑
17	17+3=20	2(17)=34	17+20+34=71	↓
18	18+3=21	2(18)=36	36+21+18=75	✓

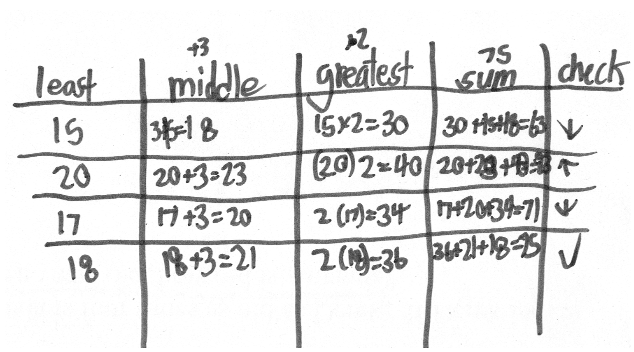
Liam reads the problem and immediately determines that the "least" number is the independent variable (*"So you'd have to work off the least…"*). His phrasing here is notable in that he has used similar process-oriented language in previous episodes (e.g., building off of, how you get the others, etc.) He proceeds to make a chart to organize his work. He chooses a first guess of 15 (“*So, let’s say the least is like 15”*). He finishes this calculation and pauses and laughs as he chooses his next guess. (“*I’ll go up to 20.”*). The use of “going up” is interesting here because it possibly evokes the idea of motion along a path. I note that this gave him 83. Liam responds “That’s too high. So I’ll go 17.” Liam notes that 71 (the result of the calculation for 17) is “a little bit too low, so 18.” He works out the calculation and sees that he has solved the problem.

After Liam solved the problem, I asked him what he based the decision to go from 17 to 18 on. He says

> Liam: ‘Cause it was only 4 too low, so, I figured that these numbers *[the inputs]* wouldn’t be that much different.

Following the solution of the problem, Liam and I discuss the problem some more. I direct his attention to his guesses for inputs of 17 and 20 and ask if there was a way he could tell how to predict his next guess based on the results of those trials.[29]

He quantifies the error for both of those guesses *[for 17 and 20]* and then thinks about things silently for a little bit. He then decides to compare the results for guesses of 15 and 20 instead of the guesses of 17 and 20 that were initially suggested.



> Liam: For, this [20] – ***I got 83, and that is – that’s 8, too high***; ***and for this is [17], 4, too low. (7 second pause***).
>
> ***I think I’ll compare these two [indicating 15 and 20], ‘right***?

> ML: Okay, so if you compared these two *[15 and 20, following Liam's suggestion],* what would you get?

> Liam: OK, so this *[63, corresponding to a guess of 15]* was – well it’s

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[29] It should be noted that I do not remember why I did this, in particular. The discussions we had around problems were open and not scripted, so this move did open the possibility for him to refine his strategy. But from my perspective, this post-solution discussion was just a discussion, not an *a priori* goal for instruction. This is one place where we see the goal of refining his strategy emerging.

too low and it was – 12 too low

ML: OK.

Liam: *This [83, corresponding to a guess of 20] was only 8, too high.*

So, this [63] was ***– I don't know how you'll put it in words, but this*** *[pointing to 63, corresponding to a guess of 15]* ***was lower***

and this *[moving his hand down to 83, corresponding to a guess of 20]* ***was too high****.*

So, it would go *[moving his hand down the sum column and then picking it up to go to the "least" column] –*

It would be further *[starts at 15 and gestures down the "least" column], '*

'cause I know that it'll be in between these two already *[between 15 and 20, pointing to both simultaneously with index and third finger of one hand].*

So, it would be higher up *[moves hand from 15 down to 20 in the "least" column]* ***instead of being right in the middle, it'll be higher up****.*

In the last line, we hear him explicitly referring to half as a benchmark or reference point. Note that "halfway" in between 15 and 20 would be 17.5. But he knows that the answer has to lie closer to 20 because 83 *[output of 20]* is closer to 75 *[target output]* than 63 *[output for a guess of 15].* Note that this reasoning path is *not* the one that he used to solve the problem initially as previously, he chose 17 and not 18 as his next guess after 15.

The post-solution reflection on this problem continues as I return to the guesses of 17 and 20 and ask him what would happen if he "moved his guess by one." He pauses to think silently for a while (9 seconds). We then determine that the guess of 20 was 8 too high and the guess of 17 was 4 too low. I then point out that this means that when the output went down 12, the input went down 3. I then ask him what would happen if we moved down 1 from 20 to 19. He thinks for a little bit (4 seconds) and then says

Liam: ***It would – wait. So, that moved up*** *3 [touching input of 17, in the comparison of 17 to 20].* ***Then*** *this [output resulting from a guess of 17]* ***moved up 12, so*** *(7 second pause****) you would have to move this down 4*** *[from 20 to 19].*

I verify that he means that ***each time you would move down four***. This very brief, but additional emphasis on the iterative effect of the increment provokes Liam to immediately say ***"Oh, so you can tell from that."*** Though he immediately asserts that you could determine the answer based on this information, at this point we do not (yet) know what he has in mind by that assertion. We will have the opportunity to unpack this more in the next episode.

This post-solution reflection is a key moment in the arc of learning that is described. For one thing, as we have just seen constructed above, this is the first place in the sessions where the idea of the "unit worth" of one guess and the effect of iterating this incremental worth arises. A strong source of evidence for the claim that Liam is constructing this idea on the spot is the amount of wait time while Liam thinks about the question of what would happen if he moved his guess down by one (from 20 to 19). Also, he stops himself as he begins an explanation, in order to think things through step-wise, *["So, that moved up 3 [input of 17], then this moved up 12. So (long pause) you would have to move this down 4."]* This step-by-step construction indicates that this is *not* pre-compiled knowledge.

In this episode, Liam has talked about controlling the variation in terms of "going up/down" (e.g., after his first guess of 15 resulted in an output that was too low, he said "I'll go up to 20.") The language of "going up/down" was not used prior to this episode, but it was used by both Liam and myself later in the episode and *may* play a supporting role in the conceptual development in this episode that led to the construction of the linear interpolation/extrapolation strategy.

As in the episodes prior to this, the chart appears to serve a mainly communicative function in that it plays the role of a prop to which Liam can point as he is explaining his solution process to me, pointing out "these numbers" for example. This allows him to give very compact descriptions of his rationale and to explain quickly what he is focusing on or thinking about without needing to expand his description in ways that could compromise conciseness and clarity.

As might be evident from the number of new ideas that were introduced in the previous episode and post-solution reflection (e.g., going up/down, concatenation of errors to reveal a linked increment in X and Y, scaling the increment to a unit, iterating this unit), episode four was one of the sites of significant learning within the arc of knowledge construction that we focus on. The summary discussion that follows of activated knowledge, control of variation scheme, and micro-developments will take place in two phases: the independent solving phase of the episode and the post-solution reflection phase.

Snapshot of the strategy system (Independent solving phase)

In this phase, a summary "snapshot" of the strategy system is:

1. **Strategic path.** The strategic path that Liam chooses is guided by MEA, as in the earlier episodes.

2. **Conceptual categories.** The conceptual categories remain the same as in previous episodes: (1) adjustable input, (2) resulting output, (3) target value, (4) error between output and target.
3. **Relations between conceptual categories.** *Co-variation schemes* that are invoked to guide choices for next guesses in this episode include:
    a. Increasing X (a little bit) → Increasing Y (a little bit)
    b. Decreasing X (a little bit) → Decreasing Y (a little bit).
    c. Sandwiching was also invoked in this episode, as is successive sandwiching (or "iterated application of in-betweenness" below)

Organization of the control of variation scheme

The control of variation scheme now looks like:

Change in X → Like Change in Y

Monotonicity

Increasing X → Increasing Y *[Episode 2 & 3]*
Decreasing X → Decreasing Y *[Episode 2]*

Proportionality

- Increasing X (by a lot) → Increasing Y (by a lot) *[Episode 3]*
- Decreasing X (by a lot) → Decreasing Y (by a lot) *[Episode 2]*

- Increasing X (by a little bit) → Increasing Y (by a little bit) *[Episodes 1, 2 & 4]*
- Decreasing X (by a little bit) → Decreasing Y (by a little bit) *[Episode 1, 2, & 4]*

- Decreasing X (by about a factor of 2) → Decreasing Y (by about a factor of 2) *[Episode 2]*

In-betweenness

Somewhere in between *[Episode 2]*
Halfway in between *[Episode 3]*
Iterative application of in-betweenness *[Episode 4]*

Figure 10. Active elements in Liam's control of variation scheme in focal episode four.

We now turn to the post-solution reflection. Recall that this begins with me asking Liam if there is a way to predict the answer to the problem knowing the results of the calculations for 20 and 17. This was a spontaneous question on my part and not part of the planned instructional arc, but it did open up the possibility for Liam to refine both his conceptual and strategic knowledge. Since this is taking place *after* the problem is solved, there is no strategic frame that guides these reflections. This discussion is still focused on considering the information one would need to solve problems, so in this sense it is still situated in terms of problem solving and strategy and not a pure discussion about concepts that Liam *could* draw upon in his solution process.

Snapshot of the strategy system (Post-solution reflection)

Following the post-solution reflection discussion, Liam's strategy system included:

1. **Conceptual categories.**
    a. Adjustable input, resulting output, error and target remain conceptual categories.
    b. Because error involved overshoots and undershoots in the cases of the two trial values that Liam compares, he can concatenate these errors to form an *interval in Y such that the corresponding interval in X will contain the solution to the problem.* These two intervals and their linkage is another conceptual category that Liam can now *set* in the future by making guessing so that they over and undershoot the target.
    c. Benchmarks (e.g., halfway) become a new conceptual category in considering the linked intervals that have now been constructed from concatenating errors.
2. **Relations between conceptual categories.** In this post-solution reflection episode, three main relations between categories are formed:
    a. Closer to one than to another
    b. Worth of a unit increment
    c. Iterated application of unit increment.

The construction of these ideas will be discussed below in the section on micro-developments. In our description of strategy systems, we focused mainly on strategic path, conceptual categories, and relations between conceptual categories. However, we also noted that in implementing strategies, the knowledge activated in the moment could be of many forms (e.g., declarative facts, co-variation schemes, etc.). In this episode, based on Liam's utterances

> "*I don't know how you'll put it in words, but this [pointing to 63, corresponding to a guess of 15] was lower and this [moving his hand down to 83, corresponding to a guess of 20] was too high. So, it would go [moving his hand down the sum column and then picking it up to go to the "least" column] – It would be further [starts at 15 and gestures down the "least" column], ''cause I know that it'll be in between these two already [between 15 and 20, pointing to both simultaneously with index and third finger of one hand]. So, it would be higher up [moves hand from 15 down to 20 in the "least" column] instead of being right in the middle, it'll be higher up."*

an *image schema ("motion along a path")* could possibly be supporting Liam's reasoning in this episode.

Organization of the control of variation scheme

Below, is the state of Liam's growing control of variation scheme, after the post-solution reflection in episode four.

Change in X → Like Change in Y

Monotonicity

Increasing X → Increasing Y *[Episode 2 & 3]*
Decreasing X → Decreasing Y *[Episode 2]*

Proportionality

- Increasing X (by a lot) → Increasing Y (by a lot) *[Episode 3]*
- Decreasing X (by a lot) → Decreasing Y (by a lot) *[Episode 2]*

- Increasing X (by a little bit) → Increasing Y (by a little bit) *[Episodes 1, 2 & 4]*
- Decreasing X (by a little bit) → Decreasing Y (by a little bit) *[Episode 1, 2, & 4]*

- Decreasing X (by about a factor of 2) → Decreasing Y (by about a factor of 2) *[Episode 2]*

Worth

- Increasing X by 1 → Increasing Y by d *[Episode 4B]*
- Every time X increases by 1 → Y increases by d *[Episode 4B]*

In-betweenness

Somewhere in between *[Episode 2]*
Halfway in between *[Episode 3]*
Closer to one than to the other [Episode 4B]
Iterative application of in-betweenness *[Episode 4A]*

Figure 11. Active elements in Liam's control of variation scheme in the post-solution reflection after focal episode four.

In this section, I will describe the construction of the new conceptual categories and relations that will ultimately seed the linear interpolation/extrapolation strategy. First, I discuss the construction of new conceptual categories:

1. **New conceptual categories formed**
   a. **Construction of linked increments** (e.g., an increment of *d* in X is linked to an increment *D* in Y). In making the comparison between 17 and 20, Liam focuses first on the conceptual category of the error between the result of 17 and the target and the result of 20 and the target. He gives an exact quantification of the error in both cases (so he is now attending not only to direction of error and rough magnitude of error, but also exact quantity of error.) Through concatenation of errors from overshooting and undershooting the targe***t*** the conceptual category of "linked increments" is formed. After doing this, Liam announces that he would rather compare the results of trials for 15 and 20 than for 17 and 20. Overshooting and

undershooting the target make the X and Y intervals easier to "see" than if Liam were to compare two guesses that were on the "same side" of the error." This under and overshooting aspect of the trials that Liam chooses to compare, as well as his focus on going up and going down (along a path) from a fixed start place (base) will become consequential to the construction of the strategy

b. **Use of half as a benchmark.** After having switched to comparing the results of 15 and 20, Liam quantifies the error between the results and target for each. This shows a shift in focus to exact quantity of error and *also,* based on what Liam says later in this episode about this comparison, a possibly instrumental use of half as a benchmark. This is a new category to set (as can be seen from what was done here in the choice of comparing 15 and 20) as well as to read out. The relevance of this benchmark may be heightened by the activation of an image schema – motion along a path.

c. **Correspondence to co-variation shift.** The formation of intervals and then the focus *first* on the change in Y values and then how much that change corresponds to in X values makes sense in terms of this activity structure. However, there is another shift from the correspondence (e.g., *A change of 3 in X corresponds to a change of 12 in Y*) to thinking about increments and unit increments (e.g., *Changing the input by one unit results in changing the output by four units*).

Overview of what happened in between episode four and episode five

Between episode four and five (e.g., throughout the rest of session three of six), there were three problems that Liam solved using guessing and checking. However, in each of these cases Liam solved the problems using estimation and no new developments along the trajectory of strategy refinement were noted. In addition to these problems, the next phase of the sessions involved Liam formulating equations. In his work on several problems, Liam was asked to use a chart to organize the finding of the relevant expressions (e.g., in identifying the entities that needed to be symbolized) and then formulate an equation. Recall that this was the objective of session three: to transition from guessing and checking approaches to equation-based approaches.

I now move to discussing a focal episode in which Liam re-constructs the linear interpolation strategy while solving a problem.

Focal episode five: Re-constructing linear interpolation/extrapolation

Before discussing the following episode in which Liam constructs the linear interpolation/extrapolation strategy, I will pause for a moment to discuss where the episodes that have been discussed so far occurred in the scheme of the six sessions. The work to this point had occurred in the first three sessions out of six. The first episode discussed in this section occurred in session one, the second and third were in session two, and the last "closer to one than the other" episode occurred in the third session. A full weekend passed before Liam and I met again to discuss the following problem.

Three consecutive integers sum to 414. Find the three integers.

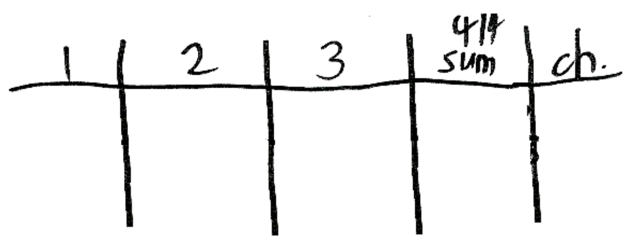
Though this episode has been discussed previously in contrast with Liam's earlier solution strategies, this time through we will take the opportunity to discuss the material leading up to the concise final description that Liam gives. This is so that one has the opportunity to see how the strategy unfolds and Liam puts everything together. The schematized treatment of the linear interpolation strategy in terms of variation primitives will follow the discussion with the line-by-line transcript.

In this discussion, Liam's artifacts have been modified so that the reader can see what his work looked like as he was solving the problem and describing his thinking.

Liam: I'm gonna guess one hundred-thirty-five.
ML: OK

It is of note that Liam's first guess is already quite close to the solution to the problem. In a similar problem that Liam solves in a later session he explains that he knows that consecutive numbers are all about the same amount and so roughly 1/3 of 414 would be a good first guess.

There is a pause in talking as Liam is engaged in making the chart template with columns labeled "1", "2", "3", "Sum", "Ch" and labeling his "target" of 414 above the "Sum" label.



ML: OK.
Liam: 408. So, ***that's a bit too low***.
ML: OK.
Liam: I guess 140. *[5 second pause]*

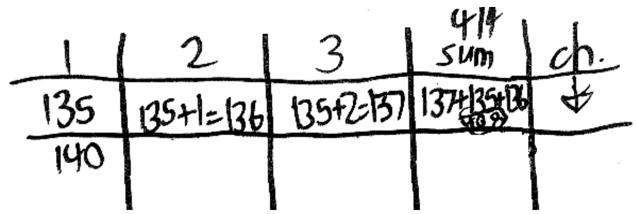
Hmmm. ***The number that I'm aiming for is//***

//Wait, you could do that when you have two… [rows], *(pause)*

Okay. 140 *[he writes 140 down in the chart as his next guess]*

In reviewing the transcripts, Liam's "Wait, you could do that when you have two…" is the first hint that Liam is starting to think of what will become the full version of his linear interpolation/extrapolation strategy. Here, we see him realizing that, based on the brief reflection we had after the last problem he'd solved using guessing and checking [an entire weekend had elapsed since that session], the problem *could* be solved with only two guesses. This comes right after noticing that his first guess was too low. It could be that this noticing of the first guess being too low prompted him to remember that he had previously thought about another way to solve the problem in which only two trial values were necessary to determine.

At this point, however, I, as the tutor, do not know that Liam remembers the impromptu reflection in the previous session and I do not realize that he is referring to his previous construction in the problem on Jabari's numbers when he says that he can "do it with two" here.



ML: So, what were you thinking of doing right there? *[when he said that he could do it when you have two].*

Liam: Well, ***compare the – like how much off this number was*** *[indicates 408, the result of his first calculation],*

and compare it to a different guess *[indicates 140 and then the whole second row – currently blank except for 140 in the "1" column – referring to the process of working through the calculation with 140]*

and look at that *[points to currently blank entry in the sum column in the second row],*

see how much off it was, then ***you could get this*** *[pointing to the sum of 414 in the problem statement].*

In the above, note that Liam's language of "how much off this number was" indicates a focus on the error between the results of a guess and the target. He then indicates that you could specify another guess and look at the result of that calculation to see how much off that guess was from the target value of 414. Liam does not yet reveal *how* he is going to get the target value of 414 from that information, but at this point, he shows that he does believe that the information from two guesses and their results is sufficient information to solve the problem.

It is important to notice in this segment, that Liam's plan for finding the solution to the problem using two guesses involves comparing the results of each trial calculation to the target and determining how far off the result of each calculation was from the target. This approach is to be explicitly contrasted with what he will do in the next segment where he looks directly at the difference between the results of the two calculations. In the next segment, he skips comparing each result to the target output to determine the error for each, and goes directly to taking the difference between the outputs.

ML: Okay, and what did you say?//

Liam: *// **Like the difference between this result** [pointing to 408 in sum column] **and this result** [indicating the second row of sum column, which is currently blank]*

and see how far apart these** [pointing to the "inputs" 135 and 140] **were,

***so you could tell how many of these** [pointing to the entries in the "1" column] **equaled how many of these** [pointing to the entries in the "Sum" column].*

In the above exchanges, Liam restates his approach, now in terms of the particular numbers that he has chosen. Liam is focusing attention on relationships between input/output pairs from the column labeled "1" (the input) and the column labeled "Sum" (the output). His gestures also indicate that he is concerned with the number that occupies this position rather than focused on the particular value that happens to be currently occupying that position.

This is an important point—he is describing a general algorithm, not specific numerical calculations. This is a clear precursor to the idea of variable – thinking about the general relations between quantities in the problem and operations on whatever value will occupy a "cell" once he has finished the relevant computation.

I ask again for clarification about what Liam wants to do and Liam says "I remember how to do it with two of them." I (who am still trying to understand Liam's proposed approach that, to me, has come out of the blue) interpret this as a reference to a previous problem that Liam has solved with a sum of *two* consecutive odd integers, not to a further elaboration of the idea of using the inputs and outputs recorded in two rows to predict the value of the first integer that will achieve the sum of 414. I suggest that for the moment we just continue with the calculation for 140. Below, Liam has completed the calculation for 140 as the first number.

Liam: Yeah. *[having verified the calculations for a guess of 140 with the calculator.]* That's a little bit too high.

ML: OK.

Liam: I said that was 408 *[results from input of 135]* and this 423 *[results from input of 140]*.

The difference of this is *[gesturing between 408 and 423 in the "Sum" column]*—***it's 15***, *[now moves hand over to column labeled "1"].*

So for every one of these *[points to entries in the "1" column],* ***it's 3 there*** *[points to "Sum" column].*

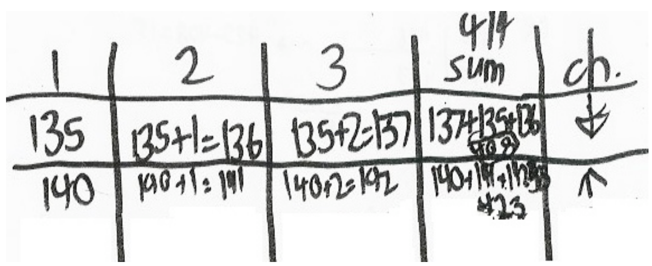
Okay, so I'm trying to get 414. So… (pauses to think).

Ah. *(pauses).* The difference between these two *[indicates 423, the result of the calculation for 140, and 414, the target value]* is 9,

So this should be less *[points to 140 in the "1" column].*

Above, Liam is now explaining how he is trying to figure out how to predict the answer while he is the process of working with specific numbers. He is mapping from his "general" idea of how this process should work, to how it works in the case with these particular values.

There is a pause in talking as Liam completes the next row of the chart using his predicted guess of 137 (since he had decided that he needed to adjust his guess by lowering it from 140 to 137). Below, there is a brief exchange about whether his calculations for 137 achieved 414.



ML: Good. So can you recap your reasoning? How did you decide that 137 was the best guess?

Liam: Well, it isn't really a guess once you get to the third one! *[gesturing between the entries in the "sum" column]*

I took 408 and 423. *[pointing to 408 and 423 in the sum column, as he is explaining]*

I have the difference between those [between 423 and 408], which is 15.

The difference between these two [pointing to 135 and 140 in the

> ***"1" column] is 5, and 15 divided by 5 is 3.***
>
> So, ***that means that for every one that this [pointing to 140 in the "1" column] changes, [moves hand over to the "sum" column] the answer changes 3.***
>
> So, then I took 423 and I subtracted that *[moves hand up to problem statement to indicate the target value of the sum: 414]*;
>
> The difference was 9.
>
> Three times three is nine*, [moves hand over to 140 in the "1" column]* so I knew it would have to be 3 less than this *[pointing again to 140 in the "1" column]*,
>
> and yeah.

This exchange is striking for the clarity of Liam's meta-observation about the generality of his algorithm. This shows that it was not just a question of Liam happening upon this method and noticing that it solved *this* problem. He is quite aware that the information from the first two guesses is all that is necessary in order to solve the problem. The final guess not really a "guess" since it's chosen so that it will solve the problem.

We will now move into the summary discussion of this episode to take a look at how the analytic framework interfaces with the data discussed in the previous episode. Because there were three distinct phases within this episode, I will discuss the state of the knowledge activated by the *end* of the episode, while aiming to communicate where certain ideas were introduced within the episode. While this episode will not contain many completely new pieces of knowledge that have not appeared previously in the episodes, this is the first time that we see Liam re-constructing the linear interpolation/extrapolation strategy on the spot as he is solving a problem. So, the way in which these pieces of knowledge get activated is novel here in addition to their role in the growing conceptual "control of variation" scheme.

Snapshot of the strategy system

1. **Strategic path.** Liam begins this episode with the same sort of strategic path as he has used in the past problem solving episodes: *Means End Analysis (MEA)* and ends the episode having constructed a version of *Linear Interpolation and then Extrapolation.*

Liam uses the relations in the problem to make a good initial first guess and notes the error between the resulting output and the target value. However, already before having even selected the second trial value, Liam is contemplating how he can use two trial values in a different way than he has in the past. There are three phases through

Liam's construction of the strategy. In the first phase, Liam merely states that two trial values would be sufficient information to solve the problem. In the second phase, he elaborates "Well – compare – like *how much off* this number was *[indicates 408, the results corresponding to his first input]* and compare it to a different guess *[indicates 140, and the whole second row that is currently blank]* and look at that *[points to the currently blank entry in the sum column in the second row]*, see how much off it was and then you could get this *[pointing to the sum of 414 in the problem statement.]"* In the third phase, he has actually worked through the calculations for 140 and describes in detail how he determines the unit worth of one increment and uses it to extrapolate to find the solution to the problem. Thus, though the episode began with Liam using MEA, the episode ended with him solving the problem by an entirely different means that he had (re-)constructed on the spot: *Linear Interpolation/Extrapolation***.**

2. **Conceptual categories.** Initially, the same categories are relevant again: adjustable input, resulting output, target, and error. However, throughout the course of this problem solving episode, several other conceptual categories became relevant to set and readout information about. For example, as opposed to thinking about error for *each* resulting output and the target, now the interval comprised of the difference between two outputs itself has increased in priority.
    a. *Difference between two results* is now a conceptual category (as opposed to forming the interval through the concatenation of errors).
    a. *Error between any output and the target value* (e.g., in order to establish a base from which to iterate the incremental worth in order to solve the problem) becomes another quantity to "set" in the course of solving problems.
    b. *Number of increments (in X and Y) that are required to solve the problem.*

3. **Relations between conceptual categories.**
    a. *The number of increments in input equals the number of increments in output required to move from the base output value to the target value.* This relation bears some similarity to the *Change in X corresponds to (like) change in Y* element. One can see this relation in play when Liam makes the determination that if the difference between an output and the target is 9 and the worth of a unit increment is 3, then it will take three increments in the X value to achieve the target Y value.
    b. Note also the difference between *correspondence* ("For every one here *[inputs],* it is three there *[outputs]")* and *co-variation* (Every time this *[inputs]* increases or decreases by one, *then* this *[outputs]* increases or decreases three).

4. **Activated auxiliary knowledge.** As we have noted before, not all of the knowledge pieces required to implement the strategy are "relations between conceptual categories" or elements of the control of variation system. Below we list two additional supporting pieces of knowledge relevant to this episode.

a. *Simple units.* A piece of knowledge that plays a role in the construction process is (as in episode four) the convenient presence of *simple units* in Liam's calculations *(*e.g., Five three's is fifteen, in this case). The determination of how much to move up or down from a base value would be much more difficult (and perhaps Liam would not see how to do it in general) if the particulars of the problem were not expressed in simple units, at least initially.
b. *Base +/- change.* In order to implement Liam's strategy, he needs to increment up/down from a "base" value. In this sense, the final form of Liam's strategy is supported by a *base +/- change* scheme. Such a scheme is like the conceptual scheme part of the *base + change* symbolic form (Sherin, 2001).

Organization of the control of variation scheme

Below, we give the current organization of the control of variation scheme. Note that this is not significantly different from the final organization after the post-reflection portion of focal episode four. However, one should note that focal episode five involved *calling upon and using the scheme* whereas focal episode four involved adding new categories and relations to the scheme.

Change in X → Like Change in Y

Monotonicity

Increasing X → Increasing Y *[Episode 2 & 3]*
Decreasing X → Decreasing Y *[Episode 2 & 5]*

Proportionality

- Increasing X (by a lot) → Increasing Y (by a lot) *[Episode 3]*
- Decreasing X (by a lot) → Decreasing Y (by a lot) *[Episode 2]*

- Increasing X (by a little bit) → Increasing Y (by a little bit) *[Episodes 1, 2 4, 5]*
- Decreasing X (by a little bit) → Decreasing Y (by a little bit) *[Episode 1, 2, 4, & 5]*

- Decreasing X (by about a factor of 2) → Decreasing Y (by about a factor of 2) *[Episode 2]*

Unit increment (Co-variation)

- Increasing X by 1 → Increasing Y by d *[Episode 4B, 5]*
- Every time X increases by 1 → Y increases by d *[Episode 4B, 5]*

In-betweenness

Somewhere in between *[Episode 2]*
Halfway in between *[Episode 3]*
Closer to one than to the other [Episode 4B]
Iterative application of in-betweenness *[Episode 4A]*

Worth (Correspondence)

For every one here [input], it is d there [output] *[Episode 4B, 5]*
Number of increments in X equals the number of increments in output required to move from the base Y value to the target value. *[Episode 5]*

Figure 12. Active elements in Liam's control of variation scheme in focal episode five.

The following are micro-developments and mechanisms of change that are observed in this episode.

1. **Construction of new conceptual categories.** As mentioned above, the new strategic plan requires the setting and reading out of several new kinds of information from the previous strategy. The difference between any two trial outputs and trial inputs become new categories, as does the difference between any output and the target value.

2. **Construction of new relations between conceptual categories.** A new relation between the linked input and result intervals shows how to *use* the "worth" or linkage of this interval in order to solve the problems. The path that Liam follows involves him *scaling* these linked intervals in order to find the worth of a unit increment and then extrapolating from that to solve the problem. The relation that he constructs is that "the number of increments in input is equal to the number of increments of the Y interval needed in order to move from the base value to the target Y value."

Liam currently has one path for determining the worth of a unit increment and also how to use it. Though not necessary for executing his strategy, eventually one would expect that Liam would come to develop other ways to determine the worth of unit increments.

In addition to *concatenation, "elision"* (See diSessa, in preparation) is a potentially important mechanism. This is how Liam (after the first time working through how the solution path worked) no longer needed to find the error from each result to the target and *instead* he realized that it was sufficient just to look for the relation between two input/output pairs.

Overview of what happened in between episode five and episode six

Following Liam's re-construction of the linear interpolation/extrapolation strategy here (beginning of episode four), the sessions turned to concentrate more on the formulation of equations for solving problems and there was no opportunity for Liam to use his algorithm throughout the rest of session four and five. The next episode that we will discuss comes from session six, the last in the series of sessions.

Focal episode six: Re-application of linear interpolation/extrapolation

The next and final episode that we will discuss in this arc of strategy/conceptual co-development gives some demonstration of the stability of Liam's construction. This episode is taken from the final session with Liam, in which the format was more of a re-cap and review of the sessions. Unlike during sessions two through five, where the method of problem solving approach was specified, in the sixth session, Liam was free to decide to solve the problems in any way he wanted. The problem that Liam was working on in this episode was:

Anne is twice as old as Paul. Bill is five less than Anne's age. Together, Anne's and Bill's ages sum to 147. How old are Paul, Bill, and Anne?

Paul	Anne	Bill	sum	ch.
35	35•2=70	70-5=65	135	↓
40	40•2=80	80-5=75	155	↑
38	38•2=76	76-5=71	147	✓

Almost immediately after the problem is read, Liam remarks on how old the protagonists in the problem are. While not surprising, given the sort of "magnitude sense" Liam has shown on other problems, it definitely shows that Liam almost reflexively attends to what range to reasonably expect the answer to be in. It could be that because of this computational facility, he doesn't think about ways to simplify his new algorithm (e.g. he could start guessing with *any* value – it is no longer necessary to choose guesses so that they get closer and closer to the solution). In this sense, vestiges of "guess and check" still remain even in the implementation of his new algorithm.

Liam: Wow, they're pretty old. Okay. Anne is twice as old. *[Working on problem]*

Okay I would do this in the guess and check chart.

ML: OK.

Liam: *[Constructing the chart with Paul, Anne, and Bill.]*

And I got that sum. *[Drawing an arc over Anne, Bill and Sum to indicate that he's finding the sum of just Anne and Bill's ages.]*

But I'm still checking for him *[points to Paul]*.

Such a spontaneous explanation of what he plans to do is probably prompted by the fact that the condition in the problem "violates" the usual pattern he's seen in other problems where all three quantities are involved in the condition. He is flagging that he still knows that Paul's age is the appropriate quantity to guess values for even though on the surface it may not appear to be involved in the test condition.

ML: I see, okay.

Liam: OK. So I'll guess Paul is *(thinking, 8 seconds)* 35.

[Works through the computations related to a guess of 35 for Paul].

Okay, so that was a bit too low. I will guess—(*thinking*, *6 seconds).*

I'll just guess 40. ***It's probably less time consuming than trying to find out the relation from this*** [Paul] ***to that*** [Sum] ***in some sort of equation.***

Here, after having guessed a value for Paul and seen that it was a bit too low, Liam decides to try at first to come up with a way to tell the relation between Paul and the Sum in "some sort of equation." However, he decides that in this case, where the relationship is not very clear to him to just choose another test value. Again, note his choice of 40 (that overshoots the target value for the sum). We cannot be sure if this is purposefully chosen to be so or is merely a consequence that he did not plan. In any case, it is notable that if he were trying to figure out the relationship between Paul and the Sum, he did not choose his next value to be 36 so that he could see how much the sum increases when the independent variable increases one.

Liam: ***Some sort of equation.*** Okay so 40 times 2 equals 80. 80 minus 5 equals 75. So that's 155. Or is that 155? 80 plus 75 equals—hold on. So it *[the result from155]* is too much.

(Thinking. 8 seconds).

Okay so the difference between these two *[between 135 and 155]* is 20. The difference between these *[35 and 40]* is 5; 20 divided by 5 is 4.

So that means every time I change this by one *[left hand touching Paul column]* **this** *[right hand descends and indicates Sum column]* ***changes four. So I should lower this [40] by two.***

Liam chooses to stick with the same version of his algorithm as before where he (1) has a guess that results in overshooting and another that undershooting the target sum, (2) where he figures out the multi-unit intervals and scales them back to units, (3) extrapolates from units to figure out that he must reduce his second guess by two in order to achieve the target sum for Anne and Bill.

When asked about his choice his comment about it being less time-consuming to just find the relation, he answers

Liam: "***Unless the equation is easier or it's something like consecutive numbers where I can easily figure out***…where it's four numbers or it's three numbers [referring to problems he's solved that have asked him to find the first of three consecutive integers given the

> sum], you'll increase or decrease by three.
>
> ***If it's harder—if it's a different equation like this [the relation given in the Paul, Bill, and Anne problem], I think it would be less time consuming to just move on and make another guess than find the difference***."

So, while this statement does affirm his focus on efficiency and saving time and this being the basis upon which he decided to try to go with the "sure-fire" three guess strategy as opposed to figuring out the relation in his head, it also shows that he does not recognize that another valid choice for a guess (and one that would be advantageous because it would involve a simpler calculation) would be a number consecutive to his guess. One interpretation is that Liam is aware of these two ways of computing the unit worth of one guess, but they are not yet coordinated for him. Recall that the impetus for the strategy construction initially began with my questioning what happens to the resulting output when the input variable is adjusted by one unit. So, the idea of "moving by one" is certainly familiar.

It appears, however, that in Liam's later usage of the strategy, he did not see (or did not consider important in this case) how to begin from this simpler way of determining the unit worth and instead relied upon calculations with two "arbitrary" values. In fact, upon inspection, the two values that he chooses are not arbitrary, but rather have the property that one guess undershoots the target and the other overshoots. This has the result of being able to find the unit rate, but by his usual longer chain of reasoning. As the activity stands, Liam knows this method of choosing two values and using their results to compute the unit worth to extrapolate from is going to work. Within the context of the activity, there is no strong impetus for him to change to a slightly more efficient version of this determination. Note also that in the excerpt from Liam directly above, he cites efficiency and which choice is more time efficient. This supports the interpretation for why there is not the conceptual coordination between the two ways to determine unit worth. However, he explicitly remarks that he finds it to be harder to find the "relation" in this problem.

After Liam finishes working on this problem, I ask him if he thinks that this strategy would always work. He says

> Liam: I think it only works when this [inaudible] when the number that you guess is involved in the equation. And if the number you guess wrong isn't involved in the equation really, then you would just pick someone else's [number].
>
> And it also might depend on how you guess. If you guess like a certain odd number that can't really be divided as easily, that might be a bit of a difficulty. Depending on what you guess, if we had a difference of like 17. You couldn't really divide it, so it'd be harder.

> I think if I did that, I would just move on to make another, more even, guess. But I should keep this in mind for the future. When I do this, I should try to guess more even, rounded numbers.

I ask *"What if the answer isn't an even, rounded number?"*

> Liam: That would be a problem. I could probably find a new method in the future.

At this point, at the end of the sessions, I mention that an assumption that we are using is that there is a common proportion between any two input-output pairs. We check this for some pairs of inputs/outputs. Liam reflects

> Liam: If it turns out that it doesn't work on certain problems, I could just start over on that problem and try to solve it in an equation or in a different way.

Snapshot of the strategy system

At this point, at the end of the sessions, Liam's functioning strategy system involves the following:

1. **Strategic path.** The strategic path that Liam uses to organize his activity in this last episode is a hybrid of MEA and Linear Interpolation/Extrapolation. Even though Liam has constructed a strategy that will work in three trial values, Liam is still interested in being able to solve the problem in fewer trials if possible. Thus, he still chooses very purposeful initial guesses with the hope that it could get close enough to solve the problem. He then repeats this, paying attention to how the functional relation interacts with increasing the input by one. Only *after* Liam attempts to solve the problem without linear interpolation/extrapolation.
2. **Conceptual categories.** The categories of "difference between two results" and "difference between two inputs" are seen to be stabilized in this episode as opposed to seeing it get constructed in focal episode five (e.g., first the Y interval was constructed from the sum of the errors between two results and the target and the later the Y interval was constructed as the *difference* between the two results).
3. **Relations between conceptual categories.** In this episode, we now see that "unit increment" is something that Liam is aware that he could determine (and he knows how to) from the beginning of the episode. Thus, it is a new "conceptual category" of information to read out or determine.

Organization of the control of variation scheme

Because there were no further conceptual changes or strategic refinements between last episode and this one, the control of variation scheme from focal episode five has remained stable. The figure below shows which co-variation schemes were used in this episode.

Change in X → Like Change in Y

Monotonicity

Increasing X → Increasing Y
Decreasing X → Decreasing Y

Proportionality

- Increasing X (by a lot) → Increasing Y (by a lot) *[Episode 3]*
- Decreasing X (by a lot) → Decreasing Y (by a lot) *[Episode 2]*

- Increasing X (by a little bit) → Increasing Y (by a little bit) *[Episodes 1, 2 4, 5]*
- Decreasing X (by a little bit) → Decreasing Y (by a little bit) *[Episode 1, 2, 4, & 5]*

- Decreasing X (by about a factor of 2) → Decreasing Y (by about a factor of 2) *[Episode 2]*

Unit increment (Co-variation)

- Increasing X by 1 → Increasing Y by d *[Episode 4B, 5, 6]*
- Every time X increases by 1 → Y increases by d *[Episode 4B, 5, 6]*

In-betweenness

Somewhere in between *[Episode 2]*
Halfway in between *[Episode 3]*
Closer to one than to the other [Episode 4B]
Iterative application of in-betweenness *[Episode 4A]*

Worth (Correspondence)

For every one here [input], it is d there [output].
Number of increments in X equals the number of increments in output required to move from the base Y value to the target value.

Figure 13. Active elements in Liam's control of variation scheme in focal episode six.

Strand 5: Micro-developmental steps in the learning process

The aim of the previous section was to use the analytic machinery that developed in reformulating both strategies and concepts as complex knowledge systems to analyze the data of Liam’s strategy construction process over the course of six focal episodes. The approach of the last chapter was to take “snapshots” of the state of Liam’s strategy system and the state of the conceptual sub-system (control of variation system) that was being organized and enriched over the course of several episodes of problem solving.

We traced changes to the strategy system (e.g., formation of new conceptual categories and relations that resulted in the formation of new strategic plans) and we also traced how Liam's growing control of (linear) variation scheme got built up over the course of the sessions. For example, we saw that a "key idea" that was instrumental in allowing the construction of Liam's linear interpolation/extrapolation strategy was that from the information generated in two trials, Liam could determine and then use the worth of a unit increment to solve problems.

In preparing for the work of this section, we also marked, episode-by-episode, changes to the system that occurred during each of the focal episodes. The aim of this section is to synthesize the work done across the focal episodes in the last section and to schematize and collect together examples of the kinds of micro-developments (in terms of both the strategic and conceptual systems) that occurred over the course of the session.

The developments listed below are drawn from the transcript in temporal order, and so they give an overview of the trajectory of development that we have already observed and discussed between Liam's initial and final strategy.

General classes of micro-developments that account for observed changes

The following are steps by which Liam's conceptual schemes were refined over the course of the sessions. These include:

1. Progressive quantification – Increasing attention to quantity in judgments.
    a. Benchmarking -- Instrumental use of ½ (or some other benchmark) as a reference point.
    b. Composition
    c. Decomposition (Scaling and Unitizing)
    d. Iteration of co-varying changes

I now briefly describe and exemplify each of the proposed micro-developmental steps of knowledge construction in this case.

Progressive quantification refers to a family of means by which Liam increasingly attended to quantity in making judgments about choosing a next guess. This transition was progressive and took place in stages, via separate mechanisms, and hence "quantification" is not a mechanism on its own. One site of progressive quantification was the nature of what was noticed about the "error from the target" for each guess. This noticing was instrumental in constructing precise intervals (as in the case of constructing "half as a reference point" or "closer to one than another" or "multi-units") that then could get scaled to unit intervals. That is, the increasing noticing of exact quantity of error eventually allowed the transformation of "more X → more Y" into "increasing X by 1 → increasing Y by m."

One example of "progressive quantification" comes in episode three when Liam (serendipitously) notices that his first guess (serendipitously) resulted in an output that was 20 too low and his second guess resulted in an output that was 20 too high. Because the error was noticed in terms of the exact amount of discrepancy, this opened up the possibility for him to make the conjecture that the variation with respect to the domain interval between 12 and 20 would work the same way. *(Reflecting on the way he had* *chosen 16 as his guess: "**I just thought it was in between these two [12 and 20] because this [56] was 20 too low and this [96] was 20 too high. And this [16] was exactly between those.")** (Episode 3).*

The next several developments are related to the idea of "progressive quantification." We will discuss (a) benchmarking, (b) composition, (c) decomposition, scaling, and unitizing, and (d) iteration of co-varying change.

Benchmarking refers to the instrumental use of ½ (or some other benchmark) as a reference point. Notice that in the discussion above, Liam noticed that the target output was halfway in between the outputs for his first two guesses. However, because this was this was not an instrumental use of ½ as a reference point this is not considered deliberate benchmarking. Episode four, however, provided an example of benchmarking in action.

> ***"So, it would go – it would be further*** *[sweeping gestures up and down input column], 'cause* ***I know that it'll be in between these two [15 and 20] already. So, it would be higher up instead of being right in the middle, it'll be higher up."*** *(Episode 4)*

In this quote, we see Liam making clear reference to "right in the middle" as a landmark or reference point for use in determining where the target input value should be. Note that we see him invoking both the idea of "in-betweenness" (he explicitly says he knows it will be in between 15 an 20] and "half as a reference point" in this utterance.

Another consequence of benchmarking in this learning trajectory is that since it involves a focus on the entire X and Y intervals defined by the two trials and also a specific landmark value on those two intervals, it is one way in which Liam's reasoning about choosing next guesses becomes increasingly attuned to quantity.

Composition is meant to refer to the process of constructing a unified object of attention (e.g., an interval) out of sub-objects (e.g., sub-intervals). The way this mechanism comes into play in this case study is in the concatenation of errors between the target and outputs from a trial that was too high and a trial that was too low. For example, if the result of trial one was $D1$ units too high with respect to the target output and the result of trial two was $D2$ units too low with respect to the target, then one could consider the entire range $D=D1+D2$ and ask how much of a change in X corresponded to a change of D in Y. That is, form the element "Increasing X by d results in increasing Y by D." This is an advance over sandwiching or bounding of the error in that instead of focusing on the bounds established by two previous guesses for the purpose of concluding that the target input must lie in between the two previous inputs, the function here is to determine how much an increase of D units in the range corresponds to in the domain. This marks a transition from global and qualitative assessments like "more $X \rightarrow$ more Y" to incremental and quantitative assessments like "changing X by $d \rightarrow$ changing Y by D."

In terms of the data in this case, notice the transition in the following episode (episode 5). In the first explanation Liam gives, he is comparing ***each*** output to the target and then ***composing*** these errors to form an interval.

> "***Compare the – like how much off this number [indicates output of 408 corresponding to 135] was, and compare it to a different guess [indicates 140 and then whole second row potentially referring the process of working through the calculation with 140]*** *and look at that [blank sum in second row], see how much off it [output corresponding to a guess of 140] was* then you could get this [indicating the target value of 414]"

Minutes later, after having worked through the calculations, when he is summarizing his process, he explains the process slightly differently:

> "***Like the difference between this result [pointing to 408 in sum column] to and this result [pointing to second row of sum column, which is currently blank]*** **and** ***see how far apart these were,*** so you could tell *how many of these [domain increments] equaled how many of these [range increments].*

In the given example, notice how in the first explanation of what he was doing, he compared the "error" of each output to the target output, whereas in the second explanation, he compares the results of his calculations *to each other*. In this way, the focus of attention becomes the difference between the two outputs as opposed to the difference between output one and the target and the difference between output two and the target separately. This was a fleeting, but consequential, transition that allowed for the formation of increments.

Decomposition (scaling and unitizing) refers to the transformation of the scheme "Changing X by d → Changing Y by D" into the scheme "Changing X by 1 → Changing Y by D/d."

> "It would – wait. ***So, that moved up 3 [touching input of 17]. Then this [corresponding output] moved up 12, so (7 second pause) you would have to move this down 4*** [to go from an input of 20 to one of 19]"

In this segment, we can see that Liam is working out how to go from the "multi-unit" increment "If I increase X by 3, Y increases by 12" to unit increment "If I decrease X by 1, Y decreases by 4." This transformation builds upon the previously constructed intervals (described above in "composition"). We do not know the exact mental process by which Liam constructed this unit increment, but we do know that the interval increment from before is implicated in the construction. Note how this process contributes to progressive quantification: "more X → more Y" has been transformed to "d more X → D more Y" which has then been transformed to "1 more X → D/d more Y."

Composition and decomposition of units are common processes that have been studied with respect to the development of students' understanding of whole number multiplication (e.g., Steffe, 1988).

Iteration of co-varying changes, in this case, is implicated in the process of instrumentally using the unit increment (or multi-unit increment) to determine how to move the domain values in order to achieve a particular target value in the range. In order to implement this strategy, one needs to choose one of the trials to iterate from and one needs to determine how many increments to iterate by.

> ***So for every one of these*** *[points to entries in the "1" column],* ***it's 3 there*** *[points to "Sum" column].* **Okay, so I'm trying to get 414. So… (pauses to think).** Ah. The difference between these two *[indicates 423, the result of the calculation for 140, and 414, the target value]* is 9, so this should be less *[points to 140 in the "1" column].*

The iteration of co-varying changes is more evident when he unpacks the statement for why it would be less and indicates in particular that it [the target input] would be three less [than the input for the chosen trial –140].

> Three times three is nine*, [moves hand over to 140 in the "1" column]* so I knew it would have to be 3 less than this *[pointing again to 140 in the "1" column].*

Here, note that he indicates that three times three is nine, so it should be three less. This is in contrast to taking the range and dividing it into three pieces. His language indicates that, at least in this initial instance, he thinks about iterating the unit three times to fill out corresponding increment of nine.

For the sake of concreteness two trajectories of "progressive quantification" are worth mentioning to help the reader with tracking the kind of incremental changes that occurred in this particular trajectory of strategy and conceptual co-development: (1) the progressive quantification of the benchmarks along the interval defined by the bound established through sandwiching and (2) the progressive quantification of co-variation schemes themselves. These two trajectories are pictured below:

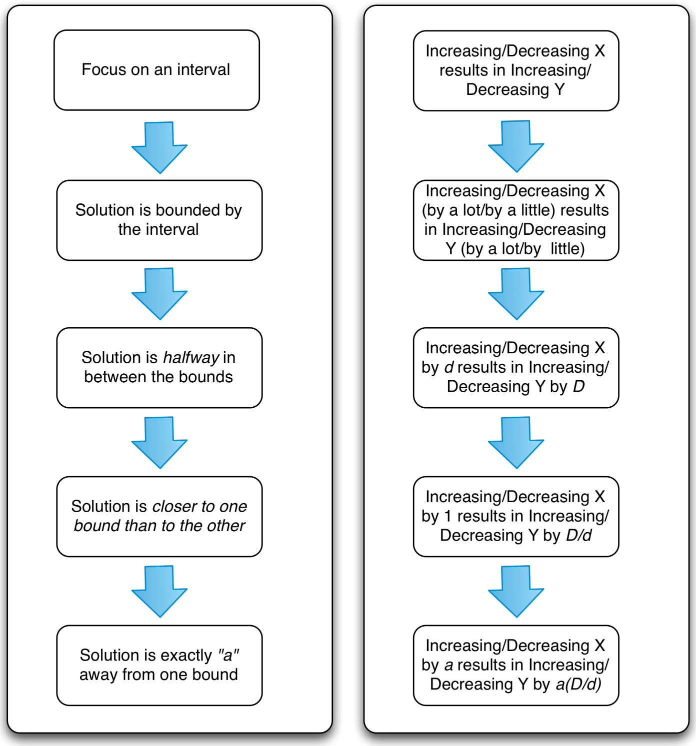


Figure 14. Two trajectories of progressive quantification. The left trajectory shows increasingly quantified abilities to express where the solution lies between bounds. The right trajectory shows the progression toward incremental co-variation.

It is important to note that these two trajectories are not unrelated. Liam is trying to solve the problems in the session, so it is natural that he is focused on considerations like bounding the search space. The reason that this is valid in this case depends on properties of the functions that underlie the problem contexts. They are all continuous and linear functions, so bounding the search is valid. Liam may have an "intuition" for why these moves are valid, but it is certain that the connection between his problem solving actions and the justification at this level is not explicit for him (and we would not really expect it to be at this point).

General classes of mechanisms by which knowledge systems develop

In this section, I reformulate some of the changes observed in the sessions in more general terms as phenomena of systems dynamics (as opposed to "idiosyncratic" changes that happened to particular schemes in interaction with a tutor, representation, activity, etc.). Framing things in more general systems terms allows us to make contact with the theoretical framework again and prepare for the final section of the analysis where we will "lift" out general features of the case study and make conjectures about the co-development of strategic and conceptual systems in general.

There are several different *types* of processes that appear to be instrumental, to a greater or lesser extent, in building up Liam's growing (linear) control of variation scheme (and in building up conceptual schemes in general). I enumerate and describe several *classes* of such processes here, in order of their prominence in the current case: (1) creating new conceptual categories and relations, (2) activation of specific knowledge that mediates processes of learning by triggering the activation of other relevant knowledge, (3) increasing/decreasing priorities of elements, and (4) increasing span and improving alignment of determinations across contexts. I will now discuss each in turn.

Creating new conceptual categories and relations

The mechanism of creating new conceptual categories and relations is especially germane to the development of strategy systems. Several instances of this mechanism of change were marked as we traced through the discussion of the focal episodes in strand four. I recall for the reader of some selected examples of these processes.

1. Formation of new conceptual categories: "Seeing" an interval that must contain the solution and coming to see quantitative or metric benchmarks along that interval (e.g., one half, closer to one than the other); worth of one unit increment becomes a conceptual category to determine, etc.
2. Refinements to the relations between conceptual categories: For example, moving from "Increasing X → Increasing Y" to "Increasing X (a little bit) → Increasing Y (a little bit)" to "Increasing X by 3 → Increasing Y by 12") is an example of a refinement or change to the relations between conceptual categories.

The creation of new conceptual categories and relations is particularly relevant for understanding the phenomenon of strategy construction and change in the activity of problem solving. On the other (conceptual) side, recall that constructing the unit increment was a conceptual advance that then *allowed* the construction of a more sophisticated strategy to determine the solution to the problem as opposed to successively make guesses for the solution.

Activation of specific knowledge

There are at least three places in the building up of this strategy where the activation of very particular knowledge supports the continued progress of strategy construction:

1. The activation of the image schema "motion along a path" is conjectured to play a role in constructing the strategy in this case.
2. The fact that the problems involved "simple units" facilitated the construction.
3. The focus on "correspondence" and then switching back to a "co-variation" perspective to leverage the correspondence/worth of one unit increment to determine the solution was possibly important.

These three examples were noted in the discussion in the previous section, but the general idea of very specific ideas "cueing" others once they have been introduced into the stream of thinking has been noted by several researchers (diSessa, 1993; diSessa, in revision; Sherin, Krakowski, & Lee, in press; Wagner, 2006).

Increasing/decreasing priorities of elements in the system

A third and rather generic mechanism for knowledge development that we discuss is the increasing/decreasing of the priorities of elements in systems. As mentioned in the previous section, as each inference is made about the effect of controlling the variation, the knowledge element that supported that inference gets added into the growing (linear) control of variation scheme. In line with the predictions of the theoretical framework, the cueing priority of inferences of the result of adjusting variation that align with Liam's expectations will increase and those inferences that do not align with his expectations will decrease in cueing priority. In this case, Liam gets only strong feedback that aligns with his expectations about how function variation works in this context. This is partially explanatory for the way the conceptual scheme gets built up over several episodes of problem solving.

Some potential examples from the sessions that could be modeled by changing priorities:

1. Going from "global" to "incremental" determinations ("global assessments" get demoted and "incremental assessments" promoted)
2. Error between each result and the target get demoted and instead the linked difference between two outputs and two inputs is promoted (as a way to determine the unit increment.)

The specific reasons for these changes in priority can be described at different levels. At one level, you could trace through interactions for evidence that particular interactions supported the move from global to incremental determinations or an increasingly quantified focus on the "error." At another level, these changes in priority are consistent within the larger frame that Liam is trying to solve problems increasingly efficiently and that each of this more local/micro moves helps him to employ a strategy that he feels achieves his objective of being efficient about his solution process.

Modeling changes in knowledge organization in terms of changes in weighting of knowledge elements is a general way to account for the dynamics of knowledge systems in activity. It is discussed in diSessa, 1993 and has been more or less explicitly named in many subsequent KiP analyses (See, for example, Kapon & diSessa, in preparation).

Increasing span and improving alignment of means of making determinations

A fourth general class of ways that knowledge systems develop is through increasing span and alignment of means of determination. This is a very general class of improvement to knowledge systems in that increasing span and alignment can happen through many different mechanisms (e.g., accumulation of situation-specific knowledge, among other ways).

In the case under study, one potential place where increasing span and improving alignment is relevant is in Liam eventually needing to develop an increased repertoire of ways to determine the unit worth of an increment. For example, in this case, one could take two values that are consecutive and compare them in order to determine the unit worth of an increment, or, if the function is linear, one can take any two values and compute the ratio of the difference in outputs to the difference in inputs. These means of determination need to be coordinated eventually. In the particular episodes under study, we see only some progress in this direction. However, we must keep in mind that the activity observed was constrained by the pragmatics of problem solving. The goal of the activity to both Liam and to the tutor was *not* maximal understanding of each of the conceptual underpinnings of the strategy he employed.

A further expected direction for improvement or change involves increasing span and alignment of means of determination *across representational contexts*. In this case study, we only looked at Liam implementing his algorithm and solving problems in a tabular environment. However, it could be interesting to observe the process of coming to recognize the features of the growing conceptual scheme in a different representational context (say, graphical)[30].

The next two sections on *functional niches* and *a priori dimensions for graded improvement* serve to elaborate some principled speculations about the nature of expected developments to knowledge systems like the ones that Liam is developing in this case.

Functional niches

The starting place for the discussion in this section is the observation that even at the end of the sessions, Liam does not use a "pure" version of the linear interpolation/extrapolation strategy. He begins problems first by trying to assess whether he can guess the answer by estimating well. However, he explicitly remarks in focal episode six that after one guess, he thinks it would be less time consuming to go ahead and make another guess as opposed to trying to figure out the relation (e.g., the path of determination and how it affects the output when the input is changed). So, while he starts out each problem with the strategy of "hoping to get lucky," he is definitely aware that if his efforts at estimating don't work out after the first couple of tries, he has a surefire solution path that will use the information that he's already collected. In this sense, Liam has a hybrid strategy system at the end of the sessions. This illustrates one way that the pragmatics and particulars of the problem-solving situation in which we observed Liam affect what we have the opportunity to observe of his strategic and conceptual knowledge.

Certainly, these problem-solving activities were never meant to be an in-depth assessment that would allow us to fully chart the rich landscape of everything that Liam (or others) might know about functional co-variation. However, studying protocols of problem solving *does* allow some window into the organization of individuals' knowledge systems. For example, as opposed to a map of a learner's conceptual ecology "in the abstract," what we do learn about instead is the nature and form of the knowledge of functional co-variation that is relevant and drawn upon by students in order to solve *these* problems. Because this knowledge is being activated for the purpose of solving problems, recognizing this also gives us insight into conceptual refinements that may have been within Liam's reach, but were not consequential or important for improving his performance on solving the problems (or were not relevant given the way the activity was framed in the sessions).

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[30] Note that the conceptual content of Liam's strategy in this context is equivalent to that statement that two points determine a line and that one can use the slope of the line to get from a point on the line to any other point.

Throughout the analysis, we referred to Liam's growing "control of variation scheme" that included increasingly organized knowledge structures (co-variation schemes) that were used in this context to guide Liam's choices for next guess. However, though we see Liam's knowledge about controlling variation "grow up" in the context of this activity, we should not expect that this is the *only* place where such knowledge is ever engaged. The particular context of solving problems using guessing and checking is a *functional niche* for Liam's conceptual knowledge related to controlling the variation of linear functions.

An expectation about knowledge in transition is that knowledge that served a particular purpose in past activity will still continue to be drawn upon as it makes sense for it to do so. We see this phenomenon, known as *functional residue*, at work as Liam transitions from guessing and checking approaches to approaches based on linear interpolation/extrapolation. As we remarked above, he used a hybrid approach because it was functional for him to do so in the context of this activity, as opposed to cleanly switching over to the "more sophisticated" linear interpolation/extrapolation approach. (See diSessa, 2004 for a discussion of functional niches and functional residues).

Dimensions of graded improvement

In this section, we move away from considering the conceptual knowledge as being activated for the purpose of solving particular classes of problems (e.g., away from the particular functional niches we have the opportunity to observe it functioning in) and we speculate here about a number of dimensions for graded improvement to a conceptual system (like the control of variation system). The discussion here is general (e.g., not particularly bound to the control of variation or other relevant systems to this analysis) and more speculative in nature. The point of this discussion is to give the reader some indications for avenues for future work and what we would expect in terms of future development to both Liam's strategic and conceptual systems from a complex systems perspective.

The graded improvement dimensions discussed here are *a priori* or expected dimensions of improvement of a knowledge system like the control of variation system that Liam is developing.[31] As we discussed when we introduced the analytic framework, in solving problems, the general case we would expect is that individuals do not just draw on knowledge from one subsystem (e.g., control of variation system), but rather it is possible that knowledge from many and diverse subsystems are engaged. That is, the conceptual knowledge that one relies upon in solving problems is a mixed inferential system.

The three dimensions we discuss here include: (1) purification, (2) exploration of inferential possibilities, and (3) justification. These three dimensions are discussed in relation to the knowledge systems of interest in this dissertation. However, as we said above, these are fairly general expectations about the process by which knowledge that

[31] This discussion of "graded improvement dimensions" and the particular components of purification, exploring inferential possibilities, and justification are based on personal communication with A. diSessa.

"grows up" in one context becomes increasingly recognizable as its own system/sub-system, as opposed to being bound to the context in which it originated.

1. **Purification.** Initially, strategy systems involve many, diverse types of knowledge all invoked together for the local purpose of solving a particular kind of problem.

We saw this in the analysis of Liam's protocols: there were conceptual categories that Liam looked to and set, relations of various sorts between these categories, and then specific pieces of conceptual knowledge, ranging from individual co-variation schemes to image schemata to declarative facts. Eventually, we'd imagine that the strategy system would become "purified" in the sense that relevant subsystems of knowledge would become more systematically organized and tightly connected. We modeled a bit of this "purification" process in following the increasing organization of the co-variation schemes within the control of variation system. Thus, though we see co-variation schemes and the control of variation scheme function only in this activity, we have reason to believe that this sub-system in particular will continue to develop on its own or in conjunction with other systems. This expectation is based on the fact that we know that people reason about variation in contexts that are not tied to the particular goal structure of the problems in these sessions. Purification is a dimension of improvement that is aimed at broadening the range of contexts in which knowledge can be applied by "freeing" it of what were initially contextual contingencies.

2. **Exploring inferential possibilities.** This dimension of graded improvement is aimed at filling out the range of means of making determinations and accomplishing goals.

While one initially may only have one means of accomplishing something or making a determination, it is clearly better to have multiple means. This is helpful for comparison and error detection/correction or also some possibilities may be better adapted to particular contexts. One example where this came up in the sessions was around the determination of the worth of a unit increment. There are many ways to make this determination (e.g., take any two values of the function that are computed and then find the ratio of the difference in output to the difference in input; take any two consecutive values and see how much the output increases from one to the next; consider the relation between input and output and think about what would change in each term if you were to increase each term by one, look at a graph of the function and read off the slope of the line, etc. All of these methods for determining need to be coordinated and they need to lead to the same result.

3. **Justification.** The third dimension of graded improvement we discuss involves knowing why what one is doing (e.g., in the case of a strategy system) works.

This is useful when strategies need to be adapted or when one needs to recognize when a strategy even applies. Note that this is important once one ventures outside of the mode of operation where there is an expected strategy to be applied to solve a particular regularly presented type of problem (e.g., the situation in this study). While this mode is fine for learning and refining strategic operation, it is only one mode of many that individuals will have to be fluent in. The justification structure will ultimately play a key role in developing that fluency.

Strand 6: Modeling the co-development of strategic and conceptual knowledge

The aim of the analysis in this dissertation has been to develop a model of how strategic and conceptual knowledge co-develop. Over the course of several strands of analysis, we have been building up to the point where we can state, in general terms, a model of this process.

Our general research strategy has been to use data of such a process of strategic and conceptual co-development as a basis for building theory about how such processes work in general. Naturally, the data that has been analyzed for this purpose has numerous contingencies given that it is case study data from a single case. To account for this, throughout the analysis, I have attempted to articulate and name both what is quite specific about the case analyzed and also what we can expect to be more general.

In studying the process by which strategic and conceptual knowledge co-develop, we used an analytic strategy that continually coordinated between bottom-up consideration of the particulars of the data under analysis and top-down considerations inherited from the general theoretical perspective guiding the analysis.

The analysis proceeded in several phases. The first phase of analysis involved figuring out how to describe and frame the change that was observed in the sessions. Even once it was established that Liam constructed a novel strategy over the course of the episodes and a preliminary description of the bookends was given, it still wasn't clear how to describe what allowed him to construct this strategy and what he "learned" through his interactions with the tutor/researcher and the task sequence. Several candidate framings for how to describe what Liam learned and how were engaged. In this chapter, we presented a discussion of two of these alternate framings and their pros and cons. This phase of activity led to the determination that modeling this learning process could be productively approached through developing a complex systems perspective.

The second phase of analysis involved the beginnings of negotiating with a particular epistemological perspective, Knowledge in Pieces, that models the conceptual understanding of individuals and how it grows in terms of complex systems, together with several other commitments (e.g., developmental perspective on the nature and form of the elements of the system, etc.) that have been found to be productive in other studies of mathematics and science learning. The process of reformulating and developing the analysis in this way was intricate work that went through several phases and continued throughout the analytic process. Reference models based on the existing literature developed from this perspective were consulted throughout the analysis, which ultimately led to the recognition of both strategies and conceptual schemes implicated in the implementation of the strategies as complex knowledge systems. The process of coming to an appropriate definition for the strategy and conceptual systems from this perspective happened in dialogue with the theory and was also constrained by the kind of phenomenology observed in the data analyzed in this case.

The third phase of the analysis approached the task of further elaborating and specifying the nature and form of the elements of the strategy and conceptual systems that were relevant to this particular analysis. This was a bottom-up program of schematizing knowledge-in-use in the sessions. The entire corpus of data (all six hours) was analyzed to exhaustively list the knowledge structures that were relevant for the strategy and conceptual co-development process.

The fourth phase of analysis involved coordinating the grounded analytic framework developed in phase three and the general formulation of the enterprise in terms of complex systems developed in phase two. Six focal episodes were purposefully selected to trace out the trajectory of strategic and conceptual co-development observed in the sessions. Through the process of putting the analytic framework and theory in contact with the data, "snapshots" of the process of change at both the strategic and conceptual level were taken and the changes to both systems were noted, episode-by-episode.

This prepared for phase five of the analysis in which the episode-by-episode changes that were observed were schematized both into specific micro-developments informed by the particular case of knowledge construction under study and general mechanisms of change that were informed by or suggested by the theoretical framing. The entire trajectory could then be understood in terms of models of the knowledge structures and mechanisms of change developed through the analysis.

The final phase of analysis involved synthesizing the work in the previous five strands in order to propose a theoretical model for how strategies and concepts co-develop.

It is now time to summarize both what we have learned and what we would conjecture, based on the theory we have been building, about the process of how strategies and conceptual knowledge co-develop.

Strategy change and conceptual change: A process of mutual bootstrapping

At the very top level, the model of strategy emergence developed in this dissertation is that new strategies emerge when then underlying conceptual knowledge and schemes that are used to implement them change in structure and organization. The strategies one uses to solve problems require the activation and use of a particular set of conceptual resources. Constructing a new strategy and exercising and using that strategy can lead to conceptual refinements. For example, Liam did not start the sessions with an understanding of how to determine the worth of a unit increment or a coordinated understanding of the many ways one could go about this. These conceptual developments were made possible (to some degree) by the activity he was engaged in. On the other hand, conceptual refinements (like the construction and instrumental use of the worth of a unit increment) allowed a novel strategy to be constructed. Thus, the process of strategic and conceptual change was bi-directional and mutually constitutive.

An important aspect of the model developed in this research is that *both* strategies and concepts are represented as complex knowledge systems. This is productive for accounting for the great diversity of aspects of what it means to "know" a strategy. In the course of the activity, recall that the knowledge system that we traced involved strategic path, conceptual categories and relations between conceptual categories. The strategic path included things like means end analysis and linear interpolation/extrapolation. The conceptual categories included adjustable input, resulting output, target, error and eventually worth of unit increment. The relations between categories included knowledge what we were calling "co-variation schemes" – knowledge structures that encode expectations about the effect on the output of "controlling the values of the input variable."

While the conceptual knowledge of co-variation of functions supported this problem solving process, at least in the sessions we observed, we did not observe it as a knowledge system that was completely autonomous and being "called" by the strategy system as a module. Rather the conceptual scheme was getting built up and enhanced through the process of solving problems. We thus conjecture that a system of knowledge around the co-variation of functions and control of variation schemes exist as independent entities to be used in other contexts, but we did not observe them to be functioning in this way in this particular analysis.

Drivers of the process of strategic and conceptual co-development

Consistent with the literature on strategy change (Siegler & Araya, 2005), *efficiency* and *accuracy* of solution approaches are considerations that drive development of novel strategies. *Progressive quantification* played an important role in this case in terms of driving conceptual and strategic development because quantifying various aspects of the problem solving process were instrumental in improving efficiency and accuracy.

However, we also discussed some general mechanisms related to the processes by which complex systems grow and change as people learn. Given that one of the strengths of the current model is that both strategies and "concepts" are presented as complex systems, it makes sense to posit these as mechanisms in the general model that we are building. These mechanisms include:

1. Creating new conceptual categories and creating and/or refining new relations between conceptual categories that result in the creation of new strategic paths
2. Changing priorities of elements – Both particular schemata and entire strategies can be promoted or demoted based on various judgments of the individual. This is related to specific interactions with representations, judgments about efficiency/accuracy of solution paths, and also related to other mechanisms below (e.g., activation or trigger ideas, etc.)
3. Increasing span and improving alignment across ways of determining (including across representational forms).
4. Activation of specific knowledge supported reasoning in context

Thus, we have now described both the statics and the dynamics of the strategy/conceptual systems, both in general terms (this section) and in specific terms (strands four and five where we elaborated the systems, their components, and traced specific changes in detail).

In the next chapter, the results of the dissertation along substantive, theoretical, and methodological dimensions are discussed, as are limitations of the current study and directions for future study.

Chapter 7: Discussion, Limitations and Future Work

The aim of this dissertation analysis has been to develop a model of the process of how strategic and conceptual knowledge co-develop. Our general research strategy has been to use data of such a process of strategic and conceptual co-development as a basis for building theory about how such processes may work in general. Naturally, the data that has been analyzed for this purpose has numerous contingencies given that it is case study data from a single case. To account for this, throughout the analysis, we have tracked the particular details of how strategies and conceptual knowledge were refined in the case of Liam, while at the same time marking what we would expect is general about the process (on the basis of the epistemological perspective that guided the analysis). One of the advantages of developing the case study in close "dialogue" with the theoretical perspective, while also staying close to the details of the particular case under study, is that we not only understand better how the process of conceptual and strategic co-development occurred in the particular case of Liam, but also, and more importantly, we have a developed an entirely new approach to the study of such processes.

Here, I discuss both the conclusions and the contingencies of the current case study. I situate this work with respect to the future work. I begin with a discussion of several questions that one might ask about the current study: (1) How representative is Liam as a subject? (2) What are some features of the empirical set-up that may limit the generality of the findings of this study? (3) What are features of the scope and sequence of the problem sequence that may limit the conclusions we can draw? (4) How concerned should one be about the breadth of data analyzed at this stage in the process of developing theory about how strategies and conceptual knowledge co-develop? We discuss each of these issues in turn.

How representative is Liam?

On one hand, Liam was exceptional among students in the sessions in that he was the only student to more or less independently come up with the algorithm we discuss here. However, Liam was an average math student in school (e.g., of three tracks of algebra, Liam was placed in the middle track in the year following this study). One way in which Liam *was* different from many of the students I studied was that he was more articulate than average about what he was doing and why, and he seemed to be constantly reflective about this. This quality made this case desirable as a context in which to study processes of strategy development because there was more access to what Liam was doing and what understandings might be supporting that.

That said, there were two qualities of Liam that may have had more of an effect on the way the particular trajectory of knowledge construction studied here unfolded. The first is that, Liam's capacity for estimation and computational fluency in general, were above average in students that I observed in the pilot classroom work. While the heavy amount of computation may have been onerous to many other students, such concerns were in the background for Liam. The second is that, throughout the sessions, Liam displayed an aesthetic for the efficiency of his solution approaches. Given such an aesthetic, it is natural that over the course of working on several similar problems using a similar approach, Liam would work to improve upon his method. Certainly, it will be of interest empirically and theoretically to conduct studies of extended episodes of problem solving across a broader range of subjects.

The co-development of strategic and conceptual knowledge in a particular context

We have studied a process of the co-development of strategic and conceptual knowledge in depth, but under very specific conditions. We would expect the process by which strategies and concepts co-develop to look rather different in cases such as this, where subjects are engaging with a problem sequence involving repeated trials with similar problems solvable by expected and similar means, as opposed to situations in which the solver needs to construct a means for solving a problem on the spot and may go through several, iterative, attempts at doing so. Though these situations are rather different, we would still expect the theoretical machinery that comes with viewing strategies and concepts as complex knowledge systems to be productive for understanding this broader range of situations.

A second aspect of the current data that should be noted is that the process of knowledge construction we studied took place in a clinical instructional context. Interactions between the tutor/researcher and the student undoubtedly shaped the particular trajectory of knowledge construction observed. However, the focus of this study was to examine the organization of the student's knowledge system and how that organization changed as the student learned. The particular "moves" of the tutor/researcher were back-grounded for the purpose of understanding the nature and form of the student's knowledge system and how it changed. This was a deliberate analytic choice.

Firstly, the particular development analyzed in this study was more of an emergent interactional phenomenon between the tutor and the student – not a "designed" part of the study. Thus, while we can be attentive to the effect of certain interactions, treating this as "purposeful instruction" is a bit misleading. Future work systematically incorporating what appeared effective in this case would be of interest in strengthening any claims about the effects of particular interactions on knowledge construction processes.

Secondly, our expectation is that generalizable knowledge about how to support students as they construct their understandings is of more central interest than the specific findings from one study about how one tutor supported one student in making particular conceptual connections. To this end, having a better-specified theory of students' knowledge systems and how they grow and change (across a wide range of interactional contexts and across a wide spectrum of content) is foundational in the larger program of work around developing theories of effective instruction that are sensitive to individual differences.

Problem sequence and scope

The problem sequence in this study was selected for the purpose of tracing out the transition from "informal" problem solving strategies (e.g., guessing and checking) to algebraic problem solving strategies (e.g., modeling problems with equations) with a particular interest in the role of a curricular tool, a Guess and Check chart in mediating this transition. This particular sequence of problems was not designed as a window into students' conceptual understanding of functions and co-variation. For this purpose, it is severely limited in the sense that, except for the problems that deal with systems of linear equations, all of the functions underlying problem contexts have to do with direct, linear variation.

How individuals develop knowledge of functional relations and co-variation became central in this study because it was a pool of knowledge that individuals drew upon in solving the problems in this study using guessing and checking approaches. Developing this analysis required taking a close look at the knowledge in this area that individuals were drawing upon and make conjectures about how it was likely organized and what likely processes of change were. A future trajectory for this work would be to make a more systematic study of how individuals understand co-variation and how this develops over time. This would be an interesting pursuit in and of itself and would also provide triangulating data for the account provided in this dissertation study.

Amount of data analyzed in the process of developing the model

One of the limitations of this study is that in addition to being a single case study design, the number of episodes and observations analyzed is admittedly few, especially in contrast to study designs that involve students solving perhaps hundreds of problems over an extended period of time. The current study focuses on a single student and a particular arc of learning that took place over six hours. Even within this six-hour span, only Liam's work on six selected focal episodes was presented in detail in the analysis.

However, the main thrust of the work in this study concerned the development of a theoretical model in close dialogue with both data and theory and thus the focus on a single case was appropriate. Certainly now that we have developed theoretical and analytic machinery for approaching the question of how strategies and conceptual knowledge co-develop, it is of interest to consider more cases across a wider range of content, subjects, and empirical set-ups.

Theoretical, methodological, and substantive contributions

A primary contribution of the analysis in this dissertation was the development of analytical tools for studying processes of mathematics thinking and learning. In the context of this analysis, this included the explicit modeling of strategies and concepts as complex knowledge systems allowing Knowledge in Pieces (KiP) as a theoretical perspective to guide the analytic process involved in trying to understand how strategies and concepts co-evolve. This approach contributes novel new tools to our repertoire for studying the phenomenon of strategy use and construction. This approach also elaborates the KiP theoretical perspective through the recognition of strategies as knowledge systems with certain characteristic functions and properties.

Developing analyses in which we make very explicit the epistemological principles guiding the analysis and also show how those principles make contact with the data allow us a basis upon which to make judgments about what good representations of thinking and learning are and how we can improve them in future work. Specific ideas that were developed included how to negotiate between a modeling language and the specifics of the knowledge that we are trying to model.

Because this is a relatively new type of analysis (at least in educational circles) the work involved in developing the methodology and the modeling language was a highly nontrivial part of the iterative analysis of the video and transcripts. As a relatively new methodology, an ongoing focus of attention within the community of researchers interested in developing such perspectives on thinking and learning processes, is to better explain and schematize the methods by which analyses are developed from data. Although quite a bit of detail was suppressed and streamlined in the presentation of the analysis, the choice to show how the model was created through layers of analysis (e.g., reformulating the concepts and strategies in question from a systems perspective, generating an analytic framework including a fine-grained description of knowledge-in-use, showing the mapping between the analytic framework and the data across the span of the learning process in question, and developing both specific and general sketches of learning mechanisms) was purposeful in that the aim was to expose some of the analytic processes that go into generating a model such as the one developed in this analysis. However, as we discussed in the previous section, there are numerous contingencies and specificities in any one case of thinking and learning and so we can expect that several such "methodological case studies" will be needed in order to better understand the spectrum of analytic strategies associated with "knowledge analysis."

One of the substantive contributions of this dissertation (e.g., on the particular content and form of knowledge implicated in the trajectory of change) is the idea that there is much more to "understanding a strategy" than just knowing the procedures that one goes through in order to solve particular problems. The interplay between strategic knowledge and the conceptual knowledge that it takes to implement strategies is important to understand. On one hand, we can use strategic performance as a (limited) window into the organization of individuals' conceptual knowledge. On the other hand, we can use this window into individuals' conceptual knowledge in order to understand what and how people learn through problem solving.

One of the major efforts of the dissertation has been to contribute to explore the productivity of modeling mathematical knowledge and knowledge construction using a complex systems perspective, guided in particular by the commitments of Knowledge in Pieces. Through contextual tuning and elaboration to engage particular questions about processes of thinking and learning, epistemological micro-modeling (diSessa, 1994) is a general and powerful approach. It is hoped that the general methods elaborated in the process of developing this particular analysis merely mark the beginning of a long line of work on developing our understanding of processes of mathematical thinking and learning.

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Appendix 1: Problems Solved in the Sessions

A complete list of the problems solved in the sessions is given together with information about the session in which the problems were worked on and the approach specified (e.g., none, guessing and checking using a chart, using a chart to transition to equation modeling, equation modeling). Problems that are in bold were selected as focal episodes in the current analysis. Problems that are starred are from the CPM Algebra Year One curriculum.

Table 9. Problems Solved in the Sessions

Session One		
Session	Specified Approach	Problem
1	**Open**	**Ben and Jerry each have a mystery number. Ben's mystery number is six more than three times Jerry's. If the sum of Ben and Jerry's numbers is 70, find the two numbers.** How would you do find the two numbers if I told you that Jerry's number was 16? What would be different about the problem?
1	N/A	Express the following relationship in a table: Anna is three more than five times as old as Juan.
1	Chart	Andre and Rosa's ages: Andre is 20 less than 3 times Rosa's age. If Andre is 25, how old is Rosa?

Session Two		
2	Chart	PAIR: Maria is one less than five times Andrew's age. If Andrew is fourteen, how old is Maria? *and* Maria is one less than five times Andrew's age. If Maria is fourteen, how old is Andrew?
2	Chart	PAIR: Amanda is four times as old as Derek. If Derek is sixteen years old, how old is Amanda? *and* Amanda is four times as old as Derek. If Amanda is sixteen years old, how old is Derek?
2	**GC-Chart**	**The base of a rectangle is three centimeters more than twice the height. The perimeter is 60 centimeters. Find the base and height of the rectangle.**
2	GC-Chart	Find three consecutive integers that sum to 126.
2	GC-Chart	Find two consecutive odd numbers whose sum is 376.*
2	GC-Chart (System)	Jennifer has only quarters, dimes, and pennies in her pocket. There are eight coins in all and the total value is 83 cents. How many of each type of coin does she have?*
	GC-Chart (System)	The drama department at Galileo High is having a production. Tickets cost $3 for members of the student body and $5 for anyone else. A total of 515 tickets were sold bringing in $1785. How many student body members attended?*
2	**GC-Chart**	**The perimeter of a triangle is 76 centimeters. The second side is twice as long as the first side. The third side is four centimeters shorter than the second side. How long is each side?**

Session Three		
3	**GC-Chart**	**Jabari is thinking of three numbers. The greatest is twice the least. The middle is three more than the least. The numbers total 75. Find the three numbers.**
3	GC-Chart	The length of a rectangle is three less than four times the width. If the length plus the width is 92 feet, find the length and width of the rectangle.
3	GC-Chart	Melanie sold 3 times as many candy bars as Jeremy. Jeremy sold four more candy bars than Elaine. Together, Melanie and Elaine sold 132 candy bars. How many candy bars did Melanie, Elaine, and Jeremy [each] sell?
3	GC-Chart	In the football game, Rocky gained 3 times as many yards as Bullwinkle. Rocky also gained 10 yards more than Boris. The 3 players gained a total of 410 yards. How many yards did Boris gain?*

Session Four		
4	**GC-Chart**	**Find three consecutive integers that sum to 414.**
4	Transition to equation	On a 520 miles trip, Gerald and Robert shared the driving. Robert drove 80 miles more than Gerald drove. How far did each person drive?
4	Transition to equation	The length of a rug is one foot less than twice the width. If it takes 34 feet of fringe to wrap around its perimeter, find the length and the width of the rug.*
4	Transition to equation	Rachel is three years older than Lauren. Lauren is twice as old as Mariah. The sum of Mariah's age and Rachel's age is 85. How old is Lauren?

Session Five		
5	Equation	Susan is buying three different colors of tiles for her kitchen floor. She is buying 25 more red tiles than beige tiles and three times as many navy blue tiles as beige tiles. If Susan was buying 435 tiles in total, how many of each color tile does she buy?
5	Equation	Margaret is twice as old as Jennie and Sarah is thirty years older than Margaret. Sarah is older than Jennie and the difference between their ages is 63. How old are Margaret, Sarah, and Jennie?
5	Equation	A rectangle has length that is 4 inches more than 5 times its length. The perimeter is 68 inches. Find the length and the width of the rectangle.
5	Equation	The State Market has 27 more apples than oranges. There are 301 apples and oranges all together. How many apples are at the market?*
5	Equation	Raisa cut a string of 112 cm. The second piece is 3 times as long as the first piece. How long is each piece?*
5	Equation	A rectangle has width that is 25 inches shorter than twice its length. If three times the length is twice the width is 83 inches, find length and width of the rectangle.

Session Six		
6	Open	The sum of three consecutive numbers is 222. What are the three numbers?
6	Open	The length of a rectangle is three more than four times its width. If the perimeter of the rectangle is 148 feet, find the dimensions of the rectangle.
6	**Open**	**Anne is twice as old as Paul. Bill is five less than Anne's age. Together, Anne's and Bill's ages sum to 147. How old are Paul, Bill, and Anne?**
6	Open	Jane has a rectangle that has a length and width that sum to 10 feet. Four times the area of her rectangle is 64 feet squared. Find the length and the width.

Appendix 2. Transcription Conventions

Punctuation	**Meaning**
…	Voice trailing off
//	Interruption
[italic brackets]	Explanatory text. Used to disambiguate unclear references or to add detail about gestures or other non-verbal actions.
(pause)	Pauses of significant duration are marked by text. (e.g., “8 second pause”)
Bold italics	Emphasis added by the analyst