THE NEGATIVE ENERGY N-BODY PROBLEM HAS FINITE DIAMETER

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Abstract. The Jacobi-Maupertuis metric provides a reformulation of the classical N-body problem as a geodesic flow on an energy-dependent metric space denoted M_E where E is the energy of the problem. We show M_E has finite diameter for $E < 0$. Consequently M_E has no "metric rays". Motivation comes from work of Burgos- Maderna and Polimeni-Terracini on the case $E \geq 0$ and from a need to correct an error made in a previous "proof". M_E is constructed by completing the JM metric, a Riemannian metric on the Hill region, a domain in configuration space. We show that M_E has finite diameter for $E < 0$ by showing that there is a constant D such that all points of the Hill region lie a distance D from the Hill boundary. (When $E \geq 0$ the Hill boundary is empty.) The proof relies on a game of escape which allows us to quantify the escape rate from a closed subset of configuration space, and the reduction of this game to one of escaping the boundary of a polyhedral convex cone into its interior.

The classical N-body problem at fixed energy E can be reformulated as a geodesic problem. The geodesics are those of the Jacobi-Maupertuis [JM] metric at that energy. Our main result is

Theorem 0.1. *The Jacobi-Maupertuis [JM]metric for the N-body problem at negative energy has finite diameter.*

The configuration space of the N-body problem is a Euclidean space endowed with a potential energy function $q \mapsto V(q)$. The Jacobi-Maupertuis metric at energy E is defined on the region $\{q : -\infty < V(q) < E\}$ whose closure, the domain $\{q : -\infty \leq V(q) \leq E\}$, we call the Hill region. The boundary $\{q : V(q) = E\}$ is called the Hill boundary and will be denoted as ∂M_E. The Hill boundary is non-empty if and only if $E < 0$. The Jacobi-Maupertuis metric degenerates to zero at the Hill boundary and so we can travel along the Hill boundary at zero (metric) cost. Write M_E for the metric completion of the Jacobi-Maupertuis metric and d_E for the corresponding metric. When forming M_E for $E < 0$ we must collapse the Hill boundary to a single point because of this zero-cost travel. We also denote this point by ∂M_E. The theorem above follows from

Theorem 0.2. *If $E < 0$ then $sup_{\{q \in M_E\}} d_E(\partial M_E, q) < \infty$*

Proof of theorem 0.1 from theorem 0.2. The theorem asserts that the boundedness of the function $d_E(\partial M_E, q)$. In other words, there is a positive constant K such for all $q \in M_E$ we have that $d_E(\partial M_E, q) < K$. Take $p, q \in M_E$. Then $d_E(p, q) \leq d_E(p, \partial M_E) + d_E(\partial M_E, q) \leq 2K$ so that the diameter of M_E is less than or equal to $2K$.

remark 1. M_E is well-known to have infinite diameter when $E \geq 0$.

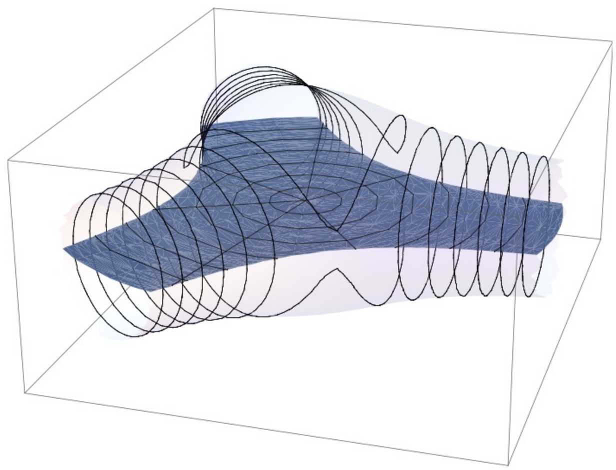

FIGURE 1. The Contour level surface $V = -1$ drawn in the planar 3-body shape space is like the surface of a plumbing fixture consisting of three pipes centered about the three binary collision rays. The Hill region $\{V \leq -1\}$ projects onto the interior of the surface. The shaded planar domain inside the Hill region is the Hill region for the collinear three-body problem at energy -1. (Courtesy of Rick Moeckel.)

0.1. **Motivation.** The JM metric perspective on N-body dynamics has recently proven particularly useful for positive energies. Maderna and Venturelli [9] combined this perspective with weak KAM methods to prove that given any initial configuration and any final asymptotic 'hyperbolic' state that there is a positive energy solution connecting the two.

Maderna and Venturelli's positive energy solutions are metric rays in M_E, $E > 0$. A *metric ray* in a metric space M is an isometric embedding of the half-line $[0, \infty) \subset \mathbb{R}$ into that metric space, the half-line being endowed with the usual metric inherited from $\mathbb{R}$. A minimizing geodesic $c : [a, b] \to M$ is an isometric embedding of the interval $[a, b]$ into M. If $\gamma : [0, \infty) \to M$ is a metric ray then the restriction of γ to any compact subinterval $[a, b] \subset [0, \infty)$ is a minimizing geodesic. (See Burago et al [4] for general definitions and results concerning geodesics in metric spaces.)

Burgos and Maderna ([6]) obtained a 'parabolic generalization" of Maderna-Venturelli, and in so doing asked the compelling question: **If the energy E is negative does M_E admit any metric rays?** Since finite diameter spaces cannot support metric rays our theorem provides an immediate answer to their question.

Corollary 0.1. *There are no metric rays in M_E for $E < 0$.*

On page 387 of [14] I had claimed to have proven theorems 0.1 and 0.2. That proof is wrong. I also wrote this article to correct that error. In detail, the error is as follows. I gave an estimate showing that any point could be connected to the Hill boundary by a Euclidean ray whose Jacobi-Maupertuis length was linear in

$\|A\|$ where A is the point where that ray pierces the Hill boundary. But $\|A\|$ is unbounded on the Hill boundary, so the claimed proof does not bound $d_E(q, \partial M_E)$.

0.2. **A conjecture.**

Conjecture 1. *$sup_{\{q \in M_E\}} d_E(\partial M_E, q) = d_E(\partial M_E, 0)$ when $E < 0$.*

This conjecture holds for $N = 2$ bodies. We can also show that it holds locally for the 3-body problem. 'Locally' here means that $q = 0$ is a local maximum for $d_E(q, \partial M_E)$, and that $d_E(q, \partial M_E) < d_E(0, \partial M_E)$ for all $\|q\|$ is sufficiently large. $d_E(\partial M_E, 0)$, can be computed explicitly in terms of 'minimal central configurations" in a manner similar to the way that the minimal action to total collision in a fixed time can be computed by 'dropping' minimal central configurations. See for example [11].

1. Set-up. Jacobi-Maupertuis metrics and the N-body equations.

Let $\mathbb{E}$ be a real inner product space and V a smooth real-valued function on $\mathbb{E}$. Together this data defines a Newton's equations:

$$\ddot{q} = -\nabla V(q). \tag{1}$$

with conserved energy

$$E(q, v) = K(v) + V(q), \text{ where } \qquad K(v) = \frac{1}{2}\langle v, v\rangle, v = \dot{q}.$$

Here the dots denote time derivatives. The inner product $\langle v, v\rangle$ used to define the kinetic energy K is the given inner product on $\mathbb{E}$. The gradient "∇" in Newton's equations is relative to this inner product so that $dV(q)(h) = \langle \nabla V(q), h\rangle$.

The function V is called the potential energy. In order to accomodate the N-body problem we allow for points q with $V(q) = -\infty$. We call these collision points. The gradient of V goes to infinity as we approach a collision point so Newton's equations break down.

Since $K \geq 0$ we have that $V(q) \leq E$ if a solution $q(t)$ has energy E. Define the Hill region at energy E to be the locus

$$\text{Hill region} = \{q \in \mathbb{E} : V(q) \leq E\}$$

and its boundary, called the Hill boundary, to be

$$\partial M_E := \{q \in \mathbb{E} : V(q) = E\}$$

All energy E solutions $q(t)$ to Newton's equations must lie in the Hill region. If the solution $q(t)$ encounters the Hill boundary at some instant $t = t_0$ then $K(\dot{q}(t_0)) = 0$ which means the solution has instantaneously stopped. We call such instants or locations along the path "brake points".

The Jacobi-Maupertuis principle asserts that the solutions to Newton's equations at energy E can be characterized as geodesics on the Hill region.

Definition 1.1. *The Jacobi-Maupertuis metric at energy E for the Newton's equation (1) is the Riemannian metric*

$$ds^2_{JM} = 2(E - V(q))\langle dq, dq\rangle \tag{2}$$

defined on the interior $\{-\infty < V < E\}$ of the Hill region for that energy, with collisions ($V = -\infty$) excluded.

Note that the conformal factor $E - V(q)$ vanishes on the Hill boundary: the metric fails to be Riemannian and paths can travel for no cost along the Hill boundary.

Theorem 1.1 (Jacobi-Maupertuis principle)**.** *Away from collisions and brake points, the energy E solutions to Newton's equations are reparameterizations of geodesics on the Hill region. Conversely, away from the collisions and the Hill boundary, geodesic for this Jacobi-Maupertuis metric are reparameterizations of energy E solutions of Newton's equations.*

See Landau-Lifshitz [8] p. 141, Knauf [7] p. 178, or Abraham-Marsden [1] p.228 for proofs of the Jacob-Maupertuis principle.

The metric completion of the interior of the Hill region will be denoted as M_E. If the Hill boundary is non-empty and connected it contains that boundary, collapsed to a point, as a single point denoted as ∂M_E. That point is typically not a manifold point. Depending on the potential V, the collision locus may, or may not be in M_E. For a detailed description of M_E see Proposition 1 on p. 383 of [14].

1.1. **The N-body equations.** We put the N-body problem into the above framework by using the set-up which Albouy and Chenciner taught me. See [2] or [3]. View the bodies as point masses. They move in d-dimensional Euclidean space $\mathbb{R}^d$. The standard choice of d is $d = 3$. The bodies, or masses, are labelled by an index $a \in [N] = \{1, 2, \ldots, N\}$. The instantaneous location of the ath body is denoted by $q_a \in \mathbb{R}^d$ so that the simulataneous positions of all N-bodies is encoded by the vector

$$q = (q_1, q_2, \ldots, q_N) \in \mathbb{E} := (\mathbb{R}^d)^N.$$

The mass of the ath body is $m_a > 0$ and the masses endow $\mathbb{E}$ with an inner product $\langle \cdot, \cdot \rangle$ which we call the mass inner product and whose associated quadratic form is :

$$\langle q, q \rangle = \Sigma m_a |q_a|^2,$$

where $|q_a|^2 = q_a \cdot q_a$ is the standard dot product on $\mathbb{R}^d$. When applied to velocities $\dot{q} = (\dot{q}_1, \ldots \dot{q}_N); \dot{q}_a := dq_a/dt$ the mass inner product yields twice the kinetic energy:

$$\begin{aligned} K(\dot{q}) &= \frac{1}{2} \langle \dot{q}, \dot{q} \rangle \\ &= \frac{1}{2} \Sigma m_a |\dot{q}_a| \end{aligned} \tag{3}$$

The potential energy $V = -U$ is the negative of

$$U(q) = G \Sigma \frac{m_a m_b}{r_{ab}}; r_{ab} = |q_a - q_b|, \qquad V = -U$$

the sum being over all distinct pairs a, b taken from $[N] = \{1, 2, \ldots, N\}$, and G being the gravitational constant. The total energy is then given by:

$$E(q, \dot{q}) = K(\dot{q}) - U(q) \tag{4}$$

Since $E \geq V \iff U \geq -E$ we have that the Hill region for energy E is the domain $\{q : U(q) \geq -E\}$ within $\mathbb{E}$. Since $U(q) > 0$ everywhere we have that the Hill region is all of $\mathbb{E}$ whenever $E \geq 0$. On the other hand, if $E < 0$ the Hill region is not all of $\mathbb{E}$ and has a non-empty boundary

$$\partial M_E = \{q : U(q) = -E\}$$

The JM metric on the interior of M_E is

$$ds_E^2 = 2(U(q) - E)\langle dq, dq \rangle.$$

We note that the metric is conformal to the flat Euclidean metric $\langle dq, dq\rangle$ but that the conformal factor is zero along the Hill boundary. It follows that we can *travel for free* along the Hill boundary. It costs nothing to move along the Hill boundary.

1.2. **Scaling Normalization.** We reduce to the case of energy $E = -1$ using the standard scaling symmetry for the N-body problem. This scaling asserts that if the curve $q(t) \in \mathbb{E}$ is a solution to (1) then, for any positive real number λ, so is $\lambda q(\lambda^{-3/2}t)$ and that if the 1st solution has energy E then the second solution has energy E/λ. At the metric level, this scaling symmetry corresponds to the fact that the scaling substitution that $q = \lambda Q$ takes the JM metric on $M_{E/\lambda}$ to $\lambda^{1/2}$ times the JM metric on M_E. By appropriate λ we can scale any negative E to $E = -1$. So, from now on, set $E = -1$, write $M_{-1} = M$ and the JM metric d_{-1} on M as d.

1.3. **Collisions.** The collision locus $\Delta \subset \mathbb{E}$ is the union

$$\Delta = \bigcup_{\text{distinct pairs}} \Delta_{ab}$$

of the linear subspaces $\Delta_{ab} = \{r_{ab} = 0\} = \{q \in \mathbb{E} : q_a = q_b\}$. The potential blows up exactly at the points of Δ and the forces, or gradients , the right hand side of Newton's equations, also blow up along the collision locus.

1.4. **Distance to collision.** The following lemma is crucial to our proof.

Lemma 1.1. *The Hill region $\{q : U(q) \geq 1\}$ lies a bounded Euclidean distance $dist(q, \Delta)$ from the collision locus: there is a positive constant k such that $U(q) \geq 1 \implies dist(q, \Delta) \leq k$.*

Here we write $dist(q, K) = inf_s(\|q - s\| : s \in K)$ for K a closed subset of $\mathbb{E}$.

The lemma uses the fact that the function $1/U(q)$ is Lipshitz equivalent to the function $dist(q, \Delta)$. In other words, there exist positive constants $c_1 < C$ such that that for all $q \in \mathbb{E}$ we have

$$c_1 dist(q, \Delta) \leq \frac{1}{U(q)} \leq C dist(q, \Delta). \tag{5}$$

Proof of lemma. The lemma follows immediately from the first inequality of inequality (5) with $k = 1/c_1$. QED

Inequality (5) is based on the distance formula [1]

$$dist(q, \Delta_{ab}) = k_{ab} r_{ab}, \text{ where } k_{ab} = \sqrt{m_a m_b/(m_a + m_b)} \tag{6}$$

from which it follows that

$$U = \Sigma \frac{\lambda_{ab}}{dist(q, \Delta_{ab})} \qquad \text{where } \lambda_{ab} = G m_a m_b k_{ab}$$

Use

$$dist(q, \Delta) = min_{ab} dist(q, \Delta_{ab})$$

to get

$$\frac{\lambda_{ab}}{dist(q, \Delta_{ab})} < U(q) \leq \Sigma \frac{\lambda_{ab}}{dist(q, \Delta)}$$

[1] This equality is stated without a derivation in the last paragraphs of my book on subRiemannian geometry. For completeness, I derive the distance formula in an Appendix.

valid for any pair a, b. Choose a pair a, b such that $dist(q, \Delta_{ab}) = dist(q, \Delta)$ and set

$$\lambda_* = min_{a \neq b} \lambda_{ab}, \qquad \text{and } \Lambda = \sum \lambda_{ab}.$$

We get

$$\frac{\lambda_*}{dist(q, \Delta)} \leq U(q) \leq \frac{\Lambda}{dist(q, \Delta)} \tag{7}$$

which yields inequality (5) with constants $c_1 = \frac{1}{\Lambda}$ and $C = \frac{1}{\lambda_*}$.

2. A game of escape

Our proof of theorem 0.2 relies on a game of escape.

Setting up the game. Select a finite-dimensional Euclidean space $\mathbb{E}$ together with a finite collection $L_1, L_2, \ldots, L_k$ of distinct linear subspaces of $\mathbb{E}$, no one of which is contained in any other.

The game. The game is to escape the union of the L_i as quickly as possible.

Write $\Delta = \cup_{i=1}^k L_i$. We measure escape in terms of $dist(q, \Delta) = min_i dist(q, L_i)$. We use it to quantify the pay-off in the game which we call the "escape rate". We have found it helpful to generalize the setting.

Let Δ be a closed subset of Euclidean space $\mathbb{E}$. For $t > 0$ let

$$N_t := N_t(\Delta) := \{q \in \mathbb{E} : dist(q, \Delta) \leq t\}$$

denote the set of points of $\mathbb{E}$ lying within a distance t of C. Note that

$$\partial N_t = \{q : dist(q, \Delta) = t\}.$$

Escape routes and their rates

Definition 2.1. *A t -escaper is a rectifiable path starting inside $N_t(\Delta)$ and exiting $N_t(\Delta)$ and along which the distance from Δ is strictly monotonic increasing as a function of arc length.*

remark In order for the distance from Δ to be strictly monotonic increasing it must be true that Δ has empty interior.

We want to talk about the escape rate of an escaper.

Definition 2.2. *Parameterize a t-escaper γ by arclength s. Suppose that*

$$dist(\gamma(s), \Delta) \geq dist(\gamma(0), \Delta) + cs \tag{8}$$

holds for some constant $c > 0$ and all s up to the escape time. Then we will say that γ has t-escape rate at least c. The largest such c will be called the escape rate of the escaper γ.

And we want to talk about the escape rate from Δ

Definition 2.3. *If there is a positive number c such that for all $t > 0$ and all $p \in N_t$ there exists a t-escaper starting at p with escape rate at least c then we will say that the escape rate from Δ is positive and at least c. The supremum of all such c's is the escape rate from Δ. When we want to distinguish this escape rate from the earlier escape rates we have defined we will refer to it as the* global escape rate.

Our goal in playing the game and making all these definitions is to prove:

Theorem 2.1. *If Δ is the union of a finite collection of proper linear subspaces of a Euclidean vector space then the global escape rate from Δ is positive and at least $1/dist(0, \partial N_1(\Delta))$. The t-escapers can be taken to be line segments.*

We postpone the proof.

3. Proof of theorem 0.2 from theorem 2.1

Take Δ to be the collision locus. Inequality (5) shows that if $dist(q, \Delta) \geq \frac{1}{C}$ then $U(q) \leq 1$. In other words, a t-escaper with $t = 1/C$ has left the Hill region $\{U \geq 1\}$ and so has crossed the Hill boundary $\{U = 1\}$ at or before escape from $N_{1/C}(\Delta)$. Theorem 2.1 applied to Δ with $t = 1/C$ guarantees a positive global t-escape rate of $c > 0$. Now $U > 1$ implies $dist(q, \Delta) < \frac{1}{C}$: the Hill region is contained in $N_{1/C}(\Delta)$. Theorem 2.1 guarantees that through any point q_* of the Hill region there is a unit speed line segment of $q(t)$ starting at q_*, satisfying $dist(q(t), \Delta) \geq dist(q_*, \Delta) + ct$ and crossing the Hill boundary. Inequality (5) now implies that $\frac{1}{U(q(t))} \geq c_1(\frac{1}{U(q_*)} + ct) \geq kt$ with $k = c_1 c$. Thus

$$U(q(t)) \leq \frac{1}{kt}.$$

It follows that along our t-escapers $q(t)$ we have $\sqrt{U(q(t)) - 1} \leq \sqrt{\frac{1}{kt} - 1}$ whenever $U(q(t) \geq 1$. For simplicity, set

$$\lambda(x) = \begin{cases} \sqrt{x-1}, x \geq 1 \\ 0, x \leq 1 \end{cases} .$$

The JM arclength integrand can then be written $\lambda(U(q(t))\|\dot{q}(t)\|dt$ and the inequality just establishes asserts that $\lambda(U(q(t)) \leq \lambda(\frac{1}{kt})$. (Our escapers $q(t)$ have escaped beyond the Hill boundary at or before the time $t_* = 1/k$.)

Along an escaper $\|\dot{q}(t)\| = 1$ so that its JM arclength satisfies

$$\ell(q) \leq \int_0^{1/k} \sqrt{\frac{1}{kt} - 1} dt = k \int_0^1 \sqrt{\frac{1}{u} - 1} du$$

This last integral is finite and less that $1/2 = \int_0^1 \sqrt{\frac{1}{u}} du$ so that the JM arclength to escape is less than $\frac{k}{2}$. We have shown that any point inside the Hill region can be connected to the Hill boundary by a path whose length is less than $k/2$, showing that $dist(q, \partial M) \leq c/2$ for all q_*.

QED

4. Proof of the escape theorem, theorem 2.1

We begin with examples of the escape game and consequent escape rates.

4.1. **Examples.**

1. If Δ is a linear subspace then the escape rate from Δ is 1. Lines orthogonal to Δ supply optimal escape routes.

2. If Δ is the union of the x and y axis in the plane then $N_t(\Delta)$ is the union of the two strips $|x| \leq t$ and $|y| \leq t$. The points $(t, t) \in \partial N_t(\Delta)$ and its mirrors $(-t, t), (t, -t), (-t, -t)$ in the other three quadrants are the furthest exit points from Δ, being the furthest points on $\partial N_t(\Delta)$ from the origin. The escape strategy used

in the proof of of the polyhedral escape theorem 4.5.1 consists of a separate strategy for each quadrant. In the first quadrant escape by translating that quadrant toward the point (t, t) at unit speed. Thus p in the first quadrant escapes along the path $p + s(1/\sqrt{2}, 1/\sqrt{2})$ The escape rate of these paths is $1/\sqrt{2}$.

3. If Δ is the union of two lines in the plane which make an angle of $\theta_* < \pi/2$ relative to each other in the plane, then the escape rate into the acute sector $0 \le \theta \le \theta_*$ is $\sin(\theta_*/2)$. The strategy within that sector is the one of the proof of of the polyhedral escape theorem 4.5.1: move according to the translation $p + sv$ where v is the unit vector along the angle bisector of this acute sector.

4. Suppose that Δ is the coordinate orthant in $\mathbb{R}^n$, by which we mean the union of the coordinate hyperplanes $x_1 = 0$, $x_2 = 0$, etc. Then the escape rate from Δ is $1/\sqrt{n}$. For escaping from the positive orthant $x_i \ge 0$ mone at unit speed in the direction parallel to the ray $x_1 = x_2 = \ldots = x_n$.

4.2. **Wrong Turns and Gradient Flows.** Originally, I had approached the problem of trying to "win" the escape game by using the gradient flow of the function $F(q) = +dist(q, \Delta)$. The trajectories of this flow, appropriately interpreted, are piecewise linear t-escapers. However, an example involving configurations near double binary collisions in the planar four-body problem shows that the escape rates of the resulting solutions to $\dot{q} = \nabla F(q)$ can be arbitrarily small, and so in the end this approach to escaping was not of help.

A successful approach which yields the correct global escape rate is based on the gradient flow for $F_t(q) = -dist(q, \partial N_t(\Delta))$. It uses the reduction to convex sets described below and the fact that the minus gradient flow to a convex set with non-empty interior has trajectories which are shortest line segments to that convex set. The strategy we describe here (see the proof of the polyhedral escape theorem 4.5.1) ended up being significantly simpler than this one, although its escapers are the limits $t \to \infty$ of the t-escapers of this alternative successful gradient flow approach.

4.3. **Reduction to Hyperplane arrangements.** We reduce the proof of theorem 2.1 to the case where the linear subspaces L_i of the theorem are hyperplanes. In order to do this, observe that if $\Delta \subset \Delta'$ are closed subsets then $N_t(\Delta) \subset N_t(\Delta')$ and $dist(q, \Delta) \ge dist(q, \Delta')$. It follows that if γ is a t-escape path for Δ' which starts in $N_t(\Delta) \subset N_t(\Delta')$ then it is also a t-escape path for Δ. Consequently, if the escape rate from Δ' is $c' > 0$ then the escape rate from Δ is also positive and is some $c \ge c'$.

Let $L_1, \ldots, L_i, \ldots, L_k$ be the linear subspaces. Choose hyperplanes $H_i \supset L_i$ and write $\Delta' = \bigcup_i H_i$. Then $\Delta \subset \Delta'$ so that $N_t(\Delta) \subset N_t(\Delta')$. It follows from the definitions that if Δ' has global escape rate c then the global escape rate from Δ is at least c.

Relabel the linear hyperplanes to call them L_i instead of H_i and their union to be Δ. We have reduced our escape problem to:

Proposition 4.1 (Escaping linear hyperplane arrangements)**.** *If $\Delta = \bigcup_{i=1}^k H_i$ is a hyperplane arrangement in a Euclidean vector space then the escape rate from Δ is finite and greater than or each to c_* where c_* is given immediately following theorem 4.1 below.*

4.4. **Reduction to a Convex polyhedral cone.** We further reduce to the problem of escaping from a convex polyhedral cone. Write each hyperplane L_i of proposition 4.1 in the form $L_i = ker(\ell_i)$ where $\ell_i \in \mathbb{E}^*$ is a nonzero linear functional.

Normalize ℓ_i to have unit length, so that

$$\ell_i(q) = \langle q, n_i \rangle$$

is the operation of inner product with respect to the unit normal vector n_i to L_i. The distance of a point q from L_i is $|\ell_i(q)|$ so that

$$dist(q, \Delta) = min_{i \in [k]} |\ell_i(q)|$$

where we use the symbol $[k]$ to mean the set $\{1, 2, 3, \ldots, k\}$.

The complement of Δ consists of a finite number of connected components:

$$\mathbb{E} \setminus \Delta = C_1 \cup C_2 \cup \ldots \cup C_M.$$

Each component C_α, $\alpha = 1, 2, \ldots, M$ is an open polyhedral cone bounded by some subcollection $L_i, i \in I = I(\alpha)$ of the hyperplanes making up our arrangement. Any t-escaper must enter into one or another of these components. Once entering it can never leave that component. This 'no exit' property follows from the fact that if a continous curve is travelling in a component C_α and then exits to travel into another component it must have crossed one of the bounding hyperplanes L_i. Crossing requires $dist(q, \Delta) = 0$ which violates the assumption that this distance function is strictly monotone increasing along escapers. So, we have reduced to escapers which escape into the interior of a single convex polyhedral cone.

In order to focus on a single convex cone we modify the definition of "escape rate" to focus on those paths escaping into the cone's interior.

Definition 4.1. *Let K be a convex cone with nonempty interior. Then by the escape rate into K we mean the global escape rate from $\Delta = \partial K$, for escapers which escape entering the interior of K.*

Theorem 4.1 (Escaping polyhedral cones)**.** *Let K be a closed polyhedral cone with non-empty interior. Then the escape rate into K is positive. The exact escape rate is $c = 1/dist(0, K_1)$ where $K_1 \subset K$ is the convex polyhedron defined by $dist(q, \partial K) \geq 1$ for $q \in K$.*

Proof of Proposition 4.1 The escape rate from the union Δ of the hyperplanes equals the minimum of the escape rates into the interiors of all the component closed convex polyhedra K^α whose interiors comprise the components of the complement of Δ. In other words, for each K^α take its associated escape rate $c_\alpha = 1/dist(0, (K_1^\alpha)$. Let $c_* = min_\alpha c_\alpha$. Since the union of the ∂K^α form $\Delta = \bigcup_i L_i$, and since any t-escaper must escape into the interior of one or another of these components, we have that the escape rate from Δ is at least c_*. QED

4.5. **Proving the polyhedral escape theorem.** We begin with some background and notation. A convex polyhedral cone $K \subset \mathbb{E}$. is defined by a finite collection of linear inequalities:

$$K = \{q : \ell_i(q) \geq 0, i = 1, \ldots, m\} \tag{9}$$

where the $\ell_i \in \mathbb{E}^*$ are unit length linear covectors. Thus $\ell_i(q) = \langle n_i, q \rangle$ where the n_i are the inward-pointing unit normal vectors to the cone's faces. Write

$$L_i = \{\ell_i = 0\}$$

for the corresponding hyperplanes so that $F_i = L_i \cap K$. The distance of a point $q \in K$ from a face F_i is $\ell_i(q)$. Then , if $q \in K$ we have that

$$dist(q, \partial K) = min_{i \in [m]} \ell_i(q), \tag{10}$$

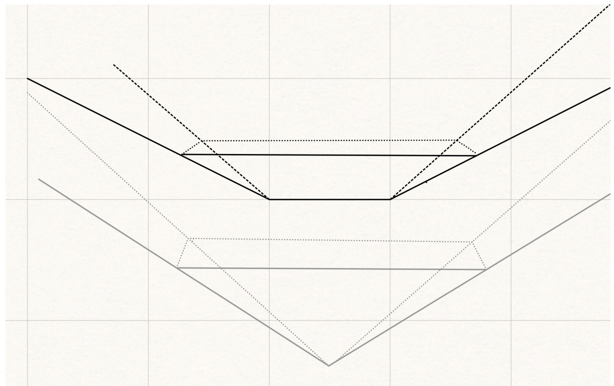

Figure 2. Cones K_0 (grey) and its equidistant K_1 (black) which is not a cone. See Appendix B for details.

where we are using the symbol $[m]$ for the set of integers $1, 2, \ldots, m$.

Define

$$K_t = \bigcap_{i \in [m]} \{q : \ell_i(q) \geq t\}. \tag{11}$$

K_t itself is a convex polyhedron, being the intersection of the finite collection of half-spaces $\ell_i \geq t$. Observe that $K_t = \{q \in K : dist(q, \Delta) \geq t\}$ and that $\partial N_t = \partial K_t$ while $K \setminus int(N_t) = K_t$.

Remark. K_1, and hence K_t, $t > 0$ is typically not a cone. See Appendix B and especially accompanying figure 2.

4.5.1. *The proof of polyhedral escape (theorem 4.1).* In what follows $\Delta = \partial K$. We will show that by translating K inward we obtain a family of t-escapers covering $N_t(\Delta)$.

Let $\mathbb{S}$ denote the unit sphere in $\mathbb{E}$. Choose any unit vector $v \in \mathbb{S} \cap int(K)$. Consider the one-parameter family of translations

$$p \mapsto \tau_s(p) = p + sv.$$

By convexity, τ_s maps K into K for $s > 0$. Write

$$c_*(v) = min_{i \in [m]} \ell_i(v)$$

I claim that for all $t > 0$, the family of rays $s \mapsto p + sv, s \geq 0$ forms t-escapers with escape rate $c_*(v)$. Indeed,

$$\ell_j(p + sv) = \ell_j(p) + s\ell_j(v) \geq dist(p, \Delta) + sc_*(v).$$

which establishes the escape rate inequality (8) for the ray $\gamma(s) = p + sv$ with escape rate $c = c_*(v)$. Fix any $t > 0$. the escape rate inequality shows that if $p \in N_t(\Delta)$ then the curve $p + sv$ has left $N_t(\Delta)$, by the time $s = t/c_*(v)$,so that the escape rate from Δ is positive and at least $c_*(v)$.

To verify the claim regarding the precise global escape rate observe that $c_*(v) = dist(v, \Delta)$. Now take the maximum of $c_*(v)$ over all $v \in \mathbb{S} \cap K$. Since both $dist(v, \Delta)$ and $\|v\|$ are homogeneous of degree 1, this maximum of $c_*(v)$ equals the maximum of $dist(v, \Delta)/\|v\|$ over $v \in K \setminus \{0\}$. This last maximum in turn is the reciprocal of the *minimum* of $\|q\|/dist(q, \Delta)$ over $q \in K \setminus \{0\}$. We can understand this minimum by setting $dist(q, \Delta) = 1$ and then minimizing $\|q\|$. But this minimum value is $dist(0, K_1)$ and is achieved as $\|q_*\|$ where q_* is the closest point to 0 on K_1. It follows that the optimal escape rate is equal to $1/dist(0, K_1)$ and that the corresponding translational ray escapers $p+sv$ are obtained by setting $v = q_*/\|q_*\|$.

To show that this value just computed for the escape rate is best possible consider the problem of escaping from the cone point $p = 0$ into K. The best $t = 1$-escaper for $p = 0$ will be the shortest path to ∂K_1 which is the segment $[0, q_*]$. Now, by homogeneity the intersection of the ray $sv, v = q_*/\|q_*\|$ with K_t, $t > 0$, yields the best t-escaper from 0, for all t.

QED

5. Appendix: On the distance formula.

Here we prove the distance formula (6) used in the paper [2].

Fix a point $q = (q_1, q_2, \ldots, q_N) \in \mathbb{E}$. The distance between q and any affine subspace S is the length of the unique line segment which hits S orthogonally. We apply this observation to $S = \Delta_{ab}$. A line segment from q to S can be written $\ell(t) = (1-t)q + ts$ for some $s \in S$.

For simplicity of notation, suppose that $a = 1, b = 2$. Write

$$q_{cm} = \frac{1}{m_1 + m_2}(m_1 q_1 + m_2 q_2)$$

for the center of mass of 1 and 2. The (unique) point of $\Delta_{12} = S$ which we will want with our line segment from q turns out to be

$$s = (q_{cm}, q_{cm}, q_3, q_4, \ldots, q_N).$$

The corresponding line segment is then $\ell(t) = (1-t)q + ts$, or, in terms of components

$$\ell_1(t) = (1-t)q_1 + tq_{cm}$$
$$\ell_2(t) = (1-2)q_2 + tq_{cm}$$

and

$$\ell_a(t) = q_a = const., \text{ for } a > 2.$$

The corresponding velocity $v = \dot{\ell} = (v_1, v_2, \ldots, v_N) \in \mathbb{E}$ is given by

$$v_1 = -q_1 + q_{cm}$$

$$v_2 = -q_2 + q_{cm}$$

and

$$v_a = 0, \text{ for } a > 2.$$

The total linear momentum of this trajectory $\ell(t)$ is zero. Indeed this total linear momentum, $\Sigma m_a v_a$ is $m_1 v_1 + m_2 v_2 = -m_1 q_1 - m_2 q_2 + m_{12} q_{cm} = 0$. Now let h be any vector in Δ_{12}. Then h is of the form $h = (a, a, h_3, h_4, \ldots, h_N)$ where $a, h_i \in \mathbb{R}^d$ are arbitrary. We compute that $< v, h >= m_1 a v_1 + m_2 a v_2 = a \cdot (m_1 \dot{v}_1 + m_2 v_2) = 0$. which shows that the line segment is orthogonal to Δ_{12}.

[2]Doubtless this formula is proved in other papers, probably in some of my own, but, not finding a derivation, I opted to give this one.

We have established that the path ℓ is the (unique) line segment joining q to Δ_{12} orthogonally. The length of ℓ is thus $dist(q, \Delta_{12})$. But this length is $\|v\|$ where $\|v\|^2 = m_1|v_1| + m_2|v_2|$. The computation is finished with some algebra.

The algebra can be streamlined by introducing the "probabilities" $p_1 = \frac{m_1}{m_1+m_2}$ and $p_2 = \frac{m_2}{m_1+m_2}$ which allow us to write

$$q_{cm} = p_1 q_1 + p_2 q_2, q_1 = p_1 q_1 + p_2 q_1, q_2 = p_1 q_2 + p_2 q_2.$$

from which we compute that

$$v_1 = -p_2(q_1 - q_2)$$

$$v_2 = p_1(q_1 - q_2)$$

and finally $\|v\|^2 = (m_1 p_2^2 + m_2 p_1^2) r_{12}^2$. A wee bit of algebra yields that $(m_1 p_2^2 + m_2 p_1^2) = \frac{m_1 m_2}{m_1+m_2}$ which is the claim.

6. A cone and its equidistant

I was temporarily seduced into the misbelief that K_t, $t > 0$, must be a cone since K_0 is a cone. This is false. See figure 2.

Correcting my misbelief corrected my intuition and helped me come up with the short correct proof given here. For a simple example where K_1 is not a cone, suppose that K_0 is the cone over the with sides $2/a$ and $2/b$ where $a, b > 0$ and $a \neq b$. Place the rectangle on the plane $z = 1$ with the cone point at the origin of the xyz plane. Then we can specify K_0 by the inequalities

$$z \geq a|x|, z \geq b|y|.$$

Its cross sections $z = z_0$ are rectangles whose sides are in the ratio $1/a : 1/b$ and which grow linearly with z_0.

To compute K_t find the four normalized linear functionals which define K_0. They are $\ell_\pm = \frac{1}{\sqrt{a^2+1}}(z \pm ax), f_\pm = \frac{1}{\sqrt{a^2+1}}(z \pm by)$. Thus K_0 is defined by $\ell_\pm \geq 0$ and $f_\pm \geq 0$, and K_t by $\ell_\pm \geq t, f_\pm \geq t$. Taking $t = 1$ and doing a bit of algebra yields that K_1 is defined by

$$z - \sqrt{a^2+1} \geq a|x|, z - \sqrt{b^2+1} \geq b|y|.$$

In particular, since the right hand side of these equations is greater than or equal to zero we have that $z \geq max\{\sqrt{a^2+1}, \sqrt{b^2+1}\}$.

For concreteness, set $a = 1$ and suppose $b < 1$ so that $z \geq \sqrt{2}$ on K_1. The cross-section K_1 with the plane $z = \sqrt{2}$ is the bounded interval $\sqrt{2} - \sqrt{b^2+1} \geq b|y|$ or $\frac{1}{b}(\sqrt{2} - \sqrt{b^2+1}) \geq |y|$. Consequently K_1 cannot be a cone. To get a 3-dimensional picture of K_1 consider the cross-sections $z = z_0$ for $z_0 > \sqrt{2}$ of K_1. These are rectangles $z_0 - \sqrt{2} \geq |x|$ and $\frac{1}{b}(z_0 - \sqrt{b^2+1}) \geq |y|$. Working out the side lengths we see that the rectangles have aspect ratio $(1 - \frac{\sqrt{2}}{z_0}) : (\frac{1}{b} - \frac{\sqrt{b^2+1}}{z_0 b})$ and asymptotes to $1 : \frac{1}{b}$, the aspect ratio of the rectangle on which K_0 is based.

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