Title
Completeness in the Shadow of Decidability

Author
Killam, Lee Makua

Publication Date
2022

Peer reviewed|Thesis/dissertation

UNIVERSITY OF CALIFORNIA,
IRVINE

Completeness In The Shadow Of Decidability

DISSERTATION

submitted in partial satisfaction of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

in Logic and Philosophy of Science

by

Lee Makua Killam

Dissertation Committee:
Associate Professor Jeremy Heis, Chair
Chancellor's Professor Jeffrey A. Barrett
Dean's Professor Kai Wehmeier
Assistant Professor Toby Meadows

2022

DEDICATION

For my mom

April 11th, 1945 —April 7th, 2022

TABLE OF CONTENTS

LIST OF FIGURES

Page

ACKNOWLEDGMENTS

Above all, I would like to thank my advisor Jeremy Heis. Jeremy believed in this project from the beginning, and stood behind it when it was out of favour. Without his trust in my ability, and his unwavering encouragement, I never would have completed it. Jeremy is also the most intellectually humble person I have ever met. His interest in the project never came with an agenda, and even the most misguided of my early drafts were met by him with a patience and respect I am not at all sure they deserved. The project would not be what it is without his discerning insight. Equally worthy of mention is Jeremy's kindness as a human being. I had many reasons to require this, and infinitely more to appreciate it.

I would like to thank my committee members Jeffrey A. Barrett, Toby Meadows, and Kai Wehmeier, for their belief and enthusiasm in this project, their generosity in answering questions and their support in ways that went beyond the scope of the dissertation itself. I also owe a huge debt to James Owen Weatherall, without whose support I would have dropped out of the program in my second year. James went out of his way to make my graduate experience easier in the face of external challenges.

Finally, among many people who have inspired and helped me along the way, thank you to Professors Andrew Irvine and Alan Richardson for believing in me as an undergraduate and helping me to get into graduate school in the first place.

VITA

Lee Makua Killam

EDUCATION

Doctor of Philosophy in Logic and Philosophy of Science **2022**
University of California *Irvine, California*

Master of Arts in Philosophy **2017**
University of Southern California *Los Angeles, California*

Bachelor of Arts in Philosophy **2013**
University of British Columbia *Vancouver, British Columbia*

TEACHING EXPERIENCE

Graduate Teaching Assistant **2014–2017**
University of Southern California *Los Angeles, California*

Graduate Teaching Assistant **2017–2022**
University of California, Irvine *Irvine, California*

ABSTRACT OF THE DISSERTATION

Completeness In The Shadow Of Decidability

By

Lee Makua Killam

Doctor of Philosophy in Logic and Philosophy of Science

University of California, Irvine, 2022

Associate Professor Jeremy Heis, Chair

This dissertation investigates the origins of the completeness theorem for first-order predicate logic in the algebraic logic work of Löwenheim and Skolem. When Gödel proved the completeness theorem in 1929, he was unaware that all the components of a completeness proof were already contained in earlier papers by Löwenheim and Skolem in which they prove the model-theoretic result known as the Löwenheim-Skolem theorem. This is not, however, a question of Gödel's completeness proof having been preempted. For neither Löwenheim nor Skolem show recognition of the result that Gödel would later make explicit.

When the similarity between the proofs was noticed in the 1950s, the fact that Skolem in particular had not put the pieces together to prove completeness before Gödel seemed a puzzling oversight. Gödel offered his own answer to the puzzle, appealing to alleged prejudices Skolem had against transfinite methods of reasoning.

Chapter One shows how the puzzle emerged and how Gödel purported to explain it. Chapters Two and Three give reconstructions of the original proofs of Löwenheim and Gödel. I analyze the meaning of Gödel's claim that "finitary prejudices" were at the heart of the failure to recognize completeness, and assess the evidence for this claim in Löwenheim and Skolem. Chapter Four reconstructs Skolem's proof of Löwenheim's theorem and establishes the technical background to understand the relation it bears to the completeness theorem.

In Chapter 5, I argue that Gödel's own answer to the puzzle rests on a false premise. When certain contextual features are accounted for, Skolem's failure to recognize a completeness theorem in his own work is not the oversight it now seems. I investigate Skolem's search for a decidability proof for first-order logic and the role this played in leading Skolem away from the discovery of completeness.

Chapter 1

Introduction

1.1 The Theorem

The completeness theorem is one of the seminal results of modern metalogic, capturing the deep relation that holds between the semantics and the syntax of first-order predicate logic. The theorem states that if a formula is true in every domain, then it is deducible from the logical axioms. Equivalently, every formula of first-order logic is either refutable or satisfiable. The question of completeness was raised and answered for propositional logic by Paul Bernays in 1918 and Emil Post in 1921. Extending this result to predicate logic was not, however, an easy task. Whereas the semantics of propositional logic is characterized by the fact that the possible models i.e., truth-assignments for a given formula are finite in number, predicate logic entails no such restriction. With the introduction of quantifiers, models can no longer be described as functions from atomic propositions into the set $\{0, 1\}$. Instead, a model of a first-order formula F is a pair $\{\Delta, \mathcal{F}\}$ where Δ is a non-empty, possibly infinite set (the "domain") and $\mathcal{F}$ is a function defined on the non-logical symbols of the language. Constant and variable symbols are mapped to members of Δ, and predicate and function symbols are mapped to subsets of Δ (subject to certain conditions). As a result, most first-order formulas have infinitely many models. And when the domain is infinite, even evaluating a formula relative to a single model may be an infinite task. The challenge of completeness becomes evident: although we can determine when a formula is not provable in the theory, there is no effective way of searching through infinitely-many models to show that the formula is therefore invalid. A new strategy had to be devised.

The completeness theorem was proven by Gödel in his dissertation (Gödel, 1929). However, Gödel's proof was neither conceptually nor technically unprecedented. The proof uses methods developed in the work of Leopold Löwenheim and Thoralf Skolem over a decade earlier. Indeed, all the mathematical components of Gödel's proof are present in Skolem's (1922), and are laid out explicitly in (Skolem, 1928). This dissertation will investigate the puzzle of why the completeness theorem for first-order predicate logic was not recognized before Gödel.

1.2 Background

1.2.1 Gödel's proof

This section briefly describes Gödel's original proof of completeness. A full reconstruction is provided in Chapter 2 below.

Gödel proves the completeness theorem in the form: Every logical expression is either satisfiable or refutable. The original proof differs from the now standard presentation based on (Henkin, 1949).[1] The first step in Gödel's proof is to fix a formal deductive system, namely, "the system given in Whitehead and Russell 1910 [and] in Hilbert and Ackermann 1928"

[1]Henkin proves completeness as a consequence of the model existence theorem, by constructing a term model for a maximally consistent set of formulas.

(Gödel, 1986, Vol. 1., p. 61). Gödel then shows how the problem can be reduced to proving completeness for a restricted class of formulas— those in "normal form". A formula is in normal form if it is prenex[2] with all universal quantifiers preceding all existential quantifiers.

To prove completeness for this restricted class of formulas, Gödel uses a method of expansion first introduced by Löwenheim in his (1915). As originally conceived, the method expands a first-order formula A into an infinite conjunction of propositional instances by systematically substituting integers for the quantified variables. Using this method, Gödel constructs an indexed sequence of formulas $A_1, A_2, \ldots, A_n$ which we call "expansions". Each A_n is the conjunction of instances formed up to the *nth* level. As n increases, the domain expands by the introduction of new variables and A_n can be considered a closer and closer approximation to the quantified formula A. Within Gödel's formal system, it can be shown, roughly, that each A_n is implied by the formula A.[3] This lemma (Theorem VI in Gödel, 1929) establishes the syntactic half of the completeness theorem: using well-known methods we can determine whether a given A_n is refutable. And if A_n is refutable for some n, then A is refutable. To establish the other half of the theorem, Gödel shows that when every A_n has a satisfying truth assignment, there is a single assignment simultaneously satisfying all the A_n. This assignment is used to determine an interpretation of the predicate letters in A such that A is satisfied in the domain of natural numbers.

1.2.2 Löwenheim, Skolem and Herbrand

The expansion method used by Gödel finds its origin in the work of Löwenheim (1915) in his proof of the model-theoretic result known as the Löwenheim-Skolem theorem (LST). The theorem has two versions. The weak version states that if a formula A is satisfiable,

[2] A prenex formula is one where all the quantifiers occur at the beginning.

[3] Strictly speaking, Gödel departs from Löwenheim by constructing the A_n to be open formulas. He therefore shows that the existential closure of each A_n is implied by A. The details are discussed in Chapter 2.

then it is satisfiable in a countable domain. The strong “subdomain” version states that if A is satisfiable in an infinite domain D, then it is satisfiable in a countable subdomain D' of D, where the predicates retain the same meaning in D' as in D, modulo the restriction. According to the most widely accepted view, Löwenheim aimed to prove the weak version of the theorem in his (1915) paper “On possibilities in the calculus of relatives”. The paper introduces the expansion method described above. However, Löwenheim's proof arguably contains a gap - he fails to justify the step from the satisfiability of every A_n to the satisfiability of the formula A. The strong version of the theorem is proven by Skolem in 1920. His proof, designed to avoid Löwenheim's “detour through the infinite”, uses the axiom of choice to obtain countably many witnesses for the quantifiers. This proof avoids the expansion method, instead using a result from Dedekind's chain theory to obtain a model through a closure operation on the witness-containing sets. Skolem gives a second proof in 1922, now proving the weaker version of the theorem but avoiding the use of choice. Unlike the 1920 proof, (Skolem, 1922) uses Löwenheim's expansion method to construct a model of A in the domain of natural numbers based on the satisfiability of the A_n. The expansion method used by Löwenheim and Skolem (1922) implicitly yields an informal refutation procedure for first-order logic. The inductive construction of the A_n yields a procedure for refuting A in finitely-many steps: each A_n can be systematically checked for satisfiability using the truth-table method from propositional logic. This procedure is complete—if a formula is unsatisfiable, then the procedure will find an n such that A_n is truth-functionally unsatisfiable. Gödel makes this result rigorous by formalizing the procedure.[4]

Meanwhile, apparently unaware of (Skolem, 1922),[5], French mathematician Jacques Herbrand took inspiration from both (Löwenheim, 1915) and (Skolem, 1920). Herbrand uses a similar expansion method to arrive at his Fundamental Theorem, recognized today as a

[4]When refutability is defined in terms of the expansion procedure, i.e., as the existence of an n such that A_n is truth-functionally unsatisfiable, the contrapositive of this claim must be shown as part of the proof of the LST.

[5]Cf. Goldfarb, introduction to (Herbrand, 1971), p. 12.

key result in proof theory (Herbrand, 1929). Unlike Skolem or Löwenheim, Herbrand explicitly formulates his theorem in terms of a formal system of quantification theory.[6] A proof of the semantic completeness of this system can be extracted from Herbrand's text (Herbrand, 1971, p. 12). Herbrand explicitly acknowledges this fact, yet does not give the proof (Herbrand, 1971, p. 165). A strict adherent of Hilbert's finitism, Herbrand considered the notions involved to be meaningless.[7]

1.3 The puzzle

Several decades later, the proximity of Skolem's 1922 proof of the LST to Gödel's completeness theorem was noted in the secondary literature:

> Since about 1950 I had been struck by the fact that all the pieces in Gödel's proof of the completeness of predicate logic had been available by 1929 in the work of Skolem (notably his [1922]) (Wang, 1996, p. 122)

Commentators agree that with minor supplementation, (Skolem, 1922) can be transformed into a completeness proof for the refutation procedure determined by the construction of the A_n ([Goldfarb, 1971], [Van Heijenoort, 1967]).

In correspondence, Hao Wang and Jean van Heijenoort question Gödel about this similarity. Gödel acknowledges that

> [t]he completeness theorem, mathematically, is indeed an almost trivial consequence of Skolem 1922. (Letter to Wang, in [Wang, 1974], p. 8)

[6] A fragment of Russell and Whitehead's Principia Mathematica.

[7] Hilbert apparently felt differently. The question of whether first-order predicate logic is semantically complete was announced as an "open" problem by Hilbert and Ackermann (1928, 68).

He then states the puzzling fact central to the project of this dissertation:

> However, the fact is that, at that time, nobody (including Skolem himself) drew this conclusion (neither from Skolem 1922, nor, as I did, from similar considerations of his own).[8] (ibid.)

The puzzle goes beyond (Skolem, 1922). In 1928, Skolem applies the basic alternative of Gödel's completeness proof to his expansion method: "The real question now is whether there are solutions of an arbitrarily high level or whether for a certain n there exists no solution of the nth level."

Skolem concludes, "In the latter case the given first order proposition contains a contradiction. In the former case, on the other hand, it is consistent." Understanding "consistent" in the sense of "satisfiable", Skolem has just stated the completeness theorem for the procedure given by his expansion method:

> *Either A is refutable by finding an A_n that is contradictory, or, A is satisfiable.*

Skolem could easily have proven this theorem using his 1922 equivalent of König's infinity lemma. But that is not what he does. Instead, he gives a syntactic argument widely recognized to be, at best, inconclusive. Gödel goes further, finding the argument "obscure" and misguided, and using this as the starting point for a theory about *why* Skolem was led astray. Gödel's explanation will be considered in detail in Chapter 5.

The gap Gödel finds in Skolem's (1928) argument[9] is the same one attributed to Löwenheim on the traditional reading of his (1915). This is the reading according to which Löwenheim proved the weak version of the LST. In Chapter 3, I argue that if this reading of Löwenheim's

[8] In 1929, Gödel was acquainted only with (Skolem, 1920). The (1922) paper, published in a Norwegian journal, did not reach a wide audience at the time.

[9] This gap is filled by Skolem's 1922 equivalent if König's infinity lemma.

theorem is correct, then Gödel's remarks on Skolem's anticipation of completeness apply equally to Löwenheim. And yet, like Skolem, Löwenheim shows no recognition of completeness as a question of interest.

The connection between syntactic provability and semantic satisfiability at the heart of the completeness theorem was not drawn explicitly until Gödel's dissertation in 1929. And yet the connection is latent in the work of Lowenheim over a decade earlier, and all the components of a proof are present in Skolem by 1922. How could Lowenheim and Skolem come so close to preempting Gödel's theorem and yet fail to do so? Call this "The Puzzle".

1.4 Views in the literature

According to the prevailing view in the secondary literature, The Puzzle is explained by the fact that prior to 1930, there was no developed notion of formal system (Bernays, 1967 [in Wang, 1996], Dreben and van Heijenoort, 1986, p. 71, Goldfarb, 1979, p. 363, Wang, 1996, p. 124, Brady, 2000, p. 163, Fenstad and Wang, 2009). This explanation will be referred to as the "formal systems" explanation. In Löwenheim's framework, the explanation is based on the absence of formal rules of inference in the calculus of relatives—"obviously, no question of completeness of a formal system could arise here" (Dreben and van Heijenoort, 1986, p. 45). Goldfarb concurs that

> what is primarily missing [from Löwenheim] is a full sense of the role of the object language in formalizations of mathematics. The absence of formal inference rules precludes the use of the relative calculus for axiomatization, in the sense of formal systems. (Goldfarb, 1979, p. 355).

Despite being the first to delineate a class of first-order expressions within a formal language, "Löwenheim does not seem to recognize the fundamental importance of the first-order fragment he had just demarcated" (Goldfarb, 1979, p. 355). In particular, he does not recognize the possibility of representing deductive relationships between sentences using a first-order formal language. As a result "he draws from his theorem none of the striking and almost paradoxical consequences that were of such concern later on" (Goldfarb, ibid.). Badesa (2004) does not dispute that "the theory of relatives [...] lacks rules of inference, and [that] Löwenheim does not define the concept of consequence:

> [Löwenheim's] presentation of the language is not as accurate as I require today; he mixes syntactic and semantic aspects, is not as explicit as one might like as to which are the symbols of the language, and gives practically no syntactic rules. (2004, p. 61)

Badesa explicitly addresses the question of completeness. Referring to a lemma used in Löwenheim's proof of the LST, Badesa writes:

> This lemma asserts the completeness of the informal procedure with respect to nonsatisfiability. In my opinion, this lemma is alien to Löwenheim, *not only because he lacks the necessary distinctions to state it,* but also because he does not prove Lemma 6.8. (emphasis added, ibid., p. 205)

The second half of this explanation concerning Lemma 6.8 will be considered in chapter 3.

Skolem, meanwhile, is alleged to have had "absolutely no interest" in completeness (Dreben and van Heijenoort, 1986, p. 45). This attitude is attributed to his opposition to the very idea of a formal system:[10][11]

[10] Cf. (Goldfarb, 1979, p. 358). Ironically, Skolem's opposition was spawned by his interpretation of the very theorem (LST) that could have yielded completeness as a corollary.

[11] The explanation can be plausibly extended to Herbrand. Herbrand did have a precise notion of formal system. Hence this is why he, unlike Skolem and Löwenheim, explicitly recognized completeness as a consequence of his work. This recognition was not impeded by the fact that he rejected the infinitary notion of

> Bernays has observed that Skolem did not think of the theorems of elementary logic as given in a formal system and, therefore, that the question of full completeness had no meaning for Skolem. (Wang, 1974, p. 10)

> Skolem was a constant opponent of all formalist and logicist foundational programs [...] [he] essentially had a completeness proof for a formal system of quantification theory. [...] Of course, Skolem never put his ideas together in this way—he would not have been interested. (Goldfarb, 1979, p. 363)

> Skolem was not much concerned with the use of formal systems as foundational tools, so in any case the completeness question for such systems was not a problem of interest to him" (Goldfarb, 1971, p. 525).

Dreben and van Heijenoort attribute the formal systems explanation to Gödel:

> [A]ccording to Gödel, the only significant difference between Skolem (1922) and Gödel 1929-1930 lies in the replacement of an informal notion of 'provable' by a formal one, hence in the establishment of [Theorem VI in Gödel , 1930][12] — and in the explicit recognition that there is a question to be answered. (Dreben and van Heijenoort, 1986, p. 52).

The authors provide no evidence from Gödel to support this attribution. However, on Gödel 's behalf, they defend this interpretation of Skolem based on his ambiguous use of "consistent" ("widerspruchsfrei", "widerspruchlos") both in (Skolem, 1922) and elsewhere.[13]

semantic validity on the basis of his commitment to finitism. For this reason he did not endorse a proof of completeness despite acknowledging its possibility (Herbrand, 1971, p. 165). The same can be said about Hilbert who was able to pose the question on the basis of his clear and metatheoretic notion of formal system.

[12] Theorem VI of (Gödel , 1930) shows that for every n, $A \rightarrow A_n$ is provable. With this theorem, if some A_n is not satisfiable, then is provable.

[13] If "consistent" is interpreted to mean "syntactically consistent", "[it] might suggest that A is not

Moore (1990, p. 125), meanwhile, blames the delay in proving completeness for first-order logic on the lack of a precise syntax/semantics distinction:

> Gödel exhibited a more profound understanding of the distinction between syntax and semantics - as well as their interrelationship - than had his predecessors. Skolem had failed to observe this distinction [...] by expressing Löwenheim's Theorem in the following form: A first-order sentence is either inconsistent or else satisfiable in a countable domain. However, Skolem demonstrated only that if a first-order sentence is satisfiable in a set M, it is satisfiable in a countable subset of M. What Gödel later established was essentially Skolem's stated theorem. Thus the completeness theorem for first-order logic arose in 1930 rather than a decade earlier. (1990, p. 125)

Moore's claim can be read as a version of the formal systems explanation. As Moore notes, Skolem's proof established only the semantic half of the completeness theorem. Gödel's contribution was to show the syntactic half using his Theorem VI, i.e. that when the sentence was unsatisfiable, its inconsistency could be demonstrated in his formal system. But Skolem was not operating within an explicitly given formal system. Thus he lacked the rules of inference on the basis of which it could be demonstrated that refutability using the expansion procedure implied refutability of the original sentence.

1.5 Gödel: Rethinking the Question

Gödel rejects the suggestion that the question of completeness is meaningless unless posed in terms of formal systems:

provable in some system left unspecified or in informal logic" (Dreben and van Heijenoort, 1986, p. 52). On this reading, Skolem has already implicitly made the conceptual connection between syntax and semantics at the heart of the completeness proof. All that is missing is a demonstration that the connection can be formalized in the system, i.e., Gödel 's Theorem VI.

> It may be true that Skolem had little interest in the formalization of logic, but this does not in the least explain why he did not give a correct proof of that completeness theorem which he explicitly stated ([Skolem, 1928], p. 134), namely that there is a contradiction at some level n if there is an informal disproof of the formula. (Wang, 1974, p. 10)

Gödel argues that the question of completeness arises naturally for *any* proposed method of proving or disproving a formula of the language. Formal or not, it is still coherent and important to ask whether a method proves all the valid formulas, or refutes all the unsatisfiable ones. Gödel's view rejects the formal systems explanation and reintroduces the puzzle: "[t]his blindness (or prejudice, or whatever you may call it) of logicians is indeed surprising" (Letter to Wang, Wang, 1974, p. 9). Both Löwenheim and Skolem introduced informal methods of refuting a first-order formula in the course of proving the LST. But they did not raise the question of completeness. Why did these prominent logicians fail to acknowledge the important fact implied by their own work?

1.5.1 The alternative

Gödel rejects the formal systems explanation in favour of an alternative:

> I think the explanation is not hard to find. It lies in a widespread lack, at that time, of the required epistemological attitude toward metamathematics and toward nonfinitary reasoning. (Wang, 1974, p. 10)

Gödel evidently has Hilbert's program in mind as representative of the epistemological prejudice he targets:

> Non-finitary reasoning in mathematics was widely considered to be meaningful

> only to the extent to which it can be 'interpreted' or 'justified' in terms of a finitary metamathematics. (Note that this, for the most part, has turned out to be impossible in consequence of my results and subsequent work.) This view, almost unavoidably, leads to an exclusion of nonfinitary reasoning from metamathematics. [...] But now the aforementioned easy inference from Skolem 1922 is definitely non-finitary, and so is any other completeness proof for the predicate calculus. Therefore these things escaped notice or were disregarded. (ibid.)

> [Skolem] was a firm believer in set theoretical relativism and in the sterility of transfinite reasoning for finitary questions (see p. 49 of his paper [1970, p. 273]). (Letter to Wang, 7 December 1967, in Wang, 1996, p. 124).

> [E]vidently because of the transfinite character of the completeness question, [Skolem] tried to eliminate it, instead of answering it (ibid.)

Gödel's proposal has been overlooked in the secondary literature, where the formal systems explanation has gone unchallenged. The inadequacy of that view, and the independent interest of Gödel's alternative, are the motivations for this dissertation.

1.6 Motivations

I began this project with the intention of answering a rather narrow research question: was Gödel correct that Skolem failed to recognize completeness because he had a prejudice against non-finitary reasoning? This question was unaddressed in the secondary literature where the formal systems explanation has gone unchallenged. And yet Gödel's critique of that explanation is compelling in its simplicity. By showing how the question of completeness can be coherently raised outside the context of a formal system, he opens an explanatory gap.

Beyond Gödel's specific targeting of Skolem lies a more general claim that "non-objectivist" philosophical viewpoints are conceptually and epistemically infertile. One reason for this, thinks Gödel, is their unwillingness to license non-finitary methods of reasoning. As evidence of this, Gödel cites the role he percieved his own "objectivism" to have played in arriving at his other discoveries, notably, his incompleteness theorems, and his consistency proof for the continuum hypothesis.

Constructivism, intuitionism, and finitism no longer occupy the place they did in mathematics up to 1930. Discoveries beginning with Gödel's incompleteness theorems forced the recognition that the promised epistemological gains of these positions were unattainable. Gödel's claim against Skolem represents a different sort of objection to such "non-objectivist" philosophies of mathematics, one that does not depend on the technical results of the 30s. Even if, for example, Hilbert's program had somehow succeeded in its goal of securing the foundations of mathematics on a finitary basis[14], Gödel's claim gives reason for thinking that this security would not have been worth the cost in terms of the conceptual limitations imposed by such frameworks. An investigation of this claim will have implications for the types of frameworks used in contemporary mathematical investigations, and for the burden of justification on those who endorse these frameworks.

These were the motivations from which the project began. But another theme emerged over the course of the investigation. The path to the discovery of the completeness theorem was arduous and convoluted. Even with all the technical components in front of them, eminent logicians of the time failed to grasp the conceptual significance of the theorem. And yet, today the theorem is something we expect introductory logic students to grasp in the course of a few lectures. My own introduction to the theorem as an undergraduate was a typical

[14] I.e., if the incompleteness theorems were not true.

example of this. I came away having learned that it is easier to prove an equivalent form of the theorem (every consistent set is satisfiable), how to reproduce Henkin's construction, and that term models had something to do with taking the symbols themselves to be the objects in the domain. However, I would have been hard pressed at the time to say *why* the theorem is important, or explain what deep connection it establishes. In other words, I could *prove* completeness but had no idea what it meant.

The latter was something I came to appreciate through an examination of the history of the theorem and its discovery. This suggests an important general motivation for the historical investigation of mathematical results. The process by which a student comes to understand a mathematical concept or proof often runs parallel to the original path of discovery in the mathematical community at large. Mathematical discoveries take place in intellectual environments where progress towards the discovery is made in many directions, and sometimes unwittingly, by many individuals at once. Various technical and conceptual components must be in place before a well-placed individual (like Gödel) is able to bring all the pieces together into a cohesive proof. From setting up the technical apparatus, to proving or disproving related results, to creating an environment in which the conceptual significance of a certain result is highlighted rather than suppressed, many of these tasks are carried out without any awareness of the discovery to which they will eventually lead.

To this end, the current project explores the impact of particular intellectual environments on the recognition of completeness (see Chapter 5). The contextual features introduced in that chapter were crucial for the historical recognition of completeness and ought not to be overlooked when teaching the completeness theorem in modern contexts.

More generally, historical investigations of this kind have important pedagogical and heuristic roles in logic and math education. Projects such as this one are needed to make accessible the original paths of discovery and remind us not to take for granted the ease by which our contemporary proofs seem to arrive at their results.

Chapter 2

Gödel

2.1 Introduction

To understand the puzzle Gödel finds in Skolem's near-anticipation of the completeness theorem, one must first understand the influence of Skolem's work on Gödel's (1930) proof of that theorem. This chapter sets the stage for that comparison by reconstructing Gödel's original proof. The latter has been widely neglected in the literature due to its relative opacity compared with the alternative due to Henkin. For good reason, Henkin's version has become the canonical proof for the completeness of first-order logic. Gödel's original, however, reveals fascinating similarities of method with earlier proofs of the Löwenheim-Skolem theorem.

2.1.1 Henkin

The version of completeness familiar to most logicians today is due to Henkin rather than Gödel. Henkin's (1949) proof is simpler than Gödel's and his methodology more widely applicable.[1] Henkin's proof of "strong completeness" yields Gödel's result as a straightforward corollary.

Henkin's Theorem

Relative to a particular formal axiomatic system, Henkin proves:

> Strong Completeness (SC): *every consistent set* Γ *of statements of a first-order language* $\mathcal{L}$ *has a model of cardinality* α*, where* α *is the cardinality of the set of primitive symbols of* $\mathcal{L}$*.*[2]

We can break down Henkin's proof into two main parts:

1. Show that every consistent set Γ can be extended to a maximally consistent set Γ' having the Henkin witness property.
2. Show that every maximally consistent set with the Henkin witness property has a canonical model.

In step 1, Henkin uses an inductive construction to obtain from Γ a maximally consistent set of closed well-formed formulas (cwff) such that for every cwff $\sigma \in \mathcal{L}$, if $\sigma \notin \Gamma'$, then $\Gamma', \sigma \vdash \perp$ but $\Gamma' \ \perp$. By adjoining to $\mathcal{L}$ an infinite set of new individual constants, the new set can also be shown to possess the Henkin witness property. This means that for every existential statement $\phi = \exists x A(x)$ in Γ, there is a witnessing constant, i.e., a constant c_ϕ in the expanded language $\mathcal{L}'$, such that $\Gamma \vdash \exists x A(x) \rightarrow A(c_\phi)$.

[1] In particular, his proof yields the important model-theoretic notion of a term model.

[2] Gödel's version follows from Henkin's by taking the set of statements to be $\Gamma \cup \{\neg\phi\}$.

In step 2, Henkin shows that Γ has a model $\mathcal{M}$ whose domain consists of the set I of individual constant symbols of the expanded language.[3]

The interpretation function assigns each constant to itself as denotation, and each predicate symbol P to the class of constants that Γ' proves to belong to P. Henkin then shows by induction on formula complexity that for every cwff $\sigma \in \mathcal{L}'$

> σ is True if $\Gamma' \vdash \sigma$ and σ is False if $\Gamma'\sigma$

This yields a truth-assignment under which every member of $\Gamma \subset \Gamma'$ is satisfied.

As an illustration, consider how the proof goes in the case where $\sigma = \forall x B$. If $\sigma \in \Gamma'$, then by the induction hypothesis and the interpretation of predicates, $\mathcal{M} \models B(t)$ for every $t \in I$. By the semantics of $\forall$, σ is true. Conversely, if $\sigma \notin \Gamma'$ then by maximal consistency, $\Gamma', \forall x B \vdash \perp$. By the deductive rules, it follows that $\Gamma' \vdash \forall x B \rightarrow \perp$ and $\Gamma' \vdash \exists x B \rightarrow \perp$. By the witness property, $\Gamma' \vdash B[x \backslash c] \rightarrow \perp$ for some constant c. Thus, if $\Gamma' \vdash B[x \backslash c]$ then $\Gamma' \vdash \perp$, contrary to assumption of consistency. So $\Gamma', B[x \backslash c]$. By the induction hypothesis, this implies that $B[x \backslash c]$ is false. By the semantics of $\forall$, $\sigma = \forall x B$ is false.

Henkin's innovative method bypasses two key features of Gödel's proof.

First, unlike Gödel, Henkin does not begin by setting up a reduction class of formulas. Noting this contrast with Gödel, Henkin later writes:

> It seems that a distinctive feature of my completeness proof for first-order logic, which distinguishes it from Gödel's, is that when a consistent set of [closed well-

[3]Henkin's original proof takes as the domain of $\mathcal{M}$ the set I itself. The standard textbook version constructs a canonical term model induced by the equivalence relation of provable equality on terms. That is, define a relation '$\sim$' by

$$s \sim t \iff \ulcorner s = t \urcorner \in \Gamma'.$$

Then the domain of $\mathcal{M}$ is the set of equivalence classes of variable-free terms t under the relation of provable equality, i.e., the set of $|t| = \{s : s \sim t\}$.

> formed formulas] is given in one language, I proceed to an extended language in which new individual constants are adjoined. But in fact, something like that is implicitly present in Gödel's proof, because he begins by reducing the problem of showing that an arbitrary [closed well-formed formula] is either satisfiable or refutable, to the case of an arbitrary [closed well-formed formula] that is in Skolem normal form. However, in a first-order language with some fixed finite set of predicate symbols, one cannot reduce every [closed well-formed formula] to one in Skolem normal form without adding new predicate symbols. (Henkin, 1996, p. 156)

This necessity is seen in Gödel's proof below with the adjunction of a new predicate R in the proof of Theorem 2.2.0.

The second feature that Henkin's proof lacks is Gödel's construction of a sequence of approximating instances to a universally-quantified statement over an infinite domain. As will be seen, this construction derives from the work of Skolem and Löwenheim in proving the Löwenheim-Skolem theorem.

2.2 Gödel's proof

Gödel's completeness theorem establishes that every valid formula is provable in the Hilbert-Ackermann (HA) system of first-order predicate logic.[4] Gödel proves the theorem in its equivalent form: *every formula is either refutable or satisfiable.* The result can be extended to any first-order predicate calculus in the usual way.

The first stage of Gödel's proof consists in showing that it suffices to prove completeness for a restricted class of formulas, namely, those in prenex normal form and of degree one. A

[4] The system consists of a formal language, axioms, and rules of inference.

formula has *degree one* if it has a single block of universal quantifiers followed by a single block of existential quantifiers. Showing this involves two steps. First, finding a prenex normal form equivalent of an arbitrary formula (this step is omitted in what follows). Second, showing that the satisfiability or refutability of any formula ϕ with k+1 alternating blocks of universal and existential quantifiers turns on the satisfiability or refutability of a formula ψ with only k alternating blocks.[5]

The first stage of the proof guarantees the sufficiency of the second stage where Gödel proves completeness for prenex normal form formulas of degree one. The proof uses an idea found in Löwenheim (1915) and Skolem (1922)—the construction of a sequence of "expansion instances" A_n that approximate to A in a finite domain. The expansions are formed by removing the quantifiers and instantiating the previously-bound variables by ordered sequences of integers. The resulting propositional instances of A are satisfiable or refutable according to the completeness of propositional logic. In the limit, the satsifying truth assignments for each A_n can be ordered in such way as to cumulatively yield a model of A in the domain of natural numbers.

2.2.1 Syntax

This section introduces the basic syntactic notions essential to Gödel's completeness proof.

Language

The primitive logical symbols are individual variables $x_0, x_1, x_2, ...$, predicate variables $F_0, F_1, ...$, propositional variables $X_0, X_1, ...$, and operators $(\vee, \neg, \forall)$. The remaining operators $(\wedge, \rightarrow, \iff, \exists)$ are defined from these in the usual way, and formulas are constructed accord-

[5]This is done by finding two formulas ψ and ψ' such that $\psi \iff \psi'$ and $\psi' \rightarrow \phi$, where ψ is of degree k. Deriving ψ from ϕ necessitates the introduction of a new predicate that takes over the role performed by one of the blocks $\forall\exists$.

ing to the standard compositional rules. Lowercase German letters $\mathfrak{x}, \mathfrak{n}, \mathfrak{u}, \mathfrak{o}$ are used as abbreviations for n-tuples of individual variables (the arity is specified in context).

Axioms

A1. $X \vee X \rightarrow X$

A2. $X \rightarrow X \vee Y$

A3. $X \vee Y \rightarrow Y \vee X$

A4. $(X \rightarrow Y) \rightarrow (Z \vee X \rightarrow Z \vee Y)$

A5. $\forall x F(x) \rightarrow F(y)$[6]

A6. $\forall x[X \vee F(x)] \rightarrow X \vee \forall x F(x)$

Rules of inference

R1. Modus ponens

R2. Rule of substitution: If t is a term and ϕ is a formula possibly containing the variable x, $\phi[t/x]$ is the result of replacing all free instances of x by t in ϕ. Then, for any ϕ and any term t, from ϕ infer $\phi[t/x]$, provided that no free variable of t occurs bound in $\phi[t/x]$.

R3. From $A(x)$ infer $\forall x A(x)$.

R4. Variables (free and bound) can be changed at will.[7]

A formula of the language is *provable* if it can be derived from axioms and rules of inference in finitely many steps. A formula is *refutable* if its negation is provable.

[6] Subject to the condition that solutions (section 5.1.2) have non-empty domains.
[7] Subject to the conditions given in (Hilbert Ackermann, 1928).

Lemmas

Gödel assumes the following lemmas without proof.

L1. For every n, and every n-tuple $\mathfrak{r}$,

(a) $\forall \mathfrak{r} F\mathfrak{r} \rightarrow \exists \mathfrak{r} F\mathfrak{r}$

(b) $\forall \mathfrak{r} F\mathfrak{r} \wedge \exists \mathfrak{r} G\mathfrak{r} \rightarrow \exists \mathfrak{r}[F\mathfrak{r} \wedge \exists G\mathfrak{r}]$

(c) $\forall \mathfrak{r} \neg F\mathfrak{r} \iff \neg \exists \mathfrak{r} F\mathfrak{r}$

L2. If n-tuples $\mathfrak{r}$ and $\mathfrak{r}'$ differ only in the order of the variables, then $\exists \mathfrak{r} F\mathfrak{r} \rightarrow \exists \mathfrak{r}' F\mathfrak{r}$ is provable.

L3. If $\mathfrak{r}$ consists entirely of distinct variables and if $\mathfrak{r}'$ has the same number of terms as $\mathfrak{r}$, then $\forall \mathfrak{r} F\mathfrak{r} \rightarrow \forall \mathfrak{r}' F\mathfrak{r}'$ is provable, even when a number of identical variables occur in $\mathfrak{r}'$.

L4. If p_i stands for either $\forall x_i$ or $\exists x_i$, and q_i stands for either $\forall y_i$ or $\exists y_i$, then

$$p_1, ..., p_n F(x_1, ..., x_n) \wedge q_1, ..., q_m G(y_1, ..., y_m) \iff \mathrm{P}[F(x_1, ..., x_n) \wedge G(y_1, ..., y_m)]$$

is provable, for every prefix P formed from the p_i and q_i, preserving the relative order of p_i amongst p_i and q_i amongst q_i but not necessarily p_i amongst q_i.[8]

1. For every formula A of the language, there is a prenex formula $P(N)$ such that $A \iff P(N)$ is provable.

[8]Gödel offers a proof sketch of L4 in 1929, using the equivalences:

L5. $A \wedge \forall x F(x) \iff \forall x[A \wedge F(x)]$

(a) $A \wedge \exists x F(x) \iff \exists x[A \wedge F(x)]$

Beginning with p_1 (or q_1, modulo the appropriate substitutions), take $q_1, ..., q_m G(y_1, ..., y_m)$ for A, and take $p_2, ..., p_n F(x_1, ..., x_n)$ for $F(x)$, apply (1) or (2) depending on whether the quantifier is $\forall$ or $\exists$, and move p_1 to the front to get $p_1[q_1, ..., q_m G(y_1, ..., y_m) \wedge p_2, ...p_n F(x_1, ..., x_n)]$. Continue in the same way by moving either p_2 or q_1. At each step there is a choice between the outermost (of the conjuncts) p_i or q_i, but the relative order amongst each is preserved.

2. Replacement rule of HA system: If $A \iff B$ is provable, so is $\mathfrak{F}(A) \iff \mathfrak{F}(B)$, where $\mathfrak{F}$ represents an arbitrary expression containing A as a part (see Hilbert Ackermann, 1928, Chapter 3).

3. Every propositional formula is either refutable or satisfiable (Post, 1921)

2.2.2 Semantics

If A is a formula in which occur the individual variables $x_0, x_1, ...x_m$, the predicate variables $F_1, ..., F_k$, and the propositional variables $X_1, ...X_l$, a *solution* S for A consists of a domain D of individuals, a set of relations $f_1, ..., f_k$ defined on D[9], and a set of truth values $w_1, ..., w_l$ such that a true proposition results when the f_i are substituted for the F_i, the x_i replaced by their indices i, and the propositional variables X_i replaced by the truth values w_i. A is *satisfiable* if there exists a solution for A. A is *valid* if and only if the negation of A is unsatisfiable.

2.2.3 First stage

Reduction to formulas of degree 1

A formula is in *Gödel normal form* if it

- (i) Has no free variables
- (ii) Is in prenex normal form (all quantifiers occur at the beginning and scope over the entire formula)
- (iii) Has a string of quantifiers beginning with $\forall$ and ending with $\exists$

[9]Where each f_i has the same arity as F_i.

Following Gödel, the notation $P(A)$ will be used to denote a formula in prenex form where P (also Q, R, etc.) represents a prefix string of quantifiers, and A is a quantifier-free matrix. The *degree* of a formula in Gödel normal form is the number of alternating blocks of quantifiers of the form $\forall\exists$ occurring in its prefix.

Gödel will show that it suffices to prove completeness for Gödel normal form formulas of degree 1 by proving:

Theorem 2.3.1

Every Gödel normal form formula of degree 1 is either satisfiable or refutable.

Theorem 2.2.0

If completeness holds for every formula in Gödel normal form of degree k, then it holds for every formula of degree $k+1$.

Theorem 2.2.1

If completeness holds for every formula in Gödel normal form, then it holds for every formula.

I prove theorems 2.2.1 and 2.2.0 here. Proving Theorem 2.3.1 is the objective of the second stage of Gödel's proof.

Proof of Thm. 2.2.1

Let A be an arbitrary formula and assume A is not in Gödel normal form. So, A fails at least one of the conditions (i)-(iii) above. The proof works by providing a chain of equivalences that preserve satisfiability/refutability between A and a formula that *does* meet each of the conditions (i)-(iii) of Gödel normal form.

(1) Suppose A violates condition (i) by containing the free variables $\mathfrak{r}$. Then A is refutable if and only if $\exists\mathfrak{r}A$ is refutable, via L1(c), R3, and A5. The same result for satisfiability holds by definition.[10]

(2) If $\exists\mathfrak{r}A$ is prenex, skip to step (3). If it is not prenex, then by L5 there is a formula $P(N)$ in prenex form such that

$$\exists\mathfrak{r}A \iff P(N)$$

where P is the quantifier string prefixed to a quantifier-free matrix N.

(3) If $P(N)$ is not yet in Gödel normal form, then the prefix P is not of the form $\forall\exists$.[11] If x and y are variables not occurring in P, and F is a predicate variable not occurring in A, then

$$P(N) \iff \forall xP\exists y[N \wedge (F(x) \vee \neg F(y)]$$

The right-hand formula now meets all the conditions (i)-(iii) and is therefore satisfiable or

[10]Gödel defines satisfiability for a formula $A(x, y, ..., w)$ with free individual variables to mean that $\exists x\exists y...\exists wA(x, y, ..., w)$ is satisfiable. (Cf. footnote 4, 1930)

[11]P is of the form $\exists\forall$, $\exists\forall\exists$, $\forall\exists\forall$, or some iteration thereof.

refutable by the hypothesis. Therefore, A is either refutable or satisfiable by the chain of equivalences:

$$A \iff \exists\mathfrak{r}A \iff P(N) \iff \forall x P \exists y[N \wedge (F(x) \vee \neg F(y))]$$

$\square$ (Theorem 2.2.1)

Proof of Thm. 2.2.0

Let $P(A)$ be a Gödel normal form formula of degree k+1. The prefix P can be rewritten as $\forall\mathfrak{r}\exists\mathfrak{n}Q$, where Q is the prefix of a degree k formula. This means that Q can be written as $\forall\mathfrak{u}\exists\mathfrak{v}R$, where R is the prefix of a degree k-1 formula.

Let E be a new predicate letter not occurring in A. Let $\mathfrak{r}'$ and $\mathfrak{n}'$ be tuples of unused variables of the same number as $\mathfrak{r}$ and $\mathfrak{n}$ respectively.

Let

$$B = \forall\mathfrak{r}'\exists\mathfrak{n}'E(\mathfrak{r}',\mathfrak{n}') \wedge \forall\mathfrak{r}\forall\mathfrak{n}[E(r,n) \rightarrow Q(A)]$$

and

$$C = \forall\mathfrak{r}'\forall\mathfrak{r}\forall\mathfrak{n}\forall\mathfrak{u}\exists\mathfrak{n}'\exists\mathfrak{v}R[E(\mathfrak{r}',\mathfrak{n}') \wedge E(r,n) \rightarrow A].$$

By L6 [replacing $Q(A)$] and two applications of L4,

$$B \iff \forall\mathfrak{r}'\exists\mathfrak{n}'E(\mathfrak{r}',\mathfrak{n}') \wedge \forall\mathfrak{r}\forall\mathfrak{n}\forall\mathfrak{u}\exists\mathfrak{v}[E(r,n) \rightarrow R(A)] = B'$$

By L4,

$$B' \iff \forall\mathfrak{r}'\forall\mathfrak{r}\forall\mathfrak{n}\forall\mathfrak{u}\exists\mathfrak{n}'\exists\mathfrak{v}[E(\mathfrak{r}',\mathfrak{n}') \wedge E(r,n) \rightarrow R(A)]$$

$$\iff \forall\mathfrak{r}'\forall\mathfrak{r}\forall\mathfrak{n}\forall\mathfrak{u}\exists\mathfrak{n}'\exists\mathfrak{v}R[E(\mathfrak{r}',\mathfrak{n}') \wedge E(r,n) \rightarrow A] = \mathrm{C}$$

Therefore,

(1) $$B \iff C$$

Moreover, it is evident that

(2) $$B \rightarrow P(A).$$

By assumption, R has degree k-1, so C has degree k. By the induction hypothesis, completeness holds for C, hence C is either satisfiable or refutable.

If C is satisfiable, i.e., true in a model M, then $P(A)$ must also be true in M, by (1) and (2). If C is refutable, then by (1), $\neg B$ is provable.

Note that in $Q(A)$ the variables $\mathfrak{r}$ and $\mathfrak{n}$ occur free (they are bound by the quantifiers that occur in P but not Q). The second occurrence of E has as arguments the same variables as $Q(A)$. In the first occurrence, $\mathfrak{r}$ and $\mathfrak{n}$ in $Q(A)$ can be replaced by $\mathfrak{r}'$ and $\mathfrak{n}'$ by R4 and L3. Then by L6, occurrences of E in $\neg B$ can be replaced by $Q(A)$, preserving provability. Thus, the provability of $\neg B$ implies the provability of

$$\neg(\forall \mathfrak{r}' \exists \mathfrak{n}' Q(A) \wedge \forall \mathfrak{r} \forall \mathfrak{n}[Q(A) \rightarrow Q(A)])$$

$\forall \mathfrak{n}[Q(A) \rightarrow Q(A)]$ is obviously provable, so if $\neg B$ is provable, then $\neg[\forall \mathfrak{r}' \exists \mathfrak{n}' Q(A)]$ must be provable. Changing variables by L3, $\neg P(A)$ is provable, that is, $P(A)$ is refutable. $\square$ (Thm. 2.2.0)

2.2.4 Second stage

Proof of Theorem 2.3.1

Every formula of degree 1 is either satisfiable or refutable

Let $P(A) = \forall u_1, u_2, ..., u_r \exists v_1, v_2, ..., v_s A(u_1, ..., u_r; v_1, ..., v_s)$ be in Gödel normal form.

The first step is to drop the quantifiers. By instantiating the variables we construct a sequence of propositional formulas that act as approximations to $P(A)$ in a finite domain. An *expansion of level* n is the conjunction of instances of A formed at the *nth* level in the construction. The domain increases at each level new variables are added to witness the existential quantifiers of $P(A)$. The idea is to obtain, as the limit of this process, a model of $P(A)$ in the domain of natural numbers.

Where r is the number of universal variables in $P(A)$, let an ordering of r-tuples of the individual variables $x_0, x_1, x_2, ...$ be given according to increasing sum of their indices, beginning with the r-tuple $(x_0, x_0, ..., x_0)$. These r-tuples in order will be denoted by $\mathfrak{r}_1, \mathfrak{r}_2, ...$

The sequence is defined inductively.

$A_1 = A(\mathfrak{r}_1; x_1, x_2..., x_s)$

$A_2 = A_1 \wedge A(\mathfrak{r}_2; x_{s+1}, ...x_{2s})$

.

.

.

$A_n = A_{n-1} \wedge A(\mathfrak{r}_n; x_{(n-1)s+1}, ...x_{ns})$

The first expansion A_1 is the result of replacing the universal variables of A by the first r-tuple $\mathfrak{r}_1 = (x_0, x_0, ..., x_0)$ of variables in the given ordering, and replacing the s-many existential variables with the first s variables distinct from those occurring in $\mathfrak{r}_1$. At each level n in the sequence, a new instance of the formula is conjoined to those already constructed—the universal variables of A are replaced by the *nth* r-tuple of variables, and the existential variables are replaced by variables distinct (w.r.t. their indices) from those in the *nth* r-tuple, *and* from variables occurring in A_i for $i < n$.

Let $P_n(A_n)$ denote the existential closure of the *nth* formula A_n. The prefix P_n is string of $\exists$ quantifiers binding all the variables of A_n.

Lemma 2.3.2

For every n, $P(A) \rightarrow P_n(A_n)$ is provable.

Proof: By induction on n.

- For $n = 1$,

$P(A) \rightarrow \forall \mathfrak{r}_1 \exists x_1, x_2, ..., x_s A(\mathfrak{r}_1; x_1, x_2, ..., x_s)$ [L3 and R4]

$\forall \mathfrak{r}_1 \exists x_1, x_2, ..., x_s A(\mathfrak{r}_1; x_1, x_2, ..., x_s) \rightarrow \exists \mathfrak{r}_1 \exists x_1, x_2, ..., x_s A(\mathfrak{r}_1; x_1, x_2, ..., x_s)$
$= P_1(A_1)$ [L1(a)]. $\square$

- Assume $P(A) \rightarrow P_n(A_n)$ holds.

  By the ordering of r-tuples according to sum of their indices, for every A_{n+1}, the variables of the nth+1 r-tuple already occur in A_n.[12] Thus, the only new variables in A_{n+1} are the $x_{(n-1)s+1}, ..., x_{ns}$.

  Define P'_n to be the portion of the prefix P_n that omits the quantifiers over the variables of the n+1th r-tuple. So, $\exists \mathfrak{r}_{n+1} P'_n = P_n$ (by L2).

Sublemma 2.3.2

For every n, $P(A) \wedge P_n(A_n) \rightarrow P_{n+1}(A_{n+1})$ is provable.

1. $P(A)$ (Premise)
2. $P_n(A_n)$ (Premise)
3. $P(A) \rightarrow \forall \mathfrak{r}_{n+1} \exists x_{ns+1)}, ..., x_{(n+1)s} A(\mathfrak{r}_{n+1}; x_{ns+1}, ..., x_{(n+1)s})$ [L3 and R4]
4. $P_n(A_n) \rightarrow \exists \mathfrak{r}_{n+1} P'_n(A_n)$ [definition]

[12]The nth+1 r-tuple is either a permutation of the nth r-tuple, or contains at most one new variable, which must have already been introduced in some A_m, for $m \leq n$ in place of an existential variable.

5. $\forall \mathfrak{r}_{n+1} \exists x_{ns+1}, ..., x_{(n+1)s} A(\mathfrak{r}_{n+1}; x_{ns+1}, ..., x_{ns}) \wedge \exists \mathfrak{r}_{n+1} P'_n(A_n)$ [from (1)-(4)]

6. $\forall \mathfrak{r}_{n+1} \exists x_{ns+1}, ..., x_{(n+1)s} A(\mathfrak{r}_{n+1}; x_{ns+1}, ..., x_{(n+1)s}) \wedge \exists \mathfrak{r}_{n+1} P'_n(A_n) \rightarrow \exists \mathfrak{r}_{n+1}[\exists x_{ns+1}, ..., x_{(n+1)s} A$
$P'_n(A_n)]$ [L1(b)]

7. $\exists \mathfrak{r}_{n+1}[\exists x_{ns+1}, ..., x_{(n+1)s} A(\mathfrak{r}_{n+1}; x_{ns+1}, ..., x_{(n+1)s}) \wedge P'_n(A_n)]$ [(5),(6)]

8. $\exists \mathfrak{r}_{n+1}[\exists x_{ns+1}, ..., x_{(n+1)s} A(\mathfrak{r}_{n+1}; x_{ns+1}, ..., x_{(n+1)s}) \wedge P'_n(A_n)] \iff P_{n+1}(A_{n+1})$

   [By definition, because $\exists \mathfrak{r}_{n+1} P'_n(A_n) = P_n(A_n)$ and,

   $P_{n+1}(A_{n+1}) = P_n(A_n) \wedge \exists x_{ns+1}, ..., x_{(n+1)s} A(\mathfrak{r}_{n+1}; x_{ns+1}, ..., x_{(n+1)s})$.]

9. $P_{n+1}(A_{n+1})$ [(7), (8)] $\Box$ (Sublemma 2.3.2)

Lemma 2.3.2 follows from Sublemma 2.3.2 by the induction hypothesis.

For $P(A)$ in Gödel normal form, the quantifier-free formula A consists of atomic components joined by propositional connectives. Assume that A is made up of the propositional variables $X_1, ..., X_l$ and the elementary formulas $F_1(x_{1_1}, ..., x_{1_p}), ..., F_k(x_{k_1}, ..., x_{k_q})$. All individual variables occurring in A are bound by quantifiers in the prefix P.

To obtain a propositional counterpart B_n of A_n, replace individual variables x_i by their integer indices $1 \leq i \leq ns$, and replace elementary formulas $F_i(x_1, ..., x_j)$ by propositional variables, different from any X_i already occurring in A_n, and preserving in this replacement the distinctness of formulas with respect to either the individual variables or the predicate letters (see example below).

Definition 2.3.3

A solution of level n is a set $S_n = \{f_1^n, f_2^n, ...f_k^n, w_1, w_2, ..., w_l\}$ consisting of relations[13] f_i^n defined on the domain $\{1, 2, ..., ns\}$[14], and truth values[15] w_i such that when in A_n each x_i is replaced by the integer i, each F_i replaced by f_i^n, and each X_i is replaced by w_i, A_n comes out true, relative to the domain in question.

Lemma 2.3.3

B_n is satisfiable if and only if there exist solutions of level n.

For the left-to-right direction of lemma 2.3.3, if there is an assignment of truth values to the propositional variables of B_n that makes the whole formula come out true, then solutions of level n are obtained by allowing the denotation of the predicate letters in A_n to be determined by the truth value assigned to the corresponding propositional variable in B_n.[16] [17]

The right-to-left direction, as well as the unsatisfiability of A_n by solutions of level $< n$, are obvious.

Theorem 2.3.3

Every B_n is either refutable or satisfiable

[13](Gödel: "functions")

[14]Where each f_i has the same arity as F_i

[15]0 or 1

[16]Note that if there do not exist solutions of level n, this must be because of the truth-functional inconsistency of the propositonal counterpart to A_n rather than the non-existence, for any atomic $F_i(x_1, ..., x_p)$, of relations f_i^n holding of the numbers $1, ..., p$. Such relations always exist since we consider all possible definitions on the domain in question.

[17]For example, if $A_n = G(x_1, x_1)$, then B_n is satisfiable by assigning truth to the single propositional variable replacing $G(x_1, x_1)$. Working backwards to find solutions of level n, if g is a binary relation defined on the domain $\{1, 2, ..., ns\}$, then $\{g\}$ is a solution for A_n if and only if $< 1, 1 >\in g$. In this case, there are as many solutions of level n as there are relations meeting this condition.

By L7 (completeness of propositional logic).

Lemma 2.3.4

Either, for every n, B_n is satisfiable or for some n, B_n is not satisfiable.

(Law of excluded middle).

Lemma 2.3.5

Either B_n is refutable for some n or every B_n has a solution of level n.

Thm. 2.3.2, Lemmas 2.3.3, 2.3.4.

Theorem 2.3.4

If B_n is refutable for some n, then $P(A)$ is refutable.

Proof: If B_n is refutable, then by R2, R3, and L1(c), $P_n(A_n)$ is refutable. It follows by Lemma 2.3.2 that $P(A)$ is refutable.

Theorem 2.3.5

If for every B_n, there is a solution of level n, then there is a solution for $P(A)$ in the domain of natural numbers.

Lemma 2.3.6

Every solution of level n+1 contains a solution of level n as a part.

By the construction of the A_i, A_n is a conjunct of A_{n+1}, so a truth assignment to the propositional variables X_i of A_{n+1} must also assign truth to the X_i in A_n. Likewise, every predicate[18] of level n+1, defined on the domain $\{1, 2..., ns+1\}$ must agree with a predicate of level n when restricted to the domain $\{1, 2, ..., ns\}$.[19]$\square$

Since the domain of every solution of level n is finite (bounded by ns), there are finitely-many different systems of relations defined on the domain, hence finitely-many possible assignments to predicate variables of A_n, and finitely-many different assignments of truth values to the resulting propositional formulas. From the fact that every A_n has only finitely-many solutions, but there are infinitely many levels n, it follows (by König's lemma) that for each n, at least one solution of level n must occur as a part in infinitely many solutions of higher level.

Therefore, by appeal to Choice (Gödel's "familiar arguments') or, alternatively, to some definable ordering on the solutions at each level since they are finite (cf. Skolem 1923), there is an infinite sequence $S = S_1, S_2, ...S_k, ...$ of solutions such that $S_k \subset S_{k+1}$.

Now we define a solution $\mathfrak{S} = \{\phi_1, \phi_2, ...\phi_k, \alpha_1, ..., \alpha_l\}$ consisting of k-many relations ϕ and l-many truth values α, as follows:

1. For $1 \leq i \leq k$, $\phi_i(z_1, ..., z_p)$ is true of the *numbers* $z_1, ..., z_p$ if and only if $\exists m \in \mathbb{N}$ such

[18] Gödel means "predicate" by "function".

[19] [In the example, at level 2 we consider all possible relations f_i^2, defined on the larger domain $\{1, 2, 3\}$, as replacements for the occurrences of G in B_2. Obviously, the set of ordered pairs defined on $\{1, 2\}$ is a subset of the ordered pairs defined on $\{1, 2, 3\}$, so each f_i^2 is an extension of some f_i^1.]

that for $f_i^m \in S_m$, $(z_1, ..., z_p) \in f_i^m$.

2. For $1 \leq i \leq l$, $\alpha_i = 1$ if and only if $\exists m \in \mathbb{N}$ such that $w_i^m \in S_m$ and $w_i^m = 1$. Otherwise, $\alpha_i = 0$.

Recall that $P(A) = \forall u_1, u_2, ..., u_r \exists v_1, v_2, ..., v_s A(u_1, ..., u_r; v_1, ..., v_s)$, so to show that $P(A)$ is satisfied by $\mathfrak{S}$ we show that for every r-tuple of integers $k_1, k_2, ..., k_r$, there is an s-tuple $j_1, j_2, ..., j_s$ such that

$$A'(k_1, ..., k_r; j_1, ..., j_s)$$

is a true proposition, where A' is the result of substituting for the predicate variables in A the ϕ_i, and for the propositional variables in A the truth values α_i.

Lemma 2.3.7

For every $\phi_i, \alpha_i \in \mathfrak{S}$,

$$\phi_i\{1, 2, ..., ns\} = f_i^n \qquad \text{and} \qquad \alpha_i = w_i^n$$

where f_i^n and w_i^n belong to the nth solution S_n in the sequence defined above.

Let $\mathfrak{v} \in \phi_i\{1, 2, ..., ns\}$. Then by definition $\exists m$ such that $\mathfrak{v} \in f_i^m$. If $m < n$, then $\mathfrak{v} \in f_i^m \to \mathfrak{v} \in f_i^n$ since $S_m \subset S_n$ by Corollary 2.3.4. If $m > n$, suppose $\mathfrak{v} \in f_i^m$ but $\mathfrak{v} \notin f_i^n$. Then $f_i^n \not\subset f_i^m$ so $S_m \not\subset S_n$, contrary to the corollary. So $\mathfrak{v} \in f_i^n$.

Similarly for the α_i.

By construction, the r-tuple indexed by the numbers $k_1, k_2, ..., k_r$ occurs in place of the universal variables of $P(A)$ at some nth place in the sequence of A_is. We have

$$A_n = A_{n-1} \wedge A(x_{k_1}, ..., x_{k_r}; x_{t_1}, ..., x_{t_s})$$

By Lemma 2.3.6, the restriction of $\mathfrak{S}$ to the domain $\{1, 2, ..., ns\}$ is a solution for A_n, so a true proposition results when the F_i^n and w_i^n are replaced by the ϕ_i and α_i respectively, and the variables x_i are replaced by their indices. Thus, $t_1, ..., t_s$ is the desired s-tuple. $\square$ **(Thm. 2.3.5)**

2.2.5 Summary

The proof can be summarized as follows. The law of excluded middle establishes a basic alternative: either

1. there is some n such that B_n is unsatisfiable, or
2. every B_n is satisfiable.

If (1) holds, then by the completeness of propositional logic, B_n is refutable. If B_n is refutable then $P(A)$ is refutable by Theorem 2.3.4 (and especially, Lemma 2.3.2).

If (2) holds, then Theorem 2.3.5, together with Lemma 2.3.3, establishes that $P(A)$ is satisfiable in the domain of natural numbers.

This suffices to prove the main theorem of part two (Theorem 2.3.1): every formula of the restricted class is either refutable or satisfiable.

Completeness follows by the results of part one. Here Gödel shows that it suffices to prove completeness for the restricted class of formulas in Gödel normal form. By Theorem 2.2.0 and the main result of part two (Theorem 2.3.1), it follows that completeness holds for every formula in Gödel normal form. It follows by Theorem 2.2.1 that completeness holds for every formula.

2.3 Conclusion

This chapter aimed to provide a background to understand proofs of the completeness theorem for first-order logic. I briefly reviewed the standard proof due to Henkin and noted salient features that differentiate it from Gödel's original. I then gave a complete reconstruction of Gödel's 1930 proof, filling a gap in the literature and hopefully rendering the proof more accessible to the modern reader. The next chapter will undertake a similar project for Löwenheim's 1915 proof of the Löwenheim-Skolem theorem.

Chapter 3

Löwenheim

3.1 Introduction

In 1915, Löwenheim proved the theorem that now bears his, and Skolem's, name. Referring to it as *the* theorem is, however, imprecise. Amongst the few commentators who have confronted the notational complexities of Löwenheim's system, there is no consensus on exactly which theorem he aimed to prove in his (1915), nor on whether he succeeded in doing so. This chapter reconstructs Löwenheim's proof of the weak version up to the point where the different interpretations diverge. I then look at different ways of finishing the proof. On two of these interpretations, the claim that Skolem's work contains the components of an informal completeness theorem can be extended to Löwenheim. The final section considers Gödel's allegation that reluctance to use non-finitary reasoning is what prevented the recognition of completeness before 1930. After making this allegation precise by defining Gödel's sense of non-finitary reasoning, I assess the challenge to his claim posed by Löwenheim and Skolem's use of such methods. I argue that Gödel's own response to this challenge is unconvincing.

3.2 Proving the theorem today

The Löwenheim-Skolem theorem (LST) as it is understood today is a more general version of the one first proven by Löwenheim in 1915. In modern terms, Löwenheim proves that if a first-order formula F is satisfiable in an infinite domain, then it is satisfiable in a domain that is at most countable. The now-standard version of the theorem extends this to arbitrary sets of formulas and arbitrary languages, splitting into "upward" and "downward" theorems depending on the cardinality of the model whose existence is guaranteed. Only the downward version is at stake in what follows:

Downward Löwenheim-Skolem Theorem

Let Γ be a set of first-order statements in a countable language. If Γ is satisfiable in an infinite domain, then Γ is satisfiable in a countable domain.

Most modern proofs of the LST derive it as a consequence of Henkin completeness. Recall Henkin's proof of:

Strong Completeness (SC): *every consistent set Γ of statements of a first-order language $\mathcal{L}$ has a model of cardinality α, where α is the cardinality of the set of primitive symbols of $\mathcal{L}$.*[1]

By Soundness, the consistency of Γ follows from its satisfiability per the hypothesis of the LST. The remainder of the proof consists in showing that expanding the language $\mathfrak{L}$ to include Henkin constants for every existentially quantified formula adds at most countably many new symbols. It follows that the Henkin term model guaranteed by (SC) is countable.[2]

[1] Gödel's version follows from Henkin's by taking the set of statements to be $\Gamma \cup \{\neg\phi\}$.

[2] This proves the weak version of LST. Recall the distinction from Chapter 1 (and repeated below): The

The proof using Henkin's method has overshadowed Löwenheim's original for the same reason that Henkin's proof overshadowed Gödel's 1929 proof of completeness. The notational complexity of Löwenheim's system is foreign and off-putting to modern readers; the following reconstruction modernizes notation wherever possible, while preserving the form and spirit of Löwenheim's original.

3.3 Löwenheim's System

Syntax Löwenheim's language includes the following logical symbols:

- "indices" *i, j, h, k, l, m*; act as variables and may be subscripted by other indices
- $1', 0'$; binary relation constants interpreted respectively as the identity and diversity relations over the first-order domain (i.e. $0' = \overline{1'}$)
- Boolean operator symbols $+, \cdot, ^{-}$; interpreted respectively as (inclusive) disjunction, conjunction, and negation operators. The operators obey the standard laws for the Boolean algebra $\{0, 1\}$.
- quantifiers Σ and Π; correspond to $\exists$ and $\forall$
- $=$; corresponds to the biconditional $\iff$
- propositional constants 1, 0; interpreted as denoting truth values

In addition to logical symbols, lowercase letters *a, b, z* denote Löwenheim's "relatives", i.e., object-language predicates or relations. Uppercase letters A, B, F etc. are metalinguistic

weak version states that if a formula A is satisfiable, then it is satisfiable in a countable domain. The strong "subdomain" version states that if A is satisfiable in an infinite domain D, then it is satisfiable in a countable subdomain D' of D, where the predicates retain the same meaning in D' as in D, modulo the restriction. In the modern proof using (SC), the Henkin model bears no relation to any model guaranteed by the satisfiability of Γ. In contrast, contemporary proofs of the strong version of the theorem typically use the Tarski-Vaught criterion to show the existence of an elementary substructure.

variables ranging over formulas. A "relative coefficient" is the result of subscripting an n-place relation symbol with an n-length sequence of either indices or terms denoting elements of the domain. Relative coefficients thus correspond to atomic formulas (with or without free variables).

A first-order formula (*Zählausdruck*) is constructed from relative coefficients, logical operators, and finitely-many quantifiers taken to range over individuals of the domain. A first-order *equation* (*Zählgleichung*) is the result of equating a formula with one of the propositional constants 0 or 1.

Note that the language defined above does not include individual constants. When a formula or equation is interpreted in a domain, the language is implicitly expanded to include canonical names for every element of the domain. These may occur in place of indices and subscripts to form atomic sentences.

The presentation here will depart from Löwenheim's subscript notation. Brackets are added where necessary. Thus,

- $A_{i_1,\ldots i_n}$ will be written as $A(i_1, \ldots i_n)$
- Π_i, Σ_i as $\Pi i, \Sigma i$
- Multiple quantifers $\prod_{i,j}, \sum_{i,j}$ as $\Pi i, j, \Sigma i, j$

Semantics The basic semantic notions needed for the proof are as follows. A *solution* in a domain D is a function that assigns truth values (0 or 1) to closed atomic formulas of the expanded language. An *assignment* in a domain D is a function that assigns elements of D to the variable terms.[3] An *interpretation* in a domain D is an assignment of truth values

[3]These include both constant indices (free variables) and special terms called "fleeing indices" introduced as part of the transformations to obtain a formula in a certain normal form. Fleeing indices are terms subscripted by universally quantified variables. Upon substitution of elements for these subscripts, the terms behave as ordinary free variables. More on this below.

to each closed formula of the language. Each solution and assignment over D determines a unique interpretation defined by recursion.

A formula is satisfied by an interpretation in a domain D if the interpretation assigns it the truth value 1. An equation is satisfied by an interpretation in a domain D if the interpretation assigns the same truth value to both sides. An equation is "identically satisfied" if every interpretation (across domains) satisfies it. Löwenheim's proof deals exclusively with expressions brought into zero form, i.e. equated to zero. Thus, $A = 0$ is identically satisfied if there is no interpretation that satisfies A, i.e. if and only if $\overline{A} = 1$ in every domain. A *fleeing equation* is one that is not identically satisfied in every domain but is identically satisfied in every finite domain.

Quantifiers Löwenheim explains the semantics of the quantifiers Σ and Π in terms of the connectives $+$ and $\cdot$. Σ and Π represent iterated sums and products over all the elements of the domain. Quantified formulas $\Sigma i_1, ..., i_r A(i_1, ..., i_r)$ or $\Pi i_1, ..., i_r A(i_1, ..., i_r)$ are taken to represent, respectively, the infinite sum (disjunction) or infinite product (conjunction) of instances $A(\mathfrak{r}_\mathfrak{k})$ for every r-tuple $\mathfrak{r}_\mathfrak{k} \in D^r$. By the Boolean definitions of sum and product, $\Pi i_1, ..., i_r A(i_1, ..., i_r)$ will be equal to 1 (i.e. true) just in case every atomic instance $A(\mathfrak{r}_\mathfrak{k})$ is equal to 1 (relative to a domain and assignment of elements to the variables). Likewise, $\Sigma i_1, ..., i_r A(i_1, ..., i_r)$ will be equal to 1 (i.e. true) just in case at least one $A(\mathfrak{r}_\mathfrak{k})$ is equal to 1. The instances $A(\mathfrak{r}_\mathfrak{k})$ obtained for each replacement of the bound variables in $\Pi i_1, ..., i_r A(i_1, ..., i_r)$ or $\Sigma i_1, ..., i_r A(i_1, ..., i_r)$ are called, respectively, the *factors* or *summands* of the quantified formula. (Note that these definitions extend laws valid for finite sums and products to infinite ones; no justification is given.)

3.4 The Proof

Löwenheim states his theorem as follows:

> If the domain is at least denumerably infinite, it is no longer the case that a first-order fleeing equation is satisfied for arbitrary values of the relative coefficients. (1915, p. 235)

By the definition of fleeing equation,

> If a first-order equation of the form $F = 0$ is identically satisfied in every finite domain but not in every domain, then it is not identically satisfied when the domain is countable.

A modern statement of the theorem is obtained by recalling that $F = 0$ is identically satisfied if there is no interpretation that satisfies the formula F:

Theorem 3.3 (Löwenheim-Skolem) If a first-order formula F is satisfiable in some infinite domain but not in any finite domain, then it is satisfiable in a countable domain.

Löwenheim's proof is open to two different readings, corresponding to a strong and weak version of the theorem.

The strong version appeals to a given infinite domain D and a given interpretation of F in D. A countable model of F is obtained as a submodel of D, with the predicates of F retaining the meaning assigned to them by the original interpretation.

The weak version of the theorem makes no appeal to a particular domain. A countable model of F is constructed from below using only the hypothesis that F is satisfiable in *some* domain. The interpretation of F in the countable model need bear no relation to the interpretation assumed in the hypothesis.

The reconstruction given here follows the standard reading according to which Löwenheim intended to prove the weak version. The revisionary reading according to which he intended to prove the strong version (Badesa, 2004) is discussed in the final subsection.

3.4.1 Outline

We prove the theorem in its modern version:

> If a first-order formula F is satisfiable in some infinite domain but not in any finite domain, then it is satisfiable in a countable domain.

The reconstruction of Löwenheim's proof of Theorem 3.3 is based on the definition given above of satisfiability for *formulas*. Löwenheim's reference to equations is dropped in what follows.

The first stage of Löwenheim's proof, like Gödel's, involves converting an arbitrary formula F into an equivalent prenex normal form (different from Gödel normal form). A formula in Löwenheim normal form has a string of existential quantifiers followed by a string of universal quantifiers prefixed to a quantifier-free matrix A.

Löwenheim's conversion process is distinctive in his introduction of "fleeing indices" —variable terms indexed by universally bound variables. Similar but not identical to Skolem functions, these terms arise when existential quantifiers are taken from within the scope of a universal quantifier and moved to the front of the formula. A special quantifier $\underline{\Sigma}$ is introduced to bind fleeing indices. After converting to the form $\Sigma\underline{\Sigma}\Pi F$, the Σ and $\underline{\Sigma}$ quantifiers can be dropped without altering the conditions under which the formula is satisfiable.

The second stage of the proof uses the assumption of satisfiability in an infinite domain to construct an interpretation that makes ΠF true in a countable domain. The interpretation will consist of (*i*) an assignment of values to the constant and fleeing indices of ΠF (i.e., the free variables created by dropping Σ and $\underline{\Sigma}$ quantifiers), and (*ii*) an assignment of truth values to the resulting atomic propositional components (relative coefficients). To this end, Löwenheim constructs, level by level, a tree consisting of the factors of ΠF conceived as an infinite product. At each level, finitely many new integers are introduced to instantiate the free variables. The nodes of the tree represent possible assignments to the variables on the finite domain constructed up to that point.

Choosing an infinite path through the tree yields (modulo a gap in the proof) an interpretation of ΠF in a domain that is at most countable. By the assumption of the theorem that F is not satisfied in any finite domain, the constructed domain must be infinite.

3.4.2 First Stage: Converting to Normal Form

Conversion to Löwenheim Normal Form

The first step is the reduction of the proof to formulas in Löwenheim normal form, in which a string of existential quantifiers is followed by a string of universal quantifiers prefixed to a quantifier-free formula F. Löwenheim's proof is complicated by the fact that he carries out the two steps—moving existential quantifiers in front of universals and obtaining an equivalent prenex form—in the opposite order to what is now standard. A modern proof would first move all quantifiers to the front by appealing to the so-called normal form theorem, and then change their order. Löwenheim begins by moving existential quantifiers in front of universals quantifiers, creating second-order quantifiers (over Löwenheim's fleeing indices) which must then be dealt with as special cases in his version of the prenex normal form theorem (see below).[4]

Löwenheim's procedure begins with an arbitrary formula F and outlines a finite recursive procedure for finding a logically equivalent formula F' in which no quantifiers of either type occur in the scope of a Π.

Löwenheim shows how to eliminate quantifiers from the "productand", i.e., the scope of the outermost Π, according to its form. Productands can take the following forms:

1. $A \cdot B$
2. ΠA
3. $A + B + ... + A_n$
4. ΣA

Löwenheim gives equations by which productands of these forms may be transformed into equivalent formulas with no quantifier in the scope of a Π. He does not address the validity of the equations used in these steps. Most are generalizations of algebraic distributivity laws taken directly from Schröder. Note that Löwenheim assumes that the productand contains at least one quantifier and that the negation operator applies only to relative coefficients. This explains the absence of a clause for negation.[5]

Form (4) introduces Löwenheim's use of "fleeing subscripts" and distinguishes Löwenheim normal form from Skolem's. The equation Löwenheim uses to transform a productand of this form is:

[4] In actual practice Löwenheim ignores this necessity because he treats $\underline{\Sigma}$ quantifiers as behaving the same as ordinary Σ quantifiers. He assumes without justification that the former quantifiers can be moved to the front of the formula by the same equations used for Σ quantifiers.

[5] This could easily be added. Schröder proves the standard equivalences for negated quantifiers. Negation symbols appearing at the beginning of a formula can then be eliminated by reverting to the equation form of the theorem and recalling that $A = 0$ if and only if $\overline{A} = 1$.

Equation 3.4 $$\Pi i \Sigma k A(i,k) = \underline{\Sigma} k_i \Pi i A(i,k_i)$$

This equation introduces a new quantifier $\underline{\Sigma}$ that ranges over what Löwenheim calls "fleeing indices", variables of the form k_i whose subindices are universally quantified variables. Löwenheim writes that the indexed term k_i is to run through "*all* subscripts, that is, through all elements of [the domain]" (p. 236). Several commentators have therefore taken $\underline{\Sigma}$ to represent a string of as many (first-order) existential quantifiers as there are elements of the domain. This reading is supported by the fact that Schröder expressly defines a quantifier with this meaning. However, this violates Löwenheim's syntactical constraints on *Zählausdruck* in which quantifiers may occur only finitely-many times. More importantly, it renders questionable the justification of equation 3.4[6]

Alternatively, $\underline{\Sigma}$ is understood as a second-order quantifier.[7] Its meaning can be stated (anachronistically) in terms of indexed families:

> $\underline{\Sigma} k_i \Pi i A(i,k_i)$ is true just in case there exists an indexed family $\{k_a | a \in D\}$ such that for all $a \in D$, $A(a,k_a)$ is true in D (relative to an interpretation).

The second reading has the virtue of confirming the validity of the above equation for changing the order of quantifiers, and does so without appeal to formulas of infinite length.[8]

By recursion on the number of logical symbols occurring in the productand, equations (1)-(4) can be used to transform a formula of the form $\Pi i_1, ..., i_n F$ into an equivalent form in which no quantifier occurs in the scope of a Π. Vacuous universal quantifiers may be added to an

[6] Which would, on this understanding, assert an equivalence between formulas of infinite length.

[7] This violates Löwenheim's constraints on *Zählausdruck*, but Löwenheim notably fails to even mention the $\underline{\Sigma}$ quantifier in his specification of syntax.

[8] In modern notation, the equivalence is:

$$\forall x \exists y A(x,y) = \exists f \forall x A(x, f(x))$$

arbitrary formula if it is not already of the form $\Pi i_1, ..., i_n F$.

The result of the procedure, according to Löwenheim, is a sum of formulas in Löwenheim normal form, i.e., $C + \Sigma D_1 + ...\Sigma D_n + \Pi E_1 + ...\Pi E_m + ...\Sigma\Pi F_1 + ...\Sigma\Pi F_r = 0$, where the Σs may also be of type $\underline{\Sigma}$. While it goes beyond the scope of the present discussion, additional transformations and variable changes are required for a rigorous recursive proof and Löwenheim does not have the technical apparatus for this.[9]

The second step in the conversion to Löwenheim normal form takes this sum (disjunction) of formulas in Löwenheim normal form and moves all the quantifiers to the front, using the equivalences:

1. $A = \Sigma i A(i) i$
2. $\Sigma i A(i) + \Sigma i B(i) = \Sigma i A(i) + B(i)$
3. $\Pi i A(i) + \Pi i B(i) = \Pi i, j[A(i) + B(j)]$

Also required (but not mentioned) is the equivalence

4. $A = \Pi i A(i)$.

Löwenheim also neglects to mention the notational changes required by these transformations, i.e. the renaming and ordering of indices (variables) to ensure that no variable is quantified more than once, and that no variable occurs both free and bound.

Using equivalence (1), a Σ can be added to any summand not already prefixed by one (with the same indices). Using (2), these Σ or $\underline{\Sigma}$ are combined to form an initial string of quantifiers

[9]Cf. (Badesa, 2004, p.125-127) for discussion and a sketch of the proof that is missing from Löwenheim.

prefixed to a sum of products. Using (3), the Πs are moved to the front (but still under the Σs) to obtain an equation of the form

$$\Sigma\Pi(F_0 + F_1 + ...F_n) = 0 \text{ or } \Sigma\Pi F = 0$$

The reconstruction offered here follows the modern statement of the theorem involving formulas rather than equations.

3.4.3 Second Stage

Constructing a tree

In what follows, lowercase German letters $\mathfrak{r}, \mathfrak{n}, \mathfrak{u}, \mathfrak{o}$ are used as abbreviations for n-tuples of individual variables (the arity is specified in context).

Let $\exists x_1, \exists x_2, ..., \exists x_n \underline{\exists} k_{\mathfrak{r}} \Pi \mathfrak{r} A(\mathfrak{r}; x_1, x_2, ..., x_n, k_{\mathfrak{r}})$ be a formula in Löwenheim normal form obtained from an arbitrary formula by the results of the first stage.[10] Let

$$\Pi F = \Pi \mathfrak{r} A(\mathfrak{r}; x_1, x_2, ..., x_n, k_{\mathfrak{r}})$$

be the result of dropping the Σ and $\underline{\Sigma}$ quantifiers. Thus, ΠF contains the variables $\mathfrak{r} = u_1, u_2, ..., u_r$ bound by Π, the free variables $x_1, x_2, ..., x_n$ (Löwenheim's "constant indices"), and the "fleeing index" $k_{\mathfrak{r}}$.

The proof begins with a recursive definition of an infinite sequence $A_1, A_2, ..., A_n, ...$ of formulas. Each A_i consists of the factors of ΠF when its universal quantifiers range over a

[10]For ease of exposition, we carry out the proof for a single $\underline{\Sigma}$ quantifier over the fleeing index $k_{\mathfrak{r}}$. In general, a fleeing index may be subscripted by any variable (or combination thereof) bound by a Π quantifier.

finite domain, increasing with n. The general description of the construction is followed by an example.

Base case A_1 is constructed as follows:

1. Replace the constant indices (free existential variables) $x_1, x_2, ..., x_n$ of ΠF by the first n integers. This assignment stays fixed across all subsequent factors of ΠF. The numerals introduced denote specific (though not necessarily distinct) elements of some domain in which ΠF is satisfied. Call this the starting domain.[11]

2. Take the universal variables $\mathfrak{r} = u_1, u_2, ..., u_r$ to range over the numerals introduced in step one. Thus, consider every r-tuple that can be formed from the numerals 1 through n. In the case where ΠF does not contain free variables, take 1 to denote an arbitrary element of the domain and replace every u_i by 1. For each r-tuple r_1^i, form the factor of ΠF that results from substituting the integers of r_1^i for the variables of $\mathfrak{r}$, including its occurrence as subindex of $k_{\mathfrak{r}}$.

3. From the substitution of the previous step, the terms $k_{r_1^i}$ now behave as ordinary free variables standing for (possibly distinct) elements of the domain. Introduce unused integers $n+1, n+2, ..., n_1$ to denote the elements represented by each $k_{r_1^i}$.

The resulting formula A_1 is the product of as many factors of ΠF as there are distinct r-tuples defined on the starting domain. Each factor is propositional in nature, being quantifier-free and containing numerals in place of the variables, free and bound, of ΠF.

[11]Löwenheim is *not* constructing a term model using the numerals introduced as constants. He is explicit about the fact that the integers denote elements of some (non-syntactic) domain. His consideration of the possible equalities and inequalities amongst the constants of a given level would make no sense if the domain consisted of the syntactic terms themselves (since every integer is distinct).

Inductive clause Assume A_n has been constructed. A_{n+1} is formed by letting the universal variables $\mathfrak{r} = u_1, u_2, ..., u_r$ range over the integer constants of A_n. For each r-tuple r^i_{n+1} that can be formed from these integers, write a factor of ΠF in which $\mathfrak{r}$ is replaced by the integers of r^i_{n+1}. The constant indices (free variables) are assigned the same numerals as in A_1. Introduce unused numerals for the fleeing terms $k_{r^i_{n+1}}$ after replacement of their subindices by the appropriate integers.

Example 3.4.3 Let $\Pi F = \Pi i A(i, k_i, j, h)$. To form A_1, the free variables j and h are replaced by 1 and 2. (Note that these variables will take the same values in every factor of ΠF. Next, the universal variable i is taken to range over the domain $\{1, 2\}$. This yields two factors of ΠF, one for $i = 1$ and one for $i = 2$. The subscript of k_i takes the same value is i in each factor. Finally, introduce distinct integers 3 and 4 for the elements denoted by k_1 and k_2 in the first and second factors. Thus,

$$A_1 = A(1, 3, 1, 2) \cdot A(2, 4, 1, 2)$$

The second formula A_2 is formed by taking the universal variable i to range over the new domain of constants introduced to form A_1, i.e., $\{1, 2, 3, 4\}$. Since this includes the domain $\{1, 2\}$, A_2 will include A_1 as one of its factors. New integers are introduced for the new terms k_3 and k_4. Thus,

$$A_2 = A(1, 3, 1, 2) \cdot A(2, 4, 1, 2) \cdot A(3, 5, 1, 2) \cdot A(4, 6, 1, 2)$$

The formula A_3 will be the factors of ΠF that can be formed when i ranges over the integers 1 through 6. And so on.

Dealing with identity In Gödel's proof, the next step would be to consider the possible assignments of truth values to atomic components that satisfy each A_n, where such assignments are subject only to the restrictions of propositional logic. Gödel assumes no prior interpretation of the predicate letters (Löwenheim's "relatives"). Their meaning is determined by the satisfying assignments of truth-values to atomic components.

However, Löwenheim's proof is for the language of first-order predicate logic *with* identity. Thus, not all the predicate symbols may be considered uninterpreted. In particular, the symbols $1', 0'$ are binary relation constants interpreted respectively as the identity and diversity relations. Assume that the formula ΠF includes at least one of these coefficients. To assign truth values to these atomic components in each A_n, one must determine the equalities and inequalities actually holding amongst the elements of the domain independent of the numerals used to denote them.

To this end, each A_n must be further differentiated into finitely many specializations $A'_n, A''_n, A'''_n, ...$ obtained by considering all possible systems of equalities on the integer constants occurring in A_n. Every system of equalities amongst the integers introduced up to and including level n determines a particular $A_n^{(v)}$ formed by replacing the integers of A_n by the lowest representative of the corresponding equivalence classes.

Example 3.4.3 Consider again the formula $\Pi F = \Pi i A(i, k_i, j, h)$ for which $A_1 = A(1,3,1,2) \cdot A(2,4,1,2)$.

For the constants 1, 2, 3, 4 occurring in A_1, there are now 14 possible systems of equalities, which are represented by the following equivalence classes:

- $\{1, 2, 3, 4\}$
- $\{1\}\{2, 3, 4\}$

- $\{2\}\{1,3,4\}$
- $\{3\}\{1,2,4\}$
- $\{4\}\{1,2,3\}$
- $\{1,2\}\{3,4\}$
- $\{1,3\}\{2,4\}$
- $\{1,4\}\{2,3\}$
- $\{1\}\{2\}\{3,4\}$
- $\{2\}\{3\}\{1,4\}$
- $\{3\}\{4\}\{1,2\}$
- $\{1\}\{4\}\{2,3\}$
- $\{2\}\{4\}\{1,3\}$
- $\{1\}\{2\}\{3\}\{4\}$

If we replace the constants in A_1 by the lowest representative of the corresponding equivalence classes, we get the following formulas (roman numerals in place of primes):

- $A_1^i = A(1,1,1,1) \cdot A(1,1,1,1)$
- $A_1^{ii} = A(1,2,1,2) \cdot A(2,2,1,2)$
- $A_1^{iii} = A(1,1,1,2) \cdot A(2,1,1,2)$
- $A_1^{iv} = A(1,3,1,1) \cdot A(1,1,1,1)$
- $A_1^{v} = A(1,1,1,1) \cdot A(1,4,1,1)$

- $A_1^{vi} = A(1,3,1,1) \cdot A(1,3,1,1)$
- $A_1^{vii} = A(1,1,1,2) \cdot A(2,2,1,2)$
- $A_1^{viii} = A(1,2,1,2) \cdot A(2,1,1,2)$
- $A_1^{ix} = A(1,3,1,2) \cdot A(2,3,1,2)$
- $A_1^{x} = A(1,3,1,2) \cdot A(2,1,1,2)$
- $A_1^{xi} = A(1,3,1,1) \cdot A(1,4,1,1)$
- $A_1^{xii} = A(1,2,1,2) \cdot A(2,4,1,2)$
- $A_1^{xiii} = A(1,1,1,2) \cdot A(2,4,1,2)$
- $A_1^{xiv} = A(1,3,1,2) \cdot A(2,4,1,2)$

A_n is satisfiable if there exists a solution for at least one $A_n^{(v)}$, i.e., an assignment of truth values to atomic components, consistent with the meaning of the identity or diversity relations $1'$ and $0'$, and such that $A_n^{(v)}$ comes out true on this assignment by the rules of propositional logic.

3.4.4 Proving the theorem

In the construction just outlined, the formulas A_n, when further differentiated into the variations $A_n^{(v)}$ based on equivalence relations over the domain, form a finitely-branching infinite tree ordered by the relation of extension between formulas. From this tree construction, the natural step towards proving the LST would be to use the hypothesis of satisfiability to show that each level must contain at least one satisfiable formula.

However, Löwenheim's exposition is unexpected. In describing the construction of the tree, he proceeds as if it is not known that ΠF is satisfiable, and at each level it is possible that

ΠF is unsatisfiable or "vanishes" (the term "vanish" (verschwinden) comes from the Boolean algebraic tradition within which Löwenheim was working. A mathematical function is said to vanish when it takes the value 0 for some argument. Similiarly in this context, a formula vanishes if it is equal to zero on every assignment of truth values to the atomic components, or in other words, is unsatisfiable).

Thus, the structure of the proof is as follows. Construct A_1 and its finitely many variants $A_1^{(v)}$. Consider all possible solutions to each $A_1^{(v)}$. If none result in a true formula, then A_1 vanishes. Without offering a proof, Löwenheim adds "ΠF will certainly vanish identically in every domain if $[A_1]$ does."

Suppose A_1 does not vanish, meaning that one of the $A_1^{(v)}$ has a solution on which it is equal to 1. Only then do we proceed to the next level, constructing A_2 and its variants. We again check whether A_2 vanishes before proceeding to construct A_3 and so on.

Whereas Gödel appeals to the completeness of propositional logic (which had not been proved in 1915), Löwenheim takes for granted that for each n, A_n is either satisfiable or, if A_n is not satisfiable, that this can be shown in a finite number of steps by running through all the finitely many possible truth assignments to the finitely many $A_n^{(v)}$.

Like Gödel, Löwenheim implicitly applies the law of excluded middle to infinite collections when he presents the alternative: either A_n vanishes for some n, or every A_n is satisfiable. In the first case, the procedure Löwenheim has just described provides a method for showing that A_n vanishes and therefore (according to Löwenheim) that ΠF vanishes as well. In the case where A_n is satisfiable for every n, Lemma 3.5.7 below purports to establish that ΠF is satisfiable.

Definition: A formula A is an *extension* of another formula B if A is of the form $B \cdot C$.

Lemma 3.5.1 *Every level n+1 formula $A^{(v)}_{n+1}$ is an extension of one and only one level n formula $A^{(v)}_n$.*

Lemma 3.5.2 *Every solution to a level n+1 formula includes as a part a solution to a level n formula.*

Lemmas 3.5.1 and 3.5.2 follow by construction of the A_n. □

Lemma 3.5.3 *If none of the level n formulas $A^{(v)}_n$ are satisfiable, then A_n is unsatisfiable.*

Lemma 3.5.4 *If none of the level n formulas $A^{(v)}_n$ are satisfiable, then no $A^{(v)}_i$ $i > n$ is satisfiable.*

From Lemmas 3.5.1 and 3.5.2.

Löwenheim then states without proof:

Lemma 3.5.5 *If A_1 is unsatisfiable, then ΠF is unsatisfiable.*

Löwenheim apparently takes this to follow immediately from the construction of A_1 and the definition of satisfaction for universally quantified formulas. When the quantifier Π is understood as an infinite product over all instances of A as $\mathfrak{r}$ ranges over a domain, it is clear that ΠF must include one of the $A^{(v)}_1$s as a factor. Any interpretation that satisfies ΠF would also satisfy this $A^{(v)}_1$. Thus, showing that all of the $A^{(v)}_1$ are unsatisfiable suffices to show that $\Pi F = 0$ on any interpretation.[12]

[12] One of the exegetical issues addressed in (Badesa, 2004) is whether Löwenheim in fact regarded quantified formulas as infinitary sums and products (the standard reading) or whether he used infinite expansions as a heuristic tool to gesture at a meaning that he lacked the technical framework to state more precisely.

Lemma 3.5.6 *If A_n is unsatisfiable for some n, then ΠF is unsatisfiable.*

This generalization of lemma 3.5.5 is also stated (p. 240) without proof.

Lemma 3.5.7 *If every A_n is satisfiable, then ΠF is satisfiable in a denumerable domain.* (Proof discussed below).

3.4.5 Basic structure of the proof

By the results of the first stage, assume that an arbitrary formula Q has been replaced by a counterpart that preserves satisfiability and has the form ΠF. Assume the antecedent of Theorem 3.3 holds, i.e. that ΠF is satisfiable but not in any finite domain. Construct the sequence of A_ns. Then by the contrapositive of Lemma 3.5.6, A_n is satisfiable for every n. By Lemma 3.5.7, ΠF is satisfiable in a countable domain.

3.5 Divergent Interpretations

Lemma 3.5.6 marks the last point of commonality amongst the different interpretations of Löwenheim's theorem. The contention centers around lemma 3.5.7: *If every A_n is satisfiable, then ΠF is satisfiable in a denumerable domain.*

Löwenheim never explicitly states this lemma, and the passage from which it is arguably derived is open to several interpretations. Each yields a different answer to the main points of controversy: which version of the theorem (weak or subdomain) did Löwenheim intend to prove, and does his proof have gaps?

On the first two interpretations described below, Löwenheim aims to prove the weak version but, at least by modern standards, the proof he actually gives has gaps. I look at ways of filling these gaps by appeal to later results. In the final subsection, I argue that on either of these interpretations, Löwenheim's proof contains all the components of a completeness theorem and is thus equally subject to the question Gödel raises about Skolem: why did Löwenheim fail to recognize this result implicit in his work? I also look at Badesa's revisionary account according to which Löwenheim aimed to prove the subdomain version of the theorem. On this interpretation, Löwenheim's proof does not contain major gaps. However, this version of the theorem does not entail completeness for reasons explained below.

3.5.1 The passage in Löwenheim

Having described the construction of the A_n and $A_n^{(v)}s$, Löwenheim concludes his proof with the following:

> If for some n (hence also for all succeeding ones) all $A_n^{(v)}$ vanish, the equation is identically satisfied. If they do not all vanish, then the equation is no longer satisfied in the denumerable domain of the first degree just constructed. For then among A'_1, A''_1, A'''_1... there is at least one $Q1$ that occurs in infinitely many of the nonvanishing $A_n^{(v)}$ as a factor (since, after all, each of the infinitely many nonvanishing $A_n^{(v)}$ contains one of the finitely many $A_1^{(v)}$ as a factor). Furthermore, among A'_2, A''_2, A'''_2... there is at least one $Q2$ that contains $Q1$ as a factor and occurs in infinitely many of the nonvanishing $A_n^{(v)}$ as a factor (since each of the infinitely many nonvanishing $A_n^{(v)}$ that contain $Q1$ as a factor contains one of the finitely many $A_2^{(v)}$ as a factor). Likewise, among A'_3, A''_3, A'''_3...there is at least one $Q3$ that contains $Q2$ as a factor and occurs in infinitely many of the nonvanishing $A_n^{(v)}$ as a factor. And so forth.

Every Q_v is $= 1$; therefore we also have

$$1 = Q_1 Q_2 Q_3, ... \text{ ad infinitum}$$

But now, for those values of the summation subscripts whose substitution yielded $Q_1, Q_2, Q_3, ...$, ΠF is $= Q_1 Q_2 Q_3, ...$, hence $= 1$. Therefore ΠF does not vanish identically. (1915/1967, p. 240)

3.5.2 A tree of satisfiable formulas

On the interpretation described in this section, the goal of the passage is to construct a product (conjunction) of formulas that, in the limit, takes the same truth value as ΠF under the same assignment to the variables. To this end, Löwenheim constructs an infinite tree whose nodes are the propositional formulas $A_n^{(v)}$ induced by different systems of equalities on the domain of A_n. Each such formula determines a different partial assignment of values to the summation indices of A, i.e., to the free variables and fleeing indices that result from dropping the Σ and $\underline{\Sigma}$ quantifiers.

Löwenheim then argues for what we now recognize as an instance of König's infinity lemma, establishing the existence of an infinite branch through any finitely-branching infinite tree.[13] His argument runs as follows:

By the hypothesis of the theorem, every level n has at least one satisfiable $A_n^{(v)}$. It follows that

1. At least one of the first level formulas $A_1^{(v)}$ is satisfiable.

2. There are infinitely many satisfiable $A_n^{(v)}$.

[13]This lemma is attributed to Hungarian mathematician Dénes König (König, 1926).

By construction, there are finitely many $A_n^{(v)}$ at each level. In particular, at the first level there are finitely many $A_1^{(v)}$. A fortiori, there are finitely many *satisfiable* $A_1^{(v)}$.

By Lemmas 3.5.1 and 3.5.2, every satisfiable $A_n^{(v)}$ has as a factor one of the satisfiable $A_1^{(v)}$ guaranteed by (1). By (2), and since there are only finitely many satisfiable level 1 formulas, at least one of the satisfiable $A_1^{(v)}$ must be a factor in infinitely many satisfiable formulas of higher level. Let Q_1 be one of the satisfiable $A_1^{(v)}$ with infinitely many satisfiable extensions.

Now consider all and only those infinitely many satisfiable formulas of level $n > 1$ that have Q_1 as a factor.[14]

For $n > 2$, Lemma 3.5.1 and the hypothesis of satisfiability together guarantee that every such A_n must have have one of the level 2 formulas $A_2^{(v)}$ as a factor in addition to Q_1.

Since there are again only finitely many $A_2^{(v)}$, and infinitely many $A_n^{(v)}$ with Q_1 as a factor, it follows that one of the $A_2^{(v)}$ has Q_1 as a factor and is a factor of infinitely many $A_n^{(v)}$.[15] Let Q_2 be such a formula.

The argument can now be repeated to show that at the third level, there must be formulas that have Q_2, and ipso facto Q_1, as factors, and have infinitely many satisfiable extensions. Let Q_3 be such a formula.

Continuing in this way yields the product

$$Q_1 \cdot Q_2 \cdot Q_3 \cdot ... \text{ ad infinitum.}$$

The product $Q_1 \cdot Q_2 \cdot Q_3 \cdot ...$ represents an infinite branch through the tree, where each Q_i for $i > 1$ includes the preceding Q_{i-1} as a factor.

[14] Since the construction rules out the possibility of factors being repeated at higher levels, any formula containing Q_i must contain it as the *ith* factor.

[15] Of course, each $A_2^{(v)}$ occurring in as a factor of satisfiable formulas must itself be satisfiable (Lemma 3.5.2).

The next step is to show that the satisfiability of the product, and hence of ΠF, follows from the satisfiability of each of the factors Q_i. On the current reading, by saying that every Q_v is equal to 1, Löwenheim means that each is *satisfiable.*

However, the fact that every Q_i is satisfiable does not suffice to find a common solution for the infinite product $Q_1 \cdot Q_2 \cdot Q_3 \cdot ....$ Nothing guarantees the satisfiability of all the factors Q_i by the same solution. As a result, the step from "every Q_v is $= 1$" to "$Q_1 Q_2 Q_3, ... = 1$" is incomplete on the present interpretation.

Filling the gap

The tree of satisfiable formulas interpretation has the virtue of closely following the original text, but entails that the proof is incomplete since the proof of Lemma 3.5.6 does not go through. However, the gap can easily be filled by appeal to the following result of Quine's, or equivalently, to the compactness theorem. Both are anachronistic with respect to Löwenheim.[16]

Lemma 3.5.8 *Law of infinite conjunction (Quine, 1959): if C is a (countably) infinite class of truth functional schemata, then either some finite conjunction of members of C is truth functionally inconsistent, or there is an assignment to the sentence letters (atomic components) that makes all members of C true.*

Proof First replace the atomic propositional components of each $S \in C$ with propositional variables (distinct variables for distinct relative coefficients) and fix an enumeration $P_1, P_2, P_3 ...$ of these variables.

Following Quine, say that an assignment of truth values to $P_1, P_2, ..., P_i$ *condemns* a given

[16]Compactness was first proven as a consequence of (Gödel, 1929).

conjunction of members of C if it makes that conjunction come out false for every possible truth assignment to the P_k for $k > i$.

Define a sequence of truth values $t_1, t_2, ...$, with $t_i \in \{0, 1\}$, according to the rule:

[(i)]Let $t_i = 1$ if the assignment $t_1, t_2, ..., t_{i-1}, 1$ to $P_1, P_2, ..., P_i$ does not condemn any conjunction of members of C. Otherwise, let $T_i = 0$.

The proof is by contraposition, showing that if the assignment of $t_1, t_2, ...$ to $P_1, P_2, ...$ does *not* make every member of C true, then some finite conjunction of members of C is inconsistent.

Suppose the assignment of $t_1, t_2, ...$ to $P_1, P_2, ...$ falsifies some member S of C. Choose j large enough so that for all $i > j$, P_i does not occur in S. Thus, S must already be falsified by the assignment $t_1, t_2, ..., t_j$ to $P_1, P_2, ..., P_j$, regardless of the values assigned to $P_{j+1}, P_{j+2}, ...$.

Then there is $h \leq j$ least such that

[(ii)]the assignment of $t_1, ..., t_h$ to $P_1, ..., P_h$ condemns some conjunction K of members of C (minimally, it condemns S).

By (i),

[(iii)]$t_h = 0$.

By (ii) and (iii),

[(iv)]the assignment of $t_1, ..., t_{h-1}, 0$ to $P_1, ..., P_h$ condemns K.

By (i) and (iii),

[(v)]the assignment of $t_1, ..., t_{h-1}, 1$ to $P_1, ..., P_h$ condemns some conjunction K' of members of C.

Suppose $h > 1$. Then by (iv) and (v), the assignment of $t_1, ..., t_{h-1}$ to $P_1, ..., P_{h-1}$ condemns the conjunction K and K' of members of C, contradicting the least-ness of h.

Therefore,

[(vi)]$h = 1$.

So, by (iv), the assignment of 0 to $P_1 (= P_h)$ condemns K. By (v), the assignment of 1 to P_1 condemns K'. The conjunction of K and K' is therefore inconsistent. □ (Lemma 3.5.6.)

To apply Quine's law, we need to ensure that no finite conjunction of Q_i is truth functionally inconsistent. Given an arbitrary finite conjunction of Q_i, let Q_n be the formula (conjunct) with the largest index. If $m < n$, $Q_n = Q_m \cdot A$ for some A, and since Q_n is satisfiable by definition, every such conjunct Q_m is also satisfiable. Hence, no finite conjunction of Q_is can be truth-functionally inconsistent.

3.5.3 A tree of solutions

An alternative interpretation takes Löwenheim's aim to be the construction of a tree whose nodes are partial solutions rather than formulas.[17] The following sketch of an argument adapts Löwenheim's passage above to reflect this interpretation.

Löwenheim has already established that at least one of the level 1 formulas $A_1^{(v)}$ must occur as a factor in infinitely many *satisfiable* formulas of higher level. Choosing one such formula, he denotes it by Q_1. Q_1 has *at least* one solution but it may have more than one solution.[18] Still,

[17] Wang (1970) and Brady (2000) endorse this reading. Wang does not elaborate. Brady sets out to give the argument but equivocates between formulas and solutions in a way that renders the result inconclusive.

[18] We may assume that the assignment of elements to the free variables in the manner described above has already been carried out, so that Q_1 is propositional. Technically, the atomic propositions of Q_1 are replaced by propositional variables which serve as arguments to the function assigning truth values (i.e. the solution).

1. Q_1 has at most finitely many solutions since it has at most finitely many atomic propositions.

2. Q_1 occurs as a factor in infinitely many satisfiable $A_n^{(v)}$s.

3. For any A_n^v that contains Q_1 as a factor, if S_n^v is a solution to A_n^v, then the restriction of S_n^v to the atomic propositions of A_1 is a solution to Q_1.

From (2) and (3), infinitely many solutions to $A_n^{(v)}$s contain as a part solutions to Q_1. From this and (1), it follows that at least one solution to Q_1 occurs as a part of infinitely many solutions to formulas of higher level. Let S_1 denote one such solution.

Now, by definition, S_1 occurs as a part of infinitely many solutions to formulas of higher level, all of which contain Q_1 as a factor. These infinitely many solutions extending S_1 must contain as a part a solution to one of the level 2 formulas $A_2^{(v)}$. There are finitely many $A_2^{(v)}$s and each has only finitely many solutions. Therefore, let S_2 denote a solution that extends S_1 and occurs in infinitely many solutions of higher level.

Continuing in this way, construct the sequence of partial solutions $S_1, S_2, S_3...$ where $S_i \subset S_{i+1}$. The axiom of dependent choice (unmentioned by Löwenheim) guarantees that the construction may be continued without end, determining an infinite branch through the tree of solutions.

The construction determines a parallel sequence of formulas $R_1, R_2, R_3, ...$, with each R_i one of the $A_i^{(v)}$, and such that S_i is a solution to R_i. Clearly, $R_i \subset R_{i+1}$ for every i.[19]

Here too, König's infinity lemma or equivalent is needed to establish the existence of an infinite path through the tree of solutions. The union of the S_i gives a solution to ΠF interpreted in the countable domain consisting of the elements named by the integer constants

[19] The solution is a function from atomic propositions to truth values; the formula can be determined from its inverse.

of the R_i.

3.5.4 The subdomain version

The final interpretation to consider is a revisionary one due to Badesa. Badesa holds that Löwenheim intended to prove the stronger "subdomain" version of the LST:

Theorem 3.7.0 *If a first-order formula* ΠF *in Löwenheim normal form is satisfiable in some infinite domain* D *but not in any finite domain, then it is satisfiable in a countable subdomain* D_0 *of* D.

Skolem gives the standard proof of this theorem in his (1920). His proof differs from Löwenheim's and is expressly designed to avoid what Skolem views as the former's "detour through the transfinite" (more on this below).

Badesa's reconstruction Badesa gives a proof of Theorem 3.7.0 intended to faithfully reconstruct Löwenheim's original.

Let S be the solution to ΠF in a domain D given by the hypothesis of the theorem. S assigns truth values to the atomic propositions[20] that result from some assignment of values to the individual variables of ΠF. To make sense of Löwenheim's tree construction, the proof does not appeal to any particular assignment to these variables. Rather, the point of constructing the tree and fixing an infinite branch is precisely to assign elements to the individual variables in such a way that the formula is satisfied under the solution S of the hypothesis.

[20] Technically, to propositional variables that replace atomic propositions.

Let C_n be the set of numerals occurring in A_n. Define

$$V_n = \{f : C_n \to D \mid S \text{ satisfies } A_n \text{ under the assignment } f\}$$

V_n is the set of assignments of elements of D to the numerals of each A_n. The domain of each f is finite, but the number of such functions at each level may by infinite, even uncountable, if D is. The tree Löwenheim constructs can now be conceived as the set $\bigcup_{n\in\omega} V_n$ ordered by the relation of strict inclusion. The goal of the proof —constructing an infinite path through the tree —is now the goal of showing the existence of a sequence $f_1, f_2, \ldots$ of partial[21] assignments from V_n, each of which is an extension of the previous one. In the limit, the union of the f_i fixes the values of the numerals in *every* A_n, and hence, in ΠF. By the definition of V_n, the assignments making up this sequence are restricted by the condition that ΠF be satisfied by the solution S fixed by the hypothesis of the theorem.

At this point, the reconstruction faces the following problem. Because the number of partial assignments at each level may be infinite, the König-style argument Löwenheim gives for the existence of the infinite branch $Q_1, Q_2, \ldots$ no longer suffices, since the choice of Q_n makes essential appeal to the finitude of the $A_n^{(v)}$s for each n. That argument can be modified. By appeal to Choice, select an assignment f_1 of the first level that has extensions at every higher level. Now restrict attention to all those (perhaps infinitely many) assignments of the second level that both extend f_1 and have extensions of every higher level. Let f_2 be one such assignment, and so forth.

For this argument to go through we need the following lemma:

Lemma 3.7.1 *For every n, the partial assignment f_n of level n has an extension f_{n+1} of level n+1.*

[21] Relative to ΠF

This Lemma does not hold in the case where ΠF has fleeing indices that depend on only some of the universally quantified variables (Badesa, p. 196).[22] To circumvent this, define the set V_n to include the condition that the partial assignments have extensions at the next level. Lemma 3.7.1 then follows by definition.

Putting everything together, the new definition of V_n ensures that the conditions are met for the argument two paragraphs above to go through. The argument establishes the existence of the infinite sequence $f_1, f_2, ...$ of assignments. The union of the f_i is a function $\mathfrak{f}$ that assigns values to the indices of ΠF and whose range is a countable subdomain D_0 of the original domain D.[23] Finally, where S is the solution which satisfies ΠF in the hypothesis of the theorem, the restriction S_0 of S to the countable subdomain D_0 is a solution that satisfies ΠF under the assignment $\mathfrak{f}$.

This interpretation, unlike the others, does not support the allegation that completeness was implicit as a corollary of Löwenheim's proof. In particular, the subdomain version does not require Löwenheim to prove Lemma 3.5.7: If every A_n is satisfiable, then ΠF is satisfiable in a denumerable domain.

Instead, Löwenheim proves that when every A_n is *true* (which of course implies satisfiability), then A is true (under the same interpretation). The contrapositive of this result does not underwrite the implicit refutation procedure discussed above. As a result, on this version of the theorem the implicit procedure lacks the guarantee that when the formula is $unsatisfiable$ (rather than untrue), then we can find an n such that A_n is demonstrably unsatisfiable.

[22] For example, suppose $\Pi F = \Pi i, j A(i, j, k_i)$. The satisfiability of A_n ensures that there is some assignment (of an element of the domain D over which the formula is satisfied by hypothesis) to k_a such that for every a, b of the domain constructed *up to level* n, $A(a, b, k_a)$ is true. But when new elements are added to the domain in the construction of A_i for $i > n$, nothing guarantees that the $A(a, j, k_a)$ will remain true for any assignment to the variable j. See Badesa for an example.

[23] Badesa introduces a further revision to Löwenheim by considering the assignments that assign values to only the fleeing indices rather than to all indices.

3.5.5 Completeness in Löwenheim

We are now in a position to answer the question: Is a completeness proof implicit in (Löwenheim, 1915)? The answer to this question depends on (a) which interpretation we adopt, and (b) whether we discern in Löwenheim an implicit "refutation" procedure for demonstrating that a formula is unsatisfiable.

The first is a straightforward technical consideration. The only interpretation that expressly rules out a completeness result is the one on which Löwenheim aimed to prove the subdomain version. In contrast, the interpretations on which Löwenheim aimed to prove the weak version leave open the possibility of establishing completeness as a corollary, modulo the alleged gaps in the proof.

Consideration (b) is more controversial. Completeness as we know it today is a property of formal systems or calculi and we have seen that Löwenheim's algebraic framework lacks many of the features we require of such systems. Some would argue that this suffices to show that completeness was not implicit in Löwenheim since he does not give us a formal calculus for which the question can even be raised.

However, the same point can be made about Skolem, whose anti-formalism is widely recognized. This represents a challenge to Gödel, threatening to deflate his puzzle. Gödel addresses the challenge explicitly:

> It may be true that Skolem had little interest in the formalization of logic, but this does not in the least explain why he did not give a correct proof of that completeness theorem which he explicitly stated (op. cit., p. 134) [...] On the basis of his lemma of 1922 this would have been quite easy [...] (Letter to Wang, In [Wang, 1970] p. 10)
>
> What [Skolem] could justly claim, but apparently does not claim, is that, in his

> 1922 paper, he implicitly proved: "Either A is provable or $\neg A$ is satisfiable" ("provable" taken in an informal sense). (Coll. Wrks. Vol 1, p. 52)

Faced with the objection that his "puzzle" has an obvious answer - what we have called the formal systems explanation - Gödel redefines the question by broadening the notion of proof to include *informal* methods that, when spelled out, constitute effective procedures for demonstrating the satisfiability/unsatisfiability of formulas.

Skolem's method is reconstructed in the next chapter. In the current section I look at the methods that can be justifiably attributed to Löwenheim considering how much remains implicit in the actual text.

Löwenheim's refutation procedure

Common to all three interpretations of Löwenheim's theorem is his method of expanding a quantified formula into quantifier-free propositional instances. This method was later appropriated by Skolem and, as a result, it bears important similarities to Gödel's method for proving completeness.

As will be seen with Skolem in 1928, Löwenheim considers the possibility that at some level in the construction of A_n, no satisfiable formulas exist. This consideration is crucial if we wish to attribute to him an awareness of a refutation procedure based on the construction.[24]

Löwenheim never describes a "refutation" procedure as such, nor does he provide what we would consider to be the requisite detail. Nonetheless, an informal procedure can be inferred from the construction of the A_n and the idea of checking whether a formula vanishes. It is reasonable to suppose that Löwenheim found an explicit presentation to be unnecessary, given the familiarity of the methods involved. This is suggested by Badesa:

[24] The consideration is unnecessary from a standpoint of simply proving the LST - witness Skolem's 1922 proof - suggesting that Löwenheim did take an interest in the refutation procedure as such.

> In order to check whether any A_n vanishes (identically) or not, Lowenheim introduces in A_n all the possible equalities between the terms of A_n. If all the resulting formulas vanish, then A_n vanishes. Lowenheim does not explain how to decide whether or not these formulas vanish, because he thinks it is a trivial matter. The formulas obtained from any A_n by means of this procedure are essentially propositional formulas (since the identity symbol and the quantifiers do not occur in them) and the method to decide whether or not a propositional formula vanishes was well known at that time. Probably, Lowenheim would first apply laws in order to simplify the formula and then, if necessary, would argue in terms of truth values. We can say that in a sense Löwenheim possesses an informal procedure for showing that any quantifier-free formula of a first-order language with identity vanishes: first, he takes into account all the possible equalities between the terms of the formula in the way explained in the proof, and then, by applying propositional methods, he checks whether all of the resulting formulas vanish. This procedure, together with the one used in order to construct the sequence $A_1, A_2, ...$, can be seen as an informal way of showing that a formula in normal form vanishes. (2004, p. 203)

If we have such a procedure for refuting any given A_n, then the leveled construction suggests the means of turning this into a refutation procedure for the original quantified formula. Systematically searching through the formulas at each level, we either reach a level n for which we can refute A_n in the manner just described, or, we proceed to level $n+1$ and repeat the process for the propositional formulas at that level.

The only thing missing is to show the connection between the refutability of A_n for some n, and the refutability of A. Gödel does this by showing that for every n, $A \to P_n(A_n)$ is provable in his formal system. In conjunction with the methods for refuting $P_n(A_n)$, this gives an explicit demonstration of the procedure by which a quantified formula can be formally refuted.

A similar demonstration is missing from Löwenheim, who simply asserts that if A_n vanishes for some level n, then the original formula is unsatisfiable. The self-evidence of this step apparently follows for him from the way the A_n are constructed (see Badesa, p. 168) and (possibly) his understanding of quantification in terms of infinite conjunction/disjunction.

Granting all these implicit steps, Löwenheim can arguably be credited with the same informal completeness result Gödel later attributes to Skolem. If some A_n is not satisfiable (i.e. vanishes), then A can be informally refuted by searching through the levels until we reach this n, showing that A_n vanishes by propositional methods, and then inferring that A vanishes by the "self-evident" step mentioned above. Conversely[25], Lemma 3.5.7 guarantees a solution to the formula in the domain of natural numbers when A_n is satisfiable for every n.

This attribution depends, of course, on our acceptance of one of the interpretations according to which Löwenheim proves the weak version of the theorem. If we accept Badesa's subdomain interpretation, then the question of why Löwenheim did not acknowledge completeness is moot. However, none of the interpretations discussed stands out as obviously correct against the others. Each has its own merits and each is faithful to the text in different respects. As a result, there is no reason to dismiss the weak interpretations, and by extension, the idea that completeness was also implicit in Löwenheim. As long as this possibility is on the table, we can raise Gödel's puzzle for Löwenheim and assess whether Gödel's own solution has bearing in this case. If Löwenheim intended to prove the weak version of his theorem, his willingness to use non-finitary reasoning weighs against Gödel's allegation that the avoidance of such reasoning is responsible for the delayed discovery of the completeness theorem.

[25] Note the use of the law of excluded middle to establish this basic alternative.

3.6 Non-finitary reasoning in Gödel and Löwenheim

In examining Gödel's criticism of the finitistic prejudices he attributes to Skolem and others, one must take into account that during the pre-1930 period in question, there was no consensus and little clarity on what counted as finitary methods. However, the history of this debate can be bracketed for the current investigation. It suffices to understand, first, what Gödel means when he says that a proof of completeness is essentially infinitary, and second, why he thought that Skolem (specifically) was against infinitary reasoning in this sense.

Infinitary reasoning in Gödel

The infinitary character of Gödel's completeness proof can be traced to two essential applications of the law of excluded middle to infinite sets. These count as non-finitary regardless of one's stance on the exact boundaries of finitism in, say, the primitive recursive functions.

The first use of the law of excluded middle occurs in setting up the basic alternative that either some A_n is unsatisfiable or every A_n has a satisfying truth assignment (see previous chapter). The second use occurs in Gödel's Theorem 2.3.5, which constructs a solution to the formula from the solutions to each B_n.

Theorem 2.3.5 is an instance of König's infinity lemma. For each level in Gödel's sequence of propositional formulas B_n, there exist finitely-many possible ways of assigning truth values to the atomic propositions to make each B_n true. Each such assignment represents a vertice of a finitely-branching tree. Vertices are connected by the relation of extension between assignments and the aim is to construct an infinite path through the tree. At each step in this construction, the law of excluded middle comes into play when a proof by contradiction is used to guarantee the existence of a vertice (assignment) with infinitely many extensions.[26]

[26]Gödel, however, does not cite König at this point, saying only that "we show, in a familiar manner, the existence of an infinite sequence of satisfying [solutions]" (1929/1986, p. 87). "Familiar manner" may

As will be seen in the next chapter, this is the step Gödel refers to as "the easy inference from Skolem [1922]".

Infinitary reasoning in Löwenheim

On any of the interpretations discussed above, both steps found in Gödel - the establishment of the basic alternative, and the construction of a single solution - are needed for Löwenheim to prove the LST.

The basic alternative is established by Löwenheim's application of the law of excluded middle to the infinite set of A_n:

> If for some n (hence also for all succeeding ones) all $A_n^{(v)}$ vanish, the equation is identically satisfied. If they do not all vanish, then the equation is no longer satisfied in the denumerable domain of the first degree just constructed.[27]

Löwenheim also needs to establish an instance of the infinity lemma, though the application varies depending on the interpretation.

The aim of the tree of formulas interpretation is the determination of an infinite sequence $Q_1, Q_2, ...$, representing a path through the tree.[28] This step assumes Choice since Löwenheim

therefore refer to a weak version of Choice (specifically, countable choice for finite families). Gödel was unaware at the time of (Skolem, 1922) in which Skolem defines an explicit ordering on the truth assignments at each level, showing Choice to be avoidable.

[27]Verschwinden für ein $[n]$ [...] sämtliche $[A_n^{(v)}$, so ist die Gleichung identisch erfüllt. Wenn nicht, so ist die Gleichung schon in dem soeben konstruierten abzählbaren Denkbereich erster Ordnung nicht mehr erfüllt. (p. 456)

[28]It was noted that a gap arises on this interpretation because the existence of the path does not suffice to show that $Q_1, Q_2, ...$ is satisfiable by a single truth assignment. The gap can arguably be explained by taking into account Löwenheim's tendency to extend laws that are valid in finite cases into the infinite. As noted, the gap is bridgeable by the compactness theorem which states that for any infinite set of formulas Γ, if every finite subset of propositional formulas Γ is satisfiable, then all of Γ is simultaneously satisfiable by a single truth assignment. Löwenheim may have been inclined to accept this result without proof, given its validity when Γ is arbitrarily large finite. This would represent an additional instance of infinitary reasoning used by Löwenheim.

(unlike Skolem in 1922) does not define an explicit ordering on the formulas at each level.

The infinity lemma step also involves the law of excluded middle via a proof by contradiction to show that at each level, there must be at least one satisfiable formula which is a factor in infinitely-many satisfiable formulas of higher level. To prove this, suppose all of the $A_1^{(v)}$ had only finitely-many satisfiable extensions. Then since every vertex of the tree must be reachable by a path going through one of these finitely-many $A_1^{(v)}$, the entire tree of satisfiable formulas must be the union of finitely-many finite sets, contradicting the hypothesis that the tree is infinite (i.e. that there are satisfiable formulas $A_n^{(v)}$ for every n).

The tree of solutions version similarly establishes the existence of an infinite sequence of solutions, the union of which determines an interpretation of the formula over the natural numbers.

The subdomain version also requires an infinite path through the tree of formulas, but differs from the first interpretation in that no gap arises. This is because the formulas at each level are satisfied by the particular solution given by the hypothesis of the theorem. The point of the path through the tree is simply to fix an assignment of elements to the variables. This assignment determines a subdomain in which the formula is satisfied by the original solution restricted to the new domain.

That Löwenheim used non-finitary reasoning of the kind found in Gödel's proof is independent of the more contentious issue of whether Löwenheim used infinitary logic or worked with formulas of infinite length. In the first half of his proof, Löwenheim expands formulas into conjunctions and disjunctions of infinite length in order to prove that Löwenheim normal form preserves satisfiablility.[29] He uses the expansions to illustrate —arguably, define —the second-order equivalences that allow existential quantifiers to be moved in front of universal quantifiers. By failing to give an actual proof of these equivalences[30], Löwenheim has been accused of applying laws to infinite domains without justification, the validity of these laws having only been established for finite domains.[31]

Whether or not this accusation is fair, Löwenheim gives no indication of any conceptual difficulties involved in working with infinite expansions. Quite generally, Löwenheim exhibits none of the epistemological concerns with infinitary reasoning that gained prominence with Hilbert and his followers. Nor is he unique among the algebraists for his use of techniques that would have been questionable from a finitist standpoint.

In sum, on the most charitable interpretations, Löwenheim's proof succeeds by implicit but intentional appeal to methods that are non-finitary in Gödel's sense. Even on interpretations according to which his proof contains gaps, his presentation shows no reluctance to restrict the proof methods available to him. The next section will look at how this, and similar evidence from Skolem, challenges Gödel's claim that reluctance to use non-finitary reasoning was responsible for the delayed recognition of completeness.

3.7 Assessing Gödel's claim

The results of the previous sections present a prima facie challenge to Gödel's allegation that reluctance to use non-finitary reasoning blocked recognition of completeness. I argued that on the most wide-spread interpretation, completeness is a straightforward corollary of Löwenheim's proof in the same way that Gödel argues it is for Skolem's. Löwenheim, like

[29]Van Heijennort and others take this as evidence that Löwenheim made essential use of infinitary logic to carry out his proof. Conversely, Badesa argues that the expansions are merely heuristic and thus eliminable.

[30]The proof makes non-constructive use of the Axiom of Choice.

[31]For example, by assuming distributive laws apply to formulas of infinite length.

Skolem, shows no recognition of this result. Conversely, Löwenheim exhibits no reluctance to use non-finitary reasoning in Gödel's sense.

This section considers Gödel's response to the similar challenge based on uses of non-finitary reasoning in (Skolem, 1922). I argue that Gödel's response is not convincing in application to Skolem, or when extended to Löwenheim. It depends on attributing to these logicians distinctions and motives for which there is no evidence. Thus, an alternative explanation is called for to explain why completeness went unrecognized despite the key non-finitary steps already being taken by both Löwenheim and Skolem in proving the LST.

3.7.1 Gödel's defense

Faced with the objection that Skolem did in fact use non-finitary reasoning in his 1922 proof of the LST, Gödel responds:

> That he used non-finitary reasoning for Löwenheim's Theorem proves nothing, because pure model theory, where the concept of proof does not come in, lies on the borderline between mathematics and metamathematics (Letter to Wang, in Wang, 1974, p. 10)

Wang concurs, adding that "Skolem probably thought he could not use the same sort of argument when considering the question of completeness, which is squarely in the domain of metamathematics" (Wang, Letter to Gödel, 19 December, 1967).

This claim can be broken down into two parts. First, the distinction between mathematical and metamathematical theorems. Second, the allegation that non-finitary methods are permissible in the former but not in the latter. I argue that both are problematic in application to Skolem and Löwenheim.

A "metamathematical" theorem

Taking Gödel's remarks at face value suggests that the completeness theorem is metamathematical because it involves the concept of proof. This is no doubt intended loosely, as it somewhat misrepresents a distinction of which Gödel himself was an originator.

More precisely, metamathematics uses mathematical methods to study mathematics itself. Particular mathematical theories and proofs are conceived as a formal objects amenable to mathematical investigation in a higher order meta-theory. Gödel's proof of completeness counts as metamathematical in this sense. It is a proof, conducted in a metatheory, to show something about proofs (refutations) in a separate object theory. If we look at his proof of Lemma 2.3.2, Gödel's appeal to syntactic features of a formal theory he specifies at the outset is an example of treating object language proofs as mathematical objects amenable to investigation in the metatheory.

Conversely, Gödel writes that pure model theory "lies on the borderline between mathematics and metamathematics". This again seems based on the idea that metatheory necessarily involves the concept of proof. Against this, model theory today *is* typically regarded as metatheoretical. It studies the relations between formal theories and their interpretations and the Löwenheim-Skolem theorem arguably fits into this category.

Definitions aside, the basis for Gödel's distinction is less important than its potential application to Skolem or Löwenheim. However, it has been widely noted that this distinction is lacking for both logicians. In particular, neither author specifies a formal theory as the intended object of study.[32] Of course, Gödel has shifted the conversation from formal to informal completeness, so the absence of an explicit formal system, at least at the object level, should not impact the coherence of his claim. Skolem's informal proofs *can* easily be formalized and made amenable to metamathematical study. But to suggest that this

[32]Though both presuppose the (semi-formal) framework of the algebraic logic tradition.

is something Skolem would or should have done is implausible since his anti-formalism is widely recognized. Meanwhile, Löwenheim evinces even less awareness of the possibility of formalizing theories and making them objects of study in a higher-order theory. Without this distinction, it is hard to see how either Skolem or Löwenheim could recognize a corresponding distinction in the permissible proof methods for each type of theory.

3.7.2 Finitism and restrictions on reasoning

According to the second part of Gödel's claim above, Skolem believed that because the completeness theorem is metamathematical (unlike the LST), the methods used to prove it must be finitary. This also attributes to Skolem a view for which no evidence exists. Even if Skolem could have recognized completeness to be metamathematical, this fact alone implies no *general* restriction on proof methods. The restriction only follows if one subscribes to the specific brand of finitism endorsed by the Hilbert school.

Skolem was familiar with Hilbert's work, at least by 1928.[33] Yet there is no indication that he shared Hilbert's views, particularly those according to which non-finitary reasoning stands in need of justification via consistency proofs in a finitary metatheory. Nor would Skolem have endorsed Hilbert's formalist methodology.

Though inaccurate, Gödel's attribution of this view to Skolem is understandable. Beginning in 1923, Skolem advocated extensively for what is now known as primitive recursive arithmetic. This attracted the interest of Hilbert because *he* considered the methods used in this arithmetic to be finitary and thus saw it as a candidate for the metatheoretical consistency proofs he sought.

Skolem, on the other hand, was never an advocate of Hilbert's program, even before it became infeasible in light of the incompleteness theorems. With the benefit of hindsight, Skolem in

[33]Skolem cites Hilbert and Ackermann's (1928) frequently from this point on.

(1944) gives an accurate but highly critical description of Hilbert's program, including its restriction to finitary methods in the metatheory. He concludes that he "cannot understand the enthusiasm with which these ideas have been met among so many mathematicians" (Skolem 1944/1970, p 525). Skolem presents his recursive arithmetic as an alternative to Hilbert's foundational attempts. He reiterates his belief that the problems facing mathematics stem from the use of unrestricted quantification which primitive recursive arithmetic avoids by using only free variables.

Skolem's rejection of unrestricted quantifiers is connected with the brand of finitism that he *does* endorse - a sceptism about axiomatic theories and a rejection of higher infinities, both prompted by his relativity thesis (see next chapter). This sort of finitism does not lend support to Gödel's claim that Skolem characterized permissible methods of proof according to whether a theorem was metamathematical or not.

If Gödel's claim has little substantiation in application to Skolem, the evidence in its favor is even less for Löwenheim. Löwenheim never mentions Hilbert and his published work shows no evidence that he was familiar with the formalist tradition. Moreover, as already noted, Löwenheim exhibits a willingness to use non-finitary methods (and possibly even infinitary logic) at multiple removes in his proof. Furthermore, there is widespread agreement that Löwenheim did not possess an object/metatheory distinction. In this he was simply following the standards of the algebraic tradition.

3.8 Conclusion

This chapter gave the framework to understand Löwenheim's (1915), and reconstructed his main theorem on three different interpretations set forth in the literature. Common to all three interpretations is Löwenheim's expansion method. This plays a crucial role in the history of the completeness theorem. Both Skolem and Gödel used versions of the same method to give, respectively, another proof of Löwenheim's theorem, and the first proof of completeness.

This shared methodology explains why it makes sense to ask whether Löwenheim, like Skolem, possessed all the technical components of a completeness proof in 1915. As seen in section 2, the answer varies depending on which interpretation of Löwenheim's theorem one adopts.

One either version of the prevailing "weak" interpretation, Löwenheim's proof arguably exhibits the key steps taken by Gödel in 1929, modulo Löwenheim's lack of a formal system. Löwenheim's expansion method yields an informal refutation procedure and the completeness of this procedure follows, albeit implicitly, as a corollary of his main theorem. This conclusion has interesting implications for Gödel's allegation that reluctance to use non-finitary reasoning blocked the recognition of completeness.

The second half of the chapter looked at the different ways Löwenheim uses such reasoning in his paper and compared these with the non-finitary steps in Gödel's completeness proof. Gödel's allegation is challenged by Löwenheim's willingness to make the same non-finitary moves as Gödel. I examined Gödel's counterargument against a similar objection based on uses of non-finitary reasoning in Skolem. Gödel's attribution of a Hilbertian version of finitism to either Skolem or Löwenheim was argued to be unsubstantiated by the textual evidence. As a result, his attempt to dismiss non-finitary reasoning in the context of Löwenheim's theorem falls short, as does his own explanation for why completeness went unrecognized. This raises the stakes for Gödel's question: if appeal to a finitistic prejudice is unconvincing, what alternative explanation can be given for why Löwenheim and Skolem came so close to preempting Gödel's proof yet failed to do so?

In Löwenheim's case, Badesa's revisionary interpretation offers a plausible account that deflates Gödel's puzzle. Badesa identifies Löwenheim's aim to be a proof of the subdomain version of the theorem. Since a correct proof of this version does not entail completeness, Badesa exempts Löwenheim from any obligation to acknowledge it.

When it comes to Skolem however, Gödel's puzzle cannot be dismissed as easily. Skolem is clear about which versions of the theorem he intends to prove. After explicitly stating and proving the subdomain version in 1920, Skolem's (1922) intentionally proves the weak version of Löwenheim's theorem in order to avoid the use of Choice.

In the next chapter, I give a reconstruction of Skolem's proofs in 1922 and 1928. This sets the stage for Chapter 5 where I give a revisionary answer to The Puzzle, arguing that Skolem had little reason to acknowledge the completeness of first order logic when he thought that a full decidability result was on the table. This account can be extended to Löwenheim on the interpretations according to which he aimed to prove the weak version of his theorem.

Chapter 4

Skolem

4.1 Introduction

The present chapter provides the background understanding of how Skolem's work bears on The Puzzle. I give a complete reconstruction of Skolem's 1922 proof of the Löwenheim-Skolem theorem. This is followed by an overview of (Skolem, 1928) and an examination of the main theorem of that paper from the perspective of the most plausible extant interpretations. These reconstructions set the stage for the examination, in Chapter 5, of Gödel's claim that finitism prevented Skolem from recognizing the completeness theorem implicit in his work.

4.2 Background

The important place occupied by Löwenheim's theorem in modern logic and model theory owes much to Skolem. In 1920, Skolem set out to give a simpler proof of the theorem that would avoid some of the notational complexities of Löwenheim's original. The comparative accessibility of Skolem's proof has contributed to the perception that he corrected the flaws in Löwenheim and deserved equal credit for the theorem.[1]

The proof Skolem gives is a proof of the subdomain version of Löwenheim's theorem: if a formula is satisfiable in an infinite domain D, then it is satisfiable in a countable subdomain of D, under the same interpretation of predicates. The proof uses the axiom of choice.

As noted in the previous chapter, this version of the theorem does not entail completeness. This is even more evident in Skolem's case: by using the axiom of choice, there is no need for the leveled construction of expansions that accounts for the similarity between Löwenheim's proof and Gödel's proof of completeness.

In 1922 however, Skolem offers a second proof of Löwenheim's theorem, this time avoiding the axiom of choice.

Skolem's overarching aim in his (1922) is critical: he argues for eight theses including the claim that axiomatized set theory cannot provide an adequate foundation for mathematics. The argument turns on what Skolem sees as the most important result of the paper —that all set-theoretic notions are unavoidably relative. He takes this claim to follow as a corollary of Löwenheim's theorem.

When generalized to countably infinite sets of formulas, Löwenheim's theorem can be applied to axiomatizations of set theory to create a "paradoxical state-of-affairs". On the one hand, the axioms guarantee the existence of transfinite cardinalities; on the other hand, the axioms have a model in the domain of integers. This shows, according to Skolem, that there is no absolute notion of transfinite cardinality because the meaning (i.e. "size") of any given cardinality varies from model to model.

This can be generalized to other set theoretic notions besides cardinality, yielding Skolem's "relativity thesis": when "sets are nothing but objects that are connected with one another

[1]In the absence of a consensus that Löwenheim's proof is flawed, or even, exactly which theorem Löwenheim was trying to prove, this assessment has more to do with the ease of Skolem's notation.

through certain relations expressed by the axioms" (1922, p. 295), then the properties sets possess are relative to the domains in which they are found. It follows that the intuitive and absolute meaning of these properties cannot be captured by any axiomatization to which Löwenheim's theorem applies.

This critical aim explains Skolem's desire for a Choiceless proof of Löwenheim's theorem:

> [H]ere, where we are concerned with an investigation in the foundations of set theory, it will be desirable to avoid the principle of choice as well. Therefore I now indicate very briefly how this can be done. It will also appear from the proof that general set-theoretical notions are unnecessary for our understanding of the content of these theorems. (Skolem, 1922, p. 293)

Skolem replaces the closure argument of 1920 with a level-by-level construction of solutions using the same method as (Löwenheim, 1915). This yields a proof of the weak version of the theorem: if a formula is satisfied in an infinite domain, then it is satisfiable in a countable domain. Unlike the 1920 version, this theorem can consistently be used to prove the relativity thesis, because it does not presuppose a standard model or appeal to the very axioms under attack by that thesis.

This is the proof that prompts Gödel's remark:

> [t]he completeness theorem, mathematically, is indeed an almost trivial consequence of Skolem [1922]. (Letter to Wang, 1967)

I unpack this claim by reconstructing the proof as faithfully to Skolem's 1922 original as possible.

4.3 The 1922 proof

4.3.1 Overview

The proof corresponds closely to what the last chapter called the "tree of solutions" interpretation. Starting with a formula in the normal form $A = \forall\exists F$, Skolem constructs a sequence of conjunctions of propositional instances of A. At each level n, the universal variables are replaced by integers from a finite domain expanding with n, and at each level, new integers are introduced to witness the existential quantifiers. As in Löwenheim's construction, the result is a sequence of propositional formulas.

Skolem begins with a formula A assumed to be satisfiable ("consistent"). This guarantees that at every level n there must be "solutions"—truth value assignments that determine the predicates and relations in such a way that the formula constructed at that level is satisfied. By a suitable ordering on the solutions of each level, Skolem bridges the gap attributed to Löwenheim[2], showing the existence of a single truth assignment that satisfies the original formula.

4.3.2 Skolem's construction

Let $A = \forall x_1, ..., x_m \exists y_1, ..., y_n F(x_1, ..., x_m, y_1, ..., y_k)$ be a first-order formula in normal form. Let A' be the result of dropping the quantifiers from A. By systematically instantiating the variables with integers, Skolem constructs an expanding sequence of propositional instances of A'. As the domain increases with each level, the conjunction of these instances represents a closer and closer approximation to A.

The construction can be described recursively:

[2] On the tree of formulas interpretation.

- To construct A_1, replace the universal variables $x_1, ..., x_m$ by a sequence of m-many 1s. Introduce new integers 2,..., $k+1$ for the existential variables $y_1, ..., y_k$. The resulting formula $A_1 = F(1, 1, ...1, 2, ..., k+1)$ is propositional.

- To construct A_{n+1}, take the (now free) universal variables $x_1, ..., x_m$ to range over all m-tuples of integers introduced up to level n. For each new m-tuple[3], form a propositional instance of A' by replacing $x_1, ..., x_m$ with the integers in the m-tuple, and introducing new integers to witness $y_1, ..., y_k$. A_{n+1} is equal to the conjunction of these instances, conjoined with A_n.[4]

Definition 4.1

Let A be a formula of first-order predicate logic. A *solution* S of level n is a function from the atomic propositions of A_n into $\{0, 1\}$ such that $A_n = 1$ according to the standard propositional rules.

Solutions are necessarily leveled; the domain increases as new instances of A' are added at each level of the construction. By the hypothesis of consistency, at each level n there must be at least one solution that satisfies A_n. Otherwise, A_n is truth-functionally inconsistent.

Definition 4.2

A solution S_n is an *extension* of a solution S_m, for $n > m$, if and only if the restriction of S_n to atomic formulas of A_m is equal to S_m.

[3]I.e.,that has not yet occurred as the x values for some A_i, $i \leq n$

[4]For example, to construct A_2, take the $x_1, ..., x_m$ to range over the integers $1, 2, ...k+1$. This means constructing an instance of A' for each of the $(k+1)^m$ permutations of this domain, with the exception of the m-tuple 1,1,...,1 already used in A_1. Taken in some order (say, lexicographic), each m-tuple is assigned unused integers for the y values. So, the first m-tuple is assigned $k+1, k+2, ..., 2k$, the second is assigned $2k+1, ..., 3k$ etc. Take the conjunction of the instances formed in this way, and conjoin with A_1. This is A_2.

Each solution of level n extends one of the solutions of level $n-1$ for all $n \geq 1$.

4.3.3 Ordering solutions

Skolem next defines an ordering '$\prec$' on the solutions at each level.

Let $R_0, R_1, R_2, \ldots$ be an enumeration of the atomic propositions of A' according to their first occurrence going left to right in the formula.

Definition 4.3

For solutions S and S' of an arbitrary level k, $S \prec S'$ if and only if the first atomic proposition R_j that is assigned different values by S and S' is assigned 0 in S and 1 in S'.

Lemma 4.1

For solutions S, S' and S'' of a single level,

$$S \prec S' \text{ and } S' \prec S'' \text{ implies } S \prec S''$$

Proof: Let R_j be the first (in the ordering) atomic proposition such that $S(R_j) = 0$ and $S'(R_j) = 1$ and let R_k be the first atomic proposition such that $S'(R_k) = 0$ and $S''(R_k) = 1$.

Suppose for reductio that $\neg(S \prec S'')$. Two cases are possible:

- **Case 1:** $S = S''$

Then $S(R_j) = S''(R_j) = 0 \neq S'(R_j)$. If R_j is the least atomic proposition on which S' and S'' differ, then $S'' \prec S$ contrary to assumption. If R_j is not the least atomic proposition on which S' and S'' differ, then there is some R_q for $q < j$ such that $S'(R_q) \neq S''(R_q)$. By leastness of R_j for S and S', $S(R_q) = S'(R_q) \neq S''(R_q)$, so $S \neq S''$.$\perp$

- **Case 2:** $S'' \prec S$

  Let R_q be least such that $S(R_q) \neq S''(R_q)$ and $S''(R_q) = 0$ and $S(R_q) = 1$. Then either $q < j < k$, or, $q = j$, or, $j < q < k$, or, $j < k \leq q$, or, $k < j$.

  - $(q < j < k)$

    Then $S(R_q) = S'(R_q) = 1$ by leastness of R_j for S and $S'.So$S'(R$_q$) $\neq S''(R_q)$ and $q < k$ contradicting leastness of k.

  - $(q = j)$.

    Then $S(R_q) = S(R_j) = 0.\perp$

  - $(j < q < k)$

    Then by leastness of q with respect to S and S'', $S(R_i) = S''(R_i)$ for all $i < q$. By leastness of k with respect to S' and S'',S'(R$_i$) $= S''(R_i)$ for $i < k$. Therefore, $S(R_i) = S'(R_i)$ for $i < q$, and $S(R_j) = S'(R_j)$..

  - $(j < k \leq q)$

    Then $S(R_i) = S'(R_i)$ for $i < k$, and therefore $S(R_j) = S'(R_j).\perp$.

  - $(k < j)$

    Then $S(R_i) = S'(R_i)$ for all $i \leq k$, so $S \prec S''$

$\square$ (Lemma 4.1)

Antisymmetry and connexivity of $\prec$ are obvious.

Corollary 4.1.1

$\prec$ is a total order.

Corollary 4.1.2

Every finite subset of solutions has a least element under $\prec$.

Lemma 4.2

For solutions S_n and S'_n of the nth level, if S_n extends the mth level solution S_m and S'_n extends the mth level solution S'_m for $m < n$, then

$$S_n \prec S'_n \text{ implies } S_m \preceq S'_m$$

Proof: Let S_n extend S_m and S'_n extend S'_m with $S_n \prec S'_n$. Let $R_0, R_1, R_2, ...$ be an enumeration of the atomic propositions that preserves their ordering by first occurrence left-to-right within a formula and the preserves the ordering of the formulas by level. Let R_i be the first atomic proposition on which S_n and S'_n differ. By the definition of '$\prec$', $S_n(R_i) = 0$ and $S'_n(R_i) = 1$. We need to show that if $S_m \neq S'_m$ then $\exists R_j$ least such that $S_m(R_j) = 0$ and $S'_m(R_j) = 1$.

Case 1: R_i occurs in a formula of level $< m$. Then R_i is already in the domain of the level m solutions S_m and S'_m. Since S_n extends S_m and S'_n extends S'_m, it follows that $S_m(R_j) = S_n(R_j)$ and $S'_m(R_j) = S'_n(R_j)$ for all $j \leq i$. So $S_m(R_i) = 0$ and $S'_m(R_i) = 1$. Moreover, R_i must be least such that $S_m \neq S'_m$ since if $\exists R_k$ such that $k < i$ and $S_m(R_k) \neq S'_m(R_k)$, then $S_n(R_k) \neq S'_n(R_k)$ contradicting leastness of R_i. So $S_m \prec S'_m$.

Case 2: R_i first occurs in a formula of level $l > m$. Then R_i not in the domain of S_m or S'_m. But then S_m must be equal to S'_m since if $S_m \neq S'_m$, then R_i is not the least atomic proposition on which S_n and S'_n differ.$\square$ (Lemma 4.2)

Definition 4.2 For integers $n, m \geq 1$, let S_n^m denote the solution of level n that is mth in the ordering $\prec$.

Convergence

The set of solutions ordered by $\prec$ forms a connected, infinite tree, that is locally finite. Skolem will show that the tree has an infinite branch. The solution determined by this branch satisfies the original formula, filling the gap in Löwenheim's proof under the tree of solutions interpretation.

The branch Skolem singles out is defined relative to the sequence of first solutions at each level, $S_1^1, S_2^1, ..., S_n^1 ...$ The desired branch is not necessarily identical with the branch defined by the union of these first solutions –this branch may terminate after a finite number of levels. However, the hypothesis of the theorem guarantees that at least one solution of each level is guaranteed to have extensions of every level.

Whenever the first solution of level n cannot be consistently extended by the first of the level $n+1$ solutions, it causes the ordering to shift right at the nth level. The first solution of the $n+1th$ level will then be a continuation of some S_n^m for $m > 1$.

From this we can see that if we consider only the first solutions, their respective restrictions to earlier levels may not agree. The theorem proven in this section is designed to show that because (i) the number of solutions at each level is finite, and (ii) the vertical branch being constructed can only shift right in the horizontal ordering under $\prec$, eventually the sequence of first solutions must converge on a particular solution of each level.

When talking about restrictions, instead of $S_k^1 A_j$, I write $S_j^{a_j^k}$. Thus, a_j^k denotes the place in the ordering of jth level solutions ($j < k$) where occurs the restriction of the $first$ of the kth level solutions to A_j. When a_j^k occurs as the superscript of a solution, I drop the subscript j since this information is contained in the subscript of the solution. E.g. $S_j^{a_j^k} = S_j^{a^k}$.

If $a_j^k > 1$, this means that there is at least one level i, $1 \leq i \leq j$, such that the first solution of the ith level has no consistent continuations at level $i+1$.

Theorem 4.3

For all levels v, there is a level $g(v)$ such that for all $n > g(v)$, $a_v^{g(v)} = a_v^n$.

Theorem 4.3 tells us that for a given level v, there is a level $g(v) > v$ such that the first solutions for levels above $g(v)$ will always agree on their restriction to the vth level, i.e., the part that assigns values to atomic propositions of A_v. Since this holds for every v, the sequence of first solutions will ultimately converge to a single truth assignment $\mathfrak{S}$. The proof requires the following lemma:

Lemma 4.3

Fix arbitrary v. The sequence $\{a_v^n\}_{n \in \omega}$ is monotone increasing.

Proof: It suffices to show that for $n' > n > v \geq 1$,

$$S_v^{a^n} \preceq S_v^{a^{n'}}$$

I.e., that $S^1_n A_v \preceq S^1_{n'} A_v$.

Proof: $S_v^{a^n}$ is extended by S^1_n and $S_v^{a^{n'}}$ is extended by $S_n^{a^{n'}}$. Clearly $S^1_n \prec S_n^{a^{n'}}$ since the restriction of $S^1_{n'}$ to the atomic propositions of A_n must be one of the nth level solutions. The lemma follows by Lemma 4.2.□

Now, the sequence $\{a_v^n\}_{n\in\omega}$ has a finite bound since there are only finitely many solutions at each level. By the monotonic sequence theorem, $\{a_v^n\}_{n\in\omega}$ converges, i.e., for each v there is some $g(v)$ such that for all $m > g(v)$, $a_v^m = a_v^{g(v)}$. □ (Theorem 4.3)

A single solution

Now form the infinite assignment $\mathfrak{S} = \bigcup_{i\in\omega} S_i^{a^{g(i)}}$. Clearly,

$$\mathfrak{S}A_v \subset \mathfrak{S}A_{v+1}$$

since if $\mathfrak{S}A_v = S_v^{a^{g(v)}}$ and $\mathfrak{S}A_{v+1} = S_{v+1}^{a^{g(v+1)}}$, then $S_{v+1}^{a^{g(v+1)}} v = S_v^{a^{g(v)}}$.

This concludes the proof of Theorem 4.1. □

4.4 Skolem 1928

Introduction

Skolem's 1928 paper "On Mathematical Logic" touches on everything from Boolean algebra and Schröder's "identical calculus", to second-order predicate logic. Skolem takes the main result to be his proposed alternative to the axiomatic development of first-order predicate calculus. Rather than deducing a formula using formal axioms and inference rules, Skolem shows how it is possible to "deal with deduction problems in a more expedient way", giving a semantic proof procedure that can effectively decide when a formula is refutable. The procedure uses the same idea found in (Löwenheim, 1915) and (Skolem, 1922) of expanding a first-order formula into truth-functional instances and checking for solutions via the methods of propositional logic.

This culminates in Skolem's statement of what is essentially the basic alternative of Gödel's completeness proof: "The real question now is whether there are solutions of an arbitrarily high level or whether for a certain n there exists no solution of the nth level." From this alternative Skolem purports to show that, in the second case, A is truth-functionally contradictory, and in the first case, A is "consistent". Thus,

Theorem 4.2.1

Either A is truth-functionally contradictory, or, A is consistent.

There is no consensus on what Skolem means by this statement, or whether his proof of it succeeds. The different interpretations put forward in the literature can be divided along three main lines of inquiry: Is the theorem semantic or syntactic? Is the proof finitistic? Is the proof complete?

Is Skolem's theorem semantic or syntactic?

Skolem's use of '"consistent" (widerspruchslos) in stating Theorem 4.2.1 is ambiguous between the semantic and the syntactic senses. This ambiguity is widespread in Skolem's work, as Goldfarb notes:

> Skolem is, as always, being heterodox, and accepting neither the usual semantic notions nor the formal proof-theoretic notions as explicative of the notion of mathematical inconsistency. (Goldfarb, 1979, p. 363)

Understanding "consistent" in a semantic sense to mean "satisfiable", the target theorem is the semantic completeness of a refutation procedure using the method of searching through solutions of each level. Skolem aims to show that either we can find an n for which the expansion A_n is truth-functionally contradictory, or, A is satisfiable.

Understanding "consistent" in a syntactic sense, a second interpretation is that Skolem intends to prove that if there is no n for which the nth level expansion A_n of A is truth-functionally contradictory, then adding the quantified formula A as an axiom in some implicit quantification theory does not result in syntactic inconsistency.

A third interpretation replaces A with the formula $A*$ that results from A when the quantifiers are dropped and the existential variables are replaced by Skolem functions. Following Goldfarb I call $A*$ the "functional form" of A. Theorem 4.2.1 can then be understood to mean that adding $A*$ to an implicit proof system (without laws for quantifiers) does not result in syntactic inconsistency.

Each interpretation fares differently with respect to the following two questions.

A finitistic proof?

Cutting across the semantic vs. syntactic question is the fact that the proof Skolem gives for Theorem 4.2.1 is apparently finitistic. Recall that in his 1922 proof of the Löwenheim-Skolem theorem, Skolem appeals to the existence of solutions for *every* n in order to construct his denumerable model.

In 1928 however, Skolem replaces the assumption of solutions for every n with the assumption of solutions for arbitrarily high (finite) n. The argument he gives attempts to show the existence of a finite upper bound on the level of the constants occurring in any contradictory proposition (see below).

A complete proof?

Depending on the interpretation one adopts, Skolem's proof may or may not suffice to establish the target theorem.

If the semantic interpretation of the theorem is correct, then the proof that Skolem gives is insufficient: a proof of completeness is inherently infinitary, requiring some equivalent of König's infinity lemma to establish the satisfiability of the formula. As noted, the (1928) proof appears to be finitary. However, commentators who support this first interpretation have pointed out that Skolem, in (1922) and (1929), *did* give infinitary proofs of exactly the step needed in order for (1928) to constitute a completeness proof.

On the first of the syntactic interpretations, Skolem's proof also has problems. This is the interpretation according to which adding the quantified formula A as an axiom of some implicit quantification theory does not result in syntactic inconsistency. In this case, a finitistic proof *is* possible, but is considerably more involved than the one given by Skolem. Also inexplicable is the fact that Skolem never gives the formal system of quantification theory presupposed by this interpretation. Although he gives examples of several rules of quantifier manipulation, he "does not go into this more deeply" turning instead to "solv[ing] deduction problems in a more expedient way".

This leaves the third interpretation according to which Skolem proves that adding the functional form $A*$ to an informal proof theory S does not result in syntactic inconsistency. Skolem's lack of a formal system is also a shortcoming for this interpretation, but by switching to the quantifier-free functional form $A*$, the system required is not as complex. Remarks in (Skolem, 1929) support his intention to prove the result for the basic system S described below.

4.4.1 Reconstructions

This section reconstructs Skolem's proof on both the first (semantic) and third (syntactic) interpretations.

Semantic completeness

The semantic interpretation is endorsed by Gödel who writes that Skolem "in his 1928 paper (at the bottom of p. 134) stated a completeness theorem (about refutation)" (Letter to Wang, 1967, in Wang 1974). I show how Skolem's construction of leveled instances of the functional form of a first-order formula can be conceived as a refutation procedure. I then sketch the steps for showing that the completeness of this procedure is a trivial consequence of the 1928 paper in conjunction with (Skolem, 1922).

Skolem's refutation procedure

Let $A = \forall x_1, ..., x_m F(x_1, ..., x_m, f_1(x_1, ..., x_m), ...,$
$f_n(x_1, ..., x_m))$ be a first-order formula in prenex normal form where the existential variables $y_1, ..., y_n$ have been replaced by Skolem functions $f_1(x_1, ..., x_m), ..., f_n(x_1, ..., x_m)$ of the scoping universal variables. For simplicity, I consider only a single block of quantifiers $\forall\exists$.[5] Let $A*$ be the result of dropping the quantifiers from A.

Skolem constructs expansions of $A*$ by defining a set of constant terms for each level n. The

[5]Though Skolem's schema allows for finitely many alternating blocks of such.

symbol “0” is the unique constant of level 0. The constants of level $n+1$ are the Skolem terms that can be formed by replacing the arguments to the Skolem functions by arbitrary m-length permutations of the constants of level n:

Definition 4.5.1

The *oth level expansion of* A is

$$A_0 = F(0, 0, ..., 0, f_1(0, 0, ..., 0), ..., f_n(0, 0, ..., 0)).$$

Definition 4.5.2

The *nth level expansion* A_n *of* A is the conjunction of instances of $F(x_1, ..., x_m, f_1(x_1, ..., x_m), ..., f_n(x_1, ..., x_m))$ when $x_1, ..., x_m$ range over the constants of level $n-1$.

For each level n, consider all assignments of m-tuples of constants of level $n-1$ to the variables. Each m-tuple generates a propositional instance of $A* = F(x_1, ..., x_m, f_1(x_1, ..., x_m), ..., f_n(x_1, ..., x_m))$. A_n is the conjunction of all such instances (including conjuncts formed at levels $q < n$). The construction of constants thus generates a parallel construction of sequences of propositional formulas.

This construction is very similar to the one found in Löwenheim.[6]

Either construction yields a refutation procedure for A. Since the expansions are made up of propositional instances of $A*$, we can consider truth values assignments to the atomic

[6]On the other hand, Skolem shows an awareness of the recursive structure of language that is absent in Löwenheim. For Skolem, the distinctness of terms is directly reflected in the syntax, i.e., the iteration of functional terms as arguments to the same functions. Also unlike Löwenheim, Skolem's constant terms are not presupposed to denote elements of a separate domain.

components that make the conjunction at each level come out true.[7] At each level, the class of possible assignments is narrowed down according to whether they can be extended to include instances of the next level.

If there exists a level n and formula A_n that cannot be satisfied by any extension of assignments to earlier levels, then the formula is shown to be unsatisfiable or "contradictory". Call this procedure $\mathcal{R}$. $\mathcal{R}$ is best illustrated with an example.

4.4.2 Example of Skolem's procedure

I demonstrate this procedure for one of Skolem's example formulas (p. 520, 1967/1928).[8]

$$\text{Let U} = \forall x \exists y [[A(x,y) \wedge B(x) \wedge \neg B(y)] \vee [A(x,x) \wedge \neg A(y,y) \wedge \neg B(y)]].$$

The first step is to drop the quantifiers, replace x with 0 and y with 1:

$$[A(0,1) \wedge B(0) \wedge \neg B(1)] \vee [A(0,0) \wedge \neg A(1,1) \wedge \neg B(1)]$$

Figure 4.1 breaks this instance into atomic components (columns) and lists all the possible truth assignments (rows) that make this instance come out true.

At the second step, consider assignments to the single[9] *new* instance when x ranges over the domain introduced in the first step:

$$[A(1,2) \wedge B(1) \wedge \neg B(2)] \vee [A(1,1) \wedge \neg A(2,2) \wedge \neg B(2)]$$

[7]Note that the atomic components of each conjunct will be distinct based on the different instantiations of the variables.

[8]Although the use of Skolem term makes the construction superficially different from the one used in (1922), in practice Skolem adopts the same strategy of replacing y-variables (in this case, Skolem functions) by integers for simplicity. See 1928/1967, pg. 519).

[9]By working with a single x and y, we avoid having to consider multiple new instances at every level.

Level 1	A(x,y)	B(x)	B(y)	A(x,x)	A(y,y)	B(y)
a_1	A(0,1) = F	B(0) = F	B(1) = F	A(0,0) = T	A(1,1) = F	B(1) = F
b_1	A(0,1) = F	B(0) = T	B(1) = F	A(0,0) = T	A(1,1) = F	B(1) = F
c_1	A(0,1) = T	B(0) = F	B(1) = F	A(0,0) = T	A(1,1) = F	B(1) = F
d_1	A(0,1) = T	B(0) = T	B(1) = F	A(0,0) = F	A(1,1) = F	B(1) = F
e_1	A(0,1) = T	B(0) = T	B(1) = F	A(0,0) = F	A(1,1) = T	B(1) = F
f_1	A(0,1) = T	B(0) = T	B(1) = F	A(0,0) = T	A(1,1) = F	B(1) = F
g_1	A(0,1) = T	B(0) = T	B(1) = F	A(0,0) = T	A(1,1) = T	B(1) = F

Figure 4.1: Level one solutions for the formula $\forall x \exists y[A(x,y) \wedge B(x) \wedge \neg B(y)] \vee [A(x,x) \wedge \neg A(y,y) \wedge \neg B(y)]$

Note that the formulas considered at each level are not U_n but rather $U_n \backslash U_{n-1}$ since U_n was defined to be the conjunction of instances of U' up to the nth level, whereas each level in Skolem's procedure operates with only the new instances. The aim of each step is to find a way of including the new instance (or instances) as an extension of a solution (in the defined sense) of the previous level. This means determining whether the new instance is compatible with any assignments to the instances of earlier levels, progressively narrowing down the possible solutions by ruling out solutions from the bottom up.

For example, at this second step we rule out the first level solutions represented by rows $a_1, c_1, e_1, f_1,$ and g_1 (Fig. 4.1). No second level assignment (Fig. 4.2) is an extension of one of these solutions because no second level assignment makes both A(1,1) and B(1) false. (Note that when checking for consistency in this example we can disregard the first column for the atomic formula A(x,y) since potential contradiction between levels can only occur with the formulas B(x) and B(y) and A(x,x,) and A(y,y).)

On the other hand, the solutions b_1 and d_1 are compatible with some second level assignments,

Level 2						
a_2	A(1,2) = F	B(1) = F	B(2) = F	A(1,1) = T	A(2,2) = T	B(2) = F
b_2	A(1,2) = F	B(1) = T	B(2) = F	A(1,1) = T	A(2,2) = T	B(2) = F
c_2	A(1,2) = T	B(1) = F	B(2) = F	A(1,1) = T	A(2,2) = T	B(2) = F
d_2	A(1,2) = T	B(1) = T	B(2) = F	A(1,1) = F	A(2,2) = F	B(2) = F
e_2	A(1,2) = T	B(1) = T	B(2) = F	A(1,1) = F	A(2,2) = T	B(2) = F
f_2	A(1,2) = T	B(1) = T	B(2) = F	A(1,1) = T	A(2,2) = F	B(2) = F
g_2	A(1,2) = T	B(1) = T	B(2) = F	A(1,1) = T	A(2,2) = T	B(2) = F

Figure 4.2: Level two solutions for the formula $\forall x \exists y [A(x,y) \wedge B(x) \wedge \neg B(y)] \vee [A(x,x) \wedge \neg A(y,y) \wedge \neg B(y)]$

specifically, those in rows d_2 and f_2 of Fig. 4.2.

Restricting our attention to these second level assignments that extend b_1 or d_1, we now ask whether any of these have extensions of the third level. Since both d_2 and f_2 assign False to A(2,2), this rules out all but two of the third level assignments (rows d_3 and e_3).

However, since d_2 and f_2 assign False to B(2) and d_3 and e_3 (Fig. 4.3) assign True to B(2), none of the second level solutions that extend a first level solution have extensions of the third level. Therefore, the procedure terminates because we have reached a point where none of the finite number of level one solutions have extensions beyond the second level. This reveals the formula to be contradictory.

Completeness of the procedure

Whether or not Skolem thought of it as such, the procedure just described constitutes a refutation procedure for A. If, for some level $n \geq 0$, none of the solutions to A_n can be

Level 3						
a_3	A(2,3) = F	B(2) = F	B(3) = F	A(2,2) = T	A(3,3) = F	B(3) = F
b_3	A(2,3) = F	B(2) = T	B(3) = F	A(2,2) = T	A(3,3) = F	B(3) = F
c_3	A(2,3) = T	B(2) = F	B(3) = F	A(2,2) = T	A(3,3) = F	B(3) = F
d_3	A(2,3) = T	B(2) = T	B(3) = F	A(2,2) = F	A(3,3) = F	B(3) = F
e_3	A(2,3) = T	B(2) = T	B(3) = F	A(2,2) = F	A(3,3) = T	B(3) = F
f_3	A(2,3) = T	B(2) = T	B(3) = F	A(2,2) = T	A(3,3) = F	B(3) = F
g_3	A(2,3) = T	B(2) = T	B(3) = F	A(2,2) = T	A(3,3) = T	B(3) = F

Figure 4.3: Level three solutions for the formula $\forall x \exists y[A(x,y) \wedge B(x) \wedge \neg B(y)] \vee [A(x,x) \wedge \neg A(y,y) \wedge \neg B(y)]$

extended to include the new conjuncts of A_{n+1}, then A_{n+1} is unsatisfiable ("contradictory"). Both Löwenheim and Skolem take it to follow immediately that A is unsatisfiable.[10] This represents one half of the following completeness theorem for the refutation procedure:

Theorem 4.5

Either A is refutable by the method of finding an n for which the expansion A_n is truth-functionally unsatisfiable, or, A is satisfiable.

When stated this way, the theorem follows as an immediate consequence of Skolem's 1922 proof of the Löwenheim-Skolem theorem (Theorem 4.1 above).

However, if Skolem intended to prove Theorem 4.5, the proof he actually gives in 1928 differs from the one just sketched and does not suffice to prove the result. He makes no

[10] A contemporary proof would require more elaboration of this connection between the unsatisfiability of the instances, and the unsatisfiability of the formula. Gödel formalizes this step as his Theorem VI in (1929, p. 113), see Lemma 1.3.2 above.

assumption of the existence of solutions for *every* level, replacing this with the assumption of "solutions for arbitrarily high n". Nor does he appeal to the 1922 result at the crucial infinitary step. This leaves a gap if Skolem's aim is to show that the formula is satisfiable, prompting commentators to endorse a different interpretation of Skolem's actual proof.

4.4.3 Syntactic consistency

This section reconstructs the interpretation on which Skolem shows that adding the functional form $A*$ to an implicit proof system S does not result in syntactic inconsistency. This reading of (Skolem, 1928) is supported by a more detailed proof in (Skolem, 1929).[11]

[11] The passage is worth quoting in full:

> If [functional form of the] formula is given as an axiom, then the theory based on it can be understood as follows:
>
> 1. The rule for generating individual terms: If $x_1, ..., x_m, z_1, ..., z_p, ...$ are individual terms, then so are $f_1(x_1, ..., x_m), ..., f_n(x_1, ..., x_m)$, $g_1(x_1, ..., x_m, z_1, ..., z_p, ...), ..., g_q(x_1, ..., x_m, z_1, ..., z_p, ...)$ etc.
> 2. If $A(t_1, ...t_u)$ is an atomic function in U, so for any individual $t_1, ..., t_u$, A should be either false or true, but not both at the same time.
> 3. For any individuals $x_1, ..., x_m, z_1, ..., z_p, ...$ the statement U should be true.
>
> The whole theory should therefore only consist in trying to determine the atomic functions for new individuals created repeatedly in accordance with 1) in such a way that 2) is fulfilled or to examine these possible determinations. [Skolem writes 2) and 3) in place of 1) and 2) which seems erroneous.] A general theorem of the theory then has the form: Let $a_1, a_2, ...$ be any individual. If $b_1, b_2, ...$ are certain symbols that can be derived from them according to 1), the statement $V(a_1, a_2, ..., b_1, b_2, ...)$ holds; Every general proposition consists in the assertion that the elementary constructed propositional function $V(a_1, a_2, ..., b_1, b_2, ...)$, in which $a_1, a_2, ...$ occur as real variables (the b are only expressions made from them) is a true statement for any choice of these individuals.
>
> According to this view, it is clear that there is no contradiction in the theory for which solutions of each level exist in the sense explained above. For if there are two contradictory provable sentences in which the indefinite symbols $a_1, a_2, ...$ appear, they remain valid if all a are replaced by 0; then the b change into symbols of the set M_v for a certain v, and no solution of the vth level can exist. (Skolem, 1970, pg. 253, translation mine)

Assume the following system S. The language of S is quantifier-free, consisting only of the truth-functional connectives $\vee, \wedge, \neg$, predicate symbols $A, B, C...$, function symbols $f, g, h...$, individual variables $x_1, x_2, \ldots, y_1, y_2 \ldots$ and constants 0,1,2... over the domain of natural numbers. The inference rules of S are the valid truth-functional inferences of propositional logic and the rule of substitution: arbitrary variable and constant terms may be substituted for variables occurring free and as arguments to functional terms.

A derivation in S is a finite sequence of formulas each of which is either an axiom or follows from preceding formulas by the inference rules.

This section sketches Skolem's proof on the interpretation[12] according to which Skolem aims to prove:

Theorem 4.6 Either we can demonstrate that A is truth-functionally contradictory, or, adding the functional form $A*$ of A as an axiom to S does not result in syntactic inconsistency.

Proof: Let A be a quantified formula containing the predicate symbols $U, V, W \ldots$. Let $A*$ be the functional form of A, formed by dropping the quantifiers and replacing each existentially quantified variable y_i by a Skolem function f_i of the universal variables scoping over y_i in A. The universal variables $x_1, ..., x_n$ of A are now free variables of $A*$.

Suppose that from $A*$ it is possible to derive in S a proposition T that is inconsistent, e.g.,the conjunction of an atomic formula (with free variables) and its negation. These contradictory atomic sentences were derived in S by propositional inferences from certain substitution instances A' and A'' of $A*$.[13] From this Skolem purports to demonstrate the existence of

[12]Cf. (Goldfarb, 1971)

[13]Each predicate symbol in $A*$ generates an atomic proposition when the variables in $A*$ are replaced by ground terms. Ground terms are formed recursively, starting with 0, by iterated application of the rule of

a level n such that the *nth* level expansion A_n has no solutions. (Note that in the 1928 construction, Skolem now considers a solution of level n to be a *cumulative* assignment of truth-values to expansion instances of every level up to n.)

The first step is to replace variables occurring anywhere in the derivation of T by 0. The result is a derivation of a variable-free sentence T_0. Terms occurring in this derivation are either 0 or can be obtained from 0 by finitely-many applications of the rule of substitution to the function symbols.[14] T_0 is inconsistent because T is.

Now, A' and A'', the instances from which the contradictory atomic sentences were derived in S, must occur somewhere in Skolem's hierarchy of expansion instances. The point of the expansion is to systematically account for all such instances in order to represent the universal quantification. The goal of Skolem's proof is to locate the lowest level in the construction by which both A' and A'' have occurred as conjuncts of the cumulative expansion up to that point.

To find a bound on the level by which both A' and A'' must have occurred, Skolem appeals to the facts that (*i*) the functional symbols f_1, f_2.... occurring in $A*$ are finite in number (since there are finitely-many existentially quantified variables of A), (*ii*) the derivation consists of finitely many steps, (*iii*) any given step may involve at most finitely many applications of the rule of substitution, and (iv) propositional inferences do not introduce new individual terms. The bound is determined by a parallel between the construction of terms (through iteration of the rule of substitution on the function symbols of A) and the construction of expansion instances.

Skolem then claims that the least level at which both A' and A'' have occurred cannot have a solution. A solution to the expansion at this level would have to assign truth to both A'

substitution in the Skolem functions.

[14]Since the rules of S do not license the introduction of new predicate symbols, the predicate symbols in T must already occur in $A*$. The free variables in T are those $x_1, ..., x_n$ already occurring free in $A*$ and terms built up from these by finitely-many applications of the rule of substitution.

and A'' and, by extension, to the atomic and its negation derivable from these instances.

This last step reveals a problem of circularity. The contradictory pair of atomics derivable from A' and A'' occur in the expansion embedded under whatever logical operations make up the formula A*. Therefore, to demonstrate the unsatisfiability of the expansion at level n requires isolating these atomics using the same propositional methods by which T was derived in S. If the intention was to show that the syntactic method only refutes unsatisfiable formulas (i.e., soundness), the proof is question-begging since the way we know a formula is unsatisfiable is ultimately by reference to the fact that it is refutable.

In defense of Skolem, this objection assumes that the explicit motive of the proof is to establish a connection - soundness - between the syntactic proof system S and the semantic method R. However, Skolem did not distinguish between syntax and semantics in a way that would support this interpretation. It is more likely that Skolem recognized the connection between S and R to be self-evident (this connection is, as Goldfarb notes, "a triviality"). The aim is not to show that S is an *independent* way of picking out the unsatisfiable formulas. Rather, the syntactic considerations are intended to supplement the semantic method R and show how it constitutes a decision procedure. This will be discussed further in Chapter 5.

Problems

This reconstruction, based on the actual proof Skolem gives, is not without problems. Wang writes in his introduction to (Skolem, 1970): "it is not clear how one can turn this suggestive argument into a convincing proof" (Wang, 1970, p. 26). I concur with this assessment. In particular, the existence of a bound on the level of the constant terms in U_0 relies on syntactical considerations about the system S that are nowhere mentioned by Skolem. Without erecting a formal framework which would be antithetical to Skolem's sensibilities, the argument is too sketchy to warrant being considered an actual proof.

In the next chapter, I argue that Skolem's intent is not to prove completeness or consistency, but to sketch a method for proving decidability.

4.5 Conclusion

This chapter began with a reconstruction of Skolem's 1922 proof of the weak version of Löwenheim's theorem. Skolem's construction resembles Löwenheim's on the "tree of solutions" interpretation. By defining an ordering on the solutions at each level, Skolem gives a convergence argument that takes the place of an infinity lemma and fills the gap in Löwenheim's proof. I then looked at the main argument of (Skolem, 1928), for Theorem 4.2.1 above. This theorem admits of at least three different interpretations. I looked at how each interpretation responds to three issues: whether Skolem uses "consistent" in a syntactical or a semantical sense, whether the proof is finitistic, and whether the proof is complete. I then reconstructed the proof on two of these interpretations, the second of which will be defended further in Chapter 5 as part of a revisionary account of Skolem's aims in 1928.

Chapter 5

Completeness in the Shadow of Decidability

5.1 Introduction

This chapter will give a revisionary account of why completeness went overlooked by Skolem. I argue that the ability to recognize completeness as an interesting and distinct property of logical systems presupposes certain contextual features that were absent in Skolem's pre-1930 context. These features are introduced in Section 2 and used to challenge a presupposition of The Puzzle. In section 3, I argue for the untenability of Gödel's explanation of Skolem's argument in 1928. I defend an alternative interpretation of Skolem's reasoning in light of his aim to prove a decidability result for all first-order logic formulas.

5.2 Completeness in context

In this section, I argue for three contextual features that promote the recognition of completeness as an important property of first-order logic. I argue that the absence of these features in Skolem challenges a presupposition of The Puzzle. Skolem's inattention to a completeness result is neither puzzling nor contemptible when his proper context is taken into account.

5.2.1 Completeness and the decision problem

An overlooked[1] but crucial step in the progression towards a completeness proof is the recognition of completeness as a property distinct from decidability. I argue that when this distinction is absent, completeness often coincides with semi-decidability, and its interest is therefore overshadowed by the interest of decidability.

The aim of the first feature is therefore to preserve the conceptual independence of completeness by way of a distinction between proof and decision procedures:

> **Feature 1** A framework in which deductive proof procedures are distinguished and investigated separately from decision algorithms.

Today, the conceptual distinction between decidability and completeness is taken for granted, usually with reference to the discovery that it is possible for theories to be complete yet undecidable, or conversely, to be decidable yet incomplete. For example, the theory of algebraically closed fields (without fixed characteristic) is a decidable but incomplete theory. Decidability in this context is a property that holds of a theory T, considered as a set of sentences closed under logical consequence, when there exists an algorithm that can effectively

[1] With the exception of Manzano and Alonso (2013).

decide the membership relation for T.

Decidability is also used to refer to the property a deductive system has when there exists an effective algorithm that can decide membership in its set of logical validities. The question of whether first-order logic possesses this property is known as the *"Entscheidungsproblem"*. As stated by Hilbert and Ackermann in 1928:[2]

> The Entscheidungsproblem is solved when we know a procedure that allows for any given logical expression to decide by finitely many operations its validity or satisfiability.[...] The entscheidungsproblem must be considered the main problem of mathematical logic.[3] (Böerger, Grädel, and Gurevich, 1997).

Today we know that completeness comes apart from decidability in this sense also: first-order logic was discovered by Gödel to be complete, and by Church and Turing (1936/37) to be undecidable.

But prior to this discovery, the decision problem for first-order logic was not always clearly distinct from the problem of completeness.

Consider the fact that any decision algorithm for validity can be transformed into a proof procedure. A decision algorithm gives a finite method that when applied to any formula F, will either determine that F is valid, or, will determine that F is invalid. Starting with any valid formula, by actually carrying out the steps of the algorithm we can construct an informal proof of that validity.[4]

[2]The origins of this problem go back to the seventeenth century with Leibniz's search for a machine that could mechanically calculate or "decide" the truth value of any mathematical statement. Meanwhile, in the algebraic tradition starting with Boole, decision problems for validity were represented and investigated as problems of algebra - deciding whether or not an equation has solutions. In the context of first-order logic, Behmann claims to have been the first to explicitly state the decision problem. He went on to solve it for monadic second-order logic (see Mancosu and Zach, 2015).

[3]A more precise statement would later be given using the notion of effective computability, formally characterized in terms general recursive functions, Turing machines, or the lambda calculus (these characterizations were shown to be equivalent).

[4]For example, suppose the decision algorithm gives a way of recursively enumerating the validities with

I argue that this sort of conflation between proof and decision procedures can devalue the question of completeness. It does so by erasing the distinction between completeness and *semi-decidability*, i.e., between asking whether a procedure can prove all the valid formulas, and asking of the same procedure whether it can decide, for any *valid* formula, that it is in fact valid.[5]

Semi-decidability, as the name implies, only yields half of what we get from full decidability. Full decidability additionally requires that the procedure can effectively determine when a formula is *not* a validity.[6]

For logicians first setting out to investigate these properties of logical systems, it would have been natural to regard a proof of the semi-decidability of validity/unsatisfiability as a methodological stepping stone towards proving decidability. It would not have been natural to regard semi-decidability as an end in itself, unless there was prior knowledge that full decidability was unattainable (see feature 3 below). Thus, when the question of completeness is only investigated as part of the decision problem, the interest of both completeness and semi-decidability becomes subordinate to that of decidability. This was the situation for Skolem writing in 1928.

Proof versus decision procedures in Skolem

Recall the procedure $\mathcal{R}$ reconstructed in Chapter 4 from Skolem's (1922) proof of the LST. Skolem adapts this procedure as part of his 1928 proof of decidability for the fragment of first-order logic with quantifier prefix $\forall x \exists y_1, ..., y_m$.

the guarantee that any valid formula will eventually appear on the list. This algorithm can generate a proof for any valid formula. The proof is a list, in order, of the valid formulas enumerated by the algorithm. The target formula is proven when it appears on the list, as the last line of the proof.

[5] Equivalently, to decide, for any unsatisfiable formula, that it is unsatisfiable.

[6] In contrast, when a formula is *not* valid, completeness does not require that the proof procedure produce a refutation. Instead, it may simply never terminate in its search for a proof. This is because there are closed formulae like $\exists x \exists y (x \neq y)$ that are not valid and yet the negation $\forall x \forall y (x = y)$ is not valid either.

As described in Section 4.4.2:

> The procedure has two possible outcomes. The first is that all possible assignments are ruled out by reaching a level n and formula A_n that cannot be satisfied by any extension of assignments to earlier levels. In this case the formula is shown to be unsatisfiable or "contradictory". The other possibility (for formulas of this prefix) is that the procedure reaches a level n such that the solutions that have extensions of level $n+1$ are the same as those that have extensions of level n. Since the assignment of truth values to the atomic components of level n is independent of the assignment of truth values to atomic components of all levels earlier than $n-1$,[7] the formula is revealed to be "consistent", i.e., satisfiable.

Note that the restriction to formulas of a certain prefix only comes into play for the second of the two possible outcomes of the procedure. In the case where a formula A is unsatisfiable, $\mathcal{R}$ can be used to show unsatisfiability regardless of quantifier prefix (provided A is in normal form).

$\mathcal{R}$ is a semantic procedure. Like the truth table method for propositional logic, it searches through "solutions" - assignments of truth values to instances of a given formula - looking for a level where no possible assignment to atomic components can render true the instance of that level.

This is in sharp contrast with the deductive procedure for which Gödel proves completeness, namely the "restricted functional calculus" of Russell and Whitehead's *Principia*[8]. The propositional fragment of this system is likewise the object of Post's (1922) completeness proof for propositional logic. The system is based on syntactical formation and inference rules that apply to formulae as *uninterpreted* objects. The burden of the completeness

[7] Again, only for formulas with this prefix.

[8] Gödel follows the presentation given in (Hilbert and Ackermann, 1928)

theorem is to show how the purely formal calculus connects up with the semantics, i.e., the truth or satisfiability of the formulae.

The semantic nature of Skolem's procedure therefore seems to place it in opposition to the notion of proof involved in standard formulations of the completeness theorem. Despite this, $\mathcal{R}$ is the basis for the claim made by Gödel, Wang, Goldfarb and others that Skolem had a "refutation procedure" for which he should have proved informal completeness.[9]

That is, in proving the LST, Skolem should have recognized the following corollary:

> Informal Completeness of $\mathcal{R}(IC)$ *For a given formula F, either F is refutable by $\mathcal{R}$, or, F is satisfiable.*

It will be argued in Section 3 that (IC) misrepresents Skolem's aim in 1928. What Skolem actually wishes to show is that $\mathcal{R}$ constitutes a decision algorithm for validity. As a result, there is no distinction *for Skolem* between the decision procedure he hoped would yield decidability, and the procedure that Gödel and others attribute to Skolem as a refutation procedure designed to disprove formulae. This means that feature (1) does not hold: the deductive proof procedure attributed to Skolem is *not* distinct from the decision algorithm he hopes to use to identify both the valid and invalid formulas.

5.2.2 Summary

This section argued that the absence of feature (1) - the separation between proof and decision procedures - makes it difficult to recognize completeness as a property distinct from semi-decidability and therefore, as being of interest outside the context of an anticipated decidability result. This argument applies to Skolem. Feature (1) is notably absent from his

[9] See for example Wang, 1970, p. 21-22.

(1928), the main aim of which was to show not completeness but decidability. The informal procedure $\mathcal{R}$ for which Skolem could have proven completeness was conceptualized by him as a decision procedure, making completeness synonymous with semi-decidability.

There is also a positive argument for feature (1). When the distinction between proof and decision procedures is present, it introduces a conceptual gap between the set of validities identified by the decision algorithm, and the set of theorems derivable in the proof calculus. This separation leads naturally to the question of completeness: does the proof procedure of the calculus allow us to prove all the valid formulae identified by the decision algorithm? The next section introduces another key distinction that gives rise to this question.

5.2.3 Semantics vs. Syntax

The second contextual feature also aims to differentiate validity from theoremhood:

> **Feature 2** A clear distinction between semantics and syntax.

Unlike the first feature, (2) has been widely mentioned in the literature. Moore, for example:

> In his doctoral dissertation, which established the completeness theorem for first-order logic, Gödel exhibited a more profound understanding of the distinction between syntax and semantics - as well as their interrelationship - than had his predecessors. Skolem had failed to observe this distinction, especially as it concerned consistency and satisfiability, by expressing Löwenheim's Theorem in the following form: A first-order sentence is either inconsistent or else satisfiable in a countable domain. However, Skolem demonstrated only that if a first-order sentence is satisfiable in a set M it is satisfiable in a countable subset of M.

> What Gödel later established was essentially Skolem's stated theorem. Thus the completeness theorem for first-order logic arose in 1930 rather than a decade earlier. (Moore, 1979, p. 125)[10]

As Moore observes, the clear syntax/semantics distinction drawn by Gödel allows him to adapt Skolem's proof of the LST by taking the notion of inconsistency in a syntactic sense, and contrasting it with the semantic notion of satisfiability. This gap between the notions of semantic and syntactic consequence is then bridged by his completeness proof.

The discussion of feature (2) in the literature has focused predominantly on what Gödel achieved by recognizing the distinction. This section looks at how the *absence* of feature (2) affects the question of completeness for Skolem.

The following argument, like the one given in the preceding section, is based on the idea that key distinctions are needed to differentiate completeness from other results. Gödel and others have used the proximity between a completeness proof and Skolem's proof of the LST to argue that Skolem ought to have recognized the easy inference connecting the two theorems. I argue that the proximity between completeness and the LST was not an aid but a hindrance to Skolem's recognition of the former. Feature (2) - the semantics/syntax distinction - is absent in Skolem's informal setting, but is crucial to establish completeness as conceptually distinct from the LST.

In order to claim that Skolem ought to have recognized completeness, Gödel maintains that the notion of completeness does not depend on a formal framework:

[10] Moore's target here is Skolem's (1928) statement of the LST - the same statement that arguably implies (IC) above. The observation that Skolem fails to distinguish semantic and syntactic senses of "inconsistent" fits with our claim that the "refutation" procedure attributed to Skolem is more charitably understood as a *semantic* decision procedure.

> It may be true that Skolem had little interest in the formalization of logic, but this does not in the least explain why he did not give a correct proof of that completeness theorem which he explicitly stated (op. cit., p. 134), namely that there is a contradiction at some level n if there is an informal disproof of the formula. (Letter to Wang, 1967, in [Wang, 1970] p. 10)

I agree with Gödel that a formal concept of proof is not essential in order to raise the question of completeness.[11] However, an unforeseen consequence of transposing completeness to the informal setting is that it removes the basis - via feature (2) - for distinguishing it from a result already familiar to Skolem. This can be seen by looking at the role of feature (2) in Gödel's proof of completeness.

Gödel's semantics vs. syntax distinction

The crucial step that characterizes Gödel's proof is his lemma 1.3.2 (see chapter 1 above). This lemma states, roughly, that for every n, the implication $A \to A_n$ is provable in the object theory.[12] It is the only step in Gödel's proof that cannot be meaningfully stated in the absence of feature (2). It is also the step that differentiates the content of Gödel's theorem from that of Skolem and Löwenheim's proofs of the LST. Apart from this lemma, Gödel's proof consists in formalizing the steps taken by Skolem in 1922 where he shows that the satisfiability of every A_n implies the satisfiability of A.

Lemma 1.3.2 relies on an explicit distinction between the formal, *uninterpreted* calculus in which the implication $A \to A_n$ is provable for every n, and the semantic content of what the implication expresses - that the truth of A implies the truth of each of the approximating

[11] Quite generally, this is the question of the adequacy of a procedure to prove all of a given set of sentences (in this case, the validities). This is conceptually independent of the particular procedure and corresponding definition of "proof", though both evidently bear on how one actual proves (in the metatheory) that the procedure has the desired property.

[12] For Gödel, the system of *Principia*.

instances A_n. [13] This distinction in Gödel is made possible by the formal proof calculus laid out explicitly as the object of investigation. In Gödel's system, the refutability of a formula A means that it is possible to derive the negation of A by the axioms and inference rules of the "restricted functional calculus" (see above). Recall that the aim of completeness is to show that for any formula A, if A is not satisfiable, then A is refutable in this sense.

On Gödel's definition of what it is to refute a formula, merely constructing the formulae A_n and checking their satisfiability does not suffice as a means to refute A. Finding an n such that A_n is unsatifiable yields at most a derivation of $\neg A_n$. By the syntactic definition, this is not identical with a derivation of $\neg A$. This is where lemma 1.3.2 comes in, enabling the move, inside the proof system, from the refutability of some A_n, to the refutability of the original quantified formula A.

To summarize, the conceptual gap between unsatisfiability and refutability is brought to light by Gödel's formal notion of refutation. The gap is bridged by his lemma 1.3.2. This lemma is the locus of the semantics/syntax distinction in Gödel and what differentiates his completeness proof from earlier proofs of the LST.

Defining away the gap between semantics and syntax

Refutability has a quite different meaning in Skolem's informal setting and relative to his procedure $\mathcal{R}$. To say that A is refutable by $\mathcal{R}$ means that we can find a finite level n such that the *nth* level expansion A_n has no satisfying interpretations (Cf. [Goldfarb, 1971, p. 522]).

[13] Juliette Kennedy writes,

> This lemma is the main step missing from the various earlier attempts at the proof due to Löwenheim and Skolem, and, in the context of the completeness theorem for first order logic, renders the connection between syntax and semantics completely explicit.

Neither Löwenheim nor Skolem attempted completeness proofs. Kennedy is likely referring to the proximity with the LST mentioned above.

Note that this is not Skolem's own definition of refutation[14], but the definition foisted on him by those, including Gödel, who think that he should have recognized $\mathcal{R}$ as a refutation procedure, rather than a semantic decision procedure.

When refutation is defined in Skolem's way, an analog to Gödel's lemma becomes unnecessary because a refutation of A_n yields a refutation of A by definition. This brings Skolem even closer to a proof of informal completeness since, granting this step, all the component parts of the proof are already available in his proof of the LST. But in consequence, it becomes difficult to see anything very interesting or novel in the question of the completeness of the informal procedure $\mathcal{R}$. "Every formula is either refutable by $\mathcal{R}$ or is satisfiable" becomes:

> For every formula A, either we can find an n such that the *nth* expansion of A is truth-functionally unsatisfiable, or, A is satisfiable.

Compare,

> If for every n the *nth* expansion A_n *is* satisfiable, then A is satisfiable.

With the added specification that the domain in which A is satisfiable is countable, this is just the Löwenheim-Skolem theorem. The only thing separating these two theorems is the distinction between being unable to *find* an n for which A_n is unsatisfiable, and the non-existence of such an A_n. This is a distinction that Skolem glossed over. In doing so, he was either taking the completeness of his refutation procedure for granted, or recognizing it as following trivially from the LST.

[14] A concept Skolem never explicitly mentions.

A defensible notion?

On the one hand, the argument just given is favourable to Skolem in that it challenges Gödel's assumption that Skolem failed to recognize completeness. Skolem may have recognized the latter as a corollary and yet neglected to state what he took to be a trivial restatement of the LST.

On the other hand, it could be countered that endorsing a concept of refutation that obviates the need for Gödel's lemma is itself an intellectually blameworthy move. Skolem can still be charged with failing to recognize the distinction between showing some A_n to be unsatisfiable, and showing the formula A to be unsatisfiable, a distinction that is defined out of existence by his notion of refutation. For this reason, Skolem ought to have rejected that notion as untenable.

In Skolem's defense, I claim that the coherence of this counterattack presupposes feature (2), the semantics/syntax distinction. When and only when the latter distinction is in place does it become inappropriate to bridge the resulting gap by definition.

In a context where feature (2) is absent however, Skolem's informal notion of refutation is justified by implicit arguments that were standard at the time. Skolem (and Löwenheim) take it to be self-evident that the unsatisfiability of one of the propositional A_n implies the unsatisfiability of the quantified formula A. This self-evidence stems from an implicit semantic argument based on the conception of quantified formulas as infinite conjunctions/disjunctions of instances formed in the manner of the A_n.

Today we would be inclined to dismiss such justifications as "semantic" and as extraneous to the question of the inferential connections between certain formulae - the question addressed by Gödel's lemma 1.3.2. But this objection is anachronistic in Skolem's context. For one, Skolem's proof procedure is also, from a modern perspective, "semantic", making it unclear what type of justification one could give *other* than an appeal to interpretations of the formulae.

But even this misses the point. Without feature (2), there is nothing to distinguish these sorts of arguments *as* semantic in the way we now understand that term. As Manzano and Alonso write:

> To speak of semantics in the 1920s can be seen [...] as a misconception largely motivated by the mere transposition of the structure and organization that these notions have at present. (p. 56)

Thus, it is not surprising that despite frequently invoking the concept of satisfiability, Skolem never gives an explicit definition of semantic consequence.[15]

The charge of anachronism is even stronger in speaking about Skolem's "syntax". Skolem did not present his arguments in the framework of a formal axiomatic system or formal language. As a result, the notion of syntax as dealing with the formation and inference rules for such languages is not applicable. The closest Skolem comes to specifying a formal language is his adoption of Schröder's notation in his (1922). Even here, he fails to distinguish between the mathematical operations and the symbols that denote them and does not mention rules of inference.

There is no evidence that this imprecision ever leads Skolem's arguments astray.[16] At most

[15]The closest Skolem comes to defining it is in his 1928 discussion of problems": If U and V are first order propositions and we pose the question whether V follows from U, this is equivalent to asking whether $[U \wedge \neg V]$ is a contradiction or not. (1928, p. 517)

It is evident from the context and the lack of a formal system that Skolem intends "contradiction" in the semantic sense: $U \wedge \neg V$ is a contradiction if it is unsatisfiable, i.e., if it is impossible for U to be true and V to be false at the same time. This definition could easily be generalized to sets of formulas, and made precise in terms of the concept of satisfiability already at play in Skolem's work. Skolem himself though, shows no inclination to do this.

[16]It is uncharitable to label as "conflations" the ways in which Skolem fails to align with modern terminology. The strict separation of semantics from pure syntax is a pervasive feature of contemporary logic and this gives the impression of its necessity. However, the early decades of the twentieth century did not share

it results in the glossing of steps that we would now expect to be spelled out in formal detail. Gödel's lemma is one of these.

Summary To summarize the argument of this section, the place where the syntax/semantics distinction is most in evidence in Gödel's proof of completeness is his Lemma 1.3.2. No analogous lemma can be attributed to Skolem because, like Löwenheim before him, Skolem did not recognize any gap between showing some finite expansion A_n to be refutable (unsatisfiable), and showing the original quantified formula A to be unsatisfiable. In Skolem's proper context, this need not be viewed as an oversight. Justifications for the immediacy of this step can be given by appeal to the interpretation of the quantifiers and our contemporary reasons for prohibiting such appeals do not come into play in the informal setting where feature (2) is absent.

However, adapting the concept of completeness to apply in Skolem's informal context ultimately obscures the distinction between it and the LST. Even if we accept the implicit semantic arguments that make lemma 1.3.2 unnecessary, there is still the question of why Skolem would recognize completeness as a conceptually distinct result: without lemma 1.3.2 to distinguish it, informal completeness coincides with the proof already given of the Löwenheim-Skolem theorem.

---

the viewpoint that came to dominate logic after 1930. As mathematician Andrej Bauer writes:

> [T]he mindsets of the early 20th century logicians were closer to that of categorical logic than first-order logic and model theory. Or to put it another way, the conflation of syntax and semantics was *not a mistake*. Using the word "conflation" to describe what they did betrays a very syntactically minded view of logic rooted in philosophy of language–which came later to dominate logic for several decades. It is easy to argue in the opposite direction and present the dichotomy as a flaw: the preoccupation with pure syntax by logicians of the mid 20th century demonstrates their inability to think abstractly, and can be likened to the historic period of mathematical analysis during which "function" was equated with "expression". (Correspondence [url: https://mathoverflow.net/q/319921], Math Stack Exchange, 2019)

5.2.4 Undecidability

The third feature does not have the benefit of historical confirmation. Nonetheless, I speculate that Church's undecidability theorem, had it been discovered during the 1920s, might have helped distinguish completeness even in contexts where feature (1) - the distinction between proof and decision procedures - is not fulfilled. Whereas completeness in the shadow of decidability appears as a weak corollary, completeness in the shadow of *undecidability* appears as a surprising and optimal result. Surprising, because the notion of validity that is seemingly out of our reach via finite decision procedures is nonetheless accessible as the counterpart of what is finitely provable. Optimal, simply because this is the only way we can access it. Hence:

> **Feature 3** Proof of the undecidability of the class of formulae for which completeness is at stake.

This feature potentially applies even in the absence of a sharp distinction between decision and proof procedure. For example, the original aim of Skolem was to find a decision procedure that would construct a model if the formula is satisfiable, or find a contradiction if the formula is unsatisfiable. In this context, if it was discovered that finding a model is not a recursive procedure, then the informal completeness of the same procedure might emerge as a significant finding. The sought-after effective method for constructing a model is replaced by the construction yielded by the Löwenheim-Skolem theorem (using non-finitary reasoning) because it is recognized that the former goal is impossible.

5.2.5 Summary

In this section, I argued that each of the contextual features introduced above is conducive to the framing of the completeness theorem as a distinct and interesting result. None of these features are claimed as necessary conditions on the recognition of completeness. However, the absence of all three in Skolem's context lends support to my claim that Skolem's failure to acknowledge completeness does not have the inexplicable character that Gödel attributes to it. This claim is backed up by the argument given in the following section, where I argue for the untenability of Gödel's own answer to his puzzle, and give an alternative interpretation of (Skolem, 1928) to fill the resulting explanatory gap.

5.3 Skolem 1928

Recall Gödel's explanation for why Skolem failed to acknowledge a completeness theorem implicit in his (1928) when supplemented with the (1922) proof of the LST:

> I think the explanation is not hard to find. It lies in a widespread lack, at that time, of the required epistemological attitude toward metamathematics and toward nonfinitary reasoning. [...]
>
> [Skolem] was a firm believer in set theoretical relativism and in the sterility of transfinite reasoning [...]
>
> [E]vidently because of the transfinite character of the completeness question, [Skolem] tried to eliminate it, instead of answering it (Letter to Wang, 7 December 1967, in Wang, 1996, p. 124).

In chapter 3, I identified the sense of non-finitary reasoning specific to Gödel's claim. A completeness proof is "essentially infinitary" in that it requires applying the law of excluded middle to - and therefore quantifying over - the actually infinite set of A_n. First, to obtain the basic alternative, and again in the infinity lemma step that Gödel's proof shares with Skolem's (1922) proof of the LST. "Finitism", in Gödel's sense, is the intentional avoidance of such quantifications, allegedly motivated by epistemological concerns.

In the following section, I argue against the impression that the syntactic argument Skolem gives on pg. 519 is evidence of finitism in Gödel's sense. There is no basis for thinking that Skolem avoids taking infinitary steps[17] for *this* reason. In place of Gödel's explanation, I give a new defense of the syntactic interpretation discussed in Section 4.4.3. I argue that, in addition to being the most charitable of the interpretations put forward, it is consistent with Skolem's aim of extending a limited decidability result to formulae of any prefix.

5.3.1 Evidence of finitism

In 1928, Skolem states the following:

> The real question now is whether there are solutions of an arbitrary high level or whether for some n there exists no solution of the nth level. In the latter case the given first order proposition contains a contradiction. In the former case, on the other hand, it is consistent.

.

In Chapter 4, I considered two interpretations of this statement as theorems 4.5 and 4.6 respectively. For neutrality, I will refer to the above statement simply as "The Theorem". The argument Skolem gives for The Theorem is, as discussed in Chapter 4, the main point of contention in the 1928 paper.

There are two main pieces of evidence that have prompted Gödel and others to label Skolem's reasoning as "finitistic". The first is Skolem's statement of The Theorem itself, the second is the syntactic nature of the argument Skolem gives for it.

[17] Steps already taken in his (1922)

Arbitrary versus Every

The phrase "solutions of arbitrary high level" used in Skolem's statement of The Theorem is suggestive. Why does Skolem not say "solutions of *every* level"? It would be understandable to read into this change of language the views on quantification that Skolem expresses in his [1923]. In that paper, Skolem takes a skeptical stance towards unbounded quantification over infinite domains. He propounds what is now called primitive recursive arithmetic (PRA) as a quantifier-free alternative foundation for mathematics. These views also characterize the sort of finitism that Gödel is argued (in chapter 3) to have in mind when he claims that the proof of completeness is essentially infinitary. I identified Gödel's non-finitary reasoning with the quantification over infinite totalities occurring in the applications of law of excluded middle in his proof of completeness.

There is, however, no evidence that Skolem extended the [1923] views to his [1928]. The latter paper makes no reference to the earlier paper, nor to the "problem" of quantification.

Moreover, it is not unprecedented for Skolem to use ambiguous (indeed,"potentialist") language when discussing quantification, apparently without intending any philosophical commitment. This appears frequently throughout his work, including his [1922] which predates Skolem's concerns about quantification.[18] For example, Skolem in [1922] describes the existence of solutions for all n in terms of the ability to "indefinitely extend" a given solution. The hypothesis of the LST guarantees that "it must be possible to continue the process [of finding solutions] in this way indefinitely" (p. 294). He then gives a convergence argument to show how the solutions guaranteed at *each* level form a satisfying interpretation for the quantified formula. This step, equivalent to König's infinity lemma, requires an application of the law of excluded middle to the completed set of solutions for *every* level n. Either we

[18]More accurately, the paradox Skolem develops on the basis of the 1922 proof of the LST is one of the motivations for the skepticism about quantification that leads Skolem to endorse recursive arithmetic without quantifiers.

charge Skolem with the large oversight of having failed to recognize that such quantification is required for the step in question, or we acknowledge that for Skolem it was legitimate to pass directly from a method for constructing solutions for arbitrary n ("indefinite extensibility"), to the existence of solutions for *all* n.

This same tendency is evidenced elsewhere in [1928]. When he proves decidability for formulas of prefix $\forall x \exists y_1, ..., y_m$, Skolem describes his decision procedure in schematic terms, avoiding a quantification over the totality of integers: "we recognize that it is then possible, for each new argument sequence of the individual variables ... to form a corresponding [solution] that is consistently compatible with the [solutions] set up for the earlier argument sequences" (p. 520). From this schematic solution, Skolem concludes immediately to the satisfiability of the formula.

But when Skolem reformulates the problem for a specific example, he includes the quantification:

> In the arithmetic formulation the problem is: Investigate whether it is possible to determine the function $A(x, y)$, whose values shall be restricted to 0 and 1, in such a way that for all $n, n = 0, 1, 2, \ldots, max(A(n, n+1), min(A(n, n), 1 - A(n+1, n+1))) = 1$.

Skolem goes on to show how the specific problem is solved "for all n" using the schematic procedure above, thus ignoring the apparent distinction between "all n" and "arbitrary high n".

For these reasons, little weight can be placed on Skolem's wording in the formulation of The Theorem.

Syntactical Reasoning

The language Skolem uses when talking about quantification is one of two arguments in support of Gödel's charge of finitism.[19] The other is Skolem's syntactical turn in the passage on pg. 519. This results in an argument which is indeed finitary in the sense that it appeals only to syntactical features of formulas as *finite* sequences of symbols. In this way, it avoids any applications of the law of excluded middle to infinite sets. However, the argument is opaque and, according to commentators, inconclusive. Alternatively, by appealing to the infinity lemma used in the LST, Skolem could have inferred directly from the existence of solutions for arbitrary high n to the satisfiability of the formula as he did in [Skolem 1922]. Gödel sees no other explanation for Skolem's turn to syntax than a desire (that was absent in [1922]) to avoid the lemma on account of its non-finitary character (in the sense explained in chapter 3, section 3.3.2).

I argue that Gödel's explanation disregards other reasons Skolem could have for not using the 1922 lemma in this particular context. Immediately following the syntactical argument, Skolem writes: "to be sure, the procedure ... is infinite, but can be made finite in certain cases". He goes on to describe these cases and thereby prove decidability for the formulas of prefix $\forall x \exists y_1, ..., y_m$. This proof presents the most telling evidence against Gödel's interpretation. —

5.3.2 Appealing to the infinity lemma to prove decidability

In the decidability proof for the class of formulas with prefix $\forall x \exists y_1, ..., y_m$, Skolem shows how his procedure $\mathcal{R}$ can be adapted to give a decision procedure for determining which of the disjuncts of the basic alternative is true: either for some n there are no solutions of level

[19]These arguments are made on behalf of Gödel who does not explicitly give any support for his allegation.

n, or, there are solutions for every n.[20]

The general idea of the procedure is as follows. Let $U = \forall x \exists y_1, ..., y_m F$. Recall that $\mathcal{R}$ searches for inconsistencies amongst the atomic propositions that are formed when the argument variables of propositional functions in F are replaced by ordered sequences of integers (Skolem's "argument sequences"). The resulting formulas are analyzed for truth-functional consistency by the methods of propositional rather than predicate logic. Argument sequences assigned to the variables determine the identities between atomic propositions and these constrain the assignments of truth values that constitute solutions to the formulas of a given level.

For example, if $B(x, x)$ and $B(y, y)$ are atomic propositional functions occurring in F, the assignment of integers to x and y will differ for each level, but the relevant feature of these assignments is whether the instantiation of $B(x, x)$ at one level is identical with the instantiation of $B(y, y)$ at another. This depends on whether the argument sequences of different levels[21] have common elements. In fact, Skolem's construction guarantees such common elements - integers are assigned to ensure that every new y-value introduced will become a value of x higher up.[22] The resulting duplications of atomic propositions across levels raise the possibility of contradiction and must be accommodated when determining consistent extensions of solutions from one level to the next.

What makes the decision problem tractable for formulas of prefix $\forall x \exists y_1, ..., y_m$ is that each argument sequence of level n ($n > 0$) will have only a single element in common with the argument sequences of level $n - 1$, and no elements in common with levels below $n - 1$. As a result, duplications of atomic propositions will only occur between consecutive levels. Consecutive levels will also have the same pattern of contradictions or consistencies amongst

[20] Each formula generates its own instance of this alternative, which is indexed to the construction of expansions A_n.

[21] Each level after the first will have multiple argument sequences.

[22] This is how the instances approximate to the quantified formula in the limit.

atomic propositions.[23]

For this reason it suffices to look for the first level n such that, of the paths through the tree of solutions that reach level n, all can be consistently extended to the next level. The pattern of consistency between the solutions at levels n and $n+1$ will then hold for any level $m > n$, contradictory assignments having already been eliminated at earlier levels.

The existence of such a level can be decided in finite time. All paths through the tree of solutions must start from one of finitely-many solutions of level 0. These solutions are progressively narrowed down as the paths terminate at higher levels. The method just described has two possible, mutually exclusive outcomes. One, it eliminates all level 0 solutions, reaching a level after which none of the level 0 solutions can be extended. Two, it reaches the level n described above where all the level 0 solutions with extensions through the nth level can be consistently extended through the $n+1th$ level. In the former case, the formula is contradictory; in the latter, it is possible to describe a solution that holds for the nth level formula, for arbitrary n. Decidability follows by the infinity lemma step taken in (Skolem, 1922) and the conflation mentioned in the previous section: if there are solutions for all n, then the original formula U has a solution defined by ordering the solutions at each level.[24]

Implications for Gödel's argument

This result cannot be ignored when assessing Skolem's motives in the syntactical passage. It challenges Gödel's explanation in two respects.

First, if Skolem is willing to use the 1922 lemma to prove decidability for the subset of

[23] For example, if $B(x, x)$ and $\neg B(y, y)$ are atomic functions occurring in F, a contradiction will arise at every consecutive pair of levels between the propositional values of these functions, because y at level n will equal x at level $n+1$.

[24] In the second situation, Skolem simply concludes: "The formula is consistent", using "consistent" here in the (for him) standard semantic sense of "satisfiable".

formulas with prefix $\forall x \exists y_1, ..., y_m$, why would he shy away from it for the theorem on pg. 519? Attributing this reluctance to Skolem's "finitistic prejudices" requires that these prejudices be confined, inexplicably, to a single result in the 1928 paper.[25] What is it about The Theorem in particular that explains a "finitistic" approach that is not upheld elsewhere in the paper?

Second, this is the only place in the paper where Skolem explicitly mentions anything to do with finitism. But the comment is not in the direction that supports Gödel's reading.

As quoted above, following the syntactical passage, Skolem writes: "to be sure, this procedure is infinite, but it can be made finite in certain cases". He then gives the proof of decidability discussed above for formulas with prefix $\forall x \exists y_1, ..., y_m$. (It is not clear *which* procedure Skolem is calling infinite in this quote, but most plausibly it is the procedure $\mathbb{R}$ rather than the procedure described in the syntactical passage. This is because the latter procedure *is* arguably finite, and furthermore, it is not this one but the procedure $\mathbb{R}$ that Skolem goes on to "make finite" for certain cases.)

Although "finitary procedure" was not a precise notion in 1928 (awaiting the definitions of recursive function and effective computability in the 1930s), the procedure Skolem gives for the restricted class of prefixes is what we would expect when using "finitary" in the intuitive sense required by the criteria of decidability, i.e., a procedure that can be executed by a human in a practicable number of steps. This is not the sense of non-finitary we distinguished in chapter 2 in connection with Gödel's proof of completeness. As analyzed in that chapter, the sense in which the LST, like Gödel's proof of completeness, is non-finitary is that its proof makes essential use of the law of excluded middle applied to infinite totalities.

[25] Varying degrees of rigor with respect to different results in a single paper are sometimes justified by differences in the nature of the results being proven. This defense on Gödel's behalf presupposes that Skolem *was* attempting something like a completeness proof, as distinct from the proof of decidability he gives immediately following the pg. 519 passage. If both proofs are conceived as decidability results, one a subset of the other (as I argue below), then the methodological rigor demanded by the nature of the result should be the same in both cases.

Skolem's comment does not fit well with the latter meaning of non-finitary, which applies to the nature of a proof (at the meta-level, as we would recognize it) rather than a procedure.

In the next section, I give a new argument in support of one of the interpretations of the passage on pg. 519 already discussed in Chapter 4. This interpretation is argued to consist with Skolem's aim of proving decidability and to better explain the two features just noted.

5.3.3 A new argument

Gödel's puzzle is premised on the idea that Skolem's Theorem is a completeness theorem. This interpretation is understandable in light of the fact that Skolem standardly uses "consistent" to mean "satisfiable". Skolem appears to equivocate on this meaning by arguing for syntactic consistency. Gödel naturally assumes that his intention was to argue for satisfiability, thereby proving completeness, but that philosophical considerations forced him to approach that theorem syntactically.

I argue instead that Skolem never intended to prove completeness, although it would have followed as a corollary of the decidability result he did intend to prove. Skolem's avoidance of the infinity lemma can be seen as motivated not by Gödel's sort of "finitistic prejudices", but by the recognition that the lemma is not helpful to prove decidability in the case of general prefixes. This is because the lemma works in application to the basic alternative, and in the case of general prefixes, there is no decision procedure for determining which of the disjuncts of this alternative holds (for simplicity, I will imprecisely refer to the latter as the "decidability of the basic alternative"). [26]

Of course, Skolem's search for a decision procedure in 1928 carries with it finitism of *one*

[26] The distinction between completeness and decidability, though obscured by Skolem's framework, reemerges in this avoidance of the infinity lemma. The lemma *can* be used to prove completeness, because it does not require deciding whether there exist solutions for all n - this can be simply assumed as the hypothesis.

sort, namely, the pre-existing but vague requirement that the procedure be executable in finite time. This sort of finitism is consistent with his turn to syntax to bypass deciding the basic alternative directly. If he could prove that each of the disjuncts of the basic alternative was equivalent with a syntactic counterpart, the latter may have seemed more tractable candidates for a decision procedure using the finitary character of syntax. This would indirectly enable us to decide the basic alternative. From there, Skolem *could* use the infinity lemma to prove that satisfiability is decidable, in the same way he does for the restricted classes of prefixes. Unfortunately for Skolem, this attempt is destined to fail. For general prefixes, determining which of the disjuncts of the basic alternative holds is an essentially undecidable problem because first-order logic is undecidable.

Decidability and the basic alternative

The decision procedure Skolem gives, with reference to the construction of expansions A_n, works in cases where argument sequences[27] of consecutive levels have a limited number of common elements, and those of non-consecutive levels do not have any common elements. The formulas for which the argument sequences meet these criteria are, roughly, those without multiple blocks of alternating quantifiers.

The method works by determining which atomic components can be assigned truth values independently of the assignments at other levels, and which have to be checked for consistency with assignments to the same components at lower levels. When common elements are restricted to consecutive levels, Skolem can show that it suffices to look for the first level n at which the existing paths from level 0 through level n can be consistently extended to level $n+1$. If such a level exists, it is straightforward to define a general solution to the formula for arbitrary n that is guaranteed not to encounter contradictions further up. As discussed,

[27] Recall that for a formula $U = \forall x \exists y_1, ..., y_m F$, the argument sequences are the sequences of integers that replace the argument variables of propositional functions in F to form expansion instances of U. Each sequence represents an assignment to the variables of a single instance of F.

Skolem takes this to imply a solution to the formula for all n, thereby deciding the basic alternative.

Skolem extends this method to formulas with any finite number of universal quantifiers followed by any finite number of existential quantifiers. But the method breaks down for more elaborate prefixes. With multiple alternating quantifier blocks, argument sequences may have elements in common with multiple arguments sequences across multiple levels. The rapid increase in complexity, and the necessity of treating each variation separately, would have undoubtedly motivated Skolem to search for a more manageable alternative.

A syntactic alternative

There is a natural analogy between the method of locating these propositional inconsistencies (an atomic and its negation) between the expansions of different levels, and the question Skolem asks according to the third interpretation of the passage on pg. 519. This is the interpretation on which Skolem aims to prove that adding the functional form A* of A as an axiom of a rudimentary system S does not result in syntactic inconsistency (see Chapter 4, Theorem 4.6).

In the reconstruction described in chapter 4, the functional form A* works as an axiom schema. In place of the individual conjuncts of expansions in Skolem's construction of A_n, we have the possible instances of A* formed by replacing the variables by individual terms. These terms are formed by arbitrary (finite) applications of the rule of substitution in functional terms.

I speculate that Skolem saw the idea of determining the syntactic consistency of A* in the system S as a way of approaching the insolubility of the basic alternative for general prefixes.

On this interpretation, Skolem's aim is to show:

- A* is syntactically consistent if there are solutions for all A_n
- A* is is syntactically inconsistent if there is a level n with no solutions

One direction may have seemed self-evident to Skolem. Notice that each A_n can be represented as a conjunction of instances of A* formed via the rule of substitution. When A* is added as an axiom, all such instances are implied. If we assume that the system S is sound (see below), then the syntactic consistency of A* would imply that none of the A_n are unsatisfiable.[28]

This leaves the converse implication, that the existence of solutions for all A_n implies the syntactic consistency of A* (contrapositive of the second implication above). I argue that this is what Skolem tries to prove in the passage on pg. 519 (see the reconstruction in Chapter 4).

If Skolem's proof were successful in showing that there are solutions for all A_n if and only if A* is syntactically consistent, this does not get us any closer to a decidability result. It simply poses the question from a different angle. However, if Skolem could go on to show that the syntactic consistency of A* in S was a decidable property, this would suffice to decide by proxy which of the disjuncts of the the basic alternative holds. Skolem could then appeal to the 1922 argument to prove that satisfiability is decidable.

Skolem, however, gives no indication of pursuing a decision procedure that would determine the syntactic consistency A* in S. Immediately following his attempted proof, he shifts attention to the cases for which "the procedure" can be made finite, referring to the method $\mathbb{R}$ of checking solutions until we either reach a level with no solutions, or one at which the solutions of that level coincide with those of the next. Perhaps this turn back to the original

[28] This is what Goldfarb means when he writes that "the connection between $\mathbb{R}$ and S is a triviality" (Goldfarb, 1971, p. 524). Once variables have been substituted for, derivation in S is a matter of using propositional rules to break down instances of A* into atomic components. If we can derive an atomic and its negation in this way, the formula is inconsistent. This is the same criterion by which $\mathbb{R}$ determines inconsistency in the *semantic* sense of unsatisfiability.

procedure $\mathbb{R}$ (which "is infinite" for the general case), indicates that Skolem had his own doubts about the viability of the syntactic approach absent a more specific framework in which to expound it.[29]

5.3.4 Summary

This section argued against the impression that Skolem in 1928 intends to reason "finitistically", understood in Gödel's sense as the intentional avoidance of the law of excluded middle applied to completed infinite totalities. Gödel's explanation does not hold up in the face of an alternative explanation for Skolem's reasoning in the passage on p. 519. The alternative I give is based on the interpretation of this passage put forward by Goldfarb. According to this interpretation, Skolem aims to show that adding the functional form A* of A as an axiom of a rudimentary system S does not result in syntactic inconsistency. This interpretation better explains the two features of Skolem's argument that raise problems for Gödel. First, there is no need to explain why Skolem would avoid the infinity lemma step in this particular case but not in others since we have argued that Skolem did not in fact have the qualms about its non-finitary character that Gödel alleges. Instead, Skolem simply never reached the point in the proof of decidability for the general prefixes where the infinity lemma would have been applicable. Second, his comment regarding the infinitary character of the procedure accurately represents the sense in which he *was* concerned with finitism. Not, as Gödel contends, as a requirement at the metalevel on the methods of available reasoning, but as a general (and at this point, still imprecise) requirement on object level decision procedures.[30]

[29] When Skolem returns to the idea 1929, he gives essentially the same argument but starts off by specifying the system we have called S above. This end result is no less opaque than the 1928 version, but the addition suggests some recognition that the 1928 proof is not fleshed out.

[30] To view it as such, there is no need to impute an understanding of this distinction to Skolem.

5.4 Conclusion

This chapter began by introducing contextual features conducive to the recognition of completeness as a significant property of logical systems. These features are notably absent for the pre-1930 context of Skolem. It was argued that Skolem had no reason to acknowledge completeness as a separate result, given its proximity to theorems that he had already proven (the Löwenheim-Skolem theorem) or was in the process of proving (decidability).

In part 2, I returned to the puzzle raised by Gödel. I argued against the impression that the contentious passage in (Skolem, 1928) is finitistic in the sense Gödel alleges. I then gave an interpretation of Skolem's (1928) argument that takes account of his actual context, specifically, his aim of finding a procedure to decide the validity of all first-order logic formulas. Unlike Gödel's explanation, mine does not come into conflict with Skolem's use of non-finitary reasoning elsewhere in his work, is maximally charitable to Skolem's intentions, and sheds new light on the relation between decidability and completeness in the pre-1930 context.

Bibliography

AVIGAD, JEREMY

[2010] Gödel and the metamathematical tradition. Retrieved from http://www.andrew.cmu.edu/user/avigad/Papers/goedel.pdf

ASPRAY, WILLIAM AND KITCHER, PHILIP (EDS.)

[1988] History and philosophy of modern mathematics. Minneapolis: University of Minnesota Press.

BADESA, CALIXTO.

[2004] The Birth of Model Theory: Löwenheim's Theorem in the Frame of the Theory of Relatives. Princeton University Press, 2004.

BAYS, TIMOTHY

[2001] On Putnam and his Models, *The Journal of Philosophy*, Vol. 98, pp. 331–50.

[2007a] The Mathematics of Skolem's Paradox, in Jacquette (ed.) [2007], pp. 615-648.

[2007b] More on Putnam's Models: A Response to Bellotti, *Erkenntnis*, Vol. 67, pp. 119–135.

[2008] Two Arguments against Realism, *The Philosophical Quarterly*, Vol. 58, pp. 193–213.

BAUER, ANDREJ

[2019] Nonconstructive reasoning in Skolem's proof of the Löwenheim-Skolem Theorem, URL (version: 2019-10-22): https://mathoverflow.net/q/344342.

BELLOTI, L.

[2005] Putnam and Constructibility, *Erkenntnis*, Vol. 62, pp. 395–409.

[2006] Skolem, the Skolem 'Paradox' and Informal Mathematics, *Theoria*, Vol. 72, pp. 177–220.

BÖERGER, EGON, ERICH GRÄDEL, AND YURI GUREVICH

[1997] The Classical Decision Problem. Heidelberg: Springer.

BRADY, GERALDINE

[1997] From the algebra of relations to the logic of quantifiers. In Houser and van Evra (eds.) [1997], pp. 173–192.

[2000] From Peirce to Skolem: A Neglected Chapter in the History of Logic. Amsterdam: Elsevier.

BROWN, FRANK

[2009] George Boole's deductive system. *Notre Dame Journal of Formal Logic,* Vol. 50, Number 3, pp. 303-330.

BURTON DREBEN

[1952] On the completeness of quantification theory. *Proceedings of the National Academy of Sciences of the United States of America*, vol. 38, pp. 1047–1052.

DREBEN, BURTON AND VAN HEIJENOORT, JEAN

[1986] Introductory note to 1929, 1930, and 1930a. In Gödel [1986], pp. 44–59.

EBBINGHAUS, HEINZ-DIETER

[2007] Ernst Zermelo: An Approach to His Life and Work. Berlin Heidelberg: Springer-Verlag.

FEFERMAN, SOLOMON

[2008] Lieber Herr Bernays!, Lieber Herr Gödel! Gödel on finitism, constructivity and Hilbert's program. *Dialectica, 62*(2), pp. 179-203.

FEFERMAN, SOLOMON AND ALFRED TARSKI.

[1953] Review. H. Rasiowa and R. Sikorski. A proof of the Skolem-Löwenheim theorem. Fundamenta mathematicae, Vol. 38, pp. 230–232. *Journal of Symbolic Logic, 18*(4), 339-340.

FENSTAD, JENS ERIK

[1996] Thoralf Albert Skolem 1887-1963: A Biographical Sketch. *Nordic Journal of Philosophical Logic* Vol. 1, pp. 99-106.

FENSTAD, JENS ERIK AND HAO WANG

[2009] Thoralf Albert Skolem. In Dov M. Gabbay and John Woods (eds.) [2009], pp. 127-194.

FRANCHELLA, MIRIAM

[1997] On the Origins of Dénes König's Infinity Lemma. *Archive for History of Exact Sciences,* Vol. 51, pp. 3-27.

DOV M. GABBAY AND JOHN WOODS (EDS.)

[2009] Logic from Russell to Church, Volume 5, 1st Edition. North Holland: Elsevier.

GAUTIER, YVONNE

[1994] Hilbert and the Internal Logic of Mathematics. *Synthese,* Vol. 101, pp. 1-14.

GEORGE, ALEXANDER

[1985] Skolem and the Löwenheim-Skolem Theorem: A Case Study of the Philosophical Significance of Mathematical Results. *History and Philosophy of Logic*, 6, pp. 75-89.

Gödel, KURT

[1929] Über die Vollständigkeit des Logikkalküls . Doctoral dissertation, University of Vienna. Printed and translated into English in G¨odel [1986], pp. 60–101.

[1930] Die Vollständigkeit der Axiome des logischen Funktionenkalküls. Monatshefte für Mathemaik und Physik , v. 37, pp. 349–360. Reprinted and translated into English in Gödel [1986], pp. 102–123.

[1964] Letter to Van Heijenoort. 14 August 1964. In Gödel [1986].

[1986] Collected works. Vol I: Publications 1929–1936. Edited by Solomon Feferman, John W. Dawson Jr., Stephen C. Kleene, Gregory H. Moore, Robert M. Solovay, and Jean van Heijenoort. Oxford: Oxford University Press.

GEORGE, ALEXANDER

[1985] Skolem and the Löwenheim-Skolem Theorems, *History and Philosophy of Logic*, Vol. 6, pp. 75–89.

GIAQUINTO, M.

[2002] *The Search for Certainty*, Oxford: Oxford University Press.

GOLDFARB, WARREN D.

[1971] Review of Skolem 1970. *The Journal of Philosophy* 68, pp. 520-530.

[1979] Logic in the Twenties: The Nature of the Quantifier. *Journal of Symbolic Logic,* 44, pp. 351-368.

HERBRAND, JACQUES

[1930] Recherches sur la théorie de la démonstration . Thesis at the University of Paris. Reprinted in Herbrand [1968], pp. 35–153. English version in Herbrand [1968], pp. 44–202. Chap. 5 is translated into English in van Heijenoort (ed.) [1967], pp. 525–581.

[1931] Unsigned note on Herbrand's thesis written by Herband himself. In [1971], pp. 271-276.

[1968] Écrits logiques. Paris: Presses Universitaires de France.

[1971] Logical writings. Warren Goldfarb (ed). Translation of Herbrand [1968]. Cambridge: Harvard University Press.

HESSELING, DENNIS E.

[2003] Gnomes in the Fog: The Reception of Brouwer's Intuitionism in the 1920s. Basel, Birkhäuser, 448 pages.

HINTIKKA, J.

[1998] Truth Definitions, Skolem Functions and Axiomatic Set Theory. *The Bulletin of Symbolic Logic, 4*, pp. 303-337.

HOUSER, NATHAN AND VAN EVRA, JAMES W. (EDS.)

[1997] Studies in the logic of Charles Sanders Peirce. Bloomington: Indiana University Press.

JACQUETTE, DALE (ED.)

[2007] *Philosophy of Logic*, London: Elsevier.

JANÉ, IGNACIO

[2001] Reflections on Skolem's Relativity of Set-Theoretical Concepts, *Philosophia Mathematica*, Vol. 3, pp. 129–153.

KNEALE, WILLIAM AND KNEALE, MARTHA

[1962] The development of logic. Oxford: Clarendon Press.

König, DÉNES

[1926] Sur les correspondances multivoques des ensembles. *Fundamenta Mathematicae,* 8, pp. 114-134.

[1927] Über eine Schlussweise aus dem Endlichen ins Unendliche. Acta litterarum ac scientiarum Regiae Universitatis Hungaricae Francisco-Josephinae, Sectio scientiarum mathematicarum , vol. 3, pp. 121–130.

LANGFORD, C. H.

[1926] Some theorems on deducibility, *Annals of mathematics,* Vol. 2, p. 16-40.

[1927] On propositions belonging to logic, *Mind,* Vol. 36, p. 342-346.

LARGEAULT, JEAN (ED.)

[1972] Logique mathématique: Textes. Paris: Librairie Armand Colin.

Löwenheim, LEOPOLD

[1911] Rezension von Ernst Schröder, Abriss der Algebra der Logik, zweiter Teil. Archiv der Mathematik und Physik , v. 17, pp. 71–73. [1915] Über Möglichkeiten im Relativkalkül. Mathematische Annalen , v. 76, pp. 447–470. English translation in van Heijenoort (ed.) [1967], pp. 228–251.

[1915] On Possibilities in the Calculus of Relatives, in van Heijenoort (ed.) [1967], pp. 228–251.

[1940] Einkleidung der Mathematik in Schröderschen Relativkalkul. Journal of Symbolic Logic , v. 5, pp. 1–15.

[1971] Malʹt͡sev, A. I., and Benjamin F. Wells. *The Metamathematics of Algebraic Systems, Collected Papers: 1936-1967.* Amsterdam: North-Holland Pub. Co, 1971. Print.

MANDELKERN, MARK

[1988] Limited Omniscience and the Bolzano-Weierstrass Principle*, Bulletin of the London Mathematics Society,* Vol. 20, pp. 319-320.

MARION, MATHIEU

[1991] Quantification and Finitism: a Study in Wittgenstein's Philosophy of Mathematics, (dissertation). Retrieved from:

MCCARTY, CHARLES, AND NEIL TENNANT

[1987] Skolem's Paradox and Constructivism. *Journal of Philosophical Logic* 16, no. 2, pp. 165-202. http://www.jstor.org/stable/30227199.

MCINTOSH, C.

[1979] Skolem's Criticisms of Set Theory, *Nous*, Vol. 13, pp. 313–334.

MILNE, PETER

[1999] Tarski, Truth and Model Theory. *Proceedings of the Aristotelian Society, New Series, 99*, pp. 141-167

MOORE, G. H.

[1980] Beyond first-order logic: The historical interplay between mathematical logic and axiomatic set theory. History and Philosophy of Logic, v. 1, pp. 95–137.

[1988] The emergence of first-order logic. In W. Aspray and P. Kitcher (eds.) [1988], pp. 95–135.

[1997] Hilbert and the emergence of modern mathematical logic. Theoria , v. 12, pp. 65–90.

MULLER, F. A.

[2005] Deflating Skolem, *Synthese*, Vol. 143, pp. 223–253.

MYHILL, JOHN

[1967] On the Ontological Significance of the Löwenheim-Skolem Theorem, in *Contemporary Readings in Logical Theory*, I. Copi and J. Gould, (eds.), New York: Macmillan, pp. 40–51.

POST, EMIL

[1921] Introduction to a general theory of elementary propositions. *American Journal of Mathematics, 43*, pp. 163–185.

QUINE, W. V.

[1955] A proof procedure for quantification theory. Journal of Symbolic Logic , v. 20, pp. 141–149. Reprinted in Quine [1995], pp. 196–211.

[1972] The methods of logic; 3rd edition. New York: Holt, Rinehart & Winston.

[1995] Selected logic papers; 2nd edition. Cambridge: Harvard University Press.

RESNIK, MICHAEL

[1966] On Skolem's Paradox, *The Journal of Philosophy*, Vol. 63, pp. 425–438.

[1969] More on Skolem's Paradox, *Noûs*, Vol 3, pp. 185–196.

SKOLEM, THORALF

[1920] Logisch-kombinatorische Untersuchungen über die Erfüllbarkeit oder Beweisbarkeit mathematischer Sätze nebst einem Theoreme über dichte Mengen. Videnskapsselskapets skrifter, I. Matematisk-naturvidenskabelig klasse , no. 4, pp. 1–36. English translation in van Heijenoort (ed.) [1967], pp. 252–263.

[1923a] Einige Bemerkungen zur axiomatischen Begründung der Mengenlehre. *Matematikerkongressen i Helsingfords den 4–7 Juli 1922*, *Den femte skandinaviska matematikerkongressen*, Redogörelse . Helsinki: Akademiska Bokhandeln, 1923, pp. 217–232. English translation in van Heijenoort (ed.) [1967], pp. 290–231.

[1923] Begriindung der elementaren Arithmetik durch die rekurrierende Denkweise ohne Anwendung scheinbarer Veranderlichen mit unendlichem Ausdehnungsbereich, Skrifter utgit av Videnskapsselskapet i Kristiania, I. Matematisk-naturvidenskabelig klasse, no. 6, 1-38; reprinted in Skolem 1970, 153-188; English translation by Stefan Bauer-Mengelberg in van Heijenoort (ed.) [1967], pp. 302-333.

[1928] Über die mathematische Logik. Norsk matematisk tidsskrift , v. 10, pp. 125–142. English translation in van Heijenoort (ed.) [1967], pp. 508–524.

[1929] Über einige Grundlagenfragen der Mathematik. Skrifter utgitt av Det Norske Videnskaps–Akademi i Oslo, I. Matematisk-naturvidenskapelig klasse , no. 4, pp. 1–49. In Skolem [1970], pp. 227–273.

[1938b] Sur la portée du théoreme de Löwenheim-Skolem. In *Les entretiens de Zurich sur les fondements et la méthode des sciences mathématiques,* 6–9 1938 , edited by Ferdinand Gonseth. Zurich: 1941, pp. 25–52. In Skolem [1970], pp. 455–482.

[1944] Some remarks on the comparison between recursive functions. *Det Kongelige Norske Videnskabers Selskab, Forhandlinger*, vol. 17, pp. 126–129.

[1950] Some remarks on the foundation of set theory. *Proceedings of the International Congress of Mathematicians, Cambridge, Massachusetts, U.S.A., August 30-September 6, 1950*, American Mathematical Society, Providence Vol. I, pp. 695–704.

[1955a] A critical remark on foundational research. *Kongelig Norske Videnskabsselskabs Forhandlinger*, Trondheim, XXVIII, no. 20, pp. 100-105. Reprinted in Skolem [1970] pp. 581-586.

[1955b] Peano's axioms and models of arithmetic.
Studies in Logic and the Foundation of Mathematics, pp. 1-14.
Reprinted in Skolem [1970], pp. 587-600.

[1958] *Une relativisation des notions mathematiques fondamentales*. In *Colloques internationaux des Centre National de la Recherche Scientique*, Paris 1958, pp. 13-18. Reprinted in Skolem [1970], pp. 633-638.

[1970] *Selected works in logic* . Edited by Jens Erik Fenstad. Oslo: Universitetsforlaget

TAIT, WILLIAM W.

[1968] Constructive Reasoning", in van Rootselaar, B., Stall, J. R, eds., *Logic, Methodology and Philosophy of Science III*, Amsterdam, North-Holland, pp. 185-199.

[1981] Finitism, *Journal of Philosophy*, Vol. 78, pp. 524-546

[1983] Against Intuitionism: Constructive Mathematics is part of Classical Mathematics, in *Journal of Philosophical Logic,* Vol. 12, pp. 173- 195.

[2010a] Gödel on Intuition and on Hilbert's finitism, In S. Feferman, C. Parsons, & S. Simpson (Eds.), *Kurt Gödel: Essays for his Centennial*, Cambridge: Cambridge University Press, pp. 88-108.

[2010b] Primitive Recursive Arithmetic and its Role in the Foundations of Arithmetic: Historical and Philosophical Reflections, *Epistemology versus Ontology: Essays on the Philosophy and Foundations of Mathematics in Honour of per Martin-Lof,* pp. 161-180.

TARSKI, ALFRED AND ROBERT L. VAUGHT

[1957] Arithmetical extensions of relational systems, *Compositio Mathematica,* Vol. 13, pp. 81-102

TENNANT, NEIL AND McCARTY, CHARLES

[1987] Skolem's Paradox and Constructivism, *The Journal of Philosophical Logic*, Vol. 16, pp. 165–202.

THIEL, CHRISTIAN

[1977] *Leopold Löwenheim: life, work, and early influence.* In Logic Colloquium 76. R. O. Gandy and M. Hyland (eds.). Amsterdam: North-Holland. 235–252.

THOMAS, WILLIAM J.

[1968] Platonism and the Skolem Paradox, *Analysis*, Vol 28, pp. 193–196.

WANG, HAO

[1970] A survey of Skolem's work in logic . In Skolem [1970], pp. 17–52.

[1974] From mathematics to philosophy. London: Routledge & Kegan Paul.

[1996] Skolem and Gödel. *Nordic Journal of Philosophical Logic,* Vol. 1, No. 2, pp. 119-132.

VAN DALEN, D.

[1974] Review of Jacques Herbrand: Logical Writings. by Warren D. Goldfarb. *The Journal of Philosophy, 71*(15), 544-549.

VAN HEIJENOORT, JEAN

[1967a] Logic as calculus and logic as language. In *Boston studies in the philosophy of science 3* , edited by Robert S. Cohen and M. W.Wartofsky. Dordrecht: Reidel, pp. 440–446.

[1974] Historical development of modern logic. Edited by Irving H. Anellis: *Modern Logic* , v. 3, 1992, pp. 242–255.

[1977] Set-theoretics semantics. In *Logic Colloquium* 76 , edited by R. O. Gandy and M. Hyland. Amsterdam: North Holland, pp. 183–190.

[1982] L'oeuvre logique de Jacques Herbrand et son contexte historique. In *Proceedings of the Herbrand symposium: Logic colloquium* '81. Amsterdam: North-Holland, pp. 57–85.

VAN HEIJENOORT, JEAN (ED.)

[1967b] From Frege to Gödel; a source book in mathematical logic, 1879-1931. Cambridge, Harvard University Press.

VAUGHT, R. L.

[1974] Model theory before 1945. In *Proceedings of the Tarski Symposium,* edited by L. Henkin, J. Addison, Ch. Ch. Chang, D. Scott, R. L. Vaught, and W. Craig. *Proceedings of Symposia in Pure Mathematics,* no. 25. Providence: American Mathematical Society, pp. 153–172.

VON PLATO, JAN

[2007] In the Shadows of the Löwenheim-Skolem Theorem: Early Combinatorial Analyses of Mathematical Proofs. *The Bulletin of Symbolic Logic, 13*(2), 189-225.

[2014] Generality and existence: quantificational logic in historical perspective. *The Bulletin of Symbolic Logic,* vol. 20, no. 4, pp. 417-448.

[2017] The Point of Constructivity. In *The Great Formal Machinery Works: Theories of Deduction and Computation at the Origins of the Digital Age*. Princeton, Oxford: Princeton University Press, pp. 140-184.

WHITNEY, HASSLER

[1933] Characteristic functions and the algebra of logic. *Annals of Mathematics,* vol. 34, pp. 405–14.

ZACH, RICHARD

[1999] Completeness before Post: Bernays, Hilbert, and the Development of Propositional Logic. *The Bulletin of Symbolic Logic,* vol. 5, no. 3, pp. 331-366.