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Geometric links between exceptional root lattices and the cohomology of theta divisors

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Geometric links between exceptional root lattices and the cohomology of theta divisors

A dissertation submitted in partial satisfaction of the requirements for the degree Doctor of Philosophy

in

Mathematics

by

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The dissertation of Jonathan Conder is approved, and it is acceptable in quality and form for publication on microfilm and electronically:

Chair

University of California San Diego

2019

Dedication

This dissertation is dedicated to my wife, and best friend, Jennifer.

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List of symbols

$\mathbf{1}$	identity morphism or matrix
$\mathbf{N}$	natural numbers (including zero)
$\mathbf{Z}$	integers
$\mathbf{Q}$	rational numbers
$\mathbf{R}$	real numbers
$\mathbf{C}$	complex numbers
$\mathbf{A}^d$	complex affine space of dimension $d \in \mathbf{N}$
$\mathbf{P}^d$	complex projective space of dimension $d \in \mathbf{N}$
M^i_j	entry in the ith row and jth column of the matrix M
$\binom{z}{k}$	binomial coefficient for $z \in \mathbf{C}$ and $k \in \mathbf{N}$ [GKP94, (5.1)]
$\binom{S}{k}$	subsets of S which have $k \in \mathbf{N}$ elements
$\left\langle {n \atop k} \right\rangle$	eulerian number [GKP94, (6.38)]
$[f(t)]_{t^k}$	coefficient of t^k in the Maclaurin series for the C^∞ function f
$\langle -, - \rangle$	bilinear form (typically an intersection pairing)
$L^\vee$	dual of the lattice L
$L(k)$	lattice obtained from L after multiplying the form by $k \in \mathbf{Z}$
F_x	stalk of the sheaf F at the point x
$F\|_x$	fibre of the coherent sheaf F at the point x
T_X	tangent sheaf of the space X
$T_x X$	Zariski tangent space of X at x (caution: $T_x X \neq T_X\|_x$ in general)
Ω_X	cotangent sheaf of the space X
Ω^p_X	pth exterior power of Ω_X
$N_{Z/X}$	normal sheaf of the subspace $Z \subseteq X$
$H^k(X, F)$	cohomology of the sheaf F on the space X
$H^k(X, R)$	singular cohomology of the space X with coefficients in R
$H^k_{\mathrm{pr}}(X, R)$	primitive cohomology of the compact Kähler manifold X [Voi02, §6.2]
$H^k(X, Z, R)$	relative singular cohomology of the pair (X, Z) with coefficients in R
$H_k(X, R)$	singular homology of the space X with coefficients in R
$H_k(X, Z, R)$	relative singular homology of the pair (X, Z) with coefficients in R
$h^k(F)$	rank of $H^k(X, F)$ for the sheaf F on a space X

$h^k(X)$	rank of $H^k(X, \mathbf{Q})$ for the space X	
$\chi(X)$	Euler characteristic of the sheaf or space X	
$\mathrm{CH}_d(X)$	Chow group of d-dimensional cycle classes on the scheme X	
$\mathrm{Pic}(X)$	Picard group of the space X	
$\mathrm{Pic}^0(X)$	identity component of $\mathrm{Pic}(X)$	
$\mathrm{Pic}^d(X)$	degree d component of $\mathrm{Pic}(X)$ for the curve X	
$W^r_d(X)$	rth Brill-Noether locus in $\mathrm{Pic}^d(X)$ [ACGH85, IV]	
g^r_d	linear system of dimension r and degree d	
$G^r_d(X)$	set of every g^r_d on the curve X [ACGH85, IV]	
$\mathcal{G}^r_d(\mathcal{X})$	relative version of $G^r_d(X)$ for a family of curves $\mathcal{X}$ [ACG11, XXI]	
$X^{(d)}$	dth symmetric power of the space X	
$\mathrm{c}(F)$	total Chern class of the coherent sheaf F	
$\mathrm{c}_k(F)$	kth Chern class of the coherent sheaf F	
$\mathrm{ch}(F)$	Chern character of the coherent sheaf F	
$\mathrm{td}(F)$	Todd class of the coherent sheaf F	
$\mathrm{td}(X)$	$\mathrm{td}(T_X)$ for the space X	
Hom	sheaf Hom	
Ext	sheaf Ext	
$\mathcal{M}_g$	moduli space of smooth curves of genus g	
$\overline{\mathcal{M}}_g$	moduli space of stable curves of genus g	
$\mathcal{A}_g$	moduli space of principally polarised abelian varieties of dimension g	
$\overline{\mathcal{R}}_g$	moduli space of allowable double covers $\widetilde{X} \to X$ with $X \in \overline{\mathcal{M}}_g$	
A	abelian variety.	4
g	dimension of A, typically five.	4
Θ	symmetric theta divisor of A.	4
Θ_α	$\Theta + \alpha$ for $\alpha \in A$	
ι	embedding $\Theta \hookrightarrow A$.	4
K	primal cohomology of Θ.	4
σ	negation on A and Θ.	4
G	cyclic group $\langle \sigma \rangle$.	4
$L^\pm$	± 1-eigenmodules for the $\mathbf{Z}[G]$-module L.	4
X^+	orbit space of the G-space X	
π	quotient map $X \to X^+$ for a G-space X	
$\widetilde{\Theta}$	blowup of Θ at its 2-torsion points.	4
β	projection $\widetilde{\Theta} \to \Theta$.	4

Δ	exceptional divisor of $\widetilde{\Theta}$.	4
θ	class of Θ in A.	4
$\widetilde{X}$	étale double cover of the curve X.	18
π	covering projection $\widetilde{X} \to X$ for a curve X.	18
σ	covering involution of $\widetilde{X}$ for a curve X.	18
x'	image of a point $x \in \widetilde{X}$ under σ.	18
$\mathcal{P}(X)$	Prym variety of the cover $\widetilde{X} \to X$.	18
$\mathcal{P}^-(X)$	degree zero Prym torsor of the cover $\widetilde{X} \to X$.	19
ι	canonical embedding $\widetilde{X} \hookrightarrow \mathcal{P}^-(X)$ for a good cover $\widetilde{X} \to X$.	19
$[x, y]$	$\iota(x) + \iota(y)$ for $x, y \in \widetilde{X}$.	19
$\varkappa$	a Prym-embedding $\widetilde{X} \hookrightarrow \mathcal{P}(X)$ for a good cover $\widetilde{X} \to X$.	19
$\mathcal{P}_\omega(X)$	even semicanonical Prym torsor of the cover $\widetilde{X} \to X$.	19
$\mathcal{P}_\omega^-(X)$	odd semicanonical Prym torsor of the cover $\widetilde{X} \to X$.	19
$\mathcal{P}_\omega^r(X)$	rth Brill-Noether locus in $\mathcal{P}_\omega(X)$ or $\mathcal{P}_\omega^-(X)$.	19
$\Theta_\omega(X)$	canonical theta divisor of $\mathcal{P}_\omega(X)$.	19
W_x	$\mathcal{P}_\omega^2(X) + \iota(x)$ for a cover $\widetilde{X} \to X$.	20
λ	involution on the fibres of $\mathcal{P}\colon \overline{\mathcal{R}}_5 \to \mathcal{A}_4$.	20
X_λ	$\lambda(X)$ for $X \in \overline{\mathcal{R}}_5$.	20
S_X	$\mathcal{P}_\omega^2(X)$ for $X \in \overline{\mathcal{R}}_6$.	20
V_D	special subvariety of $\Theta_\omega(X)$ associated to $D \in \widetilde{X}^{(d)}$ for a cover $\widetilde{X} \to X$.	20
$\overset{\diamond}{\sim}$	the tetragonal relation.	21
B	abelian variety, typically of dimension four.	30
Ξ	theta divisor of B.	30
Ξ_α	$\Xi + \alpha$ for $\alpha \in B$	
$\mathbf{W}$	primal cohomology of $\Xi \cap \Xi_\alpha$ for some $\alpha \in B$.	31
R	Enriques surface.	34
$\widetilde{R}$	K3 double cover of R.	34
ρ	covering projection $\widetilde{R} \to R$.	34
H	very ample line bundle on R.	34
$\widetilde{H}$	$\rho^* H$.	34
$M_{\widetilde{H}}$	moduli space of sheaves on $\widetilde{R}$.	34
T	general pencil in $\lvert H \rvert$.	35
$\mathcal{C}$	family of curves over T.	35
$\widetilde{\mathcal{C}}$	family of covers of $\mathcal{C} \to T$.	35
C	curve with one node, corresponds to $0 \in T$.	35

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Abstract of the dissertation

Geometric links between exceptional root lattices and the cohomology of theta divisors

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Given a general abelian fivefold A and a symmetric principal polarisation $\Theta \subset A$, the primal cohomology of Θ is the part which is not inherited from A. We compute numerical invariants of the primal cohomology lattice, and construct surfaces inside Θ whose classes span the subspace fixed by $-\mathbf{1}$ (with rational coefficients). This gives a constructive proof of the rational Hodge conjecture for Θ. The sublattice generated by the surfaces is (up to a factor of 2) isometric to the root lattice E_6.

Introduction

A complex abelian variety A is a compact complex algebraic group, or equivalently a complex torus which admits an embedding in some projective space. Such embeddings are determined by an ample line bundle L on A. When L has a unique nonzero global section up to scaling, we say that A is principally polarised by L (or, to be precise, by $\mathrm{c}_1(L) \in H^2(A, \mathbf{Z})$). This section vanishes on a codimension one subscheme $\Theta \subset A$ called a theta divisor. See [BL04] and [Mum08] for more information about (polarised) abelian varieties.

Suppose that A has dimension $g \geq 3$ and that Θ is a nonsingular variety. The singular cohomology of Θ is inherited from A, with the exception of the middle piece $H^{g-1}(\Theta, \mathbf{Z})$, which has an extra sub-Hodge structure $\mathbf{K}$, called the primal cohomology of Θ. From the point of view of Hodge theory, $\mathbf{K}$ is interesting because the Hodge decomposition

$$\mathbf{K} \otimes_{\mathbf{Z}} \mathbf{C} \cong \oplus_{p=1}^{g-2} \mathbf{K}^{p,g-1-p}$$

has fewer pieces than expected. Grothendieck's general version of the Hodge conjecture says that (roughly speaking) $\mathbf{K} \otimes_{\mathbf{Z}} \mathbf{Q}$ should come from the cohomology of a codimension one subscheme of Θ. This conjecture is known when $g = 3$ (by the Lefschetz $(1,1)$-theorem), $g = 4$ [IvS95] and $g = 5$ [ITW17]. In the $g = 5$ case the ordinary Hodge conjecture for $\mathbf{K} \otimes_{\mathbf{Z}} \mathbf{Q}$ follows from this (via the Lefschetz $(1,1)$-theorem), i.e., every Hodge class $\alpha \in \mathbf{K}^{2,2} \cap (\mathbf{K} \otimes_{\mathbf{Z}} \mathbf{Q})$ is algebraic. Our main result is an effective version of this statement.

Theorem 1. Let Θ be a theta divisor for a very general abelian fivefold A. There are 27 smooth surfaces $S_1, \ldots, S_{27} \subset \Theta$ whose classes generate $H^{2,2}(\Theta) \cap H^4(\Theta, \mathbf{Q})$. Moreover, given a smooth cubic $X \subset \mathbf{P}^3$, there is a bijection between the S_i and the lines on X which induces an isometry between $H^2_{\mathrm{pr}}(X, \mathbf{Z})(-2)$ and the sublattice of $\mathbf{K}$ generated by the differences $[S_i] - [S_j]$.

Each surface is defined (up to translation) as a Brill-Noether locus inside a Prym variety isomorphic to A. There are 27 different ways to construct a Prym variety isomorphic to A, and hence 27 different surfaces. Apart from smoothness, which was proven by Welters [Wel85], everything we know about these surfaces follows from the fact that they are special subvarieties of A in the sense of Beauville [Bea82]. This description is enough to deduce a formula for the intersection numbers $\langle [S_i], [S_j] \rangle$ for $i \neq j$ in terms of the self-intersections $\langle [S_k], [S_k] \rangle$.

To compute the latter, we mimic [IW19] and degenerate A to a natural compactification of a $\mathbf{C}^\times$-extension of an abelian fourfold B, polarised by $\Xi \subset B$. The limit surface turns out to be birational to $\Xi \cap \Xi_\alpha$ for some $\alpha \in B$ (recall that $\Xi_\alpha := \Xi + \alpha$). There are 27 Prym-embedded curves in $\Xi \cap \Xi_\alpha$, which were studied in [Iza95] and [Krä15]. We use properties of these curves to compute the self-intersection numbers. Once we understand how the surfaces intersect, the rest of the proof is straightforward.

We begin chapter 1 by defining $\mathbf{K}$ more carefully and calculating its Hodge numbers by an Euler characteristic argument. In the next section we compute the Hodge numbers of the invariant sublattice using a similar, but more involved, argument. Finally we determine the discriminant of $\mathbf{K}$ and some related properties, using some combinatorial tricks. It would be interesting to also work out the discriminant of the invariant sublattice, as this would tell us its isometry type. However, this seems to be substantially more difficult than it was for $\mathbf{K}$.

Chapter 2 summarises the results we will need concerning Prym varieties and their subvarieties. Very little of this material is new, but all of it is important in later chapters.

In chapter 3 we review some symmetry properties of the lines on a cubic surface. Using these, and some facts from chapter 2, we compute $\langle [S_i], [S_j] \rangle$ for $i \neq j$ in terms of the self-intersection numbers. After that, a proof for theorem 1 is given under the assumption that $\langle [S_k], [S_k] \rangle = 16$ for all k. The last section proves a stronger result for the Prym-embedded curves in $\Xi \cap \Xi_\alpha$, since their self-intersection numbers were already known. Apart from the self-intersection numbers, nothing in the final section is needed later on.

Chapter 4 constructs families of abelian varieties and surfaces, in order to set up the degeneration argument. The first section uses the relative Prym construction of [AFS15] to define the family of abelian varieties and describe the limit theta divisor. Having this modular description of the families is essential when it comes to proving that the total space of theta divisors is smooth. The following section defines the family of surfaces, making several base changes in order to canonically embed it in the family of theta divisors. This has to be done with care, so that the total space of theta divisors remains smooth after each base change.

In chapter 5, the self-intersection numbers are finally computed, using intersection theory to move the calculation to a resolution of the limit theta divisor. The key is to express the classes of surfaces in the resolution in terms of Prym-embedded curves in $\Xi \cap \Xi_\alpha$, which are better understood. The first section builds up to this by working out a description of the limit surfaces, using a Hilbert polynomial argument to rule out any extra components.

The appendix consists of known results for which we could not find a reference, and the long part of the Hilbert polynomial calculation, which is needed in chapters 4 and 5.

Chapter 1: The primal cohomology lattice

(1.0.1) Let A be a principally polarised abelian variety (ppav) of dimension $g \geq 3$, with a smooth symmetric theta divisor $\Theta \overset{\iota}{\hookrightarrow} A$. The *primal* part of $H^*(\Theta, \mathbf{Z})$ is

$$\mathbf{K} \coloneqq \operatorname{Ker}\big(\iota_* : H^{g-1}(\Theta, \mathbf{Z}) \to H^{g+1}(A, \mathbf{Z})\big).$$

Since Θ is symmetric, the negation map $\sigma \coloneqq -\mathbf{1}_A$ induces an involution of Θ, which we also call σ. The cyclic group $G \coloneqq \langle \sigma \rangle$ acts on $H^*(\Theta, \mathbf{Z})$ and $H_*(\Theta, \mathbf{Z})$ via σ^* and σ_*, and the Poincaré duality isomorphism is G-equivariant. For any $\mathbf{Z}[G]$-module L, let $L^{\pm} \subseteq L$ be the submodule on which σ acts as ± 1. Our goal is to compute the numerical properties of $\mathbf{K}$ and $\mathbf{K}^+$.

(1.0.2) In order to say anything about $\mathbf{K}^+$, we want to view $H^{g-1}(\Theta, \mathbf{Q})^+$ as the cohomology of the quotient $\Theta^+ \coloneqq \Theta/G$. This leads to technical difficulties because Θ^+ is singular. To work around this, let $\widetilde{\Theta} \overset{\beta}{\longrightarrow} \Theta$ be the blowup of Θ at its 2-torsion points. The action of G lifts to $\widetilde{\Theta}$ and is free away from the exceptional divisor $\Delta \subset \widetilde{\Theta}$, on which it acts trivially. Since Δ is Cartier, the quotient $\widetilde{\Theta}^+ \coloneqq \widetilde{\Theta}/G$ is nonsingular [Wat76, (2.13)]. Moreover, the quotient map $\pi : \widetilde{\Theta} \to \widetilde{\Theta}^+$ induces an isomorphism $H^*\big(\widetilde{\Theta}^+, \mathbf{Q}\big) \xrightarrow{\sim} H^*\big(\widetilde{\Theta}, \mathbf{Q}\big)^+$ [Mac62, (1.2)].

(1.0.3) Throughout this chapter $\theta \in H^2(A, \mathbf{Z})$ is the class of $\Theta \subset A$.

1.1. Primal Hodge numbers

(1.1.1) **Lemma**. If $k < g-1$, then $\iota^* : H^k(A, \mathbf{Z}) \to H^k(\Theta, \mathbf{Z})$ is an isomorphism. When $k = g-1$, ι^* is injective and $H^k(\Theta, \mathbf{Z})$ is torsion-free. Moreover, if $k > g-1$ then the Gysin map $\iota_* : H^k(\Theta, \mathbf{Z}) \to H^{k+2}(A, \mathbf{Z})$ is an isomorphism. It is surjective when $k = g-1$.

Proof. The statements about ι^* and ι_* follow from the Lefschetz hyperplane theorem [Laz04, Theorem 3.1.17]. By the universal coefficient theorem and Poincaré duality, the torsion part of $H^{g-1}(\Theta, \mathbf{Z})$ comes from

$$H_{g-2}(\Theta, \mathbf{Z}) \cong H^g(\Theta, \mathbf{Z}) \cong H^{g+2}(A, \mathbf{Z}),$$

which is torsion-free. ☺

(1.1.2) **Proposition**. The rank of $\mathbf{K}$ is $g! - \frac{1}{g+1}\binom{2g}{g}$, and its Hodge numbers are

$$h^{p,g-1-p}(\mathbf{K}) = \left\langle {g \atop p} \right\rangle - \binom{g}{p}\binom{g-1}{p} + \binom{g}{p+1}\binom{g-1}{p-1},$$

where $\left\langle {g \atop p} \right\rangle := \sum_{k=0}^{p} \binom{g+1}{k}(-1)^k(p+1-k)^g$ is an eulerian number (see [GKP94, (6.38)]).

Proof. The Hodge number $h^{p,g-1-p}(\mathbf{K})$ appears in

$$\begin{aligned}\chi(\Omega_\Theta^p) &= \sum_{k=0}^{g-1}(-1)^k h^{p,k}(\Theta) \\ &= \sum_{k=0}^{g-1-p}(-1)^k h^{p,k}(A) + (-1)^{g-1-p}h^{p,g-1-p}(\mathbf{K}) + \sum_{k=g-p}^{g-1}(-1)^k h^{p+1,k+1}(A).\end{aligned}$$

Using a standard identity [GKP94, (5.16)], the first sum simplifies to

$$\sum_{k=0}^{g-1-p}(-1)^k\binom{g}{p}\binom{g}{k} = (-1)^{g-1-p}\binom{g}{p}\binom{g-1}{p},$$

and similarly $\sum_{k=g-p}^{g-1}(-1)^k h^{p+1,k+1}(A) = (-1)^{g-p}\binom{g}{p+1}\binom{g-1}{p-1}$. Next we will compute $\chi(\Omega_\Theta^p)$. The conormal bundle sequence for $\Theta \subset A$ induces exact sequences

$$0 \to \Omega_\Theta^{p-1}(-\Theta) \to \Omega_A^p\big|_\Theta \to \Omega_\Theta^p \to 0$$

for all $p \in \{1, \dots, g\}$, so by induction (and the triviality of Ω_A)

$$\mathrm{ch}(\Omega_\Theta^p) = \sum_{k=0}^{p}\binom{g}{k}\left(-e^{-\theta}\right)^{p-k}\Big|_\Theta.$$

Moreover $\mathrm{td}(\Theta) = \mathrm{td}(\mathcal{O}_\Theta(\Theta))^{-1} = \frac{1-e^{-\theta}}{\theta}\Big|_\Theta$. It follows that

$$\chi(\Omega_\Theta^p) = \int_\Theta \mathrm{ch}(\Omega_\Theta^p)\,\mathrm{td}(\Theta)$$

$$
\begin{aligned}
&= \int_A \sum_{k=0}^{p} \binom{g}{k} (-e^{-\theta})^{p-k} (1-e^{-\theta}) \\
&= \int_A \left(\sum_{k=0}^{p} \binom{g+1}{k} (-e^{-\theta})^{p+1-k} + \binom{g}{p} \right) \\
&= \sum_{k=0}^{p} \binom{g+1}{k} (-1)^{p+1-k} (k-p-1)^g,
\end{aligned}
$$

since $\int_A \frac{\theta^g}{g!} = \chi(\mathcal{O}_A(\Theta)) = 1$ [BL04, Theorem 3.6.1]. Putting everything together gives the result.

The rank of **K** can be computed directly from these Hodge numbers (using [GKP94, Table 169]), or by a similar argument using the fact that $\chi(\Theta) = \int_\Theta \mathrm{c}(\Theta)$. ☺

1.2. Invariant Hodge numbers

(1.2.1) The methods of the previous section can also be used to compute the Hodge numbers of $\mathbf{K}^+$. In order to do so, we first establish the following results, with the goal of understanding $\Omega_{\widetilde{\Theta}^+}$ and its exterior powers.

(1.2.2) **Lemma**. Let X be a nonsingular variety, and $C \subset X$ a smooth subscheme. If $\widetilde{X} \xrightarrow{\beta} X$ is the blowup of X along C, with $E \subset \widetilde{X}$ the exceptional divisor, then there is an exact sequence

$$0 \to \beta^* \Omega_X \to \Omega_{\widetilde{X}} \to \Omega_{E/C} \to 0. \tag{1.2.3}$$

Proof. The kernel of $\beta^*\Omega_X \to \Omega_{\widetilde{X}}$ is supported on E, but $\beta^*\Omega_X$ is torsion-free, so (1.2.3) is exact on the left. To get exactness in the other two spots, we need to show that

$$\Omega_{\widetilde{X}/X} \to \Omega_{\widetilde{X}/X}\big|_E \xrightarrow{\sim} \Omega_{E/C}$$

is an isomorphism, or equivalently that $\Omega_{\widetilde{X}/X}(-E) \to \Omega_{\widetilde{X}/X}$ is zero.

This is a straightforward but somewhat technical check. Let $\mathrm{Spec}(A) \subseteq X$ be an open affine and $I \subseteq A$ the ideal of C. By definition $\beta^{-1}(\mathrm{Spec}(A)) = \mathrm{Proj}(B)$, where B is the Rees algebra of I. Thus $\widetilde{X}$ is covered by open affines of the form $\mathrm{Spec}\big(B_{(a)}\big)$, where $a \in B_1 = I$ and $B_{(a)}$ is the homogeneous localisation of B with respect to a. On $\mathrm{Spec}\big(B_{(a)}\big)$ the homomorphism $\Omega_{\widetilde{X}/X}(-E) \to \Omega_{\widetilde{X}/X}$ corresponds to the derivation

$$B_{(a)} \xrightarrow{\mathrm{d}} \Omega_{B_{(a)}/A} \xrightarrow{a\cdot -} \Omega_{B_{(a)}/A}.$$

Since $B_{(a)}$ is generated by $\frac{1}{a}B_1$ over A, and

$$a\mathrm{d}\left(\frac{b}{a}\right) = a\mathrm{d}\left(\frac{b}{a}\right) + \frac{b}{a}\mathrm{d}(a) = \mathrm{d}\left(a\frac{b}{a}\right) = \mathrm{d}(b) = 0$$

for all $b \in I$, the Leibniz rule implies that $\Omega_{\widetilde{X}/X}(-E) \to \Omega_{\widetilde{X}/X}$ is zero on $\mathrm{Spec}\big(B_{(a)}\big)$. ☺

(1.2.4) **Corollary**. In the setting of (1.2.2), there is an exact sequence

$$0 \to T_{\widetilde{X}} \to \beta^* T_X \to T_{E/C}(E) \to 0$$

Proof. See [Ful98, Lemma 15.4]. ☺

(1.2.5) A more general version of the following result is unfortunately somewhat harder to state. Since we only need it for the case $X := \Theta$, with C the 2-torsion points, we decided to simplify the discussion.

(1.2.6) **Corollary**. In the setting of (1.2.2), if $\dim(C) = 0$, then for each $p \in \mathbf{N}$ there is an exact sequence

$$0 \to \big(\beta^*\Omega^p_X\big)((p-1)E) \to \Omega^p_{\widetilde{X}} \to \Omega^p_E \to 0. \tag{1.2.7}$$

Proof. If $p = 0$ then (1.2.7) is just the ideal sheaf sequence for E. The case $p = 1$ is (1.2.2). Now suppose $p > 1$. The injective map in (1.2.7) corresponds to a morphism $\beta^*\Omega^p_X \to \Omega^p_{\widetilde{X}}((1-p)E)$. To define one, we will show that the natural embedding $\beta^*\Omega^p_X \hookrightarrow \Omega^p_{\widetilde{X}}$ factors through the subsheaf $\Omega^p_{\widetilde{X}}((1-p)E)$. Let $n \in \mathbf{N}$ be the smallest number such that $\beta^*\Omega^p_X$ does not belong to $\Omega^p_{\widetilde{X}}(-nE)$. Since $\beta^*\Omega^p_X$ embeds in $\Omega^p_{\widetilde{X}}((1-n)E)$, the ideal sheaf sequence for E gives us the following morphism of short exact sequences:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & (\beta^*\Omega^p_X)(-E) & \longrightarrow & \beta^*\Omega^p_X & \longrightarrow & (\beta^*\Omega^p_X)\big|_E & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow{\scriptstyle \varphi_n} & & \\
0 & \longrightarrow & \Omega^p_{\widetilde{X}}(-nE) & \longrightarrow & \Omega^p_{\widetilde{X}}((1-n)E) & \longrightarrow & \Omega^p_{\widetilde{X}}((1-n)E)\big|_E & \longrightarrow & 0.
\end{array}
\tag{1.2.8}
$$

Our choice of n ensures that $\varphi_n \neq 0$. Note that $(\beta^*\Omega^p_X)\big|_E = \rho^*\big(\Omega^p_X|_C\big)$, where $\rho\colon E \to C$ is the projection. The morphism

$$\Omega^p_X|_C \to \rho_*\Big(\Omega^p_{\widetilde{X}}((1-n)E)\big|_E\Big)$$

corresponding to φ_n is also nonzero. The conormal bundle sequence for $E \hookrightarrow \widetilde{X}$ induces the following exact sequence of vector bundles on E:

$$0 \to \Omega^{p-1}_E(-E) \to \Omega^p_{\widetilde{X}}\big|_E \to \Omega^p_E \to 0. \tag{1.2.9}$$

By twisting and applying ρ_*, it follows that one of $\rho_*\big(\Omega^{p-1}_E(-nE)\big)$ or $\rho_*\big(\Omega^p_E((1-n)E)\big)$ is nonzero. One can compute the ranks of these bundles using the Bott formula [OSS11, §1.1]. For instance, if $d \coloneqq \dim(X)$ then the fibres of the former bundle are isomorphic to

$$H^0\Big(\mathbf{P}^{d-1}, \Omega^{p-1}_{\mathbf{P}^{d-1}}(n)\Big),$$

which is zero for $n < p$. Similarly, the latter bundle is zero when $n < p+2$. Therefore $n \geq p$, which gives us the required factorisation.

The Euler sequence induces the following exact sequence on $E = \mathbf{P}(N_{C/X}) = \mathbf{P}(T_X|_C)$:

$$0 \to \Omega^p_E(-pE) \to \rho^*\big(\Omega^p_X|_C\big) \to \Omega^{p-1}_E(-pE) \to 0. \tag{1.2.10}$$

Twisting (1.2.9) gives an embedding $\Omega^{p-1}_E(-pE) \hookrightarrow \Omega^p_{\widetilde{X}}((1-p)E)\big|_E$, and one can check (e.g. fibrewise) that φ_p is the composition

$$\rho^*\big(\Omega^p_X|_C\big) \twoheadrightarrow \Omega^{p-1}_E(-pE) \hookrightarrow \Omega^p_{\widetilde{X}}((1-p)E)\big|_E.$$

Set $\mathcal{C} \coloneqq \operatorname{Coker}\big(\beta^*\Omega^p_X \to \Omega^p_{\widetilde{X}}((1-p)E)\big)$. There is an exact sequence

$$0 \to \Omega^p_E(-pE) \to \mathcal{C}(-E) \to \mathcal{C} \to \Omega^p_E((1-p)E) \to 0,$$

which comes from the snake lemma applied to (1.2.8). It remains to show that $\mathcal{C}(-E) \to \mathcal{C}$ is zero, or equivalently that $\Omega^p_{\widetilde{X}}(-pE) \hookrightarrow \Omega^p_{\widetilde{X}}$ factors through $\beta^*\Omega^p_X$. This follows from the case $p = 1$, which is an easy consequence of (1.2.2). Indeed $\Omega_{\widetilde{X}} \to \Omega_E$ factors through $\Omega_{\widetilde{X}} \to \Omega_{\widetilde{X}}|_E$, so the composition $\Omega_{\widetilde{X}}(-E) \to \Omega_{\widetilde{X}} \to \Omega_E$ is zero. ☺

(1.2.11) We now switch back to our specific situation; see (1.0.2) for notation. To avoid ambiguity (as Δ is a subscheme of both $\widetilde{\Theta}$ and $\widetilde{\Theta}^+$), we set $\mathcal{O}_{\widetilde{\Theta}}(1) \coloneqq \mathcal{O}_{\widetilde{\Theta}}(-\Delta)$.

(1.2.12) **Lemma**. For each positive integer p there is an exact sequence

$$0 \to \pi^*\Omega^p_{\widetilde{\Theta}^+} \to \Omega^p_{\widetilde{\Theta}} \to \Omega^{p-1}_\Delta(1) \to 0.$$

Proof. Since $\pi^*\Omega^p_{\widetilde{\Theta}^+}$ is locally free, the kernel of $\pi^*\Omega^p_{\widetilde{\Theta}^+} \to \Omega^p_{\widetilde{\Theta}}$ (which is supported on Δ) must be zero. It remains to identify $\mathcal{C} \coloneqq \operatorname{Coker}\big(\pi^*\Omega^p_{\widetilde{\Theta}^+} \to \Omega^p_{\widetilde{\Theta}}\big)$ with $\Omega^{p-1}_\Delta(1)$. For starters

$$\mathcal{C}|_\Delta = \operatorname{Coker}\Big(\Omega^p_{\widetilde{\Theta}^+}\big|_\Delta \to \Omega^p_{\widetilde{\Theta}}\big|_\Delta\Big).$$

Note that $\Omega_{\widetilde{\Theta}^+}|_\Delta$ and $\Omega_{\widetilde{\Theta}}|_\Delta$ are extensions of Ω_Δ by the conormal bundles of $\Delta \subset \widetilde{\Theta}^+$ and $\Delta \subset \widetilde{\Theta}$, which are $\mathcal{O}_\Delta(1)$ and $\mathcal{O}_\Delta(2)$ respectively. Indeed, if $\delta_i \in \operatorname{CH}\big(\widetilde{\Theta}\big)$ is the class of a component $\Delta_i \subseteq \Delta$, then $\pi_*\delta_i$ is its class in $\widetilde{\Theta}^+$, and since $\pi^{-1}(\Delta_i) = \Delta_i$ set-theoretically

$$(\pi_*\delta_i)|_{\Delta_i} = (\pi^*\pi_*\delta_i)|_{\Delta_i} = (n\delta_i)|_{\Delta_i}$$

for some $n \in \mathbf{N}$. The projection formula says that $\pi_*\pi^* = 2$, so $n = 2$.

Since $\Omega_{\widetilde{\Theta}^+}|_\Delta \to \Omega_{\widetilde{\Theta}}|_\Delta$ is a morphism of extensions, taking exterior powers gives the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega^{p-1}_{\Delta}(2) & \longrightarrow & \Omega^{p}_{\widetilde{\Theta}^{+}}\big|_{\Delta} & \longrightarrow & \Omega^{p}_{\Delta} & \longrightarrow & 0 \\ & & \Big\downarrow \varphi_p & & \Big\downarrow & & \Big\| & & \\ 0 & \longrightarrow & \Omega^{p-1}_{\Delta}(1) & \longrightarrow & \Omega^{p}_{\widetilde{\Theta}}\big|_{\Delta} & \longrightarrow & \Omega^{p}_{\Delta} & \longrightarrow & 0 \end{array}$$

Since $\mathrm{Hom}(\mathcal{O}_\Delta(2), \mathcal{O}_\Delta(1)) \cong H^0(\Delta, \mathcal{O}_\Delta(-1)) = 0$ and hence $\varphi_p = \mathbf{1} \otimes \varphi_1 = 0$, the snake lemma implies that $\mathcal{C}|_\Delta \cong \Omega^{p-1}_\Delta(1)$.

It remains to show that $\mathcal{C} \to \mathcal{C}|_\Delta$ is an isomorphism, or equivalently that $\mathcal{C}(-\Delta) \to \mathcal{C}$ is zero. This is obvious away from Δ, so it suffices to check that, given $x \in \Delta$, the image of $\Omega^p_{\widetilde{\Theta}^+,x} \hookrightarrow \Omega^p_{\widetilde{\Theta},x}$ contains the submodule $f\Omega^p_{\widetilde{\Theta},x}$, where $f \in \mathcal{O}_{\widetilde{\Theta},x}$ generates $\mathcal{O}_{\widetilde{\Theta}}(-\Delta)_x \subset \mathcal{O}_{\widetilde{\Theta},x}$.

Let $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}_{\widetilde{\Theta},x}$. Since σ acts trivially on $\Omega_\Delta|_x$ and by $-\mathbf{1}$ on $\Omega_\Theta|_{\beta(x)}$, (1.2.2) implies that there is a basis $\left(f_i + \mathfrak{m}^2\right)$ for $\mathfrak{m}/\mathfrak{m}^2 = \Omega_{\widetilde{\Theta}}\big|_x$ such that

$$\sigma(f_1) \equiv -f_1 \text{ and } \sigma(f_i) \equiv f_i \pmod{\mathfrak{m}^2}$$

for $1 < i < g$. Moreover, we can pick $f_1 = f$, according to the local arguments of (1.2.2). Since Δ is G-invariant $\sigma(f) = uf$ for some unit $u \in \mathcal{O}_{\widetilde{\Theta},x}$, and

$$(1-u)f = f - \sigma(f) \equiv 2f \pmod{\mathfrak{m}^2}$$

so $1-u \notin \mathfrak{m}$, which means we can replace f by another lift of $f + \mathfrak{m}^2$, namely $\frac{1-u}{2}f$, which is σ-antiinvariant and also generates the ideal of Δ. Similarly, by replacing the f_i by $\frac{1}{2}(f_i + \sigma(f_i))$, we may assume that $\sigma(f_i) = f_i$. By Nakayama's lemma $\Omega^p_{\widetilde{\Theta},x}$ is generated by the forms

$$\mathrm{d}f_{i_1} \wedge \cdots \wedge \mathrm{d}f_{i_p}$$

where $1 \le i_1 < \cdots < i_p < g$. These come from $\Omega^p_{\widetilde{\Theta}^+,x}$ unless $i_1 = 1$, in which case only

$$f\,\mathrm{d}f_{i_1} \wedge \cdots \wedge \mathrm{d}f_{i_p} = \frac{1}{2}\mathrm{d}\left(f^2\right) \wedge \mathrm{d}f_{i_2} \wedge \cdots \wedge \mathrm{d}f_{i_p}$$

does. This is enough to conclude that the image of $\Omega^p_{\widetilde{\Theta}^+,x}$ contains $f\Omega^p_{\widetilde{\Theta}}(1)_x$, as required. ☺

(1.2.13) **Corollary**. There is an exact sequence

$$0 \to T_{\widetilde{\Theta}} \to \pi^* T_{\widetilde{\Theta}^+} \to \mathcal{O}_\Delta(-2) \to 0.$$

Proof. This sequence is dual to the $p = 1$ case of (1.2.12). Indeed, we can compute that

$$\mathbf{Ext}^1\big(\mathcal{O}_\Delta(1), \mathcal{O}_{\widetilde{\Theta}}\big) \cong \operatorname{Coker}\big(\mathcal{O}_{\widetilde{\Theta}}(-1) \to \mathcal{O}_{\widetilde{\Theta}}(-2)\big) \cong \mathcal{O}_\Delta(-2).$$

using the locally free resolution $0 \to \mathcal{O}_{\widetilde{\Theta}}(2) \to \mathcal{O}_{\widetilde{\Theta}}(1) \to \mathcal{O}_\Delta(1) \to 0$. ☺

(1.2.14) **Proposition**. The rank of $\mathbf{K}^+$ is

$$\frac{g!}{2} + (-1)^g\left(2^{g-2}(2^g + 1) - \binom{2g}{g + \varepsilon_{g+1}}\right),$$

and its Hodge numbers $h^{p,g-1-p}(\mathbf{K}^+)$ are

$$\frac{1}{2}\left\langle {g \atop p} \right\rangle + (-1)^g\left(\binom{g}{p}\sum_{q=0}^{g-1-p}\binom{g}{q}\varepsilon_{p+q} + \binom{g}{p+1}\sum_{q=g-p}^{g-1}\binom{g}{q+1}\varepsilon_{p+q} - \binom{g-1}{p}\frac{2^g - 1}{2}\right),$$

where $\varepsilon_k := \frac{1}{2}\big(1 + (-1)^k\big)$ is one (resp. zero) if k is even (resp. odd).

Proof. By Hirzebruch-Riemann-Roch and the preceding results

$$\begin{aligned}
2\chi\big(\Omega^p_{\widetilde{\Theta}^+}\big) &= 2\int_{\widetilde{\Theta}^+} \operatorname{ch}\big(\Omega^p_{\widetilde{\Theta}^+}\big)\operatorname{td}\big(\widetilde{\Theta}^+\big) \\
&= \int_{\widetilde{\Theta}} \pi^*\Big(\operatorname{ch}\big(\Omega^p_{\widetilde{\Theta}^+}\big)\operatorname{td}\big(\widetilde{\Theta}^+\big)\Big) \\
&= \int_{\widetilde{\Theta}} \Big(\operatorname{ch}\big(\Omega^p_{\widetilde{\Theta}}\big) - \operatorname{ch}\big(\Omega^{p-1}_\Delta(1)\big)\Big)\operatorname{td}\big(\widetilde{\Theta}\big)\operatorname{td}(\mathcal{O}_\Delta(-2)) \\
&= \int_{\widetilde{\Theta}} \Big(\operatorname{ch}\big(\beta^*\Omega^p_\Theta\big)\operatorname{ch}\big(\mathcal{O}_{\widetilde{\Theta}}(1-p)\big) + \operatorname{ch}\big(\Omega^p_\Delta\big) - \operatorname{ch}\big(\Omega^{p-1}_\Delta(1)\big)\Big)\operatorname{td}\big(\widetilde{\Theta}\big)\operatorname{td}(\mathcal{O}_\Delta(-2)).
\end{aligned}$$

The first term can be computed on Θ. To be specific:

$$\begin{aligned}
\chi_1 &:= \int_{\widetilde{\Theta}} \operatorname{ch}\big(\beta^*\Omega^p_\Theta\big)\operatorname{ch}\big(\mathcal{O}_{\widetilde{\Theta}}(1-p)\big)\operatorname{td}\big(\widetilde{\Theta}\big)\operatorname{td}(\mathcal{O}_\Delta(-2)) \\
&= \int_\Theta \operatorname{ch}\big(\Omega^p_\Theta\big)\operatorname{td}(\Theta)\cdot\beta_*\left(\operatorname{ch}\big(\mathcal{O}_{\widetilde{\Theta}}(1-p)\big)\frac{\operatorname{td}(\mathcal{O}_\Delta(-2))}{\operatorname{td}(T_\Delta(-1))}\right)
\end{aligned}$$

The Euler sequence and the ideal sheaf sequence for $\Delta \subset \widetilde{\Theta}$ imply that

$$\mathrm{td}(T_\Delta(-1)) = \frac{\mathrm{td}(\mathcal{O}_\Delta)^{g-1}}{\mathrm{td}(\mathcal{O}_\Delta(-1))} = \frac{\mathrm{td}\big(\mathcal{O}_{\widetilde{\Theta}}(1)\big)^{1-g}}{\mathrm{td}\big(\mathcal{O}_{\widetilde{\Theta}}(-1)\big)},$$

and similarly $\mathrm{td}(\mathcal{O}_\Delta(-2)) = \mathrm{td}\big(\mathcal{O}_{\widetilde{\Theta}}(-2)\big)\,\mathrm{td}\big(\mathcal{O}_{\widetilde{\Theta}}(-1)\big)^{-1}$, so

$$\chi_1 = \int_\Theta \mathrm{ch}\big(\Omega^p_\Theta\big)\,\mathrm{td}(\Theta)\cdot \beta_*\Big(\mathrm{ch}\big(\mathcal{O}_{\widetilde{\Theta}}(1-p)\big)\,\mathrm{td}\big(\mathcal{O}_{\widetilde{\Theta}}(1)\big)^{g-1}\,\mathrm{td}\big(\mathcal{O}_{\widetilde{\Theta}}(-2)\big)\Big).$$

Everything inside β_* is a polynomial in $h \coloneqq \mathrm{c}_1\big(\mathcal{O}_{\widetilde{\Theta}}(1)\big)$. Since β contracts Δ the only powers of h which survive are h^0 and h^{g-1}. Therefore

$$\chi_1 = \chi\big(\Omega^p_\Theta\big) - \binom{g-1}{p}\int_{\widetilde{\Theta}} e^{(1-p)h}\Big(\frac{h}{1-e^{-h}}\Big)^{g-1}\frac{2h}{1-e^{2h}}.$$

Since $-h$ is the class of $\Delta \subset \widetilde{\Theta}$, which has $2^{g-1}(2^g-1)$ components,

$$\chi_1 = \chi\big(\Omega^p_\Theta\big) - \binom{g-1}{p}2^{g-1}(2^g-1)\chi_2,$$

where, by (A.1.1),

$$\begin{aligned}
\chi_2 &\coloneqq -\left[e^{(1-p)t}\Big(\frac{t}{1-e^{-t}}\Big)^{g-1}\frac{2}{1-e^{2t}}\right]_{t^{g-2}}\\
&= -\left[\Big(\frac{t}{1-e^{-t}}\Big)^{g}\frac{2e^{(1-p)t}\big(1-e^{-t}\big)}{(1-e^{t})(1+e^{t})}\right]_{t^{g-1}}\\
&= \left[\Big(\frac{t}{1-e^{-t}}\Big)^{g}\frac{2e^{-pt}}{1+e^{t}}\right]_{t^{g-1}}\\
&= \frac{(-1)^p}{2^{g-1}}.
\end{aligned}$$

Grothendieck-Riemann-Roch allows us to compute the remaining terms on Δ:

$$\begin{aligned}
\chi_3 &\coloneqq \int_{\widetilde{\Theta}}\Big(\mathrm{ch}\big(\Omega^p_\Delta\big) - \mathrm{ch}\big(\Omega^{p-1}_\Delta(1)\big)\Big)\,\mathrm{td}\big(\widetilde{\Theta}\big)\frac{\mathrm{td}\big(\mathcal{O}_{\widetilde{\Theta}}(-2)\big)}{\mathrm{td}\big(\mathcal{O}_{\widetilde{\Theta}}(-1)\big)}\\
&= \int_\Delta\Big(\mathrm{ch}\big(\Omega^p_\Delta\big) - \mathrm{ch}\big(\Omega^{p-1}_\Delta(1)\big)\Big)\,\mathrm{td}(\Delta)\frac{\mathrm{td}(\mathcal{O}_\Delta(-2))}{\mathrm{td}(\mathcal{O}_\Delta(-1))}
\end{aligned}$$

Recall from (1.2.10) that $\mathrm{ch}\big(\Omega^p_\Delta\big) = \binom{g-1}{p}\,\mathrm{ch}(\mathcal{O}_\Delta(-p)) - \mathrm{ch}\big(\Omega^{p-1}_\Delta\big)$, so by induction

$$\mathrm{ch}\big(\Omega^p_\Delta\big) = \sum_{k=0}^{p} (-1)^{p-k}\binom{g-1}{k} e^{-kh},$$

where $h := \mathrm{c}_1(\mathcal{O}_\Delta(1))$. Since

$$\frac{\mathrm{td}(\mathcal{O}_\Delta(-2))}{\mathrm{td}(\mathcal{O}_\Delta(-1))} = \frac{2h}{1-e^{2h}}\frac{1-e^h}{h} = \frac{2}{1+e^h},$$

it follows by (A.1.1) and (A.1.4) that $\frac{2^{1-g}}{2^g-1}\chi_3$ is

$$\begin{aligned}
&\left[\left(\frac{t}{1-e^{-t}}\right)^{g-1}\left(\sum_{k=0}^{p}(-1)^{p-k}\binom{g-1}{k}\frac{2e^{-kt}}{1+e^t} + \sum_{k=0}^{p-1}(-1)^{p-k}\binom{g-1}{k}\frac{2e^{(1-k)t}}{1+e^t}\right)\right]_{t^{g-2}} \\
&= \sum_{k=0}^{p}\frac{(-1)^p}{2^{g-2}}\binom{g-1}{k} - \sum_{k=0}^{p-1}\frac{(-1)^p}{2^{g-2}}\binom{g-1}{k} + 2(-1)^p\delta_p \\
&= \frac{(-1)^p}{2^{g-2}}\binom{g-1}{p} + 2(-1)^p\delta_p,
\end{aligned}$$

where $\delta_0 := 0$, $\delta_{g-1} = 0$ and $\delta_p := 1$ for $0 < p < g-1$. Therefore

$$\begin{aligned}
2\chi\big(\Omega^p_{\widetilde{\Theta}^+}\big) &= \chi_1 + \chi_3 \\
&= \chi\big(\Omega^p_\Theta\big) - (-1)^p\binom{g-1}{p}(2^g-1) + (-1)^p\binom{g-1}{p}2(2^g-1) + (-1)^p 2^g(2^g-1)\delta_p \\
&= \chi\big(\Omega^p_\Theta\big) + (-1)^p(2^g-1)\left(\binom{g-1}{p} + 2^g\delta_p\right).
\end{aligned} \tag{1.2.15}$$

The description of $H^*\big(\widetilde{\Theta}, \mathbf{Q}\big)$ in terms of $H^*(\Theta, \mathbf{Q})$ [Voi02, Theorem 7.31] says that

$$\begin{aligned}
\chi\big(\Omega^p_{\widetilde{\Theta}^+}\big) &= \sum_{q=0}^{g-1}(-1)^q \dim\Big(H^{p,q}\big(\widetilde{\Theta}, \mathbf{Q}\big)^+\Big) \\
&= \sum_{q=0}^{g-1}(-1)^q \dim\big(H^{p,q}(\Theta, \mathbf{Q})^+\big) + (-1)^p 2^{g-1}(2^g-1)\delta_p.
\end{aligned} \tag{1.2.16}$$

Combining (1.2.15) and (1.2.16) allows us to express $(-1)^{g-1-p}h^{p,g-1-p}(\mathbf{K}^+)$ as

$$\frac{1}{2}\chi\big(\Omega^p_\Theta\big) + (-1)^p\frac{2^g-1}{2}\binom{g-1}{p} - \sum_{q=0}^{g-1-p}(-1)^q h^{p,q}(A)\varepsilon_{p+q} - \sum_{q=g-p}^{g-1}(-1)^q h^{p+1,q+1}(A)\varepsilon_{p+q},$$

as required. The rank of $\mathbf{K}^+$ is easy to compute from its Hodge numbers, since the terms coming from A add up to $(-1)^g\big(\sum_{k=0}^g h^{2k}(A) - h^{g+\varepsilon_{g+1}}(A)\big)$. ☺

1.3. Lattice properties

(1.3.1) When g is odd, the intersection form on $H^{g-1}(\Theta,\mathbf{Z})$ is symmetric. We determine some of its properties in this section. See §A.2 for background, and notation.

(1.3.2) **Proposition**. If g is odd, the signature of the intersection form on $\mathbf{K}$ is

$$\left(\sum_{p=0}^{\frac{g-1}{2}} h^{2p,g-1-2p}(\mathbf{K}),\ \sum_{p=1}^{\frac{g-1}{2}} h^{2p-1,g-2p}(\mathbf{K})\right).$$

Proof. If $\alpha \in \mathbf{K}\otimes_{\mathbf{Z}}\mathbf{C}$ then $\iota_*(\alpha\iota^*\theta) = \iota_*(\alpha)\theta = 0$ and hence $\alpha\iota^*\theta = 0$. This shows that $\mathbf{K}\otimes_{\mathbf{Z}}\mathbf{C}$ belongs to the primitive cohomology $H^4_{\mathrm{pr}}(\Theta,\mathbf{C})$. It follows that the intersection form on $(\mathbf{K}\otimes_{\mathbf{Z}}\mathbf{R})\cap\big(\mathbf{K}^{p,g-1-p}\oplus\mathbf{K}^{g-1-p,p}\big)$ is positive (resp. negative) definite when p is even (resp. odd) [Voi02, Theorem 6.32]. ☺

(1.3.3) **Corollary**. If $g = 5$, then the intersection form on $\mathbf{K}$ (resp. $\mathbf{K}^+$) has signature $(46,32)$ (resp. $(6,0)$). ☺

(1.3.4) **Lemma**. The subgroups $\mathbf{K}$ and $\iota^*H^{g-1}(A,\mathbf{Z})$ of $H^{g-1}(\Theta,\mathbf{Z})$ are the orthogonal complements of each other.

Proof. If $\alpha \in \mathbf{K}$ and $\beta \in H^{g-1}(A,\mathbf{Z})$ then $\langle\alpha,\iota^*\beta\rangle = \langle\iota_*\alpha,\beta\rangle = 0$, so $\mathbf{K} \subseteq \big(\iota^*H^{g-1}(A,\mathbf{Z})\big)^\perp$. Conversely, if $\alpha \in \big(\iota^*H^{g-1}(A,\mathbf{Z})\big)^\perp$ then the above equation shows that $\iota_*\alpha \in H^{g-1}(A,\mathbf{Z})^\perp$. Since $H^{g-1}(A,\mathbf{Z})$ is torsion-free, it follows by Poincaré duality that $\iota_*\alpha = 0$. This means that $\mathbf{K} = \big(\iota^*H^{g-1}(A,\mathbf{Z})\big)^\perp$.

To show that $\mathbf{K}^\perp = \iota^*H^{g-1}(A,\mathbf{Z})$, let $\alpha \in \mathbf{K}^\perp$. By the hard Lefschetz theorem, there is a positive integer n and a class $\beta \in H^{g-1}(A,\mathbf{Z})$ such that $n\iota_*\alpha = \theta\beta = \iota_*\iota^*\beta$. In particular $n\alpha - \iota^*\beta \in \mathbf{K}$ is orthogonal to $\mathbf{K}\oplus\iota^*H^{g-1}(A,\mathbf{Z})$. The hard Lefschetz argument also shows that this direct sum has finite index in $H^{g-1}(\Theta,\mathbf{Z})$, so $n\alpha - \iota^*\beta = 0$. In other words, the image of α in $\mathrm{Coker}(\iota^*)$ is torsion. However, the latter is always torsion-free [Laz04, Example 3.1.18]; in our situation this follows from the fact that $H^*(\Theta, \mathbf{Z})$ and $H^*(A, \mathbf{Z})$ are torsion-free and hence

$$\operatorname{Ext}^1(\operatorname{Coker}(\iota^*), \mathbf{Z}) \cong \operatorname{Coker}\Big(H^{g-1}(\Theta, \mathbf{Z})^\vee \xrightarrow{(\iota^*)^\vee} H^{g-1}(A, \mathbf{Z}))^\vee\Big) \cong \operatorname{Coker}(\iota_*),$$

which is zero by the weak Lefschetz theorem. In particular $\alpha \in \iota^* H^{g-1}(A, \mathbf{Z})$, as required. ☺

(1.3.5) **Lemma**. The torsion subgroup of $\operatorname{Coker}\Big(H^k(A, \mathbf{Z}) \xrightarrow{\theta \cup -} H^{k+2}(A, \mathbf{Z})\Big)$ has order

$$d_k^g := \begin{cases} \displaystyle\prod_{i=1}^{\lfloor \frac{k}{2} \rfloor} \Big(1 + \frac{1}{i}\Big)^{\binom{2g}{k-2i}} & \text{if } 0 \leq k \leq g-1, \\ \displaystyle\prod_{i=1}^{\lfloor \frac{2g-2-k}{2} \rfloor} \Big(1 + \frac{1}{i}\Big)^{\binom{2g}{k+2i+2}} & \text{if } g-1 \leq k \leq 2g-2. \end{cases}$$

Proof. Since Θ gives a principal polarisation, there is a basis $\xi_1, \ldots, \xi_{2g}$ for $H^1(A, \mathbf{Z})$ such that $\theta = \sigma_1 + \cdots + \sigma_g$, where $\sigma_i := \xi_{2i-1}\xi_{2i}$ [BL04, §3.1 and §4.1]. This extends in the obvious way to a basis Ξ for $H^*(A, \mathbf{Z})$. Given a pair (A, B) of disjoint subsets of $S := \{1, \ldots, g\}$, let $\xi_{A,B} \in \Xi$ correspond to $(2A-1) \sqcup 2B \subseteq T := \{1, \ldots, 2g\}$, and set

$$L_{A,B}^k := R\xi_{A,B} \cap H^k(A, \mathbf{Z}),$$

where $R \subseteq H^*(A, \mathbf{Z})$ is the subring generated by the σ_i. Note that $\theta L_{A,B}^k \subseteq L_{A,B}^{k+2}$. For each $\xi \in \Xi$ there is a unique pair (A, B) such that $\xi \in R\xi_{A,B}$, so there is a direct sum decomposition

$$H^k(A, \mathbf{Z}) = \bigoplus_{A \sqcup B \subseteq S} L_{A,B}^k.$$

If $k - |A \sqcup B| = 2c$ for some $c \in \mathbf{N}$ then $R^{2c} \xrightarrow{-\cup \xi_{A,B}} L_{A,B}^k$ induces a bijection

$$\binom{S \setminus (A \sqcup B)}{c} \to \Xi \cap L_{A,B}^k;$$

otherwise $L_{A,B}^k = 0$. In the former case, the bijection identifies the matrix of $L_{A,B}^k \xrightarrow{\theta \cup -} L_{A,B}^{k+2}$ with $M_{c,c+1}^{g-k+2c}$ from (A.2.5). If $k \leq g-1$ then $c + (c+1) \leq g - k + 2c$, and for each integer

$0 \le c \le \left\lfloor \frac{k}{2} \right\rfloor$ there are $\binom{g}{k-2c}2^{k-2c}$ pairs (A,B) such that $k - |A \sqcup B| = 2c$, so

$$\begin{aligned} d_k^g &= \prod_{c=0}^{\left\lfloor \frac{k}{2} \right\rfloor} \left(d_{c,c+1}^{g-k+2c}\right)^{\binom{g}{k-2c}2^{k-2c}} \\ &= \prod_{c=0}^{\left\lfloor \frac{k}{2} \right\rfloor} \prod_{i=1}^{c} \left(1+\frac{1}{i}\right)^{\binom{g-k+2c}{c-i}\binom{g}{k-2c}2^{k-2c}} \\ &= \prod_{i=1}^{\left\lfloor \frac{k}{2} \right\rfloor} \prod_{c=i}^{\left\lfloor \frac{k}{2} \right\rfloor} \left(1+\frac{1}{i}\right)^{\binom{g-k+2c}{c-i}\binom{g}{k-2c}2^{k-2c}} \\ &= \prod_{i=1}^{\left\lfloor \frac{k}{2} \right\rfloor} \left(1+\frac{1}{i}\right)^{\sum_{c=0}^{\left\lfloor \frac{k-2i}{2} \right\rfloor} \binom{g-k+2i+2c}{c}\binom{g}{k-2i-2c}2^{k-2i-2c}} \\ &= \prod_{i=1}^{\left\lfloor \frac{k}{2} \right\rfloor} \left(1+\frac{1}{i}\right)^{\binom{2g}{k-2i}}, \end{aligned}$$

where the last equality follows from the direct sum decomposition for $H^{k-2i}(A,\mathbf{Z})$. Otherwise

$$\begin{aligned} d_k^g &= \prod_{c=k-g+1}^{\left\lfloor \frac{k}{2} \right\rfloor} \left(d_{c,c+1}^{g-k+2c}\right)^{\binom{g}{k-2c}2^{k-2c}} \\ &= \prod_{c=0}^{\left\lfloor \frac{2g-2-k}{2} \right\rfloor} \left(d_{c+k-g+1,c+k-g+2}^{k-g+2c+2}\right)^{\binom{g}{2g-2-k-2c}2^{2g-2-k-2c}} \\ &= d_{2g-2-k}^g \end{aligned}$$

since $L_{A,B}^{k+2} = 0$ whenever $c+1 > g-k+2c$ (or equivalently $c < k-g+1$). ☺

(1.3.6) **Proposition**. If g is odd, the discriminant of $\iota^* H^{g-1}(A,\mathbf{Z}) \subseteq H^{g-1}(\Theta,\mathbf{Z})$ is

$$\pm \prod_{i=1}^{\left\lfloor \frac{g-1}{2} \right\rfloor} \left(1+\frac{1}{i}\right)^{\binom{2g}{g-1-2i}}.$$

Moreover $\mathbf{K}$ has the same discriminant up to a sign.

Proof. The composition $H^{g-1}(A,\mathbf{Z}) \hookrightarrow H^{g-1}(A,\mathbf{Z})^\vee \xrightarrow{\sim} H^{g+1}(A,\mathbf{Z})$ is multiplication by θ, as

$$\langle \iota^*\alpha, \iota^*\beta \rangle = \int_A \iota_*\iota^*(\alpha\beta) = \int_A \theta\alpha\beta$$

for all $\alpha,\beta \in H^{g-1}(A,\mathbf{Z})$. If we fix a basis for $H^*(A,\mathbf{Z})$ as in (1.3.5), the induced basis for

$H^{g+1}(A,\mathbf{Z})$ agrees with the induced dual basis for $H^{g-1}(A,\mathbf{Z})^\vee$. The matrix M which represents multiplication by θ with respect to these bases is therefore the Gram matrix of the intersection form on $\iota^* H^{g-1}(A,\mathbf{Z})$. Moreover $|\det(M)|$ is the product of the diagonal entries of the Smith normal form of M, which coincides with the order of $\mathrm{Coker}\Big(H^{g-1}(A,\mathbf{Z}) \xrightarrow{\theta\cup -} H^{g+1}(A,\mathbf{Z})\Big)$. The result now follows from (1.3.5) and (A.2.3b). ☺

(1.3.7) **Corollary**. If $g = 5$ then $|\det(\mathbf{K})| = 2^{44} \times 3$. ☺

(1.3.8) **Proposition**. If $g = 5$ then $H^4(\Theta,\mathbf{Z})$ is even.

Proof. Following Krämer, it suffices to show that $\alpha^2 = 0$ for all $\alpha \in H^4(\Theta,\mathbf{Z}/2)$ [Krä15, Lemma 5.2]. In characteristic 2 squaring is linear, so it defines (by Poincaré duality) a class in $H^4(\Theta,\mathbf{Z}/2)$, which we claim is zero. More generally, the Steenrod squares [Hat02, §4.L]

$$\mathrm{Sq}^i : H^*(\Theta,\mathbf{Z}/2) \to H^{*+i}(\Theta,\mathbf{Z}/2)$$

extend the squaring maps $H^i \to H^{2i}$, and restrict to forms on H^{8-i} which correspond to the Wu classes $v_i \in H^i$. These classes are determined by the Wu formula $w = \mathrm{Sq}(v)$, where w is the total Stiefel-Whitney class of Θ [MS74, Theorem 11.14]. This class is just the image modulo 2 of the total Chern class of Θ [MS74, Problem 14-B]. The normal bundle sequence for $\Theta \hookrightarrow A$ gives

$$\mathrm{c}(\Theta) = \frac{\mathrm{c}(A)}{1+\theta}\Big|_\Theta = (1+\theta)^{-1}\big|_\Theta = \big(1-\theta+\theta^2-\cdots\big)\big|_\Theta,$$

so $w = 1+\alpha+\alpha^2+\cdots$ where $\alpha \in H^2(\Theta,\mathbf{Z}/2)$ is the image of $\theta \in H^2(A,\mathbf{Z})$. In particular $v_0 = 1$ and $v_1 = w_1 - \mathrm{Sq}^1(v_0) = 0$, since $\mathrm{Sq}^0 = 1$ and Sq^i vanishes on H^k for $k < i$. Similarly $v_2 = w_2 - \mathrm{Sq}^1(v_1) = \alpha$, and $v_3 = w_3 - \mathrm{Sq}^1(v_2) = \mathrm{Sq}^1(\alpha) = \beta(\alpha)$, where β is the $\mathbf{Z}/2$ Bockstein homomorphism [Hat02, §3.E]. Since $\beta^2 = 0$

$$v_4 = w_4 - \mathrm{Sq}^1(v_3) - \mathrm{Sq}^2(v_2) = \alpha^2 + \beta^2(\alpha) + \alpha^2 = 0,$$

which is what we wanted. ☺

Chapter 2: Prym varieties and subvarieties

(2.0.1) Elliptic curves are the simplest abelian varieties. More generally, given a curve X of genus g, its jacobian $\mathrm{Pic}^0(X)$ is a ppav of dimension g. These are also very well-understood, but not every abelian variety of dimension $g > 3$ is a jacobian (since the moduli space $\mathcal{M}_g$ of curves has fewer dimensions than the moduli space $\mathcal{A}_g$ of ppavs). Prym varieties form a larger class of abelian varieties, but they are understood almost as well as jacobians are. The main references are [Mum74], [Bea77a] and [BL04]. This chapter summarises some properties of Prym varieties and their subvarieties.

(2.0.2) The Prym construction applies to any stable curve $X \in \overline{\mathcal{M}}_{g+1}$, together with a double cover $\widetilde{X} \xrightarrow{\pi} X$ such that $\widetilde{X} \in \overline{\mathcal{M}}_{2g+1}$, but only produces an abelian variety if π satisfies certain conditions [Bea77a, (5.1)]. There is a (coarse) moduli scheme $\overline{\mathcal{R}}_{g+1}$ parametrising such covers, and the Prym map $\mathcal{P} \colon \overline{\mathcal{R}}_{g+1} \to \mathcal{A}_g$ is surjective for $g \leq 5$ [DS81, Theorem 1.1].

(2.0.3) The upshot for us is that every ppav A of dimension $g \in \{4, 5\}$ can be realised as the Prym variety of some $X \in \overline{\mathcal{R}}_{g+1}$. Such an X is called a *Prym-curve* for A (and the cover will always be denoted by $\widetilde{X} \xrightarrow{\pi} X$). To simplify various arguments, we will typically work with a cover which is *good*, meaning X is smooth, π is étale and X is not hyperelliptic, trigonal or bielliptic. Since proper push-forward of divisors preserves linear equivalence, this implies that $\widetilde{X}$ is not hyperelliptic or trigonal.

(2.0.4) We denote the covering involution of $\widetilde{X}$ by σ, and if $x \in \widetilde{X}$ then $x' \coloneqq \sigma(x)$.

2.1. Prym varieties

(2.1.1) The *Prym variety* $\mathcal{P}(X)$ of a good cover $\widetilde{X} \xrightarrow{\pi} X$ is the identity component of

(2.1.2) $$\mathrm{Ker}\Big(\mathrm{Pic}(\widetilde{X}) \xrightarrow{c_1} \mathrm{CH}_0(\widetilde{X}) \xrightarrow{\pi_*} \mathrm{CH}_0(X)\Big).$$

It has a principal polarisation inherited from $\mathrm{Pic}^0(\widetilde{X})$ [Mum74, §3].

(2.1.3) It is easy to show that $\mathcal{P}(X) = \left\{\sigma^* L \otimes L^{-1} \,\middle|\, L \in \mathrm{Pic}^0(\widetilde{X})\right\}$ [Mum74, §3]. The other component of (2.1.2) is the $\mathcal{P}(X)$-torsor $\mathcal{P}^-(X) := \left\{\sigma^* L \otimes L^{-1} \,\middle|\, L \in \mathrm{Pic}^1(\widetilde{X})\right\}$ [Bea77a, (3.3)]. Note that there is a canonical map $\iota\colon \widetilde{X} \to \mathcal{P}^-(X)$: it sends $x \mapsto \mathcal{O}_{\widetilde{X}}(x - x')$, with x' as in (2.0.4). Since $\widetilde{X}$ is not hyperelliptic, ι is an embedding. A *Prym-embedding* of $\widetilde{X}$ in $\mathcal{P}(X)$ is a translate of ι by a point of $\mathcal{P}^-(X)$.

(2.1.4) Following [BD87], given $x, y \in \widetilde{X}$ we define

$$[x, y] := \iota(x) + \iota(y) = \mathcal{O}_{\widetilde{X}}(x + y - x' - y') \in \mathcal{P}(X).$$

Note that $[x, y] = \kappa(x) - \kappa(y')$ for any Prym-embedding $\kappa\colon \widetilde{X} \hookrightarrow \mathcal{P}(X)$.

(2.1.5) There are two more canonically-defined $\mathcal{P}(X)$-torsors: $\mathcal{P}_\omega(X)$ (resp. $\mathcal{P}_\omega^-(X)$) is the component of the fibre of $\pi_*\,\mathrm{c}_1$ over ω_X where h^0 is even (resp. odd) [Mum74, (6.1)].

2.2. Brill-Noether loci

(2.2.1) For each positive integer r, let $\mathcal{P}_\omega^r(X)$ be the locus in $\mathcal{P}_\omega(X) \sqcup \mathcal{P}_\omega^-(X)$ where $h^0 > r$ and $h^0 \not\equiv r \pmod 2$. Of course $\mathcal{P}_\omega^r(X) \subseteq \mathcal{P}_\omega(X)$ if r is odd; otherwise $\mathcal{P}_\omega^r(X) \subseteq \mathcal{P}_\omega^-(X)$. These subspaces have scheme structures inherited from classical Brill-Noether loci $W_{2g}^r \subseteq \mathrm{Pic}(\widetilde{X})$ [Wel85, (1.2)]. If $\widetilde{X} \to X$ is a good cover of a general curve $X \in \mathcal{M}_{g+1}$, then $\mathcal{P}_\omega^r(X)$ is smooth of dimension $g - \binom{r+1}{2}$ [Wel85, (1.11)].

(2.2.2) The locus $\Theta_\omega(X) := \mathcal{P}_\omega^1(X) \subset \mathcal{P}_\omega(X)$ defines a theta divisor in the sense that, for each $L_0 \in \mathcal{P}_\omega(X)$, the translate $\Theta_\omega(X) \otimes L_0^{-1}$ is a theta divisor for $\mathcal{P}(X)$ [Mum74, §6]. This is useful because there is no canonical choice of theta divisor for $\mathcal{P}(X)$ itself. Choosing L_0 determines a group structure on $\mathcal{P}_\omega(X)$: addition is given by $(L, M) \mapsto L \otimes M \otimes L_0^{-1}$ and hence negation is given by $L \mapsto L_0^2 \otimes L^{-1}$. In particular, if L_0 is a theta characteristic (meaning $L_0^2 \cong \omega_{\widetilde{X}}$), then $\Theta_\omega(X) \otimes L_0^{-1}$ is symmetric by Serre duality.

(2.2.3) The second Brill-Noether locus $\mathcal{P}^2_\omega(X)$ is also interesting. According to [Iza95, Proposition 3.11] it parametrises Prym-embeddings of $\widetilde{X}$ in $\Theta_\omega(X)$, in the sense that each $L \in \mathcal{P}^-_\omega(X)$ belongs to $\mathcal{P}^2_\omega(X)$ if and only if $\iota(\widetilde{X}) \otimes L \subset \Theta_\omega(X)$. Similarly, if $x \in \widetilde{X}$ then

$$W_x := \mathcal{P}^2_\omega(X) + \iota(x) = \left\{ L \in \mathcal{P}_\omega(X) \mid h^0(L(-x)) > 1 \right\} \subset \Theta_\omega(X).$$

defines an embedding of $\mathcal{P}^2_\omega(X)$ in $\Theta_\omega(X)$.

(2.2.4) By Serre duality, the involution $\rho\colon L \mapsto \omega_{\widetilde{X}} \otimes L^{-1}$ preserves $\mathcal{P}^2_\omega(X)$. The map which sends $\widetilde{X} \to X$ to the cover $\mathcal{P}^2_\omega(X) \to \mathcal{P}^2_\omega(X)/\langle\rho\rangle$ induces an involution λ of the fibres of $\mathcal{P}\colon \overline{\mathcal{R}}_5 \to \mathcal{A}_4$ [Iza95, Proposition 3.13]. Following loc. cit., we set $X_\lambda := \lambda(X)$, in which case $\widetilde{X}_\lambda = \mathcal{P}^2_\omega(X)$.

(2.2.5) If $A \in \mathcal{A}_5$ and X is a good Prym-curve for A, then $S_X := \mathcal{P}^2_\omega(X)$ is one of the surfaces we will use to generate $\mathbf{K}^+$.

2.3. Special subvarieties

(2.3.1) Given linear system g^r_d on X of dimension $r \geq 1$ and degree $2r < d < 2g+2$, the associated *special subvarieties* of the symmetric power $\widetilde{X}^{(d)}$ are the connected components S_i of the fibre product

$$\begin{array}{ccccc} S_1 \sqcup S_2 & = & S & \hookrightarrow & \widetilde{X}^{(d)} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{P}^r & \xrightarrow{\sim} & g^r_d & \hookrightarrow & X^{(d)}. \end{array}$$

If the base locus of g^r_d is reduced, then S is too [Bea82, §1]. When $d \leq r+g$, we also consider the special subvarieties $T_i \subset \widetilde{X}^{(2g-d)}$ associated to the residual linear system g^{r+g-d}_{2g-d}. After choosing the indices appropriately, for each $D \in T_i$ the image of $S_i + D \subset \widetilde{X}^{(2g)}$ under the Abel map $\widetilde{X}^{(2g)} \to \mathrm{Pic}^{2g}(\widetilde{X})$ is a subvariety $V_D \subseteq \Theta_\omega(X)$. On the other hand $S_{3-i}+D$ maps into $\mathcal{P}^-_\omega(X)$. The V_D are called *special subvarieties* of $\Theta_\omega(X)$ (and $\mathcal{P}_\omega(X)$).

(2.3.2) If $\theta \in H^2(\mathcal{P}_\omega(X), \mathbf{Z})$ denotes the class of $\Theta_\omega(X) \subset \mathcal{P}_\omega(X)$, then

$$2^{d-2r-1}\frac{\theta^{g-r}}{(g-r)!}$$

is the class of $V_D \subseteq \mathcal{P}_\omega(X)$ [Bea82, Théorème 1].

(2.3.3) For a (base-point-free) g^1_4 this procedure is called the *tetragonal construction*. Each special subvariety $\widetilde{X}_i := S_i$ of $\widetilde{X}^{(4)}$ is a smooth curve, assuming the fibres of $X \to {g^1_4}^\vee \cong \mathbf{P}^1$ have at most one ramification point, with index at most 3 [Bea82, §2]. The quotients $\widetilde{X}_i \to X_i$ induced by the covering involution $\widetilde{X}^{(4)} \xrightarrow{\sigma_*} \widetilde{X}^{(4)}$ have the same Prym variety as $\widetilde{X} \to X$ [Don92, Theorem 2.16]. For each i, the projection $S_1 \sqcup S_2 \to \mathbf{P}^1$ (which has degree $2^4 = 16$) induces a morphism $\widetilde{X}_i \to \mathbf{P}^1$ which factors through $\widetilde{X}_i \to X_i$. This determines a g^1_4 on X_i for which the associated special subvarieties of $\widetilde{X}_i^{(4)}$ are $\widetilde{X}$ and $\widetilde{X}_{3-i}$ [Don92, Lemma 2.13]. We say that the $\widetilde{X}_i \to X_i$ are *tetragonally related* to $\widetilde{X} \to X$ (and each other), or that (X, X_1, X_2) is a *tetragonal triple*. The tetragonal relation is denoted by $X \overset{\diamond}{\sim} X_i$.

(2.3.4) Suppose $X \overset{\diamond}{\sim} Y$ and $\kappa : \widetilde{X} \hookrightarrow \mathcal{P}(X)$ is a Prym-embedding. According to the proof of [BL04, Theorem 12.8.2], each embedding $V_D \subset \mathcal{P}_\omega(Y)$ of $\widetilde{X} \subset \widetilde{Y}^{(4)}$ as a special subvariety determines an isomorphism $\varphi : \mathcal{P}(X) \xrightarrow{\sim} \mathcal{P}(Y)$ and a line bundle $L_\varphi \in \mathcal{P}_\omega(Y)$ such that the following diagram commutes:

$$\begin{array}{ccc} \widetilde{X} & \longrightarrow & V_D \\ {\scriptstyle\kappa}\big\downarrow & & \big\downarrow{\scriptstyle -\otimes L_\varphi^{-1}} \\ \mathcal{P}(X) & \xrightarrow{\ \varphi\ } & \mathcal{P}(Y). \end{array}$$

(2.3.5) **Lemma**. Suppose $X \overset{\diamond}{\sim} Y$. If $\varphi : \mathcal{P}(X) \xrightarrow{\sim} \mathcal{P}(Y)$ and $\psi : \mathcal{P}(Y) \xrightarrow{\sim} \mathcal{P}(X)$ are constructed as in (2.3.4), then $\varphi = \psi^{-1}$. In particular, φ is independent of κ and D.

Proof. Let $p \in \widetilde{Y}$, and pick $q, r, s \in \widetilde{Y}$ so that $A := p + q + r + s \in \widetilde{Y}^{(4)}$ belongs to $\widetilde{X}$. This is possible because every point of Y appears in some divisor of the g^1_4. Also set

$$B := p + q + r' + s', \quad C := p + q' + r + s', \quad D := p + q' + r' + s.$$

These divisors belong to $\widetilde{X}$ (as opposed to the other curve of the triple), because they differ from A by the sum of two points in $\iota(\widetilde{Y}) \subset \mathcal{P}^{-}(Y)$. To reduce ambiguity in the notation, we denote the points of $\widetilde{X}$ corresponding to these divisors by a, b, c and d respectively. If $\kappa : \widetilde{X} \hookrightarrow \mathcal{P}(X)$ and $V_E \subset \mathcal{P}_\omega(Y)$ are the embeddings used to define φ then (2.3.4) implies that

$$(2.3.6) \qquad \varphi([a,b]) = \varphi(\kappa(a) - \kappa(b')) = L_\varphi^{-1}(A+E) \otimes L_\varphi(-B'-E) = \mathcal{O}_{\widetilde{Y}}(A-B').$$

Since $A - B' = p + q + r + s - p' - q' - r - s$ it follows that $\varphi([a,b]) = [p,q]$. Similarly $\varphi([c,d]) = [p,q']$. The upshot is that

$$(2.3.7) \qquad \varphi([a,b]+[c,d]) = [p,q]+[p,q'] = 2\iota(p).$$

Now we switch perspectives and think of $\widetilde{Y}$ as a subvariety of $\widetilde{X}^{(4)}$. The divisor on $\widetilde{X}$ corresponding to p is none other than $P := a+b+c+d$. Indeed A, B, C and D are the only elements of $\widetilde{X} \subset \widetilde{Y}^{(4)}$ in which p appears. From this point of view (2.3.7) says that

$$\varphi\big(\mathcal{O}_{\widetilde{X}}(P-P')\big) = 2\iota(p).$$

On the other hand (2.3.6) implies that

$$\psi(2\iota(p)) = \psi([p,p]) = \mathcal{O}_{\widetilde{X}}(P-P').$$

Therefore $\varphi\psi = \mathbf{1}$ on $2\iota(\widetilde{Y})$. This set generates $\mathcal{P}(Y) = \big\{\sigma^*L \otimes L^{-1} \,\big|\, L \in \mathrm{Pic}^0(\widetilde{Y})\big\}$. ☺

2.4. Comparing Prym subvarieties

(2.4.1) The results in this section describe relationships between the Brill-Noether loci and special subvarieties of $\mathcal{P}_\omega(X)$. Most of them appeared in some form in [BD87] and [Iza95]; the proofs are the same, but in some cases the statements are a little stronger. Everything else is joint work with Edward Dewey and Elham Izadi.

(2.4.2) **Lemma**. Let $\widetilde{X} \to X$ be a good cover and set $\Theta := \Theta_\omega(X)$. If $p,q,r \in \widetilde{X}$ are such that p, p', q and r are distinct, then:

(a) $\Theta \cap \Theta_{[p,q]} = V_{p+q}$.

(b) $\Theta \cap \Theta_{[p,q]} \cap \Theta_{[p,r]} = W_p \cup V_{p+q+r}$.

(c) $\Theta_{[p,q]} \cap \Theta_{[p,r]} \cap \Theta_{[q,r]} = \left(W_p + [q,r]\right) \cup V_{p+q+r}$.

(d) W_p and $W_p + [q,r]$ are algebraically equivalent in $\Theta_{[p,q]} \cap \Theta_{[p,r]}$.

Moreover, if $s \in \widetilde{X}$ is such that $\pi_*(p+q+r+s) \in X^{(4)}$ moves in a pencil, then:

(e) $V_{p+q+r} = V_{p+q+r+s} \cup V_{p+q+r+s'}$.

Proof. See the proof of [BD87, Proposition 1] for (2.4.2a) and (2.4.2b), and [Iza95, Proposition 2.4.1] for (2.4.2e). According to (2.4.2b)

$$\Theta_{[p,q]} \cap \Theta_{[p,r]} \cap \Theta_{[q,r]} = \left(\Theta_{[p,r']} \cap \Theta_{[p,q']} \cap \Theta\right) + [q,r] = \left(W_p \cup V_{p+q'+r'}\right) + [q,r]. \tag{2.4.3}$$

By definition $V_{p+q'+r'} + [q,r]$ consists of line bundles $L \in \mathcal{P}_\omega(X)$ such that

$$h^0(L(-q-r-p)) = h^0(L(-q-r+q'+r'-p-q'-r')) > 0,$$

so $V_{p+q'+r'} + [q,r] = V_{p+q+r}$ and (2.4.2c) follows from (2.4.3). The algebraic equivalence $\Theta \sim \Theta_{[q,r]}$ on $\mathcal{P}_\omega(X)$ restricts to

$$\Theta \cap \Theta_{[p,q]} \cap \Theta_{[p,r]} \sim \Theta_{[q,r]} \cap \Theta_{[p,q]} \cap \Theta_{[p,r]}$$

on $\Theta_{[p,q]} \cap \Theta_{[p,r]}$. Now (2.4.2d) follows from (2.4.2b) and (2.4.2c). ☺

(2.4.4) **Lemma**. Suppose $X \overset{\diamond}{\sim} Y$ are good Prym-curves for some ppav A. If $p \in \widetilde{X}$ corresponds to $P \in \widetilde{Y}^{(4)}$, then $\psi\left(W_p\right) = V_P$ for some isomorphism $\psi : \mathcal{P}_\omega(X) \to \mathcal{P}_\omega(Y)$.

Proof. Choose a theta characteristic $L_X \in \mathcal{P}_\omega(X)$ and an isomorphism $\varphi : \mathcal{P}(X) \to \mathcal{P}(Y)$ as in (2.3.4). There is a unique theta characteristic $L_Y \in \mathcal{P}_\omega(Y)$ such that the isomorphism

$$\psi : L \mapsto \varphi\left(L \otimes L_X^{-1}\right) \otimes L_Y$$

maps $\Theta_\omega(X)$ onto $\Theta_\omega(Y)$. Indeed, any two symmetric theta divisors differ by a point of order two [BL04, Lemma 4.6.2]. If $L \in V_P$, and $q \in \widetilde{X}$ corresponds to $Q \in \widetilde{Y}^{(4)}$, then

$$\varphi(\iota(q) - \iota(p)) \otimes L = \varphi([q, p']) \otimes L = L(Q - P) \in \Theta_\omega(Y)$$

by (2.3.6) and the fact that $h^0(L(-P)) > 0$. It follows that $\Theta_\omega(X)$ contains

$$\psi^{-1}(\varphi(\iota(q) - \iota(p)) \otimes L) = (\iota(q) - \iota(p)) \otimes \psi^{-1}(L) = \iota(q) \otimes \left(\psi^{-1}(L) - \iota(p)\right)$$

for all $q \in \widetilde{X}$, so $\psi^{-1}(L) - \iota(p) \in \mathcal{P}^2_\omega(X)$ and hence $\psi^{-1}(L) \in W_p$. This shows that $\psi\left(W_p\right) \subseteq V_P$.

By (2.3.2) (and the fact that $|\omega_X(-P)|$ is a g^{g-3}_{2g-4}), the class of V_P is $\frac{1}{3}\theta^3 \in H^6(A, \mathbf{Z})$. On the other hand V_{p+q+r} has class $\frac{2}{3}\theta^3$ whenever $q, r \in \widetilde{X}$. It follows from (2.4.2b) that $\left[\psi\left(W_p\right)\right] = [V_P]$, and hence $\psi\left(W_p\right) = V_P$. ☺

(2.4.5) **Corollary**. Let (X, Y, Z) be a tetragonal triple of good Prym-curves for some ppav. Also choose isomorphisms $\varphi\colon \mathcal{P}_\omega(Y) \to \mathcal{P}_\omega(X)$ and $\psi\colon \mathcal{P}_\omega(Z) \to \mathcal{P}_\omega(X)$ as in (2.4.4). If $x \in \widetilde{X}$, $y \in \widetilde{Y}$ and $z \in \widetilde{Z}$ then, up to algebraic equivalence,

$$[W_x] + \varphi_*\left[W_y\right] + \psi_*[W_z] = [\Theta_\omega(X)]^2.$$

Proof. If $d := p + q + r + s \in \widetilde{Y} \subset \widetilde{X}^{(4)}$ lifts a reduced divisor of the g^1_4 on X, then

$$\Theta \cap \Theta_{[p,q]} \cap \Theta_{[p,r]} = W_p \cup V_{p+q+r+s} \cup V_{p+q+r+s'} = W_p \cup \varphi(W_d) \cup \psi(W_e),$$

where $e := p + q + r + s' \in \widetilde{Z}$, by (2.4.2b) and (2.4.2e) and (2.4.4). Since W_p and W_x belong to a family of subvarieties of $\Theta_\omega(X)$ parametrised by $\widetilde{X}$, they are algebraically equivalent. Similarly $[W_d] = \left[W_y\right]$ and $[W_e] = [W_z]$. ☺

Chapter 2, in part, is currently being prepared for submission for publication. Jonathan Conder, Edward Dewey, and Elham Izadi.

Chapter 3: 27 subvarieties

(3.0.1) In this chapter we consider only the cases $g = 5$ and $g = 4$, which allows us to say more about the special subvarieties of interest. Each ppav of dimension five has only 27 Prym-curves X, and the tetragonal relation between them is isomorphic to the incidence correspondence for the lines on a smooth cubic surface [Don92, Theorem 4.2]. Consequently we are able to compute the intersection pairing between the surfaces $S_X := \mathcal{P}^2_\omega(X) \subset \Theta_\omega(X)$ in terms of their self-intersection numbers.

(3.0.2) Each ppav (B, Ξ) of dimension four has a two-parameter family of Prym-curves, but for most points $\alpha \in B$, only 27 of them admit Prym-embeddings in the surface $\Xi \cap \Xi_\alpha$ [Iza95]. In this case the self-intersection numbers are known [Krä15], and again they determine the rest of the pairing. We record the calculation for completeness, but apart from the self-intersection numbers this pairing will not be used later on.

(3.0.3) Everything in this chapter is joint work with Edward Dewey and Elham Izadi.

3.1. Lines

(3.1.1) We begin by summarising some of the useful properties of the lines on a smooth cubic surface. One way to obtain such a surface is to blow up six points $p_1, \dots, p_6 \in \mathbf{P}^2$ in general position (with respect to lines and conics) [Har77, V, Corollary 4.7]. The lines on the resulting surface can be described as follows:

- The exceptional divisor E_i over each point p_i.
- The proper transform F_{ij} of the line joining p_i to p_j, for $i < j$. If $i > j$ we set $F_{ij} := F_{ji}$.
- The proper transform G_j of the conic containing p_i for $i \neq j$ but not p_j.

Two lines meet if and only if they both belong to a triple of the form (E_i, F_{ij}, G_j) with $i \neq j$, or (F_{ij}, F_{kl}, F_{mn}) with $\{i, \dots, n\} = \{1, \dots, 6\}$ [Har77, V, Remark 4.10.1].

(3.1.2) **Lemma**. If $1 \leq n \leq 4$ or $n = 6$, the automorphism group $W(\mathsf{E}_6)$ of the incidence correspondence between the lines acts transitively on n-tuples of mutually skew lines.

Proof. The case $n = 6$ is well-known [Har77, V, Remark 4.10.1]. Every line belongs to a sixer (i.e., a sextuple of mutually skew lines) of the form $(E_1, \dots, E_6)$, $(E_i, E_j, E_k, F_{lm}, F_{ln}, F_{mn})$ or $(G_1, \dots, G_6)$, which proves the case $n = 1$.

Thus, any pair of skew lines is conjugate to one containing, say, E_i. Such a pair looks like one of (E_i, E_j), (E_i, F_{jk}) or (E_i, G_i). The first two extend to one of the above sixers, while the last extends to a sixer of the form $(E_i, G_i, F_{jk}, F_{jl}, F_{jm}, F_{jn})$. This proves the case $n = 2$.

The case $n = 3$ is similar: each skew triple is conjugate to one of the form (E_i, E_j, E_k) or (E_i, E_j, F_{kl}), which both extend to sixers as above. Finally, each skew quadruple is conjugate to either (E_i, E_j, E_k, E_l) or (E_i, E_j, E_k, F_{lm}), which are easy to extend.

The result is false for $n = 5$ because $(E_1, E_2, E_3, E_4, F_{56})$ does not extend to a sixer, so it cannot be conjugate to a quintuple that does extend. ☺

3.2. Surfaces

(3.2.1) In this section $A \in \mathcal{A}_5$ is a very general ppav with a fixed symmetric theta divisor $\Theta \subset A$. Since the Prym varieties of (covers of) hyperelliptic, trigonal and bielliptic curves are special (see [Mum74, §7], [Rec74] and [Don92, §3]), we may assume all the Prym-curves for A are good. It is well-known that the automorphisms of a very general ppav are just $\pm\mathbf{1}$ [BL04, Exercise 8.11.1].

(3.2.2) For each Prym-curve X, fix an isomorphism $\varphi_X : \mathcal{P}_\omega(X) \xrightarrow{\sim} A$ which maps $\Theta_\omega(X)$ onto Θ. If φ is another such isomorphism, then $\varphi_X \varphi^{-1}$ is an automorphism of A followed by a translation [Mum08, §4, Corollary 1], but no two translates of Θ are the same [Mum08, §16], so φ_X is unique up to $\pm\mathbf{1}$. Since Serre duality preserves the Brill-Noether loci, and hence takes each translate W_p to $W_{p'}$, the action of $-\mathbf{1}$ on Θ preserves $\varphi_X(W_p)$. The algebraic equivalence class of $\varphi_X(W_p) \subset \Theta$ is therefore independent of φ_X (and p), so we denote it simply by $[S_X]$.

(3.2.3) **Theorem**. Suppose that $\langle [S_X], [S_X] \rangle = 16$ whenever X is a Prym-curve for A. If X and Y are non-isomorphic Prym-curves, then

$$\langle [S_X], [S_Y] \rangle = \begin{cases} 12 & \text{if } X \overset{\diamond}{\sim} Y, \\ 14 & \text{otherwise.} \end{cases}$$

Proof. First, suppose that $X \overset{\diamond}{\sim} Y$ and form the tetragonal triple (X, Y, Z). Using (2.4.4), choose isomorphisms $\varphi : \mathcal{P}_\omega(Y) \to \mathcal{P}_\omega(X)$ and $\psi : \mathcal{P}_\omega(Z) \to \mathcal{P}_\omega(X)$. Since $\varphi_X \varphi = \pm\varphi_Y$ and likewise for ψ, it follows from (2.4.5) that

$$[S_X] + [S_Y] + [S_Z] = [\Theta]^2.$$

Combining this with (2.3.2) gives

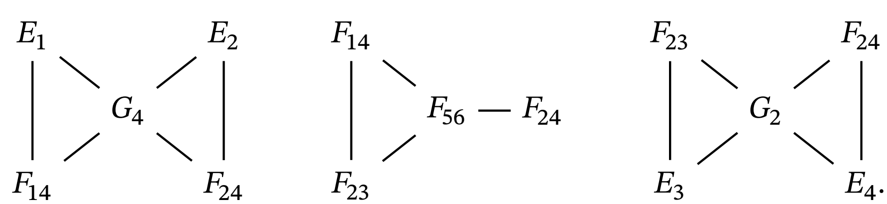
$$\text{(3.2.4)} \qquad \langle [S_W], [S_X] + [S_Y] + [S_Z] \rangle = \left\langle [S_W], [\Theta]^2 \right\rangle = \frac{1}{3}\langle \theta^3, \theta^2 \rangle = \frac{5!}{3} = 40$$

for all Prym-curves W, and hence

$$\langle [S_X], [S_Y] + [S_Z] \rangle = \langle [S_Y], [S_Z] + [S_X] \rangle = \langle [S_Z], [S_X] + [S_Y] \rangle = 40 - 16 = 24.$$

It easily follows that $\langle [S_X], [S_Y] \rangle = \langle [S_Y], [S_Z] \rangle = \langle [S_Z], [S_X] \rangle = 12$.

For the case $X \overset{\diamond}{\nsim} Y$, suppose X_1, Y_1, X_2 and Y_2 are mutually unrelated Prym-curves for A. We claim that $\left\langle [S_{X_1}], [S_{Y_1}] \right\rangle = 28 - \left\langle [S_{X_2}], [S_{Y_2}] \right\rangle$. To prove this, fix a correspondence-preserving bijection φ from the 27 lines onto $\mathcal{P}^{-1}(A)$. By (3.1.2) we may assume that φ sends (E_1, E_2, E_3, E_4) to (X_1, Y_1, X_2, Y_2). Consider the following incidence diagrams:



For notational simplicity set $e_i := \left[S_{\varphi(E_i)}\right]$ and likewise for f_{ij} and g_j. Applying (3.2.4) to the curves corresponding to the leftmost diagram gives

$$\langle e_1, e_2 + f_{24} + g_4 \rangle = 40 = \langle e_1 + f_{14} + g_4, f_{24} \rangle,$$

and since $\langle e_1, g_4 \rangle = 12 = \langle g_4, f_{24} \rangle$ it follows that $\langle e_1, e_2 \rangle = \langle f_{14}, f_{24} \rangle$. Similarly

$$\langle f_{14} + f_{23}, f_{24} \rangle = 40 - \langle f_{56}, f_{24} \rangle = 28,$$

according to the middle diagram. The rightmost diagram, by analogy with the leftmost one, implies that $\langle f_{23}, f_{24} \rangle = \langle e_3, e_4 \rangle$. Combining all three proves the claim:

$$\langle e_1, e_2 \rangle = \langle f_{14}, f_{24} \rangle = 28 - \langle f_{23}, f_{24} \rangle = 28 - \langle e_3, e_4 \rangle.$$

Now choose φ so that $\varphi(E_1) = X$ and $\varphi(E_2) = Y$ (this is possible by (3.1.2)). According to the leftmost diagram $\langle e_1, e_2 \rangle + \langle e_1, f_{24} \rangle = 28$. On the other hand

$$\langle e_1, e_2 \rangle = 28 - \langle e_5, e_6 \rangle = \langle e_1, f_{24} \rangle,$$

by the claim. Therefore $\langle [S_X], [S_Y] \rangle = \langle e_1, e_2 \rangle = 14$, as required. ☺

(3.2.5) The main theorem can be deduced from the conclusion of (3.2.3), as follows.

Proof of theorem 1. Label the Prym-curves for A as $X_1, \dots, X_{27}$, and set $S_i := S_{X_i}$. The differences $[S_i] - [S_j]$ all belong to $\mathbf{K}^+$ by (2.3.2) and the invariance of the $[S_i]$ under $-\mathbf{1}$.

We may assume that $X_1, \dots, X_7$ correspond to the lines $E_1, \dots, E_6, F_{12}$. Let $L \subseteq \mathbf{K}^+$ be the subgroup generated by $[S_i] - [S_7]$ for $i \in \{1, \dots, 6\}$. The associated morphism $\varphi : \mathbf{Z}^6 \twoheadrightarrow L$ induces a bilinear form on $\mathbf{Z}^6$ which has the following Gram matrix (by (3.2.3)):

$$\begin{pmatrix} 8 & 6 & 4 & 4 & 4 & 4 \\ 6 & 8 & 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 2 & 2 & 2 \\ 4 & 4 & 2 & 4 & 2 & 2 \\ 4 & 4 & 2 & 2 & 4 & 2 \\ 4 & 4 & 2 & 2 & 2 & 4 \end{pmatrix} = 2 \begin{pmatrix} 4 & 3 & 2 & 2 & 2 & 2 \\ 3 & 4 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 1 & 1 & 1 \\ 2 & 2 & 1 & 2 & 1 & 1 \\ 2 & 2 & 1 & 1 & 2 & 1 \\ 2 & 2 & 1 & 1 & 1 & 2 \end{pmatrix}.$$

This matrix has determinant $2^6 \times 3 \neq 0$ and represents the composition

$$\mathbf{Z}^6 \xrightarrow{\varphi} L \to L^\vee \xrightarrow{\varphi^\vee} \left(\mathbf{Z}^6\right)^\vee \xrightarrow{\sim} \mathbf{Z}^6,$$

which forces L to have rank 6. Therefore $L \otimes_{\mathbf{Z}} \mathbf{Q} = (\mathbf{K} \otimes_{\mathbf{Z}} \mathbf{Q})^+$.

The main result of [IW19] implies that $(\mathbf{K} \otimes_{\mathbf{Z}} \mathbf{Q})^-$ has no Hodge classes. Since the only Hodge classes in $H^4(A, \mathbf{Q})$ are multiples of θ^2 [BL04, Theorem 17.4.1],

$$H^{2,2}(\Theta) \cap H^4(\Theta, \mathbf{Q}) = (\mathbf{K} \otimes_{\mathbf{Z}} \mathbf{Q})^+ \oplus \mathbf{Q}[\Theta]^2$$

is generated by the $[S_i]$ (which live outside $\mathbf{K} \otimes_{\mathbf{Z}} \mathbf{Q}$ by (2.3.2)).

If X is a smooth cubic surface, it is easy to check that the Gram matrix of the intersection form on $H^2_{\text{pr}}(X, \mathbf{Z})$ with respect to the basis $([E_1] - [F_{12}], \dots, [E_6] - [F_{12}])$ is

$$-\begin{pmatrix} 4 & 3 & 2 & 2 & 2 & 2 \\ 3 & 4 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 1 & 1 & 1 \\ 2 & 2 & 1 & 2 & 1 & 1 \\ 2 & 2 & 1 & 1 & 2 & 1 \\ 2 & 2 & 1 & 1 & 1 & 2 \end{pmatrix},$$

as required. For every difference $[S_i] - \left[S_j\right]$, the corresponding class in $H^2_{\text{pr}}(X, \mathbf{Z})$ is a $\mathbf{Z}$-linear combination of the $[E_k] - [F_{12}]$. The same coefficients can be used to write $[S_i] - \left[S_j\right]$ in terms of our basis for L, because the intersection form on $\mathbf{K}^+$ is nondegenerate. ☺

(3.2.6) If $A \in \mathcal{A}_5$ is just general (as opposed to very general), then theorem 1 can be false if $H^4(A, \mathbf{Q})$ has more Hodge classes than usual. However, (3.2.3) still holds and hence the classes $[S_i] - \left[S_j\right]$ generate a full rank sublattice of $\mathbf{K}^+$ isometric to $H^2_{\text{pr}}(X, \mathbf{Z})(-2)$. To prove this, one can deform A to a very general ppav and use the fact that intersection numbers are constant in smooth families [Ful98, Corollary 10.2.2].

3.3. Curves

(3.3.1) In this section $B \in \mathcal{A}_4$ is a very general ppav with a fixed symmetric theta divisor $\Xi \subset B$. The fibre of $\mathcal{P}$ over B is two-dimensional: it is a double cover of the Fano surface of lines on some cubic threefold [Don92, Theorem 5.1; Iza95, Theorem 6.11, Lemma 6.27]. The covering involution is the map $\lambda : \overline{\mathcal{R}}_5 \to \overline{\mathcal{R}}_5$ defined in (2.2.4) [Iza95, Proposition 5.6].

(3.3.2) A general point $\alpha \in B$ determines a smooth hyperplane section of the cubic, and for each line on this cubic surface, α picks out one of the two Prym-curves $X \in \overline{\mathcal{R}}_5$ lying over it [Iza95, Lemma 5.12, Corollary 4.9]. Such an X is called an α-curve. These are precisely the Prym-curves such that $\varphi(W_p) \subset \Xi \cap \Xi_\alpha$ for some $p \in \widetilde{X}$ and an isomorphism $\varphi : \mathcal{P}_\omega(X) \xrightarrow{\sim} B$ which maps $\Theta_\omega(X)$ onto Ξ. An α-curve is *good* if X and X_λ are good Prym-curves for B.

(3.3.3) Set $L_0 := \varphi^{-1}(0)$ and define a group structure on $\mathcal{P}_\omega(X)$ by $(L, M) \mapsto L \otimes M \otimes L_0^{-1}$. By rigidity φ is a group isomorphism [Mum08, §4, Corollary 1]. In particular

$$\varphi^{-1}(\Xi_\alpha) = \Theta_\omega(X) \otimes \varphi^{-1}(\alpha) \otimes L_0^{-1}.$$

Since $\widetilde{X}_\lambda$ parametrises Prym-embeddings of $\widetilde{X}$ in $\Theta_\omega(X)$ and $\lambda^2 = \mathbf{1}$, each $L \in \mathcal{P}^-(X)$ such that $\widetilde{X}_\lambda + L \subset \varphi^{-1}(\Xi \cap \Xi_\alpha)$ belongs to $\iota(\widetilde{X}) \cap (\iota(\widetilde{X}) \otimes \varphi^{-1}(\alpha) \otimes L_0^{-1})$. In other words $\varphi^{-1}(\alpha) \otimes L_0^{-1}$ can be written as $[p, q]$ for some $p, q \in \widetilde{X}$. The embeddings $\widetilde{X}_\lambda \hookrightarrow \varphi^{-1}(\Xi \cap \Xi_\alpha)$ are then W_p and W_q [Iza95, Proposition 3.16].

(3.3.4) These curves enjoy similar properties to the surfaces of the previous section. After establishing some of these, we prove an analogue of (3.2.3) for the curves in $\Xi \cap \Xi_\alpha$. As before we fix an isomorphism $\varphi_X : \mathcal{P}_\omega(X) \xrightarrow{\sim} B$ with $\varphi_X(\Theta_\omega(X)) = \Xi$ for each α-curve X; it is unique up to a sign. The only fact we will use later is the following result of Krämer.

(3.3.5) **Proposition** (Krämer). If X is a good α-curve and $p \in \widetilde{X}$ is such that $\varphi_X(W_p) \subset \Xi \cap \Xi_\alpha$, then $\langle [\varphi_X(W_p)], [\varphi_X(W_p)] \rangle = 0$.

Proof. Since α is general $\Xi \cap \Xi_\alpha$ is smooth. Its canonical class is $[\Xi] + [\Xi_\alpha]$ by adjunction.

Moreover $W_p \cong \widetilde{X}_\lambda$ has genus 9, and its class in B is $\frac{1}{3}[\Xi]^3$ by (2.3.2). Therefore

$$\langle [\varphi_X(W_p)], [\varphi_X(W_p)] \rangle = 2(9) - 2 - \frac{2}{3}\int_B [\Xi]^4 = 16 - 16 = 0,$$

by adjunction again (see [Krä15, Lemma 7.2] for the original calculation). ☺

(3.3.6) **Lemma**. If X is a good α-curve, then the two embeddings of $\widetilde{X}_\lambda$ in $\Xi \cap \Xi_\alpha$ are algebraically equivalent.

Proof. Pick $q, r \in \widetilde{X}$ such that $\varphi_X^{-1}(\alpha) \otimes \varphi_X^{-1}(0)^{-1} = [q', r]$. The embeddings of $\widetilde{X}_\lambda$ in $\Xi \cap \Xi_\alpha$ correspond to $W_{q'}, W_r \subset \Theta \cap \Theta_{[q',r]}$, where $\Theta := \Theta_\omega(X)$. Given $p \in \widetilde{X} \setminus \{q, q', r, r'\}$, it follows that the embeddings of $\widetilde{X}_\lambda$ in

$$\left(\Theta \cap \Theta_{[q',r]}\right) + [p, q] = \Theta_{[p,q]} \cap \Theta_{[p,r]}$$

are $W_{q'} + [p, q] = W_p$ and $W_r + [p, q] = W_p + [q, r]$, which are algebraically equivalent by (2.4.2d). Now translate back by $[p, q]$ (and apply φ_X). ☺

(3.3.7) Consequently, there is a well-defined class $\left[\widetilde{X}_\lambda\right] \in H^2(\Xi \cap \Xi_\alpha, \mathbf{Z})$. It is invariant with respect to the involution $\beta \mapsto \alpha - \beta$ on $\Xi \cap \Xi_\alpha$. Since

$$\gamma_X := \left[\widetilde{X}_\lambda\right] - \frac{1}{3}[\Xi] \in H^2(\Xi \cap \Xi_\alpha, \mathbf{Q})$$

pushes forward to 0 in B, it belongs to $(\mathbf{W} \otimes_{\mathbf{Z}} \mathbf{Q})^+$, where $\mathbf{W} \subset H^2(\Xi \cap \Xi_\alpha, \mathbf{Z})$ is the analogue of $\mathbf{K}$ for the embedding $\Xi \cap \Xi_\alpha \hookrightarrow B$. Krämer showed that $\mathbf{W}^+ \cong \mathsf{E}_6(-2)$ [Krä15, Proposition 5.1], and that γ_X corresponds to a norm-minimising element of the dual lattice $\mathsf{E}_6(-2)^\vee$ [Krä15, Lemma 7.2]. From this it is straightforward to deduce some information about the pairing between the $\left[\widetilde{X}_\lambda\right]$. Our approach takes us one step further, by completely determining the pairing in terms of the tetragonal correspondence.

(3.3.8) In order to apply the argument of (3.2.3), we want to use the correspondence between α-curves and lines on a cubic surface [Iza95, Lemma 5.9, Lemma 5.12, Corollary 6.8].

Unfortunately not every α-curve is good (in general), which could create some difficulties. The next two results show that we can we can work around this issue.

(3.3.9) **Lemma**. If B and α are sufficiently general, then every α-curve is good.

Proof. Since B is general, the fibre of $\mathcal{P}$ over B contains a dense open subset of smooth Prym-curves [Iza95, Theorem 3.3, Remark 3.10]. As worked out above, a smooth Prym-curve X is a β-curve if and only if the image of β in $\mathcal{P}(X)$ belongs to the surface

$$\Sigma(X) := \{[p,q] \mid p,q \in \widetilde{X}\}; \tag{3.3.10}$$

this also holds for singular Prym-curves, but one has to take more care in defining $[p,q]$, as $\widetilde{X}$ is no longer locally factorial [Iza95, Proposition 3.16]. It follows that the correspondence

$$\{(X,\beta) \in \mathcal{P}^{-1}(B) \times B \mid X \text{ or } X_\lambda \text{ is a singular } \beta\text{-curve}\}.$$

is at most three-dimensional (by considering the projection to $\mathcal{P}^{-1}(B)$), so its image in B has positive codimension. Since $\alpha \in B$ is general, and no Prym-curve for B is hyperelliptic, trigonal or bielliptic, we may therefore assume that every α-curve is good. ☺

(3.3.11) **Lemma**. For good α-curves $X \overset{\diamond}{\sim} Y$, the other curve in the tetragonal triple (X,Y,Z) is also an α-curve, and $[\widetilde{X}_\lambda] + [\widetilde{Y}_\lambda] + [\widetilde{Z}_\lambda] = [\Xi]$ in $\Xi \cap \Xi_\alpha$.

Proof. By the argument of (2.4.5), it suffices to exhibit a reduced divisor

$$p+q+r+s \in \widetilde{Y} \subset \widetilde{X}^{(4)}$$

such that $\alpha = [p,q]$. When such a divisor exists we say (X,Y) is *good* for α. The fibre of

$$\{(Z_1,Z_2,\beta) \in \mathcal{P}^{-1}(B)^2 \times B \mid Z_1 \overset{\diamond}{\sim} Z_2 \text{ are } \beta\text{-curves}\}$$

over a pair (Z_1,Z_2) of related Prym-curves is the one-dimesional space $\Sigma(Z_1) \cap \Sigma(Z_2)$ (see (3.3.10)). On the other hand, the projection to B is generically finite. It is known that (Z_1,Z_2)

is good for almost all $\beta \in \Sigma(Z_1) \cap \Sigma(Z_2)$ [Iza95, Lemma 5.9], so the subcorrespondence

$$\left\{ (Z_1, Z_2, \beta) \in \mathcal{P}^{-1}(B)^2 \times B \;\middle|\; Z_1 \overset{\diamond}{\sim} Z_2 \text{ are bad } \beta\text{-curves} \right\}$$

has positive codimension. Since $\alpha \in B$ is general, it follows that every pair of related α-curves is good. ☺

(3.3.12) **Proposition**. If X and Y are α-curves, then

$$\langle [\widetilde{X}_\lambda], [\widetilde{Y}_\lambda] \rangle = \begin{cases} 0 & \text{if } X = Y, \\ 4 & \text{if } X \overset{\diamond}{\sim} Y, \\ 2 & \text{otherwise.} \end{cases}$$

Proof. Same as (3.2.3) with different numbers. ☺

(3.3.13) **Corollary**. The classes $\delta_i := \left[(\widetilde{X}_i)_\lambda \right] - [\widetilde{Y}_\lambda]$ freely generate $\mathbf{W}^+$ for any collection of α-curves $X_1, \ldots, X_6, Y$ such that the X_i are mutually tetragonally unrelated and Y is related to exactly two of the X_i.

Proof. If X and Y are α-curves, with E and F the associated lines on a cubic surface, then (3.3.12) says that $\langle [\widetilde{X}_\lambda], [\widetilde{Y}_\lambda] \rangle = 2(\langle E, F \rangle + 1)$. Thus, the matrix with entries $\langle \delta_i, \delta_j \rangle$ is a Gram matrix for $\mathsf{E}_6(-2)$, so its determinant is the discriminant of $\mathbf{W}^+$, namely $2^6 \times 3$. The map $\mathbf{Z}^6 \to \mathbf{W}^+$ defined by the δ_i has to be invertible for this to hold. ☺

Chapter 3, in full, is currently being prepared for submission for publication. Jonathan Conder, Edward Dewey, and Elham Izadi.

Chapter 4: Degenerating Prym varieties

(4.0.1) In order to complete the proof of theorem 1, we still need to compute the self-intersection numbers of the surfaces in (3.2.3). To do so, we adapt the rank one degeneration in [IW19], which takes a very general abelian fivefold to a singular variety that is birational to a $\mathbf{P}^1$-bundle over an abelian fourfold [Mum83, §1]. This allows us to compare the surfaces of §3.2 to the curves of §3.3.

(4.0.2) The construction employs the relative Prym variety of [AFS15], which has a modular interpretation in terms of sheaves on K3 surfaces. This can be used to show that the total space of the family of theta divisors is nonsingular, which simplifies the intersection theory in the next chapter. The idea of parametrising Prym-curves using the K3 cover of an Enriques surface goes back to [MM83].

(4.0.3) Everything in this chapter is joint work with Edward Dewey and Elham Izadi. The main difference from [IW19] is that our families parametrise line bundles of degree 10 as opposed to degree 0. Since (for our purposes) the base of these families need not be complete, we can show that their total spaces are smooth without using [IW19, Lemma 2.4].

4.1. Relative Prym torsors

(4.1.1) Let R be a very general Enriques surface and $\rho : \widetilde{R} \to R$ the associated K3 double cover. Fix a very ample line bundle H on R such that $\langle \mathrm{c}_1(H), \mathrm{c}_1(H) \rangle = 10$, and set $\widetilde{H} := \rho^* H$. Let $M_{\widetilde{H}}$ be the moduli space of $\widetilde{H}$-semistable sheaves F of pure dimension one on $\widetilde{R}$ such that $\mathrm{c}_1(F) = \widetilde{H}$ and $\chi(F) = 0$ [AFS15, (3.6)]. It is a 22-dimensional projective variety. There is a morphism $M_{\widetilde{H}} \to |\widetilde{H}| \cong \mathbf{P}^{11}$, which sends a sheaf to its (Fitting) support [AFS15, (3.8)].

(4.1.2) Pick a general pencil $T \subset |H|$. Each $C \in T$ is integral of genus 6, has at most one node, and the cover $\widetilde{C} := \rho^{-1}(C)$ is integral of genus 11 [IW19, Proposition 1.2]. Let $\widetilde{\mathcal{C}} \to \mathcal{C} \to T$ be the family of such covers, and pick a point $0 \in T$ for which $C := \mathcal{C}_0$ is singular. If $X \xrightarrow{\nu} C$ is the normalisation, with $x, y \in X$ lying over the node $z \in C$, then the normalisation $\widetilde{X} \xrightarrow{\nu} \widetilde{C}$ identifies each $p \in \pi^{-1}(x)$ with a point $q \in \pi^{-1}(y)$, and $\widetilde{X} \to X$ is a nontrivial étale double cover [IW19, Corollary 1.3]. In order to avoid pathologies, we will assume that $\widetilde{X}$ is not hyperelliptic.

(4.1.3) Let $|\widetilde{H}|^{\circ} \subset |\widetilde{H}|$ be a neighbourhood of $\widetilde{C}$ consisting of integral curves. We may assume that $|\widetilde{H}|^{\circ}$ is preserved by the covering involution σ of $\widetilde{R}$. The set $M^{\circ}_{\widetilde{H}} \subset M_{\widetilde{H}}$ of sheaves with support in $|\widetilde{H}|^{\circ}$ belongs to the stable locus [AFS15, §3.1], and is therefore nonsingular [AFS15, §3.2]. If $F \in M^{\circ}_{\widetilde{H}}$ is supported on D, then

$$\tau(F) := \mathbf{Ext}^1_{\widetilde{R}}\left(\sigma^* F, \mathcal{O}_{\widetilde{R}}\right) \cong \mathbf{Hom}_{\sigma^* D}(\sigma^* F, \omega_{\sigma^* D})$$

is supported on $\sigma^* D$ [AFS15, §3.5]. This shows that τ is an involution of $M^{\circ}_{\widetilde{H}}$. Its fixed locus $M^{\tau}_{\widetilde{H}}$ is smooth [CGP15, Proposition A.8.10].

(4.1.4) The fibre M_0 of $M_{\widetilde{H}} \to |\widetilde{H}|$ over $\widetilde{C}$ compactifies $\mathrm{Pic}^{10}(\widetilde{C})$; specifically it is the moduli space of torsion-free sheaves on $\widetilde{C}$ with rank one and degree 10 [AFS15, §3.2]. Over a smooth curve $\widetilde{Y} \in \rho^*|H|$ this moduli space is just $\mathrm{Pic}^{10}(\widetilde{Y})$, and the fibre of $M^{\tau}_{\widetilde{H}} \to |H|$ over Y has four components, two of which are $\mathcal{P}_\omega(Y)$ and $\mathcal{P}^-_\omega(Y)$ (the others differ by the 2-torsion point of $\mathrm{Pic}(Y)$ corresponding to $\widetilde{Y}$) [AFS15, (3.11)]. Our goal is to understand the limit of the $\mathcal{P}_\omega(Y)$ as Y approaches C, which we describe in (4.1.10) as one of the components of M^{τ}_0.

(4.1.5) There is a resolution $\widetilde{M}_0 \to M_0$ which parametrises *presentations* [OS79, §12]. A presentation of $F \in M_0$ is a line bundle $G \in \mathrm{Pic}^{10}(\widetilde{X})$ together with a short exact sequence

$$0 \to F \to \nu_* G \to \mathbf{C}_r \oplus \mathbf{C}_{r'} \to 0, \tag{4.1.6}$$

where $r, r' \in \widetilde{C}$ are the nodes. If $\nu^{-1}(r) = \{p, q\}$, then this sequence is determined by maps $G|_p \oplus G|_q \twoheadrightarrow \mathbf{C}$ and $G|_{p'} \oplus G|_{q'} \twoheadrightarrow \mathbf{C}$. Two presentations are isomorphic if and only if the corresponding maps have the same kernel, so $\widetilde{M}_0 \cong \mathbf{P}(E_p \oplus E_q) \times_{\mathrm{Pic}^{10}(\widetilde{X})} \mathbf{P}(E_{p'} \oplus E_{q'})$, where E_s is the line bundle with fibre $G|_s$ over $G \in \mathrm{Pic}^{10}(\widetilde{X})$. These bundles can be constructed using a Poincaré line bundle on $\widetilde{X} \times \mathrm{Pic}^{10}(\widetilde{X})$ [OS79, Proposition 12.1].

(4.1.7) We define more involutions, all denoted by τ. The first sends $G \in \mathrm{Pic}^{10}(\widetilde{X})$ to $\nu^*\omega_{\widetilde{C}} \otimes \sigma^* G^{-1}$. Note that $\tau(G)|_s \cong \mathrm{Hom}\big(G|_{s'}, \omega_{\widetilde{C}}|_{\nu(s)}\big)$. This lifts to an involution of $\widetilde{M}_0$, which sends a pair of lines $L \subset G|_p \oplus G|_q$ and $L' \subset G|_{p'} \oplus G|_{q'}$ to the lines in

$$\tau(G)|_p \oplus \tau(G)|_q = \mathrm{Hom}\big(G|_{p'} \oplus G|_{q'}, \omega_{\widetilde{C}}|_r\big)$$

and $\tau(G)|_{p'} \oplus \tau(G)|_{q'} = \mathrm{Hom}\big(G|_p \oplus G|_q, \omega_{\widetilde{C}}|_{r'}\big)$ which vanish on L' and L respectively. There is also a natural involution of $\mathrm{Pic}^8(\widetilde{X})$, which sends $L \mapsto \omega_{\widetilde{X}} \otimes \sigma^* L^{-1}$.

(4.1.8) **Lemma**. The fixed locus $\widetilde{M}_0^\tau$ is a $\mathbf{P}^1$-bundle over $\mathrm{Pic}^{10}(\widetilde{X})^\tau$. This bundle has two canonical sections, corresponding to the subbundle pairs $(E_p, E_{q'})$ and $(E_q, E_{p'})$ of $E_p \oplus E_q$ and $E_{p'} \oplus E_{q'}$, which are contracted by $\widetilde{M}_0^\tau \to M_0^\tau$ after translation by $\pm[p, q']$. Away from these sections and their image, $\widetilde{M}_0^\tau \to M_0^\tau$ is an isomorphism.

Proof. Fix a presentation as in (4.1.6) corresponding to $L \subset G|_p \oplus G|_q$ and $L' \subset G|_{p'} \oplus G|_{q'}$, and suppose that $\tau(G) \cong G$. It fits into the following diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 \to \nu_*(G(-p-q-p'-q')) & \longrightarrow & \nu_*(G(-p-q-p'-q')) & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F & \longrightarrow & \nu_* G & \longrightarrow & \mathbf{C}_r \oplus \mathbf{C}_{r'} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L_r \oplus L'_{r'} & \longrightarrow & \nu_*\big(G|_p \oplus G|_q \oplus G|_{p'} \oplus G|_{q'}\big) & \longrightarrow & \mathbf{C}_r \oplus \mathbf{C}_{r'} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0. & &
\end{array}$$

It follows that $\tau(F) = \mathbf{Hom}_{\widetilde{C}}\big(\sigma^* F, \omega_{\widetilde{C}}\big)$ is the kernel of

$$\mathbf{Hom}_{\widetilde{C}}\big(\sigma^*\nu_*(G(-p-q-p'-q')), \omega_{\widetilde{C}}\big) \to \mathbf{Ext}^1_{\widetilde{C}}\big(\sigma^*\big(L|_r \oplus L'|_{r'}\big), \omega_{\widetilde{C}}\big).$$

Since ν is an affine morphism, ν_* commutes with σ^*. By Grothendieck duality

$$\mathbf{Hom}_{\widetilde{C}}\big(\sigma^*\nu_*(G(-p-q-p'-q')), \omega_{\widetilde{C}}\big) \to \mathbf{Ext}^1_{\widetilde{C}}\Big(\sigma^*\nu_*\Big(G|_p \oplus G|_q \oplus G|_{p'} \oplus G|_{q'}\Big), \omega_{\widetilde{C}}\Big)$$

is the direct image (up to twisting) of the evaluation map of $\tau(G)$ at p, q, p' and q'. Moreover $\tau(F)$ consists of those sections of $\nu_*\tau(G)$ which vanish on L and L'. In particular, if τ fixes the presentation then $F \cong \tau(F)$.

If $F \in M_0$ is locally free, then it has a unique presentation determined by $F \to \nu_*\nu^* F$ [AK90, Lemma 13]. When F is fixed by τ, so is $\nu^* F$ since $\tau(F) = \omega_{\widetilde{C}} \otimes \sigma^* F^{-1}$. The image of the presentation by τ is a presentation of $\tau(F)$, so it is also fixed by τ.

Given a presentation as in (4.1.6), F is invertible at r if and only if the associated map $G|_p \oplus G|_q \twoheadrightarrow \mathbf{C}$ is nonzero on both summands [OS79, Proposition 12.3]. The locus in $\widetilde{M}_0$ where F is locally free is therefore a $(\mathbf{C}^\times \times \mathbf{C}^\times)$-torsor over $\mathrm{Pic}^{10}\big(\widetilde{X}\big)$ which maps isomorphically onto $\mathrm{Pic}^{10}\big(\widetilde{C}\big)$. It follows from the above that the corresponding locus in $\widetilde{M}_0^\tau$ is a $\mathbf{C}^\times$-torsor over $\mathrm{Pic}^{10}\big(\widetilde{X}\big)^\tau$ which maps isomorphically onto $\mathrm{Pic}^{10}\big(\widetilde{C}\big)^\tau$.

If $F \in M_0^\tau$ is not locally free, then it is invertible at neither r nor r'. Each presentation of F can be obtained by applying the exact functor ν_* to the twisted ideal sheaf sequence

$$0 \to G(-s-t) \to G \to G|_s \oplus G|_t \to 0 \tag{4.1.9}$$

for some $G \in \mathrm{Pic}^{10}\big(\widetilde{X}\big)$, $s \in \{p, q\}$ and $t \in \{p', q'\}$ (see [AK90, Theorem 16] for details). Note that $\tau(G) \cong G$: by Grothendieck duality

$$\tau(\nu_*(G(-s-t))) \cong \nu_* \,\mathbf{Hom}_{\widetilde{X}}\big(\sigma^*(G(-s-t)), \omega_{\widetilde{X}}\big) \cong \nu_*\big(\omega_{\widetilde{X}} \otimes \sigma^* G^{-1}(s'+t')\big),$$

and $\nu_* :\ \mathrm{Pic}^8\big(\widetilde{X}\big) \to M_0$ is an embedding [AK90, §15].

There are exactly four such presentations. Two of them, namely those for which (s, t)

is (p,q') or (p',q), are fixed by τ. Indeed $\tau(G)|_s$ is the subspace of $\tau(G)|_p \oplus \tau(G)|_q$ which vanishes on $G|_t$. The other two presentations are not important for our purposes. Note that (4.1.9) belongs to the section of $\widetilde{M}_0 \to \mathrm{Pic}^{10}(\widetilde{X})$ determined by $(E_{t'}, E_{s'})$. The other τ-invariant presentation of $G(-s-t)$ is determined by

$$G(-s'-t') \otimes [s',t'] \hookrightarrow G \otimes [s',t'],$$

so $\widetilde{M}_0^\tau \to M_0^\tau$ glues the two sections after translating one of them by $\pm[s,t]$. ☺

(4.1.10) **Corollary**. There is a canonical bijection between the connected components of M_0^τ and the four components of $\mathrm{Pic}^8(\widetilde{X})^\tau$. Each component of M_0^τ can be obtained from a $\mathbf{P}^1$-bundle over the corresponding component of $\mathrm{Pic}^8(\widetilde{X})^\tau$, by gluing two sections together after translation. If $F \in M_0^\tau$ and $L \in \mathrm{Pic}^8(\widetilde{X})^\tau$ belong to corresponding components, then $\pi_* \mathrm{c}_1(F) = \omega_C$ if and only if $\pi_* \mathrm{c}_1(L) = \omega_X$. Moreover, if $L \in \mathcal{P}_\omega(X) \sqcup \mathcal{P}_\omega^-(X)$ then

$$h^0(F) \equiv h^0(L) \pmod 2.$$

Proof. Given a presentation as in (4.1.6), its image in $\mathrm{Pic}^8(\widetilde{X})$ is defined to be $G(-p'-q)$. The first two statements follow from (4.1.8). Indeed, even if $F \in M_0^\tau$ has multiple presentations, they belong to the same component because $[p',q] \in \mathcal{P}(X)$.

If $F \in M_0^\tau$ is locally free, then $L := \nu^* F(-p'-q)$ is the corresponding element of $\mathrm{Pic}^8(\widetilde{X})^\tau$. Set $x := \pi(p)$ and $y := \pi(q)$, so that $\nu^*\omega_C \cong \omega_X(x+y)$. Since $\pi_* \mathrm{c}_1(L)$ is either $\omega_X \cong \nu^*\omega_C(-x-y)$ or $\omega_X \otimes \nu^*\eta$, where $\eta \in \mathrm{Pic}^0(C)$ defines the cover $\widetilde{C} \to C$,

$$\pi_* \mathrm{c}_1(F) = \pi_*\nu_*\nu^* \mathrm{c}_1(F) = \nu_*\pi_* \mathrm{c}_1(L(p'+q))$$

is either ω_C or $\omega_C \otimes \eta$. For every presentation with middle term $\nu_*(L(p'+q))$, the first term has the same Chern classes as F, so $\pi_* \mathrm{c}_1(F)$ agrees with $\pi_* \mathrm{c}_1(L)$ even if F is not locally free.

For the parity check, set $G := L(p'+q)$ and suppose that $h^0(G) \equiv h^0(L) \pmod 2$. This forces $h^0(G(-p-q)) = h^0(G) - 1 = h^0(G(-p'-q'))$, so the images of $H^0(\widetilde{X}, G)$ in $G|_p \oplus G|_q$ and $G|_{p'} \oplus G|_{q'}$ are lines. The corresponding presentation is uniquely determined by the property that $h^0(F) = h^0(G)$ (where F is the first term). It is also τ-invariant, by an argument similar to the start of (4.1.8) but with **Hom** and **Ext** replaced by Hom and Ext. The τ-invariant presentations with middle term $\nu_* G$ correspond to different lines, so

$$H^0\big(\widetilde{C}, \nu_* G\big) \to H^0\big(\widetilde{C}, \mathbf{C}_r \oplus \mathbf{C}_{r'}\big) \tag{4.1.11}$$

is surjective, in which case $h^0(F) = h^0(G) - 2$.

Now suppose that $h^0(G) \not\equiv h^0(L) \pmod 2$. At least one of $h := h^0(G(-p-q))$ or $h' := h^0(G(-p'-q'))$ is smaller than $h^0(G)$, because $h^0(G(-p-q')) = h^0(G) - 1$. If only one of them is, then the image of (4.1.11) is one-dimensional, which implies that $h^0(F) = h^0(G) - 1$ for any presentation involving $F \hookrightarrow \nu_* G$. When $h = h' = h^0(G) - 2$ this can fail, but only for presentations which are not τ-invariant. ☺

(4.1.12) Let $M_T^\tau \subset M_{\widetilde{H}}^\tau$ be the fibre of $M_{\widetilde{H}}^\tau \to |\widetilde{H}|^\sigma$ over $\rho^* T \cap |\widetilde{H}|^\circ$. By (4.1.10) it has a unique component A on which h^0 is even and $\pi_* \, \mathrm{c}_1$ gives the canonical class. Let $\Theta \subset A$ be the locus where $h^0 \geq 2$. Note that A is a family of Prym torsors degenerating to a component $A_0 \subset M_0^\tau$. The theta divisor of this component has the following description, which comes from [IW19, Proposition 1.5]. We give a slightly different proof.

(4.1.13) **Lemma** (Izadi-Wang). Let $\widetilde{B} \to B$ be the blowup of $B := \mathcal{P}_\omega(X)$ along $\Xi \cap \Xi_{[p,q']}$, where $\Xi := \Theta_\omega(X)$. The central fibre Θ_0 of $\Theta \to T$ is obtained from $\widetilde{B}$ by gluing the proper transforms of Ξ and $\Xi_{[p,q']}$ after translation by $\pm[p,q']$.

Proof. If $L \in B \setminus \Xi$ then $h^0(L) = 0$. On the other hand $h^0(G) \geq 2$ for $G := L(p'+q)$ by Riemann-Roch. It follows that $h^0(G) = 2$, and the proof of (4.1.10) says that there is a unique presentation with middle term $\nu_* G$ such that the first term belongs to Θ_0.

Moreover, if $L \in \Xi \setminus \Xi_{[p,q']}$ then $h^0(G(-p-q')) = 0$, which means $h^0(G) = 2$ again, so $h^0(L) = 2$ and the uniqueness statement holds again. In this case, since p' and q are base points of G, the presentation belongs to the section determined by $(E_p, E_{q'})$. Similarly, the presentations lying over $\Xi_{[p,q']} \setminus \Xi$ belong to the section determined by $(E_q, E_{p'})$.

Finally, if $L \in \Xi \cap \Xi_{[p,q']}$ then $h^0(G) \geq 3$, because (for parity reasons) p, q, p' and q' cannot all be base points of G, so the kernel of (4.1.11) is always positive-dimensional. The result now follows by the universal property of blowing up and Zariski's main theorem. ☺

(4.1.14) The last result in this section is strictly weaker than [IW19, Proposition 2.5], but much easier to prove.

(4.1.15) **Lemma** (Izadi-Wang). If $F \in \Theta_0$ then Θ is smooth at F.

Proof. If Θ_0 is smooth at F then the exact sequence of Zariski tangent spaces

$$0 \to T_F\Theta_0 \to T_F\Theta \to T_0 T$$

reveals that $\dim(T_F\Theta) \leq \dim(T_F\Theta_0) + 1 = 5$ and hence Θ is smooth at F. Otherwise $T_F\Theta_0$ is 5-dimensional [AK90, Proposition 17], so it remains to check that $T_F\Theta \to T_0T$ vanishes. Since $M^\tau_{\widetilde{H}}$ is smooth, but M^τ_0 is singular at F, the map $T_F M^\tau_{\widetilde{H}} \to T_C|H|$ is not surjective. This implies that Θ is smooth at F provided that $T_0T \subset T_C|H|$ avoids the image of $T_F M^\tau_{\widetilde{H}}$. Since the singular locus of Θ_0 is only 3-dimensional, this holds for every F (assuming T is sufficiently general). ☺

4.2. Families of surfaces

(4.2.1) Our next task is to construct families of special surfaces in the fibres of $\Theta \to T$. Given a g^2_6 on C, i.e., a net of degree 6, one can define special surfaces $S_i \subset \widetilde{C}^{(6)}$ as in chapter 2. The relative version of this does not work over T, because the fibres of $\mathcal{C} \to T$ have no canonical choice of g^2_6. We will fix this by passing to a cover $U \to T$ which parametrises curves together with a g^2_6. Ideally, the cover should be unramified, in order to preserve the smoothness of Θ. Note that a general (smooth) curve of genus six has exactly five nets of degree 6 [ACGH85, V].

(4.2.2) **Lemma**. If $C \in \overline{\mathcal{M}}_6$ is general within the boundary component whose general member is an irreducible curve, then it has exactly five nets of degree 6.

Proof. Recall that $\mathrm{Pic}^6(C) \xrightarrow{\nu^*} \mathrm{Pic}^6(X)$ is a $\mathbf{C}^\times$-torsor with fibre

$$\mathbf{P}\big(L|_x \oplus L|_y\big) \setminus \big\{L|_x, L|_y\big\}$$

over $L \in \mathrm{Pic}^6(X)$, where $x, y \in X$ lie over the node $z \in C$ [OS79, Corollary 12.4]. Given $L' \in \mathrm{Pic}^6(C)$ with $\nu^* L' \cong L$, the corresponding line in $L|_x \oplus L|_y$ is the kernel of the subtraction map $L|_x \oplus L|_y \to L'|_z$. This map corresponds to the surjection in the following presentation:

$$0 \to L' \to \nu_* L \to L'|_z \to 0. \tag{4.2.3}$$

If $h^0(L') = 3$ then (4.2.3) forces $h^0(L) = 3$ by Clifford's theorem and the generality of X. Thus we may identify the maps $H^0(X, L) \to L|_w$ and $H^0(C, L') \to L'|_{\nu(w)}$ for each $w \in X$. In particular $H^0(X, L(-x-y)) = H^0(X, L(-x)) = H^0(X, L(-y))$ is two-dimensional (as X is neither hyperelliptic nor trigonal, it has no g^2_5).

Conversely, if $h^0(L) - 1 = h^0(L(-x-y)) = 2$, then exactly one L' lying over L has $h^0(L') = 3$. Indeed, if $h^0(L') = h^0(L)$ then by (4.2.3) the following composition is zero:

$$H^0(X, L) \to L|_x \oplus L|_y \to L'|_z.$$

Since $h^0(L(-x-y)) \neq h^0(L)$ the above sequence must be exact, so the line corresponding to L' is unique (and it is neither of the summands, because X has no g^2_5).

It remains to show that exactly five $L \in W^2_6(X)$ satisfy $h^0(L(-x-y)) = 2$. On a general smooth curve Y of genus five, every g^2_6 has the form $|\omega_Y(-D)|$ for a unique $D \in Y^{(2)}$. In particular, the pair $(X, \omega_X(-x-y))$ must be general, so the image of X in $|\omega_X(-x-y)|^\vee$ is a nodal sextic [ACG11, XXI, §10]. By the genus formula it has five nodes, so there are five $D \in X^{(2)}$ such that $h^0(\omega_X(-D-x-y)) = 2$, giving five choices for $L := \omega_X(-D)$. ☺

(4.2.4) It is well-known that a general cover in $\overline{\mathcal{R}}_6$ arises from a very ample linear system on an Enriques surface [MM83]; together with (4.1.2) this allows us to assume that C satisfies (4.2.2). We may also assume without loss of generality that each pair $\big(X, \nu^* g^2_6\big)$ is general. Let $T^\circ \subset T$ be a neighbourhood of 0 such that the curves in $T^\circ \setminus \{0\}$ are smooth and have exactly five nets of degree 6. The space $U \coloneqq \mathcal{G}_6^2(\mathcal{C}_{T^\circ})$ which parametrises nets of degree 6 on the curves in T° is clearly étale of degree 5 over T°.

(4.2.5) Each fibre of the family $\mathcal{C}_U \coloneqq \mathcal{C} \times_T U \to U$ has a canonical g_6^2. These linear systems form a $\mathbf{P}^2$-bundle $\mathcal{P} \to U$ which naturally embeds in the relative symmetric power $\mathcal{C}_U^{(6)}$. The family of surfaces in $\widetilde{\mathcal{C}}_U^{(6)}$ is defined by the cartesian diagram

$$\begin{array}{ccc} \mathcal{S} & \hookrightarrow & \widetilde{\mathcal{C}}_U^{(6)} \\ \downarrow & & \downarrow \\ \mathcal{P} & \hookrightarrow & \mathcal{C}_U^{(6)}. \end{array}$$

(4.2.6) The map $\mathcal{S} \to U$ naturally factors as $\mathcal{S} \to \widetilde{U} \to U$, where $\widetilde{U}$ is the double cover of U parametrising the connected components of the fibres of $\mathcal{S} \to U$. The following fact ensures that $\widetilde{U} \to U$ is unramified, so that $\Theta_{\widetilde{U}}$ is nonsingular:

(4.2.7) **Lemma**. For each $u \in U_0$, i.e., each g_6^2 on C, $\mathcal{S}_u$ has two components.

Proof. One checks (e.g. using (4.2.3)) that $\mathcal{S}_u$ is the image of the special subvariety of $\widetilde{X}^{(6)}$ determined by $\nu^*\mathrm{g}_6^2$, which has two connected components. These components are smooth, and their images in $\widetilde{C}^{(6)}$ are disjoint, provided that the image of $C \to \mathrm{g}_6^{2\vee}$ is admissible in the sense of Welters [Wel81, (9.2), (9.6)]. It is easy to see that a general plane sextic of geometric genus five is admissible [DH88, 1(c)]. Since the five pairs $\left(X, \nu^*\mathrm{g}_6^2\right)$ are general, the result follows (see [ACG11, XXI, §10] for details). ☺

(4.2.8) **Lemma**. There is a neighbourhood $\widetilde{U}^\circ \subseteq \widetilde{U}$ of $\widetilde{U}_0$ such that $\mathcal{S}_u$ is irreducible for all $u \in \widetilde{U}^\circ$ (and smooth unless $u \in \widetilde{U}_0$). Moreover $\mathcal{S}_{\widetilde{U}^\circ}$ is integral.

Proof. There is a criterion for a special surface in $\widetilde{X}^{(6)}$ to be smooth, due to Welters [Wel81, (8.13)]. Again it suffices to show that the image of $X \to \mathrm{g}_6^{2\vee}$ is admissible, which we know from the proof of (4.2.7). The same argument applies to a general curve of genus 6 together with a g_6^2, so $\mathcal{S}_u$ is smooth for most $u \in \widetilde{U}$ provided that T and R are sufficiently general.

Since $\mathcal{S} \to \mathcal{P}$ is finite, flat and generically étale, $\mathcal{S}$ satisfies Serre's conditions R_0 and S_1 (in fact it is Cohen-Macaulay), so $\mathcal{S}$ is reduced. The previous paragraph shows that $\mathcal{S}_{\widetilde{U}^\circ}$ is irreducible. ☺

(4.2.9) Taking the g^1_4 residual to each g^2_6 determines, in a completely analogous way, a family $\mathcal{D} \to \widetilde{U}$ of 1-dimensional special subvarieties in $\widetilde{\mathcal{C}}^{(4)}_{\widetilde{U}}$. If $u \in \widetilde{U}_0$ then $\mathcal{D}_u \to g^1_4$ is generically unramified (since $C \to g^{1\vee}_4$ is too), so $\mathcal{D}_u$ is generically smooth. Given a general divisor $D \in \mathcal{D}_u$, there is a curve $\widetilde{T} \subset \mathcal{D}_{\widetilde{U}^\circ}$ containing D such that $\widetilde{T} \to \widetilde{U}^\circ$ is étale. Such a curve can be obtained, for instance, by choosing a hyperplane section of $\mathcal{D}$ (in some projective embedding) which meets the smooth locus of $\mathcal{D}_u$ transversely, then removing the ramification locus from a component which contains D.

(4.2.10) The natural embedding of $\mathcal{S}_{\widetilde{T}} \coloneqq \mathcal{S} \times_{\widetilde{U}} \widetilde{T}$ in $\widetilde{\mathcal{C}}^{(10)}_{\widetilde{T}}$ induces a rational map $\mathcal{S}_{\widetilde{T}} \to \Theta_{\widetilde{T}}$, defined on the open subset $\mathcal{S}^\circ \subset \mathcal{S}_{\widetilde{T}}$ of divisors avoiding the nodes. Note that $\mathcal{S}^\circ_E \neq \varnothing$ for all $E \in \widetilde{T}$, and $\mathcal{S}^\circ_E = \mathcal{S}_E$ unless E lies over $0 \in T$. By (B.2.9), the image of $\mathcal{S}^\circ \to \Theta_{\widetilde{T}}$ is flat over some dense open set $\widetilde{T}^\circ \subset \widetilde{T}$. Since $\mathcal{S}^\circ$ is integral, the (scheme-theoretic) closure $\mathcal{V}$ of its image in $\Theta_{\widetilde{T}}$ agrees with that of $\mathcal{S}_{\widetilde{T}^\circ}$. In particular $\mathcal{V} \to \widetilde{T}$ is flat [Har77, III, 9.8].

Chapter 4, in full, is currently being prepared for submission for publication. Jonathan Conder, Edward Dewey, and Elham Izadi.

Chapter 5: Limit surfaces

5.1. The central fibre of the family of surfaces

(5.1.1) In this section we determine the fibre $V \subset \Theta_0$ of $\mathcal{V}$ over $D \in \widetilde{T}$. There is a unique divisor in the pencil $|\pi_* D|$ passing through the node of C; the corresponding divisor on the normalisation X can be written as $\pi_* E$ where $E = p + q' + u + v$ for some $u, v \in \widetilde{X}$. Since D is general it can be identified with a divisor on $\widetilde{X}$. Replacing v by v' if necessary, we may assume that D and E belong to the same curve $\widetilde{Y} \subset \widetilde{X}^{(4)}$ (among the two curves tetragonally related to $\widetilde{X}$ via $|\pi_* D|$). The divisor $F \coloneqq p + q' + u' + v'$ also belongs to $\widetilde{Y}$.

(5.1.2) **Lemma**. If $d, e, f \in \widetilde{Y}$ correspond to $D, E, F \in \widetilde{X}^{(4)}$, then V_{d+e+f} is smooth.

Proof. Set $g^1_5 \coloneqq |\omega_Y(-\pi_*(d+e+f))|$, and suppose for a moment that $Y \to g^{1\vee}_5$ is a well-defined morphism with only simple ramification. By Welters' criterion [Wel81, (8.13)], the associated special subvarieties $S_1, S_2 \subset \widetilde{Y}^{(5)}$ are smooth. We may assume that V_{d+e+f} is the image of S_1. If some pencil in $\widetilde{Y}^{(5)}$ meets S_1, its image in $Y^{(5)}$ must be g^1_5, so the pencil must be all of S_1. This is is absurd: (B.2.3) and the formula for the class of a pencil [ACGH85, VIII, §3] imply that $S_1 \to g^1_5$ has degree 1 or 2, but the degree is 2^4 by definition. Alternatively, note that S_1 has genus 21 because $S_1 \to g^1_5$ is ramified at 72 points (4 for each of the 18 branch points). Therefore S_1 maps isomorphically onto V_{d+e+f}.

It remains to show that $Y \to g^{1\vee}_5$ has simple ramification. For this, we just need the pair $\left(Y, g^1_5\right)$, or equivalently $(Y, \pi_*(d+e+f))$, to be sufficiently general [ACG11, XXI, (11.9)]. Given $(X, \pi_*(p+q))$, there is a one-dimensional family parametrising the data of:

- a g^2_6 on X with $g^2_6(-\pi_*(p+q))$ a pencil,
- a component (namely $\widetilde{Y} \subset \widetilde{X}^{(4)}$) of the special subvariety associated to the residual g^1_4,

- a divisor $u + v \in \widetilde{X}^{(2)}$ lying over $\omega_X(-g_6^2)$, and
- a point $d \in \widetilde{Y}$ (this is the one-dimensional part).

On the other hand, for a general pair $(Y, \pi_*(d+e+f))$ the divisor $\pi_*(e+f)$ belongs to only five pencils of degree 4 (this follows from (4.2.2) by taking residual pencils). Since e and f are in the same fibre of $\widetilde{Y} \to g_4^1$, this means there are at most finitely many choices for $(X, \pi_*(p+q))$ and the extra data giving rise to $(Y, \pi_*(d+e+f))$. For dimension reasons, it follows that a general pair $(Y, \pi_*(d+e+f))$ can be obtained from the construction of (5.1.1). ☺

(5.1.3) We will show that V is birational to

$$W := \Xi_{[d,e]} \cap \Xi_{[d,f]} = \left(\Xi \cap \Xi_{[u,v]}\right) + [d,f] \subset B,$$

where $B := \mathcal{P}_\omega(X)$ and $\Xi := \Theta_\omega(X)$. Since X is general, we may assume that $[u,v] \in \mathcal{P}(X)$ and $\beta := [p,q'] \in \mathcal{P}(X)$ are as well [Iza95, 4.6]. This implies that W and $\Xi \cap \Xi_\beta$ are smooth [Krä15, 2.1]. Recall from (4.1.13) that $\widetilde{B} \to B$ is the blowup of B along $\Xi \cap \Xi_\beta$.

(5.1.4) **Lemma**. The projection $\widetilde{B} \to B$ induces isomorphisms

$$\widetilde{W} \to W,$$

$$\Delta \cap \widetilde{W} \to V_{d+e+f},$$

$$\widetilde{\Xi} \cap \widetilde{W} \to W_d,$$

$$\widetilde{\Xi}_\beta \cap \widetilde{W} \to W_d + \beta,$$

where $\Delta \subset \widetilde{B}$ is the exceptional divisor and $\widetilde{Z} \subset \widetilde{B}$ is the proper transform whenever $Z \subset B$.

Proof. By (2.3.6) and (3.3.6) $\beta = [e,f]$, so W_d and $W_d + \beta$ are the embeddings of $\widetilde{Y}_\lambda$ in W. According to (3.3.5) $W_d \cap (W_d + \beta) = \varnothing$. It follows by (2.4.2b) and (2.4.2c) that

$$\Xi \cap \Xi_\beta \cap W = \Xi \cap \Xi_{[e,f]} \cap \Xi_{[d,e]} \cap \Xi_{[d,f]} = V_{d+e+f}.$$

If $\alpha \in V_{d+e+f}$ then $\alpha \notin W_d$ or $\alpha \notin W_d + \beta$, so $\Xi \cap \Xi_\beta \cap W$ is isomorphic to $\Xi \cap W$ or $\Xi_\beta \cap W$ near

α. This implies that $\Xi \cap \Xi_\beta \cap W$ is a Cartier divisor in W, giving the first two isomorphisms.

Moreover $\big(\widetilde{\Xi} \cap \widetilde{W}\big) \setminus \Delta \to (\Xi \cap W) \setminus \big(\Xi \cap \Xi_\beta\big) = W_d \setminus V_{d+e+f}$ is an isomorphism. Since V_{d+e+f} is smooth, Ξ intersects W transversely along $V_{d+e+f} \setminus W_d$, which means $\widetilde{\Xi} \cap \widetilde{W} = \widetilde{W}_d$ set-theoretically. This also holds scheme-theoretically because $\widetilde{\Xi} \cap \widetilde{W}$ is a Cartier divisor in the smooth variety $\widetilde{W}$, in particular S_1, so it is reduced. The fourth isomorphism is similar. ☺

(5.1.5) **Lemma**. V contains the image $\overline{W} \subset \Theta_0$ of $\widetilde{W} \subset \widetilde{B}$.

Proof. First note that $W = \big(\Xi \cap \Xi_{[u,v]}\big) + [d, f] = V_{u+v} + \mathcal{O}_{\widetilde{X}}(D - F')$ by (2.4.2a) and (2.3.6). Thus, a general point $\overline{L} \in \overline{W}$ corresponds to exactly one $L \in W \setminus V_{d+e+f}$, which can be represented by

$$G + u + v + D - F' = G + D - p' - q$$

for some divisor $G \in \widetilde{X}^{(6)}$ supported away from the nodes with $\pi_* G \in \nu^* g^2_6$. By abuse of notation we can think of G as a divisor on $\widetilde{C}$, in which case $\pi_* G \in g^2_6$ and $G + D \in \mathcal{S}^\circ$. According to (4.1.10) $\mathcal{O}_{\widetilde{C}}(G + D) = \overline{L}$. This shows that V contains an open subset of $\overline{W}$, and hence all of $\overline{W}$. ☺

(5.1.6) **Proposition**. $V = \overline{W}$.

Proof. Since $\overline{W} \subseteq V$, it suffices to show that V and $\overline{W}$ have the same Hilbert polynomial with respect to Θ_0. By (B.2.9), the former is $20n^2 - 40n + 22$.

Recall that $\widetilde{B} \to B$ factors through the $\mathbf{P}^1$-bundle $P \xrightarrow{\varphi} B$ parametrising τ-invariant presentations as in (4.1.10). The divisor $\widetilde{B}$, as described in (4.1.13), moves in a pencil spanned by $B_0 \cup \varphi^{-1}\big(\Xi_\beta\big)$ and $B_\infty \cup \varphi^{-1}(\Xi)$, where $B_0, B_\infty \subset P$ are the distinguished sections [IW19, 1.2]. Therefore $\widetilde{B}$ is polarised by

$$\big(B_0 \cup \varphi^{-1}(\Xi_\beta)\big)\big|_{\widetilde{B}} = \widetilde{\Xi} \cup \big(\widetilde{\Xi}_\beta \cup \Delta\big).$$

This restricts to a divisor on $\widetilde{W}$, which can (by (5.1.4)) be identified with

$$\Phi := W_d \cup (W_d + \beta) \cup V_{d+e+f} = W_d \cup \big(\Xi_\beta \cap W\big) \subset W.$$

Write ξ for the restriction of the algebraic equivalence class $[\Xi]$ to W, and let δ be the class of a point. Note that $\langle \xi, \xi \rangle = 4!\delta = 24\delta$. The normal bundle sequence for $W \hookrightarrow B$ gives

$$\mathrm{td}(W) = \mathrm{td}(\mathcal{O}_B(\Xi))^{-2} = \left(1 + \frac{\xi}{2} + 2\delta\right)^{-2} = 1 - \xi + 14\delta.$$

Up to algebraic equivalence $[\Phi] = \xi + \omega$, where $\omega := [W_d]$. Therefore

$$\frac{\langle \Phi, \Phi \rangle}{2} = \frac{1}{2}\big(24\delta + 2\xi\omega\big) = 12\delta + \xi\omega = 20\delta$$

by Beauville's formula (2.3.2). It follows that

$$\mathrm{ch}(\mathcal{O}_W(n\Phi))\,\mathrm{td}(W) = \big(1 + (\xi + \omega)n + 20\delta n^2\big)\big(1 - \xi + 14\delta\big).$$

The coefficient of δ in this expression is

$$\chi(\mathcal{O}_W(n\Phi)) = 20n^2 - \big(\xi + \omega\big)\xi n + 14 = 20n^2 - 32n + 14.$$

The quotient map $W \xrightarrow{\psi} \overline{W}$ identifies the disjoint curves W_d and $W_d + \beta$. There is a short exact sequence

$$0 \to \mathcal{O}_{\overline{W}} \to \psi_* \mathcal{O}_W \to \psi_* \mathcal{O}_{W_d} \to 0.$$

Twisting by $\Psi := \Theta_0|_{\overline{W}}$ and using the projection formula gives

$$0 \to \mathcal{O}_{\overline{W}}(n\Psi) \to \psi_* \mathcal{O}_W(n\Phi) \to \psi_* \mathcal{O}_{W_d}(n\Phi) \to 0.$$

Since $n\Phi$ has degree $8n$ on the genus 9 curve W_d, the Hilbert polynomial of $\overline{W}$ is

$$20n^2 - 32n + 14 - (8n - 8) = 20n^2 - 40n + 22,$$

as required. ☺

5.2. Completing the proof of the main theorem

(5.2.1) **Lemma**. If $\xi \coloneqq [\Xi]$ and $\omega \coloneqq [W_d]$, then

$$[\widetilde{W}] = (\xi^2, \xi - \omega) \text{ in } H^4(\widetilde{B}, \mathbf{Z}) = H^4(B, \mathbf{Z}) \oplus H^2(\Xi \cap \Xi_\beta, \mathbf{Z}).$$

Consequently $\langle [\widetilde{W}], [\widetilde{W}] \rangle = 16$.

Proof. Let $\varphi \colon \widetilde{B} \to B$ be the blowup and $\psi \colon \Delta \to \Xi \cap \Xi_\beta$ its restriction to Δ. By standard properties of the cohomology of a blowup [Bea77b, Proposition 0.1.3(ii)]

$$[\widetilde{W}] = \big(\varphi_*[\widetilde{W}], \psi_*\big([\widetilde{W}]|_\Delta\big)\big) = ([W], [V_{d+e+f}]) = (\xi^2, \xi - \omega).$$

Therefore, by (3.3.5)

$$\langle [\widetilde{W}], [\widetilde{W}] \rangle = \int_B \xi^4 + \int_\Delta \mathrm{c}_1(\mathcal{O}_\Delta(\Delta))\psi^*(\xi - \omega)^2 = 4! - \int_{\Xi \cap \Xi_\beta} \xi^2 - 2\xi\omega + \omega^2 = 2\int_B \frac{\xi^4}{3} = 16,$$

as required. ☺

(5.2.2) **Proposition**. If $t \in \widetilde{T}$ and Θ_t is smooth, then $\langle [\mathcal{V}_t], [\mathcal{V}_t] \rangle = 16$.

Proof. The class $[\mathcal{V}] \in \mathrm{CH}_3(\Theta_{\widetilde{T}})$ defines a family of 0-cycle classes $[\mathcal{V}]^2 \in \mathrm{CH}_1(\Theta_{\widetilde{T}})$ over $\widetilde{T}$, and $\langle [\mathcal{V}_t], [\mathcal{V}_t] \rangle$ is the degree of the specialisation of $[\mathcal{V}]^2$ at t [Ful98, 10.1]. Since $\Theta_{\widetilde{T}} \to \widetilde{T}$ is flat, specialisation at t is the same as restricting to Θ_t, for any $t \in \widetilde{T}$. We can specialise $[\mathcal{V}]^2$ to the central fibre, but since Θ_0 is singular the meaning of $\langle [V], [V] \rangle$ is not clear.

To rectify this, we pass to the operational Chow ring $\mathrm{CH}^*(\Theta_{\widetilde{T}})$, which acts on $\mathrm{CH}_*(\Theta_{\widetilde{T}})$ via cap product. There is a unique "Poincaré dual" $\nu \in \mathrm{CH}^2(\Theta_{\widetilde{T}})$ such that $\nu \cap [\Theta_{\widetilde{T}}] = [\mathcal{V}]$ [Ful98, 17.4]. Using the cap product on Chow groups [Ful98, 8.1] for the inclusion

$$\iota \colon \Theta_0 = \Theta_D \hookrightarrow \Theta_{\widetilde{T}},$$

one checks that $\iota^*\nu \cap [\Theta_D] = [V]$ and $\iota^*\nu^2 \cap [\Theta_D] = \iota^*[\mathcal{V}]^2$. It follows that

$$\int_{\Theta_t} [\mathcal{V}_t]^2 = \int_{\Theta_D} \iota^*[\mathcal{V}]^2 = \int_{\Theta_D} \iota^*\nu \cap [V] = \int_{\Theta_D} \iota^*\nu \cap \psi_*[\widetilde{W}] = \int_{\widetilde{B}} \psi^*\iota^*\nu \cap [\widetilde{W}] = \int_{\widetilde{B}} (\iota\psi)^*[\mathcal{V}] \cdot [\widetilde{W}],$$

where $\psi \colon \widetilde{B} \to \Theta_0$ is the usual resolution. If $\delta := (\iota\psi)^*[\mathcal{V}] - [\widetilde{W}]$ then

$$\psi_*\delta = \psi_*\big(\psi^*\iota^*\nu \cap [\widetilde{B}]\big) - \psi_*[\widetilde{W}] = \iota^*\nu \cap [\Theta_D] - [V] = 0.$$

This means δ is supported away from the smooth locus $U := \widetilde{B} \setminus (\widetilde{\Xi} \amalg \widetilde{\Xi}_\beta)$ of Θ_0. By a diagram chase around the localisation sequences

$$\begin{array}{ccccccccc}
\mathrm{CH}(U,1) & \longrightarrow & \mathrm{CH}(\widetilde{\Xi} \amalg \widetilde{\Xi}_\beta) & \longrightarrow & \mathrm{CH}(\widetilde{B}) & \longrightarrow & \mathrm{CH}(U) & \longrightarrow & 0 \\
\| & & \downarrow & & \downarrow \psi_* & & \| & & \\
\mathrm{CH}(U,1) & \longrightarrow & \mathrm{CH}(\Xi) & \longrightarrow & \mathrm{CH}(\Theta_0) & \longrightarrow & \mathrm{CH}(U) & \longrightarrow & 0,
\end{array}$$

one can find a class $\gamma = (\gamma_1, \gamma_2) \in \mathrm{CH}_2(\Xi)^2 \cong \mathrm{CH}_2(\widetilde{\Xi} \amalg \widetilde{\Xi}_\beta)$ which maps to $\delta \in \mathrm{CH}_2(\widetilde{B})$ and $0 \in \mathrm{CH}_2(\Xi)$ (here $\mathrm{CH}(U,1)$ is one of Bloch's higher Chow groups [Blo94], but all we need is a group depending only on U, which is easy to construct with a little thought). Since $\gamma_1 + \gamma_2 = 0$,

$$\int_{\widetilde{B}} \delta \cdot [\widetilde{W}] = \int_{\Xi} \gamma_1 \cdot [W_d] + \int_{\Xi} \gamma_2 \cdot [W_d] = 0$$

by (5.1.4). Therefore $\langle [\mathcal{V}_t], [\mathcal{V}_t] \rangle = \big\langle (\iota\psi)^*[\mathcal{V}], [\widetilde{W}] \big\rangle = \big\langle [\widetilde{W}], [\widetilde{W}] \big\rangle = 16$, as required. ☺

Chapter 5, in full, is currently being prepared for submission for publication. Jonathan Conder, Edward Dewey, and Elham Izadi.

Appendix A: Combinatorics

A.1. Generating functions

(A.1.1) **Lemma**. If $g, p \in \mathbf{N}$ and $p < g$, then

$$\left[\left(\frac{t}{1-e^{-t}}\right)^g \frac{2e^{-pt}}{1+e^t}\right]_{t^{g-1}} = \frac{(-1)^p}{2^{g-1}}.$$

Proof. It is well-known [GKP94, (7.52)] that

$$\left(\frac{t}{1-e^{-t}}\right)^g = \sum_{k=0}^{\infty} \left[{g \atop g-k}\right] \binom{g-1}{k}^{-1} \frac{t^k}{k!},$$

where $\left[{g \atop g-k}\right]$ is a Stirling number of the first kind [GKP94, §6.1]. Moreover

$$\frac{2e^{-pt}}{1+e^t} = \sum_{k=0}^{\infty} E_k(-p) \frac{t^k}{k!}$$

is one definition of the Euler polynomials E_k [Dil10, 24.2.8]. Adding up the terms of degree $g-1$, our goal is to prove that

$$\sum_{k=0}^{g-1} \frac{E_k(-p)}{(g-1)!} \left[{g \atop k+1}\right] = \frac{(-1)^p}{2^{g-1}}. \tag{A.1.2}$$

There is an explicit formula for the Euler polynomials [GS08, Example 2.5], namely

$$E_k(t) = \sum_{i=0}^{k} \frac{1}{2^i} \sum_{j=0}^{i} (-1)^j \binom{i}{j} (t+j)^k.$$

When $t = 0$ this means

$$E_k(0) = \sum_{i=0}^{k} \frac{(-1)^i}{2^i} i! \left\{{k \atop i}\right\}, \tag{A.1.3}$$

where $\left\{ {k \atop i} \right\}$ is a Stirling number of the second kind [GKP94, (6.19)]. If $k > 0$ then

$$\sum_{i=1}^{k+1} \frac{(-1)^i}{2^{i-1}} (i-1)! \left\{ {k+1 \atop i} \right\}.$$

simplifies to (A.1.3) after applying the relation [GKP94, (6.3)]

$$\left\{ {k+1 \atop i} \right\} = i \left\{ {k \atop i} \right\} + \left\{ {k \atop i-1} \right\}.$$

It follows from the vanishing of $E_k(0)$ for even $k > 0$ [Dil10, 24.4.6] that

$$E_k(0) = \sum_{i=0}^{k} \frac{(-1)^{i-k}}{2^i} i! \left\{ {k+1 \atop i+1} \right\},$$

which also holds when $k = 0$. By the Stirling inversion formula [GKP94, Table 264]

$$\begin{aligned}
\sum_{k=0}^{g-1} \frac{E_k(0)}{(g-1)!} \left[{g \atop k+1} \right] &= \sum_{k=0}^{g-1} \sum_{i=0}^{k} \frac{(-1)^{i-k}}{2^i} \frac{i!}{(g-1)!} \left[{g \atop k+1} \right] \left\{ {k+1 \atop i+1} \right\} \\
&= \sum_{i=0}^{g-1} \frac{(-1)^{i+1-g}}{2^i} \frac{i!}{(g-1)!} \sum_{k=i}^{g-1} (-1)^{g-k-1} \left[{g \atop k+1} \right] \left\{ {k+1 \atop i+1} \right\} \\
&= \sum_{i=0}^{g-1} \frac{(-1)^{i+1-g}}{2^i} \frac{i!}{(g-1)!} \delta_{g,i+1} \\
&= \frac{1}{2^{g-1}},
\end{aligned}$$

as required. Since $E_k(-p) = 2(-p)^k - E_k(1-p)$ [Dil10, 24.4.2], it follows by induction that

$$\sum_{k=0}^{g-1} \frac{E_k(-p)}{(g-1)!} \left[{g \atop k+1} \right] = \sum_{k=0}^{g-1} \frac{2(-p)^k}{(g-1)!} \left[{g \atop k+1} \right] + \frac{(-1)^p}{2^{g-1}}.$$

It is well-known [GKP94, (6.11)] that

$$\sum_{k=1}^{g} (-p)^k \left[{g \atop k} \right] = (-p)(1-p) \cdots (g-1-p),$$

so (A.1.2) holds provided that $0 \le p \le g-1$. ☺

(A.1.4) **Corollary**. If g is a positive integer then

$$\left[\left(\frac{t}{1-e^{-t}} \right)^g \frac{2e^t}{1+e^t} \right]_{t^{g-1}} = 2 - \frac{1}{2^{g-1}} \text{ and } \left[\left(\frac{t}{1-e^{-t}} \right)^g \frac{2e^{-gt}}{1+e^t} \right]_{t^{g-1}} = \frac{(-1)^g}{2^{g-1}} - 2(-1)^g.$$

Proof. The argument is similar to (A.1.1), but this time $(-p)(1-p)\cdots(g-1-p) \neq 0$. Note that $E_k(1) = 2\delta_{0k} - E_k(0)$ [Dil10, 24.4.2] and $\left[{g \atop 1}\right] = (g-1)!$ [GKP94, (6.5)]. ☺

A.2. Lattices

(A.2.1) Given an abelian group L, define $L^\vee := \mathrm{Hom}(L, \mathbf{Z})$. A *lattice* is a free abelian group L of finite rank, together with a symmetric bilinear form $\langle -, - \rangle : L \otimes L \to \mathbf{Z}$ which is *nondegenerate*, meaning that the associated map $L \to L^\vee$ is injective. If this map is also surjective, then L is *unimodular*. The *Gram matrix* Q_L for L with respect to a basis (α_i) has entries $\langle \alpha_i, \alpha_j \rangle$, and the *discriminant* of L is $\det(L) := \det(Q_L)$. Since automorphisms of L have determinant ± 1, the latter is independent of the chosen basis. The *signature* of L is the pair $(p, n) \in \mathbf{N}^2$ such that the $\mathbf{R}$-bilinear extension of the form to $L \otimes_{\mathbf{Z}} \mathbf{R}$ has Gram matrix

$$\begin{pmatrix} I_p & 0 \\ 0 & -I_n \end{pmatrix}$$

with respect to some basis; it is well-defined by Sylvester's law [Art91, Theorem 2.11]. If $\langle \alpha, \alpha \rangle \in 2\mathbf{Z}$ for all $\alpha \in L$, then L is *even*; otherwise L is *odd*.

(A.2.2) **Lemma**. If L is a lattice, then its image in $L^\vee$ has index $|\det(L)|$.

Proof. Given a basis (α_i) for L, let $(\alpha_i^\vee)$ be the dual basis for $L^\vee$. Since $\sum_i \alpha_i \alpha_i^\vee(-) = \mathbf{1}_L$, the image of α_k in $L^\vee$ is $\sum_i \langle \alpha_k, \alpha_i \rangle \alpha_i^\vee$. In other words, Q_L is the matrix of $L \hookrightarrow L^\vee$ with respect to our bases. By changing both bases, we may assume Q_L is in Smith normal form (in particular, diagonal). This only modifies the determinant of Q_L by a factor of ± 1. It follows easily that $[L^\vee : L] = \left[\mathbf{Z}^{\dim(L)} : \mathrm{Im}(Q_L)\right] = |\det(Q_L)|$. ☺

(A.2.3) **Lemma**. Let $M \hookrightarrow L$ be an embedding of lattices.

(a) If $[L : M] < \infty$, then

$$|\det(M)| = |\det(L)|[L : M]^2.$$

(b) If L is unimodular and M is primitive (i.e. L/M is torsion-free), then

$$|\det(M^{\perp})| = |\det(M)|,$$

where $M^{\perp} = \mathrm{Ker}(L \xrightarrow{\sim} L^{\vee} \to M^{\vee})$ is the orthogonal complement of M.

Proof. Both statements follow from the starred formulae on [Mar03, p. 28]. Alternatively, suppose that $[L : M] < \infty$, so that L/M is a finite abelian group. There is an exact sequence

$$0 \to L^{\vee} \to M^{\vee} \to \mathrm{Ext}^1(L/M, \mathbf{Z}) \to 0$$

and hence $[M^{\vee} : L^{\vee}] = [L : M]$. Part (A.2.3a) now follows from (A.2.2) by viewing $M \hookrightarrow M^{\vee}$ as the composition $M \hookrightarrow L \hookrightarrow L^{\vee} \hookrightarrow M^{\vee}$.

For (A.2.3b), L/M is a projective $\mathbf{Z}$-module, so the dual sequence becomes

$$0 \to (L/M)^{\vee} \to L^{\vee} \to M^{\vee} \to 0.$$

In particular $L \xrightarrow{\sim} L^{\vee} \twoheadrightarrow M^{\vee} \twoheadrightarrow M^{\vee}/M$ is surjective. Its kernel is clearly $M \oplus M^{\perp}$, so

$$[L : M \oplus M^{\perp}] = [M^{\vee} : M] = |\det(M)| < \infty.$$

An easy consequence is that $L^{\vee}$ embeds in $(M \oplus M^{\perp})^{\vee}$, and hence $M \oplus M^{\perp}$ is a lattice. Applying (A.2.3a) to $M \oplus M^{\perp}$ gives

$$|\det(M)\det(M^{\perp})| = |\det(M \oplus M^{\perp})| = [L : M \oplus M^{\perp}]^2 = \det(M)^2.$$

The result follows by cancelling $|\det(M)|$. ☺

(A.2.4) The following statement seems like it should be known, but we were unable to find a reference for it. When $a + b = n$ the numbers involved grow quickly, e.g. $d^7_{3,4} = 47775744$, $d^8_{3,5} = 9760764780783360$ and $d^9_{4,5} = 70333722607339201875244530794 4960$.

(A.2.5) **Proposition**. Let $0 \le a \le b \le n$ be integers, and set $S \coloneqq \{1, \dots, n\}$. Write

$$\binom{S}{a} = \left\{A_1, \dots, A_{\binom{n}{a}}\right\} \text{ and } \binom{S}{b} = \left\{B_1, \dots, B_{\binom{n}{b}}\right\}.$$

Let $M = M^n_{a,b}$ be the $\binom{n}{a} \times \binom{n}{b}$ matrix with entries

$$M^i_j \coloneqq \begin{cases} 1 & \text{if } A_i \subseteq B_j, \\ 0 & \text{otherwise.} \end{cases}$$

The product of the diagonal entries of the Smith normal form of M is

$$d^n_{a,b} \coloneqq \begin{cases} \prod_{i=1}^{a} \left(1 + \frac{b-a}{i}\right)^{\binom{n}{a-i}} & \text{if } a+b \le n, \\ \prod_{i=1}^{n-b} \left(1 + \frac{b-a}{i}\right)^{\binom{n}{b+i}} & \text{if } a+b \ge n. \end{cases}$$

In particular, the torsion subgroup of $\operatorname{Coker}\left(M : \mathbf{Z}^{\binom{n}{b}} \to \mathbf{Z}^{\binom{n}{a}}\right)$ has order $d^n_{a,b}$.

Proof. The cases where $a + b > n$ follow from those with $a + b < n$, because $A_i \subseteq B_j$ iff $S \setminus B_j \subseteq S \setminus A_i$ and hence $M^\top = M^n_{n-b,n-a}$. If $a = 0$ (resp. $b = n$) then M has a single row (resp. column) consisting only of ones, and $d^n_{a,b} = 1$ by convention. Moreover, if $a = b$ then $M = \mathbf{1}$, and again $d^n_{a,b} = 1$. Thus, we may assume that $0 < a < b < n$. By sorting the subsets of S which belong to $T \coloneqq S \setminus \{n\}$ before those which do not, M decomposes as

$$\begin{pmatrix} M^{n-1}_{a,b} & M^{n-1}_{a,b-1} \\ 0 & M^{n-1}_{a-1,b-1} \end{pmatrix}.$$

In particular, we have the following commutative diagram of short exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{Z}^{\binom{n-1}{b}} & \longrightarrow & \mathbf{Z}^{\binom{n}{b}} & \longrightarrow & \mathbf{Z}^{\binom{n-1}{b-1}} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle M^{n-1}_{a,b}} & & \downarrow{\scriptstyle M} & & \downarrow{\scriptstyle M^{n-1}_{a-1,b-1}} & & \\ 0 & \longrightarrow & \mathbf{Z}^{\binom{n-1}{a}} & \longrightarrow & \mathbf{Z}^{\binom{n}{a}} & \longrightarrow & \mathbf{Z}^{\binom{n-1}{a-1}} & \longrightarrow & 0. \end{array}$$

If $a + b < n$, then M has more columns than rows, and the same holds for $M^{n-1}_{a-1,b-1}$. Moreover $M^{n-1}_{a,b}$ has at least as many columns as rows. By induction on n, the cokernels of the latter two matrices have orders $d^{n-1}_{a-1,b-1}$ and $d^{n-1}_{a,b}$ respectively. Since $\operatorname{Ker}\left(M^{n-1}_{a-1,b-1}\right)$ is torsion-free, it follows from the snake lemma that $\operatorname{Coker}(M)$ has order

$$\begin{aligned} d^{n-1}_{a,b}d^{n-1}_{a-1,b-1} &= \prod_{i=1}^{a}\left(1+\frac{b-a}{i}\right)^{\binom{n-1}{a-i}}\cdot\prod_{i=1}^{a-1}\left(1+\frac{b-a}{i}\right)^{\binom{n-1}{a-1-i}} \\ &= \left(1+\frac{b-a}{a}\right)\prod_{i=1}^{a-1}\left(1+\frac{b-a}{i}\right)^{\binom{n-1}{a-1-i}+\binom{n-1}{a-i}} \\ &= d^{n}_{a,b}. \end{aligned}$$

It remains to consider the case $a+b=n$. Since matrices invertible over $\mathbf{Z}$ are unimodular, it suffices to show that $|\det(M)| = d^n_{a,b}$. We may assume (by induction) that $M^{n-1}_{a,b-1}$ is invertible over $\mathbf{Q}$. Changing our basis for $\mathbf{Z}^{\binom{n}{b}}$ gives

$$M^n_{a,b} = \begin{pmatrix} M^{n-1}_{a,b-1} & M^{n-1}_{a,b} \\ M^{n-1}_{a-1,b-1} & 0 \end{pmatrix} = \begin{pmatrix} M^{n-1}_{a,b-1} & 0 \\ M^{n-1}_{a-1,b-1} & I_{\binom{n-1}{a-1}} \end{pmatrix}\begin{pmatrix} I_{\binom{n-1}{b-1}} & \left(M^{n-1}_{a,b-1}\right)^{-1}M^{n-1}_{a,b} \\ 0 & -N \end{pmatrix}, \tag{A.2.6}$$

where $N \coloneqq M^{n-1}_{a-1,b-1}\left(M^{n-1}_{a,b-1}\right)^{-1}M^{n-1}_{a,b}$. To prove that $N = \frac{b-a+1}{b-a}M^{n-1}_{a-1,b}$, fix $A \in \binom{T}{a-1}$ and $B \in \binom{T}{b}$. Let N^A_B be the entry of N corresponding to A and B, and likewise for the other matrices. Since there are $|B \setminus A'| = b-a$ sets in $\binom{B}{b-1}$ containing a given $A' \in \binom{B}{a}$,

$$\begin{aligned} N^A_B &= \sum_{A\subseteq B'\in\binom{T}{b-1}}\sum_{A'\in\binom{B}{a}}\left(\left(M^{n-1}_{a,b-1}\right)^{-1}\right)^{B'}_{A'} \\ &= \frac{1}{b-a}\sum_{A\subseteq B'\in\binom{T}{b-1}}\sum_{B''\in\binom{B}{b-1}}\sum_{A'\in\binom{B''}{a}}\left(\left(M^{n-1}_{a,b-1}\right)^{-1}\right)^{B'}_{A'}. \end{aligned}$$

On the other hand, for each B' and B'' the equation $\left(M^{n-1}_{a,b-1}\right)^{-1}M^{n-1}_{a,b-1} = \mathbf{1}$ gives

$$\sum_{A'\in\binom{B''}{a}}\left(\left(M^{n-1}_{a,b-1}\right)^{-1}\right)^{B'}_{A'} = \mathbf{1}^{B'}_{B''}.$$

Therefore $(b-a)N^A_B$ counts the number of sets $B'' \in \binom{B}{b-1}$ which contain A. If $A \not\subseteq B$ then there are no such sets, so $N^A_B = 0$. Otherwise there are $|B \setminus A| = b-a+1$ choices, so

$N_B^A = \frac{b-a+1}{b-a}$. It follows that $N = \frac{b-a+1}{b-a} M_{a-1,b}^{n-1}$, as claimed. Finally (A.2.6) gives

$$\begin{aligned}
|\det(M)| &= \left|\det\left(M_{a,b-1}^{n-1}\right) \det\left(\frac{b-a+1}{b-a} M_{a-1,b}^{n-1}\right)\right| \\
&= \left(\frac{b-a+1}{b-a}\right)^{\binom{n-1}{a-1}} d_{a,b-1}^{n-1} d_{a-1,b}^{n-1} \\
&= \left(\frac{b-a+1}{b-a}\right)^{\binom{n-1}{a-1}} \prod_{i=1}^{a}\left(\frac{i+b-a-1}{i}\right)^{\binom{n-1}{a-i}} \cdot \prod_{i=1}^{a-1}\left(\frac{i+b-a+1}{i}\right)^{\binom{n-1}{a-1-i}} \\
&= \frac{1}{a} \prod_{i=1}^{a-1}\left(\frac{1}{i}\right)^{\binom{n-1}{a-i}+\binom{n-1}{a-1-i}} \cdot \prod_{i=2}^{a}(i+b-a-1)^{\binom{n-1}{a-i}} \cdot \prod_{i=0}^{a-1}(i+b-a+1)^{\binom{n-1}{a-1-i}} \\
&= \prod_{i=1}^{a}\left(\frac{1}{i}\right)^{\binom{n}{a-i}} \cdot \prod_{i=1}^{a-1}(i+b-a)^{\binom{n-1}{a-i-1}} \cdot \prod_{i=1}^{a}(i+b-a)^{\binom{n-1}{a-i}} \\
&= \prod_{i=1}^{a}\left(\frac{i+b-a}{i}\right)^{\binom{n}{a-i}} \\
&= d_{a,b}^{n},
\end{aligned}$$

as required. ☺

Appendix B: Algebraic geometry

B.1. Cohomology of symmetric powers of curves

(B.1.1) Given a smooth curve X of genus g and a positive integer d, the cohomology ring $H^*(X^{(d)}, \mathbf{Q})$ of the symmetric power $X^{(d)} := X^d/S_d$ is identified (via pullback) with the invariant subring $H^*(X^d, \mathbf{Q})^{S_d} \subset H^*(X^d, \mathbf{Q})$ [Mac62, (1.2)]. Let $\alpha_1, \dots, \alpha_{2g}$ be a symplectic basis for $H^1(X, \mathbf{Q})$, in the sense that

$$\langle \alpha_{2i-1}, \alpha_{2i} \rangle = -\langle \alpha_{2i}, \alpha_{2i-1} \rangle = 1$$

and $\langle \alpha_i, \alpha_j \rangle = 0$ for all other indices i and j. Also let $\beta \in H^2(X, \mathbf{Q})$ be the class of a point. The invariant classes $\xi_i := \pi_1^*\alpha_i + \cdots + \pi_d^*\alpha_i$ and $\eta := \pi_1^*\beta + \cdots + \pi_d^*\beta$ generate $H^*(X^{(d)}, \mathbf{Q})$ [Mac62, (3.1)]. In fact, the monomials $\eta^p \xi_{i_1} \cdots \xi_{i_q}$ of degree $r := 2p + q$ for which $i_1 < \cdots < i_q$ and $p \geq r - d$ form a basis for $H^r(X^{(d)}, \mathbf{Q})$ [Mac62, (3.2)]. The remaining monomials can be expressed in this basis using the following relation. In order to easily state the relation and deduce it from well-known relations, we first introduce some notation.

(B.1.2) An *index* $I = (p_I, Q_I)$ for $X^{(d)}$ consists of $p_I \in \mathbf{N}$ and a subset $Q_I \subseteq \{1, \dots, 2g\}$. If $i_1 < \cdots < i_q$ are the elements of Q_I, we write

$$\xi_I := \eta^{p_I} \xi_{i_1} \cdots \xi_{i_q}$$

and set $r_I := \mathrm{Deg}(\xi_I) = 2p_I + q$. If $p_I \geq r_I - d$, we say I is *basic* (because then ξ_I belongs to our basis for $H^{r_I}(X^{(d)}, \mathbf{Q})$). Let $B_I \subseteq \{1, \dots, g\}$ be the set of indices i such that $2i-1, 2i \in Q_I$, and $A_I \subseteq Q_I$ the set of "unpaired" indices, i.e. those i such that $i - (-1)^i \notin Q_I$. Finally, set $a_I := |A_I|$ and $b_I := |B_I|$. If J is another such pair, then $J \leq I$ means that $r_J = r_I$, $A_J = A_I$ and $B_J \subseteq B_I$. In other words, ξ_J can be obtained from ξ_I by replacing factors of the form $\sigma_i := \xi_{2i-1}\xi_{2i}$ by η.

(B.1.3) **Proposition**. If I is an index which is not basic then

$$\xi_I = \sum_{J \le I \text{ basic}} \binom{p_I - r_I + d}{p_J - r_I + d} \xi_J.$$

Proof. The relation $\xi_{(p_I, A_I)}(\sigma_{i_1} - \eta) \cdots (\sigma_{i_b} - \eta) = 0$, where $i_1 < \cdots < i_b$ are the elements of B_I, is well-known [Mac62, (6.31)]. Since η and the σ_i are central in $H^*(X^d, \mathbf{Q})$, it expands to give

$$\sum_{J \le I} (-1)^{p_J - p_I} \xi_J = 0.$$

By (reverse) induction on p_I, the coefficient of a basic index $J < I$ in $\xi_I = -\sum_{J<I}(-1)^{p_J - p_I} \xi_J$ is $(-1)^{p_J - p_I + 1} + c_J$, where

$$c_J := \sum_K (-1)^{p_K - p_I + 1} \binom{p_K - r_K + d}{p_J - r_K + d}$$

and the sum is taken over nonbasic indices $J < K < I$. Each such K corresponds to a set $B_J \subset B_K \subset B_I$ such that $|B_I \setminus B_K| = p_K - p_I$. There are $\binom{p_J - p_I}{p_K - p_I}$ such sets (and $r_K = r_I$), so

$$c_J = \sum_{p = p_I + 1}^{r_I - d - 1} (-1)^{p - p_I + 1} \binom{p - r_I + d}{p_J - r_I + d} \binom{p_J - p_I}{p - p_I}.$$

The identity $\binom{n}{k} = (-1)^k \binom{k-n-1}{k}$ gives

$$\binom{n}{k} = (-1)^k \binom{k-n-1}{k} = (-1)^k \binom{k-n-1}{-n-1} = (-1)^{k-n-1} \binom{-k-1}{-n-1} \tag{B.1.4}$$

for all integers $k > n$. It follows that

$$\begin{aligned} c_J &= (-1)^{p_J - p_I} \sum_{p = p_I + 1}^{r_I - d - 1} \binom{r_I - p_J - d - 1}{r_I - p - d - 1} \binom{p_J - p_I}{p - p_I} \\ &= (-1)^{p_J - p_I} \sum_{p=1}^{r_I - p_I - d - 1} \binom{r_I - p_J - d - 1}{r_I - p_I - p - d - 1} \binom{p_J - p_I}{p}. \end{aligned}$$

The sum can be simplified using the Chu-Vandermonde identity [GKP94, (5.22)]:

$$(-1)^{p_I - p_J} c_J = \binom{r_I - p_I - d - 1}{r_I - p_I - d - 1} - \binom{r_I - p_J - d - 1}{r_I - p_I - d - 1} \binom{p_J - p_I}{0} = 1 - \binom{r_I - p_J - d - 1}{r_I - p_I - d - 1}.$$

Applying (B.1.4) again gives

$$c_J = (-1)^{p_J - p_I} + \binom{p_I - r_I + d}{p_J - r_I + d}$$

and hence $\binom{p_I - r_I + d}{p_J - r_I + d}$ is the coefficient of ξ_J in ξ_I, as required. ☺

(B.1.5) Let $\pi : \widetilde{X} \to X$ be a finite morphism of degree m, with $\widetilde{X}$ a smooth curve of genus $\tilde{g}$. If $\pi^{(d)} : \widetilde{X}^{(d)} \to X^{(d)}$ is induced by π, then ${\pi^{(d)}}^{*} : H^*\big(X^{(d)}, \mathbf{Q}\big) \to H^*\big(\widetilde{X}^{(d)}, \mathbf{Q}\big)$ is easily computed given π^*. The Gysin push-forward $\pi_*^{(d)} : H^*\big(\widetilde{X}^{(d)}, \mathbf{Q}\big) \to H^*\big(X^{(d)}, \mathbf{Q}\big)$ is likewise determined by π_*, but in a more complicated way, which we work out below. In the following $\tilde{\beta}$ and $\widetilde{\alpha}_1, \dots, \widetilde{\alpha}_{2\tilde{g}}$ are the analogues of β and the α_i for $\widetilde{X}$, and likewise for $\tilde{\eta}$, etc.

(B.1.6) **Proposition**. Let $a_{\widetilde{Q}}^{Q}$ be the matrix of

$$\textstyle\bigwedge^* \pi_* : \ \bigwedge^* H^1(\widetilde{X}, \mathbf{Q}) \to \bigwedge^* H^1(X, \mathbf{Q}),$$

with $Q \subseteq \{1, \dots, 2g\}$ indexing our basis for $\bigwedge^* H^1(X, \mathbf{Q})$, and likewise for $\widetilde{Q}$. If I is a basic index for $\widetilde{X}^{(d)}$, then

$$\pi_*^{(d)} \tilde{\xi}_I = \sum_{J \le I} \sum_{K \text{ basic}} c_{KJ} \sum_{L \le K} (-1)^{p_L - p_K} \xi_L,$$

where

$$c_{KJ} := \begin{cases} m^{d - r_J + p_J} a_{Q_J}^{Q_K} & \text{if } p_K = p_J \text{ (and } r_K = r_J\text{)}, \\ 0 & \text{otherwise,} \end{cases}$$

for each basic index K for $X^{(d)}$.

Proof. In order to continue working with invariant classes, we need to check that the following diagram commutes. The leftmost map is well-defined because the Gysin pushforward is S_d-equivariant.

$$\begin{array}{ccc} H^*\big(\widetilde{X}^d, \mathbf{Q}\big)^{S_d} & \xleftarrow{\ \sim\ } & H^*\big(\widetilde{X}^{(d)}, \mathbf{Q}\big) \\ \Big\downarrow \pi_*^d & & \Big\downarrow \pi_*^{(d)} \\ H^*\big(X^d, \mathbf{Q}\big)^{S_d} & \xleftarrow{\ \sim\ } & H^*\big(X^{(d)}, \mathbf{Q}\big) \end{array}$$

This can be accomplished by composing with the pushforward $H^*(X^d, \mathbf{Q})^{S_d} \to H^*(X^{(d)}, \mathbf{Q})$, which is (up to a factor of $d!$) the inverse of the pullback (by the projection formula).

If I is a basic index for $\widetilde{X}^{(d)}$, with $p := p_I$ and $i_1 < \cdots < i_q$ the elements of Q_I, then by definition

$$\tilde{\xi}_I = \sum_{j_1=1}^{d} \pi_{j_1}^* \tilde{\beta} \dots \sum_{j_p=1}^{d} \pi_{j_p}^* \tilde{\beta} \sum_{k_1=1}^{d} \pi_{k_1}^* \widetilde{\alpha}_{i_1} \dots \sum_{k_q=1}^{d} \pi_{k_q}^* \widetilde{\alpha}_{i_q}.$$

The terms for which the indices j_i and k_i are pairwise distinct are essentially cross products of classes in $H^*(\widetilde{X}, \mathbf{Q})$. The cross product is natural for morphisms of even relative dimension (up to a sign in general) [Spa95, 5.3.10, 5.6.21]. To evaluate π_*^d on such a term, first apply a permutation so that the indices are in order (this may introduce a factor of -1). For each missing index $i \in \{1, \dots, d\}$, insert $\pi_i^* 1$ in the appropriate spot. Now push forward, and undo the permutation (which cancels the -1 we may have picked up before). The resulting class is

$$m^{d-p-q} \pi_{j_1}^* \beta \cdots \pi_{j_p}^* \beta \cdot \pi_{k_1}^* (\pi_* \widetilde{\alpha}_{i_1}) \cdots \pi_{k_q}^* (\pi_* \widetilde{\alpha}_{i_q}), \tag{B.1.7}$$

because $\pi_* \tilde{\beta} = \beta$ and there are $d - p - q$ copies of 1, which pushes forward to $\mathrm{Deg}(\pi) = m$.

The other terms in $\tilde{\xi}_I$ are typically zero, because $H^k(\widetilde{X}, \mathbf{Q}) = 0$ for $k \geq 3$, while $\widetilde{\alpha}_i \widetilde{\alpha}_j = 0$ unless $j = i - (-1)^i$. Since $\pi_i^* \widetilde{\alpha}_{2j-1} \pi_i^* \widetilde{\alpha}_{2j} = \pi_i^* \tilde{\beta}$, we can remove repeated indices from the nonzero terms (thereby increasing p_I, and removing the pairs $\{2j-1, 2j\}$ from Q_I). This gives an expression

$$\tilde{\xi}_I = \sum_{J \leq I} \tilde{\varepsilon}_J,$$

where each $\tilde{\varepsilon}_J$ is the sum of the terms in $\tilde{\xi}_J$ for which the indices j_i and k_i are distinct.

Given a basic index K for $X^{(d)}$, define ε_K in the same way (using ξ_K instead of $\tilde{\xi}_K$). It is not hard to show (by reverse induction on p and the binomial theorem) that

$$\varepsilon_K = \sum_{L \leq K} (-1)^{p_L - p_K} \xi_L.$$

Thus, it remains to show that

$$\pi_*^d \tilde{\varepsilon}_I = \sum_{K \text{ basic}} c_{KI} \varepsilon_K.$$

Using (B.1.7), we compute that

$$\begin{aligned}
\pi^d_* \tilde{\varepsilon}_I &= m^{d-p-q} \sum_{j_1,\dots,j_p,k_1,\dots,k_q} \pi^*_{j_1}\beta \cdots \pi^*_{j_p}\beta \cdot \pi^*_{k_1}\left(\pi_*\widetilde{\alpha}_{i_1}\right) \cdots \pi^*_{k_q}\left(\pi_*\widetilde{\alpha}_{i_q}\right) \\
&= m^{d-p-q} \sum_{j_1,\dots,j_p,k_1,\dots,k_q} \pi^*_{j_1}\beta \cdots \pi^*_{j_p}\beta \sum_{l_1,\dots,l_q} \pi^*_{k_1}\left(a_{l_1 i_1}\alpha_{l_1}\right) \cdots \pi^*_{k_q}\left(a_{l_q i_q}\alpha_{l_q}\right) \\
&= m^{d-p-q} \sum_{l_1,\dots,l_q} a_{l_1 i_1} \cdots a_{l_q i_q} \sum_{j_1,\dots,j_p,k_1,\dots,k_q} \pi^*_{j_1}\beta \cdots \pi^*_{j_p}\beta \cdot \pi^*_{k_1}\alpha_{l_1} \cdots \pi^*_{k_q}\alpha_{l_q}, \qquad \text{(B.1.8)}
\end{aligned}$$

where $j_1, \dots, j_p, k_1, \dots, k_q \in \{1, \dots, d\}$ are required to be distinct but $l_1, \dots, l_q \in \{1, \dots, 2g\}$ are arbitrary. If $l_i = l_j$ then the transposition $\left(k_i\ k_j\right)$ acts on

$$\pi^*_{j_1}\beta \cdots \pi^*_{j_p}\beta \cdot \pi^*_{k_1}\alpha_{l_1} \cdots \pi^*_{k_q}\alpha_{l_q}$$

as multiplication by -1, so the terms with $k_i < k_j$ cancel with those for which $k_i > k_j$. Hence, we may assume the l_i are distinct in (B.1.8). Collecting the terms in (B.1.8) which share a given subset $\left\{l_1, \dots, l_q\right\} \subseteq R \coloneqq \{1, \dots, 2g\}$ gives

$$\begin{aligned}
\pi^d_* \tilde{\varepsilon}_I &= m^{d-p-q} \sum_{Q \in \binom{R}{q}} \sum_{\sigma \in S_q} a_{l_{\sigma(1)} i_1} \dots a_{l_{\sigma(q)} i_q} \sum_{j_1,\dots,j_p,k_1,\dots,k_q} \pi^*_{j_1}\beta \dots \pi^*_{j_p}\beta \cdot \pi^*_{k_1}\alpha_{l_{\sigma(1)}} \dots \pi^*_{k_q}\alpha_{l_{\sigma(q)}} \\
&= m^{d-p-q} \sum_{Q \in \binom{R}{q}} \sum_{\sigma \in S_q} (-1)^\sigma a_{l_{\sigma(1)} i_1} \dots a_{l_{\sigma(q)} i_q} \sum_{j_1,\dots,j_p,k_1,\dots,k_q} \pi^*_{j_1}\beta \dots \pi^*_{j_p}\beta \cdot \pi^*_{k_1}\alpha_{l_1} \dots \pi^*_{k_q}\alpha_{l_q} \\
&= m^{d-p-q} \sum_{Q \in \binom{R}{q}} \sum_{\sigma \in S_q} (-1)^\sigma a_{l_{\sigma(1)} i_1} \dots a_{l_{\sigma(q)} i_q} \varepsilon_{(p,Q)} \\
&= m^{d-p-q} \sum_{Q \in \binom{R}{q}} a_{Q Q_I} \varepsilon_{(p,Q)} \\
&= \sum_{K \text{ basic}} c_{KI} \varepsilon_K,
\end{aligned}$$

as required (note that $p + q = r_I - p$). ☺

B.2. Algebraic cycles on symmetric powers of curves

(B.2.1) It is well-known that $\eta \in H^2\left(X^{(d)}, \mathbf{Q}\right)$ is algebraic, being the class of $p + X^{(d-1)}$ for any $p \in X$ [Mac62, (14.4)]. There is another algebraic class $\theta \in H^2\left(X^{(d)}, \mathbf{Q}\right)$, namely the pullback of a theta divisor on $\mathrm{Pic}^d(X)$. In our notation

$$\theta = \sum_{i=1}^{g} \sigma_i.$$

The above formulae can be simplified by working in the subring of $H^*(X^{(d)}, \mathbf{Q})$ generated by η and θ. For general X this is equal to the algebraic part of $H^*(X^{(d)}, \mathbf{Q})$ [ACGH85, VIII, §5].

(B.2.2) **Corollary**. If $p, q \in \mathbf{N}$ and $p + 2q > d$ then

$$\eta^p \theta^q = \sum_{j=0}^{d-p-q} \binom{g-j}{q-j} \binom{d-p-2q}{d-p-q-j} \frac{q!}{j!} \eta^{p+q-j} \theta^j.$$

Proof. Note that $\theta^q = q! \sum_{B \in \binom{G}{q}} \sigma_B$, where $G := \{1, \dots, g\}$ and $\sigma_B := \prod_{b \in B} \sigma_b$. If we define $I_B := (p, (2B-1) \cup 2B)$ for $B \subseteq G$, then by (B.1.3)

$$\eta^p \theta^q = q! \sum_{B \in \binom{G}{q}} \xi_{I_B} = q! \sum_{B \in \binom{G}{q}} \sum_{J \leq I_B \text{ basic}} \binom{d-p-2q}{d+p_J-2p-2q} \xi_J.$$

A basic index J with $A_J = \varnothing$ appears in the sum $\binom{g-b_J}{q-b_J}$ times, because this is the number of sets $B \in \binom{G}{q}$ which contain B_J. The result now follows from the equation $p_J + b_J = p + q$. ☺

(B.2.3) **Corollary**. If $\pi : \widetilde{X} \to X$ is an étale double cover, then

$$\pi_*^{(d)}(\tilde{\eta}^p \tilde{\theta}^q) = 2^{d-p-q} \sum_{l=0}^{q} \binom{g-1}{q-l} \frac{q!}{l!} \eta^{p+q-l} \theta^l,$$

where $\theta \in H^2(X, \mathbf{Q})$ is the class of a theta divisor coming from $\mathrm{Pic}^d(X)$ and likewise for $\tilde{\theta}$.

Proof. Let $\alpha_1, \dots, \alpha_{2g}$ be a symplectic basis for $H^1(X, \mathbf{Q})$. The topological space underlying $\widetilde{X}$ can be constructed from X by cutting it along a loop representing (the Poincaré dual of) α_2 and gluing two copies along the resulting boundaries [BL04, Proposition 12.4.2]. This gives a symplectic basis for $H^1(\widetilde{X}, \mathbf{Q})$, which consists of a loop $\widetilde{\alpha}_1$ lying over both copies of α_1 (which were glued together after cutting at a point), a loop $\widetilde{\alpha}_2$ lying over one copy of α_2, and two loops

$\widetilde{\alpha}_i, \widetilde{\alpha}_{i+2g-2}$ lying over α_i for each $i \in \{3, \dots, 2g\}$. In this basis π_* has a simple description:

$$\pi_* \widetilde{\alpha}_i = \begin{cases} 2\alpha_1 & \text{if } i = 1, \\ \alpha_i & \text{if } 1 < i \leq 2g, \\ \alpha_{i-2g+2} & \text{otherwise.} \end{cases} \tag{B.2.4}$$

It follows that for each $\widetilde{Q} \subseteq \{1, \dots, 2\tilde{g}\}$ there is at most one $Q \subseteq \{1, \dots, 2g\}$ such that $a^Q_{\widetilde{Q}} \neq 0$ (using notation from (B.1.6)). Specifically

$$Q = \Big\{ i \in \{1, \dots, 2g\} \,\Big|\, i \in \widetilde{Q} \text{ or } 2g < i + 2g - 2 \in \widetilde{Q} \Big\},$$

and $a^Q_{\widetilde{Q}} = 0$ only if $\widetilde{Q}$ contains both i and $i + 2g - 2$ for $i > 2$. For each Q there are therefore

$$2^{|Q| - \delta_{1Q} - \delta_{2Q}}$$

choices for $\widetilde{Q}$ with $a^Q_{\widetilde{Q}} \neq 0$, where

$$\delta_{iQ} = \begin{cases} 1 & \text{if } i \in Q, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover $a^Q_{\widetilde{Q}} = \pm 2^{\delta_{1Q}}$ in these cases.

Setting $I_B := (p, (2B-1) \cup 2B)$ for $B \subseteq \widetilde{G} := \{1, \dots, \tilde{g}\}$, a similar argument shows that each basic index K with $A_K = \varnothing$ (and $b_K \leq q$) appears

$$2^{b_K - \delta_{1B_K}} \binom{\tilde{g} - b_K}{q - b_K}$$

times in

$$\pi_*^{(d)}\big(\tilde{\eta}^p \tilde{\theta}^q\big) = q! \sum_{B \in \binom{\widetilde{G}}{q}} \sum_{J \leq I_B} \sum_{K \text{ basic}} c_{KJ} \sum_{L \leq K} (-1)^{p_L - p_K} \xi_L.$$

Moreover $c_{KJ} = +2^{d-2p-2q+p_K+\delta_{1B_K}}$ every time, essentially because the σ_i commute with each other. It follows that

$$\pi^{(d)}_*(\tilde{\eta}^p\tilde{\theta}^q) = q!\sum_K 2^{d-2p-2q+p_K+b_K}\binom{\tilde{g}-b_K}{q-b_K}\sum_{L\le K}(-1)^{p_L-p_K}\xi_L,$$

where the first sum is taken over basic indices K with $A_K = \varnothing$. Collecting these indices for a fixed basic L gives the following equation (again the sum is over basic L with $A_L = \varnothing$)

$$\pi^{(d)}_*(\tilde{\eta}^p\tilde{\theta}^q) = q!2^{d-p-q}\sum_L\sum_{k=b_L}^{q}\binom{\tilde{g}-k}{q-k}\binom{g-b_L}{k-b_L}(-1)^{k-b_L}\xi_L.$$

This can be rewritten in terms of η and θ, as follows.

$$\begin{aligned}
\pi^{(d)}_*(\tilde{\eta}^p\tilde{\theta}^q) &= 2^{d-p-q}\sum_{l=0}^{q}\sum_{k=l}^{q}\binom{\tilde{g}-k}{q-k}\binom{g-l}{k-l}(-1)^{k-l}\frac{q!}{l!}\eta^{p+q-l}\theta^l \\
&= 2^{d-p-q}\sum_{l=0}^{q}\sum_{k=0}^{q-l}\binom{\tilde{g}-k-l}{q-k-l}\binom{g-l}{k}(-1)^{k}\frac{q!}{l!}\eta^{p+q-l}\theta^l \\
&= 2^{d-p-q}\sum_{l=0}^{q}\sum_{k=0}^{q-l}\binom{q-\tilde{g}-1}{q-k-l}\binom{g-l}{k}(-1)^{q-l}\frac{q!}{l!}\eta^{p+q-l}\theta^l \\
&= 2^{d-p-q}\sum_{l=0}^{q}\binom{q-l+g-\tilde{g}-1}{q-l}(-1)^{q-l}\frac{q!}{l!}\eta^{p+q-l}\theta^l \\
&= 2^{d-p-q}\sum_{l=0}^{q}\binom{\tilde{g}-g}{q-l}\frac{q!}{l!}\eta^{p+q-l}\theta^l,
\end{aligned}$$

where the penultimate line follows from the Chu-Vandermonde identity [GKP94, (5.22)]. ☺

(B.2.5) Let X be a smooth curve of genus 6 and $\widetilde{X} \to X$ an étale double cover, with (A,Θ) the associated Prym variety. Fix a g^2_6 on X and suppose the associated special subvariety $S \subset \widetilde{X}^{(6)}$ is smooth. We name the morphisms involved as below.

$$\begin{array}{ccc}
S & \xhookrightarrow{\tilde{\iota}} & \widetilde{X}^{(6)} \\
\big\downarrow{\scriptstyle \rho} & & \big\downarrow{\scriptstyle \pi^{(6)}} \\
g^2_6 & \xhookrightarrow{\iota} & X^{(6)}.
\end{array}$$

(B.2.6) **Proposition**. If $n \in \mathbf{Z}$ then $\chi(\mathcal{O}_S(n\Theta)) = 40n^2 - 80n + 44$.

Proof. We will compute $\chi(\mathcal{O}_S(2n\Theta)) = \chi\big(\mathcal{O}_S\big(n\widetilde{\Theta}\big)\big)$ for all $n \in \mathbf{N}$, where $\widetilde{\Theta}$ is a theta divisor for $\mathrm{Pic}^6(\widetilde{X})$. It suffices to show that $\chi\big(\mathcal{O}_S\big(n\widetilde{\Theta}\big)\big) = 160n^2 - 160n + 44$.

Since $\pi^{(6)}$ is an affine morphism, $\pi^{(6)}_*$ commutes with arbitrary base change. Therefore $\iota^*\pi^{(6)}_* H = \rho_*\tilde{\iota}^* H$ for $H \coloneqq \mathcal{O}_{\widetilde{X}^{(6)}}\big(n\widetilde{\Theta}\big)$. Since $R^i\rho_* = 0$ for $i > 0$

$$\chi\big(\mathcal{O}_S\big(n\widetilde{\Theta}\big)\big) = \chi\big(\iota^*\pi^{(6)}_* H\big) = \int_{\mathfrak{g}^2_6} \mathrm{ch}\big(\iota^*\pi^{(6)}_* H\big)\,\mathrm{td}\big(\mathfrak{g}^2_6\big) = \int_{\mathfrak{g}^2_6} \iota^*\,\mathrm{ch}\big(\pi^{(6)}_* H\big)\,\mathrm{td}\big(\mathfrak{g}^2_6\big). \tag{B.2.7}$$

By Grothendieck-Riemann-Roch

$$\mathrm{ch}\big(\pi^{(6)}_* H\big) = \pi^{(6)}_*\big(\mathrm{ch}(H)\,\mathrm{td}(\widetilde{X}^{(6)})\big) \cdot \mathrm{td}\big(X^{(6)}\big)^{-1}. \tag{B.2.8}$$

The Chern classes of symmetric products are well-known [ACGH85, VII, (5.4)]. In particular

$$\mathrm{c}\big(\widetilde{X}^{(6)}\big) = 1 - 4\tilde{\eta} - \tilde{\theta} + 10\tilde{\eta}^2 + 5\tilde{\eta}\tilde{\theta} + \frac{1}{2}\tilde{\theta}^2 + \cdots,$$

and hence

$$\mathrm{td}\big(\widetilde{X}^{(6)}\big) = 1 - 2\tilde{\eta} - \frac{1}{2}\tilde{\theta} + \frac{13}{6}\tilde{\eta}^2 + \frac{13}{12}\tilde{\eta}\tilde{\theta} + \frac{1}{8}\tilde{\theta}^2 + \cdots.$$

Since

$$\mathrm{ch}(H) = 1 + n\tilde{\theta} + \frac{1}{2}n^2\tilde{\theta}^2 + \cdots,$$

it follows that

$$\mathrm{ch}(H)\,\mathrm{td}\big(\widetilde{X}^{(6)}\big) = 1 - 2\tilde{\eta} + \Big(n - \frac{1}{2}\Big)\tilde{\theta} + \frac{13}{6}\tilde{\eta}^2 + \Big({-2n} + \frac{13}{12}\Big)\tilde{\eta}\tilde{\theta} + \Big(\frac{1}{2}n^2 - \frac{1}{2}n + \frac{1}{8}\Big)\tilde{\theta}^2 + \cdots,$$

so by (B.2.3)

$$\begin{aligned}\pi^{(6)}_*\big(\mathrm{ch}(H)\,\mathrm{td}(\widetilde{X}^{(6)})\big) = 64 + (160n - 144)\eta + (32n - 16)\theta + \Big(160n^2 - 320n + \frac{484}{3}\Big)\eta^2 \\ + \Big(80n^2 - 112n + \frac{112}{3}\Big)\eta\theta + (8n^2 - 8n + 2)\theta^2 + \cdots.\end{aligned}$$

Again a general formula gives

$$\mathrm{c}\big(X^{(6)}\big) = 1 + \eta - \theta + \frac{1}{2}\theta^2 + \cdots,$$

and hence

$$\operatorname{td}\left(X^{(6)}\right)^{-1} = 1 - \frac{1}{2}\eta + \frac{1}{2}\theta + \frac{1}{6}\eta^2 - \frac{1}{3}\eta\theta + \frac{1}{8}\theta^2 + \cdots.$$

Using (B.2.8)

$$\begin{aligned}\operatorname{ch}\left(\pi_*^{(6)}H\right) &= 64 + (160n - 176)\eta + (32n + 16)\theta + \left(160n^2 - 400n + 244\right)\eta^2 \\ &\quad + \left(80n^2 - 48n - 48\right)\eta\theta + \left(8n^2 + 8n + 2\right)\theta^2 + \cdots.\end{aligned}$$

The class of a linear series in $X^{(d)}$ can be computed using a special case of the secant plane formula [ACGH85, VIII, (3.2)]. In particular, the class of g^2_6 in $X^{(6)}$ is

$$10\eta^4 - 4\eta^3\theta + \frac{1}{2}\eta^2\theta^2.$$

It follows that the degree of $\iota^* \operatorname{ch}\left(\pi_*^{(6)}H\right)$ is $160n^2 - 400n + 244$. The class of a pencil in g^2_6 is

$$-5\eta^5 + \eta^4\theta,$$

so the intersection of $\iota^* \operatorname{ch}\left(\pi_*^{(6)}H\right)$ with a line has degree $160n - 176$. The class of a point is obviously η^6, so the codimension 0 term of $\iota^* \operatorname{ch}\left(\pi_*^{(6)}H\right)$ has degree 64. Therefore

$$\iota^* \operatorname{ch}\left(\pi_*^{(6)}H\right) = 64 + (160n - 176)h + \left(160n^2 - 400n + 244\right)h^2,$$

where $h := c_1\left(\mathcal{O}_{g^2_6}(1)\right)$. Finally (B.2.7) gives

$$\begin{aligned}\chi\left(\mathcal{O}_S\left(n\widetilde{\Theta}\right)\right) &= \int_{g^2_6} \left(64 + (160n - 176)h + \left(160n^2 - 400n + 244\right)h^2\right)\left(1 + \frac{3}{2}h + h^2\right) \\ &= \int_{g^2_6} 64 + (160n - 80)h + \left(160n^2 - 160n + 44\right)h^2 \\ &= 160n^2 - 160n + 44,\end{aligned}$$

as required. ☺

(B.2.9) **Corollary**. If (A, Θ) is a general ppav, $S_i \subset S$ one of the components and $V_i \subset \Theta$ any of its images in Θ, then $\chi\left(\mathcal{O}_{S_i}(n\Theta)\right) = 20n^2 - 40n + 22$, and the latter is the Hilbert polynomial of V_i with respect to Θ.

Proof. Since A has only 27 Prym-curves, each of which has five nets of degree 6 by (4.2.2), we may assume that S_i and V_i are smooth (see (4.2.8)). It follows that $R\varphi_* \mathcal{O}_{S_i} = \mathcal{O}_{V_i}$, where $\varphi: S_i \to V_i$. Therefore $\chi\big(\mathcal{O}_{S_i}(n\Theta)\big) = \chi\big(\mathcal{O}_{V_i}(n\Theta)\big)$ for all $n \in \mathbf{Z}$.

If (X, Y, Z) is a tetragonal triple, then (B.2.6) (together with the fact that translates of Θ are algebraically equivalent) says that the Hilbert polynomials of any embeddings $S_Y \hookrightarrow \Theta$ and $S_Z \hookrightarrow \Theta$ add up to $40n^2 - 80n + 44$. The result follows by permuting the triple. ☺

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