University of California
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Hamiltonian Systems and Gibbs Measures

A dissertation submitted in partial satisfaction
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Doctor of Philosophy in Mathematics

by

Samantha Xu

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Abstract of the Dissertation

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Professor Rowan Killip, Chair

Consider the radial nonlinear wave equation $-\partial_t^2 u + \Delta u = u^3$, $u : \mathbb{R}_t \times \mathbb{R}_x^3 \to \mathbb{R}$, $u(t,x) = u(t,|x|)$. In this thesis, we construct a Gibbs measure for this system and prove its invariance under the flow of the NLW. In particular, we are in the infinite volume setting.

For the finite volume analogue, specifically on the unit ball with zero boundary values, an invariant Gibbs measure was constructed by Burq, Tvetkov, and de Suzzoni [9, 12] as a Borel measure on super-critical Sobolev spaces.

We first show that this finite volume Gibbs measure is supported on a space of weighted Hölder continuous functions. Next, we show that the NLW is locally well-posed there, a counter-point to the Sobolev super-criticality noted by Burq and Tzvetkov. Furthermore, the flow of the NLW leaves this measure invariant.

We use a multi-time Feynman–Kac formula to construct the infinite volume limit measure by computing the asymptotics of the fundamental solution of an appropriate parabolic PDE. We use finite speed of propagation and results from descriptive set theory to establish invariance of the infinite volume measure.

The dissertation of Samantha Xu is approved.

Terence Tao

Frederic (Rick) Paik Schoenerg

James Ralston

Rowan Killip, Committee Chair

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To Erik Walsberg,

thank you for you

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Vita

2008	B.S. (Mathematics), University of Arizona
2011	M.S. (Mathematics), UCLA, Los Angeles, California.

Publications

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CHAPTER 1

Introduction

In this thesis, we consider the 3D radial, defocusing, cubic, nonlinear wave equation (NLW)

$$\begin{cases} -\partial_t^2 u + \Delta u = u^3, \\ u : \mathbb{R}_t \times \mathbb{R}_x^3 \to \mathbb{R}, \\ u(t,x) = u(t,|x|) = u(t,r) \end{cases}$$

with Hamiltonian $H(u) = \int_{\mathbb{R}^3} \frac{1}{4}|u|^4 + \frac{1}{2}|\nabla u|^2 + \frac{1}{2}|u_t|^2$. We wish to construct the Gibbs measure, which we informally write as

$$\begin{aligned} &\frac{1}{Z} e^{-H(u)} d\text{“}(u, u_t)\text{”} \\ &\quad \text{“}:=\text{”} \frac{1}{Z_1} \exp\left(- \int_{\mathbb{R}^3} \frac{|u|^4}{4} + \frac{|\nabla u|^2}{2} \, dx \right) \text{“}du\text{”} \otimes \frac{1}{Z_2} \exp\left(- \int_{\mathbb{R}^3} \frac{|u_t|^2}{2} \, dx \right) \text{“}du_t\text{”} \end{aligned}$$

and show that the flow of the NLW is defined for all time on the support of the Gibbs measure, and leaves this measure invariant.

This problem is a specific case of the investigation of invariant Gibbs measures on the phase space of various differential equations. This area has seen an increasing focus of research in recent years, partially due to the fact that Gibbs measure allow for global existence results at regularities which are not available in the deterministic setting.

We introduce this area via a sequence of examples, beginning with the classical case of Liouville's theorem.

1.1 The Finite Dimensional Case

Theorem 1.1 (Liouville)**.** *Let $H = H(p_1, \ldots, p_d, q_1, \ldots, q_d) \in C^2(\mathbb{R}^{2d})$. Then the flow of*

$$\frac{dp_k}{dt} = \frac{\partial H}{\partial q_k} \quad \text{and} \quad \frac{dq_k}{dt} = -\frac{\partial H}{\partial p_k}, \quad \text{for } k = 1, \ldots, d \tag{1.1}$$

preserves the Lebesgue measure $\prod_{k=1}^d dp_k dq_k$ on $\mathbb{R}^{2d}$.

Furthermore, the partially normalized Gibbs measure $\exp(-H) \prod_{k=1}^d dp_k dq_k$ is invariant under the flow given by (1.1).

Remark. If $Z := \int_{\mathbb{R}^{2d}} \exp(-H) \prod_{k=1}^d dp_k dq_k < \infty$, then the Gibbs measure

$$\frac{1}{Z} \exp(-H) \prod_{k=1}^{d} dp_k dq_k$$

is also preserved by (1.1).

Proof. We write $\vec{x} \in \mathbb{R}^{2d}$ in coordinates as $\vec{x} = (p_1, \ldots, p_d, q_1, \ldots, q_d)$. Then, equality of mixed partials shows that the vector field

$$\vec{V}(p_1, \ldots, p_d, q_1, \ldots, q_d) := \left(\frac{\partial H}{\partial q_1}, \ldots, \frac{\partial H}{\partial q_d}, -\frac{\partial H}{\partial p_1}, \ldots, -\frac{\partial H}{\partial p_d} \right)$$

on $\mathbb{R}^{2d}$ is divergence free.

Let $B \subseteq \mathbb{R}^{2d}$ be a bounded, open set with smooth boundary. For $t \in \mathbb{R}$, denote by B_t the flow of B under (1.1) at time t. Writing $|B_t|$ for Lebesgue measure of B_t, then the divergence theorem gives

$$\frac{d|B_t|}{dt} = \int \cdots \int_{\partial B_t} \vec{V} \cdot \vec{n} \, dS = \int \cdots \int_{B_t} \nabla \cdot \vec{V} \prod_{j=1}^{d} dp_j dq_j = 0,$$

where $\vec{n}$ denotes the unit normal vector, ∂B_t denotes the boundary of B_t, and dS denotes the induced surface measure on ∂B_t. Namely, the flow with respect to a divergence free vector preserves Lebesgue measure. It follows that $\prod_{k=1}^{d} dp_k dq_k$ is invariant under (1.1).

Furthermore, applying the chain rule and (1.1) shows

$$\frac{dH}{dt} = \sum_{k=1}^{d} \left(\frac{\partial H}{\partial p_k} \frac{dp_k}{dt} + \frac{\partial H}{\partial q_k} \frac{dq_k}{dt} \right) = \sum_{k=1}^{d} \left(\frac{\partial H}{\partial p_k} \frac{\partial H}{\partial q_k} - \frac{\partial H}{\partial q_k} \frac{\partial H}{\partial p_k} \right) = 0.$$

This proves the second assertion, as the Radon–Nikodym derivative $\exp(-H)$ is also invariant under (1.1). □

We illustrate Liouville's theorem with some simple examples.

The most trivial example of a Hamiltonian ODE is $H \equiv 0$. In this case, the flow of (1.1) leaves each point fixed. Clearly, every Borel measure on $\mathbb{R}^{2d}$ is preserved by this flow.

Another example is the Hamiltonian $H : \mathbb{R}^2 \to \mathbb{R}$ given by

$$H(p, q) = \tfrac{1}{2} \left(p^2 + q^2 \right).$$

By Liouville's theorem, the flow of the ODE

$$\frac{dp}{dt} = \frac{\partial H}{\partial q} = q, \qquad\qquad \frac{dq}{dt} = -\frac{\partial H}{\partial p} = -p$$

preserves Lebesgue measure on $\mathbb{R}^2$. Indeed, another way to recognize this result is as follows: the flow defined above is a rotation in $\mathbb{R}^2$ about the origin. In particular, for fixed $t \in \mathbb{R}$, the flow is a linear, orthogonal transformation on $\mathbb{R}^2$ and thus preserves Lebesgue measure. Furthermore, the Gibbs measure for this system is

$$\frac{1}{Z} \exp(-H(p, q)) dp dq = \frac{1}{2\pi} \exp \left(-\frac{1}{2} \left(p^2 + q^2 \right) \right) dp dq,$$

which is also the standard Gaussian measure on $\mathbb{R}^2$. In the following section, we observe an infinite dimensional version of this system.

1.2 A Simple Infinite Dimensional Case

Let $L > 0$ and let $B(0, L) \subseteq \mathbb{R}^3$ denote the ball in $\mathbb{R}^3$ of radius L and with center at 0. Let $\Delta_L = \Delta_{L,Dir}$ denote the Laplacian on $B(0, L)$ with Dirichlet boundary conditions.

In this section, we consider the linear wave equation

$$\begin{cases} -\partial_t^2 u + \Delta_L u = 0, \\ u : \mathbb{R}_t \times B(0, L) \to \mathbb{R}, \\ u(t, x) = u(t, |x|) = u(t, r), \\ u|_{\mathbb{R}_t \times \partial B(0,L)} \equiv 0 \end{cases} \tag{1.2}$$

and its associatied Hamiltonian,

$$H = H(u, u_t) = \int_{B(0,L)} \frac{1}{2}|\nabla u|^2 + \frac{1}{2}|u_t|^2. \tag{1.3}$$

We would like to make sense of the Gibbs measure

$$\begin{aligned} dM_L(u, u_t) &:= \frac{1}{Z} \exp(-H) \text{“}d(u, u_t)\text{”} \\ &= \frac{1}{Z} \exp\left(-\int_{B(0,L)} \frac{1}{2}|\nabla u|^2\right) \text{“}du\text{”} \otimes \frac{1}{Z'} \exp\left(-\int_{B(0,L)} \frac{1}{2}|u_t|^2\right) \text{“}du_t\text{”} \end{aligned} \tag{1.4}$$

in a manner similar to the previous section. An initial obstacle is the fact that Lebesgue measure does not exist in infinite dimensions, which is why we write du and du_t in quotes. Specifically, we observe the following result.

Proposition 1.2. *Let X be a separable, infinite dimensional Banach space and let μ be a translation-invariant, positive, Borel measure on X. If $\mu(B) < \infty$ for* *any ball $B \subseteq X$ of finite radii, then μ is the trivial measure.*

Proof. Let $B \subseteq X$ be a ball of radius R such that $\mu(B) < \infty$. Let us denote by $B/2$ the ball with the same center as B, but with radius $R/2$. Since X is infinite dimensional, there exists a sequence $\{p_j\}_{j \geq 1} \subseteq B/2$ such that

$$\|p_j - p_k\| \geq R/4$$

for all $j \neq k$, by Riesz's lemma (cf., [35, pg. 47]). The infinite sequence of balls $\{B(p_j, R/100)\}_{j \geq 1}$ are pair-wise disjoint and

$$\bigcup_{j=1}^{\infty} B(p_j, R/100) \subseteq B. \tag{1.5}$$

By translation invariance of μ, we have

$$\mu(B(p_1, R/100)) = \mu(B(p_2, R/100)) = \cdots .$$

Given this equality, as well as the assumption $\mu(B) < \infty$, the hypothesis that μ is a positive measure, and (1.5), we have

$$0 = \mu(B(p_1, R/100)) = \mu(B(p_2, R/100)) = \cdots . \tag{1.6}$$

Let $\{x_n\}_{n=1}^{\infty}$ be a countable dense subset of X. By (1.6) and translation invariance of μ, we have

$$0 = \mu(B(x_1, R/100)) = \mu(B(x_2, R/100)) = \cdots .$$

Since $X = \bigcup_{n=1}^{\infty} B(x_n, R/100)$ and μ is a positive measure, we have

$$\mu(X) = 0$$

and that μ is the trivial measure. $\square$

The second obstacle is that, even if we have a "Lebesgue measure" on an infinite dimensional space, the Radon–Nikodym derivative $\exp(-H)$ would be almost surely 0 (which in turn forces the normalization constants to be infinite). However, we may make sense of the aggregate expression in (1.4) as a Gaussian measure. The key observation is that we can diagonalize Δ_L in this setting.

1.2.1 A Free Gibbs Measure as a Gaussian Measure

Recall that the radial Laplacian can also be written as $\Delta = \partial_r^2 + \frac{2}{r}\partial_r$ and that the normalized Lebesgue measure on $B(0, L)$ is given by $\frac{1}{4\pi} r^2 dr d\sigma_{S^2}$. Consider the following orthonormal basis for $L^2_{rad}(B(0, L))$ consisting of eigenfunctions of Δ_L:

$$f_{n,L}(r) := \frac{\sqrt{2/L}\sin(n\pi r/L)}{r}, \qquad n = 1, 2, \dots \tag{1.7}$$

with eigenvalues $(n\pi/L)^2$, $n = 1, 2, \dots$, respectively.

Let us express radial functions on $B(0, L)$ via their Fourier expansion: for example, $u \in L^2_{rad}(B(0, L))$ is written $u = \sum_{n=1}^{\infty} x_n f_{n,L}$ with $\{x_n\} \in \ell^2$. The expression in (1.4) can then be formally interpreted as

$$\frac{1}{Z}\exp\left(-\int_{B(0,L)} \frac{1}{2}|\nabla u|^2\right) \text{“}du\text{”} \approx \frac{1}{Z}\exp\left(-\frac{1}{2}\sum_{n=1}^{\infty}(n\pi/L)^2|x_n|^2\right)\prod_{n=1}^{\infty} dx_n$$

and

$$\frac{1}{Z'}\exp\left(-\int_{B(0,L)} \frac{1}{2}|u_t|^2\right) \text{“}du_t\text{”} \approx \frac{1}{Z'}\exp\left(-\frac{1}{2}\sum_{n=1}^{\infty}|x_n|^2\right)\prod_{n=1}^{\infty} dx_n$$

The latter expression can be simply read as the standard Gaussian measure on an infinite dimensional Hilbert space, while the former include the Fourier multipliers arising from taking a derivative. It follows that a natural setting to make a rigorous definition of (1.4) is as a measure on Sobolev spaces.

Definition 1.3. For each $s \in \mathbb{R}$, consider the radial, homogeneous Sobolev space

$$\dot{H}^s_{rad,0}(B(0,L) \to \mathbb{C}) := \left\{ g = \sum_{n=1}^{\infty} c_n f_{n,L} \;\middle|\; c_n \in \mathbb{C}, \|g\|^2_{\dot{H}^s_{rad,0}} := \sum_{n=1}^{\infty} |n\pi/L|^{2s} |c_n|^2 < \infty \right\},$$

and $\dot{H}^s_{rad,0}(B(0,L) \to \mathbb{R})$, the sub-space with real coefficients. We equip these spaces with the usual Borel σ-algebra.

Letting $s < \frac{1}{2}$, we define the measure M_L in (1.4) to be the image measure on

$$\dot{H}^s_{rad,0}(B(0,L) \to \mathbb{R}) \times \dot{H}^{s-1}_{rad,0}(B(0,L) \to \mathbb{R})$$

under the map

$$\omega \mapsto \left(\sum_{n=1}^{\infty} \frac{a_n(\omega)}{n\pi/L} f_{n,L}(r), \sum_{n=1}^{\infty} b_n(\omega) f_{n,L}(r) \right) \qquad a_n(\omega), b_n(\omega) \sim \mathcal{N}_{\mathbb{R}}(0,1) \text{ i.i.d.} \tag{1.8}$$

where ω is an element of some suitable probability space $(\Omega, \mathcal{F}, P)$ and a_n, b_n are i.i.d. standard Gaussian random variables.

Indeed, the restriction $s < \frac{1}{2}$ is due to the fact that

$$\mathbb{E}^{\omega} \left[\Big\| \sum_{n=1}^{\infty} \frac{a_n(\omega)}{n\pi/L} f_{n,L}(r) \Big\|^2_{\dot{H}^s_{rad,0}} + \Big\| \sum_{n=1}^{\infty} b_n(\omega) f_{n,L}(r) \Big\|^2_{\dot{H}^{s-1}_{rad,0}} \right] < \infty$$

if and only if $s < \frac{1}{2}$ and so M_L is actually supported on such Sobolev spaces.

In the next two sections, we discuss changes of variables that will reappear throughout this thesis, as well as how change of variables aid in proving invariance of measures on infinite dimensional spaces of functions. In particular, we prove invariance of the measure above in Proposition 1.6.

1.2.2 Reduction to One Dimensions and Complexification

At this point, we make several remarks to motivate our changes of variables. First, since we are restricting our attention to radially symmetric functions, the phenomena that we observe is essentially one dimensional.

Also, we observe that M_L is not quite supported on a space of functions. Indeed, $\dot{H}^s$ for $s < -\frac{1}{2}$ is only defined as a space of distributions. Ultimately, we wish to work on a space of (equivalence classes of) functions, and possibly even on a space where we may evaluate the functions at a point. To this end, we complexify the wave equation, and apply the corresponding change of variables to M_L as well.

First, let us recall our 3D linear wave equation

$$\begin{cases} -\partial_t^2 u + \Delta_L u = 0, \\ u : \mathbb{R}_t \times B(0,L) \to \mathbb{R}, \\ u(t,x) = u(t,r), r = |x| \\ u|_{\mathbb{R}_t \times \partial B(0,L)} \equiv 0 \end{cases}$$

The change of variables $v = ru$ is bijective correspondence to solutions of the the 1D linear wave equation

$$\begin{cases} -\partial_t^2 v + \partial_r^2 v = 0, \\ v : \mathbb{R}_t \times [0,L] \to \mathbb{R}, \\ v(t,0) = v(t,L) = 0 \end{cases}$$

Observe that the boundary condition $v(t,0) = 0$ arises from the fact that we are multiplying by r. The changes of variables $w = v + i|\partial_r|^{-1}\partial_t v$ is a bijective correspondence to solutions of the complexified 1D linear wave equation

$$\begin{cases} -i\partial_t w + |\partial_r| w = 0, \\ w : \mathbb{R}_t \times [0, L] \to \mathbb{C}, \\ w(t, 0) = w(t, L) = 0 \end{cases} \tag{1.9}$$

Here, $|\partial_r| = \sqrt{-\partial_r^2}$ can be understood in terms of the Fourier expansion, cf., (1.11) below. The composition of these changes of variables is

$$u \longmapsto ru + i|\partial_r|^{-1}\partial_t(ru) \qquad \text{or} \qquad (u, u_t) \longmapsto ru + i|\partial_r|^{-1} ru_t \tag{1.10}$$

Let us observe how this change of variables affects the corresponding the function spaces, the Gibbs measure, and the Hamiltonian structure.

Regarding function spaces, consider the following orthonormal basis of $L^2_r([0, L])$ consisting of Dirichlet eigenfunctions of ∂_r^2:

$$e_{n,L}(r) := \sqrt{2/L}\sin(n\pi r/L), \qquad n = 1, 2, \dots,$$

with eigenvalues $(n\pi/L)^2$, $n = 1, 2, \dots$, respectively. For each $s \in \mathbb{R}$, we define the operators $|\partial_r|^s = \sqrt{-\partial_r}^{\,s}$ via

$$|\partial_r|^s \sin(n\pi r/L) = |n\pi/L|^s \sin(n\pi r/L) \tag{1.11}$$

for each integer $n \geq 1$.

Definition 1.4. For each $s \in \mathbb{R}$, consider the homogeneous Sobolev space

$$\dot{H}^s_0([0, L] \to \mathbb{C}) := \left\{ g = \sum_{n=1}^{\infty} c_n e_{n,L} \mid c_n \in \mathbb{R}, \|g\|^2_{H^s_0} := \sum_{n=1}^{\infty} |n\pi/L|^{2s} |c_n|^2 < \infty \right\}.$$

and $\dot{H}^s_0([0, L] \to \mathbb{R})$, the sub-space with real coefficients. We equip these spaces with the usual Borel σ-algebra.

Recalling (1.7), observe that $e_{n,L} = rf_{n,L}$. In general, $f(r) \mapsto rf(r)$ is (up to a constant multiple) an isometry from $L^2_{rad}(B(0,L)) \to L^2_r([0,L])$, because

$$\int_{S^2}\int_0^L |f(r)|^2\, r^2 dr \frac{dS}{4\pi} = \int_0^L |rf(r)|^2\, dr.$$

Furthermore, multiplication by r is an isometry $\dot{H}^s_{rad,0}(B(0,L) \to \Lambda) \to \dot{H}^s_0([0,L] \to \Lambda)$, where $\Lambda = \mathbb{R}$ or $\mathbb{C}$. It follows that the map

$$(f_1, f_2) \longmapsto rf_1 + i|\partial_r| rf_2 \tag{1.12}$$

motivated by (1.10), is a homeomorphism

$$\dot{H}^s_{rad,0} \times \dot{H}^{s-1}_{rad,0}(B(0,L) \to \mathbb{R}) \longrightarrow \dot{H}^s_0([0,L] \to \mathbb{C}).$$

Regarding the Gibbs measure, The analogue of the randomization in (1.8) under the change of variables (1.12) is

$$\omega \mapsto \sum_{n=1}^{\infty} \frac{a_n(\omega) + ib_n(\omega)}{n\pi/L} e_{n,L}(r), \qquad a_n(\omega), b_n(\omega) \sim \mathcal{N}_{\mathbb{R}}(0,1), \text{ i.i.d.} \tag{1.13}$$

where, again, ω is an element of some suitable probability space $(\Omega, \mathcal{F}, P)$.

Definition 1.5. Fixing $s < \frac{1}{2}$, let us denote by μ_L the image measure on $\dot{H}^s_0([0,L] \to \mathbb{C})$ under the map in (1.13). To separate the randomizations, let $\mu_{L,1}$ and $\mu_{L,2}$ denote the image measure on $\dot{H}^s_0([0,L] \to \mathbb{R})$ under the maps

$$\omega \mapsto \sum_{n=1}^{\infty} \frac{a_n(\omega)}{n\pi/L} e_{n,L}(r) \quad \text{and} \quad \omega \mapsto \sum_{n=1}^{\infty} \frac{b_n(\omega)}{n\pi/L} e_{n,L}(r), \tag{1.14}$$

respectively.

The connection between $\mu_L, \mu_{L,1}$, and $\mu_{L,2}$ is as follows: for Borel sets $A_1, A_2 \subseteq$

$\dot{H}_0^s([0,L] \to \mathbb{R})$,

$$\mu_L\big(\{g \mid \mathrm{Re}(g) \in A_1, \mathrm{Im}(g) \in A_2\}\big) = \mu_{L,1}(A_1)\mu_{L,2}(A_2). \tag{1.15}$$

Note that (1.12) is a measure preserving isomorphism

$$\Big(\dot{H}^s_{rad,0} \times \dot{H}^{s-1}_{rad,0}(B(0,L) \to \mathbb{R}), M_L\Big) \longrightarrow \Big(\dot{H}_0^s([0,L] \to \mathbb{C}), \mu_L\Big)$$

essentially by construction.

Regarding the Hamiltonian structure, we apply (1.10) to (1.3) to see that the Hamiltonian corresponding to (1.9) is

$$H(w) = \int_0^L \frac{1}{2}\big(|\partial_r| w\big)^2 \, dr. \tag{1.16}$$

Furthermore, let $w_0(r) \in \dot{H}_0^s([0,L] \to \mathbb{C})$ with Fourier expansion

$$w_0(r) = \sum_{n=1}^{\infty} (p_n(0) + iq_n(0)) e_{n,L}(r).$$

The $w(t,r)$ solution of (1.9) with initial datum $w(0,r) = w_0(r)$ is given by

$$w(t,r) = e^{-it|\partial_r|} w(0,r) = \sum_{n=1}^{\infty} (p_n(t) + iq_n(t)) e_{n,L}(r)$$

where

$$p_n(t) + iq_n(t) = e^{-it(n\pi/L)}(p_n(0) + iq_n(0)).$$

Observe that

$$H(w) = \sum_{n=1}^{\infty} \frac{1}{2}(n\pi/L)^2 (p_n^2 + q_n^2). \tag{1.17}$$

Differentiating p_n and q_n with respect to time and applying (1.17) gives, formally,

$$\frac{dp_n}{dt} = (n\pi/L)^{-1}\frac{\partial H}{\partial q_n}, \qquad \frac{dq_n}{dt} = -(n\pi/L)^{-1}\frac{\partial H}{\partial p_n}, \qquad n \geq 1. \tag{1.18}$$

1.2.3 Invariance of the Gibbs Measure for the Linear Wave Equation

We use Liouville's theorem, (1.18), as well as some considerations from descriptive set theory to prove invariance of μ_L under the free propagator $e^{-it|\partial_r|}$.

Proposition 1.6. *Fix $s < \frac{1}{2}$. The flow $e^{-it|\partial_r|}$ leaves $(\dot{H}_0^s([0, L] \to \mathbb{C}), \mu_L)$ invariant. Namely, for each $t \in \mathbb{R}$ and for each Borel subset $A \in \dot{H}_0^s([0, L] \to \mathbb{C})$, the set*

$$e^{-it|\partial_r|}A = \{e^{-it|\partial_r|}g : g \in A\}$$

is a measurable subset of $\dot{H}_0^s([0, L] \to \mathbb{C})$ and

$$\mu_L(e^{-it|\partial_r|}A) = \mu_L(A).$$

Proof. First, we observe that $e^{-it|\partial_r|}$ is defined for all t and is an isometry from $\dot{H}_0^s([0, L] \to \mathbb{C})$ to itself. It follows that $e^{-it|\partial_r|}$ preserves measurability.

Since $e^{-it|\partial_r|}$ is a bijection, we may reduce our analysis of invariance from all Borel sets $A \in \dot{H}_0^s([0, L] \to \mathbb{C})$ to those of the form

$$A = \left\{ g = \sum_{n=1}^{\infty} c_n e_{n,L} : c_j \in B_j, j = 1, \ldots, N \right\} \tag{1.19}$$

where $N \geq 1$ is a natural number and $B_1, \ldots, B_N \subseteq \mathbb{C}$ are Borel sets. Sets of the form (1.19) are known as cylinder sets and generate the Borel σ-algebra (cf., Proposition A.4). So let us fix such an A of the form (1.19).

Let D_N denote the usual Dirichlet projection, defined by

$$D_N \sum_{n=1}^{\infty} c_n e_{n,L} = \sum_{n=1}^{N} c_N e_{n,L}.$$

By (1.18) and Liouville's theorem, we have

$$[D_N^* \mu_L](e^{-it|\nabla_r|} D_N A) = [D_N^* \mu_L](D_N A),$$

where $[D_N^* \mu_L](B) := \mu_L(D_N^{-1} B)$ denotes the push-forward (or image) measure on $\operatorname{span}_{\mathbb{C}}(e_{1,L}, \ldots, e_{N,L})$. Since A is a cylinder set, we have

$$\begin{aligned} A &= D_N^{-1} D_N A \\ e^{-it|\partial_r|} A &= D_N^{-1} e^{-it|\partial_r|} D_N A. \end{aligned}$$

The result follows. $\square$

Observe that invariance of the Gibbs measure for the original linear wave equations follows from undoing the change of variables outlined in Section 1.2.2.

1.3 The Burq–Tzvetkov–de Suzzoni example

In this section, we consider a non-linear version of previous system. Let $L > 0$ and consider the cubic non-linear wave equation[1]

$$\begin{cases} -\partial_t^2 u + \Delta_L u = u^3, \\ u : \mathbb{R}_t \times B(0, L) \to \mathbb{R}, \\ u(t, x) = u(t, |x|) = u(t, r), \\ u|_{\mathbb{R}_t \times \partial B(0,L)} \equiv 0 \end{cases} \tag{1.20}$$

[1]Burq and Tzvetkov actually considered the Gibbs measure for NLW $-\partial_t^2 u + \Delta_L u = |u|^p u$ when $p < 3$, though invariance of the Gibbs measure was only shown in the case $p = 2$ by de Suzzoni.

As before, we make a change of variables to reduce down to one dimension, and to complexify. Namely, the changes of variables

$$v := ru$$

and, respectively,

$$w := v + i|\partial_r|^{-1}\partial_t v = ru + i|\partial_r|^{-1} r\partial_t u$$

is a bijective correspondence to solutions of

$$\begin{cases} -\partial_t^2 v + \partial_r^2 v = \frac{v^3}{r^2}, \\ v : \mathbb{R}_t \times [0, L] \to \mathbb{R}, \\ v(t, 0) = v(t, L) = 0 \end{cases} \tag{1.21}$$

and, respectively,

$$\begin{cases} -i\partial_t w + |\partial_r| w = -|\partial_r|^{-1}\left(\frac{(\operatorname{Re} w)^3}{r^2}\right), \\ w : \mathbb{R}_t \times [0, L] \to \mathbb{C}, \\ w(t, 0) = w(t, L) = 0. \end{cases} \tag{1.22}$$

We clarify what we mean by a solution.

Definition 1.7 (Strong Solution for (1.22))**.** Let B be a Banach space of complex-valued functions on $[0, L]$. We say that $w(t, r) : [-T, T] \times [0, L] \to \mathbb{C}$ is a *strong solution of* (1.22) *on* $[-T, T]$ *with initial datum* $g \in B$ if

1. $w \in C_t^0 B([-T, T] \times [0, L] \to \mathbb{C})$, which is to say: for fixed $t \in [-T, T]$ we have $w(t, r) \in B$ and that the B-norm of w varies continuously in time.

2. $w(0, r) = g(r)$, and

3. $w(t,r)$ obeys, as a distribution, the corresponding Duhamel formula

$$w(t,r) = \left[e^{-it|\partial_r|}g\right](r) - i\int_0^t \left[\frac{e^{-i(t-\tau)|\partial_r|}}{|\partial_r|}\operatorname{Re}\left(\frac{w(\tau,\cdot)^3}{(\cdot)^2}\right)\right](r)d\tau,$$

for each $t \in [-T,T]$.

Furthermore, if w is the unique strong solution on $[-T,T]$ with initial datum $g \in B$, then we write

$$\mathrm{Flow}_L(t,g)(r) := w(t,r), \qquad |t| \le T.$$

In this section, we shall consider the case that

$$B = \dot{H}_0^s([0,L] \to \mathbb{C}),$$

for some fixed $s < \frac{1}{2}$. In later sections, we consider the space of Hölder continuous functions.

1.3.1 The Gibbs Measure

Fix $0 \le s < \frac{1}{2}$ and $L > 0$. Recall that μ_L denote the Borel measure on $\dot{H}_0^s([0,L] \to \mathbb{C})$ from Definition 1.5. We wish to make a suitable definition (and construction) for the Gibbs measure for (1.22).

Now, the Hamiltonian for (1.22) is

$$H(w) = \int_0^L \frac{1}{2}(|\partial_r|w)^2 + \frac{1}{4}(\mathrm{Re}(w))^4\, r^{-2}dr.$$

So, we wish to make sense of the expression

$$\frac{1}{Z}\exp(-H(w))\ \text{``}dw\text{''} = \frac{1}{Z}\exp\left(-\int_0^L \frac{1}{2}(|\partial_r|w)^2 + \frac{1}{4}(\mathrm{Re}(w))^4\, r^{-2}dr\right)\ \text{``}dw\text{''}.$$

Recall, the expression

$$\frac{1}{Z'} \exp\left(-\frac{1}{2}\int_0^L (|\partial_r|w)^2 dr\right) \text{ ``}dw\text{''}$$

is simply μ_L. Thus, we define the Gibbs measure for (1.22) as a Borel measure on $\dot{H}_0^s([0,L] \to \mathbb{C})$ given by

$$\frac{1}{Z_L} \exp\left(-\frac{1}{4}\int_0^L (\operatorname{Re} g(r))^4 r^{-2}\, dr\right) d\mu_L(g). \tag{1.23}$$

First, let us show that the measure in (1.23) is actually well-defined. In particular,

1. Is the Radon–Nikodym derivative actually a measurable function?

2. Is the Radon–Nikodym derivative not μ_L-almost surely zero?

For the first question, let

$$K_N(g) := \int_0^L \frac{(\operatorname{Re} g(r))^4 \wedge N^4}{r^2 \vee (1/N)}\, dr.$$

Here $a \wedge b = \min(a,b)$ and $a \vee b = \max(a,b)$. Observe that

$$\begin{aligned}
|K_N(g) - K_N(h)| &\leq N \int_0^L \left|(\operatorname{Re} g)^4 \wedge N^4 - (\operatorname{Re} h)^4 \wedge N^4\right| dr \\
&\lesssim N \int_0^L \big(|\operatorname{Re}(g-h)| \wedge N\big) \cdot \big(|\operatorname{Re}(g)|^3 \wedge N^3 + |\operatorname{Re}(h)|^3 \wedge N^3\big)\ dr \\
&\lesssim N^4 \int_0^L |g-h|\, dr \\
&\lesssim_L N^4 \|g-h\|_{\dot{H}_0^s},
\end{aligned}$$

and so K_N is actually a continuous function on $\dot{H}_0^s([0,L] \to \mathbb{C})$. Indeed, it is even

Lipschitz. It follows that

$$\exp\left(-\frac{1}{4}\int_0^L (\operatorname{Re} g(r))^4 r^{-2}\, dr\right) = \lim_{N\to\infty} \exp\left(-\frac{1}{4}K_N(g)\right)$$

is a point-wise limit of continuous functions, and is hence measurable.

To answer the second question, we employ the following result.

Lemma 1.8 (Khinchin)**.** *Let $a_n(\omega) \sim \mathcal{N}_{\mathbb{R}}(0,1)$ i.i.d., and $\{c_n\}_n \in \ell^2$. There is $C > 0$, such that for $q \geq 2$, $\mathbb{E}^\omega\Big[\big|\sum_{n=1}^\infty a_n(\omega)c_n\big|^q\Big] \leq C^q q^{\frac{q}{2}}\Big(\sum_{n=1}^\infty |c_n|^2\Big)^{q/2}$.*

We use Tonelli's theorem, Khinchin's inequality, and Minkowski's theorem for integrals to obtain

$$\begin{aligned}
&\mathbb{E}^\omega\left[\int_0^L \left|\sum_{n=1}^\infty \frac{a_n(\omega)+ib_n(\omega)}{n\pi/L} e_{n,L}(r)\right|^4 r^{-2} dr\right] \\
&\quad \lesssim_L \int_0^L \left(\sum_{n=1}^\infty \left|\frac{e_{n,L}(r)}{n\pi/L}\right|^2\right)^2 r^{-2} dr \\
&\quad \lesssim_L \left(\sum_{n=1}^\infty n^{-2}\left[\int_0^L |e_{n,L}(r)|^4 r^{-2} dr\right]^{1/2}\right)^2
\end{aligned}$$

Using the inequality $|\sin(x)| \lesssim |x|^{1/3}$, we obtain

$$\int_0^L |e_{n,L}(r)|^4 r^{-2} dr \lesssim (n\pi/L)^{\frac{4}{3}} \int_0^L r^{-\frac{2}{3}}\, dr \lesssim_L n^{\frac{4}{3}}.$$

It follows that

$$\mathbb{E}^\omega\left[\int_0^L \left|\sum_{n=1}^\infty \frac{a_n(\omega)+ib_n(\omega)}{n\pi/L} e_{n,L}(r)\right|^4 r^{-2} dr\right] \lesssim_L \left(\sum_{n=1}^\infty n^{-\frac{4}{3}}\right)^2 < \infty$$

and so $\exp\left(-\frac{1}{4}\int_0^L (\operatorname{Re} g(r))^4 r^{-2}\, dr\right)$ is μ_L-almost surely positive. We have proven the following result.

Proposition 1.9. *The measure ν_L given by*

$$d\nu_L(g) := \frac{1}{Z_L} \exp\left(-\frac{1}{4} \int_0^L (\operatorname{Re} g(r))^4 r^{-2} \, dr \right) d\mu_L(g)$$

is a well-defined Borel measure on $\dot{H}_0^s([0, L] \to \mathbb{C})$ for $0 \leq s < \frac{1}{2}$. It is mutually absolutely continuous with respect to μ_L.

1.3.2 Connection with 3D Gibbs Measure

Let us recall the 3D non-linear wave equation

$$\begin{cases} -\partial_t^2 u + \Delta_L u = u^3, \\ u : \mathbb{R}_t \times B(0, L) \to \mathbb{R}, \\ u(t, x) = u(t, |x|) = u(t, r), \\ u|_{\mathbb{R}_t \times \partial B(0,L)} \equiv 0 \end{cases}$$

and its (formally defined) Gibbs measure

$$dm_L(u, u_t) := dm_{L,1}(u) \otimes dm_{L,2}(u_t) \tag{1.24}$$

where

$$\begin{aligned} dm_{L,1}(u) &= \frac{1}{Z_1} \exp\left(-\int_{B(0,L)} \frac{1}{4}|u|^4 + \frac{1}{2}|\nabla u|^2 \right) \text{“}du\text{”} \\ dm_{L,2}(u_t) &= \frac{1}{Z_2} \exp\left(-\int_{B(0,L)} \frac{1}{2}|u_t|^2 \right) \text{“}du_t\text{”} \end{aligned}$$

Observe that, formally speaking,

$$dm_L(u, u_t) = \frac{1}{Z} \exp\left(-\int_{B(0,L)} \frac{1}{4}|u|^4 \right) dM_L(u, u_t), \tag{1.25}$$

where M_L is as in Definition 1.3. In [9], it was noted that

$$\left\| \sum_{n=1}^{\infty} \frac{a_n(\omega)}{n\pi/L} f_{n,L}(r) \right\|_{L^4(B(0,L))}^4 < \infty, \qquad \omega \text{ a.s.} \tag{1.26}$$

In particular, the Radon–Nikodym derivative

$$(u, u_t) \mapsto \exp\left(-\frac{1}{4} \int_{B(0,L)} |u|^4 \right) \tag{1.27}$$

is positive M_L almost surely, and so the expression in (1.25) is indeed well-defined. In particular, m_L is mutually absolutely continuous with respect to the free measure M_L.

Now, we apply the change of variables

$$(f_1, f_2) \mapsto r f_1 + i|\partial_r|^{-1}(r f_2). \tag{1.28}$$

The image of the M_L under this map is μ_L. Recall that the Radon–Nikodym derivative in (1.27) only depends on the first component and that, for Borel measurable functions $\Psi : \dot{H}_0^s([0,L] \to \mathbb{R}) \to \mathbb{R}$, we have

$$\int_{\dot{H}_0^s([0,L])} \Psi(f)\, d\mu_{L,1}(f) = \int_{\dot{H}^s_{rad,0}(B(0,L))} \Psi(rf)\, d\tilde{m}_{L,1}(f).$$

In other words, letting

$$\Psi(f) = \exp\left(-\frac{1}{4} \int_0^L |f(r)|^4 r^{-2}\, dr \right)$$

then $\Psi(rf) = \exp\left(-\frac{1}{4} \int_0^L |f(r)|^4 r^2\, dr \right) = \exp\left(-\frac{1}{4} \int_{B(0,L)} |f(r)|^4 \right)$, which recovers (1.27). Thus, under the change of variables (1.28), the corresponding Gibbs measure for (1.22) is indeed

$$d\nu_L(g) := \frac{1}{Z_L} \exp\left(-\frac{1}{4}\int_0^L (\operatorname{Re} g(r))^4 r^{-2}\, dr\right) d\mu_L(g),$$

with Z_L being a normalization constant.

1.3.3 Invariance of the 1D Gibbs Measure, ν_L

In a similar spirit as the linear, we first express the formal Hamiltonian structure of (1.22). Let

$$w_0(r) = \sum_{n=1}^{\infty} [p_n(0) + iq_n(0)] e_{n,L}(r)$$

admit a unique, global, strong solution $w(t,r)$, and write

$$w(t,r) = \sum_{n=1}^{\infty} [p_n(t) + iq_n(t)] e_{n,L}(r).$$

Writing the Hamiltonian in Fourier series,

$$H(w) = \frac{1}{2}\sum_{n=1}^{\infty} (n\pi/L)^2 [p_n^2 + q_n^2] + \frac{1}{4}\int_0^L \left[\sum_{n=1}^{\infty} p_n e_{n,L}\right]^4 r^{-2} dr.$$

If we apply (1.22) and equate coefficients, we formally obtain

$$\frac{dp_n}{dt} = (n\pi/L)^{-1}\frac{\partial H}{\partial q_n}, \qquad \frac{dq_n}{dt} = -(n\pi/L)^{-1}\frac{\partial H}{\partial p_n}, \qquad n \geq 1.$$

Given this (formal) Hamiltonian structure and the definition of ν_L, we expect $\text{Flow}_L(t, \cdot)$ to preserve ν_L. Indeed, the following was proven by Burq, Tzvetkov, and de Suzzoni.

Theorem 1.10 (Burq–Tvetkov–de Suzzoni[2], [9,12])**.** *Let $s < \frac{1}{2}$, let $L > 0$, and*

[2]Strictly speaking, the results in [9,12] were stated in the 3-dimensional, complexified setting. Theorem 1.10 is the restatement of their result after multiplication by r.

let ν_L be as in (1.23). *There exists a measurable set $\Pi_L \subseteq \dot{H}^s_0([0,L] \to \mathbb{C})$ with the following properties*

1. $\nu_L(\Pi_L) = 1$.

2. *Each $g \in \Pi_L$ admits a unique, global, strong solution in $\dot{H}^s_0$. Namely, $\mathrm{Flow}_L(t,g)$ is defined for all $t \in \mathbb{R}$, and, for each $T \in [0,\infty)$, we have*

$$\mathrm{Flow}_L(t,g) \in C^0_t \dot{H}^s_0([-T,T] \times [0,L] \to \mathbb{C}).$$

3. *For each measurable set $A \subseteq \Pi_L$ and for each $t \in \mathbb{R}$, the set $\mathrm{Flow}_L(t,A) = \{\mathrm{Flow}_L(t,g) \mid g \in A\}$ is a measurable subset of $\dot{H}^s_0([0,L] \to \mathbb{C})$ and*

$$\nu_L(\mathrm{Flow}_L(t,A)) = \nu_L(A).$$

Though there are many similarities with the linear case, the proof of this result is significantly more difficult. We outline the proof of this result, and refer the reader to [9,12] for the details. Also, we note that Burq, Tzvetkov, and de Suzzoni proved their results in the 3 dimensional, complexified setting, whereas we state the results in the analogous results in the one dimensional, complexified setting.

Let $\chi \in C^\infty_C(\mathbb{R} \to \mathbb{R})$ such that $\chi \equiv 1$ on $(-1/2,1/2)$ and is zero on $\mathbb{R}\setminus(-1,1)$. Rather than using the sharp Fourier cut-offs D_N as in the linear case, Burq and Tzvetkov used the operator S_N given by

$$S_N\left(\sum_{n=1}^{\infty} c_n e\right) := \sum_{n=1}^{\infty} \chi(n^2/N^2) c_n e_{n,L}$$

Similarly to the linear case, we first examine a closely related finite dimensional system, establish invariance of the corresponding Gibbs measure, and pass to an infinite dimensional limit. To begin, consider the system

$$\begin{cases} i\partial_t w - |\partial_r| w = S_N \left(|\partial_r|^{-1} \frac{(S_N \operatorname{Re} w)^3}{r^2} \right) \\ w : \mathbb{R} \times [0, L] \to \mathbb{C} \\ w(t, 0) = w(t, L) = 0 \end{cases}$$

with initial data

$$w(0, r) = w_0(r) \in E_N := \operatorname{span}(e_{1,L}, \dots, e_{N,L})$$

and Hamiltonian

$$H(w) = \int_0^L \frac{1}{2} (|\partial_r| w)^2 \, dr + \int_0^L \frac{1}{4} (S_N \operatorname{Re} w)^4 r^{-2} \, dr.$$

Since this is a finite dimensional system and all of the terms in the Hamiltonian are positive, we may use conservation of the Hamiltonian to repeatedly apply Picard iteration to obtain global existence. Let us denote the initial data to solution map $w_0(r) \mapsto w(t, r)$ by

$$\mathrm{Flow}_{N,L}(t, w) := w(t, r).$$

We record the result below.

Lemma 1.11. $\mathrm{Flow}_{N,L}$ *is globally defined on* E_N.

Define $\mu_{N,L}$ as the image measure on E_N given by the randomization

$$\omega \longmapsto \sum_{n=1}^{N} \frac{a_n(\omega) + i b_n(\omega)}{n\pi/L} e_{n,L},$$

where ω is an element of some suitable probability space $(\Omega, \mathbb{P})$. Define $\nu_{N,L}$ via

$$d\nu_{N,L}(g) := \frac{1}{Z} \exp \left(-\frac{1}{4} \int_0^L ([S_N(g)](r))^4 r^{-2} \, dr \right) d\mu_{N,L}(g).$$

These are the finite dimensional analogues of μ_L and ν_L, respectively. Again, since we are working with a finite dimensional system, we may apply Liouville's theorem to prove the following result.

Lemma 1.12. *$\nu_{N,L}$ is invariant under* $\mathrm{Flow}_{N,L}$.

To pass to the infinite dimensional limit and obtain almost sure global existence of solutions, we wish to estimate the measure of the sets on which we apply contraction mapping. To this end, Burq and Tzvetkov proved the following result.

Lemma 1.13 (Burq–Tzvetkov, [9])**.** *Let $2 \leq p < 6$ and $s < \frac{1}{2}$. There exists some $c_p, c_s, C > 0$ such that*

$$\mu_L\left(\left\{g \in \dot{H}^s_0 \;\middle|\; \left\|r^{\frac{2-p}{p}} e^{-it|\partial_r|} g(r)\right\|_{L^p_t L^p_r([0,2]\times[0,L])} > \lambda\right\}\right) \leq Ce^{-c_p\lambda^2}$$

and

$$\mu_L\left(\left\{g \in \dot{H}^s_0 \;\middle|\; \|g\|_{\dot{H}^s_0} > \lambda\right\}\right) \leq Ce^{-c_s\lambda^2}$$

for every $\lambda > 0$.

As a sketch of the proof, we apply Markov's inequality: for X a random variable and for $q > 0$,

$$\mathbb{P}(|X| > \lambda) \leq \frac{\mathbb{E}[|X|^q]}{\lambda^q}.$$

In our case, the random variable amounts a random Gaussian series, to which we may apply Khinchin's inequality 1.8. To close the argument, we then specialize to an optimal choice of q, which, in this case, happens to be $q = \lambda^2/2$.

In particular, we note that similar estimates hold for $\mu_{N,L}$, where we may choose the constants c_p, c_s, C to be independent of N. Using invariance of $\mu_{N,L}$ under $\mathrm{Flow}_{N,L}$ and the large deviation estimates, Burq and Tzvetkov iterated the flow map to obtain long term growth estimates.

Lemma 1.14. *For every integer $N \geq 1$, there exists a $\mu_{N,L}$ measurable set $\sigma_N^i \subseteq E_N$ such that*

1. $\mu_{N,L}(E_N \setminus \sigma_N^i) \leq 2^{-i}$

2. *For every $i, N \in \mathbb{N}$, every $w_0 \in \Sigma_N^i$,*

$$\left\| r^{\frac{2-p}{p}} e^{-it|\partial_r|} w_0(r) \right\|_{L_t^p L_r^p([0,2]\times[0,L])} + \| \operatorname{Flow}_{N,L}(t, w_0) \|_{\dot{H}_0^s([0,L])} \lesssim \sqrt{i + \log(1 + |t|)}$$

for all $t \in \mathbb{R}$, where the implicit constant does not depend on i, N, nor w_0.

Using the long term estimates on $\operatorname{Flow}_{N,L}$ and the fact we may close a contraction mapping argument whenever

$$\left\| r^{\frac{2-p}{p}} e^{-it|\partial_r|} w_0(r) \right\|_{L_t^p L_r^p([0,2]\times[0,L])} + \|w_0\|_{\dot{H}_0^s} < \infty,$$

Burq and Tzvetkov were able to establish assertions (1) and (2) in Theorem 1.10.

To establish invariance, we wish to show that $\operatorname{Flow}_{N,L}$ converge uniformly, in some sense, to Flow_L on the sets in which we run the contraction mappings, which is the content of the following result by de Suzzoni.

Lemma 1.15. *Fix $0 < s < \frac{1}{2}$, $4 < p < 6$. Fix $\lambda > 0$ and let*

$$A(\lambda) = \{w_0 \in \dot{H}_0^s([0,L] \to \mathbb{C}) : \left\| r^{\frac{2-p}{p}} e^{-it|\partial_r|} w_0(r) \right\|_{L_t^p L_r^p([0,2]\times[0,L])} \leq \lambda, \|w_0\|_{\dot{H}_0^s} \leq \lambda\}.$$

Let $\sigma := \frac{3}{2} - \frac{4}{p}$. There exists a time $T = T(D, L)$ such that for all $\varepsilon > 0$, there exists $N_0 > 0$ so that

$$\| \operatorname{Flow}_L(t, w_0) - \operatorname{Flow}_{N,L}(t, w_0) \|_{C_t^0 \dot{H}^\sigma([-T,T]\times[0,L])} < \varepsilon$$
$$\| r^{-\frac{2}{p}} (\operatorname{Flow}_L(t, w_0) - \operatorname{Flow}_{N,L}(t, w_0)) \|_{L_t^p L_t^{\frac{2p}{p-2}}([-T,T]\times[0,L])} < \varepsilon$$

for all $N \geq N_0$ and for all $w_0 \in A(\lambda)$.

Note that $\sigma > \frac{1}{2} > s$ and so $\dot{H}_0^\sigma \hookrightarrow \dot{H}_0^s$. Using this local uniform convergence and measure theoretic considerations, de Suzzoni was able to prove assertion (3) of Theorem 1.10.

1.4 Further Examples

Invariant measures for Hamiltonian PDEs were first considered by Lebowitz, Rose, and Speer in [21], and refined by Bourgain in [1]. In these papers, they considered a focusing, non-linear Schrödinger (NLS) equation on the circle and constructed an L^2-truncated Gibbs measure.

Indeed, the method used by Burq, Tzvetkov, and de Suzzoni was first pioneered by Bourgain: applying a frequency truncation (to obtain a finite dimensional system), invoking Liouville's theorem for Hamiltonian ODEs, and using uniform probabilistic estimates to remove the truncations, Bourgain proved global existence of solutions on a set of full measure and the invariance of the Gibbs measure under the NLS. Prior to Bourgain's result, only local well-posedness results were available in that setting.

One benefit of randomization is that one may work in systems with super-critical scaling. Data with ill-behaved solutions generally lie in null sets of these measures, and the invariance of the Gibbs measure can be used as a conservation law to upgrade local in time existence to global existence. Indeed, the scaling that preserves solutions of (1.20) also preserves the $\dot{H}^{\frac{1}{2}}$ norm, and so μ_L is supported on super-critical Sobolev spaces.

Furthermore, we may construct the Gibbs measure for $-\partial_t^2 u + \Delta_L u = |u|^p u$ for all $p < 4$. In [9], Burq and Tzvetkov show almost surely global existence on a set of full Gibbs measure in the case $p < 3$. The analogous result in the case $3 \leq p < 4$ was proven by Bourgain and Bulut in [7] by working in the context of $X^{s,b}$ spaces. In the case $p = 2$, de Suzzoni established invariance of the Gibbs measure under the flow of the corresponding NLW. Bourgain and Bulut also analyzed the case of the power type non-linear Schrödinger equation on the unit ball in two dimensions, [5], and three dimensions, [6].

Another example is [26], where Nahmod, Pavlović, and Staffilani considered the 2D and 3D Navier–Stokes equations with randomized initial data in super-critical spaces. This is not in a Hamiltonian setting, and hence one cannot expect invariant measures via Liouville's theorem on the Fourier truncations. Nevertheless, they randomized about a fixed initial datum, and applied large deviation estimates, to obtain almost sure global existence of weak solutions, with uniqueness in the 2D case.

Another recent work is [28], where Nahmod and Staffilani proved almost sure local well-posedness for a 3D quintic NLS on $\mathbb{T}^3$ with data below $H^1(\mathbb{T}^3)$, also in a supercritical regime. Other works on finite volume spatial domains include [4], where Bourgain used Wick ordering to construct an invariant Gibbs measure for a 2D NLS on the torus; [29], where Oh proved invariance of mean 0 white noise for the 1D KdV equation on the circle; [11], where Colliander and Oh showed almost sure global existence of solutions of 1D cubic NLS with initial datum in $H^s(\mathbb{T})$, $-\frac{1}{12} < s < 0$.

This approach was also applied to the Gross–Pitaevskii hierarchy on $\mathbb{T}^3$ and closely related systems. In [37], Sohinger and Staffilani used randomization on the collision (or contraction) operator to extend the space-time evolution estimates to a lower regularity setting than in the deterministic case (cf., [17]).

Further works in the finite volume setting include [2, 5, 6, 8, 25, 27, 30, 33, 36, 40, 42, 43], etc, and references therein.

A case in which an infinite volume invariant measure is constructed is in [24], where Mckean and Vaninsky constructed a Gibbs measure for a 1D NLW on the half-line. Using techniques from stochastic analysis, they reduced the construction of such a measure to computing the asymptotics of the fundamental solution of a parabolic PDE with time-independent coefficients. Indeed, the measure was realized as a stationary diffusion on the half-line. They also constructed invariant measures for analogous finite volume NLW and used finite speed of propagation to upgrade to invariance of the infinite volume measure.

In contrast to [24], our measure does not correspond to a stationary diffusion. The fact that we are not in a strictly one dimensional setting means that our parabolic PDE has time dependent coefficients with singularities as we approach the space-time origin (cf., (4.8)).

The fact that our finite volume systems are posed with zero boundary values also makes the measure theory more delicate: generic paths in the support of the finite volume measures are ignored by the infinite volume measure. This is in contrast to [24], where the finite volume systems were considered with periodic boundary values.

Further results for invariant measure in the infinite volume setting include the works of de Suzzoni in [14], Cacciafesta and de Suzzoni in [10], and Rider in [34].

Other work in infinite volume settings include: [3], where Bourgain analyzed a 1D periodic NLS with uniform estimates on arbitrarily large intervals; [39], where Thomann randomized coefficients of eigenfunctions of a Schrödinger operator with a confining potential, and showed almost sure global existence of solutions of power-type NLS on $\mathbb{R}^d$; [23], where Lührmann and Mendelson fixed an initial datum and randomized with respect to its Littlewood–Paley pieces; [13], where de Suzzoni obtains almost sure global existence of NLW on $\mathbb{R}^3$ via the Penrose transform. Note that these works do not consider (infinite volume) Gibbs measures.

CHAPTER 2

Setup of the Infinite Volume Problem

Consider the 3D radial, defocusing, cubic, nonlinear wave equation (NLW)

$$\begin{cases} -\partial_t^2 u + \Delta u = u^3, \\ u : \mathbb{R}_t \times \mathbb{R}_x^3 \to \mathbb{R}, \\ u(t,x) = u(t,|x|) = u(t,r) \end{cases} \tag{2.1}$$

with Hamiltonian

$$H(u) = \int_{\mathbb{R}^3} \frac{1}{4}|u|^4 + \frac{1}{2}|\nabla u|^2 + \frac{1}{2}|u_t|^2.$$

We wish to construct the Gibbs measure $m_\infty = m_{\infty,1} \otimes m_{\infty,2}$ for this system, which we informally write as

$$\begin{aligned} dm_\infty(u,u_t) &= \frac{1}{Z}\exp\left(-H(u)\right) \text{“}d(u,u_t)\text{”} \\ dm_{\infty,1}(u) &= \frac{1}{Z_1}\exp\left(-\int_{\mathbb{R}^3} \tfrac{1}{4}|u|^4 + \tfrac{1}{2}|\nabla u|^2\right) \text{“}du\text{”} \\ dm_{\infty,2}(u_t) &= \frac{1}{Z_2}\exp\left(-\int_{\mathbb{R}^3} \tfrac{1}{2}|u_t|^2\right) \text{“}du_t\text{”} \end{aligned}$$

and show that the flow of (2.1) is defined for all time on the support of m_∞, and leaves m_∞ invariant.

To prove this theorem, we again change variables to reduce down to a one dimensional, complexified wave equation. As before,

$$v(t,r) := ru(t,r)$$

and, respectively,

$$w(t,r) := v(t,r) + i|\partial_r|^{-1}\partial_t v(t,r) = ru(t,r) + i|\partial_r|^{-1} r\partial_t u(t,r)$$

is a bijective correspondence to solutions of

$$\begin{cases} -\partial_t^2 v + \partial_r^2 v = \frac{v^3}{r^2}, \\ v : \mathbb{R}_t \times [0,\infty) \to \mathbb{R}, \\ v(t,0) = 0 \end{cases}$$

and, respectively,

$$\begin{cases} -i\partial_t w + |\partial_r| w = -|\partial_r|^{-1}\left[\frac{(\operatorname{Re} w)^3}{r^2}\right], \\ w : \mathbb{R}_t \times [0,\infty) \to \mathbb{C}, \\ w(t,0) = 0 \end{cases} \tag{2.2}$$

We shall construct the Gibbs measure for the latter system as a suitable infinite volume limit of ν_L. To this end, we shall employ techniques from the theory of stochastic processes. In particular, we seek to work in a space where evaluation at a point is a well-defined linear functional. In particular, we first modify the result of Burq, Tvetkov, and de Suzzoni to suit our purposes.

Ultimately, we shall state our main result in three parts:

- Theorem 2.1, which modifies the finite volume Gibbs measure
- Theorem 2.3, which constructs the infinite volume limit measure (note: we shall actually prove a slightly expanded version, which is given in Theorem 4.6)
- Theorem 2.5, which proves the invariance of the infintie volume measure under the flow of (2.2).

We motivate these results in the subsequent sections, and we shall prove the theorems in Chapters 3, 4, and 5, respectively. The construction and invariance of the Gibbs measure for (2.1) can be recovered by applying the reverse change of variables

$$w \longmapsto \big(r^{-1}\operatorname{Re}(w), r^{-1}|\partial_r|\operatorname{Im}(w)\big).$$

2.1 Revisiting the Burq–Tzvektov–de Suzzoni Example

Let us recall Definition 1.5. Then $\mu_{L,1}$ and $\mu_{L,2}$ both have the law of the random series

$$f^\omega(r) = \sum_{n=1}^{\infty} \frac{a_n(\omega)}{n\pi/L}\sqrt{\frac{2}{L}}\sin(n\pi r/L), \qquad a_n(\omega) \sim \mathcal{N}_{\mathbb{R}}(0,1) \text{ i.i.d.} \tag{2.3}$$

By Mercer's theorem (cf., [18] or [22]), the series almost surely converges uniformly and has law of the standard Brownian bridge from $(0,0)$ to $(L,0)$. Namely, $f^\omega(r)$ is a Gaussian process in r of mean zero with

$$\mathbb{E}[f^\omega(r)f^\omega(r')] = r\left(1 - \frac{r'}{L}\right), \qquad 0 \le r \le r' \le L. \tag{2.4}$$

In particular, $f^\omega(r)$ is almost surely s-Hölder continuous for every $s \in [0, \frac{1}{2})$. It follows that $\mu_{L,1}$ and $\mu_{L,2}$ are supported on

$$C_0^s([0,L] \to \mathbb{R}) := \{f : [0,L] \to \mathbb{R} \mid f \text{ is } s\text{-Hölder continuous and } 0 = f(0) = f(L)\},$$

equipped with the usual Hölder norm

$$\|f\|_{C^s([0,L])} := \sup_{r\in[0,L]} |f(r)| + \sup_{\substack{r,r'\in[0,L]\\ r\neq r'}} \frac{|f(r) - f(r')|}{|r-r'|^s}.$$

In particular, μ_L is supported on $C_0^s([0,L] \to \mathbb{C})$. By mutual absolute continuity, ν_L is also supported on $C_0^s([0,L] \to \mathbb{C})$.

We will show that (1.22) is locally well-posed in $C_0^s([0,L] \to \mathbb{C})$, thus making it an excellent space for the analysis of this random initial data problem.

Also, we slightly extend the measure theory in the following sense: we complete the Borel σ-algebra on $C_0^s([0,L] \to \mathbb{C})$ with respect to the measure ν_L, and we call a set ν_L-*measurable* if it is an element of this larger σ-algebra. By abuse of notation, we also denote the extension of the measure to this larger σ-algebra by ν_L. As we shall see in Chapter 5, this setting is convenient because for every Borel set $A \subseteq C_0^s([0,L] \to \mathbb{C})$ and for every $0 < R < L$, the set

$$\tilde{A} = \left\{ f : [0,L] \to C \mid \exists g \in A \text{ such that } g|_{[0,R]} \equiv f|_{[0,R]} \right\}$$

is not necessarily Borel, but is still ν_L-measurable. Indeed, we shall see that $\tilde{A}$ is analytic (i.e., a continuous image of a Borel set). Sets of this form arise naturally when we seek to apply finite speed of propagation arguments.

The proof of the following modified result, as well as the discussion of the proof method, is the content of Chapter 3.

Theorem 2.1. *Let* $\frac{1}{3} < s < \frac{1}{2}$, *let* $L > 0$. *Then* (1.22) *is locally well-posed in* $C_0^s([0,L] \to \mathbb{C})$. *Let* ν_L *be as in* (1.23). *Then, there exists a Borel measurable set* $\Omega_L \subseteq C_0^s([0,L] \to \mathbb{C})$ *such that*

1. $\nu_L(\Omega_L) = 1$.

2. *Each* $g \in \Omega_L$ *admits a unique global, strong solution in* C_0^s. *Namely,* $\mathrm{Flow}_L(t,g)$ *is defined for all* $t \in \mathbb{R}$, *and, for each* $T \in (0,\infty)$, *we have*

$$\mathrm{Flow}_L(t,g) \in C_t^0 C_r^s([-T,T] \times [0,L] \to \mathbb{C}).$$

3. *For each* ν_L*-measurable set* $A \subseteq \Omega_L$ *and for each* $t \in \mathbb{R}$, *the set*

$$\mathrm{Flow}_L(t,A) := \{\mathrm{Flow}_L(t,g) \mid g \in A\}$$

is also ν_L-measurable subset of $C_0^s([0, L] \to \mathbb{C})$ and

$$\nu_L(\mathrm{Flow}_L(t, A)) = \nu_L(A).$$

Moreover, if A is Borel, then so is $\mathrm{Flow}_L(t, A)$.

Remark. Observe that the scaling $w \mapsto w_\lambda(t, r) := w(\lambda t, \lambda r)$ preserves solutions of (1.22). Thus, scaling-invariant space is C_r^0. It follows that C_r^s, with $\frac{1}{3} < s < \frac{1}{2}$, are sub-critical spaces with respect to this scaling.

2.2 Defining the Infinite Volume Limit Measure

We turn to the infinite volume setting. Given that the finite volume measures ν_L are supported on $C_0^s([0, L] \to \mathbb{C})$, we shall construct the infinite volume limit measure as a Borel measure on the space

$$C_{loc}^s([0, \infty) \to \mathbb{C}) := \{f : [0, \infty) \to \mathbb{C} \mid \|f\|_{C^s([0,L])} < \infty \text{ for all } L > 0\},$$

where we equip this space with the metric

$$d(g_1, g_2) = \sum_{n=1}^{\infty} 2^{-n} \frac{\|g_1 - g_2\|_{C^s([0,n])}}{1 + \|g_1 - g_2\|_{C^s([0,n])}} \tag{2.5}$$

as well as the induced metric topology and Borel structure. Recall,

Definition 2.2. Fix $0 < R < \infty$ and let $C^s([0, R] \to \mathbb{C})$ be the space of s-Hölder continuous functions on $[0, R]$, equipped with the Hölder norm, the corresponding norm topology, and the induced Borel σ-algebra. For $0 < R < L \leq \infty$, we define the restriction map $\rho_R^L : C_0^s([0, L]) \to C^s([0, R])$ in usual manner:

$$\rho_R^L f := f|_{[0,R]}.$$

Define the (image) Borel measure $\nu_L|_{[0,R]}$ on $C^s([0,R] \to \mathbb{C})$ by

$$\nu_L|_{[0,R]}(A) := \nu_L((\rho_R^L)^{-1}(A))$$

for Borel measurable subsets $A \subseteq C^s([0,R] \to \mathbb{C})$.

We say that a Borel measure ν_∞ on $C^s_{loc}([0,\infty) \to \mathbb{C})$ is *an infinite volume limit of* $\{\nu_L\}_{L>0}$ if

$$\lim_{L\to\infty} \nu_L|_{[0,R]}(A) = \nu_\infty|_{[0,R]}(A),$$

for all $A \subseteq C^s([0,R] \to \mathbb{C})$ Borel.

The proof of the following result, as well as the discussion of the proof method, is the content of Section 4.

Theorem 2.3. *Fix* $0 \leq s < \frac{1}{2}$. *There exists a unique Borel probablity measure* ν_∞ *on* $C^s_{loc}([0,\infty) \to \mathbb{C})$ *such that for each* $R > 0$ *and for each Borel measurable* $A \subseteq C^s([0,R] \to \mathbb{C})$,

$$\lim_{L\to\infty} \nu_L|_{[0,R]}(A) = \nu_\infty|_{[0,R]}(A). \tag{2.6}$$

As before, $\nu_L|_{[0,R]}$ *and* $\nu_\infty|_{[0,R]}$ *denote the (Borel, probability) image measures on* $C^0([0,R] \to \mathbb{C})$ *given by the image, under the restriction map* $g \mapsto g|_{[0,R]}$, *of* ν_L *and* ν_∞, *respectively.*

Moreover, for each $L > 0$, *the measures* $\nu_L|_{[0,R]}$ *and* $\nu_\infty|_{[0,R]}$ *are mutually absolutely continuous. Let* $\mathcal{F}_R$ *denote the completion of the Borel* σ*-algebra on* $C^s([0,R])$ *with respect to any of these measures. Then* (2.6) *holds for every* $A \in \mathcal{F}_R$.

Remark. As mentioned before, we shall prove a slightly expanded version of this result, which is expressed by Theorem 4.6.

2.2.1 Comparison With the Linear System

Before we proceed, we illustrate why it was not possible to construct the Gibbs measure for our non-linear system by appending a suitable Radon–Nikodym derivative to the Gibbs measure corresponding to the linear system.

Namely, consider the first order linear wave equation

$$\begin{cases} -i\partial_t w + |\partial_r| w = 0, \\ w : \mathbb{R}_t \times [0,\infty) \to \mathbb{C}, \\ w(t,0) = 0 \end{cases} \tag{2.7}$$

Recall, the Gibbs measure μ_L for the linear wave equations on the interval $[0, L]$ for $L < \infty$ were constructed by an appropriately randomizing the Fourier coefficients. To do this, we needed an $L^2([0, L])$-orthonormal basis of eigenfunctions of the Laplacian. On the half-line $[0, \infty)$, there are no such L^2 eigenfunctions of the Laplacian. So, we require an alternate approach.

Recall (2.4), which is the covariance structure for the law of $\mu_{L,1}$ and $\mu_{L,2}$. If we formally take $L \uparrow \infty$, then we obtain the covariance structure

$$\mathbb{E}[f^\omega(r) f^\omega(r')] = \min(r, r'),$$

which is precisely that of Brownian motion. We can rigorously establish this fact.

Fix $0 \le s < \frac{1}{2}$. Let W_1, W_2 denote independent copies of Wiener measure on $C^s_{loc}([0,\infty) \to \mathbb{R})$. In Section 4.6, we will show that, for every $R > 0$ and for every Borel set $A \subseteq C^s([0, R] \to \mathbb{R})$,

$$\lim_{L\to\infty} \mu_{L,j}|_{[0,R]}(A) = W_j|_{[0,R]}(A)$$

for $j = 1, 2$. In particular, define the Borel measure μ_∞ on $C^s_{loc}([0,\infty) \to \mathbb{C})$ by

$$\mu_\infty(\{g \in C^s_{loc}([0,\infty) \to \mathbb{C}) : \mathrm{Re}(g) \in A_1, \mathrm{Im}(g) \in A_2\}) = W_1(A_1)W_2(A_2)$$

for Borel sets $A_1, A_2 \subseteq C^s_{loc}([0,\infty) \to \mathbb{R})$. Then we have

$$\lim_{L\to\infty} \mu_L|_{[0,R]}(A) = \mu_\infty|_{[0,R]}(A)$$

for every Borel set $A \subseteq C^s([0,R] \to \mathbb{C})$.

Now, suppose we wish to construct the Gibbs measure for the non-linear wave equation by the expression

$$\frac{1}{Z} \exp\left(-\frac{1}{4}\int_0^\infty (\mathrm{Re}(g)(r))^4 r^{-2}\, dr\right) d\mu_\infty(g). \tag{2.8}$$

Note that $[\mathrm{Re}(g)](r)$ has the law of Brownian motion, which we denote by $B(r)$. Fix $0 < a < 1$. The change of variables $r \mapsto a^2 r$ gives

$$\int_0^\infty (B(r))^4 r^{-2}\, dr = a^2 \int_0^\infty (a^{-1}B(a^2 r))^4 r^{-2}\, dr$$

By scaling invariance, $a^{-1}B(a^2 r)$ also has the law of Brownian motion and so

$$\begin{aligned}
\mathbb{E}\left[\exp\left(-\frac{1}{4}\int_0^\infty (B(r))^4 r^{-2}\, dr\right)\right] &= \mathbb{E}\left[\exp\left(-\frac{a^2}{4}\int_0^\infty (a^{-1}B(a^2 r))^4 r^{-2}\, dr\right)\right] \\
&= \mathbb{E}\left[\exp\left(-\frac{a^2}{4}\int_0^\infty (B(r))^4 r^{-2}\, dr\right)\right] \\
&= \mathbb{E}\left[\left(\exp\left(-\frac{1}{4}\int_0^\infty (B(r))^4 r^{-2}\, dr\right)\right)^{a^2}\right]
\end{aligned}$$

Observe that $\exp\left(-\frac{1}{4}\int_0^\infty (B(r))^4 r^{-2}\, dr\right)$ is a random variable taking values in

$[0, 1]$. Given $0 < a < 1$, we have

$$\exp\left(-\frac{1}{4}\int_0^\infty (B(r))^4 r^{-2}\, dr\right) \le \left(\exp\left(-\frac{1}{4}\int_0^\infty (B(r))^4 r^{-2}\, dr\right)\right)^{a^2} \tag{2.9}$$

Since the expected values of these random variables is the same, it follows that we must have almost sure equality in (2.9). In particular, both sides of (2.9) must be almost surely zero or one. Since Brownian motion is almost surely not identically zero, it follows that both sides of (2.9) must actually be zero. It follows that the Radon–Nikodym derivative

$$\exp\left(-\frac{1}{4}\int_0^\infty (\mathrm{Re}(g)(r))^4 r^{-2}\, dr\right)$$

is μ_∞ almost surely 0, and thus the expression in (2.8) is ill-defined.

In particular, this proves that the measure ν_∞ for the non-linear wave equation is singular with respect to μ_∞.

2.2.2 Setup of the Invariance Result

Recall, the one dimensional, complexified wave equation

$$\begin{cases} -i\partial_t w + |\partial_r| w = -|\partial_r|^{-1}\left(\frac{(\mathrm{Re}\, w)^3}{r^2}\right) \\ w(t,r) : \mathbb{R}_t \times [0,\infty) \to \mathbb{C} \\ w(t,0) = 0 \end{cases} \tag{2.10}$$

which we obtained from (2.1) via the change of variables $u \mapsto w(t,r) := ru(t,r) + ir\left(|\partial_r|^{-1}\partial_t u\right)(t,r)$. We make a precise definition of a strong solution of (2.10).

Definition 2.4 (Strong Solution for (2.10))**.** Let $T \in [0,\infty)$ and let $g \in C^s_{loc}([0,\infty) \to \mathbb{C})$. We say that $w(t,r) : [-T,T] \times [0,\infty) \to \mathbb{C}$ is a *strong solution of* (2.10) *on* $[-T,T]$ *with initial datum* g if

1. For each $R > 0$, we have $w \in C_t^0 C_r^s([-T, T] \times [0, R] \to \mathbb{C})$,

2. $w(0, r) = g(r)$, and

3. $w(t, r)$ obeys the corresponding Duhamel formula

$$w(t, r) = \left[e^{-it|\partial_r|} g\right](r) - i \int_0^t \left[\frac{e^{-i(t-\tau)|\partial_r|}}{|\partial_r|} \operatorname{Re}\left(\frac{w(\tau, \cdot)^3}{(\cdot)^2}\right)\right](r) d\tau,$$

   for each $t \in [-T, T]$.

Furthermore, if w is the unique strong solution on $[-T, T]$ with initial datum g, then we write

$$\mathrm{Flow}_\infty(t, g)(r) := w(t, r), \qquad |t| \leq T. \tag{2.11}$$

As with Theorem 2.1, we complete the Borel σ-algebra on $C_{loc}^s([0, \infty) \to \mathbb{C})$ with respect to the measure ν_∞, and we call a set ν_∞-*measurable* if it is an element of this larger σ-algebra. By abuse of notation, we also denote the extension of the measure to this larger σ-algebra by ν_∞.

The proof of the following result, as well as the discussion of the proof method, is the content of Section 5.

Theorem 2.5. *Fix $\frac{1}{3} < s < \frac{1}{2}$. There exists a Borel measurable set $\Omega_\infty \subseteq C_{loc}^s([0, \infty) \to \mathbb{C})$ such that*

1. $\nu_\infty(\Omega_\infty) = 1$;

2. *Each $g \in \Omega_\infty$ admits a unique global, strong solution:* $\mathrm{Flow}_\infty(t, g)$ *is defined for all $t \in \mathbb{R}$ and, for each $T > 0$ and $R > 0$, we have*

$$\mathrm{Flow}_\infty(t, g)|_{[0,R]} \in C_t^0 C_r^s([-T, T] \times [0, R] \to \mathbb{C}).$$

3. *For each* ν_∞*-measurable subset* $A \subseteq \Omega_\infty$ *and for each* $t \in \mathbb{R}$*, the set*

$$\mathrm{Flow}_\infty(t, A) := \{\mathrm{Flow}_\infty(t, g) \mid g \in A\}$$

is also a ν_∞*-measurable subset of* $C^s_{loc}([0,\infty) \to \mathbb{C})$ *and*

$$\nu_\infty(\mathrm{Flow}_\infty(t, A)) = \nu_\infty(A).$$

Moreover, if A *is Borel, then so is* $\mathrm{Flow}_\infty(t, A)$*.*

CHAPTER 3

Finite Volume Invariant Measures

In this chapter, we prove Theorem 2.1. The main new ingredients are a local well-posedness theory on $C_0^s([0,L] \to \mathbb{C})$, as well as some measure theoretic considerations from Appendix A.

3.1 Local Well-Posedness in C^s, $\frac{1}{3} < s < \frac{1}{2}$

We first establish the local well-posedness of (1.21) in $C_0^s([0,L] \to \mathbb{R})$, as the free propagator in this setting can be written down explicitly. Afterwards, we show that the complexified wave equation (1.22) is locally well-posed in $C_0^s([0,L] \to \mathbb{C})$.

To obtain explicit formulas for the linear evolution on $[0,L]$ with Dirichlet boundary values, we apply the usual odd reflections, and use d'Alembert's formula. For $L \geq 2$, for $0 \leq t \leq 1$, and for $0 < r < L$, we have

$$[\cos(t|\partial_r|)f](r) = \begin{cases} \frac{1}{2}(f(r+t) - f(t-r)) & r-t < 0, \\ \\ \frac{1}{2}(f(r+t) + f(r-t)) & 0 \leq r-t \leq r+t \leq L, \\ \\ \frac{1}{2}\left(f(r-t) - f(2L-r-t)\right) & L < r+t \end{cases} \tag{3.1}$$

and

$$\left[\frac{\sin(t|\partial_r|)}{|\partial_r|}g\right](r) = \begin{cases} \dfrac{1}{2}\displaystyle\int_{t-r}^{t+r} g(\rho)\,d\rho & r-t<0, \\ \dfrac{1}{2}\displaystyle\int_{r-t}^{r+t} g(\rho)\,d\rho & 0 \le r-t \le r+t < L, \\ \dfrac{1}{2}\displaystyle\int_{r-t}^{2L-r-t} g(\rho)\,d\rho & L \le r+t \end{cases} \tag{3.2}$$

with similar formulas when $-1 \le t \le 0$.

First, we establish some estimates on the linear propagator itself.

Lemma 3.1. *Fix $s \in [0,1)$. Let $f \in C_0^s([0,L] \to \mathbb{R})$. For $0 < T < \infty$, we have*

$$[\cos(t|\partial_r|)f](r), [\sin(t|\partial_r|)f](r) \in C_t^0 C_r^s([-T,T] \times [0,L] \to \mathbb{R}).$$

Furthermore, there exists $C = C(s) > 0$ such that

$$\|[\cos(t|\partial_r|)f](r)\|_{C_t^0 C_r^s([-T,T]\times[0,L])} \le C\|f\|_{C_0^s} \tag{3.3}$$

$$\|[\sin(t|\partial_r|)f](r)\|_{C_t^0 C_r^s([-T,T]\times[0,L])} \le C\|f\|_{C_0^s}. \tag{3.4}$$

Also, for every $t \in [-T,T]$, we have

$$0 = [\sin(t|\partial_r|)f](0) = [\sin(t|\partial_r|)f](L) = [\cos(t|\partial_r|)f](0) = [\cos(t|\partial_r|)f](L)$$

Proof. The assertions for $\cos(t|\partial_r|)$ follow immediately from (3.1).

Observe that, by (3.1) and (3.2), we have

$$\partial_r\left[\frac{\sin(t|\partial_r|)}{|\partial_r|}f\right](r) = [\cos(t|\partial_r|)f](r),$$

and so

$$\left\| \partial_r \left[\frac{\sin(t|\partial_r|)}{|\partial_r|} f \right] (r) \right\|_{C_t^0 C_r^s([-T,T]\times[0,L])} \lesssim \|f\|_{C_0^s}$$

Now, the operator $|\partial_r|(\partial_r)^{-1}$ is the finite volume Hilbert transform and, by Privalov's theorem (cf., [45]), is a bounded linear map from $C^s([0,L])$ to itself. Thus, (3.4) follows.

Finally, recall that the Fourier series of Hölder continuous functions converge uniformly to the original function. For fixed t, the function $\sin(t|\partial_r|)f$ is s-Hölder continuous, and the Fourier series of $\sin(t|\partial_x|)f$ is still a sine series. Thus,

$$0 = [\sin(t|\partial_x|)f](0) = [\sin(t|\partial_x|)f](L). \qquad \square$$

We use Lemma 3.1 to establish a local well-posedness for the second order equation (1.21). Similarly to Definition 1.7, we say that $v(t,r) : [-T,T]\times[0,L] \to \mathbb{R}$ is a strong solution of (1.21) on $[-T,T]$ with initial data (f_1, f_2) if

1. $v(t,r) \in C_t^0 C_r^s([-T,T]\times[0,L] \to \mathbb{R})$,
2. $(v, v_t)|_{t=0} = (f_1, f_2)$,
3. $v(t,r)$ obeys the Duhamel formula

$$v(t,r) = [\cos(t|\partial_r|)f_1](r) + \left[\frac{\sin(t|\partial_r|)}{|\partial_r|} f_2 \right] (r) - [K(v)](t,r),$$

where

$$[K(v)](t,r) := \int_0^t \left(\frac{\sin(t-\tau)|\partial_r|}{|\partial_r|} \frac{[v(\tau,\cdot)]^3}{(\cdot)^2} \right) (r) d\tau.$$

Proposition 3.2. *Fix $\frac{1}{3} < s < \frac{1}{2}$ and fix $L > 2$. Let $f_1 \in C_0^s([0,L] \to \mathbb{R})$ and let f_2 be a distribution supported on $[0,L]$ such that $|\partial_r|^{-1} f_2 \in C_0^s([0,L] \to \mathbb{R})$. There exists $T \in (0,1)$, whose value depends on L, $\|f_1\|_{C^s}$, $\||\partial_r|^{-1} f_2\|_{C^s}$, and a unique strong solution $v(t,r)$ of (1.21) on $[-T,T]$ with initial data (f_1, f_2).*

Moreover,

$$\|v\|_{C^0_t C^s_r([-T,T]\times[0,L]\to\mathbb{R})} \lesssim_s \|f_1\|_{C^s_0} + \left\||\partial_r|^{-1} f_2\right\|_{C^s_0} \tag{3.5}$$

and for every $\sigma \in [0,1)$,

$$v(t,r) - [\cos(t|\partial_r|)f_1](r) - \left[\frac{\sin(t|\partial_r|)}{|\partial_r|} f_2\right](r) \in C^0_t C^\sigma_r([-T,T]\times[0,L]\to\mathbb{R}). \tag{3.6}$$

Proof. We use the abbreviation $C^0_t C^s_r$ in place of $C^0_t C^s_r([-T,T]\times[0,L]\to\mathbb{R})$, where T is a constant to be specified later. We want to show that the mapping

$$v(t,r) \mapsto [\cos(t|\partial_r|)f_1](r) + \left[\frac{\sin(t|\partial_r|)}{|\partial_r|} f_2\right](r) - [K(v)](t,r), \tag{3.7}$$

admits a unique fixed point in $C^0_t C^s_r$ with zero boundary values by showing that it is a contraction for sufficiently small T.

For $(t,r) \in [-T,T]$, we denote $D(t,r)$ to be the domain of dependence from (t,r). For example, if $r - t < 0$, then

$$\begin{aligned} D(t,r) = \{(\tau,\rho) \mid t - r \le \tau \le t \text{ and } r' - t + \tau \le \rho \le t + r' - \tau\} \cup \\ \{(\tau,\rho) \mid 0 \le \tau \le t - r \text{ and } t - r' - \tau \le \rho \le t + r' - \tau\}. \end{aligned}$$

We use two key estimates. Recalling $u(t,0) = 0$, the first estimate is

$$|v(t,r)| \le r^s \|v\|_{C^0_t C^s_r}, \tag{3.8}$$

which shall give us integrability in the Duhamel terms. Recalling $3s > 1$, the second estimate is

$$b^{3s} - a^{3s} \lesssim_L b - a, \qquad 0 \le a \le b \le L \tag{3.9}$$

since the function $f(x) = x^{3s}$ is Lipschitz on bounded domains.

For the remainder of this proof, let us assume $0 \le r' < r \le L$. We treat only the case $t \ge 0$; the negative time case is similar.

Case 1, $t \le \frac{r-r'}{2}$. In this case, $D(t,r) \cap D(t,r') = \emptyset$. So, we simply use

$$|[K(v)](t,r) - [K(v)](t,r')| \le |[K(v)](t,r)| + |[K(v)](t,r')| \tag{3.10}$$

and estimate each term separately. We seek a bound of the form

$$|[K(v)](t,r) - [K(v)](t,r')| \lesssim_L \|v\|^3_{C^0_t C^s_r} t,$$

as we may then use the estimate $t \lesssim t^{1-\sigma}(r-r')^\sigma$ for every $\sigma \in [0,1)$. In particular, specializing to $\sigma = s$ and taking t sufficiently small will lead us to the desired fixed point of (3.7). We only estimate $[K(v)](t,r)$; the case $[K(v)](t,r')$ is similar.

Subcase 1.1, $r - t < 0$. In particular, $0 < r < t$. Then (3.2), (3.8), and (3.9) gives

$$|[K(v)](t,r)| \lesssim \left[\int_0^{t-r}\int_{(t-\tau)-r}^{r+(t-\tau)} + \int_{t-r}^{t}\int_{r-(t-\tau)}^{r+(t-\tau)}\right] \frac{|v(\tau,\rho)|^3}{\rho^2}\, d\rho\, d\tau$$

$$\begin{aligned}
&\lesssim \|v\|^3_{C^0_t C^s_r} \left[\int_0^{t-r}\int_{(t-\tau)-r}^{r+(t-\tau)} + \int_{t-r}^{t}\int_{r-(t-\tau)}^{r+(t-\tau)}\right] \rho^{3s-2}\, d\rho\, d\tau \\
&\lesssim_s \|v\|^3_{C^0_t C^s_r} \left[(t+r)^{3s} - (t-r)^{3s} \underbrace{-2(r)^{3s}}_{\le 0}\right] \\
&\lesssim_{s,L} \|v\|^3_{C^0_t C^s_r} r \\
&\lesssim_{s,L} \|v\|^3_{C^0_t C^s_r} t
\end{aligned}$$

Subcase 1.2, $0 \le r - t \le r + t \le L$. Then (3.2), (3.8), and (3.9) gives

$$
\begin{aligned}
|[K(v)](t,r)| &\lesssim \|v\|^3_{C^0_t C^s_r} \int_0^t \int_{r-(t-\tau)}^{r+(t-\tau)} \rho^{3s-2}\, d\rho\, d\tau \\
&\lesssim_s \|v\|^3_{C^0_t C^s_r} \big[(r+t)^{3s} - r^{3s} + \underbrace{(r-t)^{3s} - r^{3s}}_{\le 0}\big] \\
&\lesssim_{s,L} \|v\|^3_{C^0_t C^s_r} t
\end{aligned}
$$

Subcase 1.3, $L < r + t$. In particular, $L - r < t$ and $r - t < L - t < 2L - r - t$. Then (3.2), (3.8), and (3.9) gives

$$
\begin{aligned}
|[K(v)](t,r)| &\lesssim \|v\|^3_{C^0_t C^s_r} \left[\int_0^{t-(L-r)} \int_{r-(t-\tau)}^{2L-r-(t-\tau)} + \int_{t-(L-r)}^{t} \int_{r-(t-\tau)}^{r+(t-\tau)}\right] \rho^{3s-2}\, d\rho\, d\tau \\
&\lesssim_s \|v\|^3_{C^0_t C^s_r} \big(2(L^{3s} - r^{3s}) + \underbrace{(r-t)^{3s} - (2L-r-t)^{3s}}_{\le 0}\big) \\
&\lesssim_{s,L} \|v\|^3_{C^0_t C^s_r} (L-r) \\
&\lesssim_{s,L} \|v\|^3_{C^0_t C^s_r} t.
\end{aligned}
$$

Case 2, $\frac{r-r'}{2} < t$. In this case, $D(t,r) \cap D(t,r') \neq \emptyset$. Using (3.8), we have

$$
\big|[K(v)](t,r') - [K(v)](t,r)\big| \le \|v\|^3_{C^0_t C^s_r} \iint_{D(t,r)\triangle D(t,r')} \rho^{3s-2}\, d\rho\, d\tau, \quad (3.11)
$$

where $A \triangle B := (A \setminus B) \cup (B \setminus A)$ denotes the symmetric difference set. Here, we seek an estimate of the form

$$
\iint_{D(t,r)\triangle D(t,r')} \rho^{3s-2}\, d\rho\, d\tau \lesssim_{s,L} r - r'
$$

as we may then use the estimate $r - r' \lesssim t^{1-\sigma}(r-r')^{\sigma}$ for every $\sigma \in (0,1)$.

Subcase 2.1, $0 \le r' - t$ and $r + t \le L$. The domain of dependence is sketched in

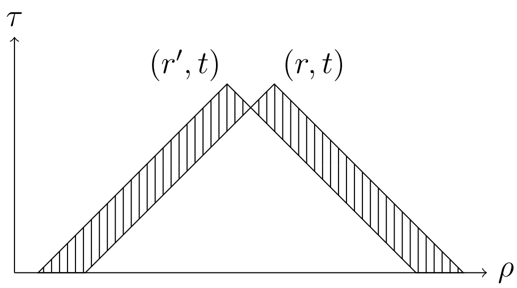

Figure 3.1: Subcase 2.1: shaded region is $D(t, r) \triangle D(t, r')$.

Figure 3.1. In this case, $D(t, r') \triangle D(t, r) \subset R_1 \cup R_2$, where

$$R_1 = \{(\tau, \rho) \mid 0 \leq \tau \leq t \text{ and } r' + t - \tau \leq \rho \leq r + t - \tau\},$$
$$R_2 = \{(\tau, \rho) \mid 0 \leq \tau \leq t \text{ and } r' - t + \tau \leq \rho \leq r - t + \tau\}.$$

Using (3.9), we have

$$\begin{aligned} \iint_{R_1} \rho^{3s-2} \, d\rho \, d\tau &= \int_0^t \int_{r'+t-\tau}^{r+t-\tau} \rho^{3s-2} \, d\rho \, d\tau \\ &\lesssim -(r)^{3s} + (r+t)^{3s} + (r)^{3s} - (r'+t)^{3s} \\ &\lesssim_L r - r'. \end{aligned}$$

and

$$\begin{aligned} \iint_{R_2} \rho^{3s-2} \, d\rho \, d\tau &= \int_0^t \int_{r'-t+\tau}^{r-t+\tau} \rho^{3s-2} \, d\rho \, d\tau \\ &\lesssim_s (r)^{3s} - (r')^{3s} + \underbrace{(r'-t)^{3s} - (r-t)^{3s}}_{\leq 0} \\ &\lesssim_{s,L} r - r'. \end{aligned}$$

Subcase 2.2, $r' - t < 0 \leq \frac{r'+r}{2} - t$ and $r + t \leq L$. The domain of dependence is sketched in Figure 3.2. In this case, $D(t, r') \triangle D(t, r) \subset R_1 \cup R_2 \cup R_3$,

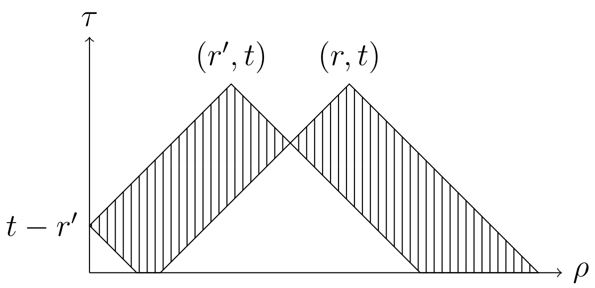

Figure 3.2: Subcase 2.2: shaded region is $D(t,r) \triangle D(t,r')$.

where

$$R_1 = \{(\tau, \rho) \mid 0 \le \tau \le t \text{ and } r' + t - \tau \le \rho \le r + t - \tau\},$$
$$R_2 = \{(\tau, \rho) \mid t - r' \le \tau \le t \text{ and } r' - t + \tau \le \rho \le r - t + \tau\},$$
$$R_3 = \{(\tau, \rho) \mid 0 \le \tau \le t - r' \text{ and } t - r' - \tau \le \rho \le t + r' - \tau\}.$$

Similarly to Subcase 2.1, we have $\iint_{R_1 \cup R_2} \rho^{3s-2}\, d\rho\, d\tau \lesssim_L r - r'$. Using (3.9) and that $t \le \frac{r'+r}{2}$, we have

$$\begin{aligned}
\iint_{R^3} \rho^{3s-2}\, d\rho\, d\tau &\lesssim \int_0^{t-r'} \int_{t-r'-\tau}^{t+r'-\tau} \rho^{3s-2}\, d\rho\, d\tau \\
&\lesssim_s -(2r')^{3s} + (t+r')^{3s} \underbrace{-(t-r')^{3s}}_{\le 0} \\
&\lesssim_{s,L} t - r' \\
&\lesssim_{s,L} r - r'.
\end{aligned}$$

Subcase 2.3, $\frac{r'+r}{2} - t < 0 \le r - t$ and $r + t \le L$. The domain of dependence is

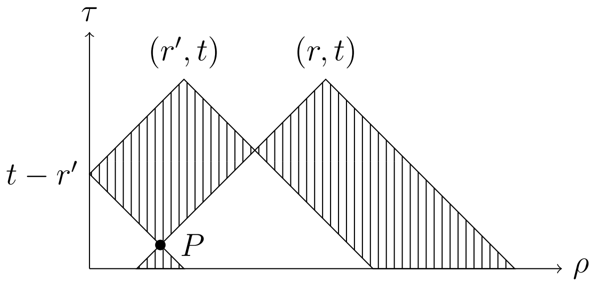

Figure 3.3: Subcase 2.3: shaded region is $D(t,r)\triangle D(t,r')$, and $P = \left(t - \frac{r+r'}{2}, \frac{r-r'}{2}\right)$.

sketched in Figure 3.3. In this case, $D(t,r')\triangle D(t,r) \subset \bigcup_{j=1}^{4} R_j$, where

$$
\begin{aligned}
R_1 &= \{(\tau,\rho) \mid 0 \le \tau \le t \text{ and } r' + t - \tau \le \rho \le r + t - \tau\},\\
R_2 &= \{(\tau,\rho) \mid t - r' \le \tau \le t \text{ and } r' - t + \tau \le \rho \le r - t + \tau\},\\
R_3 &= \{(\tau,\rho) \mid t - \tfrac{r'+r}{2} \le \tau \le t - r' \text{ and } 0 \le \rho \le r - r'\},\\
R_4 &= \{(\tau,\rho) \mid 0 \le \tau \le t - \tfrac{r'+r}{2} \text{ and } r - t + \tau \le \rho \le t - r' - \tau\}.
\end{aligned}
$$

Similarly to Subcase 2.1, we have $\iint_{R_1 \cup R_2} \rho^{3s-2}\, d\rho\, d\tau \lesssim_L r - r'$. Using (3.9), we have

$$
\begin{aligned}
\iint_{R^3} \rho^{3s-2}\, d\rho\, d\tau &\lesssim \int_{t-\frac{r'+r}{2}}^{t-r'} \int_0^{r-r'} \rho^{3s-2}\, d\rho d\tau\\
&\lesssim_{s,L} r - r'.
\end{aligned}
$$

Also, we have

$$
\begin{aligned}
\iint_{R^4} \rho^{3s-2}\, d\rho\, d\tau &\lesssim \int_0^{t-\frac{r'+r}{2}} \int_{r-t+\tau}^{t-r'-\tau} \rho^{3s-2}\, d\rho\, d\tau\\
&\lesssim_s -\left(\frac{r-r'}{2}\right)^{3s} + (t-r')^{3s} - \left(\frac{r-r'}{2}\right)^{3s} + (r-t)^{3s}.
\end{aligned}
$$

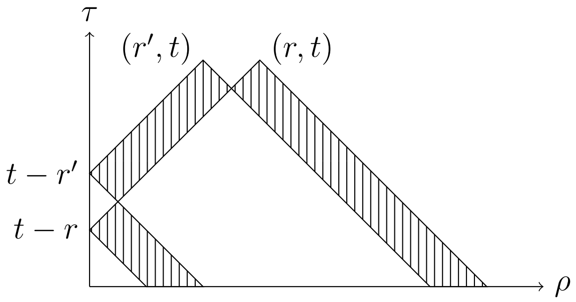

Figure 3.4: Subcase 2.4: shaded region is $D(t,r)\triangle D(t,r')$.

Using (3.9) and the fact that $t \leq r$ means $t - \frac{r}{2} \leq \frac{r}{2}$, we have

$$-\left(\frac{r-r'}{2}\right)^{3s} + (t-r')^{3s} \lesssim_L t - \frac{r}{2} - \frac{r'}{2} \lesssim_L r - r'.$$

Using (3.9) and the fact that $-t < -\frac{r+r'}{2}$, we have

$$-\left(\frac{r-r'}{2}\right)^{3s} + (r-t)^{3s} \lesssim_L r - t - \frac{r-r'}{2} \lesssim_L r - r'.$$

Subcase 2.4, $r - t < 0$ and $r + t \leq L$. The domain of dependence is sketched in Figure 3.4. In this case, $D(t,r')\triangle D(t,r) \subseteq \bigcup_{j=1}^{4} R_j$, where

$$R_1 = \{(\tau,\rho) \mid 0 \leq \tau \leq t \text{ and } r' + t - \tau \leq \rho \leq r + t - \tau\},$$

$$R_2 = \{(\tau,\rho) \mid t - r' \leq \tau \leq t \text{ and } r' - t + \tau \leq \rho \leq r - t + \tau\},$$

$$R_3 = \{(\tau,\rho) \mid t - r \leq \tau \leq t - r' \text{ and } 0 \leq \rho \leq r - r'\},$$

$$R_4 = \{(\tau,\rho) \mid 0 \leq \tau \leq t - r \text{ and } t - r' - \tau \leq \rho \leq t - r - \tau\}.$$

Similarly to Subcase 2.1, we have $\iint_{R_1 \cup R_2} \rho^{3s-2}\, d\rho\, d\tau \lesssim_L r - r'$. Similarly to Subcase 2.3, we also have $\iint_{R^3} \rho^{3s-2}\, d\rho\, d\tau \lesssim_L r - r'$. Using

(3.9), we have

$$\iint_{R^4} \rho^{3s-2}\, d\rho\, d\tau \lesssim \int_0^{t-r} \int_{t-r'-\tau}^{t-r-\tau} \rho^{3s-2}\, d\rho\, d\tau$$
$$\lesssim_s -(r-r')^{3s} + (t-r')^{3s} - (t-r)^{3s}$$
$$\lesssim_{s,L} r - r'.$$

Subcase 2.5, $0 \le r' - t$ and $\frac{r+r'}{2} + t \le L < r + t$. This case follows from reflecting the domains in Subcase 2.2 across the line $\rho = L/2$ and noting that ρ^{3s-2} is a decreasing function for $\rho \ge 0$.

Subcase 2.6, $0 \le r' - t$ and $r' + t \le L < \frac{r+r'}{2} + t$, which follows from reflecting the domains in Subcase 2.3 across the line $\rho = L/2$.

Subcase 2.7, $r' + t > L$, which follows from reflecting the domains in Subcase 2.4 across the line $\rho = L/2$.

Combining all the results from these cases, we have

$$|K(v)(t,r) - K(v)(t,r')| \lesssim_L \|v\|^3_{C^0_t C^s_r} (r-r')^\sigma t^{1-\sigma}$$

for every $\sigma \in (0,1)$, which is to say,

$$\big\|[K(v)](t,r)\big\|_{C^0_t C^\sigma_r([-T,T]\times[0,L])} \lesssim_L T^{1-\sigma} \|v\|^3_{C^0_t C^s_r}. \tag{3.12}$$

Using the fact that $|a^3 - b^3| \lesssim |a-b|(|a|^2 + |b|^2)$, a similar computation shows that

$$\big\|[K(v)](t,r) - [K(\tilde{v})](t,r)\big\|_{C^0_t C^\sigma_r} \lesssim_L T^{1-\sigma} \|v - \tilde{v}\|_{C^0_t C^s_r} \left(\|v\|^2_{C^0_t C^s_r} + \|\tilde{v}\|^2_{C^0_t C^s_r} \right). \tag{3.13}$$

Let us specialize to $\sigma = s$. Then, for T sufficiently small, the map in (3.7) is a self-mapping of the closed ball

$$\Big\{v \in C_t^0 C_r^s \mid \|v\|_{C_t^0 C_r^s} \leq (C+1)\Big(\|f_1\|_{C_0^s} + \big\||\partial_r|^{-1} f_2\big\|_{C_0^s}\Big)\Big\}$$

as well as a contraction. Here, C is the constant from Lemma 3.1. By contraction mapping, (3.7) admits a unique fixed point v, which is also the desired strong solution. □

Proposition 3.3. *Fix $\frac{1}{3} < s < \frac{1}{2}$ and fix $L > 2$. For each $\Lambda > 0$, there exists $T \in (0,1)$, whose value depends on s, L, and Λ, with the following properties:*

1. *Let $g \in C_0^s([0,L] \to \mathbb{C})$ such that $\|g\|_{C^s([0,L])} \leq \Lambda$. Then $\mathrm{Flow}_L(t,g)$ is defined for all $t \in [-T,T]$. Furthermore,*

$$\|\operatorname{Flow}_L(t,g)\|_{C_t^0 C_r^s([-T,T]\times[0,L])} \lesssim_{s,L} \|g\|^3_{C^s([0,L])} + \|g\|_{C^s([0,L])}, \tag{3.14}$$

   and for each $\sigma \in [0,1)$,

$$\mathrm{Flow}_L(t,g) - e^{-it|\partial_r|} g \in C_t^0 C_r^\sigma([-T,T]\times[0,L] \to \mathbb{C}). \tag{3.15}$$

2. *Let g be as above. If $\tilde{g} \in C_0^s([0,L])$ is such that $\|g - \tilde{g}\|_{C^s([0,L])}$ is sufficiently small, depending on Λ, s, L, then $\mathrm{Flow}_L(t,\tilde{g})$ also exists for all $t \in [-T,T]$ and*

$$\|\operatorname{Flow}_L(t,g) - \operatorname{Flow}_L(t,\tilde{g})\|_{C_t^0 C_r^s([-T,T]\times[0,L])} \lesssim_{s,L} \|g-\tilde{g}\|_{C^s}(\|g\|^2_{C^s} + \|\tilde{g}\|^2_{C^s}). \tag{3.16}$$

Proof. Let $g \in C_0^s([0,L] \to \mathbb{C})$ such that $\|g\|_{C^s([0,L])} \leq \Lambda$. Observe that the pair $(\mathrm{Re}(g), |\partial_r| \operatorname{Im}(g))$ obeys the hypotheses of Proposition 3.2. Hence there is a time interval $[-T,T]$, depending on Λ, s, L, and a unique strong solution $v(t,r) : [-T,T] \times [0,L] \to \mathbb{R}$ of (1.21) with initial data $(\mathrm{Re}(g), |\partial_r| \operatorname{Im}(g))$. By

(3.5),

$$\|v\|_{C_t^0 C_r^s([-T,T]\times[0,L])} \lesssim_{s,L} \|g\|_{C^s([0,L])}. \tag{3.17}$$

Observe that

$$|\partial_r|^{-1}\partial_t v = -\sin(t|\partial_r|)\operatorname{Re}(g) + \cos(t|\partial_r|)\operatorname{Im}(g) - \int_0^t \frac{\cos((t-\tau)|\partial_r|)}{|\partial_r|}\frac{(v(\tau,\cdot))^3}{(\cdot)^2}\,d\tau.$$

We claim that

$$\left(|\partial_r|^{-1}\partial_t v\right)(t,r) \in C_t^0 C_r^s([-T,T]\times[0,L]\to\mathbb{C})$$

and

$$0 = \left(|\partial_r|^{-1}\partial_t v\right)(t,0) = \left(|\partial_r|^{-1}\partial_t v\right)(t,L).$$

Assuming this claim, then

$$w := v + i|\partial_r|^{-1}\partial_t v \tag{3.18}$$

would be the unique[1] strong solution of (1.22).

We write $\|F\|_{C_t^0 C_r^s} := \|F\|_{C_t^0 C_r^s([-T,T]\times[0,L]\to\mathbb{C})}$. By Lemma 3.1, we have

$$\|-\sin(t|\partial_r|)\operatorname{Re}(g) + \cos(t|\partial_r|)\operatorname{Im}(g)\|_{C_t^0 C_r^s} \lesssim \|g\|_{C^s([0,L]\to\mathbb{C})}. \tag{3.19}$$

Letting

$$[\tilde{K}(v)](t,r) := \int_0^t \left[\frac{\cos((t-\tau)|\partial_r|)}{|\partial_r|}\frac{(v(\tau,\cdot))^3}{(\cdot)^2}\right](r)\,d\tau,$$

we shall prove $[\tilde{K}(v)](t,r) \in C_t^0 W_r^{1,p}([-T,T]\times[0,L]\to\mathbb{C})$ for large, but finite, p and apply Sobolev embedding.

First, we realize $|\partial_r|^{-1}$ as a convolution operator: for $f \in L^p([0,L]\to\mathbb{C})$ with $p \in (1,\infty]$, we extend it to $[-L,L]$ via $f(-r) = -f(r)$ for $r \in [0,L]$, and then

[1]Uniqueness follows from $(v, v_t) = (\operatorname{Re}(w), |\partial_r|\operatorname{Im}(w))$ and the uniqueness aspect of Proposition 3.2.

extend to $\mathbb{R}$ via $2L$ periodicity. Having this extension, then

$$\left[|\partial_r|^{-1} f\right](r) = (f * h)(r) = \int_{-L}^{L} f(r-\rho) h(\rho)\, d\rho, \tag{3.20}$$

where

$$h(r) = \sum_{n=1}^{\infty} (n\pi/L)^{-1} \cos(n\pi r/L) = \operatorname{Re}\left(\log(1 - e^{i\pi r/L})\right).$$

Note that $h(r) \in L^q([-L, L] \to \mathbb{C})$ for every $q \in [1, \infty)$.

Next, we check boundary conditions. By (3.8), we have

$$\|(v(t,r))^3 r^{-2}\|_{C_t^0 L_r^p([-T,T]\times[0,L])} \lesssim_p \|v\|^3_{C_t^0 C_r^s}. \tag{3.21}$$

for every $p \in [1, \frac{1}{2-3s})$. Given (3.20), (3.21), and the fact that both $\cos(n\pi(-\rho)/L)$ and $\cos(n\pi(L-\rho)/L) = (-1)^n \cos(n\pi\rho/L)$ are even in ρ, we have

$$\left[|\partial_r|^{-1} \frac{(v(t,\cdot))^3}{(\cdot)^2}\right](0) = 0 = \left[|\partial_r|^{-1} \frac{(v(t,\cdot))^3}{(\cdot)^2}\right](L)$$

for all t. Given (3.1), it follows that $0 = [\tilde{K}(v)](t, 0) = [\tilde{K}(v)](t, L)$, as well.

In view of (3.1), we also have

$$\left\| \left[\cos((t-\tau)|\partial_r|) \frac{(v(\tau,\cdot))^3}{(\cdot)^2}\right](r) \right\|_{C_\tau^0 L_r^p([0,t]\times[0,L])} \lesssim \|v\|^3_{C_t^0 C_r^s}.$$

for every t. By Hölder's inequality,

$$\int_0^t \int_{-L}^{L} |h(r-\rho)| \left| \left[\cos((t-\tau)|\partial_r|) \frac{(v(\tau,\cdot))^3}{(\cdot)^2}\right](\rho) \right| \, d\rho\, d\tau \lesssim_L \|v\|^3_{C_t^0 C_r^s}, \tag{3.22}$$

where we extend the integrands from $[0, L]$ to $\mathbb{R}$ in the manner above. Hence,

$$\left\| [\tilde{K}(v)](t,r) \right\|_{C_t^0 C_r^0([-T,T]\times[0,L])} \lesssim \|v\|^3_{C_t^0 C_r^s}. \tag{3.23}$$

Furthermore, by Fubini's Theorem, we may also write

$$[\tilde{K}(v)](t,r) = \left[|\partial_r|^{-1}F(t,\cdot)\right](r) \tag{3.24}$$

where

$$F(t,r) := \int_0^t \left[\cos((t-\tau)|\partial_r|)\frac{(v(\tau,\cdot))^3}{(\cdot)^2}\right](r)\, d\tau.$$

If $0 \le r-t < r+t \le L$, then (3.1) and (3.8) gives

$$\begin{aligned}
|F(t,r)| &\lesssim \int_0^t \left|\frac{(v(\tau,r+(t-\tau))^3}{(r+(t-\tau))^2}\right| + \left|\frac{(v(\tau,r-(t-\tau))^3}{(r-(t-\tau))^2}\right| d\tau \\
&\lesssim \|v\|^3_{C_t^0C_r^s}\int_0^t (r+(t-\tau))^{3s-2} + (r-(t-\tau))^{3s-2}\, d\tau \\
&\lesssim_{s,L} \|v\|^3_{C_t^0C_r^s}.
\end{aligned}$$

When $r-t<0$ or when $r+t>L$, we may similarly show $|F(t,r)| \lesssim_{s,L} \|v\|^3_{C_t^0C_r^s}$, and thus

$$\|F(t,r)\|_{C_t^0C_r^0} \lesssim_{s,L} \|v\|^3_{C_t^0C_r^s}. \tag{3.25}$$

Finally, the operator $\partial_r|\partial_r|^{-1}$ (i.e., finite volume Hilbert transform) is a bounded linear operator from $L_r^p([0,L])$ to itself for every $p \in [1,\infty)$. Thus, (3.23), (3.24), and (3.25) gives

$$\left\|[\tilde{K}(v)](t,r)\right\|_{C_t^0W_r^{1,p}([-T,T]\times[0,L])} \lesssim_{p,L} \|v\|^3_{C_t^0C_r^s} \tag{3.26}$$

for every $p \in [1,\infty)$. By Sobolev embedding, we have

$$\tilde{K}(v) \in C_t^0C_r^\sigma([-T,T]\times[0,L]\to\mathbb{C}) \tag{3.27}$$

for every $\sigma \in [0,1)$ and

$$\left\|[\tilde{K}(v)](t,r)\right\|_{C_t^0C_r^\sigma([-T,T]\times[0,L])} \lesssim_{\sigma,L} \|v\|^3_{C_t^0C_r^s}. \tag{3.28}$$

Specializing to $\sigma = s$ and combining (3.17), (3.19), (3.28), this proves

$$\|w\|_{C_t^0 C_r^s} \lesssim \|g\|_{C^s}^3 + \|g\|_{C^s}.$$

In particular, we have established (3.14) and that w is the unique strong solution of (1.22) with initial data g. (3.15) follows from (3.6) and (3.27).

The second part of the proposition follows from the fact that

$$|a^3 - b^3| \lesssim |a - b|(|a|^2 + |b|^2)$$

and arguing as above (and replacing T by $T/2$, if necessary). $\square$

An immediate corollary is the following continuity result.

Corollary 3.4. *Let $L > 2$ and let $0 \leq R \leq L - 2$. Let $g_k \in C_0^s([0, L] \to \mathbb{C})$, $1 \leq k \leq \infty$, such that each g_k admits a unique strong solution of* (1.22) *for $|t| \leq 1$. If*

$$\lim_{k\to\infty} \|g_k - g_\infty\|_{C^s([0,R+1])} = 0,$$

then

$$\lim_{k\to\infty} \|\operatorname{Flow}_L(t, g_k) - \operatorname{Flow}_L(t, g_\infty)\|_{C_t^0 C_r^s([-1,1]\times[0,R])} = 0. \tag{3.29}$$

Furthermore, if

$$\lim_{k\to\infty} \|g_k - g_\infty\|_{C^s([0,L])} = 0,$$

then

$$\lim_{k\to\infty} \|\operatorname{Flow}_L(t, g_k) - \operatorname{Flow}_L(t, g_\infty)\|_{C_t^0 C_r^s([-1,1]\times[0,L])} = 0. \tag{3.30}$$

Proof. For each $\lambda \in [0, L - 1]$, we define the "linear cut-off" operator Ψ_R :

$C_0^s([0,L]) \to C_0^s([0,L])$ via

$$[\Psi_\lambda g](r) = \begin{cases} g(r) & 0 \le r \le \lambda, \\ g(\lambda)(\lambda + 1 - x) & \lambda \le r \le \lambda + 1, \\ 0 & \lambda + 1 \le r \le L. \end{cases}$$

Indeed,

$$\|\Psi_\lambda g\|_{C_0^s([0,L])} \lesssim_{s,L} \|g\|_{C_0^s([0,L])} \tag{3.31}$$

where the implicit constant is independent of the choice of λ. Also, we clearly have

$$\lim_{k\to\infty} \|\Psi_{R+1} g_k - \Psi_{R+1} g_\infty\|_{C^s([0,L])} = 0. \tag{3.32}$$

Letting

$$\Lambda := \|\operatorname{Flow}_L(t, g_\infty)\|_{C_t^0 C_r^s([-1,1]\times[0,L])},$$

then (3.31) gives

$$\|\Psi_{R+1} g_\infty\|_{C^s([0,L])} \lesssim_{s,L} \|g_\infty\|_{C^s([0,L])} \le \Lambda.$$

By Proposition 3.3, there exists a time T, depending upon s, L, and Λ, such that $\operatorname{Flow}_L(t, \Psi_{R+1} g_\infty) \in C_t^0 C_s^r([-T,T]\times[0,L] \to \mathbb{C})$. Proposition 3.3 also gives

$$\lim_{k\to\infty} \|\operatorname{Flow}_L(t, \Psi_{R+1} g_k) - \operatorname{Flow}_L(t, \Psi_{R+1} g_\infty)\|_{C_t^0 C_s^r([-T,T]\times[0,L])} = 0.$$

By finite speed of propagation,

$$\lim_{k\to\infty} \|\operatorname{Flow}_L(t, g_k) - \operatorname{Flow}_L(t, g_\infty)\|_{C_t^0 C_s^r([-T,T]\times[0,R+1-T])} = 0.$$

We now seek to iterate the argument.

As above, we have

$$\|\Psi_{R+1-T}\operatorname{Flow}_L(T,g_\infty)\|_{C^s([0,L])} \lesssim_{s,L} \|\operatorname{Flow}_L(T,g_\infty)\|_{C^s([0,L])} \leq \Lambda$$

and

$$\lim_{k\to\infty} \|\Psi_{R+1-T}\operatorname{Flow}_L(T,g_k) - \Psi_{R+1-T}\operatorname{Flow}_L(T,g_\infty)\|_{C^r_s([0,L])} = 0.$$

Thus, we may argue as above, using Proposition 3.3, to conclude that

$$\lim_{k\to\infty} \|\operatorname{Flow}_L(t,g_k) - \operatorname{Flow}_L(t,g_\infty)\|_{C^0_t C^r_s([0,2T]\times[0,R+1-2T])} = 0.$$

We may iterate this argument approximately $2\lfloor 1/T \rfloor$ times, using the value Λ as a persistent bound, to establish (3.29). To prove (3.30), we argue as above, again using Λ as a persistent bound, but without using the operator Ψ_R. $\square$

3.2 Proof of Theorem 2.1

By Theorem A.3, the set $C^s_0([0,L]\to\mathbb{C})$ is a Borel subset of $\dot{H}^s_0([0,L]\to\mathbb{C})$ and the Borel σ-algebras on $C^s_0([0,L]\to\mathbb{C})$ generated by the s-Hölder norm and by the Sobolev norm must agree. Recall from the discussion in Section 2.1 that $\nu_L(C^s_0([0,L]))=1$.

Let Π_L be as in Theorem 1.10, then $\Pi_L \cap C^{0,s}_0([0,L]\to\mathbb{C})$ has full ν_L measure. By Theorem 1.10, the set

$$\Omega_L := \bigcap_{t\in\mathbb{Q}} \operatorname{Flow}_L\left(t, \Pi_L \cap C^s_0\right)$$

is a well-defined Borel subset of $C^s_0([0,L]\to\mathbb{C})$ with ν_L measure 1. This gives the first assertion of Theorem 2.1.

For the second assertion, let $T\in(0,\infty)$ and let $g\in\Omega_L$. By Theorem 1.10,

$\mathrm{Flow}_L(t,g)$ is defined globally in time and we have $\mathrm{Flow}_L(t,g) \in C_t^0 \dot{H}_0^s([-T,T] \times [0,L] \to \mathbb{C})$. The fact that $\mathrm{Flow}_L(t,g) \in C_t^0 C_r^s([-T,T] \times [0,L] \to \mathbb{C})$ follows from Proposition 3.3 and the definition of Ω_L.

The fact that Flow_L preserves Borel measurability follows from Theorem 1.10 and Theorem A.3. Moreover, by Theorem 1.10, Flow_L preserves the measure of all Borel sets.

Finally, let $A \subseteq C_0^s([0,L] \to \mathbb{C})$ be ν_L-measurable with $\nu_L(A) = 0$. Recalling Proposition A.7, for every $n > 0$, there exists an open set $U_n \supset A$ such that $\nu_L(U_n) < \frac{1}{n}$. By the previous paragraph, $\mathrm{Flow}_L(t, U_n)$ is Borel measurable with $\nu_L(\mathrm{Flow}_L(t,U_n)) < \frac{1}{n}$. It follows that $\bigcap_{n=1}^{\infty} \mathrm{Flow}_L(t,U_n)$ is a Borel set of measure 0 which contains $\nu_L(\mathrm{Flow}_L(t,A))$. Thus $\nu_L(\mathrm{Flow}_L(t,A))$ is ν_L-measurable with measure 0. As every ν_L-measurable set is the union of a Borel set and a ν_L-null set, Theorem 2.1 follows.

CHAPTER 4

Construction of the Infinite Volume Measure

In this chapter, we prove a lengthier version Theorem 2.3, which we express in Theorem 4.6.

Recall the definition of the (finite volume) measures $\mu_L, \mu_{L,1}$, and $\mu_{L,2}$ from Definition 1.5 and recall the definition of ν_L from Proposition 1.9. Also recall the definition of the infinite volume limit measure from Defintion 2.2.

To construct ν_∞, we shall separate ν_L into its "real" and "imaginary" components and compute the corresponding infinite volume limit of each. Namely, let

$$d\nu_{L,1}(f) := \frac{1}{Z_L} \exp\left(-\frac{1}{4}\int_0^L |f(r)|^4 r^{-2}\, dr\right) d\mu_{L,1}(f) \tag{4.1}$$

and let

$$\nu_{L,2} = \mu_{L,2}. \tag{4.2}$$

Then the connection between $\nu_L, \nu_{L,1}, \nu_{L,2}$ is as follows: for Borel measurable $A_1, A_2 \subseteq \dot{H}_0^s([0, L] \to \mathbb{R})$,

$$\nu_L(\{g \mid \mathrm{Re}(g) \in A_1, \mathrm{Im}(g) \in A_2\}) = \nu_{L,1}(A_1)\nu_{L,2}(A_2). \tag{4.3}$$

After constructing the infinite volume limits of $\nu_{L,1}$ and $\nu_{L,2}$ separately, we shall piece these limit measures back together in a way similar to (4.3).

Ultimately, this construction will follow from an analysis of the long-time asymptotics of the fundamental solution of a particular parabolic PDE (cf., Definition 4.5 and (4.8) below). We reduce the measure theoretic problem to the parabolic PDE computation in the following manner: first observing the equivalence of the Borel and the cylinder σ-algebras (see Definition 4.1 below) on $C_0^s([0, L] \to \mathbb{C})$ and on $C_{loc}^s([0, \infty) \to \mathbb{C})$, and then seeking to utilize the Kolmogorov consistency and continuity theorem.

Definition 4.1. Let $\Lambda = \mathbb{C}$ or $\mathbb{R}$. Let $I \subseteq \mathbb{R}$ be an interval and let X be a subset of $\Lambda^I := \{f : I \to \Lambda\}$. The *cylinder set σ-algebra on X* is the σ-algebra generated by sets of the form

$$\{f \in X \mid f(r) \in B\}$$

where $r \in I$ and $B \subseteq \Lambda$ is Borel. We say that a cylinder set probability measure μ on X is supported on a cylinder measurable subset $A \subseteq X$ if $\mu(A) = 1$.

Proposition 4.2. *Let $\Lambda = \mathbb{C}$ or $\mathbb{R}$. The Borel and cylinder σ-algebras on $C_0^s([0, L] \to \Lambda)$ coincide. Also, endow $C_{loc}^s([0, \infty) \to \Lambda)$ with the metric*

$$d(f, g) = \sum_{n=1}^{\infty} 2^{-n} \frac{\|f - g\|_{C^s([0,n])}}{1 + \|f - g\|_{C^s([0,n])}} \tag{4.4}$$

and the induced metric topology. Then $C_{loc}^s([0, \infty) \to \Lambda)$ is a Polish space (i.e., separable, completely metrizable). Furthermore, the Borel σ-algebra generated by (4.4) *and the cylinder σ-algebra on $C_{loc}^s([0, \infty) \to \Lambda)$ coincide.*

Remark. The full strength of the fact that $C_{loc}^s([0, \infty) \to \mathbb{C})$ is Polish will not be used until Section 5. We record the result for convenience in the following proof.

Proof. Note that convergence in the $C_0^s([0, L] \to \mathbb{C})$ norm implies uniform convergence, and hence point-wise convergence. It follows that evaluation at a point is continuous with respect to this norm, which then implies that the cylinder σ-algebra is contained in the Borel σ-algebra on $C_0^s([0, L] \to \mathbb{C})$.

For the reverse containment, let us fix $f_0 \in C^s_0([0,L] \to \mathbb{C})$ and $\lambda > 0$. Then

$$\begin{aligned}
\overline{B_L(f_0,\lambda)} &:= \{f \in C^s_0([0,L] \to \mathbb{C}) \mid \|f - f_0\|_{C^s([0,L])} \le \lambda\} \\
&= \bigcap_{\substack{r,r' \in \mathbb{Q}\cap[0,L],\\ r' \le r}} \{f \mid \big|[f(r) - f_0(r)] - [f(r') - f_0(r')]\big| \le \lambda (r - r')^s\} \\
&= \bigcap_{\substack{r,r' \in \mathbb{Q}\cap[0,L],\\ r' \le r}} \left\{ f \mid f(r') \in \mathbb{C}, f(r) \in \overline{B_{\mathbb{C}}\Big(f(r') + f_0(r') - f_0(r), \lambda(r - r')^s\Big)} \right\},
\end{aligned}$$

which is a countable intersection of cylinder sets. Therefore,

$$B_L(f_0,\lambda) := \{f \mid \|f - f_0\|_{C^s([0,L])} < \lambda\} = \bigcup_{k=1}^{\infty} \overline{B_L(f_0, \lambda(1 - 2^{-k}))},$$

from which it follows that every Borel subset of $C^s_0([0,L] \to \mathbb{C})$ is cylinder measurable.

Fix $n \in \mathbb{N}$ and recall that $C^s([0,n] \to \mathbb{C})$ is Polish. Let $\{f_{n,m} \mid m \in \mathbb{N}\}$ be a countable dense subset of $C^s([0,n] \to \mathbb{C})$. We define

$$\tilde{f}_{n,m}(r) = \begin{cases} f_{n,m}(r) & \text{if } r \in [0,n] \\ f_{n,m}(n) & \text{if } r \in (n,\infty) \end{cases}$$

Then $\{\tilde{f}_{n,m} \mid n,m \in \mathbb{N}\}$ is a countable dense subset of $C^s_{loc}([0,\infty) \to \mathbb{C})$.

Also, a Cauchy sequence in $C^s_{loc}([0,\infty) \to \mathbb{C})$ must also be Cauchy with respect to each semi-norm $\|\cdot\|_{C^s([0,n])}$. Completeness of $C^s_{loc}([0,\infty) \to \mathbb{C})$ then follows from the completeness of $C^s([0,n] \to \mathbb{C})$.

Note that convergence in the metric d implies local uniform convergence, and hence point-wise convergence. Similarly to above, the cylinder σ-algebra is hence contained in the Borel σ-algebra induced by d.

Arguing as above shows that, for $n \ge 1$, for $f_0 \in C^s_{loc}([0,\infty) \to \mathbb{C})$ and for

$\lambda > 0$,

$$B_n(f_0, \lambda) := \{f \in C^s_{loc}([0,\infty) \to \mathbb{C}) \mid \|f - f_0\|_{C^s([0,n])} < \lambda\} \tag{4.5}$$

is also cylinder measurable. As open sets of the form (4.5) constitute a sub-basis for the topology on $C^s_{loc}([0,\infty) \to \mathbb{C})$ and this space is separable, it follows that every open ball (and hence every Borel set) is cylinder measurable. Finally, observe that all of the arguments also hold if we replace $\mathbb{C}$ by $\mathbb{R}$. $\square$

To express the Kolmogorov theorem, we first recall a definition.

Definition 4.3. Let $\Lambda = \mathbb{C}$ or $\mathbb{R}$. Let I be an infinite index set, and for each finite sub-index $A \subseteq I$, let P_A be some Borel probability measure on $\Lambda^{|A|}$. We say that the collection $\{P_A\}_{A\subseteq I, |A|<\infty}$ is a *consistent family of finite dimensional distributions indexed on* I if, for every finite $A \subseteq I$ and every $r \in I \setminus A$, we have

$$P_A(B) = P_{A\cup\{r\}}(B \times \Lambda)$$

for every Borel set $B \subseteq \Lambda^{|A|}$.

The proof of the Kolmogorov theorem be found in [32] and in [38].

Theorem 4.4 (Kolmogorov Continuity and Consistency)**.** *Let $\Lambda = \mathbb{C}$ or $\mathbb{R}$. Let $\{P_{r_1,\ldots,r_n}\}$ be some consistent family of finite dimensional distributions indexed on some interval $I \subseteq \mathbb{R}$. Then there exists a unique (cylinder) probability measure P on $\Lambda^I = \{f : I \to \Lambda\}$ such that for Borel sets $B_1, \ldots, B_n \subseteq \Lambda$ and for $r_1, \ldots, r_n \in I$,*

$$P(f(r_j) \in B_j, j = 1, \ldots, n) = P_{r_1,\ldots,r_n}(B_1 \times \cdots \times B_n).$$

Let $\beta, \gamma > 0$. If, for each compact sub-interval $K \subseteq I$, there exists a $C_K < \infty$ such that for all $r, s \in K$,

$$\mathbb{E}^P\left[|f(r) - f(s)|^\beta\right] \leq C_K|r - s|^{1+\gamma},$$

then P is supported on $C^0_{loc}(I \to \Lambda)$. Furthermore, for every $0 \le s < \gamma/\beta$, P is also supported on $C^s_{loc}(I \to \Lambda)$.

In order to apply Kolmogorov's theorem to construct the infinite volume Gibbs measure, denoted ν_∞, we shall reduce the problem to computing the asymptotics of the fundamental solution of a certain parabolic PDE. The main tool used in this reduction is a multi-time Feynman–Kac formula (cf., Theorem B.3).

Definition 4.5. Let $C_1(r,x)$, $C_2(r,x)$, and $C_3(r,x)$ be functions from $[0,\infty)\times\mathbb{R}$ to $\mathbb{R}$ and consider the equation

$$L\phi := -\partial_r\phi + C_1(r,x)\partial_x^2\phi + C_2(r,x)\partial_x\phi + C_3(r,x)\phi = 0.$$

Let $\phi(r,x;s,y)$ be a function on the following domain: $x,y\in\mathbb{R}$, $s\ge 0$, and $r>s$. We say that ϕ is the *fundamental solution of $L\phi = 0$ at (s,y)* if it obeys both of the following conditions:

1. ϕ is continuously differentiable once in r and twice in x and satisfies, as a function of r and x, the equation $L\phi = 0$ (in the classical sense).
2. $\lim_{r\downarrow s}\phi(r,x;s,y) = \delta_{x-y}$ as linear functionals on $C_0(\mathbb{R})$: for $f\in C_0(\mathbb{R})$,

$$\lim_{r\downarrow s}\int_{\mathbb{R}}\phi(r,x;s,y)f(x)\,dx = f(y). \tag{4.6}$$

If $\phi(r,x;s,y)$ is the fundamental solution of $L\phi = 0$ at every $(s,y)\in\mathbb{R}^{\ge 0}\times\mathbb{R}$, then we simply say that *ϕ is the fundamental solution of $L\phi = 0$*.

For example, the heat kernel

$$\phi_0(r,x;s,y) := \frac{1}{\sqrt{2\pi(r-s)}}\exp\left(-\frac{(x-y)^2}{2(r-s)}\right) \tag{4.7}$$

is the fundamental solution of the heat equation $-\partial_r\phi_0 = \frac{1}{2}\partial_x^2\phi_0$.

Recalling the definitions of $\nu_L, \nu_{L,1}$, and $\nu_{L,2}$ in (1.23), (4.1), and (4.2). We now state the main result of this chapter.

Theorem 4.6. *1. There exists a (strictly positive) function $\phi(r, x; s, y)$ which is the fundamental solution of*

$$-\partial_r \phi + \frac{1}{2}\partial_x^2 \phi - \frac{1}{4}\frac{x^4}{r^2}\phi = 0 \tag{4.8}$$

at each $(s, y) \in (0, \infty) \times \mathbb{R}$ and at $(s, y) = (0, 0)$.

2. *For each $L > 1$, the measure $\nu_{L,1}$ obeys the following law: let $0 < r_1 < \cdots < r_N < L$ and let $B_1, \ldots, B_N \subseteq \mathbb{R}$ be Borel sets, then, with ϕ as above,*

$$\begin{aligned}&\mathbb{P}_{\nu_{L,1}}(f(r_j) \in B_j, j = 1, \ldots, N)\\ &= \int_{B_1} \cdots \int_{B_N} \frac{\phi(L, 0; r_N, x_N)}{\phi(L, 0; 0, 0)} \prod_{j=2}^{N} \phi(r_j, x_j; r_{j-1}, x_{j-1})\phi(r_1, x_1; 0, 0)\, dx_n \cdots dx_1.\end{aligned}$$

3. *There is a positive, bounded, continuous function $F(s, y) : (0, \infty) \times \mathbb{R} \to \mathbb{R}$ such that, for fixed s,*

$$\lim_{L\to\infty} \left\| \frac{\phi(L, 0; s, y)}{\phi(L, 0; 0, 0)} - F(s, y) \right\|_{C_y^0} = 0.$$

4. *There exists a unique cylinder probability measure $\nu_{\infty,1}$ on $C^0_{loc}([0, \infty) \to \mathbb{R})$ such that for $0 < r_1 < \cdots < r_N$ and for Borel sets $B_1, \ldots, B_N \subseteq \mathbb{R}$,*

$$\begin{aligned}&\mathbb{P}_{\nu_{\infty,1}}(f(r_j) \in B_j, j = 1, \ldots, N)\\ &= \int_{B_1} \cdots \int_{B_N} F(r_N, x_N) \prod_{j=2}^{N} \phi(r_j, x_j; r_{j-1}, x_{j-1})\phi(r_1, x_1; 0, 0)\, dx_N \cdots dx_1.\end{aligned}$$

   Furthermore, for every $s \in [0, \frac{1}{2})$, $\nu_{\infty,1}$ is supported on $C^s_{loc}([0, \infty) \to \mathbb{R})$.

5. *Fix $s \in [0, \frac{1}{2})$. Let $1 < R < L$ and let $\nu_{\infty,1}|_{[0,R]}$ and $\nu_{L,1}|_{[0,R]}$ denote the image*

measure (or push-froward measure) of $\nu_{\infty,1}$ *and* $\nu_{L,1}$*, resp., on* $C^s([0,R] \to \mathbb{R})$ *under the restriction map* $f \mapsto f|_{[0,R]}$*. Then* $\nu_{\infty,1}|_{[0,R]}$ *and* $\nu_{L,1}|_{[0,R]}$ *are mutually absolutely continuous, with Radon–Nikodym derivative*

$$\frac{d\nu_{L,1}|_{[0,R]}}{d\nu_{\infty,1}|_{[0,R]}}(f) = \frac{\phi(L,0;R,f(R))}{F(R,f(R))\phi(L,0;0,0)}.$$

For every Borel subset $A \subseteq C^0([0,R] \to \mathbb{R})$*, we have*

$$\lim_{L\to\infty} \nu_{L,1}|_{[0,R]}(A) = \nu_{\infty,1}|_{[0,R]}(A). \tag{4.9}$$

6. *Let* W *denote the Wiener measure on* $C^s_{loc}([0,\infty) \to \mathbb{R})$*. Let* ν_∞ *be the Borel probability measure on* $C^s_{loc}([0,\infty) \to \mathbb{C})$ *given by*

$$\nu_\infty(\{g \mid \mathrm{Re}(g) \in A_1, \mathrm{Im}(g) \in A_2\}) := \nu_{\infty,1}(A_1)W(A_2) \tag{4.10}$$

for Borel measurable sets $A_1, A_2 \subseteq C^s_{loc}([0,\infty) \to \mathbb{R})$*.*

Let $1 < R < L$*. Then* $\nu_\infty|_{[0,R]}$ *and* $\nu_L|_{[0,R]}$ *are mutually absolutely continuous measures on* $C^s([0,R] \to \mathbb{C})$*. For every Borel subset* $A \subseteq C^s([0,R] \to \mathbb{C})$*, we have*

$$\lim_{L\to\infty} \nu_L|_{[0,R]}(A) = \nu_\infty|_{[0,R]}(A). \tag{4.11}$$

Furthermore, ν_∞ *is the unique probability measure which obeys* (4.11)*.*

Let $\mathcal{F}_R$ *denote the completion of the Borel* σ*-algebra on* $C^s([0,R])$ *with respect to any of these measures. Then* (4.11) *holds for every* $A \in \mathcal{F}_R$*.*

We will break up the proof of Theorem 4.6 into several sections.

4.1 Theorem 4.6, Part 1: The Parabolic PDE

The main difficulty in constructing a fundamental solution of (4.8) lies with the coefficient $-\frac{1}{4}\frac{x^4}{r^2}$. It is neither bounded in x nor is it uniformly Hölder continuous. For $x \neq 0$, this coefficient also goes to $-\infty$ point-wise as r goes to 0.

To handle these issues, we apply suitable cut-offs. For real numbers $a, b \in \mathbb{R}$, let

$$a \vee b := \max(a, b) \qquad \text{and} \qquad a \wedge b := \min(a, b).$$

Let us consider the cut-off equations

$$L_n\phi := -\partial_r\phi + \frac{1}{2}\partial_x^2 u - \frac{1}{4}\frac{x^4 \wedge n}{r^2 \vee \frac{1}{n}}\phi = 0, \qquad n = 1, 2, \ldots \tag{4.12}$$

For each $n \in \mathbb{N}$, it is not hard to see that the coefficient $-\frac{1}{4}\frac{x^4 \wedge n}{r^2 \vee 1/n}$ is bounded and uniformly Hölder continuous in x (indeed, Lipschitz). By the parametrix method (cf., [16, pg. 14–20] or [20]) there exists a there exists a unique fundamental solution $\phi_n(r, x; s, y)$ of the cut-off PDE $L_n\phi = 0$ at all $(s, y) \in [0, \infty) \times \mathbb{R}$ and obeys the Duhamel formula

$$\begin{aligned} &\phi_n(r, x; s, y) \\ &\quad = \phi_0(r, x; s, y) - \frac{1}{4}\int_s^r \int_{\mathbb{R}} \phi_0(r, x; \rho, w)\frac{w^4 \wedge n}{\rho^2 \vee \frac{1}{n}}\phi_n(\rho, w; s, y)\, dw\, d\rho. \end{aligned} \tag{4.13}$$

Furthermore, $\phi_n > 0$ for each $n \in \mathbb{N}$. Note that ϕ_0 denotes the standard heat kernel (cf., (4.7)), which is consistent with (4.12).

Lemma 4.7. *Let ϕ_n be as above. Then*

$$\phi_0 \geq \phi_1 \geq \phi_2 \geq \cdots \geq 0. \tag{4.14}$$

Furthermore, the function

$$\phi(r,x;s,y) := \lim_{n\to\infty} \phi_n(r,x;s,y) \tag{4.15}$$

is well defined and obeys the estimate

$$0 \le \phi(r,x;s,y) \le \phi_0(r,x;s,y) \tag{4.16}$$

as well as the Duhamel formula

$$\phi(r,x;s,y) = \phi_0(r,x;s,y) - \frac{1}{4}\int_s^r \int_{\mathbb{R}} \phi_0(r,x;\rho,w)\frac{w^4}{\rho^2}\phi(\rho,w;s,y)\, dw\, d\rho \tag{4.17}$$

for $(s,y) = (0,0)$ *or* $(s,y) \in (0,\infty)\times\mathbb{R}$*. Moreover,* ϕ *is the fundamental solution of* (4.8) *at* $(s,y) = (0,0)$ *and all* $(s,y) \in (0,\infty)\times\mathbb{R}$*. Finally, we have* $\phi > 0$*.*

Proof. Observe that $r^2 \vee \frac{1}{n} \ge r^2 \vee \frac{1}{n+1}$ and $x^4 \wedge n \le x^4 \wedge (n+1)$. It follows that, for $n \in \mathbb{N}$,

$$-\frac{x^4 \wedge (n+1)}{r^2 \vee \frac{1}{n+1}} \le -\frac{x^4 \wedge n}{r^2 \vee \frac{1}{n}}.$$

By the comparison principle (cf., [16, p. 45-46]), we have (4.14). As decreasing sequences that are bounded below must tend to a limit, the function ϕ given by (4.15) is well-defined and clearly obeys (4.16).

To establish the Duhamel formula, we first recall a heat semi-group-like identity. Let $B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ be the beta function, and let $\lambda > 0$. For $-\infty < \alpha, \beta < \frac{3}{2}$, we recall that

$$\begin{aligned}\int_s^r \int_{\mathbb{R}} &(r-\rho)^{-\alpha} \exp\left(-\frac{\lambda|x-w|^2}{2(r-\rho)}\right)(\rho-s)^{-\beta}\exp\left(-\frac{\lambda|w-y|^2}{2(\rho-s)}\right)\, dw\, d\rho \\ &= \left(\frac{2\pi}{\lambda}\right)^{1/2} B(\tfrac{3}{2}-\alpha, \tfrac{3}{2}-\beta)\cdot(r-s)^{\frac{3}{2}-\alpha-\beta}\exp\left(-\frac{\lambda|x-y|^2}{2(r-s)}\right).\end{aligned} \tag{4.18}$$

Indeed, the proof of this identity can be found in [16, pg. 15].

We first consider the case $(s,y)=(0,0)$. For each n, (4.14) gives

$$\begin{aligned}\frac{w^4\wedge n}{\rho^2\vee\frac{1}{n}}\phi_n(\rho,w;0,0) &\lesssim \frac{1}{\rho^{1/2}}\left[\frac{w^4}{\rho^2}\exp\left(-\frac{w^2}{2\rho}\right)\right] \\ &\lesssim \frac{1}{\rho^{1/2}}\exp\left(-\frac{w^2}{8\rho}\right).\end{aligned} \tag{4.19}$$

Thus, (4.18) gives

$$\begin{aligned}&\left|\int_0^r\int_{\mathbb{R}}\phi_0(r,x;\rho,w)\frac{w^4\wedge n}{\rho^2\vee\frac{1}{n}}\phi_n(\rho,w;0,0)\,dwd\rho\right| \\ &\quad\lesssim \int_0^r\int_{\mathbb{R}}\frac{\exp\left(-\frac{(x-w)^2}{8(r-\rho)}\right)}{(r-\rho)^{1/2}}\frac{\exp\left(-\frac{w^2}{8\rho}\right)}{\rho^{1/2}}\,dwd\rho \\ &\quad\lesssim \sqrt{r}\exp\left(-\frac{x^2}{8r}\right),\end{aligned} \tag{4.20}$$

where the implicit constants are independent of n. Thanks to (4.19), dominated convergence implies that the first integral in (4.20) converges as n goes to $+\infty$. Using (4.13) and (4.15), we have

$$\begin{aligned}\phi(r,x;0,0) &= \lim_{n\to\infty}\phi_n(r,x;0,0) \\ &= \phi_0(r,x;0,0)-\frac{1}{4}\int_0^r\int_{\mathbb{R}}\phi_0(r,x;\rho,w)\frac{w^4}{\rho^2}\phi(\rho,w;0,0)\,dw\,d\rho,\end{aligned}$$

which establishes (4.17) in the case $(s,y)=(0,0)$.

We consider the case $(s,y)\in(0,\infty)\times\mathbb{R}$. For each $n\in\mathbb{N}$ and $\rho>s$, (4.14) gives

$$\begin{aligned}\frac{w^4\wedge n}{\rho^2\vee\frac{1}{n}}\phi_n(\rho,w;s,y) &\le \frac{w^4}{\rho^2}\frac{1}{(\rho-s)^{1/2}}\exp\left(-\frac{(w-y)^2}{2(\rho-s)}\right) \\ &\lesssim \frac{1+y^4}{(\rho-s)^{1/2}}\left(\frac{1}{\rho^2}+\frac{(w-y)^4}{(\rho-s)^2}\right)\exp\left(-\frac{(w-y)^2}{2(\rho-s)}\right) \\ &\lesssim_{s,y} \frac{1}{(\rho-s)^{1/2}}\exp\left(-\frac{(w-y)^2}{8(\rho-s)}\right).\end{aligned} \tag{4.21}$$

Thus, applying (4.18) gives

$$\left| \int_s^r \int_{\mathbb{R}} \phi_0(r,x;\rho,w) \frac{w^4 \wedge n}{\rho^2 \vee \frac{1}{n}} \phi_n(\rho,w;s,y)\, dw\, d\rho \right| \lesssim_{y,s} e^{-\frac{(x-y)^2}{8(r-s)}} \sqrt{r-s}. \tag{4.22}$$

Again, the estimate (4.22) is uniform in n. Using (4.13), (4.14), and (4.15), dominated convergence gives

$$\begin{aligned}\phi(r,x;s,y) &= \lim_{n\to\infty} \phi_n(r,x;s,y) \\ &= \phi_0(r,x;s,y) - \frac{1}{4}\int_0^r \int_{\mathbb{R}} \phi_0(r,x;\rho,w)\frac{w^4}{\rho^2}\phi(\rho,w;s,y)\, dw\, d\rho,\end{aligned}$$

which establishes (4.17) in the case $(s,y) \in (0,\infty)\times\mathbb{R}$.

An immediate corollary of the proof of (4.17) is the delta function property (4.6) of fundamental solutions. Indeed, (4.17), (4.20), and (4.22) imply

$$\lim_{r\downarrow s} \phi(r,x;s,y) = \lim_{r\downarrow s} \phi_0(r,x;s,y) = \delta_{x-y}.$$

Next, we use (4.17) to prove that ϕ is continuously differentiable twice in x and once in r. Clearly, $\phi_0(r,x;s,y) = \frac{1}{\sqrt{2\pi(r-s)}} e^{-\frac{(x-y)^2}{2(r-s)}}$ is infinitely differentiable in every variable. Letting

$$D(r,x;s,y) := \frac{1}{4}\int_s^r \int_{\mathbb{R}} \phi_0(t,x;\rho,w)\frac{w^4}{\rho^2}\phi(\rho,w;s,y)\, dw\, d\rho$$

we claim that D is continuously differentiable twice in x and once in r, and

$$\partial_x D(r,x;s,y) = \frac{1}{4}\int_s^r \int_{\mathbb{R}} \partial_x \phi_0(r,x;\rho,w)\frac{w^4}{\rho^2}\phi(\rho,w;s,y)\, dw\, d\rho \tag{4.23}$$

$$\partial_x^2 D(r,x;s,y) = \frac{1}{4}\int_s^r \int_{\mathbb{R}} \partial_x^2 \phi_0(r,x;\rho,w)\frac{w^4}{\rho^2}\phi(\rho,w;s,y)\, dw\, d\rho \tag{4.24}$$

$$\partial_r D(r,x;s,y) = \frac{1}{4}\frac{x^4}{r^2}\phi(r,x;s,y) + \frac{1}{2}\partial_x^2 D(r,x;s,y). \tag{4.25}$$

If we accept this claim for now, then the relation $\phi = \phi_0 - D$ gives

$$\partial_r \phi = \partial_r \phi_0 - \partial_r D = \frac{1}{2}\partial_x^2 \phi_0 - \frac{1}{4}\frac{x^4}{r^2}\phi - \frac{1}{2}\partial_x^2 D = \frac{1}{2}\partial_x^2 \phi - \frac{1}{4}\frac{x^4}{r^2}\phi,$$

which shows that ϕ is indeed the fundamental solution of (4.8), and hence finishes the proof of Theorem 2.1, part 1.

We first establish (4.23). The mean-value theorem gives

$$\begin{aligned} &\frac{D(r, x+h; s, y) - D(r, x; s, y)}{h} \\ &\qquad = \frac{1}{4}\int_s^r \int_{\mathbb{R}} \partial_x \phi_0(r, x + \theta_h h; \rho, w) \frac{w^4}{\rho^2} \phi(\rho, w; s, y)\, dw\, d\rho \end{aligned} \tag{4.26}$$

for some $\theta_h \in [0, 1]$. From (4.19) and (4.21), we have

$$\frac{w^4}{\rho^2}\phi(\rho, w; s, y) \lesssim_{s,y} \frac{1}{(\rho - s)^{1/2}} \exp\left(-\frac{(w-y)^2}{8(\rho - s)}\right). \tag{4.27}$$

Also, we have

$$|\partial_x \phi_0(r, x + \theta_h h; \rho, w)| \lesssim \frac{1}{r - \rho} \exp\left(-\frac{(x + \theta_h h - w)^2}{8(r - \rho)}\right). \tag{4.28}$$

Combining (4.26), (4.27), (4.28) together, and applying (4.18),

$$\begin{aligned} &\left|\frac{D(r, x+h; s, y) - D(r, x; s, y)}{h}\right| \\ &\qquad \lesssim_{s,y} \int_s^r \int_{\mathbb{R}} \frac{\exp\left(-\frac{(x+\theta_h h - w)^2}{8(r-\rho)}\right)}{r - \rho} \frac{\exp\left(-\frac{(w-y)^2}{8(\rho - s)}\right)}{(\rho - s)^{1/2}}\, dw d\rho \\ &\qquad \lesssim_{s,y} \exp\left(-\frac{(x + \theta_h h - y)^2}{8(r - s)}\right). \end{aligned} \tag{4.29}$$

Since the right hand side of (4.29) converges as $h \to 0$, the generalized dominated convergence theorem implies that D is differentiable once in x and satisfies (4.23). The same proof also shows that D is continuously differentiable in x. Furthermore, (4.28) and (4.29) imply that, for each $R > 0$,

$$|\partial_x \phi(r, x; s, y)| \lesssim_{R,s,y} \frac{\exp\left(-\frac{(x-y)^2}{8(r-s)}\right)}{r-s}, \tag{4.30}$$

for $s < r \leq s + R$.

We now establish (4.24). We again apply the mean-value theorem to obtain

$$\begin{aligned} &\frac{\partial_x D(r, x+h; s, y) - \partial_x D(r, x; s, y)}{h} \\ &\quad = \frac{1}{4} \int_s^r \int_{\mathbb{R}} \partial_x^2 \phi_0(r, x + \theta_h h; \rho, w) \frac{w^4}{\rho^2} \phi(\rho, w; s, y)\, dw\, d\rho \end{aligned} \tag{4.31}$$

for some $\theta_h \in [0, 1]$. Unfortunately, the estimate

$$\begin{aligned} &|\partial_x^2 \phi_0(r, x + \theta_h h; \rho, w)| \\ &\quad \lesssim (r-\rho)^{-\left(\frac{3}{2}-\beta\right)} |x + \theta_h h - w|^{-2\beta} \exp\left(-\frac{(x+\theta_h h - w)^2}{8(r-\rho)}\right), \end{aligned} \tag{4.32}$$

for $\beta \geq 0$, cannot be easily used with (4.18), as the resulting singularity in ρ turns out to not be integrable. Instead, we use integration by parts to move the singularity in ρ to other factors: first recall that

$$\partial_x \phi_0(r, x + \theta_h h; \rho, w) = -\partial_w \phi_0(r, x + \theta_h h; \rho, w) \tag{4.33}$$

$$\partial_x^2 \phi_0(r, x + \theta_h h; \rho, w) = \partial_w^2 \phi_0(r, x + \theta_h h; \rho, w) \tag{4.34}$$

Using (4.34), integrating by parts once in w, and then using (4.33) gives

$$\begin{aligned} &\int_s^r \int_{\mathbb{R}} \partial_x^2 \phi_0(r, x + \theta_h h; \rho, w) \frac{w^4}{\rho^2} \phi(\rho, w; s, y)\, dw\, d\rho = \\ &\quad \int_s^r \int_{\mathbb{R}} \partial_x \phi_0(r, x + \theta_h h; \rho, w) \frac{4w^3}{\rho^2} \phi(\rho, w; s, y)\, dw\, d\rho \\ &\qquad + \int_s^r \int_{\mathbb{R}} \partial_x \phi_0(r, x + \theta_h h; \rho, w) \frac{w^4}{\rho^2} \partial_w \phi(\rho, w; s, y)\, dw\, d\rho \end{aligned} \tag{4.35}$$

Using (4.16), (4.30), and arguing as in (4.21), we have, for $s < \rho$,

$$\left|\frac{4w^3}{\rho^2}\phi(\rho, w; s, y)\right| + \left|\frac{w^4}{\rho^2}\partial_w \phi(\rho, w; s, y)\right| \lesssim_{s,y} \frac{\exp\left(-\frac{(w-y)^2}{16(\rho-s)}\right)}{\rho - s}. \tag{4.36}$$

Combining (4.28), (4.31), (4.35), (4.36) together, and then applying (4.18),

$$\begin{aligned} &\left|\frac{\partial_x D(r, x+h; s, y) - \partial_x D(r, x; s, y)}{h}\right| \\ &\quad\lesssim_{s,y} \int_s^r \int_{\mathbb{R}} \frac{\exp\left(-\frac{(x+\theta_h h - w)^2}{16(r-\rho)}\right)}{r - \rho} \frac{\exp\left(-\frac{(w-y)^2}{16(\rho-s)}\right)}{\rho - s} \, dw d\rho \\ &\quad\lesssim_{s,y} \frac{1}{(r-s)^{1/2}} \exp\left(-\frac{(x+\theta_h h - y)^2}{16(r-s)}\right). \end{aligned} \tag{4.37}$$

Applying a similar generalized dominated convergence as above establishes (4.24) and shows that D is continuously differentiable twice in x. Furthermore, (4.32) and (4.37) imply that, for each $R > 0$,

$$\left|\partial_x^2 \phi(r, x; s, y)\right| \lesssim_{R,s,y} \frac{\exp\left(-\frac{(x-y)^2}{16(r-s)}\right)}{(r-s)^{3/2}}, \tag{4.38}$$

for $s < r \leq s + R$.

Now, we establish (4.25). For $h > 0$, the mean value theorem gives,

$$\begin{aligned} &\frac{D(r+h, x; s, y) - D(r, x; s, y)}{h} \\ &\quad= \frac{1}{4}\int_{\mathbb{R}} \phi_0(r+h, x; r+\theta_h h, w) \frac{w^4 \phi(r+\theta_h h, w; s, y)}{(r+\theta_h h)^2} \, dw \\ &\quad+ \frac{1}{4}\int_s^r \int_{\mathbb{R}} \partial_r u_0(r+\theta_h h, x; \rho, w) \frac{w^4}{\rho^2} \phi(\rho, w; s, y) \, dw \, d\rho \end{aligned}$$

for some $\theta_h \in [0, 1)$. As $h \downarrow 0$, the first term converges to $\frac{1}{4}\frac{x^4}{r^2}\phi(r, x; s, y)$ by the delta function property of ϕ_0. For second term, observe that $\partial_r \phi_0(r, x; s, y) = \frac{1}{2}\partial_x^2 \phi_0(r, x; s, y)$, and therefore the second term converges to $\frac{1}{2}\partial_x^2 D(r, x; s, y)$ as $h \downarrow 0$. A similar argument can be made for $h < 0$. Also, (4.25), (4.27), and (4.37) imply that, for each $R > 0$,

$$|\partial_r \phi(r, x; s, y)| \lesssim_{R,s,y} \frac{\exp\left(-\frac{(x-y)^2}{16(r-s)}\right)}{(r-s)^{3/2}}, \tag{4.39}$$

for $s < r \le s + R$.

Finally, $\phi > 0$ follows immediately from the maximum principle (c.f., [16, p. 39]). This finishes the final claim. $\square$

Remark. Observe that, in the proof of the Duhamel formula (4.17), the Gaussian bounds are ineffective when $s = 0$ and $y \neq 0$; in particular, the expression $-\frac{1}{4}\frac{x^4}{r^2}\phi(r, x; 0, y)$ admits suitable bounds only when $y = 0$.

4.2 Theorem 4.6, Part 2: Applying Feynman–Kac

We revisit $\mu_{L,1}$ and $\nu_{L,1}$. As noted above, $\mu_{L,1}$ is the measure corresponding to the standard Brownian bridge from $r = 0$ to $r = L$ and

$$d\nu_{L,1}(f) = \frac{1}{Z_L} \exp\left(-\frac{1}{4}\int_0^L |f(r)|^4 r^{-2}\, dr\right) d\mu_{L,1}(f).$$

Recall that $\phi_n(r, x; s, y)$ is the fundamental solution of

$$-\partial_r \phi + \frac{1}{2}\partial_x^2 \phi - \frac{1}{4}\frac{x^4 \wedge n}{r^2 \vee (1/n)}\phi = 0.$$

Let

$$dP_n(f) := \exp\left(-\frac{1}{4}\int_0^L \frac{(f(r))^4 \wedge n}{r^2 \vee (1/n)}\, dr\right) d\mu_{L,1}(f)$$

be a Borel measure on $C^0([0, L] \to \mathbb{R})$, not necessarily a probability measure. By the multi-time Feynman–Kac formula with respect to Brownian bridges (cf., Theorem B.3), P_n obeys the following law: for Borel sets $B_1, \ldots, B_N \subseteq \mathbb{R}$ and for

$0 < r_1 < r_2 < \cdots < r_N < L$, we have

$$P_n(f(r_j) \in B_j, j = 1, \ldots, N) \tag{4.40}$$
$$= \int_{B_1} \cdots \int_{B_n} \frac{\phi_n(L, 0; r_N, x_N)}{\phi_0(L, 0; 0, 0)} \prod_{j=1}^{N} \phi_n(r_j, x_j; r_{j-1}, x_{j-1})\, dx_N \cdots dx_1,$$

with $(x_0, r_0) := (0, 0)$. Using (4.14) and (4.15) and applying dominated convergence,

$$\lim_{n\to\infty} RHS(4.40)$$
$$= \int_{B_1} \cdots \int_{B_n} \frac{\phi(L, 0; r_N, x_N)}{\phi_0(L, 0; 0, 0)} \prod_{j=1}^{N} \phi(r_j, x_j; r_{j-1}, x_{j-1})\, dx_N \cdots dx_1.$$

Another application of dominated convergence gives

$$\lim_{n\to\infty} LHS(4.40) = P(f(r_j) \in B_j, j = 1, \ldots, N)$$

where

$$dP(f) := \exp\left(-\frac{1}{4}\int_0^L \frac{(f(r))^4}{r^2}\, dr\right) d\mu_{L,1}(f).$$

Let $Z_L := \frac{\phi(L,0;0,0)}{\phi_0(L,0;0,0)}$ be a normalization constant, which is non-zero because of Theorem 4.6, Part 1 and because $\phi_0(L, 0; 0, 0) = (2\pi L)^{-\frac{1}{2}}$. Then we have

$$\frac{1}{Z_L} dP(f) = d\nu_{L,1}(f),$$

which obeys the desired multi-time law.

4.3 Theorem 4.6, Part 3: The Asymptotics

To compute $\lim_{L\to\infty} \frac{\phi(L,0;s,y)}{\phi(L,0;0,0)}$, we seek to compute the asymptotics of each factor separately. The main obstruction is that the coefficient $-\frac{1}{4}\frac{x^4}{r^2}$ gives a quartic restoring force that decays with time, and contributes significantly to the asymptotics: it turns out that ϕ is neither of polynomial decay in r (cf., heat kernel) nor exponential decay (cf., Mehler kernel).

To handle these issues, we change variables to remove r-dependence from the significant terms: the function

$$\Phi(r, x; s, y) := \tfrac{s}{3}\phi\left(\tfrac{r^3}{27}, \tfrac{xr}{3}; \tfrac{s^3}{27}, \tfrac{ys}{3}\right) \tag{4.41}$$

is the fundamental solution of

$$-\partial_r \Phi + \frac{1}{2}\partial_x^2 \Phi - \frac{1}{4}x^4 \Phi + \frac{x}{r}\partial_x \Phi = 0 \tag{4.42}$$

at each $(s, y) \in (0, \infty) \times \mathbb{R}$.

At this point, we set up the separation of variables. Let

$$H := -\tfrac{1}{2}\partial_x^2 + \tfrac{1}{4}x^4,$$

This is an essentially self–adjoint operator with a discrete spectrum (cf., [41, Section 5.14]). By Sturm–Liouville theory, H has simple eigenvalues, which we list as

$$\lambda_0 < \lambda_1 < \cdots < \lambda_k < \cdots$$

Furthermore, each ψ_k (eigenfunction of H corresponding to λ_k) is Schwartz and ψ_0 is sign-definite. Without loss of generality, ψ_0 is positive.

Recall that the eigenvalues of the harmonic oscillator $H_0 = -\frac{1}{2}\partial_x^2 + \frac{1}{2}x^2$ are $k + \frac{1}{2}$, $k \geq 0$. By the min-max principle (cf., [31, Ch. XIII]) and the fact that $\frac{1}{4}x^4 \geq \frac{1}{2}x^2 - \frac{1}{4}$, we have

$$\lambda_k \geq k + \tfrac{1}{4}. \tag{4.43}$$

As usual, the function

$$e^{-(r-s)H}(x,y) := \sum_{k=0}^{\infty} e^{-(r-s)\lambda_k}\psi_k(x)\psi_k(y)$$

is the fundamental solution of

$$-\partial_r f + \tfrac{1}{2}\partial_x^2 f - \tfrac{1}{4}x^4 f = 0.$$

It turns out that the first-order term $\frac{x}{r}\partial_x\Phi$ still gives a large contribution to the asymptotics. To handle this term, first observe that

$$x\partial_x = -\tfrac{1}{2} + \left(x\partial_x + \tfrac{1}{2}\right)$$

is the decomposition of $x\partial_x$ into its self-adjoint and anti-self-adjoint parts. Rewrite (4.42) as

$$\begin{aligned} 0 &= -\partial_r\Phi + \frac{1}{2}\partial_x^2\Phi - \frac{1}{4}x^4\Phi + \frac{x}{r}\partial_x\Phi \\ &= -\partial_r\Phi + \left(\frac{1}{2}\partial_x^2 - \frac{1}{4}x^4 - \frac{1}{2r}\right)\Phi + \left(\frac{x}{r}\partial_x + \frac{1}{2r}\right)\Phi \end{aligned}$$

Note that $(s/r)^{\frac{1}{2}}e^{-(r-s)H}(x,y)$ is the fundamental solution of

$$-\partial_r f + \left(\frac{1}{2}\partial_x^2 - \frac{1}{4}x^4 - \frac{1}{2r}\right)f = 0.$$

With this in mind, the corresponding Duhamel formula is

$$\begin{aligned} \Phi(r,x;s,y) &= \left(\frac{s}{r}\right)^{\frac{1}{2}} e^{-(r-s)H}(x,y) \\ &\quad + \int_s^r \int_{\mathbb{R}} \left(\frac{\rho}{r}\right)^{\frac{1}{2}} e^{-(r-\rho)H}(x,w)\left[\frac{w}{\rho}\partial_w + \frac{1}{2\rho}\right]\Phi(\rho,w;s,y)\,dwd\rho. \end{aligned} \tag{4.44}$$

This turns out to the correct setting to compute the asymptotics of Φ. The main result of this section is the following proposition.

Proposition 4.8. *For every $(s, y) \in (0, \infty) \times \mathbb{R}$,*

$$\lim_{r\to\infty} \left(\frac{r}{s}\right)^{\frac{1}{2}} e^{(r-s)\lambda_0}\Phi(r, 0; s, y) = G(s, y)\psi_0(0), \tag{4.45}$$

where

$$G(s, y) = \psi_0(y) + \int_s^\infty \int_{\mathbb{R}} \left(\frac{\rho}{s}\right)^{\frac{1}{2}} e^{-(s-\rho)\lambda_0}\psi_0(w) \left[\frac{1}{2\rho} + \frac{w}{\rho}\partial_w\right] \Phi(\rho, w; s, y)\, dw\, d\rho$$

and obeys $0 \le G(s, y) \lesssim \psi_0(y) + s^{-\frac{1}{2}}$. There exists an M such that for all $s \ge M$, $G(s, y)$ is strictly positive when $|y| < 1$. For fixed s, the convergence is uniform in y; in particular, G is continuous in y.

Assuming Proposition 4.8 is valid, let us finish the proof of Theorem 4.6, part 3. Let M be as above, and, for $s \ge 0$, let

$$N := \max(s + 1, \tfrac{M^3}{27}).$$

For $L > N$, inverting the change of variables in (4.41) gives

$$\phi(L, 0; N, y) = N^{-\frac{1}{3}}\Phi\left(3L^{\frac{1}{3}}, 0; 3N^{\frac{1}{3}}, yN^{-\frac{1}{3}}\right).$$

Also, recall the identity

$$\phi(L, 0; s, y) = \int_{\mathbb{R}} \phi(L, 0; N, w)\phi(N, w; s, y)\, dw.$$

Because $0 < \phi(N, w; s, y) \le \frac{1}{\sqrt{2\pi(N-s)}} e^{-\frac{(w-y)^2}{2(N-s)}}$, the following estimate is independent of s and y:

$$\int_{\mathbb{R}} \phi(N, w; s, y)\, dw = \|\phi(N, w; s, y)\|_{L^1_w} \le 1, \tag{4.46}$$

and so Proposition 4.8 and Hölder's inequality gives

$$\begin{aligned}\lim_{L\to\infty} L^{\frac{1}{6}} e^{3\lambda_0 L^{1/3}} \phi(L,0;s,y) &= \lim_{L\to\infty} \int_{\mathbb{R}} L^{\frac{1}{6}} e^{3\lambda_0 L^{1/3}} \phi(L,0;N,w)\phi(N,w;s,y)\,dw \\ &\approx_N \int_{\mathbb{R}} G\left(3N^{\frac{1}{3}}, wN^{-\frac{1}{3}}\right)\phi(N,w;s,y)\,dw. \qquad (4.47)\end{aligned}$$

For fixed s, the convergence is uniform in y, by Proposition 4.8 and by (4.46). The limit is finite because, by (4.46) and the fact that ψ_0 is Schwartz,

$$RHS(4.47) \lesssim \int_{\mathbb{R}} \left[\psi_0\big(wN^{-\frac{1}{3}}\big) + 1\right]\phi(N,w;s,y)\,dw \lesssim 1,$$

Furthermore,

$$RHS(4.47) \geq \int_{|w|<N^{1/3}} G\left(3N^{\frac{1}{3}}, wN^{-\frac{1}{3}}\right)\phi(N,w;s,y)\,dw,$$

hence, by the positivity aspect of Proposition 4.8 and the fact that $\phi > 0$ (cf., Theorem 4.6, Part 1), the limit is also strictly positive. In particular,

$$\lim_{L\to\infty} L^{\frac{1}{6}} e^{3\lambda_0 L^{1/3}} \phi(L,0;0,0) = C > 0.$$

It follows that

$$F(s,y) := \lim_{L\to\infty} \frac{\phi(L,0;s,y)}{\phi(L,0;0,0)} = C^{-1} s^{\frac{1}{6}} e^{\lambda_0 s^{1/3}} \psi_0(0) G\big(3s^{\frac{1}{3}}, ys^{-\frac{1}{3}}\big)$$

is a well-defined, strictly positive function that is bounded in s and y. For fixed s, the convergence is also uniform in y and so F is continuous in y. This finishes the proof of Theorem 4.6, part 3.

We now focus on the proof of Proposition 4.8. We will compute the asymptotics of the two terms in (4.44) separately.

Lemma 4.9. *For every $x, y \in \mathbb{R}$ and $s > 0$,*

$$\lim_{r\to\infty} \left(\frac{r}{s}\right)^{\frac{1}{2}} e^{(r-s)\lambda_0} \left[\left(\frac{s}{r}\right)^{\frac{1}{2}} e^{-(r-s)H}(x,y)\right] = \psi_0(x)\psi_0(y).$$

For fixed s, the convergence is uniform in x and y.

Proof. The identity $(-\frac{1}{2}\partial_x^2 + \frac{1}{4}x^4)\psi_k = \lambda_k\psi_k$ gives $\int_{\mathbb{R}} \frac{1}{2}|\partial_x\psi_k|^2 + \frac{1}{4}|x^2\psi_k|^2 = \lambda_k$, and so $\|\partial_x\psi_k\|_2 \lesssim \sqrt{\lambda_k}$. Observe that

$$\|\psi_k^2\|_\infty \le \|\partial_x(\psi_k^2)\|_1 \le \|\psi_k\|_2\|\partial_x\psi_k\|_2 = \|\partial_x\psi_k\|_2,$$

and so

$$\|\psi_k\|_\infty \lesssim (\lambda_k)^{\frac{1}{2}}. \tag{4.48}$$

Thus, for all $x, y \in \mathbb{R}$ and for all $r \ge s+1$, the fact that $\lambda_k \ge k$ (cf., (4.43)) gives

$$\left|\sum_{k=1}^\infty e^{-(r-s)\lambda_k}\phi_k(x)\phi_k(y)\right| \lesssim e^{-(r-s)\lambda_1}\sum_{k=1}^\infty \lambda_k e^{-(r-s)(\lambda_k-\lambda_1)} \lesssim e^{-(r-s)\lambda_1}.$$

For fixed s, it follows that

$$\lim_{r\to\infty} e^{(r-s)\lambda_0}\sum_{k=1}^\infty e^{-(r-s)\lambda_k}\psi_k(x)\psi_k(y) = 0,$$

uniformly in x and y, which in turn gives the result. □

The asymptotics for the other term in (4.44) is significantly more delicate and requires several sets of additional, a priori, estimates, which we call short term and long term estimates.

4.3.1 Short Term Estimates

We use Gaussian bounds to obtain rational function bounds in r, s. The goal is to obtain bounds so that integrating various expressions in r from s to $s+1$ is finite. For example, (4.16) and changing variables give

$$\Phi(r, x; s, y) \lesssim \frac{s}{\sqrt{r^3 - s^3}} \exp\left(-\frac{3(xr - ys)^2}{2(r^3 - s^3)}\right),$$

which gives

$$\|\Phi(r, x; s, y)\|_{L_x^2} \lesssim \frac{s}{r^{\frac{1}{2}}(r^3 - s^3)^{\frac{1}{4}}}. \tag{4.49}$$

Another application the comparison principle gives the bound

$$e^{-(r-s)H}(x, y) \lesssim (r - s)^{-\frac{1}{2}} \exp\left(-\frac{(x - y)^2}{2(r - s)}\right), \tag{4.50}$$

which implies

$$\|e^{-(r-s)H}(x, y)\|_{L_x^2} \lesssim \frac{1}{(r - s)^{\frac{1}{4}}} \quad \text{and} \quad \|e^{-(r-s)H}(x, y)\|_{L_y^2} \lesssim \frac{1}{(r - s)^{\frac{1}{4}}}. \tag{4.51}$$

To handle the terms with derivatives, we have the following result.

Lemma 4.10. *Let $0 < s < r \leq s + 1$. Then, for all $x, y \in \mathbb{R}$,*

$$\|x\partial_x e^{-(r-s)H}(x, y)\|_{L_x^2} \lesssim \frac{1 + |y|^5}{(r - s)^{\frac{3}{4}}} \quad \textit{and} \quad \|y\partial_y e^{-(r-s)H}(x, y)\|_{L_y^2} \lesssim \frac{1 + |x|^5}{(r - s)^{\frac{3}{4}}}$$

Proof. Note that $e^{-(r-s)H}(x, y) = \sum_{k=0}^{\infty} e^{-(r-s)\lambda_k}\psi_k(x)\psi_k(y)$ is symmetric in x and y, so it suffices to establish the result for $\|x\partial_x e^{-(r-s)H}(x, y)\|_{L_x^2}$.

Using the Duhamel formula

$$e^{-(r-s)H}(x, y) = \phi_0(r, x; s, y) - \frac{1}{4}\int_s^r \int_{\mathbb{R}} \phi_0(r, x; s, w) w^4 e^{-(\rho-s)H}(w, y)\, dw\, d\rho,$$

then, a similar computation to (4.23) gives

$$\begin{aligned}\partial_x e^{-(r-s)H}(x,y) &= \partial_x \phi_0(r,x;s,y) \\ &\qquad - \frac{1}{4}\int_s^r \int_{\mathbb{R}} \partial_x \phi_0(r,x;s,w) w^4 e^{-(\rho-s)H}(w,y)\, dw\, d\rho \\ &=: (A) + (B).\end{aligned}$$

It is not hard to see that

$$|(A)| \lesssim \frac{x-y}{(r-s)^{\frac{3}{2}}} e^{-\frac{(x-y)^2}{2(r-s)}} \lesssim \frac{1}{r-s} e^{-\frac{(x-y)^2}{4(r-s)}}.$$

By Lemma 4.18, (4.50), and the hypothesis $s \le \rho \le r \le s+1$ (in particular, $1 \le \frac{1}{\sqrt{\rho-s}}$), we have

$$\begin{aligned}|(B)| &\lesssim \int_s^r \int_{\mathbb{R}} \frac{x-w}{(r-s)^{\frac{3}{2}}} e^{-\frac{(x-w)^2}{2(r-\rho)}} \frac{w^4}{\sqrt{\rho-s}} e^{-\frac{(w-y)^2}{2(\rho-s)}}\, dw\, d\rho \\ &\lesssim \int_s^r \int_{\mathbb{R}} \frac{x-y-w}{(r-s)^{\frac{3}{2}}} e^{-\frac{(x-y-w)^2}{2(r-\rho)}} \frac{(1+y^4)(1+w^4)}{\sqrt{\rho-s}} e^{-\frac{w^2}{2(\rho-s)}}\, dw\, d\rho \\ &\lesssim (1+y^4) \int_s^r \int_{\mathbb{R}} \frac{x-y-w}{(r-s)^{\frac{3}{2}}} e^{-\frac{(x-y-w)^2}{2(r-\rho)}} \frac{1}{\sqrt{\rho-s}} e^{-\frac{w^2}{2(\rho-s)}}\, dw\, d\rho \\ &\qquad + (1+y^4) \int_s^r \int_{\mathbb{R}} \frac{x-y-w}{(r-s)^{\frac{3}{2}}} e^{-\frac{(x-y-w)^2}{2(r-\rho)}} \frac{1}{\sqrt{\rho-s}} \left[\frac{w^4}{(\rho-s)^2} e^{-\frac{w^2}{2(\rho-s)}}\right] dw\, d\rho \\ &\lesssim (1+y^4) \int_s^r \int_{\mathbb{R}} \frac{1}{r-s} e^{-\frac{(x-y-w)^2}{4(r-\rho)}} \frac{1}{\sqrt{\rho-s}} e^{-\frac{w^2}{4(\rho-s)}}\, dw\, d\rho \\ &\lesssim (1+y^4) e^{-\frac{(x-y)^2}{4(r-s)}}.\end{aligned}$$

Combining both estimates gives

$$|\partial_x e^{-(r-s)H}(x,y)| \lesssim \frac{1+y^4}{r-s} e^{-\frac{(x-y)^2}{4(r-s)}},$$

and so

$$
\begin{aligned}
\left\|x\partial_x e^{-(r-s)H}(x,y)\right\|_{L^2_x}^2 &\lesssim \frac{1+y^8}{(r-s)^2}\int_{\mathbb{R}} x^2 \exp\left(-\frac{(x-y)^2}{2(r-s)}\right)\,dx \\
&\lesssim \frac{1+y^{10}}{(r-s)^2}\int_{\mathbb{R}} (1+x^2)\exp\left(-\frac{x^2}{2(r-s)}\right)\,dx \\
&\lesssim \frac{1+y^{10}}{(r-s)^{\frac{3}{2}}}.
\end{aligned}
$$

□

4.3.2 Long Term Estimates

Here, we seek exponential decay estimates in $r-s$ whenever $r \geq s+1$.

Lemma 4.11. *Let* $(s,y) \in [1,\infty) \times \mathbb{R}$ *be fixed. For all* $r \geq s+1$*,*

$$
\|\Phi(r,x;s,y)\|_{L^2_x} \lesssim \left(\frac{s}{r}\right)^{\frac{1}{2}} e^{-\lambda_0(r-s)}.
$$

Furthermore, let $P_0^\perp$ *denote orthogonal projection onto* $(\mathrm{Span}(\psi_0))^\perp$*. For all* $r \geq s+1$*,*

$$
\|[P_0^\perp\Phi](r,x;s,y)\|_{L^2_x} \lesssim \left(\frac{s}{r}\right)^{\frac{1}{2}} e^{-\lambda_1(r-s)}.
$$

Proof. Observe that

$$
\begin{aligned}
\partial_x\Phi(r,x;s,y) &= \tfrac{rs}{3}\phi_x\left(\tfrac{r^3}{27},\tfrac{xr}{3};\tfrac{s^3}{27},\tfrac{ys}{3}\right) \\
\partial_x^2\Phi(r,x;s,y) &= \tfrac{r^2s}{9}\phi_{xx}\left(\tfrac{r^3}{27},\tfrac{xr}{3};\tfrac{s^3}{27},\tfrac{ys}{3}\right) \\
\partial_r\Phi(r,x;s,y) &= \tfrac{r^2s}{9}\phi_r\left(\tfrac{r^3}{27},\tfrac{xr}{3};\tfrac{s^3}{27},\tfrac{ys}{3}\right) + \tfrac{xs}{3}\phi_x\left(\tfrac{r^3}{27},\tfrac{xr}{3};\tfrac{s^3}{27},\tfrac{ys}{3}\right)
\end{aligned}
$$

By (4.16), (4.30), (4.38), and (4.39), each of the functions

$$
\partial_r\Phi,\quad \partial_x^2\Phi,\quad x\partial_x\Phi,\quad \text{and}\quad x^4\Phi,
$$

obey Gaussian bounds in x (with coefficients depending on r,s,y). In particular, each of the functions are in L^2_x.

Writing $\langle \cdot, \cdot \rangle$ for the L^2_x inner product, then

$$\partial_r \langle \Phi, \Phi \rangle = 2\langle \partial_r \Phi, \Phi \rangle = 2\langle \tfrac{1}{2}\partial_x^2 \Phi - \tfrac{1}{4}x^4 \Phi, \Phi \rangle + 2\left\langle \frac{x}{r}\partial_x \Phi, \Phi \right\rangle$$

Observe that, for fixed r,

$$\begin{aligned} \left\langle (\tfrac{1}{2}\partial_x^2 - \tfrac{1}{4}x^4)\Phi, \Phi \right\rangle &= -\left\langle \sum_{k=0}^{\infty} \langle \Phi, \psi_k \rangle \lambda_k \psi_k, \sum_{k=0}^{\infty} \langle \Phi, \psi_k \rangle \psi_k \right\rangle \\ &= -\sum_{k=0}^{\infty} \lambda_k \langle \Phi, \psi_k \rangle^2 \\ &\leq -\sum_{k=0}^{\infty} \lambda_0 \langle \Phi, \psi_k \rangle^2 = -\lambda_0 \langle \Phi, \Phi \rangle \end{aligned} \tag{4.52}$$

and that

$$2\left\langle \frac{x}{r}\partial_x \Phi, \Phi \right\rangle = \left\langle \frac{x}{r}\partial_x \Phi, \Phi \right\rangle - \frac{1}{r}\left\langle \Phi, (1 + x\partial_x)\Phi \right\rangle = -\frac{1}{r}\langle \Phi, \Phi \rangle. \tag{4.53}$$

Combining (4.52) and (4.53), we see that, as a function of r, $\langle \Phi, \Phi \rangle$ is a subsolution of the ODE $\partial_r f = -\left(2\lambda_0 + \frac{1}{r}\right) f$. At $r = s + 1$, we have the initial condition $\|\Phi(s+1, x; s, y)\|^2_{L^2_x}$, which is uniformly bounded in s and y by (4.49). Therefore,

$$\langle \Phi, \Phi \rangle \leq \|\Phi(s+1, x; s, y)\|^2_{L^2_x} \left(\frac{s+1}{r} e^{-2\lambda_0(r-(s+1))} \right) \lesssim \frac{s}{r} e^{-2\lambda_0(r-s)},$$

where we used the bound $s \geq 1$ to conclude $s + 1 \lesssim s$. The result for $P_0^{\perp}\Phi$ follows from the fact that we may write all the sums beginning at $k = 1$, and then use λ_1 in place of λ_0. $\square$

Lemma 4.12. *Let s, y be fixed. For all $r \geq s + 1$,*

$$\|e^{-(r-s)H}(x, y)\|_{L^2_x} \lesssim e^{-\lambda_0(r-s)} \quad \textit{and} \quad \|x\partial_x e^{-(r-s)H}(x, y)\|_{L^2_x} \lesssim e^{-\lambda_0(r-s)}$$

Proof. Indeed, for $r \geq s+1$, (4.48) gives

$$\begin{aligned}
\|e^{-(r-s)H}(x,y)\|_{L^2_x} &= \sum_{k=0}^{\infty} e^{-(r-s)\lambda_k} |\psi_k(y)| \cdot \|\psi_k(x)\|_{L^2_x} \\
&\lesssim e^{-\lambda_0(r-s)} \sum_{k=0}^{\infty} e^{-(r-s)(\lambda_k-\lambda_0)} (\lambda_k)^{\frac{1}{2}} \\
&\lesssim e^{-\lambda_0(r-s)}.
\end{aligned}$$

The identity $-\frac{1}{2}\partial_x^2\psi_k + \frac{1}{2}x^4\psi_k = \lambda_k\psi_k$ implies the inequality

$$-\tfrac{1}{2}x^2\psi_k\partial_x^2\psi_k \leq \lambda_k x^2\psi_k^2$$

Integrating both sides by parts gives

$$\|x\partial_x\psi_k\|_2^2 \lesssim \lambda_k \int x^2\psi_k^2 - \int (\partial_x\psi_k)(x\psi_k) \lesssim \lambda_k \|x\psi_k\|_2^2 + \|\partial_x\psi_k\|_2 \|x\psi_k\|_2.$$

Furthermore $\|\partial_x\psi_k\|_2 \lesssim (\lambda_k)^{\frac{1}{2}}$ and $\|x\psi_k\|_2 \leq \|\psi_k\|_2 + \|x^2\psi_k\|_2 \lesssim (\lambda_k)^{\frac{1}{2}}$. It follows that

$$\|x\partial_x\psi_k\|_2 \lesssim \lambda_k \tag{4.54}$$

A similar computation as above gives the second result. $\square$

At this point, we have all the necessary short term and long term estimates.

Proof of Proposition 4.8. Recall the Duhamel formula,

$$\Phi(r,0;s,y) = \left(\frac{s}{r}\right)^{\frac{1}{2}} e^{-(r-s)H}(0,y) + \int_s^r \int_{\mathbb{R}} \left(\frac{\rho}{r}\right)^{\frac{1}{2}} \sum_{k=0}^{\infty} J_k(\rho,w;s,y)\, dw\, d\rho,$$

with

$$J_k(\rho,w;s,y) := e^{-(r-\rho)\lambda_k}\psi_k(0)\psi_k(w) \left[\frac{1}{2\rho} + \frac{w}{\rho}\partial_w\right] \Phi(\rho,w;s,y).$$

In view of Lemma 4.9, we first seek to show that

$$\lim_{r\to\infty}\left(\frac{r}{s}\right)^{\frac{1}{2}} e^{(r-s)\lambda_0}\int_s^r\int_{\mathbb{R}}\left(\frac{\rho}{r}\right)^{\frac{1}{2}}\sum_{k=1}^{\infty} J_k(\rho,w;s,y)\,dw\,d\rho=0,$$

which is to say, that the higher eigenvalues do not contribute to the asymptotic.

Note that $(\rho/r)^{1/2}\sum_{k=1}^{\infty} e^{-(r-\rho)\lambda_k}\psi_k(x)\psi_k(w)$ develops a singularity as ρ goes to r, and that $\left[\frac{1}{2\rho}+\frac{w}{\rho}\partial_w\right]\Phi(\rho,w;s,y)$ also develops a singularity as ρ goes to s. We split the integral

$$\int_s^r\int_{\mathbb{R}}\left(\frac{\rho}{r}\right)^{\frac{1}{2}}\sum_{k=1}^{\infty} J_k(\rho,w;s,y)\,dw\,d\rho=I_1(r;s,y)+I_2(r;s,y)+I_3(r;s,y)$$

into three parts, with

$$\begin{aligned}
I_1(r;s,y)&:=\int_s^{s+1}\int_{\mathbb{R}}\left(\frac{\rho}{r}\right)^{\frac{1}{2}}\sum_{k=1}^{\infty} J_k(\rho,w;s,y)\,dw\,d\rho\\
I_2(r;s,y)&:=\int_{s+1}^{r-1}\int_{\mathbb{R}}\left(\frac{\rho}{r}\right)^{\frac{1}{2}}\sum_{k=1}^{\infty} J_k(\rho,w;s,y)\,dw\,d\rho\\
I_3(r;s,y)&:=\int_{r-1}^{r}\int_{\mathbb{R}}\left(\frac{\rho}{r}\right)^{\frac{1}{2}}\sum_{k=1}^{\infty} J_k(\rho,w;s,y)\,dw\,d\rho
\end{aligned}$$

and consider the asymptotics of each part separately.

Before analyzing these integrals, we first record a useful estimate on $\int\sum_k J_k dw$. Since $\frac{1}{2}+w\partial_w$ is anti-self-adjoint, we obtain

$$\int_{\mathbb{R}}\psi_k(w)\left(\tfrac{1}{2}+w\partial_w\right)\Phi(\rho,w;s,y)\,dw=\int_{\mathbb{R}}\left[-\left(\tfrac{1}{2}+w\partial_w\right)\psi_k(w)\right]\Phi(\rho,w;s,y)\,dw$$

Applying Cauchy–Schwarz, (4.43), and (4.54),

$$\left|\int_{\mathbb{R}}\psi_k(w)\left(\tfrac{1}{2}+w\partial_w\right)\Phi(\rho,w;s,y)\,dw\right|\lesssim\lambda_k\|\Phi(\rho,w;s,y)\|_{L^2_w},$$

Using the previous estimate and the definition of $J_k(\rho, w)$,

$$\left| \int_{\mathbb{R}} \sum_{k=1}^{\infty} J_k(\rho, w; s, y)\, dw \right| \lesssim \sum_{k=1}^{\infty} e^{-(r-\rho)\lambda_k} (\lambda_k)^{\frac{3}{2}} \frac{1}{\rho} \|\Phi(\rho, w; s, y)\|_{L^2_w}$$
$$\lesssim \frac{1}{\rho} e^{-(r-\rho)\lambda_1} \|\Phi\|_{L^2_w} \sum_{k=1}^{\infty} e^{-(r-\rho)(\lambda_k - \lambda_1)} (\lambda_k)^{\frac{3}{2}}.$$

Thus, whenever $s \leq \rho \leq r-1$,

$$\left| \int_{\mathbb{R}} \sum_{k=1}^{\infty} J_k(\rho, w; s, y)\, dw \right| \lesssim \frac{1}{\rho} e^{-(r-\rho)\lambda_1} \|\Phi(\rho, w; s, y)\|_{L^2_w}. \tag{4.55}$$

The asymptotics for I_1 and I_2 follow quickly from (4.55).

Indeed, first applying (4.55) and then applying (4.49),

$$|I_1(r; s, y)| \lesssim \int_s^{s+1} \left(\frac{\rho}{r}\right)^{\frac{1}{2}} \frac{1}{\rho} e^{-(r-\rho)\lambda_1} \frac{s}{\rho^{\frac{1}{2}} (\rho^3 - s^3)^{\frac{1}{4}}}\, d\rho \lesssim \frac{1}{(rs)^{\frac{1}{2}}} e^{-(r-s)\lambda_1}.$$

For fixed s, it follows that

$$\lim_{r \to \infty} \left(\frac{r}{s}\right)^{\frac{1}{2}} e^{(r-s)\lambda_0} I_1(r; s, y) = 0,$$

with uniform convergence in y (recall, $\lambda_1 > \lambda_0$). Applying (4.55) and Lemma 4.11,

$$|I_2(r; s, y)| \lesssim \int_{s+1}^{r-1} \left(\frac{\rho}{r}\right)^{\frac{1}{2}} \frac{1}{\rho} e^{-(r-\rho)\lambda_1} \left(\frac{s}{\rho}\right)^{\frac{1}{2}} e^{-(\rho-s)\lambda_0}\, d\rho$$
$$\lesssim \left(\frac{s}{r}\right)^{\frac{1}{2}} e^{-\lambda_0(r-s)} \left[\int_{s+1}^{\frac{r}{2}} e^{-(r-\rho)(\lambda_1-\lambda_0)} \frac{d\rho}{\rho} + \int_{\frac{r}{2}}^{r-1} e^{-(r-\rho)(\lambda_1-\lambda_0)} \frac{d\rho}{\rho} \right]$$
$$\lesssim \left(\frac{s}{r}\right)^{\frac{1}{2}} e^{-(r-s)\lambda_0} \left[\frac{e^{-\frac{r}{2}(\lambda_1-\lambda_0)}}{s+1} + \frac{2}{r} \right].$$

Again, for fixed $s > 0$,

$$\lim_{r\to\infty} \left(\frac{r}{s}\right)^{\frac{1}{2}} e^{(r-s)\lambda_0} I_2(r; s, y) = 0$$

with uniform convergence in y.

Using the fact that $\frac{1}{2} + w\partial_w$ is anti-self-adjoint and applying Cauchy–Schwarz,

$$\begin{aligned}
&|I_3(r; s, y)| \\
&\leq \int_{r-1}^{r} \left(\frac{\rho}{r}\right)^{\frac{1}{2}} \frac{1}{\rho} \left\| \left(\tfrac{1}{2} + w\partial_w\right) \sum_{k=1}^{\infty} e^{-(r-\rho)\lambda_k} \psi_k(0)\psi_k(w) \right\|_{L^2_w} \|\Phi(\rho, w; s, y)\|_{L^2_w} \, d\rho
\end{aligned}$$

For $r - 1 \leq \rho < r$, applying (4.51) and Lemma 4.10 gives

$$\begin{aligned}
&\left\| \left[\tfrac{1}{2\rho} + w\partial_w\right] \sum_{k=1}^{\infty} e^{-(r-\rho)\lambda_k} \psi_k(0)\psi_k(w) \right\|_{L^2_w} \\
&\lesssim \left\| \left[\tfrac{1}{2} + w\partial_w\right] e^{-(r-\rho)H}(0, w) \right\|_{L^2_w} + \left\| \left[\tfrac{1}{2} + w\partial_w\right] e^{-(r-\rho)\lambda_0} \psi_0(0)\psi_0(w) \right\|_{L^2_w} \\
&\lesssim (r-\rho)^{-\frac{3}{4}}
\end{aligned} \tag{4.56}$$

Combining (4.56) and Lemma 4.11,

$$\begin{aligned}
|I_3(r; s, y)| &\lesssim \int_{r-1}^{r} \left(\frac{\rho}{r}\right)^{\frac{1}{2}} \frac{1}{(r-\rho)^{\frac{3}{4}}} \frac{1}{\rho} \left(\frac{s}{\rho}\right)^{\frac{1}{2}} e^{-(\rho-s)\lambda_0} \, d\rho \\
&\lesssim \frac{1}{r} \left(\frac{s}{r}\right)^{\frac{1}{2}} e^{-(r-s)\lambda_0}.
\end{aligned}$$

It follows that, for fixed s,

$$\lim_{r\to\infty} \left(\frac{r}{s}\right)^{\frac{1}{2}} e^{(r-s)\lambda_0} I_3(r; s, y) = 0$$

with uniform convergence in y.

At this point, we have established (4.45), with uniform convergence in y. Our next goal is to establish the positivity results. In particular, we shall show that the integral term in

$$G(s,y) := \psi_0(y) + \int_s^\infty \left(\frac{\rho}{s}\right)^{\frac{1}{2}} e^{-(s-\rho)\lambda_0}\psi_0(w)\left[\frac{1}{2\rho} + \frac{w}{\rho}\partial_w\right]\Phi(\rho, w; s, y)\; dw\; d\tau$$

converges, uniformly in y, to 0 as s goes to ∞ (recall that $\psi_0(y) > 0$).

Applying Cauchy–Schwarz and (4.49),

$$\begin{aligned}
&\left|\int_s^{s+1}\int_{\mathbb{R}} \left(\frac{\rho}{s}\right)^{\frac{1}{2}} e^{-(s-\rho)\lambda_0}\psi_0(w)\left[\frac{1}{2\rho} + \frac{w}{\rho}\partial_w\right]\Phi(\rho, w; s, y)\; dw\; d\rho\right| \qquad (4.57)\\
&\quad\lesssim \frac{\|(\frac{1}{2} + w\partial_w)\psi_0(w)\|_2}{s}\int_s^{s+1}\|\Phi(\rho, w; s, y)\|_{L^2_w}\; d\rho\\
&\quad\lesssim \frac{1}{s}\int_s^{s+1}\frac{s}{\rho^{\frac{1}{2}}(\rho^3 - s^3)^{\frac{1}{4}}}\; d\rho\\
&\quad\lesssim \frac{1}{s}.
\end{aligned}$$

For the integral over $(s+1, \infty)$, first recall that $\left[\left(\frac{1}{2} + w\partial_w\right)\psi_0\right](w)$ is perpendicular to $\psi_0(w)$. Integrating by parts, applying Cauchy–Schwarz, and applying Lemma 4.11,

$$\begin{aligned}
&\left|\int_{s+1}^{\infty}\int_{\mathbb{R}} \left(\frac{\rho}{s}\right)^{\frac{1}{2}} e^{-(s-\rho)\lambda_0}\psi_0(w)\left[\frac{1}{2\rho} + \frac{w}{\rho}\partial_w\right]\Phi(\rho, w; s, y)\; dw\; d\rho\right| \qquad (4.58)\\
&\quad= \left|\int_{s+1}^{\infty}\int_{\mathbb{R}} \left(\frac{1}{s\rho}\right)^{\frac{1}{2}} e^{-(s-\rho)\lambda_0}\left[\tfrac{1}{2} + w\partial_w\right]\psi_0(w)\Phi(\rho, w; s, y)\; dwd\rho\right|\\
&\quad\lesssim \int_{s+1}^{\infty}\int_{\mathbb{R}} \left(\frac{1}{s\rho}\right)^{\frac{1}{2}} e^{-(s-\rho)\lambda_0}\|[P_0^\perp\Phi](\rho, w; s, y)\|_{L^2_w}\; d\rho\\
&\quad\lesssim \int_{s+1}^{\infty}\int_{\mathbb{R}} \left(\frac{1}{s\rho}\right)^{\frac{1}{2}} e^{\lambda_0(\rho-s)}\left(\frac{s}{\rho}\right)^{\frac{1}{2}} e^{-\lambda_1(\rho-s)}\; d\rho\\
&\quad\lesssim \frac{1}{s}\int_{s+1}^{\infty} e^{-(\lambda_1-\lambda_0)(\rho-s)}d\rho\\
&\quad\lesssim \frac{1}{s}.
\end{aligned}$$

Since ψ_0 is strictly positive and continuous, we have $\inf_{|y|<1}\psi_0(y) > 0$. In view of (4.57) and (4.58), there exists some M such that $G(s,y) > 0$ for all $(s,y) \in$

$[M, \infty) \times (-1, 1)$. The same estimates also show that $|G(s, y)| \lesssim \psi_0(y) + s^{-\frac{1}{2}}$, which finishes the proof of Proposition 4.8. $\square$

4.4 Theorem 4.6, Part 4: Support is on Hölder Spaces

Our main tool will be the Kolmogorov Continuity and Consistency theorem (cf., Theorem 4.4), which allows us to upgrade a consistent family of finite dimensional distributions to a (cylinder) measure on path space.

We construct our consistent family of measures. For $0 < r_1 < r_2 < \cdots < r_N$ and for Borel sets $B_0, B_1, \ldots, B_N \subseteq \mathbb{R}$, let

$$P_{r_1,\ldots,r_N}(B_1 \times \cdots \times B_N) := \int_{B_1} \cdots \int_{B_N} F(r_N, x_N) \prod_{j=2}^{N} \phi(r_j, x_j; r_{j-1}, x_{j-1}) \phi(r_1, x_1; 0, 0)\, dx_N \cdots dx_1.$$

and let

$$P_{0,r_1,\ldots,r_N}(B_0 \times B_1 \times \cdots \times B_N) := \delta_0(B_0) P_{r_1,\ldots,r_N}(B_1 \times \cdots \times B_N)$$

where $\delta_0(B) = 1$ if $0 \in B$ and $\delta_0(B) = 0$ otherwise.

The consistency of this family follows from the semi-group property of ϕ, given in (4.59) below. First, recall that ϕ_n is the fundamental solution of the cut-off parabolic PDEs in (4.12). Fix $\rho > s$. For $r > \rho$, both $\int_{\mathbb{R}} \phi_n(r, x; \rho, w)\phi_n(\rho, w; s, y)\, dx$ and $\phi_n(r, x; s, y)$ solve the Cauchy problem $L_n u(r, x) = 0$ with initial data $u(\rho, w) = \phi_n(\rho, w; s, y)$. By uniqueness of bounded solutions of such parabolic PDEs (cf., [16, Section 1.9]), we must have

$$\int_{\mathbb{R}} \phi_n(r, x; \rho, w)\phi_n(\rho, w; s, y)\, dx = \phi_n(r, x; s, y).$$

Applying (4.15), (4.16), and dominated convergence gives

$$\int_{\mathbb{R}} \phi(r,x;\rho,w)\phi(\rho,w;s,y)\,dx = \phi(r,x;s,y). \tag{4.59}$$

Having this semi-group property, then, for all $r > r_N$,

$$\begin{aligned}
\int_{\mathbb{R}} F(r,x)\phi(r,x;r_N,x_N)\,dx &= \lim_{L\to\infty}\int_{\mathbb{R}} \frac{\phi(L,0;r,x)}{\phi(L,0;0,0)}\phi(r,x;r_N,x_N)\,dx \\
&= \lim_{L\to\infty} \frac{\phi(L,0;r_N,x_N)}{\phi(L,0;0,0)} \\
&= F(r_N,x_N).
\end{aligned}$$

By Theorem 4.4, there exists a unique cylinder measure, which we denote $\nu_{\infty,1}$, on $R^{[0,\infty)}$ with the desired finite dimensional distributions.

We now show that $\nu_{L,1}$ is supported on the space of continuous functions and, in particular, on locally s-Hölder continuous functions, with $s < \frac{1}{2}$. Fix $R > 1$ and let $p > 2$. For $0 < s < r < R$,

$$\begin{aligned}
&\mathbb{P}_{\nu_{\infty,1}}(|f(r)-f(s)|>\lambda) \\
&\quad= \int_{\mathbb{R}}\int_{|x-y|>\lambda} F(r,x)\phi(r,x;s,y)\phi(s,y;0,0)\,dx\,dy \\
&\quad= \int_{\mathbb{R}}\int_{|x-y|>\lambda}\int_{\mathbb{R}} F(R,w)\phi(R,w;r,x)\phi(r,x;s,y)\phi(s,y;0,0)\,dw\,dx\,dy
\end{aligned}$$

Because $F(R,w) \lesssim_R 1$ and $\phi(R,w;r,x) \lesssim \frac{1}{\sqrt{R-r}}\exp\left(-\frac{(w-x)^2}{2(R-r)}\right)$, therefore

$$\int_{\mathbb{R}} F(R,w)\phi(R,w;r,x)\,dw \lesssim_R \int_{\mathbb{R}} \frac{1}{\sqrt{R-r}}\exp\left(-\frac{(w-x)^2}{2(R-r)}\right)\,dw \lesssim_R 1.$$

and, by (4.16),

$$\int_{|x-y|>\lambda} \phi(r,x;s,y)\,dx \lesssim \int_{\lambda}^{\infty} \frac{1}{\sqrt{r-s}}\exp\left(-\frac{x^2}{2(r-s)}\right)\,dx \lesssim \exp\left(-\frac{\lambda^2}{2(r-s)}\right)$$

and, again by (4.16),

$$\int_{\mathbb{R}} \phi(s, y; 0, 0)\ dy \leq 1.$$

Putting this all together,

$$\mathbb{P}_{\nu_{\infty,1}}(|f(r) - f(s)| > \lambda) \lesssim_R \exp\left(-\frac{\lambda^2}{2(r-s)}\right)$$

It follows that

$$\begin{aligned}
\mathbb{E}^{\nu_{\infty,1}}\left[|f(r) - f(s)|^p\right] &= \int_0^\infty \lambda^{p-1}\mathbb{P}_{\nu_{\infty,1}}(|f(r) - f(s)| > \lambda)\ d\lambda \\
&\lesssim_R \int_0^\infty \lambda^{p-1} \exp\left(-\frac{\lambda^2}{2(r-s)}\right)\ d\lambda \lesssim_{p,R} (r-s)^{\frac{p}{2}}.
\end{aligned}$$

Thus, for all $s \in \left[0, \frac{p-2}{2p}\right)$, Theorem 4.4 shows that the measure $\nu_{L,1}$ gives measure one to $C^s_{loc}([0,\infty) \to \mathbb{R})$. Finally, note that as $p \to \infty$, we have $\frac{p-2}{2p} \uparrow \frac{1}{2}$, and so we may choose any $s < \frac{1}{2}$ as our Hölder exponent.

4.5 Theorem 4.6, Part 5: Restriction to Bounded Intervals

By definition, the measures $\nu_{\infty,1}|_{[0,R]}$ and $\nu_{L,1}|_{[0,R]}$ are Borel measures on $C^s([0,R] \to \mathbb{R})$ that obey the following laws: for $0 < r_1 < r_2 < \cdots < r_N := R$ and Borel sets $B_1, \ldots, B_N \subseteq \mathbb{R}$,

$$\begin{aligned}
&\mathbb{P}_{\nu_{L,1}|_{[0,R]}}(f(r_j) \in B_j, j = 1, \ldots, N) \\
&\quad = \int_{B_1} \cdots \int_{B_N} \frac{\phi(L, 0; R, x_N)}{\phi(L, 0; 0, 0)} \prod_{j=2}^{N} \phi(r_j, x_j; r_{j-1}, x_{j-1})\phi(r_1, x_1; 0, 0)\ dx_N \cdots dx_1.
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{P}_{\nu_{\infty,1}|_{[0,R]}}(f(r_j) \in B_j, j = 1, \dots, N) \\
&\quad = \int_{B_1} \cdots \int_{B_N} F(R, x_N) \prod_{j=2}^{N} \phi(r_j, x_j; r_{j-1}, x_{j-1}) \phi(r_1, x_1; 0, 0) \, dx_N \cdots dx_1.
\end{aligned}$$

Let P be the Borel measure on $C^s([0, R])$ given by

$$dP(f) := \frac{F(R, f(R))\phi(L, 0; 0, 0)}{\phi(L, 0; R, f(R))} d\nu_{L,1}|_{[0,R]}(f) \tag{4.60}$$

Recall that, by Theorem 4.6, part 1, we have $\phi > 0$. Furthermore, as $L > R$, division by $\phi(L, 0; R, f(R))$ is well-defined. It is not hard to see that

$$P(f(r_j) \in B_j, j = 1, \dots, N) = \mathbb{P}_{\nu_{\infty,1}|_{[0,R]}}(f(r_j) \in B_j, j = 1, \dots, N).$$

It follows that the finite dimensional distributions of P and $\nu_{\infty,1}|_{[0,R]}$ are identical, and by the uniqueness aspect of Theorem 4.4, we have

$$\nu_{\infty,1}|_{[0,R]} = P.$$

As the Radon–Nikodym derivative in (4.60) is strictly positive everywhere, a similar argument shows that $\nu_{L,1}|_{[0,R]}$ is absolutely continuous with respect to $\nu_{\infty,1}|_{[0,R]}$, with Radon–Nikodym derivative $\frac{\phi(L,0;R,f(R))}{F(R,f(R))\phi(L,0;0,0)}$.

Let $A \subseteq C^s([0, R] \to \mathbb{R})$ be a Borel set. Note that, for each $f \in C^s([0, R] \to \mathbb{R})$,

$$\left| \chi_A(f) \frac{\phi(L, 0; R, f(R))}{F(R, f(R))\phi(L, 0; 0, 0)} \right| \leq \frac{\phi(L, 0; R, f(R))}{F(R, f(R))\phi(L, 0; 0, 0)}$$

and

$$\lim_{L\to\infty} \frac{\phi(L, 0; R, f(R))}{F(R, f(R))\phi(L, 0; 0, 0)} = 1$$

Also, for each $L > R$,

$$\int_{C^s([0,R])} \frac{\phi(L,0;R,f(R))}{F(R,f(R))\phi(L,0;0,0)} d\nu_{\infty,1}|_{[0,R]}(f) = \int_{C^([0,R])} d\nu_{L,1}|_{[0,R]}(f) = 1$$

and so the generalized dominated convergence theorem (cf., [15, Exer. 2.20]) gives

$$\begin{aligned}
\lim_{L\to\infty} \nu_{L,1}|_{[0,R]}(A) &= \lim_{L\to\infty} \int_{C^s([0,R])} \chi_A(f) \frac{\phi(L,0;R,f(R))}{F(R,f(R))\phi(L,0;0,0)} \, d\nu_{\infty,1}|_{[0,R]}(f) \\
&= \int_{C^s([0,R])} \chi_A(f) \, d\nu_{\infty,1}|_{[0,R]}(f) \\
&= \nu_{\infty,1}|_{[0,R]}(A),
\end{aligned}$$

as desired.

4.6 Theorem 4.6, Part 6: Construction of ν_∞

Recall the heat kernel $\phi_0(r,x;s,y) = \frac{1}{\sqrt{2\pi(t-s)}} \exp\left(-\frac{(x-y)^2}{2(t-s)}\right)$ and recall that W, the standard Wiener measure, obeys the following law: for $0 < r_1 < r_2 < \cdots < r_N$ and for Borel sets $B_1, B_2, \ldots, B_N \subseteq \mathbb{R}$,

$$\begin{aligned}
&\mathbb{P}_W(f(r_j) \in B_j, j = 1, \ldots, N) = \\
&\qquad \int_{B_1} \cdots \int_{B_N} \prod_{j=2}^{N} \phi_0(r_j, x_j; r_{j-1}, x_{j-1}) \phi_0(r_1, x_1; 0, 0) \, dx_n \cdots dx_1.
\end{aligned}$$

Also, recall that $\nu_{L,2} = \mu_{L,2}$ obeys the same finite dimensional distributions as the Brownian bridge from 0 to L, i.e.,

$$\begin{aligned}
&\mathbb{P}_{\nu_{L,2}}(f(r_j) \in B_j, j = 1, \ldots, N) = \\
&\quad \int_{B_1} \cdots \int_{B_N} \frac{\phi_0(L,0;r_N,x_N)}{\phi_0(L,0;0,0)} \prod_{j=2}^{N} \phi_0(r_j, x_j; r_{j-1}, x_{j-1}) \phi_0(r_1, x_1; 0, 0) \, dx_n \cdots dx_1.
\end{aligned}$$

We denote by $W|_{[0,R]}$ the image measure of W on $C^s([0,R] \to \mathbb{R})$, with the Borel σ-algebra, under the restriction map $f \mapsto f|_{[0,R]}$ and similarly for $\nu_{L,2}|_{[0,R]}$. By an argument similar to Section 4.4, the measures $W|_{[0,R]}$ and $\nu_{L,2}|_{[0,R]}$ are mutually absolutely continuous, with the Radon–Nikodym derivative

$$\frac{d\nu_{L,2}|_{[0,R]}}{dW|_{[0,R]}}(f) = \frac{\phi_0(L,0;R,f(R))}{\phi_0(L,0;0,0)}.$$

Mutual absolute continuity of $\nu_\infty|_{[0,R]}$ and $\nu_L|_{[0,R]}$ follow from the tensor product structure of ν_∞ and the mutual absolute continuity of each of its components.

As $\lim_{L\to\infty} \frac{\phi_0(L,0;R,f(R))}{\phi_0(L,0;0,0)} = 1$ for every $f \in C^s([0,R] \to \infty)$, a similar argument as in Section 4.5 shows that for every Borel set $A \subseteq C^s([0,R] \to \mathbb{R})$, we have

$$\lim_{L\to\infty} \nu_{L,2}|_{[0,R]}(A) = W|_{[0,R]}(A). \tag{4.61}$$

Furthermore, (4.11) follows from (4.9), (4.61), and the fact that ν_∞ is essentially a tensor product of $\nu_{\infty,1}$ and W.

Finally, as the measures $\nu_L|_{[0,R]}$ and $\nu_\infty|_{[0,R]}$ are mutually absolutely continuous on $C^s([0,R] \to \mathbb{C})$, the completion of the Borel σ-algebra with respect to each of these measures must coincide. Let us denote this σ-algebra by $\mathcal{F}_R$. Also, each set $A \in \mathcal{F}_R$ is the union of a Borel set and a null set (cf., Proposition A.7). Thus, (4.11) also holds for each $A \in \mathcal{F}_R$.

CHAPTER 5

Almost sure global existence and invariance

In this chapter, we prove Theorem 2.5. Let us recall the first order NLW in (2.10) and the definition of Flow_∞ from (2.11). Furthermore, by Proposition 4.2, $C_0^s([0,\infty) \to \mathbb{C})$ is a Polish space, and hence we may utilize results from Appendix A. As before, let $\rho_R^L : C_0^s([0,L]) \to C^s([0,R])$ and $\rho_R^\infty : C_{loc}^s([0,\infty)) \to C^s([0,R])$ denote the restriction maps $g \mapsto g|_{[0,R]}$.

The first two assertions of Theorem 2.5 are quite immediate.

Proposition 5.1. *There exists a Borel subset $\Omega_\infty \subseteq C_{loc}^s([0,\infty) \to \mathbb{C})$ such that*

1. $\nu_\infty(\Omega_\infty) = 1$;
2. *For every $g \in \Omega_\infty$, $\mathrm{Flow}_\infty(t,g)$ is defined globally in time. For each $T > 0$ and $R > 0$, we have $\mathrm{Flow}_\infty(t,g)|_{[0,R]} \in C_t^0 C_r^s([-T,T] \times [0,R] \to \mathbb{R})$.*

Proof. Let Ω_L as in Theorem 2.1 and let

$$\begin{aligned}\tilde{\Omega}_\infty &:= \bigcap_{L=2}^{\infty} (\rho_{\lfloor L/2 \rfloor}^\infty)^{-1} \circ \rho_{\lfloor L/2 \rfloor}^L(\Omega_L) \\ &= \{g \in C_{loc}^s([0,\infty)) \mid \forall L \geq 2, \exists g_L \in \Omega_L \text{ s.t. } g_L|_{[0,\lfloor L/2 \rfloor]} \equiv g|_{[0,\lfloor L/2 \rfloor]}\}.\end{aligned}$$

Here, $\lfloor L/2 \rfloor$ denotes the largest integer less than or equal to $L/2$. As restriction is a continuous map, therefore $\rho_{\lfloor L/2 \rfloor}^L(\Omega_L)$ is an analytic set. By Proposition A.6, it follows that $(\rho_{\lfloor L/2 \rfloor}^\infty)^{-1} \circ \rho_{\lfloor L/2 \rfloor}^L(\Omega_L)$ is analytic and, hence, $\tilde{\Omega}_\infty$ is also analytic. By Proposition A.6, the set $\tilde{\Omega}_\infty$ is ν_∞-measurable.

By Theorem 2.1, we have $\nu_L(\Omega_L) = 1$. As $\Omega_L \subseteq (\rho^L_{\lfloor L/2 \rfloor})^{-1} \circ \rho^L_{\lfloor L/2 \rfloor}(\Omega_L)$, it follows that $\nu_L|_{[0,\lfloor L/2 \rfloor]}(\rho^L_{\lfloor L/2 \rfloor}(\Omega_L)) = 1$. By mutual absolute continuity of $\nu_L|_{[0,\lfloor L/2 \rfloor]}$ and $\nu_\infty|_{[0,\lfloor L/2 \rfloor}$ (cf, Theorem 2.3), we have $\nu_\infty|_{[0,\lfloor L/2 \rfloor]}(\rho^L_{\lfloor L/2 \rfloor}(\Omega_L)) = 1$. In other words,

$$\nu_\infty \left((\rho^\infty_{\lfloor L/2 \rfloor})^{-1} \circ \rho^L_{\lfloor L/2 \rfloor}(\Omega_L) \right) = 1.$$

Thus, $\nu_\infty(\tilde{\Omega}_\infty) = 1$. In particular, $C^s_{loc}([0,\infty)) \setminus \tilde{\Omega}_\infty$ is measure 0, and so there is some Borel set A of measure 0 that contains $C^s_{loc}([0,\infty)) \setminus \tilde{\Omega}_\infty$. We define

$$\Omega_\infty = C^s_{loc}([0,\infty)) \setminus A.$$

Observe that $\Omega_\infty \subseteq \tilde{\Omega}_\infty$ and $\nu_\infty(\Omega_\infty) = 1$.

We now prove the second assertion. Fix $g \in \Omega_\infty$ and fix $T > 0$. For each $L \geq 2$, let $g_L \in \Omega_L$ such that $g|_{[0,\lfloor L/2 \rfloor]} \equiv g_L|_{[0,\lfloor L/2 \rfloor]}$. By finite speed of propagation, we have

$$\mathrm{Flow}_L(t, g_L)|_{[0,\lfloor L/2 \rfloor - t]} = \mathrm{Flow}_{L+k}(t, g_{L+k})|_{[0,\lfloor L/2 \rfloor - t]}$$

for all $L > 2T$, all $t \in [-T, T]$, and all $k \geq 0$. We define $\mathrm{Flow}_\infty(t, g)$ to be the unique function such that for each $R > 0$,

$$\mathrm{Flow}_\infty(t, g)|_{[0,R]} \equiv \mathrm{Flow}_L(t, g_L)|_{[0,R]} \text{ for all } L > 2(R + T), |t| \leq T \tag{5.1}$$

and note that it obeys the regularity conditions asserted above. $\square$

We next turn to the invariance assertions of Theorem 2.5. To do this, we first show invariance on the fixed time interval $[-1, 1]$, and then iterate the flow map to achieve global invariance.

Lemma 5.2. *For each $t \in [-1, 1]$, the map*

$$\mathrm{Flow}(t, \cdot) : \Omega_\infty \to C^s_{loc}([0,\infty) \to \mathbb{C})$$

is continuous with respect to (4.4)*. Furthermore, for each Borel subset $A \subseteq \Omega_\infty$, the set* $\mathrm{Flow}_\infty(t, A) = \{\mathrm{Flow}_\infty(t, g) \mid g \in A\}$ *is a Borel subset of* $C^s_{loc}([0,\infty))$.

Proof. For $1 \leq k \leq \infty$, let $g_k \in \Omega_\infty$ such that $d(g_k, g_\infty) \to 0$. Observe that convergence in this metric is equivalent to convergence in each semi-norm.

Fix $n \in \mathbb{N}$. For each $1 \leq k \leq \infty$, choose $\tilde{g}_k \in \Omega_{2n+2}$ such that

$$\tilde{g}_k|_{[0,n+1]} \equiv g_k|_{[0,n+1]}.$$

In particular, $\lim_{n\to\infty} \|\tilde{g}_k - \tilde{g}_\infty\|_{C^s([0,n+1])} = 0$. By Corollary 3.4, we have

$$\lim_{k\to\infty} \| \mathrm{Flow}_{2n+2}(t, \tilde{g}_k) - \mathrm{Flow}_{2n+2}(t, \tilde{g}_\infty)\|_{C^0_t C^s_r([-1,1]\times[0,n])} = 0.$$

By finite speed of propagation,

$$\lim_{k\to\infty} \| \mathrm{Flow}_\infty(t, g_k) - \mathrm{Flow}_\infty(t, g_\infty)\|_{C^0_t C^s_r([-1,1]\times[0,n])} = 0. \tag{5.2}$$

As (5.2) holds for each n, it follows by the above observation that

$$\lim_{k\to\infty} d(\mathrm{Flow}_\infty(t, g_k), \mathrm{Flow}_\infty(t, g_\infty)) = 0$$

for each $t \in [-1, 1]$, proving the continuity assertion.

For fixed $t \in [-1, 1]$, we extend $\mathrm{Flow}_\infty(t, \cdot)$ to $C^s_{loc}([0,\infty) \to \mathbb{C})$ by

$$\mathrm{Flow}_\infty(t, g) := 0$$

for $g \in C^s_{loc}([0,\infty) \to \mathbb{C}) \setminus \Omega_\infty$. This defines a Borel measurable map from $C^s_{loc}([0,\infty) \to \mathbb{C})$ to itself. Furthermore, the restriction of $\mathrm{Flow}_\infty(t, \cdot)$ to Ω_∞ is an injective map, as the flow is reversible. By the Lusin–Souslin Theorem (cf., Theorem A.2), $\mathrm{Flow}_\infty(t, \cdot)$ maps Borel subsets of Ω_∞ to Borel subsets of

$C^s_{loc}([0,\infty) \to \mathbb{C})$. $\square$

Lemma 5.3. *Let $K \subseteq C^s_{loc}([0,\infty) \to \mathbb{C})$ be a closed set such that $K \subseteq \Omega_\infty$. For every $t \in [-1,1]$, we have*

$$\nu_\infty(K) \leq \nu_\infty(\mathrm{Flow}_\infty(t, K)).$$

Proof. Let $0 < R \ll L$, and recall that $(\rho^L_{R+1})^{-1} \circ \rho^\infty_{R+1}(K)$ is an analytic subset of $C^s([0,L] \to \mathbb{C})$. By Theorem 2.1, we have

$$\begin{aligned} \nu_L((\rho^L_{R+1})^{-1} \circ \rho^\infty_{R+1}(K)) &= \nu_L\left(\Omega_L \cap \left((\rho^L_{R+1})^{-1} \circ \rho^\infty_{R+1}(K)\right)\right) \\ &= \nu_L\left(\mathrm{Flow}_L\left[t, \Omega_L \cap (\rho^L_{R+1})^{-1} \circ \rho^\infty_{R+1}(K)\right]\right) \end{aligned} \tag{5.3}$$

for each $t \in \mathbb{R}$.

Next, we claim that, for $t \in [-1,1]$,

$$\rho^L_R\left(\mathrm{Flow}_L\left[t, \Omega_L \cap \left((\rho^L_{R+1})^{-1} \circ \rho^\infty_{R+1}(K)\right)\right]\right) \subseteq \rho^\infty_R\left(\mathrm{Flow}_\infty(t, K)\right) \tag{5.4}$$

Indeed, let $g \in \mathrm{Flow}_L\left[t, \Omega_L \cap (\rho^L_{R+1})^{-1} \circ \rho^\infty_{R+1}(K)\right]$. Then there exists some $f \in (\rho^L_{R+1})^{-1} \circ \rho^\infty_{R+1}(K) \cap \Omega_L$ such that $\mathrm{Flow}_L(t,f) = g$. Furthermore, there exists $\tilde{f} \in K$ such that $\tilde{f}|_{[0,R+1]} \equiv f|_{[0,R+1]}$ and thus, by finite speed of propagation,

$$\mathrm{Flow}_\infty(t, \tilde{f})|_{[0,R]} \equiv \mathrm{Flow}_L(t,f)|_{[0,R]} \equiv g|_{[0,R]}.$$

The claim follows.

Given (5.4), we also have

$$\begin{aligned} &\mathrm{Flow}_L\left[t, \Omega_L \cap \left((\rho^L_{R+1})^{-1} \circ \rho^\infty_{R+1}(K)\right)\right] \\ &\quad \subseteq (\rho^L_R)^{-1} \circ \rho^L_R\left(\mathrm{Flow}_L\left[t, \Omega_L \cap \left((\rho^L_{R+1})^{-1} \circ \rho^\infty_{R+1}(K)\right)\right]\right) \\ &\quad \subseteq (\rho^L_R)^{-1} \circ \rho^\infty_R\left(\mathrm{Flow}_\infty(t, K)\right) \end{aligned} \tag{5.5}$$

Furthermore, $\mathrm{Flow}_\infty(t,K)$ is a Borel subset of $C^s_{loc}([0,\infty)\to\mathbb{C})$ by Lemma 5.2, and thus $(\rho^L_R)^{-1}\circ\rho^\infty_R\left(\mathrm{Flow}_\infty(t,K)\right)$ is analytic (and ν_L-measurable) by Proposition A.6.

Putting (5.5) into (5.3), then

$$\nu_L((\rho^L_{R+1})^{-1}\circ\rho^\infty_{R+1}(K))\leq\nu_L\left((\rho^L_R)^{-1}\circ\rho^\infty_R\left(\mathrm{Flow}_\infty(t,K)\right)\right),$$

which is to say,

$$\nu_L|_{[0,R+1]}\left(\rho^\infty_{R+1}(K)\right)\leq\nu_L|_{[0,R]}\left(\rho^\infty_R(\mathrm{Flow}_\infty(t,K))\right).$$

Sending $L\to\infty$, Theorem 2.3 gives

$$\nu_\infty|_{[0,R+1]}\left(\rho^\infty_{R+1}(K)\right)\leq\nu_\infty|_{[0,R]}\left(\rho^\infty_R(\mathrm{Flow}_\infty(t,K))\right),$$

or

$$\nu_\infty\left((\rho^\infty_{R+1})^{-1}\circ\rho^\infty_{R+1}(K)\right)\leq\nu_\infty\left((\rho^\infty_R)^{-1}\circ\rho^\infty_R(\mathrm{Flow}_\infty(t,K))\right). \tag{5.6}$$

Similar to above, $(\rho^\infty_R)^{-1}\circ\rho^\infty_R(\mathrm{Flow}_\infty(t,K))$ is an analytic subset of $C^s_{loc}([0,\infty))$ and thus it is indeed ν_∞-measurable.

Note that (5.6) holds for arbitrary $R>0$. Furthermore, we have

$$(\rho^\infty_R)^{-1}\circ\rho^\infty_R(\mathrm{Flow}_\infty(t,K))\supseteq(\rho^\infty_{R+1})^{-1}\circ\rho^\infty_{R+1}(\mathrm{Flow}_\infty(t,K))$$

and similarly for K. By dominated convergence,

$$\nu_\infty\left(\bigcap_{R=2}^\infty(\rho^\infty_R)^{-1}\circ\rho^\infty_R(K)\right)\leq\nu_\infty\left(\bigcap_{R=1}^\infty(\rho^\infty_R)^{-1}\circ\rho^\infty_R(\mathrm{Flow}_\infty(t,K))\right). \tag{5.7}$$

We claim that

$$\mathrm{Flow}_\infty(t,K) = \bigcap_{R=1}^{\infty} (\rho_R^\infty)^{-1} \circ \rho_R^\infty(\mathrm{Flow}_\infty(t,K)). \tag{5.8}$$

The containment $\subseteq$ is obvious. Now, let $g \in \bigcap_{R=1}^{\infty} (\rho_R^\infty)^{-1} \circ \rho_R^\infty(\mathrm{Flow}_\infty(t,K))$. For each integer $R \geq 1$, there exists $f_R \in K$ such that

$$\mathrm{Flow}_\infty(t,f_R)|_{[0,R]} \equiv g|_{[0,R]} \tag{5.9}$$

By finite speed of propagation, it follows that for each $n \geq 0$ and for each $R \geq 1$,

$$f_R|_{[0,R-1]} \equiv f_{R+n}|_{[0,R-1]}.$$

Thus, the sequence $\{f_R\}_{R=1}^{\infty}$ converges to some $f \in C^s_{loc}([0,\infty) \to \mathbb{C})$, with

$$f|_{[0,R-1]} \equiv f_R|_{[0,R-1]}$$

for each $R \geq 1$. Indeed, as K is closed, we also have $f \in K$. Given (5.9) and finite speed of propagation, we have

$$\mathrm{Flow}_\infty(t,f)|_{[0,R-2]} \equiv g|_{[0,R-2]}$$

for each $R \geq 2$, and thus $\mathrm{Flow}_\infty(t,f) = g$. This proves the reverse containment.

As K is closed, a similar converging sequence argument as above shows

$$K = \bigcap_{R=2}^{\infty} (\rho_R^\infty)^{-1} \circ \rho_R^\infty(K). \tag{5.10}$$

Putting (5.8) and (5.10) into (5.7) finishes the proof. $\square$

The above proof actually shows that Flow_∞ preserves closed subsets K of

$C^s_{loc}([0,\infty) \to \mathbb{C})$ which are contained[1] in Ω_∞, though we will not use this fact for later results. Finally, we prove the invariance assertion of Theorem 2.5.

Proposition 5.4. *For each ν_∞-measurable subset $A \subseteq \Omega_\infty$ and for each $t \in \mathbb{R}$, the set* $\mathrm{Flow}_\infty(t, A)$ *is also ν_L-measurable and* $\nu_\infty(\mathrm{Flow}_\infty(t,A)) = \nu_\infty(A)$.

Proof. We first consider the case when $t \in [-1,1]$ and when A is Borel. In view of Theorem A.7, let

$$K_1 \subseteq K_2 \subseteq K_3 \subseteq \cdots \subseteq A$$

be compact sets such that $\nu_\infty(A \setminus K_n) \leq \frac{1}{n}$ for each n. By Lemma 5.3, we have $\nu_\infty(K_n) \leq \nu_\infty(\mathrm{Flow}_\infty(t, K_n))$ and so

$$\nu_\infty(A) = \nu_\infty\Big(\bigcup_{n=1}^{\infty} K_n\Big) \leq \nu_\infty\Big(\mathrm{Flow}_\infty\Big[t, \bigcup_{n=1}^{\infty} K_n\Big]\Big) \leq \nu_\infty(\mathrm{Flow}_\infty(t,A)). \tag{5.11}$$

A similar argument also shows

$$\nu_\infty(\Omega_\infty \setminus A) \leq \nu_\infty(\mathrm{Flow}_\infty(t, \Omega_\infty \setminus A)). \tag{5.12}$$

Since

$$1 = \nu_\infty(A) + \nu_\infty(\Omega_\infty \setminus A) \leq \nu_\infty(\mathrm{Flow}_\infty(t,A)) + \nu_\infty(\mathrm{Flow}_\infty(t, \Omega_\infty \setminus A)) \leq 1,$$

all of the inequalities in (5.11) and (5.12) must actually be equalities.

The case for all $t \in \mathbb{R}$ follows from iterating the flow, and intersecting with Ω_∞ if necessary. The case for all ν_∞-measurable A follows from a similar argument as in Section 3.2. $\square$

[1] Inner regularity of ν_∞ guarantees existence of such closed sets. Indeed, we may find compact sets $K \subseteq C^s_{loc}([0,\infty) \to \mathbb{C})$ contained in Ω_∞ with measure arbitrarily close to 1.

APPENDIX A

Some Descriptive Set Theory

In this appendix, we recall some defintions and facts from descriptive set theory that will be useful for our paper. In particular, we use these results in Section 3.2 and Section 5. We refer to [19] for proofs.

Definition A.1. A topological space is said to be a *Polish space* if it is completely metrizable, and separable with respect to this metric.

One of the more useful results in this setting is a theorem by Lusin and Souslin, which states that injective Borel maps (in particular, continuous embeddings) between Polish spaces are Borel isomorphisms onto their image.

Theorem A.2 (Lusin–Souslin)**.** *Let X, Y be Polish spaces, and let $f : X \to Y$ be Borel measurable. If $A \subseteq X$ is Borel and $f|_A$ is injective, then $f(A) \subseteq Y$ is Borel.*

Indeed, let us deduce the following result as a corollary:

Proposition A.3. *Fix $L \in (0, \infty)$ and $s \in [0, \infty)$. For each of the Banach spaces*

$$\begin{aligned} &X = L^p([0, L]), \text{ with } 1 \leq p < \infty, \text{ or} \\ &X = \dot{H}^s_0([0, L]), \text{ or} \\ &X = C^0([0, L]), \end{aligned}$$

the following statement holds:

Endow X with its usual norm topology, as well as the corresponding Borel σ-algebra, $\mathcal{B}_X$. Then $C_0^s([0,L]) \in \mathcal{B}_X$. Furthermore, the restriction of $\mathcal{B}_X$ to $C_0^s([0,L])$ coincides with the standard Borel σ-algebra induced by the s-Hölder norm.

Proof. Recall that all of the Banach spaces mentioned above are separable, and hence are Polish spaces. Furthermore, $C_0^s([0,L])$ embeds continuously into each Banach space X above. By Theorem A.2, it follows that $C_0^s([0,L]) \in \mathcal{B}_X$ for each X above, and that the σ-algebras mentioned above must coincide. □

Proposition A.4. *Fix $s \in \mathbb{R}$, and fix $L > 0$. Let $\Lambda = \mathbb{R}$ or $\mathbb{C}$ and let $\dot{H}_0^s([0,L] \to \Lambda)$ be as in Definition 1.3. We define the Fourier cylinder σ-algebra to be the σ-algebra generated by sets of the form*

$$A_N = \{g = \sum_{n=1}^{\infty} c_n e_{n,L} \in \dot{H}_0^s([0,L] \to \Lambda) : c_N \in B\}$$

where $B \subseteq \Lambda$ is Borel. Then the Fourier cylinder σ-algebra coincides with the usual Borel σ-algebra on $\dot{H}_0^s([0,L] \to \Lambda)$.

Proof. Let

$$\Lambda^{\mathbb{N}} = \{\vec{c} = (c_1, c_2, \ldots) : c_n \in \Lambda\}$$

denote the the space of sequences in Λ. We equip this space with the metric

$$d(\vec{c}, \vec{d}) := \sum_{n=1}^{\infty} 2^{-n} \frac{|c_n - d_n|}{1 + |c_n - d_n|}$$

as well as the corresponding topology and Borel σ-algebra. Observe that open cylinder sets, i.e. sets of the form

$$\{\vec{c} = (c_1, c_2, \ldots) : c_N \in \mathcal{O}\},$$

where $\mathcal{O} \subseteq \Lambda$ is open, form a sub-basis for this topology, and so they also generate the Borel σ-algebra. It follows that the standard cylinder σ-algebra on $\Lambda^{\mathbb{N}}$ coincides with the Borel σ-algebra.

We consider the map

$$\dot{H}_0^s([0, L] \to \Lambda) \longrightarrow \Lambda^{\mathbb{N}}$$

sending each Sobolev "function" to its corresponding sequence of Fourier coefficients. Clearly the map is injective. Furthermore, the map is continuous since convergence in $\dot{H}^s$ implies convergence in each Fourier coefficient, which then implies convergence in the metric above. The result follows from Theorem A.2. $\square$

In this paper, we also consider general continuous images of Borel sets, such as restriction maps. These sets are not necessarily Borel, but still obey nice measure theoretic properties.

Definition A.5. Let X be a Polish space.

1. A set $A \subseteq X$ is called *analytic* if there is a Polish space Y, a Borel subset $B \subseteq Y$, and a continuous function $f : Y \to X$ such that $f(B) = A$.

2. Let μ be a σ-finite Borel measure on X, and let $\mathcal{F}$ be the completion of the Borel σ-algebra with respect to μ. A set $A \subseteq X$ is called *μ-measurable* if $A \in \mathcal{F}$.

3. Finally, a set $A \subseteq X$ is called *universally measurable* if A is μ-measurable for every σ-finite Borel measure μ on X.

Proposition A.6 (Properties of analytic sets)**.** *Let X be a Polish space.*

1. *If $A_n \subseteq X$ is analytic for $n \in \mathbb{N}$, then $\bigcap_{n\in\mathbb{N}} A_n$ and $\bigcup_{n\in\mathbb{N}} A_n$ are also analytic.*

2. *Let Y be another Polish space and let $f : X \to Y$ be Borel measurable. If $A \subseteq X$ is analytic, then $f(A) \subseteq Y$ is analytic. If $B \subseteq Y$ is analytic, then $f^{-1}(B) \subseteq X$ is analytic.*

3. *(Lusin) Analytic sets are universally measurable.*

Finally, we recall a generalization of regularity results for Lebesgue measure to the Polish space setting.

Proposition A.7 (Regularity of Borel Measures)**.** *Let X be a Polish space and let μ be a finite Borel measure on X. Then μ is regular: for any μ-measurable set $A \subseteq X$,*

$$\begin{aligned}\mu(A) &= \sup\{\mu(K) \mid K \subseteq A, K \textit{ compact}\} \\ &= \inf\{\mu(U) \mid U \supseteq A, U \textit{ open}\}.\end{aligned}$$

In particular, a set $A \subseteq X$ is μ-measurable if and only if there exists an F_σ set $F \subseteq A$ (resp., G_δ set $G \supseteq A$) such that $\mu(A \setminus F) = 0$ (resp., $\mu(G \setminus A) = 0$).

APPENDIX B

A Multi-time Feynman–Kac formula

In this appendix, we make various modifications to the classical Feynman–Kac formula. To this end, we make use of the fact that fundamental solutions of parabolic PDEs (cf., Definition 4.5) also have nice properties in their secondary variables. We list the relevant results below, and refer to [16] for proofs.

Lemma B.1. *Let $V(r,x) : \mathbb{R}^{\geq 0} \times \mathbb{R} \to \mathbb{R}$ be bounded and continuous, and let $\phi(r,x;s,y)$ be the fundamental solution of*

$$\partial_r \phi = \frac{1}{2}\partial_x^2 \phi + V(r,x)\phi$$

Then, as a function of (s,y), we also have $-\partial_s \phi = \frac{1}{2}\partial_y^2 \phi + V(s,y)\phi$. Furthermore, for $g \in C_0(\mathbb{R})$ and for fixed $r > 0$, the unique solution of

$$\begin{cases} -\partial_s \psi = \frac{1}{2}\partial_y^2 \psi + V(s,y)\psi, & (s,y) \in [0,r) \times \mathbb{R} \\ \psi(r,\cdot) = g(\cdot) \end{cases}$$

of sub-exponential growth in y is given by

$$\psi(s,y) = \int_{\mathbb{R}} g(x)\phi(r,x;s,y)\, dx.$$

Let $W_{s,y}$ denote the Wiener measure for Brownian motion starting from time s and at point y. We use B to denote a generic function in the support of $W_{s,y}$.

As before,

$$\phi_0(r,x;s,y) := \frac{1}{\sqrt{2\pi(r-s)}}\exp\left(-\frac{(x-y)^2}{2(r-s)}\right)$$

is the heat kernel.

Theorem B.2 (Multi-time Feynman–Kac, Brownian Motion)**.** *Let $V(r,x) : \mathbb{R}^{\geq 0}\times\mathbb{R}\to\mathbb{R}$ be bounded and continuous, and let $\phi(r,x;s,y)$ be the fundamental solution of*

$$\partial_r\phi = \frac{1}{2}\partial_x^2\phi + V(r,x)\phi.$$

Let $L\in(0,\infty)$, $s\in[0,L)$, and $y\in\mathbb{R}$. Let $s<r_1<\cdots<r_N<L$ and let $f_1,\ldots,f_N,g:\mathbb{R}\to\mathbb{R}$ be bounded, measurable functions. With $(r_0,x_0):=(s,y)$, we then have

$$\mathbb{E}^{W_{s,y}}\left[\exp\left(\int_s^L V(\rho,B(\rho))\,d\rho\right)g(B(L))\prod_{j=1}^N f(B(r_j))\right] \tag{B.1}$$
$$= \int_{\mathbb{R}}\cdots\int_{\mathbb{R}} g(x)\phi(L,x;r_N,x_N)\prod_{k=1}^N f(x_j)\phi(r_j,x_j;r_{j-1},x_{j-1})\,dx dx_N\cdots dx_1$$

Proof. The case $N=0$ is the usual Feynman–Kac formula, which we shall assume as a base case (see [44] for a proof). For the inductive step, we shall mimic the technique of this proof.

Let us write the left hand side of (B.1) as

$$\mathbb{E}^{W_{s,y}}\left[\left(\sum_{n=0}^\infty A_n\right)\exp\left(\int_{r_1}^L V(\rho,B(\rho))\,d\rho\right)g(B(L))\prod_{j=1}^N f(B(r_j))\right],$$

where

$$A_n := \frac{1}{n!}\int_s^{r_1}\cdots\int_s^{r_1} V(\rho_1,B(\rho_1))\cdots V(\rho_n,B(\rho_n))\,d\rho_n\cdots d\rho_1$$

for $n \geq 1$ and $A_0 = 1$. Observe that, if we let

$$\mathcal{R}_n := \{(\rho_1, \dots, \rho_n) \in \mathbb{R}^n : s < \rho_1 < \cdots < \rho_n < r_1\}$$

and let $d\vec{\rho} := d\rho_n \cdots d\rho_1$, then, for $n \geq 1$,

$$A_n = \int_{\mathcal{R}^n} \prod_{j=1}^{n} V(\rho_j, B(\rho_j))\, d\vec{\rho}.$$

By hypothesis and by Fubini's theorem, one may interchange expectations, sums, and integrals as desired. In particular,

$$LHS(\text{B.1}) = \sum_{n=0}^{\infty} J_{s,y}(n)$$

where

$$J_{s,y}(0) := \mathbb{E}^{W_{s,y}}\left[\exp\left(\int_{r_1}^{L} V(\rho, B(\rho))\, d\rho\right) g(B(L)) \prod_{j=1}^{N} f(B(r_j))\right]$$

and, for $n \geq 1$,

$$J_{s,y}(n) := \int_{\mathcal{R}^n} \mathbb{E}^{W_{s,y}}\left[\prod_{j=1}^{n} V(\rho_j, B(\rho_j)) \exp\left(\int_{r_1}^{L} V(\rho, B(\rho))\, d\rho\right) g(B(L)) \times \right.$$
$$\left. \prod_{j=1}^{N} f(B(r_j))\right] d\vec{\rho}.$$

Now, let

$$h(r_1, x_1) := \mathbb{E}^{W_{r_1,x_1}}\left[\exp\left(\int_{r_1}^{L} V(\rho, B(\rho))\, d\rho\right) g(B(L)) \prod_{j=2}^{N} f(B(r_j))\right],$$

which, by induction, is equal to

$$\int_{\mathbb{R}} \cdots \int_{\mathbb{R}} g(x)\phi(L, x; r_N, x_N) \prod_{k=2}^{N} f(x_k)\phi(r_k, x_k; r_{k-1}, x_{k-1})\, dx dx_N \cdots dx_2. \quad \text{(B.2)}$$

Applying the Markov property of Brownian motion and (B.2),

$$J_{s,y}(0) = \int_{\mathbb{R}} h(r_1, x_1) f(x_1)\phi_0(r_1, x_1; s, y)\, dx_1.$$

and, for $n \geq 1$,

$$J_{s,y}(n) = \int_{\mathcal{R}^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}} h(r_1, x_1) f(x_1)\phi_0(r_1, x_1; \rho_n, w_n) \prod_{j=2}^{n} V(\rho_j, w_j)\phi_0(\rho_j, w_j; \rho_{j-1}, w_{j-1}) V(\rho_1, w_1)\phi_0(\rho_1, w_1; s, y) dx_1 d\vec{w} d\vec{\rho}$$

For the next step, observe that

$$-\partial_s \phi_0(r, x; s, y) = \frac{1}{2}\partial_y^2 \phi_0(r, x; s, y)$$

and so

$$-\partial_s J_{s,y}(0) = \frac{1}{2}\partial_y^2 J_{s,y}(0)$$

For $n \geq 1$, differentiating $J_{s,y}(n)$ in s produces two terms: one from the bounds in $\mathcal{R}_n$ and one from differentiating $\phi_0(\rho_1, w_1; s, y)$; namely, for $n \geq 1$,

$$-\partial_s J_{s,y}(n) = V(s, y) J_{s,y}(n-1) + \frac{1}{2}\partial_y^2 J_{s,y}(n)$$

In particular, $\sum_{n=0}^{\infty} J_{s,y}(n)$ obeys[1] the backwards Kolmogorov equation

$$-\partial_s \psi = \frac{1}{2}\partial_y^2 \psi + V(s, y)\psi$$

[1] Indeed, $\sum_n J_{s,y}(n)$ represents the iterated Duhamel expansion for the solution of this PDE.

with terminal condition $\psi(r_1, \cdot) = h(r_1, \cdot)f(\cdot)$. Given Lemma B.1, we have

$$LHS(\text{B.1}) = \sum_{n=0}^{\infty} J_{s,y}(n) = \int_{\mathbb{R}} h(r_1, x_1)f(x_1)\phi(r_1, x_1; s, y)\, dx_1.$$

In view of (B.2), we are done. $\square$

Let $BB_{L,x;s,y}$ denote the measure for the Brownian bridge that starts at time s and at point y, with ending at time L and at point x. We use BB to denote a generic element in the support of $BB_{L,x;s,y}$. The proof of the above theorem also applies, mutatis mutandis, to the following setting.

Theorem B.3 (Multi-time Feynman–Kac, Brownian Bridge)**.** *Let $V(r,x) : \mathbb{R}^{\geq 0} \times \mathbb{R} \to \mathbb{R}$ be bounded and continuous, and let $\phi(r, x; s, y)$ be the fundamental solution of*

$$\partial_r \phi = \frac{1}{2}\partial_x^2 \phi + V(r,x)\phi.$$

Let $L \in (0, \infty)$, $s \in [0, L)$, and $y \in \mathbb{R}$. Let $s < r_1 < \cdots < r_N < L$ and let $f_1, \ldots, f_N : \mathbb{R} \to \mathbb{R}$ be bounded, measurable functions. With $(r_0, x_0) := (s, y)$, we then have

$$\mathbb{E}^{BB_{L,x;s,y}}\left[\exp\left(\int_s^L V(\rho, BB(\rho))\, d\rho\right) f_1(BB(r_1)) \cdots f_N(BB(r_N))\right]$$
$$= \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \frac{\phi(L, x; r_N, x_N)}{\phi_0(L, x; s, y)} \prod_{j=1}^{N} f(x_j)\phi(r_j, x_j; r_{j-1}, x_{j-1})\, dx_N \cdots dx_1.$$

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