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Black Hole Entropy from BMS Symmetry at the Horizon

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Abstract

Near the horizon, the obvious symmetries of a black hole spacetime—the horizon-preserving diffeomorphisms—are enhanced to a larger symmetry group with a BMS_3 algebra. Using dimensional reduction and covariant phase space techniques, I investigate this augmented symmetry, and show that it is strong enough to determine the black hole entropy.

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1 Introduction

A striking feature of black hole thermodynamics is the universality of the Bekenstein-Hawking entropy. Black holes, black strings, black rings, black branes, and black Saturns, in any dimension, with any charges and spins, with horizons arbitrary distorted by external fields, all have entropies given by the same simple expression,

$$S_{BH} = \frac{A_{hor}}{4G\hbar} \quad (1.1)$$

where A_{hor} is the horizon area. Changing the action can change this formula, but only by another universal term.

The mystery deepens when one notes that many different models of the quantum black hole, from string theory to loop quantum gravity to induced gravity, can all yield the same entropy, even though they appear to count very different microstates [1]. Even in the elegant analysis of BPS black holes in string theory [2], a separate computation is needed for each choice of dimension and each set of charges. It seems clear that some underlying structure is yet to be found.

A natural guess for this deeper structure is that the degrees of freedom responsible for the entropy live on the horizon [3]. But this is not enough: while it may explain the proportionality of entropy to area, there is no obvious reason for the coefficient of $1/4$ to be universal. A more elaborate idea, first suggested (I believe) in [4], is that the entropy is governed by a horizon symmetry. Two-dimensional conformal symmetry, in particular, has similar universal properties—the Cardy formula fixes the asymptotic behavior of the density of states in terms of a few parameters, independent of the details of the theory [5]—and the possibility of a connection is appealing.

This possibility was first confirmed for the (2+1)-dimensional BTZ black hole in 1998 [6,7], and attempts to extend it to higher dimensions soon followed [8,9]. But while these efforts have had significant successes—see [10] for a review—they have been plagued by several serious limitations:

- The symmetries are almost always taken to be either at infinity or at a timelike “stretched horizon” (although with rare exceptions [11]). While physics at infinity is very powerful, especially for asymptotically anti-de Sitter spaces, the symmetries by themselves cannot distinguish a black hole from, for instance, a star. The stretched horizon more directly captures the properties of the black hole, but while the entropy has a well-defined limit at the horizon, other parameters typically blow up [12,13] (again with occasional exceptions [14]). Moreover, the definition of the stretched horizon is not unique, and different choices can lead to different entropies [15,16].
- The standard canonical approach fails in what should be the simplest case, two-dimensional dilaton gravity. Symmetry generators are defined at boundaries of spatial slices, and the zero-dimensional boundary of a one-dimensional slice is simply too small. There are ad hoc fixes—lifting the theory to three dimensions [17] or artificially introducing an integral over time [18]—but none is convincing.
- In higher dimensions, the relevant symmetries are those of the “ r - t plane” picked out by the horizon. But to obtain a well-behaved symmetry algebra, one must introduce an extra ad hoc angular dependence of the parameters that has no clear physical justification.

Here* I show how to fix these problems. The basic mistake, I argue, has been to try to force the horizon symmetry into the form of a two-dimensional conformal symmetry. This was understandable: until recently, such a symmetry was the only one known to be powerful enough to control the density of states. But it has been recently discovered that a BMS₃ (or Galilean conformal) symmetry has similar universal properties, including a generalized Cardy formula for the asymptotic density of states [19].

By using covariant phase space methods, introduced in this context in [20] and elaborated in [21], I show that the symmetry generators can be expressed as integrals along the horizon [22], with no need for “stretching.” I then demonstrate that a BMS₃ symmetry appears in a completely natural way on the horizon, circumventing the problems of previous efforts, and that it gives the correct counting of states.

2 Dilaton gravity with null dyads

The horizon Δ of a stationary black hole in any dimension has a preferred null direction, determined by the geodesics that generate the horizon. A neighborhood of Δ also has a preferred spatial coordinate, the proper distance from the horizon. Together, these define a two-dimensional r - t plane, in which most of the interesting physics is expected to take place, since transverse derivatives are red-shifted away near the horizon. Hawking radiation, for instance, can be obtained by dimensional reduction to this plane [24].

Upon dimensional reduction and a field redefinition, the Einstein-Hilbert action becomes [25]

$$I = \frac{1}{16\pi G} \int_M (\varphi R + V[\varphi]) \epsilon \quad (2.1)$$

where ϵ is the volume two-form. The scalar field φ , the dilaton, is the remnant of the transverse geometry, essentially the transverse area. The resulting equations of motion are

$$E_{ab} = \nabla_a \nabla_b \varphi - g_{ab} \square \varphi + \frac{1}{2} g_{ab} V = 0 \quad (2.2a)$$

$$R + \frac{dV}{d\varphi} = 0 \quad (2.2b)$$

where the second equation follows from the divergence of the first.

Let us choose a null dyad (ℓ_a, n_a) , with $\ell^2 = n^2 = 0$, normalized so that $\ell \cdot n = -1$. For notational convenience, define $D = \ell^a \nabla_a$, $\bar{D} = n^a \nabla_a$. The metric and Levi-Civita tensor are then

$$g_{ab} = -(\ell_a n_b + n_a \ell_b) \quad \epsilon_{ab} = (\ell_a n_b - n_a \ell_b) \quad (2.3)$$

The dyad is determined only up to a local Lorentz transformation, $\ell^a \rightarrow e^\lambda \ell^a$, $n^a \rightarrow e^{-\lambda} n^a$. We can partially fix this freedom by choosing n_a to have vanishing acceleration, $n^b \nabla_b n^a = 0$; the remaining transformations are those for which $n^a \nabla_a \lambda = 0$. With this choice,

$$\begin{aligned} \nabla_a \ell_b &= -\kappa n_a \ell_b & \nabla_a \ell^a &= \kappa \\ \nabla_a n_b &= \kappa n_a n_b & \nabla_a n^a &= 0 \end{aligned} \quad (2.4)$$

* An expanded version of this work will appear in [23].

where κ will be the surface gravity at a horizon. Under variation of the dyad, (2.4) is preserved if

$$\begin{aligned}\bar{D}(\ell^c \delta n_c) &= (D + \kappa)(n^c \delta n_c) \\ \delta \kappa &= -D(n^c \delta \ell_c) + \kappa \ell^c \delta n_c + \bar{D}(\ell^c \delta \ell_c)\end{aligned}\tag{2.5}$$

By considering the commutator $[\nabla_a, \nabla_b] \ell^b$, one may easily show that

$$R = 2\bar{D}\kappa\tag{2.6}$$

Below, I will also frequently use two identities:

$$[D, \bar{D}] = -\kappa \bar{D}\tag{2.7}$$

$$df = -Df n_a - \bar{D}f \ell_a \quad \text{for any function } f\tag{2.8}$$

where in the latter I am treating n_a and ℓ_a as one-forms. Eqn. (2.8) will be useful for integration by parts along the horizon.

3 The covariant canonical formalism and symplectic structure

The idea underlying the covariant canonical formalism is that for a theory with a unique time evolution, the phase space, viewed as the space of initial data, can be identified with the space of classical solutions [26, 27]. This observation, which can be traced back to Lagrange (see [26]), means that we can formulate all the usual ingredients of a Hamiltonian approach without ever having to break general covariance by choosing a time slicing.

Consider a theory in an n -dimensional spacetime with fields Φ^A (for us, φ and g) and a Lagrangian density $L[\Phi]$, which we view as an n -form. Under a general variation of the fields, $L[\Phi]$ changes as

$$\delta L = E_A \delta \Phi^A + d\Theta[\Phi, \delta \Phi]\tag{3.1}$$

where the equations of motion are $E_A = 0$ and the last ‘‘boundary’’ term comes from integration by parts. The symplectic current ω is defined as

$$\omega[\Phi; \delta_1 \Phi, \delta_2 \Phi] = \delta_1 \Theta[\Phi, \delta_2 \Phi] - \delta_2 \Theta[\Phi, \delta_1 \Phi]\tag{3.2}$$

and the symplectic form is

$$\Omega[\Phi; \delta_1 \Phi, \delta_2 \Phi] = \int_{\Sigma} \omega[\Phi; \delta_1 \Phi, \delta_2 \Phi] = \int_{\Sigma} \omega_{AB} \delta_1 \Phi^A \wedge \delta_2 \Phi^B\tag{3.3}$$

where Σ is a Cauchy surface. In keeping with the covariant phase space philosophy, $\Omega[\Phi; \delta_1 \Phi, \delta_2 \Phi]$ depends on a classical solution Φ , which fixes a point in phase space, and is a two-form on the phase space. The variations $\delta \Phi$ are thus tangent vectors to the space of classical solutions, that is, solutions of the linearized equations of motion. The integral (3.3) may depend on the choice of Cauchy surface, but only weakly: the symplectic current is a closed form, so integrals over two Cauchy surfaces Σ_1 and Σ_2 differ only by boundary terms that arise if $\partial \Sigma_1 \neq \partial \Sigma_2$.

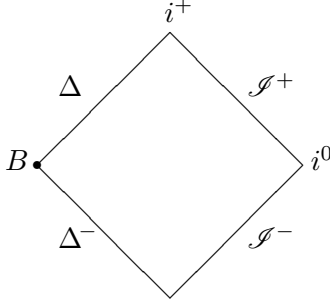


Figure 1: Penrose diagram for the exterior of a black hole

As in ordinary mechanics, the symplectic form determines Poisson brackets and Hamiltonians. In particular, given a family of transformations $\delta_\tau \Phi^A$ labeled by a parameter τ , the Hamiltonian $H[\tau]$ is determined by the condition

$$\delta H[\tau] = \Omega[\delta\Phi, \delta_\tau \Phi] \quad (3.4)$$

for an arbitrary variation $\delta\Phi$. Indeed, this is just Hamilton’s equation of motion,

$$\delta_\tau \Phi^A = (\omega^{-1})^{AB} \frac{\delta H[\tau]}{\delta \Phi^B} \quad (3.5)$$

The Poisson bracket of two Hamiltonians is then

$$\{H[\tau_1], H[\tau_2]\} = \Omega[\delta_{\tau_1} \Phi, \delta_{\tau_2} \Phi] \quad (3.6)$$

Specializing to dilaton gravity and using (2.6), it is straightforward to show that

$$\Omega[(\varphi, g); \delta_1(\varphi, g), \delta_2(\varphi, g)] = \frac{1}{8\pi G} \int_\Sigma [\delta_1 \varphi \delta_2(\kappa n_a) + \delta_1(\bar{D}\varphi) \delta_2 \ell_a] - (1 \leftrightarrow 2) \quad (3.7)$$

where I am again treating ℓ_a and n_a as one-forms on the (one-dimensional) Cauchy surface Σ .

4 Horizons and boundary conditions

For dilaton models obtained by dimensional reduction, φ is essentially the transverse area, and the natural definition of a local “nonexpanding horizon” Δ —a null surface with vanishing expansion [28]—is that $D\varphi = 0$ on Δ . This correctly determines the horizon from the purely two-dimensional viewpoint as well: on shell, Δ is a Killing horizon [25] and the boundary of a trapped region [29]. Exact black hole solutions in two dimensions have such horizons, with essentially the same Penrose diagrams as those in higher dimensions [30].

To study horizon symmetries in the covariant phase space formalism, we shall incorporate Δ as part of our Cauchy surface. Let us focus on the exterior region of an asymptotically flat black hole, with the Penrose diagram of figure 1. Take Σ to be the union of the future horizon Δ and future null infinity \mathcal{I}^+ , with ends at the bifurcation point B and spacelike infinity. The details of \mathcal{I}^+ are unimportant; the analysis below would be unchanged for asymptotically de Sitter or anti-de Sitter spaces.

Define \triangleq to mean “equal on Δ ,” where the horizon Δ is now determined by the requirement that $D\varphi \triangleq 0$. We shall impose three “boundary conditions” at this horizon:

1. $DR \triangleq 0$. This is a requirement of stationary geometry on Δ . In higher dimensions, this condition follows automatically from the Raychaudhuri equation; here it must be imposed by hand, though it holds identically on shell.
2. The conformal class of the metric is fixed on Δ , so $\ell^a \delta \ell_a \triangleq 0$ and $n^a \delta n_a \triangleq 0$. This is in keeping with physical picture of conformal fluctuations of the metric as the relevant degrees of freedom. I believe this condition can be relaxed, but at the cost of some complication.
3. The integration measure n_a is fixed on Δ . In view of condition 2, this is the additional requirement that $\ell^a \delta n_a \triangleq 0$. This is really a gauge-fixing condition, which can always be achieved by a suitable local Lorentz transformation. Again, it may be possible to relax this requirement [23].

These conditions simplify the symplectic form (3.7) considerably: for the portion lying on the horizon,

$$\Omega_{\Delta}[(\varphi, g); \delta_1(\varphi, g), \delta_2(\varphi, g)] = \frac{1}{8\pi G} \int_{\Delta} [\delta_1 \varphi \delta_2 \kappa - \delta_1 \varphi \delta_2 \kappa] n_a \quad (4.1)$$

One subtlety remains, though. A variation of φ will typically “move the horizon,” changing the locus of points $D\varphi = 0$. This will not matter for the symplectic form, since Ω_{Δ} is independent of the integration contour. More precisely, if δ_{ζ} is a transverse diffeomorphism generated by a vector field $\zeta^a = \bar{\zeta} n^a$,

$$\Omega_{\Delta}[(\varphi, g); \delta(\varphi, g), \delta_{\zeta}(\varphi, g)] = -\frac{1}{8\pi G} \int_{\Delta} \bar{\zeta} (D + \kappa) \bar{D}\varphi \delta n_a + \frac{1}{8\pi G} (\bar{\zeta} \bar{D}\varphi) \Big|_{\partial\Delta} \quad (4.2)$$

The bulk term vanishes by virtue of the boundary condition $\delta n_a = 0$, and the boundary term will vanish provided that $\bar{\eta} = 0$ at $\partial\Delta$.

For the variation of an object such as a Hamiltonian defined as an integral over Δ , however, we shall have to take this change into account. The diffeomorphism needed to “move the horizon back” is determined by the condition that

$$\delta(D\varphi) + \zeta^a \nabla_a (D\varphi) \triangleq 0 \Rightarrow \zeta^a = \bar{\zeta} n^a = -\frac{D\delta\varphi}{\bar{D}D\varphi} n^a \quad (4.3)$$

and hence

$$\delta \int_{\Delta} \mathcal{H} n_a = \int_{\Delta} (\delta \mathcal{H} + \zeta^a \nabla_a \mathcal{H}) n_a \quad (4.4)$$

5 Symmetries and approximate symmetries

The action (2.1) is, of course, invariant under diffeomorphisms, including horizon “supertranslations” [31] generated by vector fields $\xi^a = \xi \ell^a$. By condition 3 of the preceding section, we must supplement such diffeomorphisms by local Lorentz transformations $\delta\lambda = D\xi$ to ensure that $\ell^a \delta_{\xi} n_a = 0$. By (2.5), this requires that $\bar{D}\xi \triangleq 0$. We thus have an invariance

$$\begin{aligned} \delta_{\xi} \ell^a &= 0, & \delta_{\xi} n^a &= -(D + \kappa) \xi n^a \\ \delta_{\xi} g_{ab} &= -(D + \kappa) \xi g_{ab} \\ \delta_{\xi} \varphi &= \xi D\varphi \quad \text{with } \bar{D}\xi \triangleq 0 \end{aligned} \quad (5.1)$$

As noted long ago, though [22], the action also has an *approximate* invariance under a certain shift of the dilaton near a black hole horizon, with an approximation that can be made arbitrarily good by restricting the transformation to a small enough neighborhood of Δ . Consider a variation

$$\hat{\delta}_\eta \varphi = \nabla_a(\eta \ell^a) = (D + \kappa)\eta \quad \text{with } \bar{D}\eta \triangleq 0 \quad (5.2)$$

(where the hat in $\hat{\delta}$ distinguishes it from a supertranslation). The action transforms as

$$\hat{\delta}_\eta I = \frac{1}{16\pi G} \int_M \left(R + \frac{dV}{d\varphi} \right) \hat{\delta}_\eta \varphi \epsilon = -\frac{1}{16\pi G} \int_M \eta \left[DR + \frac{d^2 V}{d\varphi^2} D\varphi \right] \epsilon \quad (5.3)$$

But $D\varphi$ and DR both vanish at the horizon, so the variation (5.3) can be made as small as one wishes by choosing η to fall off fast enough away from Δ .

This is not quite enough: while the transformation (5.2) does not directly act on the curvature, the change of φ moves the horizon, and DR may no longer vanish at the new location. The displacement of the horizon is characterized by the diffeomorphism (4.3), and can be compensated with a “small” (order $D\varphi$) Weyl transformation of the metric to restore the condition $DR \triangleq 0$:

$$\hat{\delta}_\eta g_{ab} = \hat{\delta}\omega_\eta g_{ab} \quad \text{with } \hat{\delta}\omega_\eta = X \frac{D\varphi}{DD\varphi}, \quad \bar{\zeta} \bar{D}DR + 2D(D + \kappa)X \triangleq 0 \quad (5.4)$$

On shell, a short calculation gives an explicit expression for X :

$$\bar{\zeta} \bar{D}DR + 2D(D + \kappa)X \triangleq \frac{d^2 V}{d\varphi^2} D(D + \kappa)\eta + 2D(D + \kappa)X \triangleq 0 \Rightarrow X \triangleq -\frac{1}{2} \frac{d^2 V}{d\varphi^2} \eta \quad (5.5)$$

Like (5.2), the Weyl transformation (5.4) changes the action only by terms proportional to $D\varphi$, which can be made arbitrarily small by choosing η to fall off fast enough away from Δ .

We must also check the variation of the equations of motion (2.2a)–(2.2b). These are, of course, preserved by diffeomorphisms, so we need only consider the transformations (5.2) and (5.4). Since we are assuming that η falls off rapidly away from the horizon, it is enough to check the variations at Δ . Note that after varying an equation of motion, we can put the system on shell—a variation of an equation of motion by symmetry need only vanish up to equations of motion.

A straightforward computation then shows that on shell,

$$g^{ab} \hat{\delta}_\eta E_{ab} \triangleq 2(D + \kappa) \bar{D} \hat{\delta}_\eta \varphi + \frac{dV}{d\varphi} \hat{\delta}_\eta \varphi \triangleq 0 \quad (5.6a)$$

$$n^a n^b \hat{\delta}_\eta E_{ab} \triangleq \bar{D}^2 \hat{\delta}_\eta \varphi - \bar{D}\varphi \bar{D} \hat{\delta}_\eta \omega \triangleq 0 \quad (5.6b)$$

$$\hat{\delta}_\eta \left(R + \frac{dV}{d\varphi} \right) \triangleq \hat{\delta}_\eta R + \frac{d^2 V}{d\varphi^2} \hat{\delta}_\eta \varphi \triangleq 0 \quad (5.6c)$$

This leaves the variation $\ell^a \ell^b \hat{\delta}_\eta E_{ab}$, which is not zero, but instead matches the anomalous variation of the stress-energy tensor in a conformal field theory. Indeed, if we set $E_{ab} = 8\pi G T_{ab}$, we find

$$\ell^a \ell^b \hat{\delta}_\eta T_{ab} \triangleq \frac{1}{8\pi G} (D - \kappa) D(D + \kappa) \eta \quad (5.7)$$

which is just the anomaly for a conformal field theory with a central charge proportional to $1/G$. One might worry that the anomaly could spoil the covariant phase space construction of section 3, since the closure of the symplectic current (3.2) relies on the classical field equations. Fortunately, this is not a problem: the only dangerous term is proportional to $\eta n^a \delta n_a$, which is zero at the horizon because of our boundary conditions and falls off like η away from the horizon.

6 Canonical generators and their algebra

At the horizon, the two symmetries of the preceding section obey an algebra

$$\begin{aligned} [\delta_{\xi_1}, \delta_{\xi_2}]f &\triangleq \delta_{\xi_{12}}f & \text{with } \xi_{12} &= -(\xi_1 D\xi_2 - \xi_2 D\xi_1) \\ [\hat{\delta}_{\eta_1}, \hat{\delta}_{\eta_2}]f &\triangleq 0 \\ [\delta_{\xi_1}, \hat{\delta}_{\eta_2}]f &\triangleq \hat{\delta}_{\eta_{12}}f & \text{with } \eta_{12} &= -(\xi_1 D\eta_2 - \eta_2 D\xi_1) \end{aligned} \quad (6.1)$$

This may be recognized as a BMS_3 algebra, or equivalently a Galilean conformal algebra [32]. We must now ask whether these transformations can be realized canonically as in (3.4), that is, whether there exist generators that satisfy

$$\delta L[\xi] = \frac{1}{8\pi G} \int_{\Delta} [\delta\varphi \delta_{\xi}\kappa - \delta_{\xi}\varphi \delta\kappa] n_a = \frac{1}{8\pi G} \int_{\Delta} [\delta\varphi D(D + \kappa)\xi - \xi D\varphi \delta\kappa] n_a \quad (6.2a)$$

$$\delta M[\eta] = \frac{1}{8\pi G} \int_{\Delta} [\delta\varphi \hat{\delta}_{\eta}\kappa - \hat{\delta}_{\eta}\varphi \delta\kappa] n_a = \frac{1}{8\pi G} \int_{\Delta} \left[-\delta\kappa(D + \kappa)\eta + \frac{1}{2} \frac{D\delta\varphi}{DD\varphi} \eta \frac{d^2V}{d\varphi^2} D\varphi \right] \quad (6.2b)$$

where variations of the generators must include the horizon displacement described by (4.4), and the covariant phase space formalism allows us to impose the equations of motion after variation. Such generators exist:

$$L[\xi] = \frac{1}{8\pi G} \int_{\Delta} [\xi D^2\varphi - \kappa\xi D\varphi] n_a \quad (6.3a)$$

$$M[\eta] = \frac{1}{8\pi G} \int_{\Delta} \eta \left(D\kappa - \frac{1}{2}\kappa^2 \right) n_a \quad (6.3b)$$

Using (3.6), we find Poisson brackets[†]

$$\{L[\xi_1], L[\xi_2]\} = L[\xi_{12}] \quad (6.4a)$$

$$\{M[\eta_1], M[\eta_2]\} \triangleq 0 \quad (6.4b)$$

$$\{L[\xi_1], M[\eta_2]\} \triangleq M[\eta_{12}] + \frac{1}{16\pi G} \int_{\Delta} (D\xi_1 D^2\eta_2 - D\eta_2 D^2\xi_1) n_a \quad (6.4c)$$

with ξ_{12} and η_{12} as in (6.1). The canonical generators thus give a representation of the symmetry algebra, but with an added off-diagonal central term.

7 Modes, zero-modes, and entropy

It is well known that for a unitary theory with a two-dimensional conformal symmetry, the asymptotic density of states—the entropy—is determined, via the Cardy formula, by the central charge [5]. The same is true for a theory with a BMS_3 algebra [19]: with a mode decomposition

$$\begin{aligned} i \{L_m, L_n\} &= (m - n)L_{m+n} \\ i \{M_m, M_n\} &= 0 \\ i \{L_m, M_n\} &= M_{m+n} + c_{LM} m(m^2 - 1)\delta_{m+n,0} \end{aligned} \quad (7.1)$$

[†]The first of these holds even if $D\varphi \neq 0$. The second and third do not—the η transformations are symmetries only on a horizon—but the deviations are of order $(D\varphi)^2$.

the asymptotic behavior of the entropy is

$$S \sim 2\pi h_L \sqrt{\frac{c_{LM}}{2h_M}} \quad (7.2)$$

where h_L and h_M are the eigenvalues of L_0 and M_0 .

To use this result, we first need a mode decomposition. For a black hole with constant surface gravity, the relevant modes take the form $e^{in\kappa v}$, where v is the advanced time along the horizon, normalized so that $l^a \nabla_a v = 1$. We can generalize this by defining a phase ψ such that

$$D\psi \triangleq \kappa, \quad \bar{D}\psi \triangleq 0 \quad \Rightarrow \quad d\psi \triangleq -\kappa n_a \quad (7.3)$$

The modes are then

$$\zeta_n \triangleq \frac{1}{\kappa} e^{in\psi} \quad (\text{where } \zeta \text{ is either } \xi \text{ or } \eta) \quad (7.4)$$

with a prefactor chosen so $\{\zeta_m, \zeta_n\} = \zeta_m D\zeta_n - \zeta_n D\zeta_m = -i(m-n)\zeta_{m+n}$. With this moding,

$$\frac{1}{16\pi G} \int_{\Delta} (D\xi_m D^2\eta_m - D\eta_n D^2\xi_m) n_a = \frac{i}{8\pi G} \int_{\Delta} mn^2 e^{i(m+n)\psi} d\psi \quad (7.5)$$

If we take the integral to be over a single period—essentially mapping the problem to a circle, as is standard in conformal field theory—we obtain a central charge in (6.4c) of

$$c_{LM} = \frac{1}{4G} \quad (7.6)$$

We also need the zero-modes of L and M . For M , this is straightforward: from (6.3b),

$$h_M = M[\eta_0] = -\frac{1}{16\pi G} \int_{\Delta} \kappa n_a = \frac{1}{16\pi G} \int d\psi = \frac{1}{8G} \quad (7.7)$$

For L , the “bulk” contribution to $L[\xi_0]$ vanishes. But L , unlike M , has a boundary contribution. Indeed, the variation leading to (6.2a) involves integration by parts, with a boundary term

$$\delta L[\xi] = \dots + \frac{1}{8\pi G} (\xi D\delta\varphi - (D + \kappa)\xi \delta\varphi) \Big|_{\partial\Delta} \quad (7.8)$$

From (4.2)–(4.3), we must set $D\delta\varphi$ to zero at $\partial\Delta$, but we should certainly not hold φ itself fixed, since that would fix φ along the entire horizon, eliminating the η symmetry. Instead, we should fix the conjugate variable κ at $\partial\Delta$. This gives a boundary contribution at the bifurcation point of

$$h_L = \frac{1}{8\pi G} \varphi (D + \kappa)\xi_0 \Big|_{\partial\Delta} = \frac{\varphi_+}{8\pi G} \quad (7.9)$$

where φ_+ is the value of φ at the bifurcation point B of figure 1. Inserting (7.6), (7.7), and (7.9) into (7.2), we finally obtain

$$S = \frac{\varphi_+}{4G} \quad (7.10)$$

which is precisely the correct Bekenstein-Hawking entropy.

8 Conclusions

We have seen that black hole entropy is indeed governed by horizon symmetries. In contrast to previous attempts, this derivation requires no stretched horizon and no extra angular dependence or other ad hoc ingredients. The main assumptions are merely that dimensional reduction is possible and that the horizon obeys the “boundary conditions” of section 4.

How should we think about the resulting BMS symmetry? It is not a gauge symmetry: our counting arguments imply that states are not invariant, but transform under high-dimensional representations. Nor is it quite a standard asymptotic symmetry: while we can view the horizon as a sort of boundary, it is a boundary that exists only for a restricted class of field configurations. Physically, we are asking a question of conditional probability—*if* a black hole is present, what are its properties?—and the symmetries reflect this condition.

There are obvious directions for generalization. Dimensional reduction focuses our attention on the relevant parts of the geometry, but it would be good to explicitly lift the argument to higher dimensions. We should clarify the relationship between the symmetries of this paper and other appearances of BMS symmetry at the horizon [31, 33–36], as well as the related horizon symmetry used by Wall to prove the generalized second law [37]. It should be feasible to significantly relax the boundary conditions of section 4. It may also be possible to make the concept of “approximate symmetry” in section 5 more precise. In this regard, recall that the shift parameter η appears in the variation of the action with no transverse derivatives, and can also be rescaled by a constant without changing the algebra, so both its value and its support can be made arbitrarily small.

Finally, if this symmetry is really responsible for the universal properties of black hole entropy, one might expect to find it hidden in other derivations of entropy. Preliminary steps in this direction have been taken for loop quantum gravity [38], for induced gravity [39], and perhaps for near-extremal black holes in string theory [40], but none of these attempts has exploited the full BMS symmetry. Ideally, we could hope to do more: perhaps this symmetry can be used to couple the black hole to matter and obtain Hawking radiation, as Emparan and Sachs did for the (2+1)-dimensional black hole [41].

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