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ABSTRACT

We examine the optimal trading strategy for an investment fund which in the absence of transactions costs would like to maintain assets in exogenously fixed proportions, e.g. 60/30/10 in stocks, bonds and cash. Transactions costs are assumed to be proportional, but may differ with buying and selling, and may include a (positive) capital gains tax component.

We show that the optimal policy involves a no-trade region about the target stock proportions. As long as the actual proportions remain inside this region, no trading should occur. When proportions are outside the region, trading should be undertaken to move the ratio to the region's boundary. We compute the optimal multi-asset no-trade region and resulting annual turnover and tracking error of the optimal strategy. Almost surely, the strategy will require trading just one risky asset at any moment, although which asset is traded varies stochastically through time. Compared to the current practice of periodic rebalancing of all assets to their target proportions, the optimal strategy will reduce turnover by almost 50%.

The optimal response to a capital gains tax is to allow proportions to substantially exceed their target levels before selling. When an asset' s proportion exceeds a critical level, selling should occur to bring it back to that critical level. Capital gains taxes lead to lower optimal initial investment levels. Similarly, it is optimal to invest less initially in asset classes that have high transactions costs, such as emerging markets.

Keywords: trading strategy, transactions costs, capital gains tax

OPTIMAL PORTFOLIO MANAGEMENT WITH

TRANSACTIONS COSTS AND CAPITAL GAINS TAXES

I. Introduction

Many investors, both institutional and private, state their investment strategy in terms of *desired asset proportions*, such as a 60/40 ratio of stocks to bonds, or 40/40/20 proportions of domestic assets, foreign assets, and cash. As asset values move randomly, asset ratios diverge from their targets. But when asset returns follow a diffusion process, it is well known (e.g. Leland [1985]) that an infinite amount of trading is required to keep assets *continuously* at their target proportions. This creates a problem: frequent readjustments to keep assets close to their target levels will incur high trading costs. But infrequent revision will create "tracking error" relative to the ideal portfolio.

Typical current practice is to "rebalance" to desired target proportions on a periodic basis, e.g. quarterly or annually. Less frequent rebalancing lowers the expected amount of trading, but creates higher average deviations from the desired asset ratios. But the conventional strategy of periodic rebalancing to the target ratios will not be optimal when transactions costs are proportional to the dollar amounts traded. Work by Magill and Constantinides [1976], Taksar, Klass, and Assaf [1988], and Davis and Norman [1990] on the single risky asset case shows that the optimal strategy is characterized by a "no trade" interval about the target risky asset proportion. When the proportion varies randomly within this interval, no trading is needed. When the risky asset ratio moves outside the no-trade interval, it should be

adjusted back to the nearest edge of the interval--*not* to the target proportion.[1] Dixit [1991], Dumas [1991], and Shreve and Sohner [1994] provide further mathematical results for this and related problems with a single risky asset, based on work by Harrison and Taksar [1983], Harrison [1985] and others on regulated Brownian motion.[2] Akian, Menaldi, and Sulem [1996] consider a multi-dimensional version of Davis and Norman [1990]. Dixit [1997] and Eberly and Van Mieghem [1997] examine the related problem of a profit-maximizing firm facing partially-irreversible investment decisions in multiple factors of production.

Our work differs from previous work in several ways. First, our ***focus is on managing portfolios that have a given set of target or "ideal" asset ratios.*** Many large investors, including pension funds, formulate their investment strategies in terms of desired long-run asset allocation ratios. These ratios are based on perceived risks and returns, for example using a mean-variance optimization approach, and typically do not adjust for transactions costs. [3] Rather than assuming a specific utility function over wealth (which investment managers can rarely specify), we postulate a "loss function" that is natural to many portfolio managers: the sum of trading costs and the costs associated with *tracking error--* divergences from the desired target ratios. This permits a possible distinction between risk aversion for asset selection and risk aversion towards tracking error, a distinction that many practitioners consider

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[1] The intuition behind this result is as follows. The loss L from diverging from the optimal ratio is (approximately) U-shaped. Because it is flat at the bottom, very little loss reduction results from moving the last small amount to the optimum ratio: the gain is of second order, and insufficient to justify the (first order) trading costs.

[2] Related problems include the optimal cash management problem examined by Connor and Leland [1995] and the option replication problem in the presence of transactions costs (see, e.g., Leland [1985] and Hodges and Neuberger [1989]).

[3] An ad hoc approach has been to adjust the mean return of an asset or asset class downwards to reflect trading costs. This approach is erroneous, as discussed in Section X.

important.[4]

Second, we develop a **technique for estimating the expected turnover and the expected tracking error** of arbitrary policies. This allows an investor to assess the tradeoff between the required volume of trading and the tracking accuracy of alternative efficient policies, and to compute the loss associated with following sub-optimal strategies. There are important cost savings to be realized from following optimal rebalancing strategies rather than traditional periodic rebalancing strategies. For the same average tracking error, the optimal strategy will reduce trading costs by almost 50%.

Third, we examine **the effects of capital gains taxes on optimal trading strategies**. Capital gains taxes can be deferred by not selling—but not selling an asset may lead the portfolio to become dangerously over-invested in that asset. We determine an investor's optimal strategy when facing a capital gains tax.

Fourth, we can use our results to **examine important tax and regulatory policy questions**. For example, how would a "transactions tax" on portfolio trading, or a capital gains tax on sales, affect the long-term average volume of trading? We show below that, under reasonable assumptions, a 2% transactions tax could cut trading by over 40%.

Finally, we consider how **trading costs affects the optimal initial investments in different asset classes**. Popular wisdom holds that trading costs should be modeled *as reducing the expected return* of an asset,

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[4] See, for example, Grinold and Kahn (1995).

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with a consequent scaling back of the amount invested in that asset.   But we show that this will not be a correct approach. Initial investments that *exceed* ideal levels may be optimal for an asset that incurs trading costs, if those costs are smaller than those of other correlated assets. Furthermore, if an investor should initially hold a *larger* position than the target amount, it may well be optimal to retain that position. Our results permit a rigorous examination of the impact of trading costs on the optimal holding of assets.

II. Asset Price and Proportion Dynamics

Consider dollar holdings S_i of asset i which evolve as a regulated logarithmic Brownian motion, $\tau \geq t$:

(1) $$dS_i(\tau) \;=\; \mu_i\,S_i\,d\tau + \sigma_i\,S_i\,dZ_i(\tau) + dL_i(\tau) - dM_i(\tau) \qquad\qquad i = 0,\,...,\,N,$$

with initial values

$$S_i(t^-) \;=\; S_{i\,t} \qquad\qquad\qquad i = 0,\,...,\,N.$$

where the dZ_i are the increments to a joint Wiener process with correlations ρ_{ij}. $S_i(t^-)$ denotes the left-hand limit of the process S_i at time t. By assumption, asset i = 0 is riskfree, with $\sigma_0 = 0$ and $\mu_0 = r$. $L_i(\tau)$ and $M_i(\tau)$ are right-continuous and nondecreasing cumulative *dollar purchases and sales* of asset i on [t, τ], respectively, with $L_i(t^-) = M_i(t^-) = 0$. Note that *initial trades* (purchases or sales) of asset i are given by $L_i(t)$ or $M_i(t)$.

For simplicity, it is assumed that any transactions costs incurred will be paid by additional contributions

to the fund. With this exception, there are no net contributions or withdrawals from the investor's holdings, implying a self-financing constraint

$$\Sigma_{i\,0}\,(dL_i(\tau) - dM_i(\tau)) \;=\; 0, \qquad \text{for all } \tau.$$

Let μ denote the column vector (μ_1,...,μ_N) of instantaneous expected rates of return of the risky assets, and V denote the instantaneous variance covariance matrix of risky rates of return, with elements $\rho_{ij}\,\sigma_i\,\sigma_j$, i, j = 1, …, N. Let $\Sigma_{i\,0}$ denote the summation operator over all assets, i = 0,..., N, and Σ_i denote the summation operator over risky assets i = 1,...,N.

Define the following:

$W(\tau) = \Sigma_{i\,0}\,S_i(\tau)$:   investor wealth at time τ, assumed strictly positive for all $\tau \geq t$.[5]

$w_i(\tau) = S_i(\tau)/W(\tau)$:   the proportion of wealth held in risky asset i

at time τ,   i = 0, …,N.

Note $\Sigma_{i\,0}\,w_i(\tau) = 1$.

$w(\tau)$:   the vector of the risky assets proportions $w_i(\tau)$,   i = 1, …, N.

x: the vector of initial asset proportions $w_i(t-) = S_{it}/W(t)$,   i = 1, …, N.

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[5] Sufficient conditions for nonnegative wealth are that no short positions or borrowing be allowed. While satisfied by the examples we construct below, these are not necessary conditions.

w_i^*:   the (given) target proportion of wealth in asset i,   i = 0, …, N.

$$\text{Note } \Sigma_{i\,0}\, w_i^* = 1.$$

w^*:   the vector of target risky asset proportions w_i^*,   i = 1, ..., N.

Then

$$dW(\tau)/W(\tau) \quad = \Sigma_{i\,0}\, dS_i(\tau)/W(\tau)$$

$$= \Sigma_{i\,0}\, \mu_i\, (S_i(\tau)/W(\tau))d\tau \ + \ \Sigma_{i\,0}\, \sigma_i\, (S_i(\tau)/W(\tau))dZ_i + \ \Sigma_{i\,0}\, (dL_i(\tau) - dM_i(\tau))$$

$$= \Sigma_{i\,0}\, \mu_i\, w_i(\tau)d\tau \ + \ \Sigma_{i\,0}\, \sigma_i\, w_i(\tau)dZ_i\,,$$

$$= (r + \Sigma_i\, (\mu_i - r)w_i(\tau)d\tau + \ \Sigma_i\, \sigma_i\, w_i(\tau)dZ_i$$

where the first line follows from the definition of $W(\tau)$, the second line uses equation (1), the third utilizes the definition of $w_i(\tau)$ and the self-financing constraint, and the last follows from $w_0(\tau) = 1 - \Sigma_i\, w_i(\tau)$.

The weights $w_i(\tau)$ will be changing through time, but typically will be close to w_i^* when assets are optimally traded. Thus to a close order of approximation,

(2) $dW(\tau)/W(\tau)$ $= \ \Sigma_{i0}\, \mu_i\, w_i^*\, d\tau \ + \ \Sigma_{i0}\, \sigma_i\, w_i^*\, dZ_i\,.$

                                       $= \ (r + \Sigma_i\, (\mu_i - r)w_i^*)d\tau + \ \Sigma_i\, \sigma_i\, w_i^*\, dZ_i\,.$

Define

$$\mu_W \quad = \quad r + \Sigma_i\, (\mu_i - r)w_i^*$$

$$\sigma_W^2 \quad = \quad \Sigma_i\, \Sigma_j\, w_i^*\, w_j^*\, \rho_{ij}\sigma_i\sigma_j$$

$$\sigma_{iw} \quad = \quad \sigma_i \Sigma_j w_j^* \rho_{ij} \sigma_j$$

Observe

$$\mu_w \, d\tau \quad = \quad E[dW/W]$$

$$\sigma_w^2 \, d\tau \quad = \quad E[(dW/W)^2]$$

and, when there is no trading ($dL_i(t) = dM_i(t) = 0$),

$$\sigma_{iw} \, d\tau \quad = \quad E[(dS_i/S_i)(dW/W)]$$

Since $w_i(\tau) = S_i(\tau)/W(\tau)$, it follows from Ito's Lemma that

(3) $\quad dw_i(\tau) \; = \; (\mu_i - \mu_w + \sigma_w^2 - \sigma_{iw}) \, w_i \, d\tau + \sigma_i \, w_i \, dZ_i - (\Sigma_j \sigma_j w_j^* \, dZ_j) \, w_i + \delta w_i(\tau),$

where $\quad \delta w_i(\tau) \; = \; (dL_i(\tau) - dM_i(\tau))/W(\tau).$

The nonnegative process $w(\tau)$ is thus right continuous with left-hand limit. Define

$$a_i \quad = \quad \mu_i - \mu_w + \sigma_w^2 - \sigma_{iw}$$

$$q_{ij} \quad = \quad \sigma_i \sigma_j \rho_{ij} - \sigma_{iw} - \sigma_{jw} + \sigma_w^2,$$

and let **a** and **Q** represent the (1xN) vector and (NxN) matrix with elements a_i and q_{ij}, respectively.[6]

Note that, when there is no trading ($dL_i(\tau) = dM_i(\tau) = 0$),

$$a_i \, w_i \, d\tau \quad = \quad E[dw_i]$$

$$q_{ij} \, w_i \, w_j \, d\tau \quad = \quad E[dw_i dw_j].$$

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[6] Observe that the dynamics of the $w_i(\tau)$ can vary significantly from the dynamics of the $S_i(\tau)$. For example, if there are two *positively* correlated risky assets whose weights sum to one (implying $w_0(\tau) = 0$), then $dw_i(\tau)/w_i(\tau)$ will be *perfectly negatively* correlated, $i = 1,2$.

III. The Investor's Objective

The investor wishes to hold risky assets in target proportions w^*. Divergence between the actual ratios $w(\tau)$ and w^* creates an expected utility loss resulting from "tracking error". This loss can be reduced by trading more frequently—but more frequent trading will lead to greater transactions costs. The optimal investment strategy will minimize the sum of tracking error costs and trading costs.

III.a. Tracking Error

The incremental loss $dU(\tau)$ of utility at time τ, measured in dollar terms, is assumed proportional to the variance of tracking error—the variance of rate of return from holding assets in proportion $w(\tau)$ rather than w^* over the interval $d\tau$:

$$(4) \qquad dU(\tau) \quad = \quad \lambda\, (w^* - w(\tau))'\, V\, (w^* - w(\tau))d\tau,$$

where primes denote transposes and λ is the investor's "price of tracking error."[7]

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[7] Equation (4) can be justified by assuming mean-variance preferences over *rates of return* to wealth. In that case, (4) can be expressed as the difference between mean-variance utility at w^* and at $w(\tau)$:

$$L(\tau) \quad = [(\mu - r)'\, w^*\, d\tau - \lambda\, w^{*'}\, V\, w^*\, d\tau] - [(\mu - r)'\, w(\tau)\, d\tau - \lambda\, w(\tau)'\, V\, w(\tau)\, d\tau]$$
$$= \lambda\, (w^* - w(\tau))'\, V\, (w^* - w(\tau))d\tau,$$

where λ is the price of portfolio risk (variance), and the second line uses the mean-variance optimal portfolio condition that $w^* = (1/2\lambda)V^{-1}(\mu - r)$.

Note that this rationale would imply that the same "λ" is used to choose the optimal portfolio and to price tracking error. More generally, however, we can separate the two. Revealed behavior by portfolio managers suggests that tracking error is often accorded a higher cost than the λ implied by the selection of asset

III.b. Trading Costs

Over an infinite horizon, the investor wishes to minimize the discounted integral of tracking error losses $dU(\tau)$, plus the discounted trading costs associated with adjusting asset proportions $w(\tau)$. Trading costs are assumed to be proportional to the dollar amount of trades. The analysis immediately below assumes that buying and selling costs are the same. Later, costs are allowed to vary with the type of transaction.

Define

$\quad$ k_i : $\quad$ the transactions cost per dollar of asset i traded[8],

$\quad$ k: $\quad$ the vector of costs k_1, ..., k_N.

We assume the riskless asset $(i = 0)$ is costless to trade.

Recalling $\delta w_i(\tau) = (dL_i(\tau) - dM_i(\tau))/W(\tau)$, the incremental trading costs $dT_i(\tau)$ of asset i at time t will be

$$dT_i(\tau) \;=\; k_i\, W(\tau)|\delta w_i(\tau)|.$$

Since the loss (4) is expressed in terms of the tracking error on portfolio rates of return (i.e. dollar return

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proportions.

[8] Alternatively, proportional transactions costs which vary with buying and selling could be represented by k_{i+} for purchases dL_i, and k_{i-} for sales dM_i. See, for example, Section VIII(b) below.

divided by wealth), we also normalize dollar transactions cost by wealth $W(\tau)$. Define

$$dTC_i(\tau) = k_i |\delta w_i(\tau)|.$$

Therefore, at any time t, the change $dC(\tau)$ in cost per unit wealth of tracking error plus trading cost will

be given by

(5) $\qquad\qquad dC(\tau) = dU(\tau) + \Sigma_i dTC_i(\tau)$

$$= \lambda (w(\tau) - w^*)' V (w(\tau) - w^*) d\tau + k' |\delta w(\tau)|.$$

III.c. Total Discounted Costs

The present value of future expected costs $J(x; \beta)$ at time t depends on the chosen trading strategy β

(defined more specifically below) and on the initial asset proportions x before trade, where $x = w(t^-)$. J is

the discounted integral of expected future costs $dC(\tau)$, $t \leq \tau \leq \infty$:

$$J(x; \boldsymbol{b}) = E[\int_t^\infty e^{-r(t-t)} dC(\boldsymbol{t})) | x, \boldsymbol{b}]$$

$$= E[\{\int_t^\infty e^{-r(t-t)} \boldsymbol{l}(w(\boldsymbol{t}) - w^*)'V(w(\boldsymbol{t}) - w^*)dt + \int_t^\infty e^{-r(t-t)} k' | \boldsymbol{dv}(\boldsymbol{t})) |\} | x, \boldsymbol{b}]$$

$J(x; \beta)$ is convex in x as the instantaneous cost function dC is strictly convex and the cost of trading is

proportional to trade size (see Harrison and Taksar [1983] and Dumas [1991]).

Following the insights of Magill and Constantinides [1976] and the results of Davis and Norman [1990]

for a single control variable, and the results of Akian, Menaldi, and Sulem [1996] and Eberly and Van

Mieghem [1997] for N-dimensional control, the optimal trading strategy will describe a connected

compact ***no-trade (or "continuation") region*** $\mathbf{c} \ \mathbf{e} \ \mathbf{R}^N$ when J is convex in x. In the case of a single

risky asset, where w is a scalar, the no-trade region is an interval: $\chi = [w_{min}, w_{max}]$.

When $x \ \varepsilon \ \chi$, no initial trading takes place and $w(t) = w(t^-) = x$. If $x \notin \chi$, initial trading is required and

$w(t) = x + \delta w(t) \ \varepsilon \ \chi.$[9] Similar to the results of the previously cited papers, when transactions costs are

proportional, trading will always move asset ratios to a point on the *boundary* of the no-trade region χ.

After a potentially large initial trade, subsequent trades will be infinitesimal in size as the (continuous)

diffusion process governing the movement of asset ratios $w(\tau)$ will not carry these ratios "far" outside the

boundary before trading back to the boundary occurs. Harrison and Taksar [1983] label this situation as

one of "instantaneous control."

Our objective is to characterize χ and detail the nature of portfolio trading in the presence of proportional

transactions costs and capital gains taxes. The boundary of χ will be denoted $\beta(\chi)$, a subset of χ with

elements ("points") denoted $\{w^\beta\}$. With a single risky asset, $\beta(\chi) = \{w_{min}, w_{max}\}$. Determining the

boundary β completely determines χ, given the assumptions above. Hereafter the boundary of the no-

trade region, $\beta = \{w^\beta\}$, is treated as the investor's choice variable. Thus the investor's objective is to find

the boundary $\beta = \beta^*$ of the no-trade region that minimizes discounted expected costs $J(x; \beta)$.

IV. Determining the Function J(x; $\mathbf{b}$) for a given Boundary $\mathbf{b}$

First consider the case where the no-trade region χ and its boundary β are given, but are not necessarily

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[9] The optimal solution will specify how the trades $\delta w(\tau)$ are determined.

optimal. When initial asset proportions x ε χ, there is no trading at t, and w(t) = x. The only cost incurred over dt is the cost of diverging from the optimal w*.

From the definition of J(x; β),

(7)
$$J(x; \boldsymbol{b}) = E[\int_{\boldsymbol{t}=t}^{t+dt} \boldsymbol{1}(w(\boldsymbol{t}) - w^*)'V(w(\boldsymbol{t}) - w^*)d\boldsymbol{t} + e^{-rdt}J(x + dw, \boldsymbol{b}) \mid w(t) = x]$$
$$= \boldsymbol{1}(x - w^*)'V(x - w^*)dt + E[e^{-rdt}J(x + dw(t), \boldsymbol{b})]$$

when x ε χ.

Expanding the expectation term of (7), and simplifying gives the partial differential equation

(8)
$$x'(a \otimes J_x(x; \boldsymbol{b})) + .5(x'[Q \otimes J_{xx}(x; \boldsymbol{b})]x) + \boldsymbol{1}(x - w^*)'V(x - w^*) - rJ(x; \boldsymbol{b}) = 0$$

where

$J_x = (J_1, ..., J_N)$ is the vector of first partial derivatives of J(x; β) w.r.t. x;

J_{xx} is the N x N matrix of cross partials with elements J_{ij};

1 is the N-dimensional unit vector;

$(a \otimes J_x)$ is the vector with elements $\{a_i J_i\}$; and

$[Q \otimes J_{xx}]$ is the N x N matrix with elements $\{q_{ij} J_{ij}\}$.

As J is convex, $J_{xx}(x; β)$ is positive semi-definite and positive definite for x in the interior of χ. For x ∉ χ, the strategy chooses δw(t) to instantly move the asset ratios to a boundary point $w^β(x) \in β$. For x outside the no-trade region χ, it follows immediately that

$$J(x, \beta) \quad = \quad J(w^\beta(x); \beta) + k' \, |x - w^\beta(x)|.$$

Associating a boundary point $w^\beta(x) \in \beta$ with arbitrary points x outside the boundary is examined in Appendix A. From (A.1a) – (A.1c), for points $x \notin \chi$,

(9) $\qquad J_i(w^\beta(x); \beta) \quad = \quad k_i \quad$ when $x_i > w_i^\beta(x)$

$\qquad\qquad\qquad\qquad\qquad\quad = \quad -k_i \quad$ when $x_i < w_i^\beta(x)$

$\qquad |J_i(w^\beta(x), \beta)| \quad < \quad k_i \quad$ only if $x_i = w_i^\beta(x)$.

Conditions (9) hold for arbitrary no-trade regions χ and their associated boundaries $\beta(\chi) = \{w^\beta\}$. Dumas (1991) terms these "value matching" conditions. As will be seen below, "most" boundary points will be characterized by (9) holding for a single i. But the 2^N "corner" points on the boundary for which $J_i(w^\beta; \beta) = \pm k_i$, for all i, will be of considerable computational importance. Let $M(\beta) \, \varepsilon \, \beta$ denote such points.

Now consider the problem of determining the *optimal* boundary β. The case of a single risky asset (N = 1) builds from the analysis of Magill and Constantinides [1979], Davis and Norman [1990], and Dumas [1991].[10] The general case with an arbitrary number of risky assets N can subsequently be examined.

V. Determining the Optimal No-Trade Region in the Single Risky Asset Case (N = 1)

When there is a single risky asset, the optimal strategy moves x to the boundary point w_{min} if $x < w_{min}$, and to

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[10] Our one-dimensional case differs from Dumas in that the dynamics (3) of the asset weight w follows a loJArithmic Brownian motion. It differs from Constantinides in the form of the loss function.

w$_{max}$ if x > w$_{max}$. (We drop the subscript "1" for the single risky asset case). Therefore w$^\beta$(x) = w$_{max}$ for all x > w$_{max}$, and w$^\beta$(x) = w$_{min}$ for all x < w$_{min}$. Observe that M(β) = β = {w$_{min}$, w$_{max}$}: every boundary point belongs to M(β), which is not true for N ≥ 2. At the boundary points when w$_{min}$ and w$_{max}$, it follows from (9) that

(10) $J_1(w_{min}; w_{min}, w_{max}) = -k$

(11) $J_1(w_{max}; w_{min}, w_{max}) = k$

where J$_n$ (•; •, •) is the derivative of J with respect to the nth argument.[11] In the single risky asset case, equation (8) is an ordinary differential equation with solution

(12) $$J(x; w_{min}, w_{max}) = \mathbf{l}\,\mathbf{s}^2 \left[\frac{x^2}{r - 2a - Q} - \frac{2xw^*}{r - a} + \frac{w^{*2}}{r} \right] + C_1 x^{c_{11}} + C_2 x^{c_{12}}$$

where c$_{11}$ and c$_{12}$ are uniquely determined as

(13) c_{11} $= (-a + Q/2 + [(a-Q/2)^2 + 2Qr]^{.5})/Q;$

                        c_{12} $= (-a + Q/2 - [(a-Q/2)^2 + 2Qr]^{.5})/Q,$

where from (3)

                $a = (1 - w^*)(\mu - r - \sigma^2 w^*);$ $Q = \sigma^2 (1 - w^*)^2$

For a given β = {w$_{min}$, w$_{max}$}, the constants C$_1$ and C$_2$ are determined by the boundary conditions (10) and (11). The *optimal* boundary β^* = {w$^*_{min}$, w$^*_{max}$} is determined by the "super contact" conditions, which require that J(x; w$_{min}$, w$_{max}$) be minimized w.r.t. w$_{min}$ and w$_{max}$. This provides the two final

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[11] Dumas [1991] shows that these conditions are not smooth-pasting optimality conditions, but rather the limit of a "value matching" condition. The conditions (sometimes termed "super contact") associated with the optimal boundary β^* are given by (16) and (17).

conditions needed for optimization, that at the optimal w_{min} and w_{max}

(14) $\qquad J_2(w_{min} ; w_{min} , w_{max}) = 0,$

(15) $\qquad J_3(w_{max} ; w_{min} , w_{max}) = 0.$

Following Dumas [1991], it can in turn be shown that these conditions imply

(16) $\qquad J_{11} (w_{min} ; w_{min} , w_{max}) = 0,$

(17) $\qquad J_{11} (w_{max} ; w_{min} , w_{max}) = 0.$

Solving (8) subject to the conditions (10), (11), (16), and (17) generates solutions for the optimal strategy parameters w^*_{min} and w^*_{max} , and for the constants C_1 and C_2 of equation (12), thereby uniquely determining $J(x; \beta^*) = J(x; w^*_{min} , w^*_{max}).$

VI. Determining the Optimal No-Trade Region with Multiple Risky Assets (N ₃ 2)

The solution to the (partial) differential equation (8) is much more difficult when $N \geq 2$, since the boundary set β is now described by an infinite number of points rather than the two points $\{w_{min} , w_{max} \}$. From equations (A.1a) – (A.1c), recall that at every boundary point $w^\beta,$

(18) $\qquad |J_i (w^\beta, \beta)| \leq k_i, \qquad\qquad i = 1, ..., N,$

with equality holding for at least one i. In addition, the maximizing conditions equivalent to (16) and

(17) are that whenever $|J_i(w^\beta; \beta)| = k_i$, then

$$(19) \qquad\qquad\qquad J_{ii}(w^\beta; \beta) = 0.$$

When $N \geq 2$, we are unaware of closed form solutions to equation (8) that satisfy conditions (18) and (19) at all points of the boundary $\beta = \{w^\beta\}$ of the no-trade region. So we turn now to finding an approximation of the optimal strategy, which we term the quasi-optimal strategy.

VI(a). Determining A Quasi-Optimal No-Trade Region

Our strategy is to find a quasi-optimal solution JA(x; B) which satisfies the p.d.e. equation (8), and which satisfies conditions (18) and (19) , with JA replacing J, at a *finite* set of boundary points B. Given JA, we can then construct the remaining boundary points β between these points B by an algorithm described in Appendix D.[12] In general, these in-between boundary points will be constructed to satisfy either (18) or (19), but will not (except by chance) satisfy both. We develop a measure of how well this quasi-optimal solution approximates an exact solution, and show that it will be highly accurate for realistic choices of parameters. Clearly the quasi-optimal function JA(x; B) will depend on the choice of the points B at which both conditions (18) and (19) will be satisfied. The solution to equation (8) can be represented by the sum of a homogeneous solution and a particular solution. From Appendix B, there exists a homogeneous solution in the form

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[12] Since there are only two boundary points when $N = 1$ (B = $\{w_{min}, w_{max}\}$), it follows immediately that the quasi-optimal solution is the fully optimal solution in this case.

$$H(x; B) = \sum_{k=1}^{K} \{ C^k(B) \left[\prod_{i=1}^{N} x_i^{c_{ik}} \right] \}$$

where the coefficients $\{c_{ik}\}$ satisfy equation (B5) and the coefficients $\{C^k\}$ depend upon the boundary conditions.[13]

The actual number of terms K needed will depend upon the number of boundary points at which both conditions (18) and (19) are presumed to be satisfied. A natural set of boundary points B are the 2^N "corner" points where conditions (18) hold *with equality for all i*, as well as conditions (19). Each corner point has N dimensions, so there are 2^N N variables characterizing the points $w^B \varepsilon$ B to be determined. In addition, we must determine the K constants $\{C^k\}$ in the homogeneous solution. The total number of variables to be determined therefore is $2^N N + K$.

The equations to be satisfied at each of the 2^N corner points are the N equations (18)--which hold with equality at $w^B \varepsilon$ B--and the N equations (19). Thus 2N conditions must hold at each of the 2^N points, for a total of $2^{N+1}N$ equations. For the number of equations to equal the number of unknowns, it follows that

$K = 2^N N$. We introduce this number of power functions in the homogeneous term (B.2), where the exponents $\{c_{i\,k}\}$ of each function k = 1, …, K satisfy equation (B.5). A particular solution exists of form (B.9) in Appendix B. The quasi-optimal function JA(x; B) is given by the sum of the homogeneous solution and the particular solution, satisfying the equations (18) with equality and (19) at the boundary

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[13] In the one risky asset case (N = 1), the coefficients c_{ik} are uniquely determined. When N > 1, an infinite number of possible coefficients exist, subject to constraint (B5) being satisfied. We can always find a solution for the corner points when the c_{ik} are all possible combinations of zero or one.

corner points $w^B \, \varepsilon \, B$. These boundary conditions jointly determine the boundary corner points and the

constants $\{C^k\}$, $k = 1, \ldots, K$.

We can use two alternative techniques to construct the remaining boundary points connecting adjacent

corner points $w^B \, \varepsilon \, B$, given the function $JA(x; B)$. The first assures that conditions (18) for JA are met

along this boundary; the second assures that conditions (19) are satisfied. Appendix D develops these

algorithms. Let $\beta 1^*$ denote the set of boundary points $\{w^{\beta 1^*}\}$ determined by the first technique, and $\beta 2^*$

denote the set of boundary points $\{w^{\beta 2^*}\}$ determined by the second technique. Because the second

technique proved more tractable computationally, we used it to perform all calculations that follow.

The calculation of the boundary $\beta 2^*$ using (19) or (D.3) also offers a measure of accuracy of our

approximation. Equation (20) will in general not be satisfied: $|JA_j(w^{\beta 2^*}, B)| - k_j \neq 0$. But for each point

$w^{\beta 2^*}$ on the boundary we can compute the transactions cost $k_j(w^{\beta 2^*})$ which would make this term zero,

i.e. $k_j(w^{\beta 2^*}) \equiv |JA_j(w^{\beta 2^*}, B)|$. For such transactions costs, the quasi-optimal boundary would be fully

optimal, since it would satisfy both conditions (18) and (19). The maximal error over the entire

boundary $\beta 2^*$,

$$(22) \qquad\qquad E = \underset{\{w^{\beta 2^*}, j\}}{\text{Max}} \; [\, \text{Max} \; [(|k_j - k_j(w^{\beta 2^*})|)] \,]$$

indicates the *maximal* amount by which transactions costs would have to vary for the quasi-optimal

solution to be the optimal solution. For realistic parameters and modest asset correlations ($< .30$), this

number is usually small: less than .0005. Thus if the quasi-optimal solution is based on transactions

costs of (say) 1%, then the quasi-optimal solution is *exact* when transactions costs range appropriately (as $w^{\beta2*}$ varies) between .95% and 1.05%. Since it is rare to have exact estimates of actual transactions costs, this range seems tolerable for most practical situations.

But as asset correlations increase, we observe from the examples in Section IX below that errors become larger—reaching a maximum of .0025 when transactions costs are 1 percent. The solution in this case is exactly optimal only if transactions costs were to vary appropriately between 0.75% and 1.25%, a fairly broad range. To further reduce errors, a straightforward extension of the approximation method outlined above is now considered.

VI(b). Greater Accuracy of the Quasi-Optimal Solution

The accuracy of the quasi-optimal solution can be improved by finding a JA solution that satisfies the appropriate conditions (18) and (19) at more than just the corner points.[14]

In two dimensions, for example, we could require matching the appropriate optimality conditions at a finite number of points along the boundary segments (xy, yz, zv, vx) as well as at the corner points (x, y, z, v). An obvious set of additional points would be the four midpoints of the boundary segments. We would need an additional four terms in the homogeneous sum (B.2) with exponents satisfying (B.5). Thus K = 12 in this situation. Matching the boundary conditions at more and more points B leads to greater and greater accuracy as measured by (22), at the cost of the extra computational requirements as K rises.

In Section IX, a set of examples examine the differences between solutions with K = 8 and solutions with K = 12. In the latter solutions, optimality conditions are met at the midpoints of the boundary segments as well as at the corners. For asset return correlations of 0.2 or less, the no-trade regions are virtually indistinguishable, with the extreme points (x,y,z,v) differing by less than 1% in both dimensions and the segments joining these points continuing to appear as virtually "straight" lines. However, the maximal error E defined by (22) falls to about half its previous level. When asset correlation rises to 0.7, the no-trade regions differ perceptively, and the error of the "corners only" solution (K = 8) rises to .0025. The maximal error falls to .0005 for the K = 12 solution.

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[14] A subset of coordinates for these additional points will be fixed (e.g., the midpoint of a boundary segment joining two corner points). At these points, a subset of conditions (18) and (19) will hold: See Appendix D.

VII. Costs, Turnover, and Tracking Error of Optimal Trading Strategies

We have developed techniques for determining the quasi-optimal trading strategies, as expressed by a "no trade" region with boundary $\beta^* \varepsilon R^N$, where $\beta^* = \beta_1^*$ or β_2^*, depending on whether technique 1 or 2 is used to determine the boundary. Now consider a method, first introduced by Leland and Connor [1995] in the single risky asset case, to estimate the expected present value of transactions costs and annual expected turnover from following the optimal strategy. We also develop a measure of the expected tracking error of the optimal strategy.

Consider the function $T(x; \beta) \equiv J(x; \beta|\lambda = 0)$ that satisfies the value-matching conditions (18) on the boundary β. This is the expected present cost of the trading strategy with no-trade region determined by boundary β when there is *no cost of tracking error,* since $\lambda = 0$. Therefore $J(x; \beta|\lambda = 0)$ measures *the expected net present value of trading costs* when the no-trade region has boundary β. By setting $\beta = \beta^*$, the boundary of the optimal no-trade region for the original problem, the resulting function $T(x; \beta^*)$ will measure the expected net present value of trading costs for the optimal strategy.

When $N > 1$, there is no known closed form solution for T, so we must use the approximation methods we used previously. Let TA(x; B) denote the approximate cost function, given the boundary determined by optimizing JA(x; B). It will satisfy the differential equation (8) with value-matching boundary conditions as in (18') holding with equality at $w^\beta \varepsilon B$. (It will not satisfy conditions (19), since the boundary points are optimal for JA, not TA).

The solution $TA(x; B)$ will be the sum of a homogeneous solution to the p.d.e. (8); plus a particular solution whose coefficients are all zero and hence can be ignored. The homogeneous equation will again be of the form (B.2), with exponents satisfying (B.5). The $2^N N$ coefficients C^k, $k = 1,...,K$ must be chosen such that the $2^N N$ value-matching conditions (18') hold with equality at $w^\beta \in B$.

$TA(x; B)$ therefore gives the discounted expected total trading costs, from $\tau = t$ to infinity, of the quasi-optimal trading strategy associated with $JA(x; B)$. The *annualized* expected trading costs ATC are simply

(23)
$$ATC = r\, TA(x; B).$$

If all assets have the same trading costs $k_i = k$, $i = 1,..., N$, then the annualized expected (one-way) turnover is

(24)
$$\text{Turnover} = ATC / k. \quad [15]$$

The (approximate) expected discounted tracking error cost TE comprises the residual cost: $TE = JA - TA$. The annualized tracking error is rTE, and the annualized variance AV associated with this cost is simply

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[15] When k_i varies across assets, it is still possible to estimate turnover associated with the quasi-optimal strategy $JA(x; B)$. We construct a function $TA^*(x; B)$ as above, but which satisfies the value-matching conditions
$|TA_i^*(x; w^B)| = k$ for all i, where is k is an arbitrary *constant* across all assets i. The solution is the expected discounted transactions costs associated with B, when k is the (common) cost of trading each asset. Annualized turnover is given by rTA^*/k.

(25)
$$\text{AV} \quad = \text{r TE} / \lambda$$

$$= \text{r (JA - TA)} / \lambda.$$

VIII. Optimal Policies with a Single Risky Asset: Some Examples

Consider the following base parameters for asset returns:

Risky asset:	μ	$=$	.125
	σ^2	$=$	.040
Riskless interest rate:	r	$=$	.075
Target proportion of risky asset:	w*	$=$	.60.

We first assume that the cost of selling and buying are identical, and equal to k.

Table I lists the optimal no-trade boundaries $\{w_{min}, w_{max}\}$, with percent turnover and percent standard deviation of tracking error in parentheses below, for a range of transactions costs k and cost per unit of tracking error variance λ.

Table I examines a range of trading costs, from a low of 0.1% to a high of 10%. For comparison, two values for the price of tracking error λ are considered: 1, the value which would also lead a mean-variance investor to choose a target proportion of (approximately) 60% in the risky asset, given its return and risk; and 10, a larger value which often seems to characterize the actions of investors trying to track a target ratio. The upper numbers in each cell are w_{min} and w_{max}. The lower numbers in parentheses are

annual turnover in percent (from equation (24)) and annualized standard deviation of tracking error in percent (the square root of AV from equation (25)).

TABLE I

	k				
	.001	**.005**	**.01**	**.05**	**.10**
$\lambda = 1$	.562/.633	.533/.655	.513/.669	.436/.725	.381/.775
	(3.24/.41)	(1.85/.70)	(1.44/.88)	(0.80/1.5)	(0.60/1.92)
$\lambda = 10$	.583/.616	.571/.627	.562/.633	.533/.655	.513/.669
	(7.05/.19)	(4.10/.32)	(3.24/.41)	(1.85/.70)	(1.44/.88)

Note that the table cells are identical when the ratio k/λ is the same.[16] Therefore higher transactions costs or greater aversion to tracking error play an inverse role in determining the optimal strategy. For small k (and/or large λ), the size of the optimal no-trade interval is approximately proportional to the cube root of k.[17] Thus doubling transactions costs will increase the no-trade interval by a factor of about

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[16] This result also holds when $N \geq 2$: The quasi-optimal boundary, turnover, and tracking accuracy are homogeneous of degree zero in the vector of trading costs and λ.

[17] A heuristic demonstration of this result is as follows. For small k, $J(w;\beta^*)$ is approximately quadratic with a minimum at w^*, implying $J_1(w^*) = J_{111}(w^*) = 0$. Using a Taylor series expansion, $J_1(w_{max}) - J_1(w^*) = J_{11}(w_{max})(w_{max} - w^*) + (1/2)J_{111}(w_{max}) (w_{max} - w^*)^2$. Using (11) and (17), $J_1(w_{max}) = k$ and $J_{11}(w_{max}) = 0$, implying $k = (1/2)J_{111}(w_{max}) (w_{max} - w^*)^2$. From a Taylor series expansion, $J_{111}(w_{max}) = J_{111}(w^*) + J_{1111}(w^*)(w_{max} - w^*) = J_{1111}(w^*)(w_{max} - w^*)$, implying $k = (1/2) J_{111}(w_{max}) (w_{max} - w^*)^3$. Thus $(w_{max} - w^*) = Zk^{1/3}$, where $Z = 2/J_{1111}(w^*))^{1/3}$.
By symmetry, $(w^* - w_{min}) = (w_{max} - w^*)$. So the size of the no-trade interval $(w_{max} - w_{min})$ is approximately $2Zk^{1/3}$.

$2^{1/3} = 1.26$. Similarly, turnover will fall by this same factor. Tracking error at the optimum also rises by a factor $k^{1/3}$. We may summarize our results as follows: for k/λ small (e.g. $\leq .1$), then to a close order of approximation the following propositions hold:

(i)   The size of the optimal no-trade interval (w_{max} - w_{min}) is

proportional to the cube root of transactions costs.

(ii)   Turnover and the size of the optimal no-trade interval

are inversely proportional, implying that turnover is

inversely proportional to the cube root of transactions costs.

(iii)   The standard deviation of tracking error and the size of the optimal

no-trade interval are proportional, implying that the standard deviation

of trading error is inversely proportional to the cube root of transactions costs.

Propositions (iii) and (iv) immediately suggest

(iv)   The standard deviation of tracking error and turnover are inversely proportional.

Sensitivity to changes in tracking error aversion λ follows immediately from the invariance of the optimal strategy for constant k/λ., implying a change in λ has the same effect as a change in $(1/k)$.

VIII(a).   The Effect of a "Turnover Tax" on Optimal Trading Volume

The results above can be used to assess the impact of proposed taxes on trading by funds seeking to keep assets in given proportions. If other trading costs averaged 0.5%, a 2% tax on portfolio trades, raising total costs to 2.5%, would reduce trading volume from optimal rebalancing by $(1-(.005/.025)^{1/3})$, or 42%.

VIII(b). Asymmetric Buying/Selling Costs and Optimal Strategies
in the Presence of Capital Gains Taxes

We now examine the case where the cost of selling (k_s) differs from the cost of buying (k_b). These costs may differ simply because of market conditions, or because there are capital gains taxes associated with selling. How does the presence of capital gains taxes, and therefore of asymmetric selling and buying costs, affect the optimal trading strategy?[18]

Assume that selling incurs capital gains taxes (e.g. $k_s = .10$, i.e. half the sales price is taxable at a 20% rate), whereas buying incurs only a trading cost (e.g. $k_b = .01$). Consider the following conjecture: the optimal strategy is to choose w_{max} as in the case when both buying and selling costs are .10, and w_{min} as in the case when both are .01. If $\lambda = 10$, from Table 1 this would imply

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[18] Since the time horizon is infinite, capital gains taxes can be avoided if assets are never sold. Some caution is needed in extending our results to capital gains. First, we assume that capital losses are immediately realized and assets are replaced by repurchasing the (equivalent of) the current position. (This ignores wash sale rules.) Second, our stationary model assumes that capital gains tax rates will remain constant for future asset sales. (Note that the dynamics of buying and selling could change the average cost basis and hence the effective tax rate on sales revenue). The assumption of constant selling costs also precludes a difference between short- and long-term capital gains which Constantinides [1984] showed could create an additional reason for realizing long-term gains.

$$w_{max} = .669, \qquad w_{min} = .562.$$

But this conjecture is incorrect. When we solve for the optimal policy with the boundary conditions

$$J_1(w_{min}; w_{min}, w_{max}) = -k_b,$$

$$J_1(w_{max}; w_{min}, w_{max}) = k_s,$$

plus the optimizing conditions (16) and (17), we find

$$w_{max} = .661, \; w_{min} = .534.$$

While the upper trade ratio w_{max} is similar to that when both trading costs are 10%, the lower trade ratio w_{min} is substantially lower than that when both trading costs are 1%. The reason for this becomes clear: the probability of incurring capital gains in the long run depends (roughly) on the distance between the midpoint of the no-trade interval and the upper boundary. By lowering the lower boundary of the no-trade interval, this distance can be increased while keeping the average exposure closer to w^*.[19]

Because w_{min} is a decreasing function of the capital gains tax rate, an investor commencing with a cash-only position should *initially* invest less in the risky asset (to a level w_{min}) as the tax rate rises. In the long run, however, the *average* risky asset proportion will approximate w^* when the optimal strategy is followed.

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[19] Even if the cost of buying were zero, it would still not be optimal to buy whenever $w < w^*$, if the cost of selling is positive. If $k_b = 0$, and $k_s = .10$, the optimal trading strategy for the example above would be $w_{min} = .536$. $w_{max} = .660$.

Our results should be contrasted with those of Constantinides [1983], who argues that capital gains never should be realized. But this conclusion presumes that investors can sell short to offset long positions ("shorting the box") in order to avoid realizing capital gains. When this is not possible, the diversification arguments that are important here cannot be ignored: some capital gains must be realized to keep tracking error within bounds. Even if shorting the box or a highly correlated security is possible, it is generally expensive. Synthetically "selling" the original security using these techniques can be incorporated in our analysis, using their (high) transactions cost rather than the (even higher) capital gains costs.

VIII(c). *Turnover of the Optimal Strategy vs. Periodic Rebalancing*

Consider now rebalancing periodically at a time interval δt. At the end of each rebalancing period, the random asset proportions $w(t + \delta t)$ are readjusted back to the desired proportions w^*. For equal tracking error, we are interested in the turnover associated with this strategy relative to the optimal strategy considered above.

Appendix C gives formulae for the average annual tracking error and turnover associated with a periodic rebalancing strategy, when there is a single risky asset.

Consider the example from above with $w^* = .60$ and $\sigma^2 = .04$. From Table I, with $\lambda = 10$ and $k = .01$ (or equivalently, $\lambda = 1$ and $k = .001$), annual tracking error standard deviation is 0.41%. Using formula

(C5), setting $\delta t = 0.357$ (rebalancing approximately three times per year) gives an identical tracking error. But plugging this value of δt into (C7) gives an expected annual turnover of 6.36%, in comparison with the optimal strategy's turnover of 3.24%. Thus optimal trading reduces turnover by 49%. Comparisons with other cells in Table I give similar savings of about 50%.

IX. Optimal Policies with a Multiple Risky Assets: Some Examples

We now find quasi-optimal trading strategies for the case with two risky assets plus a riskless asset. While higher-dimensional examples could be constructed with the techniques of Section VII above, they are numerically intensive and most salient points can be seen in the two risky asset case.

We consider first a symmetric "base" case, with

Risky assets:	$\mu_1 = \mu_2$	$= .125$
	$\sigma_1^2 = \sigma_2^2$	$= .040$
	ρ	$= .200$
Riskless interest rate:	r	$= .075$
Target proportions of risky assets:	$w_1^* = w_2^*$	$= .40$
Transactions costs:	$k_1 = k_2$	$= .01$
Tracking error aversion:	λ	$= 1.30$

The tracking error aversion factor of 1.30 is chosen to equal the variance aversion of the investor choosing optimally to invest 40% in each risky asset, given the distribution of returns in the base case

above. Alternatively, we later allow the tracking error aversion factor to exceed the investor's variance aversion (see footnote 5).

Figure 1 shows the no-trade region for this λ using the "corners only" (K = 8) solution.   The corner point coordinates are X = {.462, .462}, Y = {.478, .322}, Z = {.332, .332}, and V = {.322, .478}. Turnover is 3.2% and the standard deviation of tracking error is 1.13% per year. The K = 12 solution differs by no more than .001 in any coordinate, and is not shown here.

Although the segments XY, YZ, etc. appear to be linear, they are not exactly. This would be the case only if $JA(x; B)$ were a quadratic function in the interior of χ. Nonetheless, once the boundary points {X,Y,Z,V } have been found, "connecting the dots" seems a reasonable approximation for most parameter choices.

The quasi-optimal policy is not fully optimal, as discussed above. Since we constructed the boundary segments XY, YZ, etc. using the conditions $JA_{11}(w^\beta, B) = 0$, $JA_{22}(w^\beta, B) = 0$, etc., we can measure the error using equation (22), which determines the maximal absolute error $|JA_i - k_i|$ along the entire boundary. Figure 1A shows the error along the boundary XY, where w_2 varies from $Y_2 = .332$ to $X_2 = .462$. Figure 1B shows the error along the boundary ZY, where w_1 varies from $Y_1 = .322$ to $X_1 = .462$. The maximal error is approximately .0003, or about .03%. (Maximal errors along the other segments ZV and VX are similar.)   We conclude that the quasi-optimal strategy would be fully optimal if transactions costs ranged appropriately between 0.97% and 1.03% along the boundary β^*.   The K = 12 solution gives even tighter bounds.

When λ = 10, the no-trade region shrinks to x = {.432, .432}, y = {.438, .361}, z = {.367. .367}, and v = {.361, .438}.[20] Turnover rises to 6.8% and the standard deviation of tracking error falls to 0.56% per year.

Increasing the correlation between the assets to 0.7 increases the "skewness" of the no-trade region, as indicated in the K = 8 solution in Figure 2.[21] Maximal errors, however, are substantially larger using the K = 8 solution. As can be seen in Figures 2A and 2B, these errors reach .0025, or a quarter of the actual transactions costs of 1%. Figure 2(i) illustrates the no-trade region using the K = 12 solution. It is even more "skewed" than the K = 8 solution. Figures 2A(i) and 2B(i) show that maximal errors are reduced to .0005, or 5% of the actual transactions costs. Note that even more accuracy could be achieved by matching the optimality conditions (18) and (19) at even more points along the boundaries—at the cost of increasing K and computational complexity.

When transactions costs for all risky assets are scaled by a factor ϕ, the size of the no-trade region is approximately proportional to $\phi^{1/3}$, as in Proposition (i) for the single risky asset case. Turnover and tracking error also behave in the fashion outlined by Propositions (ii)-(iv) above.

Not surprisingly, reducing transactions costs of a particular risky asset will shrink the size of the no-trade

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[17] We continue to assume that w* = (.40, .40). Thus the aversion to tracking error is higher than the aversion to risk which determined the optimal investment proportions. See footnote 5.

[18] We keep the investor's aversion to variance of tracking error at λ = 1.30 for purposes of comparison .
This *tracking error* aversion parameter was initially selected because it was also consistent with the risk aversion towards *portfolio variance* of a mean-variance investor choosing w* = (.40, .40), given the base case distribution. Note, however, that a different risk aversion to portfolio variance would be required for the investor to keep w* = (.40, .40) given the new distribution of asset returns, with higher correlation. See also footnote 5.

region in that dimension. Less obvious is the fact that the no-trade interval will shift for assets whose transactions costs remain unchanged.

Figure 3 illustrates the case where the trading cost of asset 2 falls to 0.1 percent.[22] Other parameters remain as in the base case. The optimal no-trade region has corner points X = {.471, .421}, Y = {.473, .360}, Z = {.324. .378}, and V = {.323, .443}.   Contrast the point Z here with Z = (.367, .367), optimal when transactions costs are both 1 percent. As expected, from a start with zero holdings of both risky assets, more of asset 2 should be bought initially since its transactions costs are lower. But less of asset 1 should be bought, reflecting the positive correlation of the two assets. Positive correlation implies that they are (partial) "substitutes." Buying more of the less-costly asset permits a smaller purchase of the more-costly asset.   This illustrates the dependency of the no-trade region in one dimension upon transactions costs in other dimensions.

X.   The Effect of Transactions Costs on Initial Asset Investments

Consider an investment fund which initially has cash holdings only. It wishes to invest in two assets with means and variances as given in the base case example of Section IX.   The optimal no-trade region is illustrated in Figure 1.

How do transactions costs affect initial asset investments? Because the initial allocation x = (0, 0) is in Region V, the initial investments will move asset proportions to the point Z = (.332, .332).   When transactions costs are equal, equal investment will occur in both assets, but less than the desired

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[22] We use K = 12 for greater accuracy.

proportions (.40, .40). Since initial investment starts at a lower level than desired, the average asset exposure over a short period of time will be less than 40 percent to each asset. Only as the time horizon becomes infinite will average investment proportions approach 40 percent each.

Figure 3 illustrates the case with asset correlation 0.7, and transactions costs of the two assets are 1 percent and 0.1 percent, respectively. The aversion to tracking error remains at $\lambda = 1.30$. The no-trade region is computed with $K = 12$, and errors are given in Figures 3A and 3B.

Starting from an all-cash position, the optimal strategy is to purchase initial fractions given by the point $Z = (.331, .412)$. Not surprisingly, the amount of asset 1, the high-cost asset, is scaled back substantially from the ideal 40% proportion. But the initial investment in asset 2, the low-trading-cost asset, is 41.2%, which *exceeds* the desired proportion of 40%. The presence of trading costs can lead to a higher initial purchase of an asset. In the example above, it is optimal to purchase more of the low-cost asset 1 to replace the lesser amount of asset 2 because the assets are highly correlated and therefore act as substitutes. In the long run, of course, the *average* asset proportions will approximate their desired proportions when the optimal trading strategy is followed.

A currently-popular technique for dealing with transactions costs is to decrease the mean rate of return on assets, in proportion to their transactions costs. This strategy yields allocation fractions that are *permanently* less than target levels.[23] And an investor would never make an initial investment that

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[23] The actual trading strategy, after these "adjusted" target allocations are initially established, is not articulated. If actual proportions are returned to the adjusted targets on a periodic basis, further inefficiencies result as in Section VIII(c).

exceeds the ideal target proportion—although that is required by the optimal policy in Figure 3. The mean-adjustment technique clearly has important shortcomings, and will be costly to follow relative to the optimal strategy.

XI. Extensions

XI(a). Nonconstant Target Proportions

Our analysis has examined the case where dw/w moves relative to a fixed target w^*. But the analysis could be extended to cases where the target w^* itself follows a stochastic process. For example, say $w_i^* = w_i^*(Z(\tau))$, $i = 1,...,N$, $\tau \geq t$, where $Z(\tau)$ is a vector of variables following a joint diffusion process. $Z(\tau)$ may include other variables (e.g. macroeconomic factors whose levels follow a diffusion) as well as stock prices. The vector $w(\tau) - w^*(Z(\tau))$ will itself follow a diffusion process with mean and volatility determined by Ito's Lemma. If the process for $Z(\tau)$ has time independent parameters, $w(\tau) - w^*(Z(\tau))$ will also have time-independent parameters, and our analysis will be directly applicable with the vector a and the matrix Q of equation (3) being appropriately modified.

Examples of such $w^*(Z(\tau))$ processes include the case when the target ratios w_i^* are functions of macroeconomic variables (such as aggregate consumption or interest rates) which follow a time-independent diffusion, or when the target ratios are functions of relative stock price levels. Time-independent strategies such as constant proportion portfolio insurance (e.g. Black and Jones [1988]) are also included, but option-replication strategies are not, due to the time dependence of the target delta.

Our analysis could also be extended to alternative time-independent stochastic processes for the risky asset values S_i, for example an Ornstein-Uhlenbeck diffusion process exhibiting mean reversion.

XI(b). Fixed Components of Transactions Costs

Consider now a fixed cost component in addition to a proportional component of transactions costs. Work by Dixit [1991] suggests that the optimal trading strategy will be characterized by two regions surrounding w^*, one nested within the other. The outer boundary will define the no-trade region. The inner boundary will be the asset levels optimal to trade to, when asset proportions move outside the larger no-trade boundary.   Dixit [1991] shows how to compute outer and inner regions (intervals) in the single risky asset case. Even in one dimension the determination of outer and inner intervals is not easy, and we do not pursue the multi-dimensional extension here.

XII. Conclusions

We have considered optimal rebalancing strategies in the presence of proportional transactions costs. For both single and multiple risky assets, the optimal strategy is characterized by a no-trade region. When asset proportions lie within this no-trade region, the optimal policy will be to do nothing. When asset proportions move outside the region, trades should be undertaken to move the asset proportions to an appropriate point on the no-trade region's boundary. It is not optimal to trade periodically to the target asset proportions, despite the popularity of such a strategy. The optimal strategy with identical tracking error reduces expected turnover by almost 50%.

We developed an exact solution for the single risky asset case, following Constantinides [1979] and others, and then considered policies for the multi-asset case. A technique for obtaining nearly-optimal strategies was presented. Our examples suggest that the multi-dimensional no-trade boundary consists of

almost-planar surfaces connecting 2^N vertices. We showed that optimal strategies almost always require buying or selling just one risky asset at any trading date. (The risky asset that must be traded will change stochastically through time).

We also developed a technique for estimating the turnover and tracking error associated with both optimal strategies and traditional periodic rebalancing strategies. This allows investment managers to consider the tradeoff between turnover and tracking error in choosing their desired strategy, and to estimate turnover savings in moving from sub-optimal to optimal strategies.

Capital gains (with some caveats) can be included in the analysis. An important conclusion is that the presence of capital gains not only affects the no-trade region's upper limits, but also importantly affects the minimum limits. If synthetic selling strategies are not possible, some capital gains will be realized by the optimal strategy, in contrast with Constantinides [1984].

A number of *ad hoc* approaches are currently used to determine the impact of transactions costs on the amount invested in risky sectors. A popular approach is to reduce the *mean* return of an asset to reflect its transactions costs.[24] This will lead to less investment than the target amount in the asset, both initially *and in the long run*. Our analysis suggests that this approach is seriously flawed. While an investor starting with an all-cash position may optimally invest *less* than the target amount in an asset with transactions costs, we found reasonable examples where it is optimal to initially invest *more*. (Very) long run average investment ratios will approach the ideal proportions when the optimal policy is

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[24] The typical adjustment suggests reducing the mean by product of the annual turnover rate of the asset, and the fractional trading costs of the asset. This is ad hoc, in that an optimal policy must determine the turnover endogenously.

followed.

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APPENDIX A

Characterizing the Boundary Set **b**

We wish to characterize the set β of boundary points $w^\beta(x) \; \varepsilon \; R^N$ to which an arbitrary set of asset proportions x outside of β will trade.

The cost of moving from x to an arbitrary point w^β is $k'|x - w^\beta|$. The optimal boundary point to trade to will be

$$w^\beta(x) \quad = \quad \underset{w^\beta}{ArgMin} \;\; \{J(w^\beta, \beta) + k'|x - w^\beta|\}$$

As $J_{ww}(w^\beta, \beta)$ is positive (semi-) definite, it follows directly that the optimal w^β will satisfy the Kuhn-Tucker conditions

(A.1a) $J_i(w^\beta(x), \beta)$ $= k_i$ if $x_i > w_i^\beta(x)$

(A.1b) $= -k_i$ if $x_i < w_i^\beta(x)$

(A.1c) $|J_i(w^\beta(x), \beta)|$ $< \; k_i$ only if $x_i = w_i^\beta(x)$

Figure 1 shows a hypothetical no-trade region and associated boundary β when N = 2, and the 4 points $\{X,Y,Z,V\} \; \varepsilon \; M(\beta)$ at which $\{J_1 = \; \pm k_1, \; J_2 = \; \pm k_2\}$. These points in turn define the (curvi-) linear segments XY, YZ, ZV, and VX.

Along the segment XY, $J_1(w^\beta,\beta) = k_1$ at all points w^β, and $J_2(w^\beta, \beta)$ ranges from $+k_2$ (at $w^\beta = x$) to $-k_2$ (at $w^\beta = y$), with $|J_2(w^\beta, \beta)| < k_2$ except at the endpoints of the segment. From the first order condition (A.1c), it must be that $w_2^\beta(x) = x_2$ along the interior of the segment XY.

Thus initial asset proportions x which lie to the right of the boundary segment XY (i.e., points in Region II of Figure 1) will be changed to a point on the segment XY with the same amount of the second risky asset. The optimal trade will sell only risky asset 1, and will not trade risky asset 2. Visually, any initial proportions in Region II will trade to the boundary point that is directly to the left of the original point. Similarly, initial asset proportions x lying in Region VI will buy only asset 1 and will not trade asset 2, while initial asset proportions lying in Region IV (Region VIII) will buy (sell) only asset 2 and will not trade asset 1.

Initial asset holdings in Regions I, III, V, and VII will require that both assets be traded, with resulting post-trade holdings represented by the points X, Y, Z, and V, respectively.

Note that, once the no-trade region is reached, random vibrations of the asset proportions will almost surely lead to the next trade being to some boundary point other than X, Y, Z, or V. Thus almost surely subsequent trades will require only one asset to be traded at any given moment.

APPENDIX B

Solution to the Partial Differential Equation (8)

As is standard, we write the solution to the partial differential equation as the sum of a homogenous solution(s) plus a particular solution. From equation (8), the homogeneous term H(x; β) of the solution must satisfy

(B1) $x'(a \otimes H_x) + .5 (x' [Q \otimes H_{xx}]x) - rH = 0$

We postulate that functions of the form

(B2) $$H(x;B) = \sum_{k=1}^{K} \left\{ C^k(B) \left[\prod_{i=1}^{N} x_i^{c_{ik}} \right] \right\}$$

satisfy equation (B1), for arbitrary constants $C^k(\beta)$, k = 1,...,K, when the exponents $\{c_{ik}\}$ are correctly chosen. Consider an arbitrary term in the sum (B2), denoted H^k, where

(B3) $$H^k(x;B) = C^k(B) \left[\prod_{i=1}^{N} x_i^{c_{ik}} \right]$$

 (We suppress the dependence of H^k and C^k on β).

Let c^k denote the vector $(c_{1k}, ..., c_{Nk})$. It readily verified that for any H^k,

(B4) $x'(a \otimes H^k_x) = (a'c^k) H^k;$

 $x'(Q \otimes H^k_{xx}) x = (c^{k'} Q c^k - \Sigma_i c_{ik}q_{ii}) H^k.$

Plugging these into (B1) yields the (scalar) quadratic equation in the vector c^k:

(B5) $a'c^k + .5(c^{k'}Qc^k - \Sigma_i c_{ik}q_{ii}) - r = 0,$

a quadratic equation which has two solutions for c_{ik} when the remaining N-1 $\{c_{jk}\}$ (j =/ i) are fixed at arbitrary values. When N = 1, Q is a scalar and there are two solutions to equation (B5):

(B6) $c_{11} = [-(a-Q/2) + ((a-Q/2)^2 + 2Qr)^{0.5}]/Q,$

$c_{12} = [-(a-Q/2) - ((a-Q/2)^2 + 2Qr)^{0.5}]/Q,$

as we derived in Section V above. Note that $K = 2$ here.

When $N = 2$, there are an infinite number of solutions to (B5). We may set c_{2k} at any value, and solve (B5) for two possible values for c_{1k}:

(B7)

$$c_{1k} = \frac{-(a_1 + c_{2k}q_{12} - .5q_{11}) \pm \sqrt{(a_1 + c_{2k}q_{12} - .5q_{11})^2 - 2q_{11}(a_2c_{2k} + .5q_{22}c_{2k}^2 - .5q_{22}c_{2k} - r)}}{q_{11}}$$

Conversely, we could set c_{1k} at any value, and derive two possible values for c_{2k}.
In the example in Section VI, we set $c_{21} = c_{22} = 0$, and $c_{23} = c_{24} = 1$ for $k = 1, 2, 3,$ and 4, respectively.
We can then use (B7) to derive
Finally, we set $c_{15} = c_{16} = 0$, $c_{17} = c_{18} = 1$, and derive $c_{25},...,c_{28}$ in a similar manner.

The homogeneous term $H(x, \beta)$ is the weighted sum of the H^k functions whose exponents each satisfy

(B8)
$$c_{11} = \frac{-(a_1 - .5q_{11}) + \sqrt{(a_1 - .5q_{11})^2 + 2q_{11}r}}{q_{11}}$$

$$c_{12} = \frac{-(a_1 - .5q_{11}) - \sqrt{(a_1 - .5q_{11})^2 + 2q_{11}r}}{q_{11}}$$

$$c_{13} = \frac{-(a_1 + q_{12} - .5q_{11}) + \sqrt{(a_1 + q_{12} - .5q_{11})^2 - 2q_{11}(a_2 - r)}}{q_{11}}$$

$$c_{14} = \frac{-(a_1 + q_{12} - .5q_{11}) - \sqrt{(a_1 + q_{12} - .5q_{11})^2 - 2q_{11}(a_2 - r)}}{q_{11}}$$

equation (B5), and therefore H will satisfy B5 as well. The weights $C^k(\beta)$ will be determined by the boundary conditions.

We seek a particular solution in the form

(B.9) $P(w) = \phi_0 + \Sigma_i \phi_i x_i + \Sigma_{i,j} \phi_{ij} x_i x_j$

which satisfies the partial differential equation (8).

The solution is tedious but straightforward. For example, when $N = 2$, the coefficients satisfy

$$(B.10) \qquad \phi_0 \quad = \quad \lambda(w_1^{*2}\sigma_1^2 + 2w_1^* w_2^* \rho_{12}\sigma_1\sigma_2 + w_2^{*2}\sigma_2^2)/r$$

$$\phi_1 \quad = \quad 2\lambda(w_1^*\sigma_1^2 + w_2^*\rho_{12}\sigma_1\sigma_2)/(a_1 - r)$$

$$\phi_2 \quad = \quad 2\lambda(w_2^*\sigma_2^2 + w_1^*\rho_{12}\sigma_1\sigma_2)/(a_2 - r)$$

$$\phi_{11} \quad = \quad -\lambda\sigma_1^2/(q_{11} + 2a_1 - r)$$

$$\phi_{12} \quad = \quad -\lambda\rho_{12}\sigma_1\sigma_2/(q_{12} + a_1 + a_2 - r)$$

$$\phi_{22} \quad = \quad -\lambda\sigma_2^2/(q_{22} + 2a_2 - r)$$

The solution to the p.d.e. (8) is the sum of the homogeneous solution and the particular solution, with the coefficients of the homogeneous solution chosen to fit the boundary conditions.

APPENDIX C

C(A). Tracking Error of Periodic Rebalancing

Consider the expected net present value of tracking error in the single risky asset case, where Σ denotes the rebalancing strategy:

$$(C1) \qquad L(t) = E[\int_{t=t}^{\infty} (w(t) - w^*)^2 s^2 \, dt / \Sigma]$$

Assume that the program commences at $w(t) = w^*$, and rebalancing to w^* occurs at times $\{t + \delta t, t + 2\delta t, \dots\}$. δt (measured in years) is the rebalancing interval. It follows that

$$(C2) \qquad L(t) = s^2 (E[\int_{t=t}^{t+d} e^{-r(t-t)}(w(t) - w^*)^2 \, dt + \int_{t=t+d}^{t+2d} e^{-r(t-t)}(w(t) - w^*)^2 \, dt + \dots / w(t) = w^*, w(t + d) = w^*, \dots])$$

From time homogeneity, we may write (C2) as

$$L(t) = s^2 \{ E[\int_{t=t}^{t+d} e^{-r(t-t)}(w(t)-w^*)^2 dt \mid w(t)=w^*] +$$

(C3)

$$e^{-rd} E[\int_{t=t}^{t+d} e^{-r(t-t)}(w(t)-w^*)^2 dt \mid w(t)=w^*] +$$

$$e^{-2rd} E[\int_{t=t}^{t+d} e^{-r(t-t)}(w(t)-w^*)^2 dt \mid w(t)=w^*] + ...\}$$

or using the formula for the sum of an infinite series,

$$L(t) = \frac{s^2 (E[\int_{t=t}^{t+d} e^{-r(t-t)}(w(t)-w^*)^2 dt \mid w(t)=w^*])}{(1-e^{-rd})}$$

From (3), $w(\tau - t)/w^*$ is lognormal with mean $e^{a(\tau-t)}$ and variance $e^{2a(\tau-t)}(e^{Q(\tau-t)} - 1)$. Therefore

(C4) $$Z \equiv E[\int_{t=t}^{t+d} e^{-r(t-t)}(w(t)-w^*)^2 dt \mid w(t)=w^*]$$

$$= w^{*2} (\int_{t=t}^{t+d} e^{-r(t-t)}(e^{(2a+Q)(t-t)} - 2e^{a(t-t)} + 1)dt)$$

$$= w^{*2} (\frac{2}{h_1}(1-e^{h_1 d}) - \frac{1}{h_2}(1-e^{h_2 d}) + \frac{1}{r}(1-e^{-rd}))$$

where
$$h_1 = a - r; \quad h_2 = 2a + Q - r.$$

Substituting this into (C3) gives

(C5) $$L(t) = s^2 w^{*2} (\frac{1}{1-e^{-rd}})Z$$

C(B). Trading Required for Periodic Rebalancing

Trading will occur at the end of each rebalancing period, i.e. at $t + \delta t$, $t + 2\delta t$, etc., and will restore the risky asset proportion to w^*. The expected cost of trading (normalized by wealth W) at the end of each

period will equal $ET = kE[|w(t + \delta t) - w^*|]$, the expected absolute value of the change in w over the rebalancing period times the cost of trading k. The present value of the first period trading will be $e^{-r\delta t}ET$, and the capitalized value of the trading cost will be $T = e^{-r\delta t}ET/(1 - e^{r\delta t})$. Annualized trading cost will be rT.

Thus we must derive the expected absolute move of $w(\tau) - w^*$ at the end of the rebalancing period δt. It can be shown that[25]

(C6) $E[|w(t + \delta t) - w^*|] = w^* (N(-z_1) - N(z_1) + e^{a\delta t}(N(z_2) - N(-z_2)))$,

where $N(\bullet?)$ is the cumulative uniform normal distribution, and

$$z_1 \quad = \quad (a - .5Q)\delta t/(Q\delta t)^{1/2}$$

$$z_2 \quad = \quad z_1 + (Q\delta t)^{1/2}.$$

The discounted expected trading costs T of rebalancing at intervals δt is therefore

(C7) $T = ke^{-r\delta t} (w^* (N(-z_1) - N(z_1) + e^{a\delta t}(N(z_2) - N(-z_2)))/(1 - e^{-r\delta t})$,

and the expected annualized (one-way) turnover is given by

(C8) $Turnover = rT/k$.

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[25] The author thanks Peter DeMarzo and Hui Ou-Yang for this result.

APPENDIX D: CALCULATING THE QUASI-OPTIMAL BOUNDARY GIVEN JA(x; B)

Given the function JA(x; B), the algorithm works as follows. For a given (arbitrarily small) step size γ_i, consider the vector of values $M_i = (-k_i, -k_i+\gamma_i, -k_i+2\gamma_i,..., +k_i)$, $i = 1,...,N$.

Define the product space

$$M_{-j} = \prod_{i \neq j}^{N} M_i$$

and let m_{-j} denote an arbitrary point in M_{-j}. The ith component $(i \neq j)$ of the vector m_{-j} will be denoted $m_{i, -j}$. Now choose an arbitrary j, $1 \leq j \leq N$. Solve the following equations for x:

(D.1) $JA_j(x; B) = k_j$

$JA_i(x; B) = m_{i, -j},$ $i = 1, ..., N, \ i \neq j.$

As JA is strictly convex for x, there will be a unique solution $x = w(k_j, m_{-j})$ for each possible $m_{-j} \ \varepsilon \ M_{-j}$. There will be another unique solution $w(-k_j, m_{-j})$ when $JA_j(x; B) = -k_j$. The boundary set β will then be given by

$$\beta = \{w^\beta = w(\pm k_j, m_{-j}) | m_{-j} \ \varepsilon \ M_{-j}, j = 1,...,N\}.$$

Note that for all $w^\beta \ \varepsilon \ \beta$, the conditions

(D.2) $|JA_i(w^\beta; B)| \leq k_i,$ $i = 1, ..., N$

will be satisfied, with equality holding for at least one i. Also note that as the steps $\{\gamma_i\}$ become small,

the relative number of points w^β in β for which (D.2) holds with equality for $i \neq j$ will become equally

small. In the limit, almost surely conditions (D.2) will hold with strict inequality for $i \neq j$. From

Appendix A, this in turn implies that only asset j will be traded in moving to the boundary. We conclude

that once we are inside or on the boundary, the quasi-optimal trading strategy will almost surely require

trading no more than one risky asset at any moment. Which asset is traded will vary stochastically

through time.

The second technique for determining the boundary replaces equations (D.1) with

(D.3)
$$JA_{jj}(x; B) = 0,$$
$$JA_i(x; B) = m_{i,-j}, \qquad\qquad i = 1, ..., N, \ \ i \neq j.$$

There will be two values of x satisfying (D.3) for each point m_{-j}, corresponding to opposite sides of the

no-trade boundary. The solution set $\beta^* = \{w^{\beta^*}\}$ satisfying (D.3) for all $\{i, j\}$ will be slightly different

than the solution set $\beta = \{w^\beta\}$ satisfying (D.1) except at the points B. However, examples show that the

"patched in" boundaries will be virtually identical. Conditions (D.3) are used to determine the no-trade

boundary in our examples.

Notes: Multbst0 and Multbst2 are basic Mathematica programs, Multbst3 and 5 add intermediate pts.
Figure A.1 is drawn with FIGa-1. This is best.
Bal1dim5 does the analysis of the one-dimensional case.

Side as well as corner point (K = 12) Mathematica programs are given in series "Multitc-try-"

The original solution K = 8 can be found in Multibest5 and rho-7.
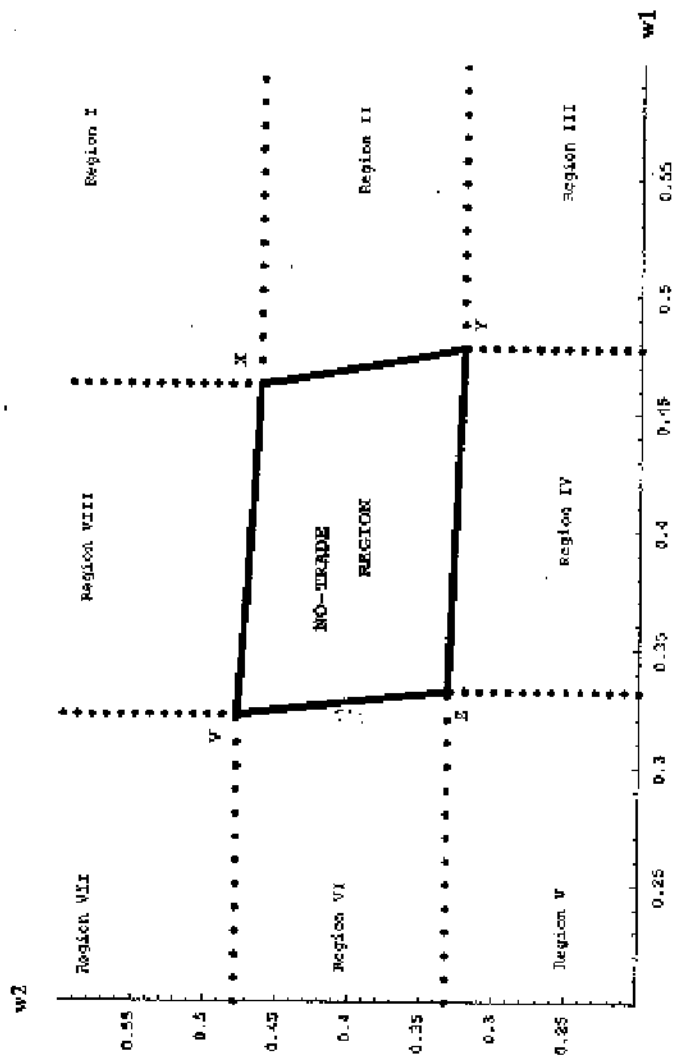
FIGURE 1: Corr. = 0.2, k1 = .01, k2 = .01
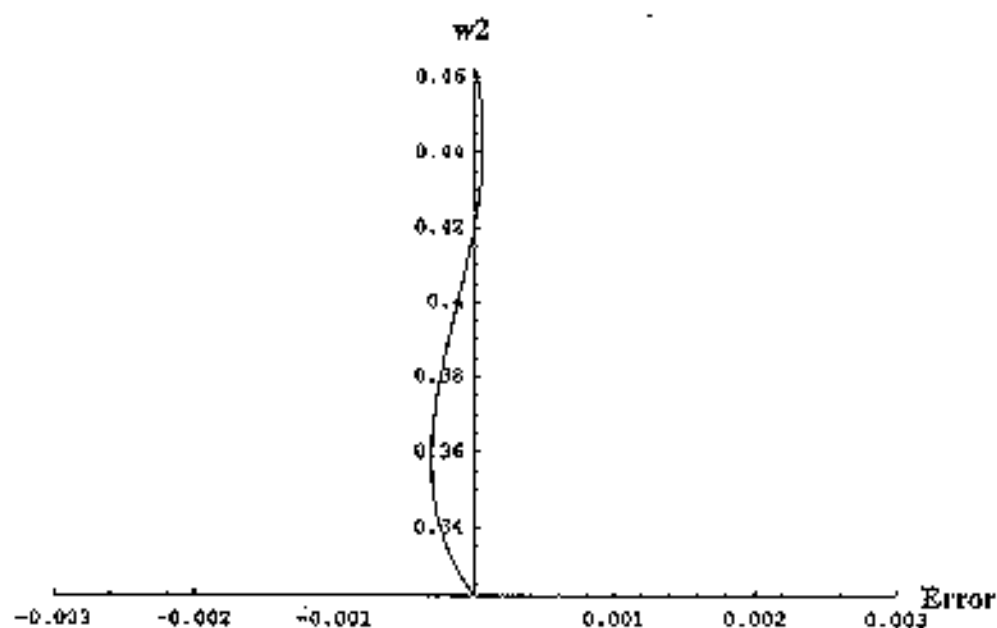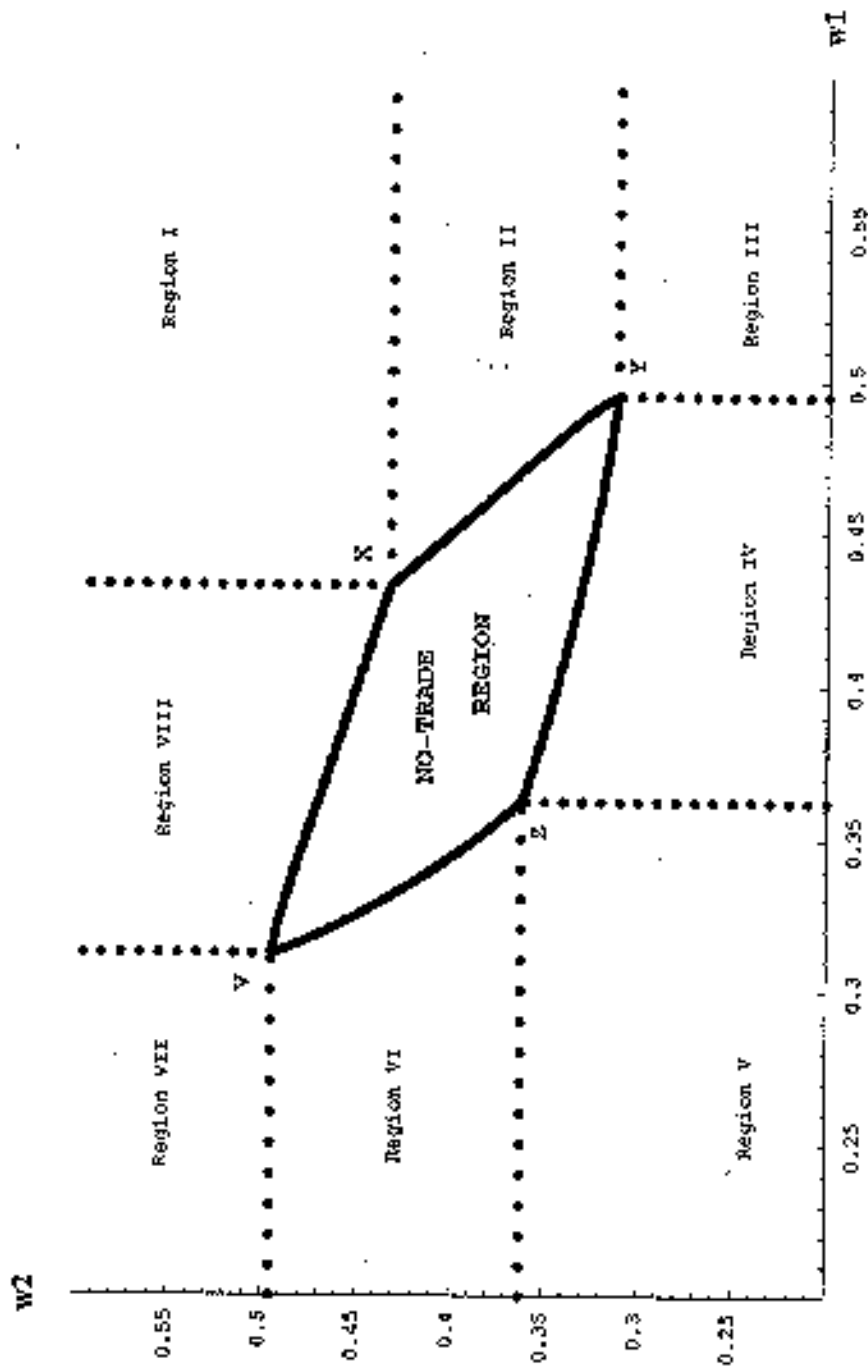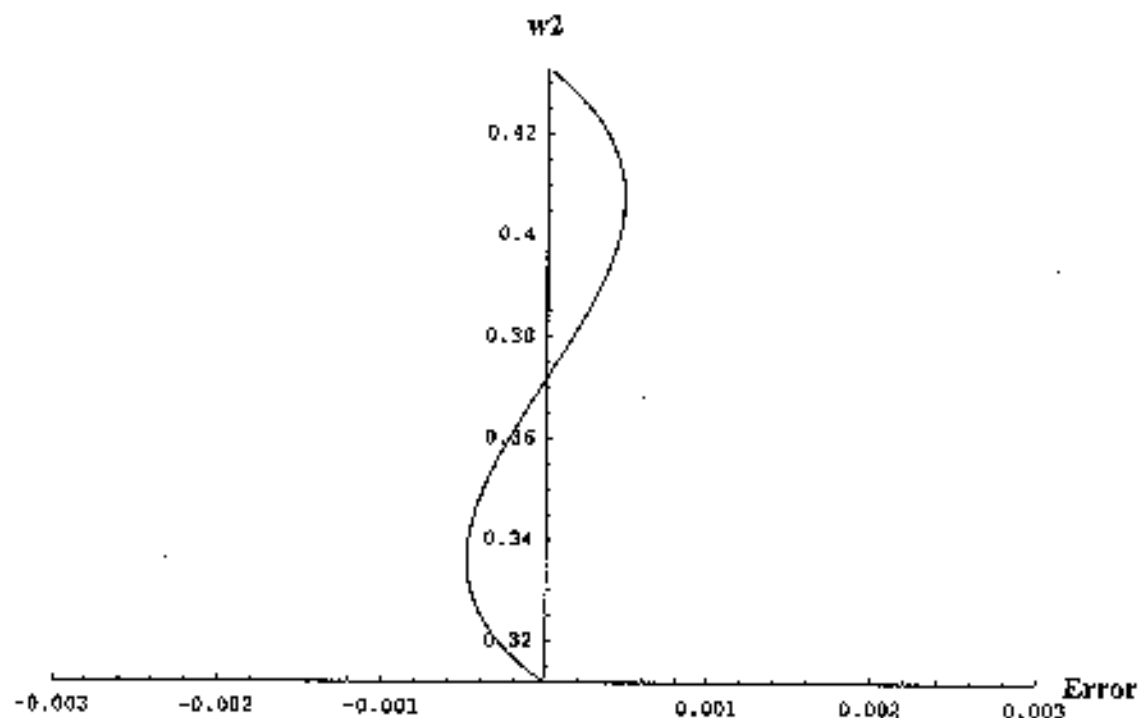
FIGURE 1A:
Error (JA1 − k1) Along Segment XY

FIGURE 1B:
Error (JA2 + k2) Along Segment ZY
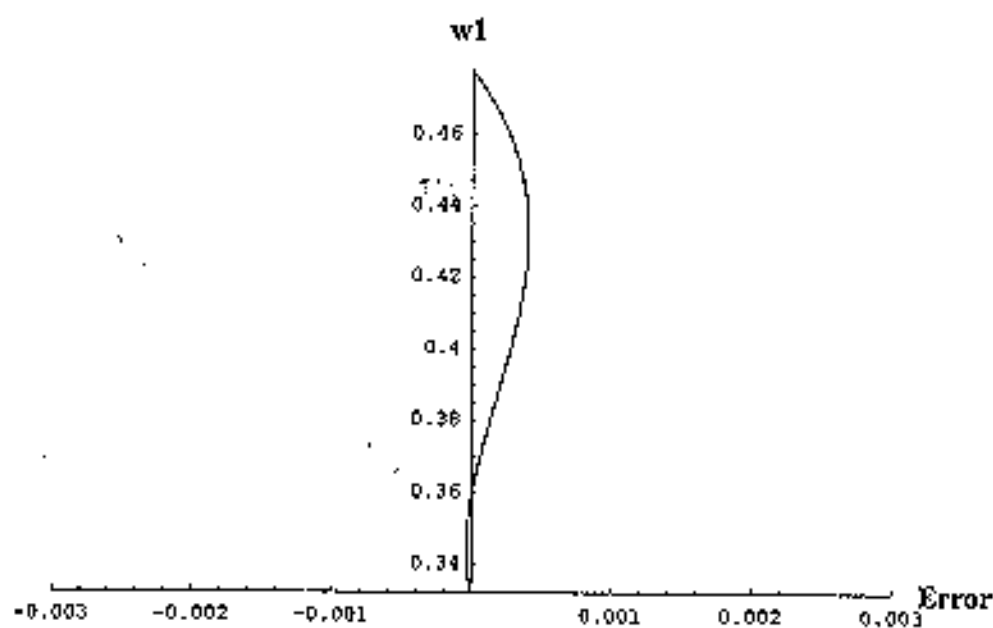
FIGURE 2: Corr. = 0.7, k1 = .01, k2 = .01
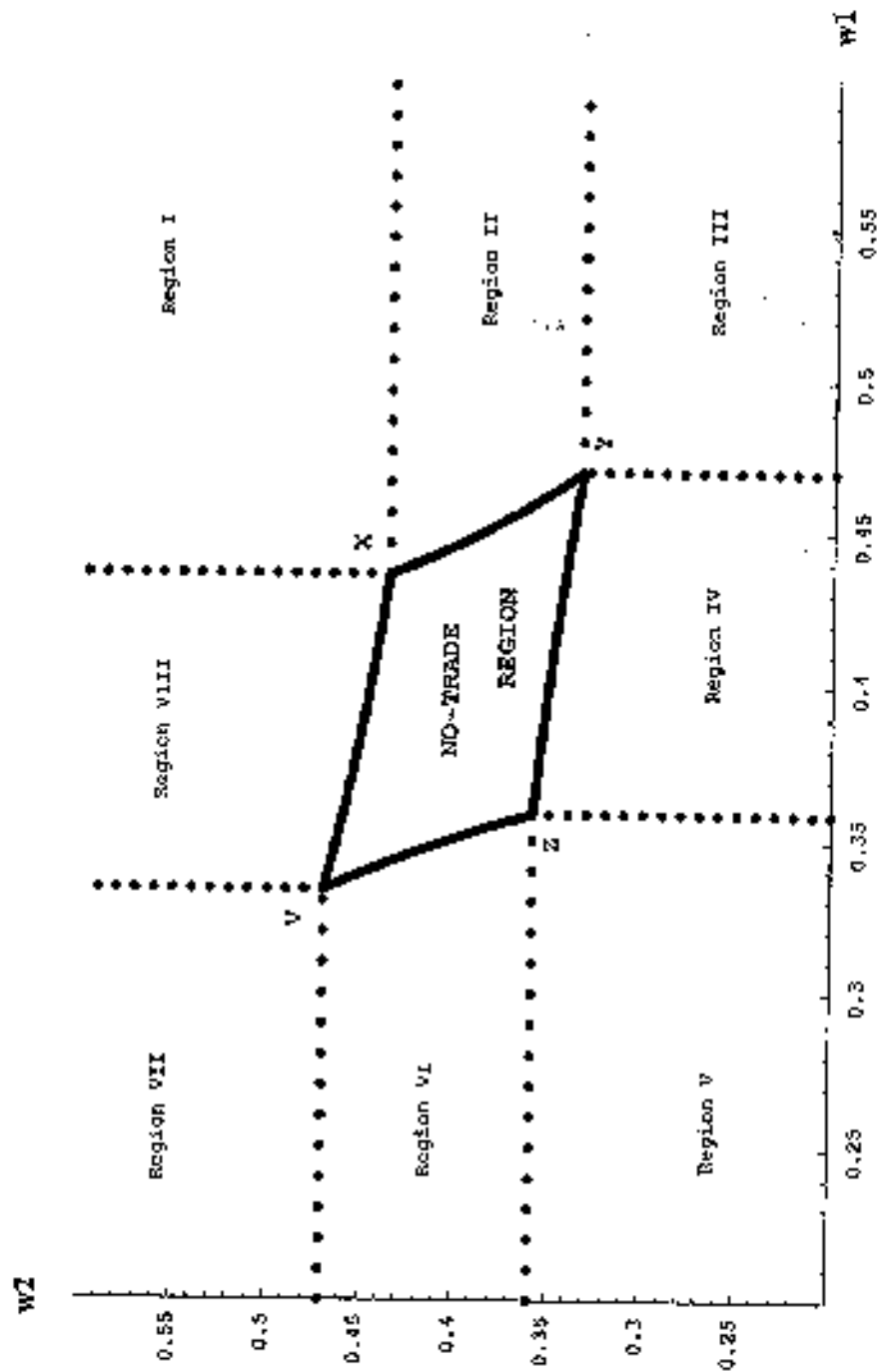
FIGURE 2A:
Error (JA1 − k1) Along Segment XY
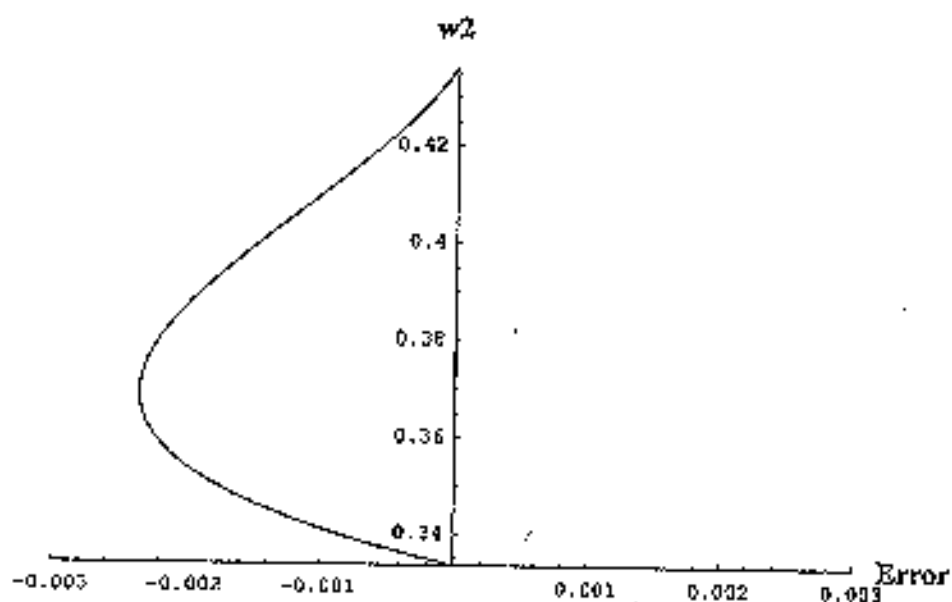
FIGURE 2B:
Error (JA2 + k2) Along Segment ZY
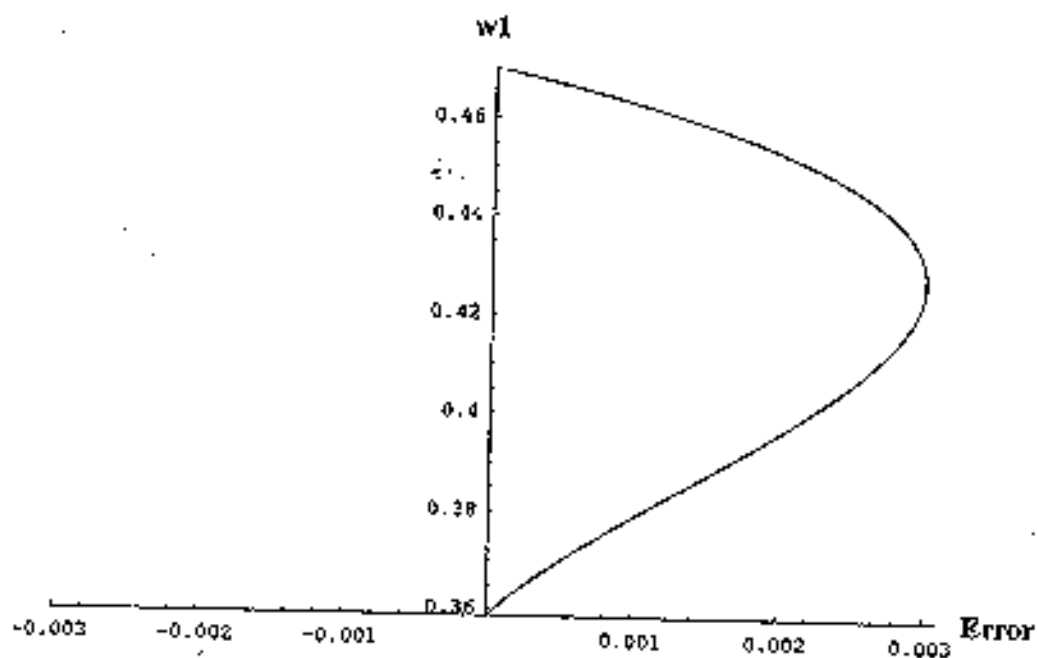
FIGURE 2(I): Corr. = 0.7, k1 = .01, k2 = .01

FIGURE 2A(1):
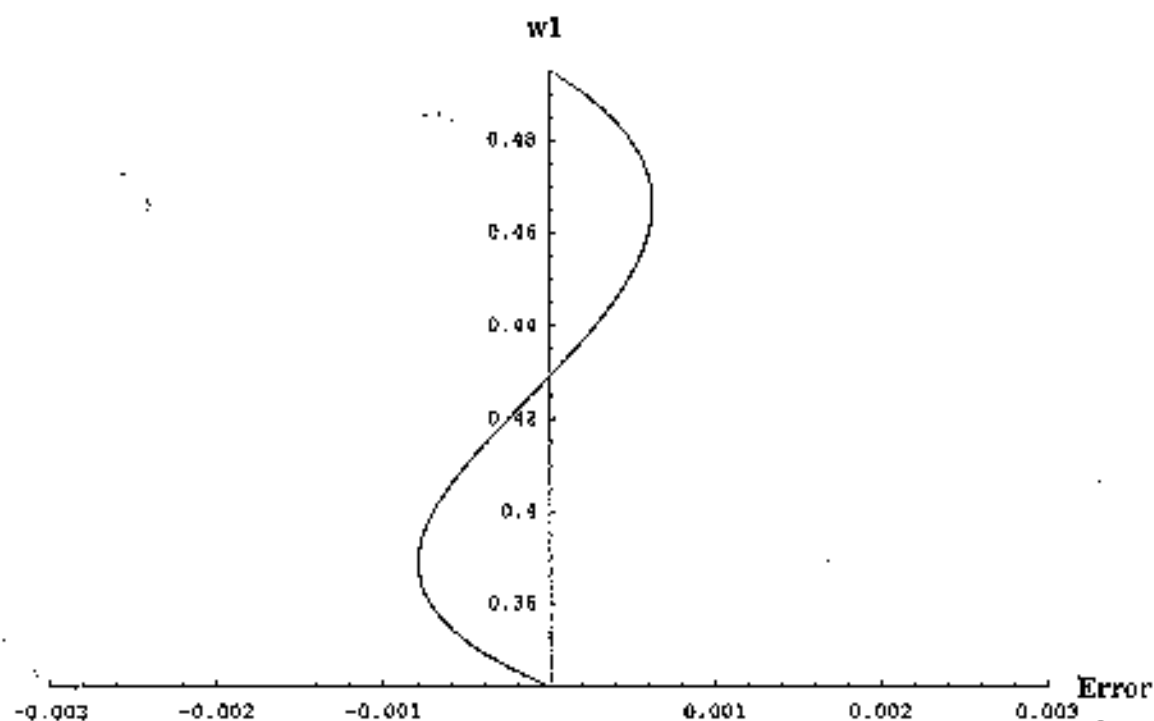## Error (JA1 − k1) Along Segment XY

FIGURE 2B(1):
Error (JA2 + k2) Along Segment ZY

FIGURE 3: Corr. = 0.7, k1 = .01, k2 = .001