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CULVERT WHISTLERS

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ABSTRACT

I describe easily observable phenomena that are the acoustical analogue of electromagnetic whistlers that occur in the earth's atmosphere. Then I give a simple derivation of the dispersive behavior of waves in a wave guide, and compare sound modes and electromagnetic modes in cylindrical wave guides.

OBSERVATION OF CULVERT WHISTLERS

One morning last May (1970) I was romping with my two children on a beach near Bolinas, California, when, during a lapse of my attention, they disappeared. But I could still hear shrill laughter and weird echoes apparently emerging from a sand dune, and I soon found them by following my ears. Thus I found myself peering into one end of a concrete culvert about 4 ft in diameter and 200 ft long, open at both ends, passing under the dune.

After retrieving the kids, it was my turn to play. Naturally I began by generating a delta function--a handclap. I thought perhaps that I might hear some chaotic combination of resonant frequencies, as well as the reflection of my handclap back from the other end. Indeed, I heard the expected sharp reflection about a third of a second after the original clap. But then to my astonishment I heard, not resonances, but a loud descending "zroom" that commenced at high pitch at the same time as the reflected sharp handclap and descended within a few tenths of a second to a rather low pitch where it continued to sound for several seconds as it gradually faded away. For the next few minutes I clapped and banged on the inside culvert wall with sticks and stones. Depending on how I clapped or banged I could emphasize somewhat different frequency components in the descending whistle, but the final long-drawn-out sound always had the same pitch.

After many minutes of puzzling over this striking effect I recalled that there are electromagnetic phenomena called "whistlers." Could it be that my culvert whistlers were the acoustical analogue of these electromagnetic whistlers? I knew that in an electromagnetic whistler a lightning stroke (handclap!) in the northern hemisphere produces radiation that travels along the earth's magnetic lines of force as if the radiation were in a wave guide (culvert!). When the radiation reaches the southern hemisphere it reflects there, travels back along the magnetic lines and returns to its starting point near the lightning stroke, where it can be heard on an ordinary hi-fi audio amplifier equipped with a 200-ft antenna.[1] Because of the dispersive behavior of the earth's ionosphere, the high-frequency components of the lightning-stroke delta function come back first; the lower frequencies take longer, and one hears a swiftly descending whistle. Now, as everyone knows, the ionosphere is dispersive for radio waves, but air is not dispersive for sound waves. Does that mean the analogy breaks down? No. The radio waves that propagate as whistlers are actually of such low frequency (a few kilocycles) that they are "below cutoff" and would not propagate at all in the ionosphere if it were not for the earth's magnetic field. This field lowers the cutoff frequency for one circular-polarization component and allows it to propagate along the field lines as if in a wave guide. Similarly the sound waves would not go through the sand dune without the culvert wave guide, and it is the propagation in a wave guide that gives the dispersion.

DISPERSION OF WAVES IN A WAVE GUIDE

The dispersive behavior of electromagnetic waves traveling in wave guides is well known to most physics students.[2] Less familiar is the similar dispersive behavior of sound waves in a wave guide.[3] We can understand this dispersive behavior simply as follows. (The derivation to follow may also be used for electromagnetic waves in a wave guide. I have not seen this approach used to derive wave guide propagation modes, but it is familiar in optics.) Start with the handclap at one end of the culvert. Fourier analyze it into its wide spectrum of frequencies. For each frequency we have approximately a spherical wave spreading from the handclap. Consider the "rays" along various propagation directions. One small bundle of rays travels nearly down the center of the culvert and reaches the other end without hitting the walls. However, most rays are multiply reflected off the walls, provided the culvert is long compared with its diameter. Consider the special case of a ray that passes through the axis of the cylinder; suppose it makes an angle θ with that axis. After bouncing specularly first off one wall and then off the opposite wall, this twice-reflected ray will be traveling in its original direction. Now draw a plane-wave front perpendicular to original ray. This plane interesects both the original ray and the twice-reflected ray. (See Fig. 1.) For certain special angles θ these two points of intersection will be exactly in phase. For those special directions we will have a constructive-interference maximum, and thus a large wave amplitude. At angles that do not give constructive interference the wave amplitude is comparatively negligible, especially if there are a large number of multiply-reflected rays to interfere. The angles θ that give constructive interference are called "propagation directions."

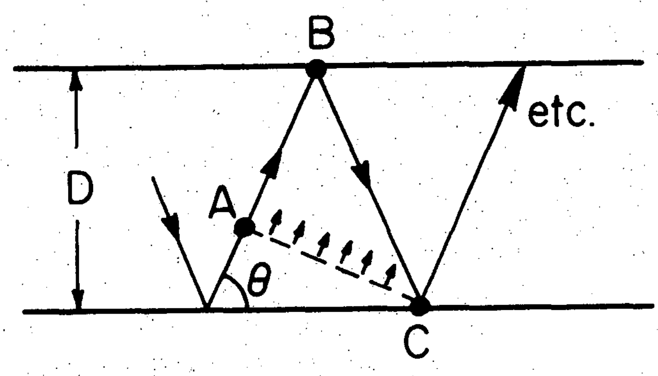

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Fig. 1. A ray traveling at angle θ to the axis of the guide and passing through points A, B, and C. The dotted line A–C is a portion of a wave front. The condition for A and C to be in phase is that path A–B–C be one wavelength. Arrows on the rays give their propagation directions. Short arrows on the wave front give its propagation direction.

We now derive the relation between propagation direction θ and wavelength λ under the approximation that only the rays passing through the axis are important. Later we will give the exact relation, in Eq. (9). The condition for constructive interference is that a twice-reflected ray travel exactly one wavelength farther than the original ray before it intersects their common wave front. (We are considering only the lowest dispersive mode. For higher modes it travels two wavelengths farther, or three, etc.) That is, we demand that path A—B—C in Fig. (1) be one wavelength. For a pipe of diameter D this condition gives (for rays passing through the axis)

$$2D \sin \theta = \lambda . \tag{1}$$

According to Eq. (1), short wavelengths propagate (i.e., give constructive interference) at small angles θ to the axis. Because θ is small these carry energy down the pipe the fastest. Longer wavelengths propagate at larger angles and take more time to zig-zag down the pipe. That is why the short-wavelength components of the handclap arrive first and the handclap becomes a descending whistle after the waves have propagated some distance down the pipe. In fact the group velocity, v_g (velocity at which energy is carried down the pipe), is just the projection of the free-space velocity of sound, c, on the axis:

$$v_g = c \cos \theta . \tag{2}$$

The largest wavelength that propagates (gives constructive interference) corresponds to rays bouncing sideways back and forth across the tube, i.e., with $\theta = 90$ deg. The group velocity is then zero. This "cutoff" wavelength is twice the tube diameter, according to Eq. (1):

$$\lambda_{max} = 2D \; . \tag{3}$$

For wavelengths longer than λ_{max} there is no angle at which multiply-reflected rays can give constructive interference. Various rays bouncing at various angles add up in such a way that, as it turns out, the wave is exponentially attenuated as it passes down the guide.[2]

Since the angle θ is not easily observable experimentally, we eliminate it from Eq. (2), using Eq. (1). Then Eq. (2) becomes [using also Eq. (3)]

$$v_g = c\sqrt{1 - (\lambda/\lambda_{max})^2} \; . \tag{4}$$

Since we actually hear a pitch rather than measure a wavelength we rewrite Eq. (4), using the relation

$$c = \lambda\nu = \lambda_{max}\nu_{min}$$

to obtain

$$v_g = c\sqrt{1 - (\nu_{min}/\nu)^2} \; . \tag{5}$$

My derivation of Eq. (1) only considered rays passing through the axis of the culvert. Rays that do not pass through the axis have a smaller sideways distance to travel between two reflections (along a chord rather than a diameter). When we average over all rays we see that to correct Eq. (1) we must replace diameter D by a somewhat smaller quantity, which turns out to be D/1.172.[4] Equations (4) and (5) remain correct, but Eq. (3) must be replaced by

$$\lambda_{max} = 2D/1.172. \qquad (6)$$

(For a rectangular culvert in a mode with rays bouncing parallel to one side, Eq. (1) is correct as it stands.)

FURTHER OBSERVATIONS IN CULVERTS

In my observations I heard the waves reflected from the other end of the culvert, so that a given frequency component of my handclap had time delay equal to twice the culvert length divided by the group velocity as given by Eq. (5). The longer the culvert the larger the time spread of the whistler and the easier it is to hear. I can barely hear the effect with a 20-foot culvert. One hundred feet works beautifully.[5]

My favorite culvert is about 100 ft long, and has a 24-in. inner diameter. It is located on North Canyon Road just across from Strawberry Canyon Swimming Pool, in Berkeley. The cutoff frequency calculated from Eq. (6) and a sound velocity of 340 m sec^{-1} (at 15°C), is 341 sec^{-1}. That is close to the note F above middle C. Using a tuning fork, I have verified that the long-drawn-out note at the end of the whistler has the expected cutoff pitch.

Many of the culverts that I find under roadways are made of corrugated metal, rather than smooth concrete. For corrugated culverts one may worry that perhaps part of the whistling sound might be a "chirper,"[6] due to the diffraction-grating effect of the corrugations, rather than a whistler, due to wave-guide dispersion. However, the chirping wavelength is very small compared with the whistler cutoff wavelength, and in fact the chirp is inaudible in the presence of the whistler, for corrugated culverts.

A more effective way than handclapping to excite whistlers is to use a bongo drum. Another fine way is to nearly cover the end of the culvert with a sheet of plywood or cardboard and then bang on the board with your hand. The resulting thud contains more low-frequency components than does a handclap. In that case one can easily hear something completely unexpected from the above discussion. One hears in addition to the now-familiar whistler, some repeated echoes from sound bouncing back and forth from one end of the culvert to the other. These echoes are nondispersed, i.e., they come back with a bang, not a whistle. Furthermore, they are easily heard to consist mostly of frequency components that lie below the cutoff frequency of the whistler. This is surprising at first, because according to Eq. (5) it is at the high-frequency limit ($\nu = \infty$) that the group velocity approaches the sound velocity c and there is no dispersion. Indeed, the high-frequency components of a sharp handclap are heard to return as a clap. However, I was amazed to hear the reflected nondispersed thud, because I expected that frequencies below cutoff would be exponentially attenuated, and thus not propagate at all. For example, I knew that there are no nondispersive electromagnetic waves that propagate in a wave guide; thus there are no waves that propagate for frequencies below the lowest cutoff frequency.[2] But sound waves are different from electromagnetic waves. One can send a plane wave of sound down the axis of a tube, at any frequency.[7] The reason it is not possible to send an electromagnetic plane wave down the axis of a metal wave guide is that the tangential electric field must be zero at the walls of the guide. For a plane wave this ensures that the field is zero everywhere across the guide; i. e., there is no wave. For sound waves the longitudinal sound vibrations of air molecules along the direction of the axis, corresponding to a plane wave traveling along the axis, are not required to be zero at the wall (except for a small boundary layer due to friction). Thus for all frequencies this "plane wave mode" exists for sound waves in a wave guide, although not for electromagnetic waves in a wave guide. (This mode does exist for electromagnetic waves on a transmission line, however.)

There should be higher whistler modes, but I have not been able to hear them in Strawberry Culvert. Maybe I will hear higher modes when I try a larger diameter culvert or tunnel (say 10 or 15 ft diameter); then the cutoff frequency for the lowest mode will be too low to be audible, so the lowest mode will not drown out higher modes.

In Appendix 1 I give the exact expressions for the lowest sound modes in a cylindrical culvert and tell how I searched for higher modes. In Appendix 2 I compare sound modes and electromagnetic modes in a cylindrical wave guide. In Appendix 3 I recall the famous factor of 1.22 from optics and recount an instructive mistake I made.

APPENDIX 1. LOWEST SOUND MODES IN A CYLINDRICAL CULVERT

Assume the culvert is infinitely long so that we only have waves traveling in the +z direction. For the plane-wave mode the sound pressure p is independent of transverse position in the guide and is given by

$$p = p_0 \cos(kz - \omega t) , \qquad (7)$$

where $k = \omega/c$. There is no dispersion for this mode; all frequencies have phase and group velocity equal to c.

For the dispersive mode with lowest cutoff frequency, which is the only one I hear, the pressure is given by[3]

$$p = p_0 \cos\phi \, J_1(k_r r) \cos(k_z z - \omega t), \qquad (8)$$

where $k_r = k \sin\theta$; $k_z = k \cos\theta$; $k = \omega/c$; r, ϕ, and z are cylindrical coordinates inside the cylinder; and θ is the propagation angle as previously defined. The boundary condition at the wall is that $\partial p/\partial r = 0$ there, i.e., that $dJ_1(k_r r)/dr = 0$ at $r = \frac{1}{2}D$. The first zero of $dJ_1(x)/dx$ is at $x = 0.5861\pi$. Using $k_r = k \sin\theta$ and $k = 2\pi/\lambda$, we obtain

$$\lambda = (2D/1.172) \sin\theta, \qquad (9)$$

which is the corrected form of Eq. (1). For $\sin\theta = 1$ we get Eq. (6).

The whistler mode with the next higher cutoff frequency has pressure given by

$$p = p_0 \cos 2\phi \, J_2(k_r r) . \qquad (10)$$

The first zero of $dJ_2(s)/dx$ occurs at $x = 0$. That doesn't count. The next occurs at $x = 0.9722\pi$. The ratio of the corresponding cutoff frequency to that of the lowest whistler mode is thus 0.9722/0.5861, which is 1.66, which is the musical interval of a major sixth. I have listened for this whistler cutoff pitch in Strawberry Culvert, without success. I also built a contraption with plywood baffles to force nodes at the four planes where $\sin 2\phi$ is zero. I inserted this into one end of the culvert, and covered the end of the culvert with baffles to let in sound pressure at the optimum places. I was hoping to both excite the second whistler mode and suppress the loud lowest mode. It didn't work; perhaps plywood was the wrong stuff to use. Nevertheless there was an unexpected dividend in that I discovered the low-frequency "plane wave" mode while accidentally banging a slab of plywood across the end of the culvert. (Once having discovered it I can hear it with low-frequency voice grunts, or box thuds.)

APPENDIX 2. COMPARISON OF SOUND MODES AND ELECTROMAGNETIC MODES

It is interesting to compare the acoustical modes with the electromagnetic modes in a cylindrical wave guide. First we consider the plane-wave mode. In electromagnetism this is called the TEM (transverse electric and magnetic) mode. It can occur for electromagnetic waves on a transmission line (two separate parallel conductors) but not in a wave guide (a single hollow conductor).[2] It can occur for sound waves in a wave guide, as we have seen.

Next we consider all the dispersive sound-wave modes. These have sound pressure that satisfies[3]

$$p = p_0 \cos n\phi \, J_n(k_r r) \cos(k_z z - \omega t), \quad n = 0, 1, 2, \cdots \tag{11}$$

with boundary condition

$$\partial p/\partial r = 0 \text{ at the wall.} \tag{12}$$

Thus the cutoff frequencies for all sound-wave modes correspond to zeroes of $dJ_n(x)/dx$.

Next we consider the so-called TE (transverse electric) electromagnetic modes. They all satisfy[2]

$$B_z = B_0 \cos n\phi \, J_n(k_r r) \cos(k_z z - \omega t), \quad n = 0, 1, 2, \cdots \tag{13}$$

with boundary condition

$$\partial B_z/\partial r = 0 \text{ at the wall.} \tag{14}$$

(This boundary condition actually results from the boundary condition that the tangential electric field vanishes at the wall, together with the necessary relation between magnetic and electric fields in free space.)

By comparison of Eq. (13) with (11), and (14) with (12), we see that the sound modes correspond exactly with the TE electromagnetic modes. For a guide of a given diameter, they have the very same cutoff wavelengths. (Of course the cutoff frequencies are greater for the electromagnetic waves than for the sound waves by a factor equal to the ratio of light velocity to sound velocity, about 10^6.) The first three sound-mode or TE-mode cutoff wavelengths correspond to

$dJ_1/dx = 0$ at $x = 0.5861\pi$, $dJ_2/dx = 0$ at $x = 0.9722\pi$, and $dJ_0/dx = 0$ at $x = 1.2197\pi$.

Next we consider the so-called TM (transverse magnetic) electromagnetic modes. They all satisfy[2]

$$E_z = E_0 \cos n\phi \, J_n(k_r r) \cos(k_z z - \omega t), \quad n = 0, 1, 2, \cdots \tag{15}$$

with boundary condition

$$E_z = 0 \text{ at the wall.} \tag{16}$$

These have no analogue in the sound modes. (Of course the reason there are twice as many possible electromagnetic modes as sound modes is that there are two independent polarization states for electromagnetic waves, and only one for sound waves.) The first two TM-mode cutoff wavelengths correspond to $J_0(x) = 0$ at $x = 0.76\pi$, and $J_1(x) = 0$ at $x = 1.2197\pi$. It happens "by accident" that $dJ_1/dx = -J_0(x)$, so that the TM modes having $E_z \sim J_0(k_r r)$ have the same cutoff frequencies as the TE modes having $B_z \sim J_1(k_r r)$. Otherwise the TM and TE mode frequencies are different.

APPENDIX 3. FAMOUS FACTOR OF 1.22

We may recall that in the familiar optics problem of Fraunhofer diffraction from a circular hole, the diffracted field amplitude is given by

$$E = \frac{A \, J_1(k_r r)}{r} ,$$

with $k_r = k \sin \theta$. The first zero of this diffraction pattern occurs at the first zero of $J_1(x)$, at $x = 1.2197\pi$. This is the famous factor 1.22 that is always occurring in optics books. This factor is so familiar and "apparently universal" that when I first calculated the expected cutoff frequency for Strawberry Culvert I assumed that the way to correct "$\lambda_{max} = 2D$" was simply to replace D by D/1.22, "as one always does for circular holes," instead of the later-discovered correct value D/1.17. (None of my learned colleagues corrected me on this.) This 5% mistake corresponds to a musical interval of about one minor second, and led me to predict a cutoff pitch of about F sharp. Yet my ear and tuning fork gave me a note definitely much closer to F. I wish I could now report that, using my ear as a guide, I rejected the factor of 1.22, discovered the factor 1.17 experimentally, and thereby was led to discover, independently, that there is more than one Bessel function. It didn't happen that way, but it could have, if I had trusted my ear.

ACKNOWLEDGMENTS

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1. Robert Helliwell, Whistlers and Related Ionospheric Phenomena (Stanford University Press, Stanford, 1965).
2. See S. Ramo and J. R. Whinnery, Fields and Waves in Modern Radio (John Wiley & Sons, Inc., New York, 1953), Chaps. 8 & 9.
3. See Philip M. Morse, Vibration and Sound (McGraw-Hill, Book Co., Inc., New York, 1948), p. 305 ff.
4. See Appendix 1.
5. The dispersion of sound waves in acoustical wave guides has been well understood for many years. However, I have not found any reference to either the prediction or the experimental observation of what I call culvert whistlers. Probably many children have heard them. Did Lord Rayleigh overlook (i.e., not overhear) them?
6. Frank S. Crawford, Chirped Handclaps, Amer. J. Phys. 38, 378 (1970).
7. After rediscovering this experimentally, I finally realized that I already knew it and it is well known (see Ref. 3.) Indeed, this

is the only kind of mode possible when sound wavelengths are large compared with the tube diameter, as is usually the case for organ pipes and mailing tubes.

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