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DEVELOPMENT OF AN S-MATRIX THEORY BASED ON THE CALCULUS OF FINITE DIFFERENCES

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SUMMARY

We investigate the possibility of constructing an S-Matrix theory in which partial wave expansions are made in terms of the spin-matrix polynomials $W_n(z) = \Gamma(z + \frac{n}{2} + \frac{1}{2})/\Gamma(z - \frac{n}{2} + \frac{1}{2})$. The technique employed to project an arbitrary function of z onto this basis is found in the calculus of finite differences. This circumstance seems to give the theory a flavor rather different from the conventional approach where projections are done by means of integrals. In this preliminary treatment we first present the mathematical

properties of the expansion, showing how analytic continuation into the complex n-plane may be done. We then go on to discuss the possibility of constructing a bootstrap dynamics based on analyticity, unitarity, and crossing symmetry.

I. INTRODUCTION

Historically, the development of S-matrix theory has been closely tied to the Legendre expansion of the scattering amplitude

$$(1) \qquad A(s, z) = \sum_{\ell=0}^{\infty} (2\ell + 1)\, a_\ell(s) P_\ell(z) \quad .$$

The reasons for this are as follows:

(1) Elastic unitarity takes a particularly simple form for $a_\ell(s)$ and lends itself naturally to the construction of an N/D method for doing practical calculation.

(2) By applying the Sommerfeld-Watson transform to eq. (1), Regge Pole contributions appear. These poles control the high energy limit in the cross channel and have therefore generated much enthusiasm in high energy phenomenology.

However the Regge representation for the scattering amplitude does not satisfy the finite energy sum rules exactly and it is difficult (if not impossible) to find a form of the Regge pole amplitude which does not suffer from the double-counting difficulties of the interference model. These problems were solved by Veneziano (1), who guessed a form of the scattering amplitude which (a) is crossing symmetric, (b) has Regge asymptotic behavior, and (c) satisfies the finite energy sum rules exactly.

It would be interesting to find an expansion of the scattering amplitude which produces Veneziano poles much like the Legendre expansion produces Regge poles. This problem was solved independently by

Veneziano (2), and by Nelson (3). The spin-matrix polynomial expansion used by the latter has the reflection properties needed to construct signatured amplitudes. In addition, there is some literature dealing with expansions in terms of these functions (4). We therefore base our work on Nelson's expansion

$$A(s, z) = \sum_{n=0}^{\infty} \frac{c_n(s)}{n!} W_n(z) \quad . \tag{2}$$

Our motivation is to pursue the remarkable property of the spin-matrix polynomials, that they introduce Veneziano poles, hopefully within the context of a new dynamical procedure in S-matrix theory.

During the course of this work, two papers have appeared in preprint form which we feel have relevance to our point of view. One, by Khuri (5), concentrates on deriving the Veneziano representation from the Regge representation. The other, by Martin (6), deduces the existence of the complex n-plane.

II. PROPERTIES OF $c_n(s)$: n = INTEGER

We consider a two-body scattering process described by the amplitude $A(s, z)$, where z and s are the center-of-mass scattering angle and total energy squared, respectively. We ignore complications due to spin, isospin, and subtractions.

We will expand the scattering amplitude in terms of spin-matrix polynomials according to

$$A(s, z) = \sum_{n=0}^{\infty} \frac{c_n(s)}{n!} W_n(\rho z) \quad , \tag{3}$$

where ρ is an energy-dependent scale factor and $W_n(\rho z)$ is given by

$$W_n(\rho z) = \Gamma(\rho z + \tfrac{n}{2} + \tfrac{1}{2})/\Gamma(\rho z - \tfrac{n}{2} + \tfrac{1}{2}) \quad . \tag{4}$$

The coefficients $c_n(s)$ are to be calculated from the formula

$$c_n(s) = \left[\nabla_\xi{}^n \left(\frac{A(s, z) + (-)^n A(s, -z)}{2}\right)\right]_{\xi = (n - 1)/2} , \tag{5}$$

where we regard the independent variables as s and $\xi = \rho z$.

Now the difference operator ∇_z acting on an arbitrary function $f(z)$ is defined to be $\nabla_z f(z) = f(z) - f(z - 1)$. It is not difficult to see that acting n times with the difference operator gives

(6) $$\nabla_z^n f(z) = \sum_{k=0}^{n} (-)^k \binom{n}{k} f(z - k) ,$$

where $\binom{n}{k}$ is just the binomial coefficient $n!/k!(n - k)!$. Let us suppose that $A(s, z)$ satisfies a fixed-s Mandelstam representation

(7) $$A(s, z) = \frac{1}{\pi} \int_{z_0}^{\infty} dz' \frac{D_t(z', s)}{z' - z} + \frac{1}{\pi} \int_{z_0}^{\infty} dz' \frac{D_u(-z', s)}{z' + z} ,$$

where we ignore subtractions. Here the discontinuity functions are defined to be $D_t = \left[A(s, z + i\epsilon) - A(s, z - i\epsilon)\right]/i2$ for positive z and $D_u = \left[A(s, z - i\epsilon) - A(s, z + i\epsilon)\right]/i2$ for negative z, in the limit $\epsilon \to 0^+$. The first term in each expression tends to the physical amplitude on the real axis, as usual. Using the result (7)

(8) $$\sum_{k=0}^{n} (-)^k \binom{n}{k} \frac{1}{z + k} = \frac{\Gamma(n + 1)\, \Gamma(z)}{\Gamma(n + 1 + z)} ,$$

and recognizing that this combination of gamma functions coincides with the beta function, it is easy to see that $\nabla_z^n A(s, z)$ has the simple form

$$(9) \qquad \nabla_z^n A(s, z) = \sum_{k=0}^{\infty} (-)^k \binom{n}{k} \frac{1}{\pi}$$

$$\times \left[\int_{z_0}^{\infty} dz' \; \frac{D_t(z', s)}{z' - z + k} - \int_{z_0}^{\infty} dz' \; \frac{D_n(-z', s)}{-(z' + z) + k} \right]$$

$$= \frac{1}{\pi} \int_{z_0}^{\infty} dz' \; B(n + 1, z' - z) D_t(z', s)$$

$$- \frac{1}{\pi} \int_{z_0}^{\infty} dz' \; B(n + 1, - z' - z) \; D_u(-z', s) \quad .$$

This reduces to eq. (7) for $n = 0$, since $\nabla_z^0 A(s, z) = A(s, z)$ and $B(1, z' - z) = (z' - z)^{-1}$. Otherwise we regard eq. (9) as a generalized form of the fixed-s Mandelstam representation associated with the integer n.

If we now return to eq. (7) and take the difference n times with respect to the variable $\xi = \rho z$, and then set $\xi = (n-1)/2$ as in eq. (5), we obtain

$$(10)\quad c_n(s) = \frac{1}{\pi}\int_{z_0}^{\infty} dz' \;\rho\left[\frac{D_z(z',s) + (-)^n D_u(-z',s)}{2}\right] B(n+1,\ \rho z' - \tfrac{n}{2} + \tfrac{1}{2})$$

$$- \frac{(-)^n}{\pi}\int_{z_0}^{\infty} dz' \;\rho\left[\frac{D_z(z',s) + (-)^n D_u(-z',s)}{2}\right] B(n+1,\ -\rho z' - \tfrac{n}{2} + \tfrac{1}{2}).$$

There is a reflection property for the beta functions involved here, namely $B(n+1,\ \rho z' - \frac{n}{2} + \frac{1}{2}) = (-)^{n+1}\, B(n+1,\ -\rho z' - \frac{n}{2} - \frac{1}{2})$. We find that eq. (10) simplifies to

$$(11)\quad c_n(s) = \frac{1}{\pi}\int_{z_0}^{\infty} dz' \;\frac{\rho^2 z' \left[D_t(z',s) + (-)^n D_u(-z',s)\right]}{(\rho z')^2 - (\frac{n}{2} + \frac{1}{2})^2}\;\frac{\Gamma(n+1)}{W_n(\rho z')}\quad ,$$

where we have used $x\,\Gamma(x) = \Gamma(x+1)$ in a couple of places and the reflection property of the spin-matrix polynomials, $W_n(-\rho z') = (-)^n W_n(\rho z')$, as well.

III. PROPERTIES OF $c_n(s)$; n COMPLEX

In this section we consider the analytic continuation of $c_n(s)\, W_n(\rho z)/n!$ into the complex n-plane. These details are needed to justify the application of the Sommerfeld-Watson transformation to A(s, z). In Ref. (3) it was assumed without proof that the transformation is justified. Substituting eq. (11) into eq. (3), we obtain

$$A(s, z) = \sum_{n=0}^{\infty} \frac{\rho}{\pi} \int_{z_0}^{\infty} dz' \quad \times \frac{\rho z' \left[D_t(z', s) + (-)^n D_u(-z', s) \right]}{(\rho z')^2 - (\frac{n}{2} + \frac{1}{2})^2} \quad \frac{W_n(\rho z)}{W_n(\rho z')} . \tag{12}$$

For large complex n, we find $|W_n(\rho z)| \sim (\frac{n+1}{2})^{n/2} \exp\left[n + 1 + \pi |\mathrm{Im}\, n|/2\right]$. As this is independent of the argument of the spin-matrix polynomial, the ratio of polynomials in ρz and $\rho z'$ appearing in eq. (12) tends to unity. However the factor $(-)^n = e^{i\pi n}$ violates the conditions of Carlson's theorem, and so $c_n(s)\, W_n(\rho z)/n!$ does not possess a unique continuation into the complex n-plane. The way around this difficulty is both standard and physical, namely we have to apply to Sommerfeld-Watson transformation to the signatured amplitudes $A^{\pm}(s, z)$, where

(13) $$A^{\pm}(s, z) = \sum_{n=0}^{\infty} \frac{c_n^{\pm}(s)}{n!} W_n(\rho z) ,$$

$$c_n^{\pm}(s) = \frac{\rho}{\pi} \int_{z_0}^{\infty} dz' \frac{\rho z' \left[D_t(z', s) \pm D_u(-z', s)\right]}{(\rho z')^2 - (\frac{n}{2} + \frac{1}{2})^2} \frac{\Gamma(n + 1)}{W_n(\rho z')} .$$

One can recover $A(s, z)$ from the signatured amplitudes by using the reflection property of the spin-matrix polynomials

(14) $$W_n(-\rho z) = (-)^n W_n(\rho z) ,$$

and this leads to

(15) $$A(s, z) = \frac{1}{2}\left[A^+(s, z) + A^+(s, -z) + A^-(s, z) - A^-(s, -z)\right] .$$

IV. THE POLES OF c(n, s)

We can split the integral in eq. (13) into two parts corresponding, in the t-channel say, to the low energy and high energy regions

$$(16)\qquad c_n^{\pm}(s) = \frac{\rho}{\pi}\int_{z_0}^{z_{max}} dz' \, \frac{\rho z' \, D_t(z', s)}{(\rho z')^2 - (\frac{n}{2}+\frac{1}{2})^2} \, \frac{\Gamma(n+1)}{W_n(\rho z')}$$

$$+ \frac{\rho}{\pi}\int_{z_{max}}^{\infty} dz' \, \frac{\rho z' \, D_t(z', s)}{(\rho z')^2 - (\frac{n}{2}+\frac{1}{2})^2} \, \frac{\Gamma(n+1)}{W_n(\rho z')}$$

$$\pm \left(D_t(z', s) \rightarrow D_u(-z', s)\right) \quad .$$

Here z_{max} defines the region where $D_t(z', s)$ exhibits Regge behavior

$$(17)\qquad D_t(z', s) \sim (z')^{\alpha(s)} \; ; \quad z' \gtrsim z_{max} \quad .$$

The large z' behavior of $W_n(\rho z')$ is proportional to $(\rho z')^n$ and therefore, if s is chosen so that the high energy integral in eq. (16) converges, we get a contribution to $c(n, s)$ like

$$(18)\qquad c^{\pm}(n, s) = \frac{1}{n - \alpha(s)} \quad .$$

Thus the high energy (i.e. inelastic) region in the crossed channel is responsible for the simple moving poles in $c(n, s)$. Nelson assumed the existence of these poles in showing that the Sommerfeld-Watson transformation of the expansion in spin-matrix polynomials leads to a Veneziano formula. (One must also assume that the sum $\alpha(s) + \alpha(t) + \alpha(u)$ has a special value, for example 2 in the process Veneziano discussed originally, $\pi\pi \to \pi\omega$.) The present discussion lends some degree of plausibility to the notion that $c(n, s)$ possesses such poles.

A few remarks concerning the low energy integral in eq. (16) are in order here. We might try to saturate $D_t(z', s)$ with a delta function corresponding to a narrow resonance in the t-channel. If we do this in the process $\pi\pi \to \pi\omega$, with the rho trajectory given by $\alpha(s) = a + bs$, then eq. (13) shows that $c(n, s)$ will possess a pole at $n = \alpha(s) - 1$ if $b(2m_\rho^2 - 3m_\pi^2 - m_\omega^2) = a$. This last condition seems to be consistent with experimental values, and combined with the Veneziano supplementary condition for this process, $\alpha(s) + \alpha(t) + \alpha(u) = 3a + b\Sigma = 2$, ensures that the trajectory passes through unity at $s = m_\rho^2$. However this kind of calculation clearly displays the diseases associated with the simple pole approximation. In addition to the pole coming from the term $\rho z' - (\frac{n}{2} + \frac{1}{2})$ in the denominator, we also get poles from $\Gamma(\rho z' - \frac{n}{2} + \frac{1}{2})$ in the numerator. The latter are displaced by integers above $\alpha(s)$ and so violate the Froissart bound. We do not know what conditions on the spectrum in the t-channel would suffice to eliminate the ancestors, but it's obvious that they will still occur if we take a finite number of fixed poles.

V. THRESHOLD BEHAVIOR OF $c_n(s)$

We have the two expansions for the scattering amplitude

$$A(s,\ z) = \sum_{\ell=0}^{\infty} (2\ell + 1)\ a_\ell(s)\ P_\ell(z) \tag{19}$$

$$= \sum_{n=0}^{\infty} \frac{c_n(s)}{n!}\ W_n(\rho z) \quad ,$$

and the behavior of $a_\ell(s)$ near threshold is known to be

$$a_\ell(s) \underset{q_s \to 0}{\sim} (q_s)^{2\ell} \quad . \tag{20}$$

One does not have to carry the process through completely to see that in re-expanding the Legendre polynomials in terms of spin-matrix polynomials, $c_n(s)$ is given by a sum over all the $a_\ell(s)$ of the same parity for $\ell \geq n$. Moreover the coefficients in the expansion are polynomials of order ℓ in ρ^{-1}. However in order to extract the Veneziano form from the Sommerfeld-Watson transformation, one must take ρ to be $2bq_s^2$. It follows therefore that the $c_n(s)$ tend to constants at threshold. The same line of reasoning applies when there are unequal masses in the final or initial state, as in the process $\pi\pi \to \pi\omega$, and one can then infer the behavior of $c_n(s)$ from the fact that the Veneziano amplitude has no singularities other than poles, meaning that the threshold singularity must be absent in $c_n(s)$. It is remarkable that we can only build Veneziano amplitudes from the expansion in spin-matrix polynomials if the trajectories are linear and have a universal slope, and that these properties, which permit us to choose ρ to be a universal constant times a kinematic quantity, are approximately true in nature.

VI. HADRON DYNAMICS BASED ON THE CALCULUS OF FINITE DIFFERENCES

In this section we discuss the possibility of constructing a bootstrap dynamics based on analyticity, unitarity, and crossing symmetry. Elastic unitarity in the s-channel

$$(21) \qquad D_s^{\pm}(s,t) = \frac{q_s}{32\,\pi^2 (s)^{1/2}} \int d\Omega_I \, A^{\pm}(s,t_{fI})\, A^{\pm *}(s,t_{Ii}) \qquad ,$$

could be used to impose restrictions on the $c_n(s)$. However it is not too difficult to show that the resulting algebraic equations in the $c_n(s)$ are not diagonal in n. Another way to proceed is to use eq. (21) in the crossed channels to compute $D_t^{\pm}(z', s)$ and $D_u^{\pm}(-z', s)$ from some input amplitude, which we would take to be a Veneziano formula. We can then use these discontinuities to compute the $c_n^{\pm}(s)$ from

$$(22) \qquad c_n^{\pm}(s) = \frac{\rho}{\pi} \int_{z_0}^{\infty} dz' \, \frac{\rho z' \left[D_t^{\pm}(z',s) \pm D_u^{\pm}(-z',s)\right]}{(\rho z')^2 - (\frac{n}{2} + \frac{1}{2})^2} \qquad \frac{\Gamma(n+1)}{W_n(\rho z')} \quad ,$$

and finally calculate $A(s, z)$ to first order from

$$A^{\pm}(s, z) = \sum_{n=0}^{\infty} \frac{c_n^{\pm}(s)}{n!} W_n(\rho z) \quad . \tag{23}$$

To close the cycle, we can analytically continue eq. (23) into the crossed channels and recompute the discontinuities. We cannot guarantee beforehand that this iterative procedure for constructing a unitary amplitude will converge. As has been emphasized by Veneziano (2), our input amplitude satisfies all necessary requirements except unitarity. We can only hope that this amplitude is sufficiently close to reality so that it will not be badly distorted by unitarity.

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