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Abstract

In an nontransferable utility (NTU) exchange economy with a continuum of agents, the Mas-Colell bargaining set coincides with the set of Walrasian equilibria. In this paper, we show that the Mas-Colell bargaining set, as well as a smaller bargaining set due to Zhou, may fail to converge to competitive outcomes in large finite NTU exchange economies.

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1 Introduction

The bargaining set was originally defined by Aumann and Maschler [2] and Davis and Maschler [3]. Mas-Colell [10] considered exchange economies with a continuum of agents but without transferable utility or smooth preferences. His definition differs from the Aumann-Davis-Maschler definition principally because it does not involve the concept of a leader. Under hypotheses similar to those of Aumann's core equivalence theorem, he showed that the Mas-Colell bargaining set coincides with the set of Walrasian allocations. Since models with a continuum of agents are thought of as idealizations of large economies, it seemed reasonable to expect that the Mas-Colell bargaining set would become approximately competitive in sequences of finite exchange economies as the number of agents increased. Zhou [13] has proposed additional restrictions on counterobjections; these restrictions are satisfied in the Aumann-Davis-Maschler definition. Since Zhou's additional restrictions make it easier to form a justified objection, they make the bargaining set smaller.

In Theorem 3.4, we show that the Mas-Colell and Zhou bargaining sets need not converge in replica sequences of economies. In the example, the measure of the set of individually rational Pareto optimal equal-treatment (IRPOET) allocations which are *not* in the Mas-Colell and Zhou bargaining sets tends to zero as the economy is replicated; in particular, the set of IRPOET bargaining set allocations converges in the Hausdorff distance to the set of *all* IRPOET allocations. The replica sequence in the example satisfies the hypotheses of the Debreu-Scarf theorem [5] and of Debreu's rate of convergence theorem for the core [4]. The cooperative game generated by the example has transferable utility and satisfies the assumptions of Shapley and Shubik [11]; thus, the Aumann-Davis-Maschler bargaining set *does* converge. The discrepancy between the behavior of the Mas-Colell bargaining set in the continuum and its behavior in sequences of large finite economies gives reason to be cautious in accepting the continuum as the proper idealization of a "large" economy.

The essence of the nonconvergence example is easy to describe. It is a replica sequence with two goods and two types of agents. Let f be an IRPOET allocation and let f_n denote the n-fold replica of f. Let the agents of type 2 be relatively favored by f, and the agents of type 1 disfavored. One first shows that if a coalition S can make a justified Mas-Colell or Zhou

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The essence of the nonconvergence example is easy to describe. It is a replica sequence with two goods and two types of agents. Let f be an IRPOET allocation and let f_n denote the n-fold replica of f. Let the agents of type 2 be relatively favored by f, and the agents of type 1 disfavored. One first shows that if a coalition S can make a justified Mas-Colell or Zhou objection to f, then S must contain all n agents of type 1; if not, one could form a counterobjection by switching one of the type 1 agents omitted from S for one of the type 1 agents included in S. There is an ideal ratio t of the number of type 2 agents to type 1 agents that maximizes the utility that can be provided to type 1 agents. For most n, it will be the case that there exist $n' < n$ and m' such that

$$\left|\frac{m'}{n'} - t\right| < \min_{m \in \mathbf{N}} \left|\frac{m}{n} - t\right|. \tag{1}$$

But then a coalition consisting of n' agents of type 1 and m' agents of type 2 can counterobject. It is only in the rare case that the best approximation to t of the form $\frac{m'}{n'}$ with $n' \leq n$ has $n' = n$ that f_n is *not* in the Mas-Colell and Zhou bargaining sets. Thus, the nonconvergence example is driven entirely by an integer problem.

Anderson [1] shows that a different bargaining set, due to Geanakoplos [8], *is* approximately competitive in large finite NTU exchange economies. In addition, in large finite NTU exchange economies with *smooth* preferences and uniformly bounded endowments, the Aumann-Davis-Maschler bargaining set is approximately competitive.

The Aumann-Davis-Maschler bargaining set differs from the Mas-Colell bargaining set only by the designation of a single individual as the leader of an objection. The nonconvergence example occurs in a replica sequence of TU economies with smooth preferences which satisfies all the assumptions of the convergence theorem for the Aumann-Davis-Maschler bargaining set in Anderson [1]. It is remarkable that the designation of a single leader should make such a profound difference in the resulting bargaining set.

2 Preliminaries

We begin with some notation and definitions which will be used throughout. Suppose $x, y \in \mathbf{R}^k, B \subset \mathbf{R}^k$. x^i denotes the ith component of x; $x \geq y$ means $x^i \geq y^i$ for all i; $x > y$ means $x \geq y$ and $x \neq y$; $x \gg y$ means $x^i > y^i$ for all i; $\|x\|_1 = \sum_{i=1}^k |x^i|$; $\|x\|_\infty = \max\{|x^1|, \cdots, |x^k|\}$; $\mathbf{R}^k_+ = \{x \in \mathbf{R}^k : x \geq 0\}$; $\mathbf{R}^k_{++} = \{x \in \mathbf{R}^k : x \gg 0\}$. If $t \in \mathbf{R}$, $\lfloor t \rfloor$ denotes the greatest integer less than or equal to t and $\lceil t \rceil$ denotes the smallest integer greater than or equal to t.

A preference is a binary relation $\succ$ on $\mathbf{R}^k_+$ satisfying the following conditions:

1. weak monotonicity: $x \gg y \Rightarrow x \succ y$;
2. continuity: $\{(x,y) \in \mathbf{R}^k_{++} : x \succ y\}$ is open;
3. transitivity: if $x \succ y$ and $y \succ z$, then $x \succ z$;
4. negative transitivity: if $x \not\succ y$ and $y \not\succ z$, then $x \not\succ z$;
5. irreflexivity: $x \not\succ x$; and
6. convexity: $\{x : x \succ y\}$ is a convex set.

Let $\mathcal{P}$ denote the set of preferences. If $\succ \in \mathcal{P}$, *define* $x \sim y$ if $x \not\succ y$ and $y \not\succ x$, $x \succeq y$ if $x \succ y$ or $x \sim y$. Note that the indifference relation $x \sim y$ is defined from the underlying strict preference relation $x \succ y$, and is not one of the primitives of the specification of the economy.

Lemma 2.1 *If $\succ$ satisfies properties 1-5 in the definition of $\mathcal{P}$, then*

1. $x \succ y \succeq z \Rightarrow x \succ z$;
2. $x \succeq y \succ z \Rightarrow x \succ z$; *and*
3. $x \geq y \succ z \Rightarrow x \succ z$.

Proof:

1. First, we prove conclusion 1. There are two cases to consider.
   (a) If $y \succ z$, then $x \succ z$ by transitivity.
   (b) If $y \sim z$, then $z \not\succ y$; if $x \not\succ z$, then $x \not\succ y$ by negative transitivity, contradiction. Therefore, $x \succ z$.
2. The proof of conclusion 2 is essentially the same as the proof of conclusion 1.
3. Now, we turn to conclusion 3. Since $x \geq y$ and $\succ$ is weakly monotone, we may find $x_n \to x$ with $x_n \gg x$, hence $x_n \gg y$, so $x_n \succ y$. If $y \succ x$, then $y \succ x_n$ for n sufficiently large, by continuity; but then $y \succ x_n \succ y$, so $y \succ y$ by transitivity, contradicting irreflexivity. Therefore, $x \succeq y$, so $x \succ z$ by conclusion 2.

■

An *exchange economy* is a map $\chi : A \to \mathcal{P} \times \mathbf{R}^k_+$, where A is a finite set of agents. For $a \in A$, let $\succ_a$ denote the preference of a (i.e. the projection of $\chi(a)$ onto $\mathcal{P}$) and $e(a)$ the initial endowment of a (i.e. the projection of $\chi(a)$ onto $\mathbf{R}^k_+$). An *allocation* is a map $f : A \to \mathbf{R}^k_+$ such that $\sum_{a \in A} f(a) = \sum_{a \in A} e(a)$; let $\mathcal{A}(\chi)$ denote the set of allocations of the economy χ. An allocation f is *individually rational* if $e(a) \not\succ_a f(a)$ for every $a \in A$. A *coalition* is a non-empty subset of A.

A *weak objection* to an allocation f is a pair (S, g), where S is a coalition, $g : S \to \mathbf{R}^k_+$, $\sum_{a \in S} g(a) \leq \sum_{a \in S} e(a)$, and

$$g(a) \succeq_a f(a) \text{ for all } a \in S \text{ with strict preference for at least one } a. \quad (2)$$

An allocation f is *strongly Pareto optimal* if there is no weak objection (S, g) to f with $S = A$. A *strong objection* to an allocation f is a pair (S, g), where S is a coalition, $g : S \to \mathbf{R}^k_+$, $\sum_{a \in S} g(a) = \sum_{a \in S} e(a)$, and

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An allocation f is *weakly Pareto optimal* if there is no strong objection (S, g) to f with $S = A$.

If (S, g) is a (weak or strong) objection to f, a *weak counterobjection* to (S, g) is a pair (T, h), where T is a coalition, $h : T \to \mathbf{R}^k_+$, $\sum_{a \in T} h(a) \leq \sum_{a \in T} e(a)$, and

$$h(a) \succeq_a \begin{cases} g(a) & \text{if } a \in T \cap S \\ f(a) & \text{if } a \in T \setminus S. \end{cases} \quad (4)$$

with strict preference for at least one agent a. If (S, g) is a (weak or strong) objection to f, a *strong counterobjection* to (S, g) is a pair (T, h), where T is a coalition, $h : T \to \mathbf{R}^k_+$, $\sum_{a \in T} h(a) = \sum_{a \in T} e(a)$ and

$$h(a) \succ_a \begin{cases} g(a) & \text{if } a \in T \cap S \\ f(a) & \text{if } a \in T \setminus S. \end{cases} \quad (5)$$

The bargaining set will be defined, roughly, as the set of all allocations such that every objection admits a counterobjection. Suppose that preferences are continuous, strongly monotonic (i.e. $x > y \to x \succ y$), transitive and negatively transitive. It is then well known that an allocation is weakly Pareto optimal if and only if it is strongly Pareto optimal. The same argument shows the following two facts:

1. if preferences are strongly monotonic, continuous, transitive and negatively transitive, and f is an allocation, then f admits a weak objection if and only if it admits a strong objection; and

2. if preferences are strongly monotonic, continuous, transitive and negatively transitive, and (S, g) is a (weak or strong) objection to f, then (S, g) admits a weak counterobjection if and only if it admits a strong counterobjection.

However, as we shall see below in Proposition 3.2, it is very hard to construct a strong objection which does not admit a counterobjection. Thus, in the definition of the bargaining set, it matters a great deal whether we require objections to be weak or strong. Accordingly, we must define more than one bargaining set. The *weak Mas-Colell bargaining set*, denoted $\mathcal{B}_w(\chi)$, is the set of all allocations to which every strong objection has a strong counterobjection; if preferences are continuous, strongly monotonic, transitive and negatively transitive, this is the same as the set of all allocations to which every strong objection has a weak counterobjection. The *strong Mas-Colell bargaining set*, denoted $\mathcal{B}_s(\chi)$, is the set of all allocations to which every weak objection has a weak counterobjection; if preferences are continuous, strongly monotonic, transitive and negatively transitive, this is the same as the set of all allocations to which every weak objection has a strong counterobjection. Observe that if preferences are continuous, strongly monotonic, transitive and negatively transitive,

$$\mathcal{B}_s(\chi) \subset \mathcal{B}_w(\chi). \tag{6}$$

If an objection has no counterobjection (weak or strong, as determined by the context), we will say that the objection is *justified.*

Zhou [13] has proposed adding three restrictions on counterobjections:

1. $T \cap S \neq \emptyset$;

2. $S \not\subset T$;

3. $T \not\subset S$.

The *Zhou bargaining set*, denoted $\mathcal{B}_Z(\chi)$, is the set of all allocations such that every weak objection admits a weak counterobjection satisfying these three restrictions.

If $T \cap S = \emptyset$, then a (weak or strong) counterobjection (T, h) would have little deterrent effect on the members of S in proposing the objection (S, g); even if the members of T do implement h, it is still feasible for the members of S to implement g. If $S \subset T$, then every member of S would be willing to put (T, h) forward as an objection *in place of* (S, g). For more comments on the motivation for these restrictions, see Zhou [13].

3 Nonconvergence Examples for the Mas-Colell and Zhou Bargaining Sets

We will consider replica sequences of economies, as defined by Edgeworth [7] and Debreu and Scarf [5]. The *base economy* is defined by $\chi : A \to \mathcal{P} \times \mathbf{R}^k_+$. The *n-fold replica of* χ is $\chi_n : A_n \to \mathcal{P} \times \mathbf{R}^k_+$ where

$$\begin{aligned} A_n &= A \times \{1, \ldots, n\} \\ \chi_n(a, j) &= \chi(a) \ (1 \leq j \leq n). \end{aligned} \tag{7}$$

The *n-fold replica of an allocation* f is $f_n : A_n \to \mathbf{R}^k_+$ with

$$f_n(a, j) = f(a) \ (1 \leq j \leq n). \tag{8}$$

In Proposition 3.2, we show that the weak bargaining set is very big in replica economies with two types of agents; in the light of the situation in continuum economies, this is not surprising. In Theorem 3.4, we show that the strong Mas-Colell bargaining set and the Zhou bargaining set may also be very big in replica economies, in sharp contrast with the situation in continuum economies.

We begin with some propositions concerning the bargaining set in the replica context.

Lemma 3.1 *Suppose f is any individually rational allocation of a base economy $\chi : A \to \mathcal{P} \times \mathbf{R}^k_+$ and f_n is the n-fold replica of f. Suppose further (S, g) is a weak objection to f_n with no weak Zhou counterobjection. Then*

1. *for all $a \in A$, either*

   (a)

$$g(a, j) \sim_a f(a) \text{ for all } j \text{ satisfying } (a, j) \in S \tag{9}$$

   or

   (b) $(a,j) \in S$ *for all* $j \in \{1,\ldots,n\}$;

2. *there exists* $a \in A$ *such that* $(a,j) \in S$ *for all* $j \in \{1,\ldots,n\}$; *and*

3. *if* f_n *is strongly Pareto optimal, then there exists* $a \in A$ *such that* $(a,j) \notin S$ *for some* j.

Proof:

1. Fix $a \in A$. If $(a,j) \notin S$ $(1 \le j \le n)$, then equation 9 is vacuously satisfied. Now suppose $(a,j) \in S$ for some j. Suppose $(a,i) \notin S$. By the definition of a weak objection, $g(a,j) \succeq_a f(a)$ for j satisfying $(a,j) \in S$. If $g(a,j) \succ_a f(a)$ for some j, let $T = (S \cup \{(a,i)\}) \setminus \{(a,j)\}$. Clearly, $S \not\subset T$ and $T \not\subset S$. There are two cases to consider:

   (a) *Case I:* $T \cap S \neq \emptyset$. Let
   $$h(b,k) = \begin{cases} g(b,k) & \text{if } (b,k) \in S \setminus \{(a,j)\} \\ \\ g(a,j) & \text{if } (b,k) = (a,i). \end{cases} \tag{10}$$
   Then (T,h) is a weak Zhou counterobjection to (S,g), contradiction.

   (b) *Case II:* $T \cap S = \emptyset$. Then $S = \{(a,j)\}$, so $e(a) \geq g(a,j) \succ_a f(a)$, so $e(a) \succ_a f(a)$ by Lemma 2.1, contradicting the assumption that f is individually rational.

   Since both cases lead to a contradiction, $g(a,j) \sim_a f(a)$ for all j such that $(a,j) \in S$.

2. If for every $a \in A$, there exists j with $(a,j) \notin S$, then $g(a,j) \sim_a f(a)$ for all $(a,j) \in S$, by item 1. But then (S,g) is not a weak objection to f_n, contradiction. Hence, there is some $a \in A$ with $(a,j) \in S$ for all $j \in \{1,\ldots,n\}$.

3. Suppose f_n is strongly Pareto optimal. Then $S \subset A_n, S \neq A_n$. Thus, there is some $a \in A$ and some $j \in \{1,\ldots,n\}$ such that $(a,j) \notin S$.

■

The following proposition shows that, in replica economies with two types of agents, the weak Mas-Colell bargaining set is extremely large. A similar phenomenon occurs in the continuum context, as noted by Mas-Colell [10].

Proposition 3.2 *Suppose $|A| = 2$, f is any individually rational allocation of a base economy $\chi : A \to \mathcal{P} \times \mathbf{R}_+^k$, $\succ_a$ is strongly monotonic for each $a \in A$, f_n is the n-fold replica of f, and $n > 1$. If f_n is weakly Pareto optimal in χ_n, then $f_n \in \mathcal{B}_w(\chi_n)$.*

Proof: Suppose (S, g) is a strong objection to f_n. Since f_n is weakly Pareto optimal, we have $S \neq A_n$, so there exists (a, j) with $(a, j) \notin S$. Relabelling, we may assume without loss of generality that $a = 1$. Since strong objections are weak objections, item 1 of Lemma 3.1 implies we are in one of two cases:

1. (S, g) has a weak counterobjection. Since preferences are continuous, strongly monotonic and transitive, (S, g) also has a strong counterobjection.

2. $g(1, j) \sim_1 f(1)$ for all j such that $(1, j) \in S$. Since (S, g) is a strong objection, we must have $(1, j) \notin S$ for $j = 1, \ldots, n$. Since $|A| = 2$, we have $\sum_{j=1}^n g(2, j) = ne(2)$. $g(2, j) \succ_2 f(2) \succeq_2 e(2)$ because f is individually rational. By Lemma 2.1, $g(2, j) \succ_2 e(2)$. By convexity, $e(2) = \frac{1}{n} \sum_{j=1}^n g(2, j) \succ e(2)$, contradicting irreflexivity.

Thus, we are always in the first case. Accordingly, every strong objection has a strong counterobjection, so $f_n \in \mathcal{B}_w(\chi_n)$. ■

Example 3.3 We consider a replica sequence of economies with two goods and two types of agents; both types have the same Cobb-Douglas utility function. The characteristics of the agents in the unreplicated economy χ are given by

$$\begin{array}{c} A = \{1, 2\} \\ e(1) = (3, 1) \qquad e(2) = (1, 3) \\ u(x, y) \;=\; \sqrt{xy} \end{array} \tag{11}$$

Let χ_n denote the n-fold replica of χ. Given $\xi \in [0, 4]$, let f_ξ denote the allocation (in the unreplicated economy)

$$f_\xi(1) = (\xi, \xi) \quad f_\xi(2) = (4 - \xi, 4 - \xi) \tag{12}$$

and let $f_{\xi n}$ denote the n-fold replica of f_ξ, i.e. $f_{\xi n}(a, j) = f_\xi(a)$. Let λ denote Lebesgue measure on $\mathbf{R}$.

Theorem 3.4 *If χ_n is the replica sequence described in Example 3.3, then*

$$\forall \xi \in [\sqrt{3}, 4-\sqrt{3}] \quad |\{n : f_{\xi n} \in \mathcal{B}_Z(\chi_n)\}| = \infty \tag{13}$$

and there is a constant C such that

$$\lambda(\{\xi \in [\sqrt{3}, 4-\sqrt{3}] : f_{\xi n} \notin \mathcal{B}_Z(\chi_n)\}) \leq \frac{C}{\sqrt{n}} \to 0. \tag{14}$$

Proof:

1. $g : S \to \mathbf{R}^2_+$ is Pareto optimal on S if and only if there exist $\lambda_a \in [0,1]$ with $\sum_{a\in S} \lambda_a = 1$ such that

$$\begin{aligned} g(a) &= \lambda_a \textstyle\sum_{a\in S} e(a) \\ u(g(a)) &= \lambda_a u\left(\textstyle\sum_{a\in S} e(a)\right) \end{aligned} \tag{15}$$

   for all a. Thus, the cooperative game generated by χ_n has transferable utility, with characteristic function $V(S) = u\left(\sum_{a\in S} e(a)\right)$.

2. Suppose $\xi \in [\sqrt{3}, 4-\sqrt{3}]$. If $\xi = 2$, then $f_{\xi n}$ is a Walrasian allocation, and hence $f_{\xi n} \in \mathcal{B}_Z(\chi_n)$ for all n. Thus, we can restrict attention to the case $\xi \neq 2$. By symmetry, we can assume without loss of generality that $\xi \in [\sqrt{3}, 2)$. Moreover, we can also assume without loss of generality that $n \geq 2$.

3. Suppose (S, g) is a weak objection to $f_{\xi n}$ with no weak Zhou counter-objection. Observe that $f_{\xi n}$ is strongly Pareto optimal. Thus, renumbering if necessary, we find we are in either

$$\begin{aligned} &\text{Case I: } (1,1) \notin S \quad \& \ (2,j) \in S \text{ for all } j \\ \text{or}& \\ &\text{Case II: } (2,1) \notin S \quad \& \ (1,j) \in S \text{ for all } j \end{aligned} \tag{16}$$

   by item 1 of Lemma 3.1.

   (a) In Case I, let m be $|\{j : (1,j) \in S\}|$. We have $u(g(1,j)) = u(f_\xi(1)) = \xi$ for all j with $(1,j) \in S$, by Lemma 3.1.

$$m\xi + \sum_{j=1}^{n} u(g(2,j)) \leq V(S) = \sqrt{(3m+n)(3n+m)}. \tag{17}$$

We can assume without loss of generality that $u(g(2,n)) \geq u(g(2,j))$ for all j. Now let T be a coalition consisting of $m+1$ agents of type 1 and $n-1$ agents of type 2, specifically $(2,1), \ldots, (2, n-1)$.

$$\begin{aligned} V(T) &= \sqrt{(3(m+1)+n-1)\,(m+1+3(n-1))} \\ &= \sqrt{(3m+n+2)\,(3n+m-2)} \qquad (18) \\ &\geq \sqrt{(3m+n)\,(3n+m)} = V(S), \end{aligned}$$

since if $m \leq n-2$,

$$\begin{aligned} (3m+n+2)+(3n+m-2) &= (3m+n)+(3n+m) \\ (3m+n) < (3m+n+2) &\leq \tfrac{(3m+n)+(3n+m)}{2} \qquad (19) \\ &\leq (3n+m-2) < (3n+m) \end{aligned}$$

and if $m = n-1$,

$$(3m+n+2) = 3n+m \text{ and } (3n+m-2) = 3m+n. \qquad (20)$$

Therefore, it is feasible in T to give the m type 1 agents in $S \cap T$ utility ξ, the type 1 agent in $T \setminus S$ utility $u(g(2,n)) > \xi$, and the type 2 agents in T the same utility level they received under g; if we do so, we produce a weak Mas-Colell counterobjection. By construction, $T \not\subset S$; moreover, $S \not\subset T$ and $S \cap T \neq \emptyset$, since $n \geq 2$, so T can form a Zhou counterobjection.

(b) Therefore, we must be in Case II, so S contains all n agents of type 1 and $m < n$ agents of type 2. Moreover, $u(g(2,j)) = 4 - \xi$ whenever $(2,j) \in S$, and we can assume without loss of generality that $u(g(1,1)) \leq u(g(1,2)) \leq \cdots \leq u(g(1,n))$.

i. Let

$$F(\xi, t) = \sqrt{(3+t)(3t+1)} - t(4-\xi) = \sqrt{3t^2+10t+3} - t(4-\xi). \qquad (21)$$

Then

$$\frac{1}{n}\sum_{j=1}^{n} u(g(1,j)) = \frac{\sqrt{(3n+m)(3m+n)}}{n} - \frac{m}{n}(4-\xi)$$
$$= \sqrt{\left(3+\frac{m}{n}\right)\left(\frac{3m}{n}+1\right)} - \frac{m}{n}(4-\xi) = F\left(\xi, \frac{m}{n}\right). \tag{22}$$

ii.

$$\frac{\partial F(\xi,t)}{\partial t} = \frac{3t+5}{\sqrt{3t^2+10t+3}} - (4-\xi)$$
$$\left.\frac{\partial F(\xi,t)}{\partial t}\right|_{t=0} = \frac{5}{\sqrt{3}} - (4-\xi) \geq \frac{8-4\sqrt{3}}{\sqrt{3}} > 0$$
$$\left.\frac{\partial F(\xi,t)}{\partial t}\right|_{t=1} = \frac{8}{\sqrt{16}} - (4-\xi) = \xi - 2 < 0 \tag{23}$$
$$\frac{\partial^2 F(\xi,t)}{\partial t^2} = \frac{3\sqrt{3t^2+10t+3}-(3t+5)\left(\frac{1}{2}\frac{1}{\sqrt{3t^2+10t+3}}(6t+10)\right)}{3t^2+10t+3}$$
$$= \frac{3(3t^2+10t+3)-(3t+5)^2}{(3t^2+10t+3)^{\frac{3}{2}}} = \frac{-16}{(3t^2+10t+3)^{\frac{3}{2}}} < 0$$

for $t \in [0,1]$. Thus, for each ξ, $F(\xi,t)$ achieves its maximum (over $t \in [0,1]$) at a unique point $t_\xi \in (0,1)$. The first order conditions imply that

$$\xi = 4 - \frac{3t_\xi + 5}{\sqrt{3(t_\xi)^2 + 10t_\xi + 3}}, \tag{24}$$

so

$$\frac{d\xi}{dt_\xi} = \frac{16}{(3(t_\xi)^2 + 10t_\xi + 3)^{\frac{3}{2}}} > 0 \tag{25}$$

for all $t_\xi \in [0,1]$. By the inverse function theorem, the map $\xi \rightarrow t_\xi$ is C^1 and has positive derivative (and thus is one to one) for all $\xi \in [\sqrt{3}, 2)$.

iii. We claim that

$$\exists \epsilon > 0 \ \ \forall \xi \in [\sqrt{3}, 4-\sqrt{3}]$$
$$|y - t_\xi| > 2|x - t_\xi|, \ \ |x - t_\xi| < \epsilon \Longrightarrow F(\xi, x) > F(\xi, y). \tag{26}$$

Note that $\frac{\partial^2 F}{\partial t^2}$ and $\left|\frac{\partial^3 F}{\partial t^3}\right|$ are continuous on $[\sqrt{3}, 4-\sqrt{3}] \times [0,1]$, so there exists $M > 0$ such that

$$\frac{\partial^2 F}{\partial t^2} < -\frac{1}{M} \text{ and } \left|\frac{\partial^3 F}{\partial t^3}\right| < M \tag{27}$$

for all $(\xi, t) \in [\sqrt{3}, 4-\sqrt{3}] \times [0,1]$. Let $\epsilon = \frac{9}{16M^2}$. Since $\frac{\partial^2 F}{\partial t^2} < 0$ for all t, $F(\xi, \cdot)$ is increasing to the left of t_ξ and decreasing to the right of t_ξ; hence, we may assume without loss of generality that $|y - t_\xi| \leq 2\epsilon$. Then

$$\begin{aligned} F(\xi, x) &= F(\xi, t_\xi) + \left.\tfrac{\partial F}{\partial t}\right|_{(\xi, t_\xi)} (x - t_\xi) \\ &+ \tfrac{1}{2} \left.\tfrac{\partial^2 F}{\partial t^2}\right|_{(\xi, t_\xi)} (x - t_\xi)^2 + \tfrac{1}{6} \left.\tfrac{\partial^3 F}{\partial t^3}\right|_{(\xi, t_x)} (x - t_\xi)^3 \end{aligned} \tag{28}$$

for some t_x between t_ξ and x. The same formula holds for $F(\xi, y)$ for some t_y between t_ξ and y. Note that $\left.\frac{\partial F}{\partial t}\right|_{(\xi, t_\xi)} = 0$. Therefore,

$$\begin{aligned} &F(\xi, x) - F(\xi, y) \\ &= \tfrac{1}{2} \left.\tfrac{\partial^2 F}{\partial t^2}\right|_{(\xi, t_\xi)} [(x - t_\xi)^2 - (y - t_\xi)^2] \\ &+ \tfrac{1}{6} \left.\tfrac{\partial^3 F}{\partial t^3}\right|_{(\xi, t_x)} (x - t_\xi)^3 - \tfrac{1}{6} \left.\tfrac{\partial^3 F}{\partial t^3}\right|_{(\xi, t_y)} (y - t_\xi)^3 \\ &> \tfrac{1}{2M} \tfrac{3}{4} (y - t_\xi)^2 - \tfrac{M}{6} 2 |y - t_\xi|^3 \\ &= (y - t_\xi)^2 \left[\tfrac{3}{8M} - \tfrac{M}{3} |y - t_\xi|\right] \\ &\geq (y - t_\xi)^2 \left[\tfrac{3}{8M} - \tfrac{M}{3} \left(\tfrac{9}{8M^2}\right)\right] = 0, \end{aligned} \tag{29}$$

which establishes Equation (26).

iv. Fix $\xi \in [\sqrt{3}, 2)$. We will show that $f_{\xi n} \in \mathcal{B}_Z(\chi_n)$ for infinitely many n. By Theorem 185 of Hardy and Wright[1] [9], there are

[1]Hardy and Wright state this only if t_ξ is irrational; however, if $t_\xi = \frac{p_0}{q_0}$ is rational, Equation (30) is satisfied for $(p, q) = (p_0, q_0), (2p_0, 2q_0), (3p_0, 3q_0), \ldots$.

infinitely many pairs of integers p, q such that

$$\left|\frac{p}{q} - t_\xi\right| < \frac{1}{q^2}. \tag{30}$$

Since $0 < t_\xi < 1$, there are infinitely many pairs (p, q) with $\frac{1}{q^2} < \epsilon$ and $5 \leq p \leq q - 5$ (and thus $q \geq 3$) satisfying Equation (30). For any such pair, consider the n-fold replica χ_n, where $n = q + 1$. Then

$$\begin{aligned} \tfrac{p}{q+1} &= \tfrac{p}{q+1} - \tfrac{p}{q} + \tfrac{p}{q} = -\tfrac{p}{q(q+1)} + \tfrac{p}{q} \\ &\leq -\tfrac{5}{\frac{4}{3}q^2} + t_\xi + \tfrac{1}{q^2} < t_\xi - \tfrac{2}{q^2}; \\ \tfrac{p+1}{q+1} &= \tfrac{p+1}{q+1} - \tfrac{p}{q} + \tfrac{p}{q} = \tfrac{q-p}{q(q+1)} + \tfrac{p}{q} \\ &\geq \tfrac{5}{\frac{4}{3}q^2} + t_\xi - \tfrac{1}{q^2} > t_\xi + \tfrac{2}{q^2}. \end{aligned} \tag{31}$$

For every $m \in \{0, \ldots, n\}$,

$$\left|\frac{m}{n} - t_\xi\right| > 2\left|\frac{p}{q} - t_\xi\right| \text{ and } \left|\frac{p}{q} - t_\xi\right| < \epsilon. \tag{32}$$

so

$$F\left(\xi, \frac{m}{n}\right) < F\left(\xi, \frac{p}{q}\right) \tag{33}$$

by Equation (26). But then a coalition T consisting of q agents of type 1 (namely, agents $(1,1), \ldots, (1,q)$) and p agents of type 2 can counterobject to (S, g), since S has n agents of type 1 and m agents of type 2. Since $m < n$, we can choose one of the p type 2 agents (say, agent $(2, j_0)$) from the complement of S. Observe that $(1,1) \in S \cap T$ (since $q > 0$), $(2, j_0) \in T \setminus S$, and $(1, n) \in S \setminus T$. Therefore, T can form a Zhou counterobjection, contradiction. Thus, we have shown that $f_{\xi n} \in \mathcal{B}_Z(\chi_n)$ for infinitely many n.

v. Now, we will show that Equation (14) holds. Fix n. By Theorem 36 of Hardy and Wright [9],

$$\forall t \in [0,1] \;\; \exists q \leq n \;\; \left|\frac{p}{q} - t\right| \leq \frac{1}{q(n+1)}. \tag{34}$$

Note that

$$\lambda\left(\left\{t\in[0,1]:\exists q\le\sqrt{n}\ \exists p\le q\ \ \left|\tfrac{p}{q}-t\right|\le\tfrac{1}{q(n+1)}\right\}\right) \\ \le \textstyle\sum_{q=1}^{\lfloor\sqrt{n}\rfloor}\tfrac{2q}{q(n+1)}\le\tfrac{2\sqrt{n}}{n+1}. \tag{35}$$

Therefore,

$$\lambda\left(\left\{t:\exists q>\sqrt{n}\ \exists p\le q\ \ \left|\frac{p}{q}-t\right|\le\frac{1}{q(n+1)}\right\}\right)\ge 1-\frac{2\sqrt{n}}{n+1}, \tag{36}$$

so

$$\lambda\left(\left\{t:\exists p\le q\le n\ \ \left|\frac{p}{q}-t\right|\le\frac{1}{\sqrt{n}(n+1)}\right\}\right)\ge 1-\frac{2\sqrt{n}}{n+1}. \tag{37}$$

However,

$$\lambda\left(\left\{t:\exists m\le n\ \ \left|\frac{m}{n}-t\right|\le\frac{2}{\sqrt{n}(n+1)}\right\}\right)\le\frac{4n}{\sqrt{n}(n+1)}, \tag{38}$$

so

$$\lambda\left(\left\{t:\exists\xi\in[\sqrt{3},2]\ t_\xi=t\ \exists m\le n\ \ F\left(\xi,\tfrac{m}{n}\right) \\ =\max_{p\le q\le n}F\left(\xi,\tfrac{p}{q}\right)\right\}\right) \\ \le\tfrac{4n}{\sqrt{n}(n+1)}+\tfrac{2\sqrt{n}}{n+1}\le\tfrac{6}{\sqrt{n}}. \tag{39}$$

Since the map $\xi\to t_\xi$ has positive derivative, there is a constant $L>0$ such that the derivative is at least L for every $\xi\in[\sqrt{3},4-\sqrt{3}]$. Therefore,

$$\lambda(\{\xi:f_{\xi n}\notin\mathcal{B}_Z(\chi_n)\})\le\frac{6}{L\sqrt{n}} \tag{40}$$

which proves Equation (14).

■

References

[1] Anderson, Robert M., "Convergence of the Aumann-Davis-Maschler and Geanakoplos Bargaining Sets," Working Paper, Department of Economics, University of California at Berkeley, January 1994.

[2] Aumann, Robert J. and Michael Maschler, "The Bargaining Set for Cooperative Games," in M. Dresher, Lloyd S. Shapley and Albert W. Tucker (eds.), *Advances in Game Theory*, Princeton: Princeton University Press, 1964, 443-476.

[3] Davis, Morton and Michael Maschler, "Existence of Stable Payoff Configurations for Cooperative Games," *Bulletin of the American Mathematical Society* 69(1963), 106-108.

[4] Debreu, Gérard, "The Rate of Convergence of the Core of an Economy," *Journal of Mathematical Economics*, 2(1975), 1-7.

[5] Debreu, Gérard and Herbert Scarf, "A Limit Theorem on the Core of an Economy," *International Economic Review* 4(1963), 236-246.

[6] Dutta, Bhaskar, Debraj Ray, Kunal Sengupta, and Rajiv Vohra, "A Consistent Bargaining Set," *Journal of Economic Theory* 49(1989), 93-112.

[7] Edgeworth, Francis Y., *Mathematical Psychics*, 1881. London: Kegan Paul.

[8] Geanakoplos, John, "The Bargaining Set and Nonstandard Analysis," Chapter 3 of Ph.D. Dissertation, Department of Economics, Harvard University, Cambridge, MA, 1978.

[9] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, Fifth Edition, Oxford, UK: Clarendon Press, 1979.

[10] Mas-Colell, Andreu, "An Equivalence Theorem for a Bargaining Set," *Journal of Mathematical Economics* 18(1989), 129-139.

[11] Shapley, Lloyd S. and Martin Shubik, "Convergence of the Bargaining Set for Differentiable Market Games," Appendix B in Martin Shubik,

A Game Theoretic Approach to Political Economy, volume II of *Game Theory in the Social Sciences*, Cambridge, MA: M.I.T. Press, 1984.

[12] Vohra, Rajiv, "An Existence Theorem for a Bargaining Set," *Journal of Mathematical Economics* 20(1991), 19-34.

[13] Zhou, Lin, "A New Bargaining Set of an N-Person Game and Endogenous Coalition Formation," preprint, Cowles Foundation, Yale University, April 1992.

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