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Geometry of affine Deligne-Lusztig varieties

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Geometry of affine Deligne-Lusztig varieties

by

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Abstract

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The purpose of this dissertation is to discuss previously known results and prove new criteria / results on affine Deligne-Lusztig varieties. Our explicit criterion on the nonemptiness pattern and dimension formula generalizes a previously known result by removing a large restriction. This new criterion has not been suggested even as a conjecture beforehand. Next, we discuss the connected components of affine Deligne-Lusztig varieties. In this question, we follow a novel approach based on the moduli space of mixed-characteristic shtukas which has not been adapted to conquer the restricted versions of the connected components problem. Finally, we study Hodge-Newton indecomposability and show an identity which gives a multiplicity-one result for special types (finite Coxeter type) of affine Deligne-Lusztig varieties.

사랑과 자유를 알려주신 어머니와
지혜와 여유를 알려주신 아버지께

Contents

List of Figures

Acknowledgments

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Chapter 1

Overview

Affine Deligne-Lusztig varieties show up naturally in the study of Shimura varieties and the computations of the L-functions, as the conjecture of Langland and Rapoport suggests. Recently, remarkable progress in p-adic geometry shows that affine Deligne-Lusztig varieties are closely related to the moduli spaces of p-adic shtukas. In this dissertation, we study the geometry of affine Deligne-Lusztig varieties, connections towards Shimura varieties and moduli spaces of p-adic shtukas, and certain affine Deligne-Lusztig varities with simple geometric structures. We note that affine Deligne-Lusztig varieties could mean mildly different objects depending on the context, so each chapter has own introduction and definitions to avoid any confusions.

In Chapter 2, we will discuss the recent progress on the nonemptiness problem of single affine Deligne-Lusztig varieties at Iwahori level in the basic case. Under a genericity condition (the *shrunken Weyl chambers* condition), an explicit criterion has been already known. However, no explicit criterion has been available without the condition even conjecturally. We conjecture a new criterion in full generality, and prove it except for a finite number of cases. As an application, a new conjectural dimension formula will be discussed. The organization of Chapter 2 will be as follows:

§2.2 is a preliminary section. We recall the setup of [GHN15] and note down some computations related to critical strips. Following *loc.cit.*, we summarize the proof of GHN B and collect the lemmas subject to be generalized.

In §2.3, we introduce the set W_x. Then, we discuss some properties of closed subsets of a root system ([ĐCH94]). Then we prove Theorem 2.1.2 and Theorem 2.1.2 (2).

In §2.4, we prove Theorem 2.1.2 (1) by handling some possibly exceptional cases. We study in detail using the exact positive coefficients from [OV90].

In §2.5, we wrap up some tedious computations postponed in §3 and §4 proving Theorem 2.1.2 (3). In the application part, we recall some facts from [MV20] and [He21] and then prove Theorem 2.1.4.

In Chapter 3, we will summarize the joint work with Ian Gleason and Yujie Xu on the set of connected components of affine Deligne–Lusztig varieties at arbitrary parahoric levels, which finishes the open problem in the mixed characteristic case. We do so by relating them to the connected components of infinite level moduli spaces of p-adic shtukas, where we use v-sheaf-theoretic techniques such as the specialization map of *kimberlites*. As applications, new results (CM lifting) on the integral models of Shimura varieties at arbitrary parahoric levels will be mentioned. The organization of Chapter 3 will be as follows:

§3.2 is a preliminary section. We start by collecting general notation and standard definitions that we omitted in this introduction. We recall the group-theoretic setup of [GHN19] necessary to discuss the Hodge-Newton decomposition for general reductive groups, and its relation to the connected components of affine Deligne–Lusztig varieties.

In §3.3 we give a brief intuitive account of the theory of kimberlites. We also review the geometry of affine Deligne–Lusztig varieties and their relation to moduli spaces of p-adic shtukas. Moreover, we discuss ad-isomorphisms, z-extensions and compatibility with products (which will be used in §3.6 to reduce the proofs of Theorem 3.1.1 and Theorem 3.1.9 to the key cases).

In §3.4 we discuss the Hodge-Newton decomposition and use it to prove the implication (2) $\implies$ (3) in Theorem 3.1.9.

In §3.5, we discuss Mumford–Tate groups. We review [Che14] and discuss the modifications needed to prove Theorem 3.1.11. We deduce the implication (3) $\implies$ (4) in Theorem 3.1.9.

In §3.6, we give a new proof of [He18, Theorem 7.1] (see Theorem 3.6.8), and complete proofs of our main results such as Theorem 3.1.1 and Theorem 3.1.9.

The final chapter, Chapter 4, will discuss the *multiplicity one* identity appearing in the study of affine Deligne-Lusztig varities with finite Coxeter parts. This certain type is interesting as it turns out that this type has a simple geometric structure such as $\mathbb{A}^n$ or $\mathbf{G}_m^n$. In order to prove that they are "simple", we need to show that there is only one *reduction path*. This can be reduced to the proof of an identity of two $\mathbf{q}$-polynomials.

In §4.2 we discuss a specific and explicit identity of two $\mathbf{q}$-polynomials and introduce our new *probabilistic* approach counting integral points on the xy-plane.

In §4.3 we consider the general case and introduce new objects such as *envelope*s. We then state some main propositions which lead us to the proof of Theorem 4.3.2 using the *probabilistic* approach.

In §4.4 we provide the proofs of the main propositions.

Chapter 2

Nonemptiness of affine Deligne-Lusztig varieties

2.1 Introduction

In the study of the special fibers of Shimura varieties, affine Deligne-Lusztig varieties show up naturally. In his seminal expository article [Rap05a], Rapoport introduced *affine Deligne-Lusztig variety* over a mixed characteristic local field as an important piece in the description of $\overline{\mathbb{F}}_p$-points of the special fiber of a certain Shimura variety with hyperspecial level structure or Iwahori level structure. This was motivated by the Langlands-Rapoport conjecture in that the p-part of the conjecture is the affine Deligne-Lusztig variety. Since then, affine Deligne-Lusztig varieties have been exploited in the study of Shimura varieties, Rapoport-Zink spaces, local Shimura varieties, and moduli spaces of local shtukas.

Many questions arise naturally including the geometric (or scheme) structure, the nonemptiness, the dimension formula, and the set of connected components, etc. These basic questions are not only interesting on their own but useful in the study of the aforementioned objects. For example, the set of connected components computed in [CKV15] is used to prove the Langlands-Rapoport conjecture in [Kis17a]. It (resp. the dimension formula) can also be used to describe the set of connected components (resp. the dimension) of Rapoport-Zink spaces ([She20, 3], [Zhu17, 3.2]). In this chapter, we focus on the nonemptiness question and the dimension formula.

2.1.1 Mazur's Inequality

Let G be a connected reductive group over $\mathbb{Z}_p$ with a maximal torus T and let $\breve{\mathbb{Q}}_p$ be the fraction field of the ring of Witt vectors $W(\overline{\mathbb{F}}_p) =: \breve{\mathbb{Z}}_p$. The Frobenius morphism on $\overline{\mathbb{F}}_p$ lifts uniquely to $\breve{\mathbb{Z}}_p$ by the universal property of the ring of Witt vectors and then extend to a bijective map (denoted by σ) on $\breve{\mathbb{Q}}_p$. Now, fix $b \in G(\breve{\mathbb{Q}}_p)$ and a dominant cocharacter

$\mu \in X_*(T)^+$. The affine Deligne-Lusztig variety is defined as

$$X_\mu(b) := \{gG(\breve{\mathbb{Z}}_p) \in G(\breve{\mathbb{Q}}_p)/G(\breve{\mathbb{Z}}_p) : g^{-1}b\sigma(g) \in G(\breve{\mathbb{Z}}_p)p^\mu G(\breve{\mathbb{Z}}_p)\},$$

where p^μ is the image of p under the cocharacter μ.

The very first result on the nonemptiness is due to Rapoport-Richartz ([RR96, Theorem 4.2]). They showed that, when G is unramified, if $X_\mu(b)$ is nonempty then *Mazur's inequality*, that is, $[b] \in B(G,\mu)$ (see definition 2.2.1) holds. Thanks to [Kot03], [Gas10], and [He14], it is now a theorem that $X_\mu(b) \neq \emptyset$ if and only if $[b] \in B(G,\mu)$ for a general reductive group G (defined over $\mathbb{Q}_p$) and a special maximal parahoric subgroup K in place of $G(\breve{\mathbb{Z}}_p)$.

Similarly, using Bruhat-Tits theory, one can consider affine Deligne-Lusztig varieties with an arbitrary parahoric level structure K as follows.

$$X(\mu,b)_K := \{gK \in G(\breve{\mathbb{Q}}_p)/K : g^{-1}b\sigma(g) \in K\dot{x}K \\ \text{for some } x \in W_K\backslash \operatorname{Adm}(\mu)/W_K\},$$

where W_K is the group generated by the simple reflections of K and $\operatorname{Adm}(\mu)$ is the μ-admissible set ([KR00]). Still, Mazur's inequality is the necessary and sufficient condition for $X(\mu,b)_K \neq \emptyset$ (see [KR03], [Win05], and [He16]).

Meanwhile, $X(\mu,b)_K$ is, from the definition, a disjoint union of several pieces (where x varies over $W_K\backslash\operatorname{Adm}(\mu)/W_K$). These pieces, therefore, can be thought of as more refined objects or building blocks, which we call as *single*[1] affine Deligne-Lusztig varieties. One can study their nonemptiness problem and, to get to the point first, it has a considerably different flavor.

We remark that, among parahoric level structures, the Iwahori level structure contains the finest information via the natural projection map from the affine flag variety (Iwahori level) to the affine partial flag variety (parahoric level) or the affine Grassmannian (hyperspecial level). From now on, we restrict ourselves to the Iwahori level.

2.1.2 Single affine Deligne-Lusztig variety at Iwahori level

Along with the notations from section 2.1.1, let I be a σ-stable Iwahori subgroup of $G(\breve{\mathbb{Q}}_p)$ stabilizing a base alcove, $\widetilde{W}$ be the Iwahori-Weyl group, and W_0 be the relative Weyl group (see section 2.2.1). For $x \in \widetilde{W}$ and $b \in G(\breve{\mathbb{Q}}_p)$, the single affine Deligne-Lusztig variety (at Iwahori level) is defined by

$$X_x(b) := \{gI \in G(\breve{\mathbb{Q}}_p)/I : g^{-1}b\sigma(g) \in I\dot{x}I\}$$

where $\dot{x}$ is an element in a subgroup of $G(\breve{\mathbb{Q}}_p)$ which is a lift of $x \in \widetilde{W}$.

The question on the nonemptiness criterion (and the dimension formula) for $X_x(b)$ first appeared in [Rap00, Problem 4.5] and some cases when $G = \mathrm{GL}_2$ were studied in

[1] This is the terminology used in the literature occasionally. See [He14] for example.

[Rap05a, Example 4.3]. For a more general setting, the first partial conjecture was posed by Reuman ([Reu04, Conjecture 7.1]) where he detected an element in W_0, now called $\eta_\sigma(x)$ (see definition 2.2.9), that gives a good amount of information on the nonemptiness (and even on the dimension formula). The conjecture was reformulated and proved partially in [GHKR10] and then completely by [GHN15, Theorem B] in the basic case. For convenience, let us name the theorem GHN B.

We note first that there is an obvious obstruction for the nonemptiness. Recall the Kottwitz map (cf. Definition 2.2.1) defined in [Kot97a, 4.5]. Then,

$$\text{if } X_x(b) \neq \emptyset \text{ then } \kappa_G(x) = \kappa_G(b),$$

because the Kottwitz map applied to the condition $g^{-1}b\sigma(g) \in I\dot{x}I$ results in $\kappa_G(b) = \kappa_G(\dot{x}) = \kappa_G(x)$. Geometrically, this means simply 'x and b are in the same connected component of the loop group $G(\breve{\mathbb{Q}}_p)$'.

We may and will reduce to the case where G is simple, quasi-split, and adjoint (Section 2.2.2). Then, GHN B tells us that, under the condition called *Shrunken Weyl chambers* (Definition 2.2.4), there is only one interesting obstruction ($\mathrm{supp}_\sigma(\eta_\sigma(x)) = \mathbb{S}$) other than the obvious one ($\kappa_G(x) = \kappa_G(b)$).

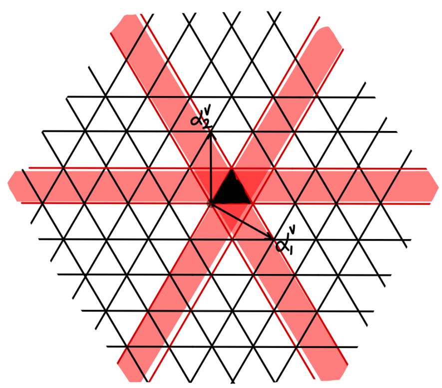

Figure 2.1: An apartment of the Bruhat-Tits building of PGL_3

GHN B. *Let b be basic. If x lies in the shrunken Weyl chambers then*

$$X_x(b) \neq \emptyset \text{ if and only if } \kappa_G(x) = \kappa_G(b) \text{ and } \mathrm{supp}_\sigma(\eta_\sigma(x)) = \mathbb{S}.$$

Here, $\mathbb{S}$ is the set of simple reflections of W_0 and supp_σ is the map sending $w \in W_0$ to the minimal σ-stable subset of $\mathbb{S}$ containing all simple reflections from any reduced expression of w (Definition 2.2.10). Shrunken Weyl chambers are, intuitively, the complement of the red strips (called *critical strips*) in fig. 2.1 and the critical strips are defined to be the strips passing through the base alcove (the black triangle).

2.1.2.1 A side remark on explicit criteria

Here, we clarify the term 'explicit criterion' mentioned in [GHN15, 1.2] and will be used in this paper at times. As this is not related to the rest of the paper, one may skip this remark.

In *loc.cit.*, we have the following theorem (name it GHN A).

GHN A. *Let b be basic. Then, $X_x(b) \neq \emptyset$ if and only if, for all pairs (J, w) such that x is a (J, w, δ)-alcove, the following holds:*

$$\kappa_{M_J}(w^{-1}x\delta(w)) \in \kappa_{M_J}\left([b] \cap M_J(\breve{\mathbb{Q}}_p)\right).$$

The Levi subgroup of G corresponding to $J \subset \mathbb{S}$ is denoted by M_J and κ_{M_J} is its Kottwitz map. We refer to definition 2.2.8 for the term (J, w, δ)-*alcove*.

Practically, GHN B is used often in applications[2] while GHN A is not. However, they solve the same problem and, actually, GHN B is more restrictive. Taking that into account, we can see that GHN B is more applicable and explicit already. Let us now see an example explaining this more clearly.

For the sake of simplicity, let G be split or residually split for a moment. Let x be a translation element t^μ. Then, GHN B directly implies that, if t^μ lies in the shrunken Weyl chambers then $X_{t^\mu}(b) = \emptyset$ always.[3] Moreover, our new (explicit) criterion will show that $X_{t^\mu}(b) \neq \emptyset$ if and only if $[t^\mu] = [b] \in B(G)$. This recovers [GHKR10, Corollary 9.2.1] for b basic.

On the other hand, it is not easy to find all pairs (J, w) such that x is a (J, w, δ)-alcove (especially definition 2.2.8 (2) is not easily manageable) even when $x = t^\mu$, which is a necessary step to apply GHN A. In addition, it is not an easy take to compute the values in $\kappa_{M_J}([b] \cap M_J(\breve{\mathbb{Q}}_p))$ afterwards.

2.1.3 Main Conjecture

Our goal is to remove the shrunken Weyl chambers condition. We will suggest a general conjecture on an explicit nonemptiness criterion in the basic case and prove it for all but finitely many x's and specify some classes of elements satisfying the conjecture. Let b be basic.

There is a new assumption in our conjecture that the following example can justify. Let $x = \mathbf{I}_{\widetilde{W}}$, the identity element in $\widetilde{W}$, and $b = 1$. Then, directly from the definition, $X_x(b) \neq \emptyset$. However, $\mathrm{supp}_\sigma(\eta_\sigma(x)) = \emptyset$ as $\eta_\sigma(x) = \mathbf{I}_{W_0}$. Hence, there are some exceptional x's not having the supp_σ-obstruction.

In the following, we denote by $\widetilde{\mathrm{supp}}_\sigma$ the σ-support function on $\widetilde{W}$.[4]

[2] See, for example, [He21, Theorem 1.1] and [MV20, Remark 3.18]. Using our new explicit criterion, we will give more applications.

[3] This is because $\eta_\sigma(x)$ is the identity element $\mathbf{I}_{W_0}$. See definition 2.2.9 for $\eta_\sigma(x)$.

[4] We found that it may cause confusion to use the same notations for two σ-support functions each defined on W_0 and $\widetilde{W}$ and decided to use $\widetilde{\mathrm{supp}}_\sigma$ instead of $\mathrm{supp}_\sigma : \widetilde{W} \to 2^{\widetilde{\mathbb{S}}}$.

Lemma 2.1.1. *Assume that* $\kappa_G(x) = \kappa_G(b)$*. If* $\widetilde{\mathrm{supp}_\sigma}(x) \neq \widetilde{\mathbb{S}}$ *then* $X_x(b) \neq \emptyset$*.*

Proof. Let ν_x be the image of x under the Newton map. The condition $\widetilde{\mathrm{supp}_\sigma}(x) \neq \widetilde{\mathbb{S}}$ implies that ν_x is central because the group $W_{\widetilde{\mathrm{supp}_\sigma}(x)}$ generated by the elements of $\widetilde{\mathrm{supp}_\sigma}(x)$ is finite in such a case. However, if ν_x is central, for a representative $\dot{x}$, we have $\kappa(\dot{x}) = \kappa(b)$ and $\bar{\nu}_{\dot{x}} = 0 = \bar{\nu}_b$ so that $[\dot{x}] = [b] \in B(G)$. Hence, $X_x(b) \neq \emptyset$. □

Lemma 2.1.1 shows that we only need to consider the case $\widetilde{\mathrm{supp}_\sigma}(x) = \widetilde{\mathbb{S}}$. Our main conjecture is the following: (we follow the notations from section 2.2)

Conjecture 1. *Let* G *be a simple and quasi-split reductive group of adjoint type. Let* x *be an element of Iwahori-Weyl group* $\widetilde{W}$ *and* $b \in \breve{G}$ *be basic. Assume that* $\kappa_G(b) = \kappa_G(x)$ *and* $\widetilde{\mathrm{supp}_\sigma}(x) = \widetilde{\mathbb{S}}$*. Then,*

$$X_x(b) \neq \emptyset \text{ if and only if } \mathrm{supp}_\sigma(\sigma^{-1}(r)\eta_\sigma(x)r^{-1}) = \mathbb{S} \text{ for all } r \in W_x,$$

where W_x *is the subset of* W_0 *defined in definition 2.3.5.*

Conjecture 1 claims that, except for the lemma 2.1.1 cases, the critical obstruction for the nonemptiness is whether or not the σ-supports of certain σ-conjugates of $\eta_\sigma(x)$ are all full. For convenience, we label by $\Leftarrow$ (resp. $\Rightarrow$) the 'if' (resp. 'only if') direction. Our main theorem is the following.

Theorem 2.1.2. *Conjecture 1 holds for all but finitely many* x*. More precisely,* $\Leftarrow$ *holds for all* x *and* $\Rightarrow$ *holds for all but finitely many cases. In addition,* $\Rightarrow$ *holds for*

1. *x lying in exactly one critical strip (see definition 2.2.4),*
2. *a translation element or vt^μ-form element (for μ dominant), and*
3. *x in type A_n under the condition of proposition 2.5.1.*

Note that GHN B takes care of infinitely many x's, but it does not apply to infinitely many cases as well. We prove $\Leftarrow$ in corollary 2.3.8 in full generality and prove $\Rightarrow$ in proposition 2.3.10 for x such that $\ell(x) \gg 0$. An efficetive bound for the length can be computed and we do so for type A_n in proposition 2.5.1.

Let us discuss Theorem 2.1.2 (3) in more detail. For simplicity, let $G = \mathrm{PGL}_n$. There are two assumptions in proposition 2.5.1. The first assumption is that $\kappa_G(x) = 0 \in \mathbb{Z}/n \simeq X^*(Z(\hat{G})^\Gamma)$. It means that x can be written as $t^\lambda w$ where $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{Z}^n_{\mathrm{sum}=0}$ and $w \in S_n$ (symmetric group). And then the second assumption is that '$\max_i \lambda_i > 1$ and $\min_i \lambda_i < -1$'.

On the other hand, any element $x = t^\lambda w$ satisfying $\widetilde{\mathrm{supp}_\sigma}(x) \neq \widetilde{\mathbb{S}}$ (cf. Lemma 2.1.1) has the property '$\max_i \lambda_i = 1$ and $\min_i \lambda_i = -1$'. Hence, there is only a small gap left in the conjecture under the assumption that $\kappa_G(x) = 0$. This way, we can view Theorem 2.1.2 (3) as an evidence for Conjecture 1 in the sense that elements in lemma 2.1.1 are the only exceptions.

Remark 1. It is worth pointing out that GHN B and Theorem 2.1.2 for x lying in the shrunken Weyl chambers are *almost* the same, but there is a subtle difference. For example, when $x = w_0$ (the longest element of W_0) as an element of $\widetilde{W}$, we apply GHN B to get $X_x(1) \neq \emptyset$. However, x does not fit into Theorem 2.1.2 as $\widetilde{\mathrm{supp}_\sigma}(x) \neq \widetilde{\mathbb{S}}$, so we should apply lemma 2.1.1 to get $X_x(1) \neq \emptyset$ here.

Remark 2. One might wonder what the condition '$\widetilde{\mathrm{supp}_\sigma}(x) = \widetilde{\mathbb{S}}$' means. Surprisingly, a simple concrete picture exists for this condition in the sense that "$\widetilde{\mathrm{supp}_\sigma}(x) \neq \widetilde{\mathbb{S}}$ if and only if the action of $x \circ \sigma$ fixes a point in the closure of the base alcove". Moreover, under the obvious obstruction (that is, $\kappa_G(x) = \kappa_G(b)$) when b is basic, it is equivalent to that $I\dot{x}I \subset [b]$. ([GHN19, Proposition 5.6 and Lemma 5.8])

Surprisingly, our result says that the abstract criterion (GHN A) can be made much stronger as follows. This was observed by Sian Nie.

Theorem 2.1.3 (Stronger GHN A)**.** *Let b be basic and suppose that $\widetilde{\mathrm{supp}_\sigma}(x) = \widetilde{\mathbb{S}}$ with $\ell(x) \gg 0$. Then, $X_x(b) \neq \emptyset$ if and only if x is not a $(J, w)_\sigma$-alcove for any proper $J \subsetneq \mathbb{S}$.*

We note that the condition '$\ell(x) \gg 0$' can be removed once Conjecture 1 is fully proved. In short, for x not intersecting with the base alcove, $X_x(b) \neq \emptyset$ if and only if x is not "coming from a proper Levi subgroup".

2.1.4 New Ideas and Sketch of Proof

The novelty of our work lies in the introduction of the set W_x which arises naturally in the following sense.

The idea is that a critical strip behaves as if it belongs to the shrunken Weyl chambers *adjacent* to the strip. Hence, if x lies in a critical strip, we can "embed" x into each shrunken Weyl chamber adjacent to the strip. Noting that, when x lies in a shrunken Weyl chamber, the rule for $X_x(b) \neq \emptyset$ is $\mathrm{supp}_\sigma(\eta_\sigma(x)) = \mathbb{S}$, we can presuppose that, in general, the rule for $X_x(b) \neq \emptyset$ would be $\mathrm{supp}_\sigma(\eta_\sigma(x')) = \mathbb{S}$ for each embedding x' of x into each shrunken Weyl chamber adjacent to the strips.

The set W_x is defined to be the set of elements in W_0 that, intuitively, embed x into those shrunken Weyl chambers. Practically, for x in the shrunken Weyl chambers, we get $W_x = \{\mathbf{I}_{W_0}\}$ and, for x in one critical strip (cf. Theorem 2.1.2 (1)), we have $W_x = \{\mathbf{I}_{W_0}, s\}$ where s is the simple reflection related to the critical strip containing x.

In order to define W_x, we make a new observation on a certain structure of the set of critical strips containing x (Proposition 2.3.4). More precisely, let Φ be the set of (relative) roots of G and, for simplicity, x lie in the dominant Weyl chamber. Next, denote by Φ_x the set of positive roots whose critical strip contains x. Then, Φ_x is "anti-closed" in the sense that if $\alpha + \beta \in \Phi_x$ for two positive roots α and β then $\alpha \in \Phi_x$ or $\beta \in \Phi_x$. Alternatively, it is equivalent to that the complement of Φ_x in the set of positive roots is closed.

Using this, we show that the set W_x is well-defined (Proposition 2.3.3) and that W_x contains exactly the elements needed for the 'σ-support test' (cf. Lemma 2.3.7). After that, we follow the strategy of [GHN15] as explained in section 2.2.4 with more careful study on the W_x-action on dominant cocharacters. We also use the positivity of the coroot-coefficients of dominant cocharacters (see Lemma 2.2.18) and work with the exact (positive) coefficients in the proof of Theorem 2.1.2 (1) and (3).

Remark 3. There has been an interesting coincidence happening while we have been working on this problem. Felix Schremmer ([Sch22]) recently defined a set called 'length-positive' in his work on the generic Newton point. Our work and Schremmer's work do not overlap, that is, the results are rather complementary. However, the length-positive set of Schremmer turns out to be a W_0-conjugate of the set W_x of ours.

2.1.5 Applications and future works

2.1.5.1 The set $B(G)_x$ and cordial elements

The question of whether $X_x(b)$ is nonempty is equivalent to the question of whether $I\dot{x}I \cap [b]$ is nonempty. In this perspective, we can denote the set of $[b] \in B(G)$ such that $I\dot{x}I \cap [b] \neq \emptyset$ by $B(G)_x$ and ask to describe it. This approach first appeared in [Bea09].

Remark 4. For clarification, we remark that the main difference between the approach using $B(G)_x$ and ours is which variable is fixed. In our approach, we fix b and study the nonemptiness, but the study of $B(G)_x$ fixes x. As we will see in a moment, these two approaches are complementary.

When x is cordial (see definition 2.5.2), the set $B(G)_x$ has the property called saturated (see lemma 2.5.3), which makes the complete description of $B(G)_x$ easier. Combining this with Theorem 2.1.2 (3), we obtain a full description of $B(G)_x$ in some special cases as follows.

Theorem 2.1.4. *Let $x = vt^\mu$ for a dominant non-central*[5] *μ and $v \in W_0$, and let $W_0(\mu)$ be the stabilizer subgroup of W_0 fixing μ. Then,*

$$W_x = \{r \in W_0(\mu) : \ell(vr^{-1}) = \ell(v) + \ell(r)\}.$$

Now, if $\mathrm{supp}_\sigma(\sigma^{-1}(r)\eta_\sigma(x)r^{-1}) = \mathbb{S}$ for all $r \in W_x$ then $B(G)_x = B(G, \mu)$.

2.1.5.2 Future works on new dimension formulas

The following dimension formula ([He14]) is known for x lying in the shrunken Weyl chambers:

$$\dim X_x(b) = \frac{1}{2}\left(\ell(x) + \ell(\eta_\sigma(x)) - \mathrm{def}_G(b)\right).$$

On the contrary, outside of the shrunken Weyl chambers, even a conjectural formula for $\dim X_x(b)$ is still mysterious. However, if more nonemptiness results are found, we can

[5] If μ is central, it is obvious that $B(G)_x = \{[t^\mu]\} = B(G, \mu)$ via lemma 2.1.1.

approach this problem with the following recursive formula ([He14, Proposition 4.2]): For $s \in \mathbb{S}$ satisfying $\ell(sx\sigma(s)) = \ell(x) - 2$,

$$\dim X_x(b) = \max\{\dim X_{sx}(b), \dim X_{sx\sigma(s)}(b)\} + 1.$$

Typically, this is a bottom-up formula for the length reason (that is, $\ell(x) = \ell(sx) + 1 = \ell(sx\sigma(s)) + 2$). However, for example, if $X_{sx\sigma(s)}(b)$ is empty then we can also compute $\dim X_{sx}(b)$ from $\dim X_x(b)$, which is top-down.

In future work, using Theorem 2.1.2 (2), we will show the following dimension formula which is new even in the rank two case:

Theorem 2.1.5. *Let G be residually split with $\mathrm{rk}G_{sc} = 2$ and b be basic. For $x \in \widetilde{W}$ lying in only one critical strip (associated to $v\alpha$), if $X_x(b) \neq \emptyset$ then*

$$\dim X_x(b) = \frac{1}{2}\left(\ell(x) + \min\{\ell(\eta_\sigma(x)), \ell(\sigma^{-1}(s_\alpha)\eta_\sigma(x)s_\alpha)\} - \mathrm{def}_G(b)\right) - \epsilon,$$

where $\epsilon = 1$ if $\eta_\sigma(x) = w_0$ and $\epsilon = 0$ otherwise.

2.2 Basics on single affine Deligne-Lusztig varieties

Let F be a nonarchimedean local field with a uniformizer t and G be a connected reductive group over F. Denote by $\breve{F} := \widehat{F^{\mathrm{nr}}}$ the completion of maximal unramified extension of F and by $\breve{G}$ the $\breve{F}$-points of G. The Frobenius map on the residue field of $\breve{F}$ lifts to that of $\breve{F}$ which we denote by σ again. The induced map on $\breve{G}$ will be denoted the same. Finally, we denote the set of σ-conjugacy classes of $\breve{G}$ by $B(G)$.

2.2.1 Iwahori-Weyl group and $B(G)$

Let S be a maximal $\breve{F}$-split torus of G defined over F and T be the centralizer of S. Note that T is a maximal torus of G as G becomes quasi-split over $\breve{F}$ by Steinberg's Theorem.

2.2.1.1 Iwahori-Weyl group $\widetilde{W}$

The Iwahori-Weyl group associated to S is defined as

$$\widetilde{W} := N_S(G)(\breve{F})/T(\breve{F})_1 \tag{2.1}$$

where $T(\breve{F})_1$ is the unique parahoric subgroup of $T(\breve{F})$. We can fit $\widetilde{W}$ into the following short exact sequence of groups ([HR08a] Definition 7)

$$0 \to X_*(T)_{\Gamma_0} \to \widetilde{W} \to W_0 \to 1$$

where Γ_0 is the absolute Galois group of $\breve{F}$ and W_0 is the relative Weyl group

$$W_0 := N_S(G)(\breve{F})/T(\breve{F}) \tag{2.2}$$

Now, we fix a σ-invariant base alcove $\mathbf{a}$ in the apartment of S and let I be the Iwahori subgroup of G corresponding to $\mathbf{a}$. By fixing a special vertex in the apartment, we get a section $W_0 \to \widetilde{W}$ (not necessarily σ-equivariant) which allows us to identify $\widetilde{W}$ with $X_*(T)_{\Gamma_0} \rtimes W_0$.

The Newton map $\nu : \widetilde{W} \to X_*(T)^{\sigma}_{\Gamma_0,\mathbb{Q}}$ is defined as follows. The action of σ on $\widetilde{W}$ is of finite order so that, given $x \in \widetilde{W}$, there exists a positive integer N such that $\prod_{i=0}^{N-1} \sigma^i(x)$ belongs to $X_*(T)^{\sigma}_{\Gamma_0}$, say μ. The Newton map is defined to send x to $\nu_x = \frac{\mu}{N}$. This does not depend on the choice of N.

Let G_{sc} be the simply connected cover of the derived subgroup of G and T_{sc} the inverse image of T via $G_{\mathrm{sc}} \to G_{\mathrm{der}} \to G$. The Iwahori-Weyl group of G_{sc} is the affine Weyl group W_a and gives rise to the following short exact sequence ([HR08a] Lemma 14):

$$1 \to W_a \to \widetilde{W} \xrightarrow{\tilde{\kappa}_G} X^*(Z(\hat{G})^{\Gamma_0}) \to 1. \tag{2.3}$$

Denoting by $\Omega \subset \widetilde{W}$ the stabilizer of the base alcove, we have an isomorphism $X^*(Z(\hat{G})^{\Gamma_0}) \simeq \Omega$ which gives a section of $\tilde{\kappa}_G$. This presents $\widetilde{W}$ as $W_a \rtimes X^*(Z(\hat{G})^{\Gamma_0})$. Now, the Bruhat order on W_a extends onto $\widetilde{W}$ by making two elements be comparable when their projections to $X^*(Z(\hat{G})^{\Gamma_0})$ agree.

Note that W_a is the affine Weyl group generated by orthogonal reflections with respect to the hyperplanes in $X_*(T_{\mathrm{sc}})_{\Gamma_0} \otimes \mathbb{R}$. Hence, by [Bou81, Ch.VI, §2.5. Proposition 8], there exists a reduced root system Σ whose affine Weyl group is canonically isomorphic to W_a. We denote by $Q^\vee$ the coroot lattice of Σ and $P^\vee$ its coweight lattice. Lastly, let $\mathbb{S}$ be the set of simple reflections of the finite Weyl group of Σ (that is, W_0) and $\widetilde{\mathbb{S}}$ the set of affine simple reflections.

The map σ on $\breve{G}$ induces an action on $\mathbb{S}$ which we will denote by σ again. We call $J \subset \mathbb{S}$ a σ-stable subset if $\sigma(J) = J$. For any σ-stable subset J, we denote $X_*(T)_{\Gamma_0} \rtimes W_J$ by $\widetilde{W}_J$ where W_J is the subgroup of W_0 generated by the simple reflections of J.

A comment on the notation: we will use the notation $v \cdot \mu$ when considering W_0-action on $X_*(T)_{\Gamma_0}$. So, for example, $vt^\mu \in \widetilde{W}$ can also be written as $t^{v \cdot \mu} v$.

2.2.1.2 $B(G)$ with Newton map and Kottwitz map

Recall the Newton map and the Kottwitz map from [Kot97a, 4.5] that give an injective homomorphism $(\bar{\nu}, \kappa_G) : B(G) \to X_*(T)^{\Gamma,+}_{\mathbb{Q}} \times X^*(Z(\hat{G})^{\Gamma})$ where Γ is the absolute Galois group of F.

Definition 2.2.1 ($B(G,\mu)$)**.** *Let $\mu \in X_*(T)^+$ be a dominant cocharacter of G. We define $B(G,\mu)$ as the subset of $B(G)$ consisting of $[b] \in B(G)$ such that*

$$\bar{\nu}_b \leq \mu^\diamond \text{ and } \kappa_G([b]) = \kappa_G([t^\mu])$$

where $\mu^\diamond$ is the Γ-average[6] *of μ and t^μ is the image of t under $\mu : \mathbb{G}_m \to T$. Mazur's inequality mentioned in section 2.1.1 refers to $\bar{\nu}_b \leq \mu^\diamond$.*

We note that the map $\tilde{\kappa}_G$ in eq. (2.3) followed by the projection $X^*(Z(\hat{G})^{\Gamma_0}) \to X^*(Z(\hat{G})^{\Gamma})$ gives $\kappa_G : \widetilde{W} \to X^*(Z(\hat{G})^{\Gamma})$ and it is compatible with κ_G on $B(G)$ via the lifting from $\widetilde{W}$ to $N_S(G)(\breve{F}) \subset \breve{G}$.

2.2.2 Single affine Deligne-Lusztig variety

The Iwahori-Bruhat decomposition says

$$\breve{G} = \bigsqcup_{x \in \widetilde{W}} I\dot{x}I$$

where $\dot{x} \in N_S(G)(\breve{F})$ is a representative of $x \in \widetilde{W}$.

Definition 2.2.2 (affine Deligne-Lusztig "variety")**.** *For $x \in \widetilde{W}$ and $b \in B(G)$, the single affine Deligne-Lusztig variety associated to x and b is*

$$X_x(b) := \{gI \in \breve{G}/I : g^{-1}b\sigma(g) \in I\dot{x}I\}.$$

A priori, it is an affine Deligne-Lusztig *set* and we do not have a natural scheme structure on it. In fact, in the case of an equal characteristic local field F, it is not difficult to identify $X_x(b)$ as the $\overline{\mathbb{F}}_q$-points of a (locally of finite type) locally closed subscheme in the affine flag variety over $\overline{\mathbb{F}}_q$. It is the mixed characteristic case where we need distinguished works of [Zhu17] and [BS17] to give a scheme structure on $X_x(b)$.

In order to study the nonemptiness pattern, it is more convenient to do some reductions. By [HZ20b, Corollary 4.4 and Section 4.3], we can reduce the nonemptiness problem to the case when G is the quasi-split inner form of an adjoint group. Finally, if $G = G_1 \times G_2$ then $B(G) = B(G_1) \times B(G_2)$ and $\widetilde{W}_G = \widetilde{W}_{G_1} \times \widetilde{W}_{G_2}$. Hence, $X_x(b) = X_{x_1}(b_1) \times X_{x_2}(b_2)$ where x_i's are the projections of x onto $\widetilde{W}_{G_i}$ and b_i's are that of b onto $B(G_i)$. Now, we may assume that G is a simple quasi-split reductive group of adjoint type.

Remark 5. In *loc.cit.*, it is assumed that G is tamely ramified over F and $p \nmid \pi_1(G^{\mathrm{ad}})$ in the equal characteristic case. However, this assumption is not necessary for the nonemptiness results as a universal homeomorphism preserves the nonemptiness.

2.2.3 Terminologies on positions of alcoves

From now on, we assume that G is a simple quasi-split reductive group of adjoint type. For simplicity, we abusively use x to denote the alcove $x\mathbf{a}$. For example, $\mathbf{I}_{\widetilde{W}}$ denotes the base alcove.

[6]To be precise, this is true only when G is quasi-split. In general, it is the average of the *dominant representatives* of Galois orbits. For a more detailed explanation, see [HN18, 2.4].

Let Φ be the set $\Phi(G,S)$ of relative roots and Φ^+ (resp. Φ^-) be the subset of positive (resp. negative) roots. Let $V := X_*(T)_{\Gamma_0} \otimes \mathbb{R}$ which is isomorphic to $X_*(T_{\mathrm{sc}})_{\Gamma_0} \otimes \mathbb{R}$ as G is semisimple. For $\alpha \in \Phi$, the hyperplane H_α in V is defined by $\{\mathbf{v} \in V : \langle \alpha, \mathbf{v} \rangle = 0\}$ and, more generally, for any integer k, we define $H_\alpha(k) := \{\mathbf{v} \in V : \langle \alpha, \mathbf{v} \rangle = k\}$. Note that $H_\alpha(k) = H_{-\alpha}(-k)$.

Definition 2.2.3 ([GHKR10, Section 2.1], k-value of an alcove with respect to a root)**.** *For $\alpha \in \Phi$ and $x \in \widetilde{W}$, let us denote by $k(\alpha, x)$ the integer k such that x is located in between the hyperplanes $H_\alpha(k)$ and $H_\alpha(k+1)$.*

Definition 2.2.4 (The critical strips and shrunken Weyl chambers)**.** *For each positive root α, we call the set of alcoves between $H_\alpha(0)$ and $H_\alpha(1)$ the critical strip associated to α and denote by C_α.*[7] *The set of alcoves which do not lie in any critical strip is called the shrunken Weyl chambers.*

The following computes the k-values explicitly. Here, the expression $t^\mu w$ is an alcove in the dominant Weyl chamber always and v is an element of W_0. Both a and α mean roots, but the latter will denote a positive root mostly.

Lemma 2.2.5. *For any root $a \in \Phi$,*

$$k(a, t^\mu w) = \begin{cases} \langle a, \mu \rangle & \text{if } w^{-1}a > 0, \\ \langle a, \mu \rangle - 1 & \text{otherwise.} \end{cases}$$

Proof. We only need to consider the case where the alcove is represented by a finite Weyl group element and, in this case, the value $\delta_{w^{-1}\alpha}$ decides whether w is in the a-direction or $-a$-direction. □

In general, $k(a, vt^\mu w) = \langle a, v\mu \rangle + \delta_{w^{-1}v^{-1}a}$ where $\delta_\alpha = 0$ if $\alpha \in \Phi^+$ or $= -1$ otherwise.

Corollary 2.2.6. *Let a be a root in (1) and a, b, and $a+b$ are roots in (2).*

1. $k(a, t^\mu w) + k(-a, t^\mu w) = -1$.
2. $k(a+b, t^\mu w) = k(a, t^\mu w) + k(b, t^\mu w)$ *or* $k(a, t^\mu w) + k(b, t^\mu w) + 1$.

Proof. (1): $w^{-1}a > 0$ implies $w^{-1}(-a) < 0$ and vice versa. (2): Apply lemma 2.2.5 noting that the cases where both $w^{-1}a$ and $w^{-1}b$ are positive (resp. negative) but $w^{-1}(a+b)$ is negative (resp. positive) are not possible. □

From lemma 2.2.5, $vt^\mu w \in C_{v\alpha}$ for some $\alpha \in \Phi^+$ if and only if $\langle \alpha, \mu \rangle + \delta_{w^{-1}\alpha} - \delta_{v\alpha} = 0$. Noting that $k(\alpha, t^\mu w) \geq 0$ when $\alpha \in \Phi^+$, we have the following corollary easily:

[7] One can define the critical strip of a negative root α to be the set of alcoves between $H_\alpha(0)$ and $H_\alpha(-1)$ and also denote by C_α. This convention will be secretly used.

Corollary 2.2.7. *Let $\alpha \in \Phi^+$ and suppose $vt^\mu w \in C_{v\alpha}$. Then $v\alpha \in \Phi^+$ and one of the following holds:*

1. *$\langle \alpha, \mu \rangle = 0$ and $w^{-1}\alpha \in \Phi^+$,*
2. *$\langle \alpha, \mu \rangle = 1$ and $w^{-1}\alpha \in \Phi^-$.*

When $vt^\mu w \notin C_{v\alpha}$, if $\langle \alpha, \mu \rangle = 0$ then $v\alpha \in \Phi^-$ and $w^{-1}\alpha \in \Phi^+$.

Now, we recall the definition of $(J, w)_\sigma$-alcove[8] from [GHN15, 3.3].

Definition 2.2.8 ($(J, w)_\sigma$-alcove)**.** *Let J be a σ-stable subset of $\mathbb{S}$ and w be an element of W_0. We say that $x \in \widetilde{W}$ is a $(J, w)_\sigma$-alcove if*

(1) $w^{-1}x\sigma(w) \in \widetilde{W}_J$ and,

(2) for any $a \in w(\Phi^+ \setminus \Phi_J^+)$, $k(a, x) \geq k(a, \mathbf{I})$.

Next, the "essential" finite part $\eta_\sigma(x)$ of x observed by Reuman is defined as follows.

Definition 2.2.9 ($\eta_\sigma(x)$, [GHN15, 3.6])**.** *Let $v_x \in W_0$ be the unique element such that $v_x^{-1}x$ is in the dominant Weyl chamber and $v_x^{-1}x = t^{\mu_x}w_x$ for $\mu_x \in X_*(T)^+_{\Gamma_0}$ and $w_x \in W_0$. We define*

$$\eta_\sigma(x) = \sigma^{-1}(w_x)v_x.$$

Finally, we recall the following from [Bou81, Ch.IV, §1.8. Proposition 7].

Definition 2.2.10 (support and σ-support)**.** *Given a Coxeter system (W, S) and $w \in W$, the support of w is defined to be the set of $s \in S$ appearing in some (equivalently each) reduced expression of w and denoted by $\mathrm{supp}(w)$. When (W, S) is equipped with an action by ξ, the minimal ξ-stable set containing $\mathrm{supp}(w)$ is called the ξ-support of w and denoted by $\mathrm{supp}_\xi(w)$.*

Remark 6. There are two Coxeter systems $(W_0, \mathbb{S})$ and $(W_a, \widetilde{\mathbb{S}})$ in this paper. We have $\xi = \sigma$ for $(W_0, \mathbb{S})$ and $\xi = \omega\sigma$ for $(W_a, \widetilde{\mathbb{S}})$ where ω is an element of Ω defined in eq. (2.3). For $x \in W_a \rtimes \Omega$ whose projection to Ω is ω_x, we use the notation $\widetilde{\mathrm{supp}_\sigma}(x)$ instead of $\mathrm{supp}_{\omega_x\sigma}(x)$ for simplicity.

[8] In [GHN15], this is denoted by (J, w, δ)-alcove. We choose to use the notation '$(J, w)_\sigma$-alcove' in this paper not just for the sake of simplicity. In fact, the induced action δ on $\mathbb{S}$ by σ is a fixed map. It is not varying, unlike J or w in the notation.

2.2.4 On the known result GHN B

Now we recall:

Theorem 2.2.11 (GHN B). *Let $b \in B(G)$ be basic and $x \in \widetilde{W}$ lie in the shrunken Weyl chambers with $\kappa(b) = \kappa(x)$. Then,*

$$X_x(b) \neq \emptyset \textit{ if and only if the } \sigma\textit{-support of } \eta_\sigma(x) \textit{ is } \mathbb{S}.$$

The proof of theorem 2.2.11 is a combination of the following lemmas.

Lemma 2.2.12 ([GHN15, Proposition 3.6.4 and Proposition 3.6.5]). *Let b be basic. Under the following two assumptions, we have $X_x(b) = \emptyset$.*

1. $\nu_{\mu_x} \neq \nu_b$ *and*
2. $\mathrm{supp}_\sigma(\eta_\sigma(x)) \neq \mathbb{S}$,

If we assume that x lies in shrunken Weyl chambers, (2) implies (1).

This proves that if $X_x(b) \neq \emptyset$ then $\mathrm{supp}_\sigma(\eta_\sigma(x)) = \mathbb{S}$ in theorem 2.2.11.[9] For the reverse direction, we need the following theorem:

Proposition 2.2.13 ([GHN15, Theorem 4.4.7]). *Let b be basic. If x satisfies NLO, then $X_x(b) \neq \emptyset$. NLO means the following: for every pair (J, w) with σ-stable J and $w \in W_0$ such that $x \in \widetilde{W}$ is a $(J, w)_\sigma$-alcove, there exists $b_J \in w\widetilde{W}_J\sigma(w)^{-1}$ such that*

1. $\kappa(b) = \kappa(b_J)$
2. $\nu_{b_J} = \nu_b$
3. $\kappa_J(w^{-1}b_J\sigma(w)) = \kappa_J(w^{-1}x\sigma(w))$

If x lies in the shrunken Weyl chamber, one can show that $\mathrm{supp}_\sigma(\eta_\sigma(x)) = \mathbb{S}$ and so, by lemma 2.2.14 below, the only J such that x is a $(J, w)_\sigma$-alcove is $J = \mathbb{S}$. For $J = \mathbb{S}$, we can let b_J be any element in the σ-straight conjugacy class of $\widetilde{W}$ corresponding to $[b] \in B(G)$ ([He14, 3.3]) so that x satisfies NLO and $X_x(b) \neq \emptyset$.

Lemma 2.2.14 ([GHN15, Proposition 4.1.1]). *Let $x \in \widetilde{W}$ lie in the shrunken Weyl chambers. If x is a $(J, w)_\sigma$-alcove for σ-stable subset J of $\mathbb{S}$, then $\mathrm{supp}_\sigma(\eta_\sigma(x)) \subset J$.*

Finally, we have the following lemmas on the properties of dominant cocharacters and the Weyl group action on roots:

[9] However, note that the proof of lemma 2.2.12 uses [GHN15, Proposition 3.5.1] ("σ-conjugacy classes never fuse") which was stated without the assumption that b is basic but in fact needs that assumption. See [GHN].

Lemma 2.2.15. *Let μ be a cocharacter and $w \in W_0$. Then, $\mu - w \cdot \mu \in Q^\vee_{\mathrm{supp}(w)}$.*

Proof. For any simple reflection s, the formula $s \cdot \mu = \mu - \langle \alpha_i, \mu \rangle \alpha_i^\vee$ tells us that $\mu - s \cdot \mu \in Q^\vee_{\{s\}}$. Note that w is a product of simple reflections and so we can think inductively on $\ell(w)$. Considering $\mu - ws \cdot \mu = \mu - s \cdot \mu + s \cdot \mu - w \cdot (s \cdot \mu) \in Q^\vee_{\{s\}} \oplus Q^\vee_{\mathrm{supp}(w)}$, it is straightforward. □

The following corollary is stated in [GHN15, 3.5 (1)] without a proof. For completeness, we give a proof for this.

Corollary 2.2.16. *Let J be a σ-stable subset of $\mathbb{S}$. If $x \in \widetilde{W}_J$, then $\nu_{v_x \cdot \mu_x} - \nu_x \in Q^\vee_{J,\mathbb{Q}}$.*

Proof. The difference is a $\mathbb{Q}$-multiple of $\sum_i (\sigma^i(v_x \cdot \mu_x) - (v_x w_x \sigma)^i (v_x \cdot \mu_x))$. Noting that $\prod_{0 \le j \le i-1} \sigma^j(v_x w_x) \in W_J$, apply lemma 2.2.15 to get the conclusion. □

Lastly, we prove:

Lemma 2.2.17. *Suppose that the Dynkin diagram of G is σ-connected. Let J be a proper σ-stable subet of $\mathbb{S}$. If $\mu \in Q^\vee_{J,\mathbb{Q}}$ is dominant, then $\mu = 0$.*

Proof. $Q^\vee_{J,\mathbb{Q}} \subset Q^\vee_{\mathbb{Q}} = P^\vee_{\mathbb{Q}}$, hence μ is a $\mathbb{Q}$-linear combination of fundamental coweights. As it is dominant, all the coefficients are non-negative. If $\mu \neq 0$, then at least one coefficient is positive so that it is a positive linear combination of coroots of a connected component by lemma 2.2.18. Hence, J contains some connected component completely. As the Dynkin diagram is σ-connected, we must have $J = \mathbb{S}$ which is contradiction. □

The following Lemma 2.2.18 has been probably known to experts for a long time. However, to the best of our knowledge, a reference for this lemma is hard to locate. So, following Michael Rapoport's suggestion, we record it here. In this paper, this is particularly needed to remedy some argument in [GHN15] using *loc.cit.* 3.5. (2) which is not correct. See Proposition 2.4.5.

Lemma 2.2.18. [10] *Let $\bar{C}$ be the closed Weyl chamber defined as $\{\mu \in V : \langle \mu, \alpha \rangle \ge 0$ for all $\alpha \in \Delta\}$ and $C^\vee$ be the obtuse Weyl chamber defined as $\{\mu \in V : \mu = \sum_{i \in \mathbb{S}} c_i \alpha_i^\vee,\ c_i \ge 0\}$. Then $\bar{C} \setminus \{\vec{0}\}$ is contained in the interior of $C^\vee$. In other words, a fundamental coweight is a positive linear combination of simple coroots.*

Proof. The fundamental coweights $\varpi_i^\vee$ form a basis of V and they satisfy $\langle \varpi_i^\vee, \alpha_j \rangle = \delta_{ij}$. Hence, $\bar{C} \setminus \{\vec{0}\} = \{\sum_{i \in \mathbb{S}} d_i \varpi_i^\vee$: not all d_i's are $0\}$. Hence, it is enough to show that $\langle \varpi_i^\vee, \varpi_j \rangle =: \pi_{ij}$, the coefficient of $\alpha_j^\vee$ in the expression of $\varpi_i^\vee$, is positive.

First of all, the coefficients from [Bou81, Ch.VI, §4.5-§4.13, (VI)] prove the dual version. For our version, recall the Cartan matrix A whose entries are $A_{ij} = \langle \alpha_i^\vee, \alpha_j \rangle$. Then, $\delta_{ik} = \langle \varpi_i^\vee, \alpha_k \rangle = \langle \sum_j \pi_{ij} \alpha_j^\vee, \alpha_k \rangle = \sum_j \pi_{ij} A_{jk}$. Therefore, π_{ij}'s are the entries of the inverse of A, which we know to be positive by [OV90, Reference Chapter. §2. Table 2] (note that one should take their transpose). For a more general situation, see [LT92, 5.]. □

[10] The letter C will be used in this lemma to follow the historical notations. The letter C appearing in this paper is always referring to the critical strip, except here. We believe there would be no confusion occurring.

2.3 Proof of main theorems

2.3.1 Closed subsets of a root system

Definition 2.3.1. *A subset Ψ of the set of roots Φ is called*[11]

1. *closed if $\alpha, \beta \in \Psi$ implies $\{\alpha + \beta\} \cap \Phi \subset \Psi$,*
2. *radical if $\Psi \cap -\Psi = \emptyset$, and*
3. *parabolic if $\Psi \cup -\Psi = \Phi$.*

Being closed or radical are invariant under any W_0-action. An example of a radical closed subset is any subset of all positive roots or W_0-conjugate of that. In fact, this is essentially the only example.

Lemma 2.3.2. *For a radical closed subset Ψ, there exists $w \in W_0$ such that $\Psi \subset w\Phi^+$.*

Proof. [Sop02, Proposition 3] (cf. [Bou81, Ch.VI, §1.7. Proposition 22]) □

The following proposition is stronger than lemma 2.3.2 and crucial to generalize our work to multiple critical strips case. The proof relies on the classification of parabolic subsets.

Proposition 2.3.3. *Let $\Psi_r \subset \Psi_p$ be two closed subsets of Φ such that Ψ_r is radical and Ψ_p is parabolic. Then, there exists $w \in W_0$ such that*

$$w\Psi_r \subset \Phi^+ \subset w\Psi_p.$$

Proof. We call a closed subset to be invertible if the complement (in the full Φ) is also closed. A parabolic subset is an invertible subset and all invertible subsets are W_0-conjugate to $\Phi_J \cup (\Phi^+ \setminus \Phi^+_{J'})$ where $J \subset J' \subset \mathbb{S}$ and $J \perp (J' \setminus J)$ by [ĐCH94, Lemma 1 and Theorem 4]. Only when $J = J'$, it is parabolic.

Hence, $\Psi_p = w_0(\Phi^+ \cup \Phi^-_J)$ for some $w_0 \in W_0$ and $J \subset \mathbb{S}$. Now, consider $w_0^{-1}\Psi_r \cap \Phi_J$ which is radical because it belongs to a radcial subset $w_0^{-1}\Psi_r$ and closed because it is the intersection of two closed subsets. Hence, there exists $w_1 \in W_J$ such that $w_0^{-1}\Psi_r \cap \Phi_J \subset w_1\Phi^+_J$ by lemma 2.3.2.

Note that $\Psi_r \subset \Psi_p = w_0(\Phi^+ \cup \Phi^-_J)$ so that

$$w_1^{-1}w_0^{-1}\Psi_r = w_1^{-1}(w_0^{-1}\Psi_r \cap \Phi_J) \cup w_1^{-1}(w_0^{-1}\Psi_r \cap (\Phi^+ \setminus \Phi^+_J))$$

But from above the first part belongs to W^+_J. The second set belongs to $w_1^{-1}(\Phi^+ \setminus \Phi^+_J)$ which belongs to Φ^+ as $w_1 \in W_J$. Obviously, $w_1^{-1}w_0^{-1}\Psi_p = w_1^{-1}(\Phi^+ \cup \Phi^-_J) \supset \Phi^+$. □

[11] The terms *closed* and *parabolic* are from [Bou81, Ch.VI, §1.7. Définition 4.] and the term *radical* is from [ĐCH94, 1.].

Now, let Φ_x be the set of positive roots α such that $x \in C_{v_x\alpha}$. The following is the most important observation towards our main theorem.

Proposition 2.3.4. *$\Phi^+ \setminus \Phi_x$ is a radical closed subset.*

Proof. It is enough to prove that Ψ_x satisfies the following:

$$\text{for } \alpha, \beta \in \Phi^+, \text{ if } \alpha + \beta \in \Psi_x \text{ then } \alpha \in \Psi_x \text{ or } \beta \in \Psi_x.$$

However, for a positive root γ, $\gamma \in \Psi_x$ is equivalent to $k(\gamma, t^{\mu_x} w_x) = k(v_x\gamma, \mathbf{I}) = 0$. Applied to $\gamma = \alpha + \beta$, we get $k(\alpha+\beta, t^{\mu_x} w_x) = k(v_x\alpha + v_x\beta, \mathbf{I}) = 0$. Using corollary 2.2.6 (2) and that $t^{\mu_x} w_x$ lies in the dominant Weyl chamber, we get $k(\alpha, t^{\mu_x} w_x) = k(\beta, t^{\mu_x} w_x) = 0$. Moreover, by corollary 2.2.6 (2) again, $0 = \delta_{v_x\alpha} + \delta_{v_x\beta} + 1$ or $0 = \delta_{v_x\alpha} + \delta_{v_x\beta}$. Hence, $\delta_{v_x\alpha} = 0$ or $\delta_{v_x\beta} = 0$, that is, $\alpha \in \Psi_x$ or $\beta \in \Psi_x$. □

2.3.2 The set W_x

We define W_x mentioned in Conjecture 1.

Definition 2.3.5. *Given $x \in \widetilde{W}$, the subset $W_x \subset W_0$ is the set of r's such that*

$$r(\Phi^+ \setminus \Phi_x) \subset \Phi^+,$$

or equivalently, by taking the negation and the complement, $r^{-1}\Phi^+ \subset \Phi^+ \cup -\Phi_x$.

Remark 7. If x lies in a shrunken Weyl chamber then $\Phi_x = \emptyset$, so $W_x = \{\mathbf{I}_{W_0}\}$. For x lying in exactly one critical strip, we have $\Phi_x = \{\alpha_x\}$ for some unique $\alpha_x \in \Phi^+$. In fact, it should be a simple root by Theorem 2.4.1. Hence,

$$W_x = \{r \in W_0 : r^{-1}\Phi^+ \subset \Phi^+ \cup \{-\alpha_x\}\} = \{id, s_x\}$$

where s_x is the simple reflection corresponding to α_x.

We note that W_x is not necessarily a subgroup, but the following lemma suggests some structure on W_x.

Lemma 2.3.6. *The set W_x is left-closed in the sense that if $w \in W_x$ and $\ell(sw) < \ell(w)$ for a simple reflection s, then $sw \in W_x$.*

Proof. We know that w can be written as sw' where $\ell(sw') = \ell(w') + 1$. Then $w'^{-1}\Phi^+ = w^{-1}s(\Phi^+) = w^{-1}(\Phi^+ \cup \{-\alpha_s\} \setminus \{\alpha_s\}) = w^{-1}\Phi^+ \cup \{-w\alpha_s\} \setminus \{w\alpha_s\}$. However, as $w^{-1}s < w^{-1}$, $-w^{-1}\alpha_s \in \Phi^+$ so that $w^{-1}\Phi^+ \cup \{-w\alpha_s\} \setminus \{w\alpha_s\} \subset w^{-1}\Phi^+ \cup \Phi^+ \subset \Phi^+ \cup -\Phi_x$ as $w \in W_x$. □

2.3.3 Lemma 2.2.14 in general

As before, b is basic. We will express x as $v_x t^{\mu_x} w_x$ where $v_x \in W_0$ is the unique element such that $v_x^{-1}x$ is in the dominant Weyl chamber.

Lemma 2.3.7. *If x is a $(J, w)_\sigma$-alcove for a σ-stable subset J of $\mathbb{S}$, then $\sigma^{-1}(r)\eta_\sigma(x)r^{-1} \in W_J$ for some $r \in W_x$.*

Similarly as before, we get the following corollary:

Corollary 2.3.8. *Let x be an arbitrary element in $\widetilde{W}$ and $b \in \breve{G}$ such that $\kappa(x) = \kappa(b)$. If $\mathrm{supp}_\sigma(\sigma^{-1}(r)\eta_\sigma(x)r^{-1}) = \mathbb{S}$ for all $r \in W_x$, then $X_x(b) \neq \emptyset$.*

Proof of lemma 2.3.7. Suppose that there exists $a \in w(\Phi^+ \setminus \Phi_J^+)$ such that $v_x^{-1}a \in -(\Phi^+ \setminus \Phi_x)$. Then, by definition of Φ_x, $-a$ satisfies $k(-a, x) \neq k(-a, \mathbf{I})$, that is, $k(a, x) \neq k(a, \mathbf{I})$. Hence, $k(v_x^{-1}a, t^{\mu_x}w_x) \gneq k(a, \mathbf{I}) \geq -1$ and so $v_x^{-1}a$ should be a positive root as $t^{\mu_x}w_x$ is in the dominant Weyl chamber. This contradicts to the assumption. Hence, $v_x^{-1}a \in \Phi^+ \cup -\Phi_x$.

This implies that $v_x^{-1}w(\Phi^+ \setminus \Phi_J^+) \subset \Phi^+ \cup -\Phi_x$, or equivalently, $\Phi^+ \setminus \Phi_x \subset v_x^{-1}w(\Phi^+ \cup \Phi_J^-)$. By proposition 2.3.4, $\Psi_r := \Phi^+ \setminus \Phi_x$ and $\Psi_p := v_x^{-1}w(\Phi^+ \cup \Phi_J^-)$ fit into the assumptions in proposition 2.3.3 so that there exists $r \in W_0$ such that

$$r(\Phi^+ \setminus \Phi_x) \subset \Phi^+ \subset rv_x^{-1}w(\Phi^+ \cup \Phi_J^-).$$

Taking the complement and the negation, we get $rv_x^{-1}w(\Phi^+ \setminus \Phi_J^+) \subset \Phi^+$ for some $r \in W_x$ by definition 2.3.5. Therefore, $rv_x^{-1}w \in W_J$ for some $r \in W_x$ and consequently $\sigma(w^{-1})\sigma(v_x)\sigma(r^{-1}) \in W_J$ as J is σ-stable. Moreover, $w^{-1}v_x w_x \sigma(w) \in W_J$ since x is a $(J, w)_\sigma$-alcove. Multiplying them together, we get $rw_x\sigma(v_x)\sigma(r^{-1}) \in W_J$ so that $\sigma^{-1}(r)\eta_\sigma(x)r^{-1} \in W_J$. □

2.3.4 Lemma 2.2.12 under some restriction

For $r \in W_x$, let us denote by $J_{r,x}$ the σ-stable subset $\mathrm{supp}_\sigma(\sigma^{-1}(r)\eta_\sigma(x)r^{-1})$.

Lemma 2.3.9. *x is a $(J_{r,x}, v_x r^{-1})_\sigma$-alcove.*

Proof. The first condition of definition 2.2.8 can be checked easily because the finite part is $rv_x^{-1}v_x w_x \sigma(v_x r^{-1}) \in W_{J_{x,r}}$. For definition 2.2.8 (2), we need to compare $k(v_x r^{-1}\alpha, v_x t^{\mu_x} w_x)$ and $k(v_x r^{-1}\alpha, \mathbf{I})$ for all $\alpha \in \Phi^+ \setminus \Phi_{J_{x,r}}^+$. The first one is $k(r^{-1}\alpha, t^{\mu_x}w_x)$ which is ≥ 0 if $r^{-1}\alpha \in \Phi^+$ and $0 \geq k(v_x r^{-1}\alpha, \mathbf{I})$ always. Hence, we only need to consider the case $r^{-1}\alpha \in \Phi^-$. However, $r(\Phi^+ \setminus \Phi_x) \subset \Phi^+$ implies (again by the negation complement) $r^{-1}\Phi^+ \subset \Phi^+ \cup -\Phi_x$. Therefore, we have $r^{-1}\alpha \in -\Phi_x$. By the definition of Φ_x, we have the same k-values. □

Proposition 2.3.10. *Let $x \in \widetilde{W}$ be an element with $\ell(x) \gg 0$. If $J_{r,x} \neq \mathbb{S}$ for some $r \in W_x$, then $X_x(b) = \emptyset$.*

Proof. By the previous lemma, x is a $(J_{r,x}, vr^{-1})_\sigma$-alcove so [GHN15, Proposition 3.6.4] tells us that

$$\nu_{r\cdot\mu_x} \in Q^\vee_{J_{r,x},\mathbb{Q}}.$$

Note that, inductively computing, when $r = s_{i_m}\cdots s_{i_1}$,

$$r\cdot\mu_x = \mu_x - \sum_{j=1}^{k}\langle s_{i_1}\cdots s_{i_{j-1}}\alpha_{i_j}, \mu_x\rangle\alpha^\vee_{i_j} \tag{2.4}$$

so that

$$\nu_{r\cdot\mu_x} = \nu_{\mu_x} - \nu_{\langle\alpha_1,\mu_x\rangle\alpha_1^\vee+\cdots+\langle s_1\cdots s_{k-1}\alpha_k,\mu_x\rangle\alpha_k^\vee}.$$

Assume that the expression of r is reduced, then $s_{i_1}\cdots s_{i_{j-1}}\alpha_{i_j}\in\Phi^+$. Moreover, $rs_{i_1}\cdots s_{i_{j-1}}\alpha_{i_j}\in\Phi^-$. Hence, $s_{i_1}\cdots s_{i_{j-1}}\alpha_{i_j}\in\Phi^+\cap r^{-1}\Phi^-\subset\Phi_x$ implying that $\langle s_{i_1}\cdots s_{i_{j-1}}\alpha_{i_j},\mu_x\rangle = 0$ or 1. Therefore, the σ-average of the sum of coroots on the right hand side is bounded. However, for x such that $\ell(x)\gg 0$, if $\mu_x = \sum_i c_i\varpi_i^\vee$, then $\sum_i c_i \gg 0$ and, by lemma 2.2.18, the coefficients in the linear combination of coroots are large enough so that $\nu_{\mu_x} - \nu_{\sum_{j=1}^k\langle s_{i_1}\cdots s_{i_{j-1}}\alpha_{i_j},\mu_x\rangle\alpha^\vee_{i_j}}$ has all positive coefficients. This is a contradiction. □

Combining proposition 2.3.10 and corollary 2.3.8, we get

Theorem 2.3.11. *Let $b\in B(G)$ be basic and suppose that $x\in\widetilde{W}$ satisfies $\kappa(b)=\kappa(x)$ and $\ell(x)\gg 0$. Then*

$$X_x(b)\neq\emptyset \text{ if and only if } \operatorname{supp}_\sigma(\sigma^{-1}(r)\eta_\sigma(x)r^{-1}) = \mathbb{S} \text{ for all } r\in W_x.$$

Proof of Theorem 2.1.2. Note that $\widetilde{\operatorname{supp}_\sigma}(x) = \widetilde{\mathbb{S}}$ condition in theorem 2.1.2 (1) is satisfied for x such that $\ell(x)\gg 0$. As there are only finitely many x with $\ell(x)$ bounded by some number, conjecture 1 holds. □

Proof of Theorem 2.1.2 (2). If $x = v_x t^{\mu_x} w_x$ is a translation element, we have $w_x = v_x^{-1}$. In this case, Φ_x consists of $\alpha\in\Phi^+$ such that $0\le\langle\alpha,\mu_x\rangle+\delta_{v_x\alpha} = \delta_{v_x\alpha}$ so that $\langle\alpha,\mu_x\rangle = 0$ and $v_x\alpha\in\Phi^+$ for all $\alpha\in\Phi_x$. For $x = v_x t^{\mu_x}$, we have $w_x = \mathbf{I}$ so that $\alpha\in\Phi_x$ satisfies $\langle\alpha,\mu_x\rangle+\delta_\alpha = \delta_{v_x\alpha}$ so that $\langle\alpha,\mu_x\rangle = 0$ and $v_x\alpha\in\Phi^+$ as well.

Hence, in both cases, we do not have $\alpha\in\Phi_x$ such that $\langle\alpha,\mu_x\rangle = 1$. Therefore, in the proof of proposition 2.3.10, all the terms $\langle s_{i_1}\cdots s_{i_{j-1}}\alpha_{i_j},\mu_x\rangle\alpha^\vee_{i_j}$ are zeros. Therefore, $\nu_{r\cdot\mu_x}$ is dominant still so that $J_{r,x}\neq\mathbb{S}$ is a contradiction assuming $X_x(b)\neq\emptyset$. □

Proof of Theorem 2.1.3. By lemma 2.3.7 and lemma 2.3.9, x is a $(J,w)_\sigma$-alcove if and only if $J = J_{r,x}$. Theorem 2.3.11 implies that, for $\ell(x)\gg 0$, $X_x(b)\neq\emptyset$ if and only if $J_{r,x} = \mathbb{S}$ for all $r\in W_x$. □

2.4 One critical strip

2.4.1 Finding a simple root

We will prove the following:

Theorem 2.4.1. *Let $b \in B(G)$ be basic and $x \in \widetilde{W}$ lie in exactly one critical strip C_a for some $a \in \Phi$. Then $v_x^{-1}a$ is a simple root and $X_x(b) \neq \emptyset$ if and only if $\kappa_G(b) = \kappa_G(x)$ and*

$$\text{both } \operatorname{supp}_\sigma(\eta_\sigma(x)) \text{ and } \operatorname{supp}_\sigma(\sigma^{-1}(s_x)\eta_\sigma(x)s_x) \text{ are } \mathbb{S},$$

where s_x is the simple reflection corresponding to $v_x^{-1}a$.

This is slightly stronger than Theorem 2.1.2 (1). More precisely, compared to Theorem 2.1.2 (1), Theorem 2.4.1 has additional claims that $v_x^{-1}a$ is a simple root and has no restriction '$\widetilde{\operatorname{supp}_\sigma}(x) = \widetilde{\mathbb{S}}$'. In fact, one can easily show that $v_x^{-1}a$ must be a simple root (lemma 2.4.2) from proposition 2.3.4. The absence of the restriction $\widetilde{\operatorname{supp}_\sigma}(x) = \widetilde{\mathbb{S}}$ is the main reason why Theorem 2.4.1 is stronger. It is explained in lemma 2.4.6.

Lemma 2.4.2. *Suppose that x belongs to exactly one critical strip associated to a. Then, $v_x^{-1}a$ is a simple root.*

Proof. By proposition 2.3.4, Φ_x is a singleton such that $\Phi^+ \setminus \Phi_x$ is closed. As simple roots generate Φ^+, we know Φ_x should contain a simple root. $\square$

We will denote by α_x the unique positive simple root such that $x \in C_{v_x\alpha_x}$.

2.4.2 Lemma 2.2.12 for one strip

Here, we prove that if $\operatorname{supp}_\sigma(\eta_\sigma(x)) \neq \mathbb{S}$ or $\operatorname{supp}_\sigma(s_x\eta_\sigma(x)\sigma(s_x)) \neq \mathbb{S}$, then $X_x(b) = \emptyset$.

Proposition 2.4.3. *Under the assumption of theorem 2.4.1, if $\operatorname{supp}_\sigma(\eta_\sigma(x)) \neq \mathbb{S}$ then $X_x(b) = \emptyset$.*

Proof. By Lemma 2.2.12, it is enough to show that $\nu_{\mu_x} \neq \bar{\nu}_b$. We follow the strategy of [GHN15, Proposition 3.6.5] here.

Let n be the order of $W_0 \rtimes \langle\sigma\rangle$ and suppose that ν_{μ_x} is central. Then $0 = \langle \nu_{\mu_x}, \beta\rangle = \frac{1}{n}\langle \mu_x + \sigma(\mu_x) + \cdots + \sigma^{n-1}(\mu_x), \beta\rangle$, but $\sigma^i(\mu_x)$ are all dominant. Hence, if β is positive then $\langle \mu_x, \beta\rangle = 0$. Consider the case β being the maximal root, we get μ_x central. This implies that $w_x = \mathbf{I}$ and $x = v_x t^{\mu_x}$. Note that $\eta_\sigma(x) = v_x$.

As we assume that there is only one critical strip containing x, it means that the number of $a \in \Phi^+$ such that $k(a, v_x t^{\mu_x}) = k(a, \mathbf{I}) = 0$ is 1. But, $k(a, v_x t^{\mu_x}) = \delta_{v_x^{-1}a}$. It means that $|\Phi^+ \cap v_x\Phi^+| = |\Phi^+| - \ell(v_x) = 1$ so that $\ell(v_x) = |\Phi^+| - 1$. However, if $\ell(v_x) \leq |\Phi^+_{\operatorname{supp}_\sigma(v_x)}| < |\Phi^+| - 1$. $\square$

Now we consider the case $\operatorname{supp}_\sigma(\sigma^{-1}(s_x)\eta_\sigma(x)s_x) \neq \mathbb{S}$.

Lemma 2.4.4. *Under the assumption of theorem 2.4.1, denote by J_x the set* $\operatorname{supp}_\sigma(\sigma^{-1}(s_x)\eta_\sigma(x)s_x)$. *Then x is a $(J_x, v_x s_x)_\sigma$-alcove.*

Proof. We have $\Phi_x = \{\alpha_x\}$, so $s_x \in W_x$ because $s_x\Phi^+ = \Phi^+ \cup \{-\alpha_x\} \setminus \{\alpha_x\}$. Now, the conclusion follows from lemma 2.3.9. $\square$

Proposition 2.4.5. *If $J_x \neq \mathbb{S}$ in lemma 2.4.4, then $X_x(b) = \emptyset$.*

Proof. x is a $(J_x, v_x s_x)_\sigma$-alcove. Hence, it is enough to prove that

$$\nu_{\lambda'} - \overline{\nu}_b \in Q^\vee_{J_x} \otimes \mathbb{Q}$$

leads to a contradiction where $\lambda' := s_x\mu_x$ following the proof of [GHN15, Proposition 3.6.4]. However, $\lambda' = s_x\mu_x$ is either $\mu_x - \alpha_x^\vee$ or μ_x. The latter case is already proved[12] in *loc.cit.*.

Now, let us consider the case $s_x\mu_x = \mu_x - \alpha_x^\vee$. We have

$$\nu_{s_x\mu_x} - \bar{\nu}_b \in Q^\vee_{J_x,\mathbb{Q}}.$$

Let $\varpi_i^\vee$ be the fundamental coweights and ϖ_i be the fundamental weights. For simplicity, denote by $\mathcal{O}$ the σ-orbit of s_x and by $\alpha_{\mathcal{O}}$ (resp. $\varpi_{\mathcal{O}}$ or $\varpi_{\mathcal{O}}^\vee$) the sum of the elements in the σ-orbit of α_x (resp. $\varpi_{\mathcal{O}}$ or $\varpi_{\mathcal{O}}^\vee$).

Taking the inner product with $\varpi_{\mathcal{O}}$, we get 0 on the right hand side. On the left hand side, we have $\langle \varpi_{\mathcal{O}}, s_x\mu_x\rangle$. It is $\langle s_x\varpi_{\mathcal{O}}, \mu_x\rangle = \langle \varpi_{\mathcal{O}} - \alpha_x, \mu_x\rangle = \langle \varpi_{\mathcal{O}}, \mu_x\rangle - 1$. However, if $\mu_x = \sum c_i\varpi_i^\vee$ where $c_i \in \mathbb{Z}_{\geq 0}$.

Let D be the connected component containing α_x. If we restrict to D of the value $\langle \varpi_{\mathcal{O}}, \mu_x\rangle$, we have $\langle \varpi_x, \varpi_x^\vee + \sum_{i\neq x} c_i\varpi_i^\vee\rangle$. Note that $\langle \varpi_i, \varpi_j^\vee\rangle$ is the (i,j)-entry of the inverse of the Cartan matrix of D which are all positive. In most cases, this already exceeds 1 by [OV90, Reference Chapter. §2. Table 2]. Note that $\langle \varpi_x, \varpi_x^\vee\rangle$ is the corresponding entry on the diagonal of the inverse matrix.

There are two cases left in large: $\langle \varpi_x, \varpi_x^\vee\rangle = 1$ or $\langle \varpi_x, \varpi_x^\vee\rangle < 1$. The first case is when x is the unique vertex of degree=1 (with a single edge) in $B_{\geq}2$, $C_{\geq}2$, $D_{\geq}4$ or the middle vertex of $A_3 = D_3$. The second case is when x is one of two degree=1 vertices in A_n. As they contain a little tedious computation, we put it off to section 2.5.1. $\square$

Combining proposition 2.4.3 and proposition 2.4.5, we get

$$X_x(b) \neq \emptyset \text{ only if } \operatorname{supp}_\sigma(\eta_\sigma(x)) = \operatorname{supp}_\sigma(\sigma^{-1}(s_x)\eta_\sigma(x)s_x) = \mathbb{S}.$$

Proof of theorem 2.4.1. Note again that $\Phi_x = \{\alpha_x\}$ so that $W_x = \{\mathbf{I}_{W_0}, s_x\}$. Now, combine corollary 2.3.8 and section 2.4.2. $\square$

[12] Note that, as mentioned before, the proof in *loc.cit.* is incorrect when referring to 'Section 3.5 (2)' and you need to use lemma 2.2.17 instead.

2.4.3 On the condition $\widetilde{\mathrm{supp}_\sigma}(x) = \widetilde{\mathbb{S}}$

Finally, the following lemma explains why the assumption $\widetilde{\mathrm{supp}_\sigma}(x) = \widetilde{\mathbb{S}}$ is not necessary in Theorem 2.1.2 (1). Note that $\widetilde{\mathrm{supp}_\sigma}(x) \neq \widetilde{\mathbb{S}}$ implies that ν_x is central.

Lemma 2.4.6. *Let x be in exactly one critical strip and suppose ν_x is central. Then both $\mathrm{supp}_\sigma(\eta_\sigma(x))$ and $\mathrm{supp}_\sigma(\sigma^{-1}(s_x)\eta_\sigma(x)s_x)$ are $\mathbb{S}$.*

Proof. Note that $\nu_x = v_x \cdot \nu_{v_x^{-1}x\sigma(v_x)}$ and $v_x^{-1}x\sigma(v_x) = t^{\mu_x}w_x\sigma(v_x) = t^{\mu_x}\sigma(\eta_\sigma(x))$[13]. Hence, $\nu_{t^{\mu_x}\sigma(\eta_\sigma(x))}$ is also central. On the other hand, similarly to corollary 2.2.16, we have $\nu_{\mu_x} - \nu_{t^{\mu_x}\sigma(\eta_\sigma(x))} \in Q^\vee_{\mathrm{supp}_\sigma(\sigma(\eta_\sigma(x))),\mathbb{Q}}$ by applying lemma 2.2.15. As μ_x is dominant, ν_{μ_x} is dominant. If μ_x is non-central, then ν_{μ_x} is also non-central and so lemma 2.2.17 enforces $\mathrm{supp}_\sigma(\sigma(\eta_\sigma(x))) = \mathbb{S}$. Similarly as above, we have

$$\nu_{s_x\cdot\mu_x} - \nu_{t^{s_x\cdot\mu_x}s_xw_x\sigma(v_x)\sigma(s_x)} \in Q^\vee_{\mathrm{supp}_\sigma(s_x\sigma(\eta_\sigma(x))\sigma(s_x)),\mathbb{Q}}.$$

However, $t^{s_x\cdot\mu_x}s_xw_x\sigma(v_x)\sigma(s_x) = (v_xs_x)^{-1}x\sigma(v_xs_x)$, but ν_x is central so the second term is central. We can now repeat the proof of proposition 2.4.5.

When μ_x is central, we have $x = v_xt^{\mu_x}$. We can repeat proposition 2.4.3 to show that $\mathrm{supp}_\sigma(\eta_\sigma(x)) = \mathbb{S}$. Moreover, the one critical strip assumption tells us that $w_0v_x^{-1}$ is a simple reflection s (corresponding to $\alpha \in \Phi^+$). Then, $\alpha_x = v_x^{-1}\alpha$ and so $v_x\alpha_x \in \Phi^+$, that is, $\ell(v_xs_x) = \ell(v_x) + 1 = \ell(w_0)$. Hence, $v_x = w_0s_x$ and $\sigma^{-1}(s_x)\eta_\sigma(x)s_x = \sigma^{-1}(s_x)w_0$ whose support is $\mathbb{S}$. □

2.5 Some computations and applications

In this section, we show computations for possibly exceptional cases (mentioned in proposition 2.4.5) when x belongs to exactly one critical strip. Moreover, we have some remarks on type A_n case and we prove Theorem 2.1.4.

2.5.1 Completion of the proof for the one critical strip case

Proof of proposition 2.4.5 (cont'd). We separate into two cases. (Note that we use [OV90, Reference Chapter. §2. Table 2] here.)

Case 1-1. x is the vertex of degree 1 (with a single edge) in $B_{\neq 2}, C_{\neq 2}, D_{\geq 4}$.

$\langle \varpi_x, \varpi_x^\vee \rangle$ is already 1. So, c_i's are all zero, that is, $\mu_x = \varpi_x^\vee$. For simplicity, we use the notation $*_1$ instead of $*_x$ where $* = s, \alpha, \varpi^\vee$ in this paragraph. For any positive root α with no support at α_1, we have $x \notin C_{v_x\alpha}$ and $\langle \alpha, \varpi_1^\vee \rangle = 0$. So we have $v_x\alpha \in \Phi^-$ and $w_x^{-1}\alpha \in \Phi^+$ by lemma 2.2.5. As $x \in C_{v_x\alpha_1}$ and $\langle \alpha_1, \varpi_1^\vee \rangle = 1$, we have $v_x\alpha_1 \in \Phi^+$ and $w_x^{-1}\alpha_1 \in \Phi^-$. Note that $s_2\alpha_1 = \alpha_1 + \alpha_2$ is different from α_1 and in Φ^+. Moreover, $\langle s_2\alpha_1, \varpi_1^\vee \rangle = 1$. Due to the uniqueness assumption of critical strips, either $v_x(s_2\alpha_1) \in \Phi^-$ or $w_x^{-1}(s_2\alpha_1) \in \Phi^+$ by lemma 2.2.5.

[13] Caution. This is not necessarily in the dominant Weyl chamber.

1. The latter case: $w_x^{-1}(\alpha_1+\alpha_2) \in \Phi^+$ so that $w_x^{-1}s_1\alpha \in \Phi^+$ for all positive simple roots α. Hence, $w_x = s_1$ so that $\sigma^{-1}(s_1)\eta_\sigma(x)s_1 = vs_1$. As $J_x = \mathbb{S} \setminus \mathcal{O}$, $v_x \in W_{\mathbb{S}\setminus\mathcal{O}}s_1$. However, $\Phi^+ \ni v_x\alpha_1 \in W_{\mathbb{S}\setminus\mathcal{O}}(-\alpha_1) \subset \Phi^-$ as $1 \in \mathcal{O}$. Contradiction.

2. The first case: Apply the same argument to w_0v_x instead of w_x^{-1} and get $v_x = w_0s_1$. Then $\sigma^{-1}(s_1)\eta_\sigma(x)s_1 = \sigma^{-1}(s_1w_x)w_0$. Consider $\sigma(w_0)w_x^{-1}s_1\alpha_1$ which is in Φ^- by above, but $\sigma(w_0)w_x^{-1}s_1 \in W_{\mathbb{S}\setminus\mathcal{O}}$ so it should be a positive root which is a contradiction.

Case 1-2. x is the middle point in A_3.

$\mu_x = \varpi_2^\vee$. Similarly as before, $v_x\alpha_2 \in \Phi^+$ and $w_x^{-1}\alpha_2 \in \Phi^-$. It means that $w_x \in s_2W_{\{1,3\}}$ or $w_x = s_2s_1s_3s_2$. Hence, $s_2w_x \in W_{\{1,3\}}$ or $s_2w_x = s_1s_3s_2$ and $\sigma^{-1}(s_2)\eta_\sigma(x)s_2 = \sigma^{-1}(s_2w_x)v_xs_2$. As it belongs to $W_{\mathbb{S}\setminus\mathcal{O}}$, we have $v_x \in W_{\mathbb{S}\setminus\mathcal{O}}s_2$ so that $v_x\alpha_2 \in \Phi^-$ which is a contradiction or $v_x \in s_2W_{\mathbb{S}\setminus\mathcal{O}}s_2$. In the latter case, note that v_x is supported at s_1 because $v_x\alpha_1 \in \Phi^-$ and so $v_x\alpha_2 \in \Phi^-$ from $v_x \in s_2W_{\mathbb{S}\setminus\mathcal{O}}s_2$.

Case 1-3. x is the vertex with an outward arrow in $B_2 = C_2$.

May assume B_2 and $\mu_x = \varpi_1^\vee$. This is same as *Case 1-1* as $s_2\alpha_1 = \alpha_1 + \alpha_2$.

Case 2. x is a vertex of degree 1 in A_n.

As the sum is 1, $D \cap \mathcal{O} = \{x\}$ necessarily and $\mu_x = \varpi_x^\vee + \varpi_y^\vee$ where y is a vertex (possibly in another connected component) of degree 1 not in $\mathcal{O}$.

For simplicity, we will call D' the connected component containing $\varpi_y^\vee$ and use $\varpi_{1,D}^\vee$ instead of $\varpi_x^\vee$ and $\varpi_{n,D'}^\vee$ instead of $\varpi_y^\vee$. Moreover, σ does not shuffle 1 and n in (distinct) connected components.

1. The case $D \neq D'$: The proof is exactly the same as *Case 1-1*.

2. The case $D = D'$: $\mu_x = \varpi_1^\vee + \varpi_n^\vee$ in the same connected component. The proof is similar. In a similar way, we have $v_x\alpha \in \Phi^-$ and $w_x^{-1}\alpha \in \Phi^+$ for α not supported at both α_1 and α_n. For $\alpha = \alpha_1$, we have the opposite situation. For the remaining case, it should not be the opposite situation.

   The method is essentially the same. Note that the goal in the above was to show that $w_x = s_1$ or $v_x = w_0s_1$. Here, the goal is to show that $w_x = s_1 \cdots s_m$ for some $m \leq n$ or $v_x = w_0s_m \cdots s_1$ for some $m \leq n$. Using that the roots are of the form $\alpha_i + \alpha_{i+1} + \cdots + \alpha_j$, one can prove that. We skip the proof.

□

2.5.2 Sharper bounds for Theorem 2.1.2 (3)

Denote by $Q_{>0}^\vee$ the set of positive $\mathbb{Q}$-linear sums of all simple coroots. Note that $\nu_{\mu_x} \in Q_{>0}^\vee$ unless μ_x is central.

Suppose that $r \in W_x$ and s is a simple reflection such that $sr \in W_x$ and $sr > r$. Considering eq. (2.4), we know that $sr \cdot \mu = r \cdot \mu - \langle r^{-1}\alpha, \mu\rangle\alpha^\vee$ where α is the simple positive root corresponding to s. Moreover, as $sr \in W_x$, $\langle r^{-1}\alpha, \mu\rangle = 0$ or 1. Hence, as long as $sr \in W_x$ (assuming $sr > r$), we know that $sr \cdot \mu$ is the same as $r \cdot \mu$ or $r \cdot \mu - \alpha^\vee$.

Therefore, if $\nu_{r'\cdot\mu_x} \in Q^\vee_{J,\mathbb{Q}}$ for some $J \subsetneq \mathbb{S}$ and $r' \in W_x$, we can find a minimal $r_0 \in W_0$ such that $\ell(r'r_0^{-1}) + \ell(r_0) = \ell(r')$ (so that $r_0 \in W_x$ by lemma 2.3.6) and $r_0 \cdot \mu \notin Q^\vee_{>0}$ in the sense that $sr_0 \cdot \mu \in Q^\vee_{>0}$ for any simple reflection s such that $sr_0 < r_0$. Note that it is not necessarily unique and we make any choice.

The minimality assumption tells us that any reduced expression $s_{i_k} \cdots s_{i_1}$ of r_0 has the same s_{i_k} (the first simple reflection should be constant). Moreover, $\langle s_{i_1} \cdots s_{i_{k-1}}\alpha_{i_k}, \mu_x\rangle = 1$ considering eq. (2.4). Finally, $\mathrm{supp}(r_0)$ should be connected. Now, the precise statement of Theorem 2.1.2 (3) is the following.

Proposition 2.5.1. *In type A_n, suppose that $\mu_x \in Q^\vee$ and assume $\langle \varpi_1, \mu_x\rangle > 1$, $\langle \varpi_n, \mu_x\rangle > 1$. Then Conjecture 1 holds.*

Proof. Let r_0 be a minimal element chosen before and m be the unique number such that $r_0\alpha_m < 0$. Similar to proposition 2.3.10, it is enough to show that $r_0 \cdot \mu_x \in Q^\vee_{\neq\mathbb{S},\mathbb{Q}}$ is a contradiction. Note that $\mu_x \in Q^\vee$, so we can assume $r_0 \cdot \mu_x \in Q^\vee_{\neq\mathbb{S}}$. Due to the minimality of r_0, we have that $\mathbb{S} \setminus \mathrm{supp}(r_0 \cdot \mu_x) = \{m\}$. Moreover, for each $i \in \mathrm{supp}(r_0 \cdot \mu_x)$, $\alpha_i^\vee$-coefficient of $r_0 \cdot \mu_x$ is a positive integer. Therefore, if $m \neq 1$ or n, we have $\langle \alpha_m, r_0 \cdot \mu_x\rangle \leq -2$. However, $r_0^{-1}\alpha_m \in -\Phi_x$ so that $\langle r_0^{-1}\alpha_m, \mu_x\rangle = -1$ which is a contradiction.

When $m = 1$ (resp. n), note that $r_0 \cdot \mu_x \in Q^\vee_J$ (where $J = \mathbb{S} \setminus \{m\}$) is weakly dominant in the sense of [Nie18, Proposition 3.1]. Therefore, $r_0 \cdot \mu$ and μ fits into the setting in [Nie18, Lemma 5.9] so that $\mu - r_0 \cdot \mu$ is the sum of some positive roots orthogonal to each other. However, no two positive roots containing α_1 (resp. α_n) can be orthogonal. This implies that the $\alpha_1^\vee$(resp. $\alpha_n^\vee$)-coefficient of μ is either that of $r_0 \cdot \mu$ (which is 0) or one larger than that. So it must be 1. This contradicts to the assumption. □

We remark that the above proof works for D_n and E_n types but need one more assumption that $\langle \varpi_{n-1}, \mu_x\rangle > 1$ where $n-1$ is the vertex of degree 1 other than the vertices 1 and n.

2.5.3 Application: Cordial elements and generic σ-conjugacy class

We will summarize some related facts first. For more details, we refer to [MV20].

Given $x \in \widetilde{W}$, let $B(G)_x$ be the set of $[b] \in B(G)$ such that $I\dot{x}I \cap [b] \neq \emptyset$. Then, $B(G)_x$ contains a unique maximal element called generic σ-conjugacy class and denoted by $[b_x]$.

Definition 2.5.2 (Cordial element). *$x \in \widetilde{W}$ is called cordial if*

$$\ell(x) - \ell(\eta_\sigma(x)) = \langle 2\rho, \nu_x\rangle - \mathrm{def}(b_x),$$

where 2ρ is the sum of all positive coroots and $\mathrm{def}(b_x)$ is the difference between the rank of G and J_{b_x}, the σ-centralizer of b_x in $\breve{G}$.

As an example, x in the antidominant Weyl chamber is cordial by *loc.cit.* Theorem 1.2.

Lemma 2.5.3. *Let x be a cordial element. Then, $B(G)_x$ is saturated in the following sense:*

> *Suppose that $[b_1]$, $[b_2] \in B(G)_x$ satisfy $[b_1] \leq [b_2]$. Then, for any $[b] \in B(G)$ satisfying $[b_1] \leq [b] \leq [b_2]$, we have $[b] \in B(G)_x$.*

Here, $\leq$ is the partial ordering defined in $B(G)$.

Hence, if the minimal and maximal elements are known, we have the full description of $B(G)_x$ which implies the complete classification of the nonemptiness of $X_x(b)$ for a fixed x.

Proof of Theorem 2.1.4. By [He21, Proposition 4.2], $x = vt^\mu$ for a dominant μ is cordial and the generic σ-conjugacy class is $[t^\mu]$. By lemma 2.5.3, we only need to describe the minimal element in $B(G)_x$.

For a central μ, we note that $[t^\mu]$ is the minimal among the elements whose image under the Kottwitz map is $\kappa_G(t^\mu)$. As it is minimal and maximal at the same time, $B(G)_x = \{[t^\mu]\} = B(G,\mu)$.

Let μ be non-central. We proved $\Phi_x = \{\alpha \in \Phi^+ : \langle \alpha, \mu \rangle = 0 \text{ and } v\alpha \in \Phi^+\}$ in the last part of section 2.3.4. Hence, μ is fixed by the reflection with respect to H_α for $\alpha \in \Phi_x$. Now, we get

$$\begin{aligned} W_x &= \{r = s_m \cdots s_1 \in W_0 : \alpha_1, s_1\alpha_2, \cdots, s_1 \cdots s_{m-1}\alpha_m \in \Phi_x\} \\ &= \{r \in W_0 : \operatorname{supp}(r) \subset W_0(\mu) \text{ and } v\alpha_1, \cdots, vs_1 \cdots s_{m-1}\alpha_m \in \Phi^+\} \\ &= \{r \in W_0(\mu) : \ell(vr^{-1}) = \ell(v) + \ell(r)\} \end{aligned}$$

Now, by Theorem 2.1.2 (2), $X_x(b_{\mathrm{b}}) \neq \emptyset$ for the basic element b_{b} satisfying $\kappa_G(x) = \kappa_G(b_{\mathrm{b}})$ if $\operatorname{supp}_\sigma(\sigma^{-1}(r)\eta_\sigma(x)r^{-1}) = \mathbb{S}$ for all $r \in W_x$. In such a case, by lemma 2.5.3, $B(G)_x = \{[b] \in B(G) : [b_{\mathrm{b}}] \leq [b] \leq [b_x] = [t^\mu]\} = \{[b] \in B(G) : [b] \leq [t^\mu]\} = B(G,\mu)$. $\square$

Chapter 3

Connected components of affine Deligne-Lusztig varieties

3.1 Introduction

3.1.1 Background

In [Rap05b], Rapoport introduced certain geometric objects called affine Deligne–Lusztig varieties (ADLVs), to study mod p reduction of Shimura varieties. Since then, ADLVs have played a prominent role in the geometric study of: Shimura varieties, Rapoport–Zink spaces, local Shimura varieties and moduli spaces of local shtukas. Moreover, results on connected components of affine Deligne–Lusztig varieties have found remarkable applications to Kottwitz' conjecture and Langlands–Rapoport conjecture, which describe mod p points of Shimura varieties in relation to L-functions, as part of the Langlands program (for more background on this, see for example [Kis17b]).

Although there have been many successful approaches [Vie08, CKV15, Nie18, HZ20a, Ham20, Nie21] to computing connected components of ADLVs in the past decade, as far as the authors know, the current article is the first one that approaches the problem using p-adic analytic geometry *à la Scholze*. As it turns out, the p-adic approach proves the most general case of Conjecture 2 of [He18] and gives a new and uniform proof to all previously known cases. More precisely, we use a combination of Scholze's theory of diamonds [Sch17], the theory of *kimberlites* due to the first author [Gle22b], the connectedness of p-adic period domains [GL22a], and the normality of the local models [AGLR22, GL22b] to compute the connected components of ADLVs. Just as diamonds are generalizations of rigid analytic spaces, kimberlites and prekimberlites are the v-sheaf-theoretic generalizations of formal schemes. Roughly speaking, they are diamondifications of formal schemes.

As is well-known to experts, affine Deligne–Lusztig varieties parametrize (at-p) isogeny classes on integral models of Shimura varieties. As an application of our main theorems, we deduce the isogeny lifting property for integral models for Shimura varieties at parahoric levels constructed in [KP18]. Moreover, we give a new CM lifting result on integral models for

Shimura varieties–which is a generalization of the classical Honda-Tate theory–for quasi-split groups at p at connected parahoric levels. This improves on previous CM lifting results, which were proved either assuming (1) $G_{\mathbb{Q}_p}$ residually split, or assuming (2) G unramified, or assuming that (3) the parahoric level is very special.

As a further application, we prove that the Newton strata of the integral models for Shimura varieties at parahoric level constructed in [PR21] satisfy p-adic uniformization, and that the Rapoport–Zink spaces considered *loc.cit.* agree with the moduli spaces of p-adic shtukas of [SW20] associated to the same data.

3.1.2 Notations

To not overload the introduction, we use common terms whose rigorous definitions we postpone till later (§3.2).

We denote by φ the lift of arithmetic Frobenius to $\breve{\mathbb{Q}}_p$. Let $\mathcal{I}$ and $\mathcal{K}_p$ be $\mathbb{Z}_p$-parahoric group schemes with common generic fiber a reductive group G. We let $K_p = \mathcal{K}_p(\mathbb{Z}_p)$, $\breve{\mathcal{I}} = \mathcal{I}(\breve{\mathbb{Z}}_p)$ and $\breve{K}_p := \mathcal{K}_p(\breve{\mathbb{Z}}_p)$. We require that $\mathcal{I}(\mathbb{Z}_p) \subseteq K_p$ and that $\mathcal{I}$ is an Iwahori subgroup of G.

Fix $S \subseteq G$, a $\mathbb{Q}_p$-torus that is maximally split over $\breve{\mathbb{Q}}_p$. Let $T = Z_G(S)$ be the centralizer of S, by Steinberg's theorem it is a maximal $\mathbb{Q}_p$-torus. Let $B \subseteq G_{\breve{\mathbb{Q}}_p}$ be a Borel containing $T_{\breve{\mathbb{Q}}_p}$, which may be defined only over $\breve{\mathbb{Q}}_p$. Let $\mu \in X_*^+(T)$ be a B-dominant cocharacter, and let $b \in G(\breve{\mathbb{Q}}_p)$. Let $\widetilde{W}$ be the Iwahori–Weyl group of G over $\breve{\mathbb{Q}}_p$. Let $\mathrm{Adm}(\mu) \subseteq \widetilde{W} = \breve{\mathcal{I}} \backslash G(\breve{\mathbb{Q}}_p)/\breve{\mathcal{I}}$ denote the μ-admissible set of Kottwitz–Rapoport [KR00].

The (closed) affine Deligne–Lusztig variety associated to (G, b, μ), denoted as $X_\mu(b)$, is a locally perfectly finitely presented $\bar{\mathbb{F}}_p$-scheme (see [Zhu17]), with $\bar{\mathbb{F}}_p$-valued points given by:

$$X_\mu(b) = \{g\breve{\mathcal{I}} \mid g^{-1}b\varphi(g) \in \breve{\mathcal{I}}\,\mathrm{Adm}(\mu)\breve{\mathcal{I}}\}. \tag{3.1}$$

By definition, $X_\mu(b)$ embeds into the Witt vector affine flag variety $\mathcal{F}\ell_{\breve{\mathcal{I}}}$, whose $\bar{\mathbb{F}}_p$-valued points are the cosets $G(\breve{\mathbb{Q}}_p)/\breve{\mathcal{I}}$. We also consider the $\mathcal{K}_p$-version $X_\mu^{\mathcal{K}_p}(b)$ with $\bar{\mathbb{F}}_p$-points:

$$X_\mu^{\mathcal{K}_p}(b) = \{g\breve{K}_p \mid g^{-1}b\varphi(g) \in \breve{K}_p\,\mathrm{Adm}(\mu)\breve{K}_p\}. \tag{3.2}$$

Let $\mathbf{b} \in B(G)$ be the φ-conjugacy class of b, and let $\boldsymbol{\mu}$ be the conjugacy class of μ. Assume $\mathbf{b}$ lies in the Kottwitz set $B(G, \boldsymbol{\mu})$. Let $\mu^\diamond \in X_*(T)_{\mathbb{Q}}^+$ denote the "twisted Galois average" of μ (see (3.17)), and let $\boldsymbol{\nu}_{\mathbf{b}} \in X_*(T)_{\mathbb{Q}}^+$ denote the dominant Newton point. Recall that $\mathbf{b} \in B(G, \boldsymbol{\mu})$ implies that $\mu^\diamond - \boldsymbol{\nu}_{\mathbf{b}}$ is a non-negative sum of simple positive coroots with rational coefficients. We say that $(\mathbf{b}, \boldsymbol{\mu})$ with $\mathbf{b} \in B(G, \boldsymbol{\mu})$ is *Hodge–Newton irreducible* (HN-irreducible) if all simple positive coroots have non-zero coefficient in $\mu^\diamond - \boldsymbol{\nu}_{\mathbf{b}}$.

Let Γ and I denote the Galois groups of $\mathbb{Q}_p$ and $\breve{\mathbb{Q}}_p$ respectively. Recall that the Kottwitz map [Kot97b, 7.4]

$$\kappa_G : G(\breve{\mathbb{Q}}_p) \to \pi_1(G)_I \tag{3.3}$$

induces bijections $\pi_0(\mathcal{F}\ell_{\breve{\mathcal{I}}}) \cong \pi_0(\mathcal{F}\ell_{\breve{K}_p}) \cong \pi_1(G)_I$. Moreover, it is known that the map induced by κ_G on connected components of ADLV,

$$\omega_G : \pi_0(X_\mu^{\mathcal{K}_p}(b)) \to \pi_1(G)_I, \tag{3.4}$$

factors surjectively onto $c_{b,\mu}\pi_1(G)_I^{\varphi} \subseteq \pi_1(G)_I$ for a unique coset element $c_{b,\mu} \in \pi_1(G)_I/\pi_1(G)_I^{\varphi}$ (see for example [HZ20a, Lemma 6.1]).

3.1.3 Main Results

In his ICM talk [He18], X. He underlines the study of connected components as an important open problem in the study of the geometric properties of ADLVs. Moreover, He suggests the following conjecture.

Conjecture 2. *If* $(\mathbf{b}, \boldsymbol{\mu})$ *is HN-irreducible, the following map is bijective*

$$\omega_G : \pi_0(X_\mu^{\mathcal{K}_p}(b)) \to c_{b,\mu}\pi_1(G)_I^{\varphi}$$

Our main theorem is the following (see Theorem 3.6.2).

Theorem 3.1.1. *For all p-adic shtuka datum* (G, b, μ) *and all parahoric subgroups* $\mathcal{K}_p \subseteq G(\mathbb{Q}_p)$*, Conjecture 2 holds.*

To state the applications to the geometry of Shimura varieties, we shall also need the following notations. Let $(\mathbf{G}, X)$ be a Shimura datum of Hodge type. Suppose $\mathbf{G}$ splits over a tamely ramified extension[1]. We shall always assume $p > 2$ and $p \nmid |\pi_1(\mathbf{G}^{\mathrm{der}})|$.[2] Let $K_p \subseteq \mathbf{G}(\mathbb{Q}_p)$ be a connected parahoric subgroup[3]. By [KP18][4], there is a normal integral model $\mathscr{S}_{\mathcal{K}_p}(\mathbf{G}, X)$, for the Shimura variety $\mathrm{Sh}_{\mathcal{K}_p}(\mathbf{G}, X)$.

The following Corollary 3.1.2 is the parahoric analogue of [Kis17b, Proposition 1.4.4] and can be obtained by combining our Theorem 3.1.1 with [Zho20, Proposition 6.5]. This is a generalization of existing results in literature to $\mathcal{K}_p$ arbitrary connected parahoric. Recall that to any $x \in \mathscr{S}_{\mathcal{K}_p}(\mathbf{G}, X)(\bar{\mathbb{F}}_p)$, one can associate a $b \in G(\breve{\mathbb{Q}}_p)$ as in [Kis17b, Lemma 1.1.12].

Corollary 3.1.2. *Let* $\mathcal{K}_p$ *be a connected parahoric. For any* $x \in \mathscr{S}_{\mathcal{K}_p}(\mathbf{G}, X)(\bar{\mathbb{F}}_p)$*, there exists a map of perfect schemes*

$$\iota_x : X_\mu^{\mathcal{K}_p}(b) \to \mathscr{S}^{perf}_{\mathcal{K}_p, \bar{\mathbb{F}}_p}(\mathbf{G}, X) \tag{3.5}$$

preserving crystalline tensors and equivariant with respect to the geometric r-Frobenius.

The following Corollary 3.1.3(1) (resp. Corollary 3.1.3(2)) is a parahoric analogue to [Kis17b, Proposition 2.1.3] (resp. [Kis17b, Theorem 2.2.3]) when $\mathbf{G}_{\mathbb{Q}_p}$ is quasi-split, and can be obtained by combining our Theorem 3.1.1 with [Zho20, Proposition 9.1] (resp. [Zho20, Theorem 9.4]). Notations as *loc.cit.*

[1] we expect that this condition can be relaxed using [KZ21].

[2] we expect the same results to hold for $p = 2$, using similar ideas from [KMP16], which only addressed the hyperspecial level integral models.

[3] Following [Zho20], we say that $\mathcal{K}_p$ is a connected parahoric if it agrees with the stabilizer group scheme of a facet in the Bruhat–Tits building.

[4] In [KP18], the authors construct parahoric integral models assuming that $\mathbf{G}_{\mathbb{Q}_p}$ splits over a tamely ramified extension. We expect that some of the technical conditions of our corollaries can be relaxed using the constructions in [KZ21] or in [PR21].

Corollary 3.1.3. *Let* $\mathbf{G}$ *be quasisplit at* p. *Let* $\mathcal{G} := \mathcal{K}_p$ *be a connected parahoric. Let* $k \subseteq \bar{\mathbb{F}}_p$ *be a finite field extension of* $\mathbb{F}_p$.

1. *the map* ι_x *in* (3.5) *induces an injective map*

$$\iota_x : I_x(\mathbb{Q})\backslash X_\mu^{\mathcal{K}_p}(b)(\bar{\mathbb{F}}_p) \times G(\mathbb{A}_f^p) \to \mathscr{S}_{\mathcal{K}_p}(G,X)(\bar{\mathbb{F}}_p), \tag{3.6}$$

where I_x *is a subgroup of the automorphism group of the abelian variety (base changed to* $\bar{\mathbb{F}}_p$*) associated to* x *fixing the Hodge tensors*[5].

2. *The isogeny class* $\iota_x(X_\mu^{\mathcal{K}_p}(b)(\bar{\mathbb{F}}_p)) \times \mathbf{G}(\mathbb{A}_f^p))$ *contains a point which lifts to a special point on* $\mathscr{S}_{\mathcal{K}_p}(G,X)$.

Theorem 3.1.1 together with [Zho20, Theorem 8.1(2)] finish the verification of the He–Rapoport axioms [HR17] for integral models of Shimura varieties.

Corollary 3.1.4. *The He–Rapoport axioms hold for* $\mathscr{S}_{\mathcal{K}_p}(\mathbf{G},X)$.

Moreover, we also obtain the following corollary by combining [HK19, Theorem 2] with Corollary 3.1.2, which allow us to verify Axiom A *loc.cit.*.

Corollary 3.1.5. *Let* $\mathbf{G}$ *be quasisplit at* p. *Let* $\mathcal{K}_p$ *be a connected parahoric. The "almost product structure" of the Newton strata in* $\mathscr{S}_{\mathcal{K}_p,\bar{\mathbb{F}}_p}(G,X)$ *holds.*

We refer the reader to [HK19, Theorem 2] for the precise formulation of the almost product structure of Newton strata.

As yet another corollary, we remove a technical assumption from the following theorem originally due to the third author [Xu21, Main Theorem][6].

Corollary 3.1.6. *Let* $\mathbf{G}$ *be quasisplit at* p. *Let* $\mathcal{K}_p$ *be a connected parahoric. The normalization step in the construction of* $\mathscr{S}_{\mathcal{K}_p}(\mathbf{G},X)$ *is unnecessary, and the closure model* $\mathscr{S}^-_{\mathcal{K}_p}(\mathbf{G},X)$ *is already normal. Therefore we obtain closed embeddings* $\mathscr{S}_{\mathcal{K}_p\mathcal{K}^p}(\mathbf{G},X) \hookrightarrow \mathscr{S}_{\mathcal{K}'_p\mathcal{K}'^p}(\mathrm{GSp},S^\pm)$. *As a further consequence, the analogous statement holds for toroidal compactifications of integral models, for suitably chosen cone decompositions.*

Also, we obtain the following corollary by combining Corollary 3.1.4 with [SYZ21, Theorem C]. See *loc.cit.* for the definition of EKOR strata.

Corollary 3.1.7. *Every EKOR stratum in* $\mathscr{S}_{\mathcal{K}_p}(\mathbf{G},X)_{\bar{\mathbb{F}}_p}$ *is quasi-affine.*

[5]As is standard in the theory of Shimura varieties, a Shimura variety of Hodge type carries a collection of Hodge tensors that "cut out" the reductive group $\mathbf{G}$. See 5 for more details

[6]The original version of this theorem is stated assuming $\mathbf{G}_{\mathbb{Q}_p}$ residually split for integral models at parahoric levels; at hyperspecial levels, this assumption is not necessary. We are now able to relax "$\mathbf{G}_{\mathbb{Q}_p}$ residually split" to "$\mathbf{G}_{\mathbb{Q}_p}$ quasi-split" thanks to our main theorem 3.1.1.

Remark 8. Our main theorem 3.1.1 is independent of the integral models of Shimura varieties that one works with. For this reason, we expect our main theorems to have similar applications as the above corollaries to the more general setup considered by Pappas–Rapoport [PR21].

Finally, we deduce that the Newton strata of the integral models of Shimura varieties considered by Pappas–Rapoport [PR21, Theorem 4.10.6] satisfy p-adic uniformization with respect to the local Shimura varieties $\mathcal{M}^{\mathrm{int}}_{\mathcal{G},b,\mu}$ of [SW20, Definition 25.1.1]. Let $(p, \mathbf{G}, X, \mathbf{K})$ be a tuple of global Hodge type [PR21, §1.3], let $\mathscr{S}_{\mathbf{K}}$ denote the integral model of [PR21, Theorem 1.3.2], let k an algebraically closed field in characteristic p and let $x_0 \in \mathscr{S}_{\mathbf{K}}(k)$. Pappas and Rapoport consider a map of v-sheaves $c : \mathrm{RZ}^{\diamondsuit}_{\mathcal{G},\mu,x_0} \to \mathcal{M}^{\mathrm{int}}_{\mathcal{G},b,\mu}$ [PR21, Lemma 4.1.0.2], where the source is a Rapoport–Zink space. Let the notations be as in [PR21, Theorem 4.10.6, §4.10.2]. We verify Conjecture (U_x) in [PR21, §4.10.2] and obtain the following.

Corollary 3.1.8. *(Corollary 3.6.3) The map $c : \mathrm{RZ}^{\diamondsuit}_{\mathcal{G},\mu,x_0} \to \mathcal{M}^{\mathrm{int}}_{\mathcal{G},b,\mu}$ is an isomorphism. Thus, $\mathcal{M}^{\mathrm{int}}_{\mathcal{G},b,\mu}$ is representable by a formal scheme $\mathscr{M}_{\mathcal{G},b,\mu}$, and we obtain a p-adic uniformization isomorphism of $O_{\breve{E}}$-formal schemes*

$$I_x(\mathbb{Q})\backslash(\mathscr{M}_{\mathcal{G},b,\mu} \times \mathbf{G}(\mathbb{A}_f^p)/\mathbf{K}^p) \to (\mathscr{S}_{\mathbf{K}} \widehat{\otimes_{O_E}} O_{\breve{E}})_{/\mathcal{I}(x)}. \tag{3.7}$$

3.1.4 Rough Sketch of the argument

Many cases of Conjecture 2 have been proved in literature under various additional assumptions[7], see for example [Vie08, Theorem 2], [CKV15, Theorem 1.1], [Nie18, Theorem 1.1], [HZ20a, Theorem 0.1], [Ham20, Theorem 1.1(3)], [Nie21, Theorem 0.2].

Previous attempts in literature used characteristic p perfect geometry and combinatorial arguments to construct enough "curves" connecting the components of the ADLV. In our approach, we use the theory of *kimberlites* and their specialization maps [Gle22b], and the general kimberlite-theoretic unibranchness result for the local models considered by Scholze–Weinstein (see [SW20, § 21.4]) recently established in [GL22b, Theorem 1], to turn the problem of computing $\pi_0(X_\mu^{\mathcal{K}_p}(b))$ into the characteristic-zero question of computing $\pi_0(\mathrm{Sht}_{(G,b,\mu,\mathcal{K}_p)})$. We remark that when μ is non-minuscule, diamond-theoretic considerations are necessary, since the spaces $\mathrm{Sht}_{(G,b,\mu,\mathcal{K}_p)}$ are not rigid-analytic spaces. Moreover, even when μ is minuscule, the theory of kimberlites is necessary here because: although $\mathrm{Sht}_{(G,b,\mu,\mathcal{K}_p)}$ is representable by a rigid-analytic space, its canonical integral model is not known to be representable by a formal scheme.

Once in characteristic zero, we are now able to exploit Fontaine's classical p-adic Hodge theory. In our approach, the role of "connecting curves" is played by "*generic* crystalline

[7]When G is split and $\mathcal{K}_p$ is hyperspecial, [Vie08, Theorem 2] applies. When G is unramified, $\mathcal{K}_p$ is hyperspecial and $\boldsymbol{\mu}$ is minuscule, [CKV15, Theorem 1.1] applies. When G is unramified, $\mathcal{K}_p$ is hyperspecial and $\boldsymbol{\mu}$ is general, [Nie18, Theorem 1.1] applies. When G is residually split or when $\mathbf{b}$ is basic [HZ20a, Theorem 0.1] applies. When G is quasi-split and $\mathcal{K}_p$ is very special, [Ham20, Theorem 1.1(3)] applies. When G is unramified and $\mathcal{K}_p$ is arbitrary, [Nie21, Theorem 0.2] applies.

representations", inspired by the ideas in [Che14] (see §3.5.1). Intuitively speaking, the action of Galois groups can be interpreted as "analytic paths" in the moduli spaces of p-adic shtukas.

More precisely, the infinite level moduli space $\mathrm{Sht}_{G,b,\mu,\infty}$ of p-adic shtukas can be realized as the moduli space of trivializations of the universal crystalline $G(\mathbb{Q}_p)$-torsor[8] over the b-admissible locus Gr_μ^b of the affine Grassmannian [SW20]. Then rational points of Gr_μ^b give rise to loops in Gr_μ^b, which produce "connecting paths" inside any covering space over Gr_μ^b (in particular the covering space $\mathrm{Sht}_{G,b,\mu,\infty}$). Thus it suffices to prove that the universal crystalline representation has enough monodromy to "connect" $\mathrm{Sht}_{G,b,\mu,\infty}$. We can then deduce our main theorem 3.1.1 at finite level $\mathrm{Sht}_{G,b,\mu,\mathcal{K}_p}$ from the analogous result at infinite level.

3.1.5 More on the arguments

We now dig in a bit deeper into the strategy for our main theorem 3.1.1, and sketch a few more results that led to our main theorem.

To each $(G, b, \mu, \mathcal{I}(\mathbb{Z}_p))$, one can associate a diamond $\mathrm{Sht}_{(G,b,\mu,\mathcal{I}(\mathbb{Z}_p))}$, which is the moduli space of p-adic shtukas at level $\mathcal{I}(\mathbb{Z}_p)$ defined in [SW20]. In [Gle22a], the first author constructed a specialization map

$$\mathrm{sp} : |\mathrm{Sht}_{(G,b,\mu,\mathcal{I}(\mathbb{Z}_p))}| \to |X_\mu(b)|. \tag{3.8}$$

By the unibranchness result of the first author joint with Lourenço [GL22b, Theorem 1.3], and the construction of certain v-sheaf local model correspondence due to the first author [Gle22a, Theorem 3], the specialization map induces an isomorphism of sets

$$\mathrm{sp} : \pi_0(\mathrm{Sht}_{(G,b,\mu,\mathcal{I}(\mathbb{Z}_p))} \times \mathrm{Spd}\,\mathbb{C}_p) \cong \pi_0(X_\mu(b)). \tag{3.9}$$

Therefore we have now turned the question on $\pi_0(X_\mu(b))$ into a characteristic zero question on the connected components $\pi_0(\mathrm{Sht}_{(G,b,\mu,\mathcal{I}(\mathbb{Z}_p))})$ of the diamond $\mathrm{Sht}_{(G,b,\mu,\mathcal{I}(\mathbb{Z}_p))}$, which we now compute.

For this purpose, we make use of the infinite level moduli space $\mathrm{Sht}_{(G,b,\mu,\infty)}$ of p-adic shtukas. Since $\mathrm{Sht}_{(G,b,\mu,\mathcal{I}(\mathbb{Z}_p))} = \mathrm{Sht}_{(G,b,\mu,\infty)}/\mathcal{I}(\mathbb{Z}_p)$, we have

$$\pi_0(\mathrm{Sht}_{(G,b,\mu,\mathcal{I}(\mathbb{Z}_p))}) = \pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)})/\mathcal{I}(\mathbb{Z}_p). \tag{3.10}$$

Let G^{ad} denote the adjoint group of G. Our main theorem 3.1.1 follows directly from the following Theorem 3.1.9 whenever G^{ad} does not have anisotropic factors. When G^{ad} is anisotropic, we give a separate argument (see the proof of Theorem 3.6.2). Let $G^\circ := G(\mathbb{Q}_p)/\operatorname{Im}(G^{\mathrm{sc}}(\mathbb{Q}_p))$ denote the maximal abelian quotient of $G(\mathbb{Q}_p)$.

Theorem 3.1.9. *(Theorem 3.6.1) Suppose that $\mathbf{b} \in B(G, \boldsymbol{\mu})$ and that G^{ad} does not have anisotropic factors. The following statements are equivalent:*

[8] For local Shimura varieties coming from Rapoport–Zink spaces, this torsor corresponds to the local system defined by the p-adic Tate module of the universal p-divisible group.

1. *The map* $\omega_G : \pi_0(X_\mu(b)) \to c_{b,\mu}\pi_1(G)_I^\varphi$ *is bijective.*

2. *The map* $\omega_G : \pi_0(X_\mu^{\mathcal{K}_p}(b)) \to c_{b,\mu}\pi_1(G)_I^\varphi$ *is bijective.*

3. *The pair* $(\mathbf{b}, \boldsymbol{\mu})$ *is HN-irreducible.*

4. *There exists a field extension* K *of finite index over* $\breve{\mathbb{Q}}_p$*, and a crystalline representation* $\xi : \Gamma_K \to G(\mathbb{Q}_p)$ *with invariants* $(\mathbf{b}, \boldsymbol{\mu})$ *for which* $G^{\mathrm{der}}(\mathbb{Q}_p) \cap \xi(\Gamma_K) \subseteq G^{\mathrm{der}}(\mathbb{Q}_p)$ *is open.*

5. *The action of* $G(\mathbb{Q}_p)$ *on* $\mathrm{Sht}_{(G,b,\mu,\infty)}$ *makes* $\pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)} \times \mathrm{Spd}\,\mathbb{C}_p)$ *into a* G°*-torsor.*

Remark 9. The implication (3) $\implies$ (5) of Theorem 3.1.9 confirms almost all cases (excluding the anisotropic groups) of a conjecture of Rapoport–Viehmann [RV14, Conjecture 4.30]. Moreover, we generalize the statement to moduli spaces of p-adic shtukas, instead of only for local Shimura varieties as *loc.cit.*

Remark 10. The implication (3) $\implies$ (5) of Theorem 3.1.9 is a more general version of the main theorem of [Gle22a], where the first author proved the statement for unramified G, and computed the Weil group and $J_b(\mathbb{Q}_p)$-actions on $\pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)} \times \mathrm{Spd}\,\mathbb{C}_p)$. One should be able to combine the methods of our current paper with those *loc.cit.* to compute the Weil group and $J_b(\mathbb{Q}_p)$-actions in the more general setup of Theorem 3.1.9.

3.1.5.1 Loop of the argument for Theorem 3.1.9

We now discuss the proof of Theorem 3.1.9. Using ad-isomorphisms and z-extensions (see §3.3.7), we reduce all statements of Theorem 3.1.9 to the case where G^{der}–the derived subgroup of G–is simply connected (see Proposition 3.6.7). In this case, $G^\circ = G^{\mathrm{ab}}(\mathbb{Q}_p)$ and, using that G^{ad} has only isotropic factors, we prove the implications

$$(1) \implies (2) \implies (3) \implies (4) \implies (5) \implies (1).$$

Let us explain the chain of implications. The implication (1) $\implies$ (2) follows from [He16, Theorem 1.1], which says that the map $X_\mu(b) \to X_\mu^{\mathcal{K}_p}(b)$ is surjective. We give a new and simple proof of this result in Theorem 3.6.8, by observing that $\mathrm{Sht}_{(G,b,\mu,\mathcal{I}(\mathbb{Z}_p))} \to \mathrm{Sht}_{(G,b,\mu,\mathcal{K}_p)}$ is automatically surjective. This again exemplifies the advantage of working on the generic fiber (of the v-sheaf $\mathrm{Sht}_\mu^{\mathcal{K}_p}(b)$). For more details, see § 3.3.4. The implication (2) $\implies$ (3) follows from the HN-decomposition (Theorem 3.4.3) and group-theoretic manipulations (Proposition 3.4.10).

The implication (3) $\implies$ (4) follows from an explicit construction that goes back to [Che14, Théorème 5.0.6] when G is unramified (Proposition 3.5.8). In §3.5.1, we push the methods *loc.cit.* and generalize the result to arbitrary reductive groups G (see also §3.1.5.3 in this introduction).

The implication (5) $\implies$ (1) follows from (3.9) (see also Proposition 3.3.7) and the identification $\pi_0(\mathrm{Sht}_{(G,b,\mu,\mathcal{I}(\mathbb{Z}_p))}) = \pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)})/\mathcal{I}(\mathbb{Z}_p)$. Indeed, using the map $\det : G \to G^{\mathrm{ab}} := G/G^{\mathrm{der}}$, combined with Lang's theorem, we reduce (5) $\implies$ (1) to the tori case which can be handled directly (see §3.6.1). For more details, see §3.6.6.

3.1.5.2 Proof for (4) $\implies$ (5) in Theorem 3.1.9

The core of the argument lies in (4) $\implies$ (5). For simplicity, we only discuss the case where G is semisimple and simply connected in the introduction (see §3.6.7 for the general argument). In this (simplified) case, G° is trivial, thus it suffices to show that $\mathrm{Sht}_{(G,b,\mu,\infty)} \times \mathrm{Spd}\,\mathbb{C}_p$ is connected. The first step is to prove that $G(\mathbb{Q}_p)$ acts transitively on $\pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)} \times \mathrm{Spd}\,\mathbb{C}_p)$ (Proposition 3.3.10). This follows from the main theorem of [GL22a] (see Theorem 3.3.9).[9] Let G_x denote the stabilizer of $x \in \pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)} \times \mathrm{Spd}\,\mathbb{C}_p)$. Since $G(\mathbb{Q}_p)$ acts transitively, it suffices to prove $G_x = G(\mathbb{Q}_p)$.

For this, it suffices to show that: (i) G_x is open (see Lemma 3.6.11); (ii) the normalizer N_x of G_x in $G(\mathbb{Q}_p)$ is of finite index in $G(\mathbb{Q}_p)$ (see Lemma 3.6.12). Indeed, since we assumed that G is semisimple and simply connected with only isotropic factors, a standard fact from [Mar91, Chapter II, Theorem 5.1] shows that $G(\mathbb{Q}_p)$ does not have finite index subgroups. Thus (ii) allows us to conclude that G_x is normal in $G(\mathbb{Q}_p)$. Moreover, the same standard fact *loc.cit.* shows that $G(\mathbb{Q}_p)$ does not have non-trivial open normal subgroups, therefore (i) implies $G_x = G(\mathbb{Q}_p)$.

To prove that $G_x \subseteq G(\mathbb{Q}_p)$ is open, we use the Bialynicki-Birula map (3.37) and the "admissible=weakly admissible" theorem [CF00]. Now, for (ii) we exploit that the actions of $J_b(\mathbb{Q}_p)$ and $G(\mathbb{Q}_p)$ commute. This, together with the key bijection of (3.9), allow us to translate the general finiteness results of [HV20] into the finiteness of $[G_x : N_x]$.

3.1.5.3 The Mumford-Tate group of "generic crystalline representations"

Let us give more detail on the construction used to prove the implication (3) $\implies$ (4) from §3.1.5.1. Fix a finite extension $K/\breve{\mathbb{Q}}_p$ with Galois group $\Gamma_K := \mathrm{Gal}(\overline{K}/K)$, and let $\xi : \Gamma_K \to G(\mathbb{Q}_p)$ denote a conjugacy class of p-adic Hodge–Tate representations.

Definition 3.1.10. *Let MT_ξ denote the connected component of the Zariski closure of the image of ξ in $G(\mathbb{Q}_p)$. This is the p-adic Mumford–Tate group attached to ξ which is well-defined up to conjugation.*

It follows from results of Serre [Ser79, Théorème 1] and Sen [Sen73, §4, Théorème 1] (see also [Che14, Proposition 3.2.1]) that $\xi(\Gamma_K) \cap \mathrm{MT}_\xi(\mathbb{Q}_p)$ is open in $\mathrm{MT}_\xi(\mathbb{Q}_p)$. Let $\mu^\eta : \mathbb{G}_m \to G_K$ be a cocharacter conjugate to μ. Suppose that (b, μ^η) defines an admissible pair in the sense of [RZ96, Definition 1.18]. Since $\mathbf{b} \in B(G, \boldsymbol{\mu})$, it induces a conjugacy class of crystalline representations $\xi_{(b,\mu^\eta)} : \Gamma_K \to G(\mathbb{Q}_p)$, and a p-adic Mumford–Tate group $\mathrm{MT}_{(b,\mu^\eta)}$ attached to $\xi_{(b,\mu^\eta)}$ (See Definition 3.5.1).

Let $\mathrm{Fl}_\mu := G/P_\mu$ denote the generalized flag variety. We say that μ^η is *generic* if the map $\mathrm{Spec}(K) \to \mathrm{Fl}_\mu$ induced by μ^η lies over the generic point.[10] Our third main theorem is the following generalization of [Che14, Théorème 5.0.6] to arbitrary reductive groups.

[9] Before [GL22a] was available, the argument for Theorem 3.1.9 relied on the results of [Ham20] which are only available when G is quasi-split.

[10] this is possible since $\breve{\mathbb{Q}}_p$ has infinite transcendence degree over $\mathbb{Q}_p$.

Theorem 3.1.11. *(Theorem 3.5.7) Let G be a reductive group over $\mathbb{Q}_p$. Let $b \in G(\breve{\mathbb{Q}}_p)$ and $\mu^\eta : \mathbb{G}_m \to G_K$ as above. Suppose that b is decent, that μ^η is generic and that $\mathbf{b} \in B(G, \boldsymbol{\mu})$. The following hold:*

1. *(b, μ^η) is admissible.*

2. *If $(\mathbf{b}, \boldsymbol{\mu})$ is HN-irreducible, then $\mathrm{MT}_{(b,\mu^\eta)}$ contains G^{der}.*

Now, Theorem 3.1.9 has as corollary a converse to Theorem 3.1.11. The following gives a p-adic Hodge-theoretic characterization of HN-irreducibility.

Corollary 3.1.12. *(Proposition 3.5.10) Assume that G^{ad} has only isotropic factors. If $\mathrm{MT}_{(b,\mu^\eta)}$ contains G^{der}, then $(\mathbf{b}, \boldsymbol{\mu})$ is HN-irreducible.*

Remark 11. Our Corollary 3.1.12 confirms the expectation in [Che14, Remarque 5.0.5] that, at least when G has only isotropic factors, HN-irreducibility is equivalent to having full monodromy.

3.2 Group-theoretic setup

Given a group G, let G^{der} denote its derived subgroup, G^{sc} the simply connected cover of G^{der}, and $G^{\mathrm{ab}} := G/G^{\mathrm{der}}$. Since G^{ab} is a torus, it admits a unique parahoric model denoted by $\mathcal{G}^{\mathrm{ab}}$.

We continue the notation from §3.1.2. Recall that S is a maximal split $\mathbb{Q}_p$-torus of G. Let $\mathcal{N} = N_G(S)$ be the normalizer of S in G. Let $W_0 := \mathcal{N}(\breve{\mathbb{Q}}_p)/T(\breve{\mathbb{Q}}_p)$ be the relative Weyl group. Recall that $T = Z_G(S)$ is the centralizer of S. Let $\mathcal{T}$ denote its unique parahoric model[11]. Denote by $\widetilde{W}$ the Iwahori-Weyl group $\mathcal{N}(\breve{\mathbb{Q}}_p)/\mathcal{T}(\breve{\mathbb{Z}}_p)$. There is a φ-equivariant exact sequence ([HR08b]):

$$0 \to X_*(T)_I \to \widetilde{W} \to W_0 \to 1 \tag{3.11}$$

Let $\mathcal{A}$ denote the apartment in the Bruhat–Tits building of $G_{\breve{\mathbb{Q}}_p}$ corresponding to S. Let $\mathbf{a} \subseteq \mathcal{A}$ denote the φ-invariant alcove determined by $\mathcal{I}(\mathbb{Z}_p)$. We choose a special vertex $\mathbf{o} \in \mathbf{a}$, and identify $\mathcal{A}$ with $X_*(T)^I \otimes \mathbb{R} = X_*(T)_I \otimes \mathbb{R}$ by sending the origin to $\mathbf{o}$. Let B be the Borel subgroup attached to $\mathbf{a}$ under this identification. Observe that the natural linear action of φ on $X_*(T)^I$ is the gradient of the affine action of φ on $\mathcal{A}$. Let $\Delta \subseteq \Phi^+ \subseteq \Phi \subseteq X^*(T)$ denote the set of simple positive roots, positive roots and roots attached to B, respectively.

The choice of $\mathbf{o}$ defines a splitting $W_0 \to \widetilde{W}$, which may not be φ-equivariant. Let $\bar{\mu}$ denote the image of μ in $X_*(T)_I$. For every element $\lambda \in X_*(T)_I$, let t_λ be its image in $\widetilde{W}$ under (3.11). Let $\mathbb{S}$ be the set of reflections along the walls of $\mathbf{a}$. Let W^{a} be the affine Weyl group generated by $\mathbb{S}$. It is a Coxeter group. There is a φ-equivariant exact sequence ([HR08b, Lemma 14]):

$$1 \to W^{\mathrm{a}} \to \widetilde{W} \to \pi_1(G)_I \to 0 \tag{3.12}$$

[11]This is the identity component of the locally of finite type Néron model of T.

This sequence splits and we can write $\widetilde{W} = W^{\mathrm{a}} \rtimes \pi_1(G)_I$. We can extend the Bruhat order $\preceq$ given on W^{a} to the one on $\widetilde{W}$ as follows: for elements $(w_i, \tau_i) \in \widetilde{W}$ with $i = 1, 2$, where $w_i \in W^{\mathrm{a}}$ and $\tau_i \in \pi_1(G)_I$, we say

$$(w_1, \tau_1) \preceq (w_2, \tau_2) \tag{3.13}$$

if $w_1 \preceq w_2$ in W^{a} and $\tau_1 = \tau_2 \in \pi_1(G)_I$. By [Hai18, Theorem 4.2], we can define the Kottwitz–Rapoport admissible set as

$$\mathrm{Adm}(\mu) = \{\tilde{w} \in \widetilde{W} \mid \tilde{w} \preceq t_{\langle} \,\mathrm{with}\, t_{\langle} = t_{w(\bar{\mu})} \,\mathrm{for}\, w \in W_0\}. \tag{3.14}$$

1. *Let $\widetilde{W}^{\mathrm{ad}}$ denote the Iwahori–Weyl group of G^{ad}. By [HR08b, Lemma 15][12], there exists an element $w^{\mathrm{ad}} \in \widetilde{W}^{\mathrm{ad}}$ such that $w^{\mathrm{ad}} \cdot \varphi(\mathbf{o}) = \mathbf{o}$ and $w^{\mathrm{ad}} \cdot \varphi(\mathbf{a}) = \mathbf{a}$. Conjugation by a lift of w^{ad} to $G^{\mathrm{ad}}(\breve{\mathbb{Q}}_p)$ gives the quasisplit inner form of G, which we denote by G^*. This defines a second action φ_0 on $G(\breve{\mathbb{Q}}_p)$ (called the L-action), whose fixed points are $G^*(\mathbb{Q}_p)$ and that satisfies $\varphi_0(\mathcal{A}) = \mathcal{A}$, $\varphi_0(\mathbf{o}) = \mathbf{o}$, $\varphi_0(B) = B$.*

2. *Let $\mu \in X_*(T)^+$ be a dominant cocharacter. Denote by $\mu^\natural \in \pi_1(G)_\Gamma$ the image of μ under the natural projection $X_*(T) \to \pi_1(G)_\Gamma$. As in [Kot97b, (6.1.1)], we define*

$$\mu^\diamond := \frac{1}{[\Gamma : \Gamma_\mu]} \sum_{\gamma \in \Gamma/\Gamma_\mu} \gamma(\mu) \in X_*(T)^+_{\mathbb{Q}}, \tag{3.15}$$

where the Galois action on $X_*(T)$ is the one coming from G^*. Via the isomorphism $X_*(T)_I \otimes \mathbb{Q} \simeq (X_*(T) \otimes \mathbb{Q})^I$ given by $[\mu] \mapsto \frac{1}{[I:I_\mu]} \sum_{\gamma \in I/I_\mu} \gamma(\mu)$, we may write $\mu^\diamond$ as follows (see [HN18, A.4]):

$$\underline{\mu} := \frac{1}{[I : I_\mu]} \sum_{\gamma \in I/I_\mu} \gamma(\mu) \tag{3.16}$$

$$\mu^\diamond = \frac{1}{N} \sum_{i=0}^{N-1} \varphi_0^i(\underline{\mu}) \tag{3.17}$$

Here N is any integer such that $\varphi_0^N(\underline{\mu}) = \underline{\mu}$, and I_μ is the stabilizer of μ associated to the action by the inertia group. Alternatively,

$$\mu^\diamond = \frac{1}{N} \sum_{i=0}^{N-1} \varphi^i(\mu)^{\mathrm{dom}}. \tag{3.18}$$

Here λ^{dom} denotes the unique B-dominant conjugate of λ for $\lambda \in X_*(T) \otimes \mathbb{Q}$.

[12] More precisely, $P^\vee$ *loc.cit.* acts transitively on the set of special vertices and σ sends a special vertex $\mathbf{o}$ to a special vertex. Thus $P^\vee$ and W_0 together make it possible to find this element w^{ad}.

3. *Recall that attached to* b*, there is a slope decomposition map*

$$\nu_b : \mathbb{D} \to G_{\breve{\mathbb{Q}}_p}, \tag{3.19}$$

where $\mathbb{D}$ *is the pro-torus with* $X^*(\mathbb{D}) = \mathbb{Q}$*. We let the Newton point, denoted as* $\boldsymbol{\nu}_{\mathbf{b}}$*, be the unique conjugate in* $X_*(T)^+_{\mathbb{Q}}$ *of* (3.19)*. Recall that there is a Kottwitz map* $\kappa_G : B(G) \to \pi_1(G)_\Gamma$ *[Kot97a, Kot97b].*

Definition 3.2.1. *Let* $\mathbf{b} \in B(G)$*.*

1. *We write* $\mathbf{b} \in B(G, \boldsymbol{\mu})$ *if* $\mu^\natural = \kappa_G(\mathbf{b})$ *and* $\mu^\diamond - \boldsymbol{\nu}_{\mathbf{b}} = \sum_{\alpha \in \Delta} c_\alpha \alpha^\vee$ *with* $c_\alpha \in \mathbb{Q}$ *and* $c_\alpha \geq 0$*.*
2. *We say* $(\mathbf{b}, \boldsymbol{\mu})$ *is HN-irreducible (Hodge–Newton irreducible) if* $\mathbf{b} \in B(G, \boldsymbol{\mu})$ *and* $c_\alpha \neq 0$ *for all* $\alpha \in \Delta$*.*

Definition 3.2.2. *[RZ96, Definition 1.8] Let* $s \in \mathbb{N}$*. We say that* $b \in G(\breve{\mathbb{Q}}_p)$ *is* s*-decent if* $s \cdot \nu_b$ *factors through a map* $\mathbb{G}_m \to G_{\breve{\mathbb{Q}}_p}$*, and the decency equation* $(b\varphi)^s = s \cdot \nu_b(p)\varphi^s$ *is satisfied in* $G(\breve{\mathbb{Q}}_p) \rtimes \langle \varphi \rangle$*. If the context is clear, we say that* b *is decent if it is* s*-decent for some* s*.*

4. *If* b *is* s*-decent, then* $b \in G(\mathbb{Q}_{p^s})$ *and* ν_b *is also defined over* $\mathbb{Q}_{p^s}$*, where* $\mathbb{Q}_{p^s}$ *is the degree* s *unramified extension of* $\mathbb{Q}_p$*. Moreover, for all* $\mathbf{b} \in B(G)$*, there exists an* $s \in \mathbb{N}$ *and an* s*-decent representative* $b \in G(\mathbb{Q}_{p^s})$ *of* $\mathbf{b}$*, such that* $\nu_b = \boldsymbol{\nu}_{\mathbf{b}}$*. Indeed, by [RZ96, 1.11], every* $\mathbf{b}$ *has a decent representative. Moreover, we can choose* s *large enough such that* G *is quasisplit over* $\mathbb{Q}_{p^s}$*, and then take an arbitrary* s*-decent element. Now, replacing* b *by a* φ*-conjugate in* $G(\mathbb{Q}_{p^s})$ *preserves decency and conjugates the map* ν_b*, thus we can assume without loss of generality that* ν_b *is dominant.*

Remark 12. One can define affine Deligne–Lusztig varieties over any local field F, and the statement of Theorem 3.1.1 is conjectured to hold in this generality. Our Theorem 3.1.1 holds when F is a finite extension of $\mathbb{Q}_p$, via a standard restriction of scalars argument (see for example [DOR10, §5&§8]). It is not clear if our method goes through in the equal characteristic case.

5. *To conclude this background section, we briefly recall some notations (see for example [Kis17b, (1.3.6)], [Zho20, Xu21]) for Corollary 3.1.3. Let* $k \subset \overline{\mathbb{F}}_p$ *be a finite extension of the residue field* κ_E *of* $E_{(v)}$*, where* $E = E(G, X)$ *is the reflex field. For* $x \in \mathscr{S}_{K_p}(G, X)(k)$*, we denote by* $I_x \subset \mathrm{Aut}_{\mathbb{Q}}(\mathcal{A}_x \otimes \overline{\mathbb{F}}_p)$ *(resp.* $I_{/k} \subset \mathrm{Aut}_{\mathbb{Q}}(\mathcal{A}_x)$*) the subgroup fixing the Hodge tensors* $s_{\alpha,\ell,x}$ *for all* $\ell \neq p$ *and the crystalline tensor* $s_{\alpha,0,x}$*. Let* $\overline{x}$ *be the* $\overline{\mathbb{F}}_p$*-point associated to* x*. Recall that the* ℓ*-adic tensors* $s_{\alpha,\ell,x} \in H^1_{\text{ét}}(\mathcal{A}_{\overline{x}}, \mathbb{Q}_\ell)^\otimes$ *cut out a group inside* $\mathrm{GL}(H^1_{\text{ét}}(\mathcal{A}_{\overline{x}}, \mathbb{Q}_\ell))$ *that is identifiable to* $G_{\mathbb{Q}_\ell}$ *via the level structure* ϵ^p*. Since the tensors* $s_{\alpha,\ell,x}$ *are fixed by the action of the geometric Frobenius* γ_ℓ *on* $H^1_{\text{ét}}(\mathcal{A}_{\overline{x}}, \mathbb{Q}_\ell)$*, we can view* γ_ℓ *as an element of* $G(\mathbb{Q}_\ell)$*. We denote by* $I_{\ell/k}$ *the centralizer of* γ_ℓ *in* $G(\mathbb{Q}_\ell)$ *and by* I_ℓ *the centralizer of* γ_ℓ^n *for sufficiently* *large n (recall from [Kis17b, 2.1.2] that the centralizers of γ_ℓ^n form and increasing sequence and stabilizes for large enough n).*

On the other hand, let $\mathscr{G}_x$ be the p-divisible group associated to x. By [KP18], the Frobenius on $\mathbb{D}(\mathscr{G}_x)$ is of the form $\varphi = \delta\varphi$ for some $\delta \in G(K_0)$ let $I_{p/k}$ be the group over $\mathbb{Q}_p$ whose R-points are given by $I_{p/k}(R) := \{g \in G(W(k)[\frac{1}{p}] \otimes_{\mathbb{Q}_p} R) | g^{-1}\delta\sigma(g) = \delta\}$. The following Corollary 3.1.3 is a parahoric analogue to [Kis17b, Propositions 2.1.5] when $\mathbf{G}_{\mathbb{Q}_p}$ is quasi-split.

Corollary 3.2.3. *Let $H^p = \prod_{\ell \neq p} I_{\ell/k}(\mathbb{Q}_\ell) \cap K^p$ and $H_p = I_{p/k}(\mathbb{Q}_p) \cap \mathcal{G}(W(k))$. Then the map (3.6) induces an injective map*

$$I_{/k}(\mathbb{Q}) \backslash \prod_\ell I_{\ell/k}(\mathbb{Q}_\ell)/H_p \times H^p \to \mathscr{S}_K(G, X)(k), \tag{3.20}$$

where $I_{/k}$ is the analogue of I_x for the abelian variety over k.
In particular, the left hand side of (3.20) is finite.

Proof. It follows by combining our Theorem 3.1.1 with [Zho20, Prop 9.1]. □

Corollary 3.2.4. *For some prime $\ell \neq p$, $I_{x,\mathbb{Q}_\ell} = I_x \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ contains the connected component of the identity in I_ℓ. In particular, the ranks of I_x and G are equal.*

Proof. Using Corollary 3.2.3, this follows similarly as [Kis17b, 2.1.7]. □

Corollary 3.2.5. *(Corollary 3.1.3(2)) The isogeny class $\iota_x(X_\mu^{\mathcal{K}_p}(b)(\bar{\mathbb{F}}_p)) \times \mathbf{G}(\mathbb{A}_f^p))$ contains a point which lifts to a special point on $\mathscr{S}_{\mathcal{K}_p}(G, X)$.*

Proof. This follows the outline from the proof of [Kis17b, Theorem 2.2.3], and can be directly obtained by combining our Theorem 3.1.1 with [Zho20, Theorem 9.4]. Note that Corollary 3.2.4 is the parahoric version of an ingredient crucially used in [Kis17b, Theorem 2.2.3]. □

3.3 Geometric background

3.3.1 v-sheaf-theoretic setup

We work within Scholze's framework of diamonds and v-sheaves [Sch17]. More precisely, we consider geometric objects that are functors

$$\mathcal{F} : \mathrm{Perf}_{\mathbb{F}_p} \to \mathrm{Sets}, \tag{3.21}$$

where $\mathrm{Perf}_{\mathbb{F}_p}$ is the site of affinoid perfectoid spaces in characteristic p, endowed with the v-topology (see [Sch17, Definition 8.1]). Recall that given a topological space T, we can define a v-sheaf $\underline{T}$ whose value on (R, R^+)-points is the set of continuous maps $|\operatorname{Spa}(R, R^+)| \to T$. We will mostly use this notation $\underline{T}$ for topological groups T.

Example 3.3.1. *$\underline{\mathcal{I}(\mathbb{Z}_p)}$ and $\underline{G(\mathbb{Q}_p)}$ are the v-sheaf group objects attached to the topological groups $\mathcal{I}(\mathbb{Z}_p)$ and $G(\mathbb{Q}_p)$.*

Conversely, to any diamond or v-sheaf $\mathcal{F}$, by [Sch17, Proposition 12.7], one can attach an underlying topological space that we denote by $|\mathcal{F}|$.

6. *Recall that in the more classical setup of Rapoport–Zink spaces [RZ96], affine Deligne–Lusztig varieties arise, via Dieudonne theory, as the perfection of special fibers of Rapoport-Zink spaces. Moreover, the rigid generic fiber of such a Rapoport-Zink space is a special case of the so called local Shimura varieties [RV14]. In this way, Rapoport-Zink spaces (formal schemes) interpolate between local Shimura varieties and their corresponding affine Deligne–Lusztig varieties. Or in other words, Rapoport-Zink spaces serve as integral models of local Shimura varieties whose perfected special fibers are ADLVs. Moreover, by [SW20], the diamondification functor*

$$\begin{aligned}\diamondsuit : \{\textit{Adic Spaces}/\operatorname{Spa}\mathbb{Z}_p\} &\longrightarrow \{\textit{v-sheaves}/\operatorname{Spd}\mathbb{Z}_p\}\\ X &\longmapsto X^{\diamondsuit}\end{aligned}$$

applied to a local Shimura variety is a locally spatial diamond that can be identified with a moduli space of p-adic shtukas (see §3.3.4).

Alternatively, one could consider the diamondification functor applied to the entire formal schemes (such as Rapoport-Zink spaces), rather than only their rigid generic fibres. The diamondification functor naturally takes values in v-sheaves, but contrary to the rigid-analytic case, these v-sheaves are no longer diamonds. Nevertheless, the v-sheaf associated to a formal scheme still has a lot of structure. Indeed, they are what the first author calls kimberlites [Gle22a, Definition 4.35], i.e. we have a commutative diagram

$$\begin{tikzcd} \{\textit{Adic Spaces}/\operatorname{Spa}\mathbb{Z}_p\} \arrow[r, "\diamondsuit"] & \{\textit{v-sheaves}/\operatorname{Spd}\mathbb{Z}_p\} \\ \{\textit{Formal Schemes}/\operatorname{Spf}\mathbb{Z}_p\} \arrow[u, hook] \arrow[r, "\diamondsuit"] & \{\textit{Kimberlites}/\operatorname{Spd}\mathbb{Z}_p\} \arrow[u, hook] \end{tikzcd} \tag{3.22}$$

Kimberlites share with formal schemes many pleasant properties that general v-sheaves do not. Let us list the main ones. Let $\mathfrak{X}$ be a kimberlite.

1. *Each kimberlite has an open analytic locus $\mathfrak{X}^{\mathrm{an}}$ (which is a locally spatial diamond by definition), and a reduced locus $\mathfrak{X}^{\mathrm{red}}$ (which is by definition a perfect scheme).*

2. *Each kimberlite has a continuous "specialization map" whose source is $|\mathfrak{X}^{\mathrm{an}}|$ and whose target is $|\mathfrak{X}^{\mathrm{red}}|$ (see 7 for details).*

3. *Kimberlites have a formal étale site and a formal nearby-cycles functor ([GL22b])*

$$R\Psi^{\mathrm{for}} : D_{\text{ét}}(\mathfrak{X}^{\mathrm{an}}, \Lambda) \to D_{\text{ét}}(\mathfrak{X}^{\mathrm{red}}, \Lambda).$$

Although we expect that every local Shimura variety admits a formal scheme "integral model" (see [PR22] for the strongest result on this direction), this is not known in full generality. Nevertheless, as the first author proved, every local Shimura variety (even the more general moduli spaces of p-adic shtukas) is modeled by a prekimberlite[13] *whose perfected special fiber is the corresponding ADLV (see Theorem 3.3.5). We shall return to this discussion in §3.3.4.*

7. *Recall that given a formal scheme $\mathcal{X}$, one can attach a specialization triple $(\mathcal{X}_\eta, \mathcal{X}^{\mathrm{red}}, \mathrm{sp})$, where $\mathcal{X}_\eta$ is a rigid analytic space (the Raynaud generic fiber), $\mathcal{X}^{\mathrm{red}}$ is a reduced scheme (the reduced special fiber) and*

$$\mathrm{sp} : |\mathcal{X}_\eta| \to |\mathcal{X}^{\mathrm{red}}| \tag{3.23}$$

is a continuous map.

Analogously, to a prekimberlite $\mathfrak{X}$ [Gle22b, Definition 4.15] over $\mathrm{Spd}(\mathbb{Z}_p)$*, one can attach a specialization triple* $(\mathfrak{X}_\eta, \mathfrak{X}^{\mathrm{red}}, \mathrm{sp})$ *where*

- *$\mathfrak{X}_\eta$ is the generic fiber (which is an open subset of the analytic locus $\mathfrak{X}^{\mathrm{an}}$ [Gle22b, Definition 4.15] of $\mathfrak{X}$).*
- *$\mathfrak{X}^{\mathrm{red}}$ is a perfect scheme over $\mathbb{F}_p$ (obtained via the reduction functor [Gle22b, §3.2]) and*
- sp *is a continuous map [Gle22b, Proposition 4.14] analogous to* (3.23).

For example, if $\mathfrak{X} = \mathcal{X}^\diamondsuit$ for a formal scheme $\mathcal{X}$, then $\mathfrak{X}$ is a kimberlite, and we have $\mathfrak{X}_\eta = \mathcal{X}_\eta^\diamondsuit$, $\mathfrak{X}^{\mathrm{red}} = (\mathcal{X}^{\mathrm{red}})^{\mathrm{perf}}$ and the specialization maps attached to $\mathcal{X}$ and $\mathfrak{X}$ agree, i.e. we have the following commutative diagram:

$$\begin{tikzcd} {|\,\mathcal{X}_\eta\,|} \arrow[r, "\cong"] \arrow[d, "\mathrm{sp}"'] & {|\,\mathfrak{X}_\eta\,|} \arrow[d, "\mathrm{sp}"] \\ {|\,\mathcal{X}^{\mathrm{red}}\,|} \arrow[r, "\cong"] & {|\,\mathfrak{X}^{\mathrm{red}}\,|} \end{tikzcd} \tag{3.24}$$

8. *A smelted kimberlite is a pair $(\mathfrak{X}, X)$ where $\mathfrak{X}$ is a prekimberlite and $X \subseteq \mathfrak{X}^{\mathrm{an}}$ is an open subsheaf of the analytic locus, subject to some technical conditions. This is mainly used when $X = \mathfrak{X}^{\mathrm{an}}$ or when X is the generic fiber of a map to* $\mathrm{Spd}\,\mathbb{Z}_p$ *that is not p-adic.*

Given a smelted kimberlite $(\mathfrak{X}, X)$ and a closed point $x \in |\mathfrak{X}^{\mathrm{red}}|$, one can define the tubular neighborhood $X_x^{\circledcirc}$ ([Gle22b, Definition 4.38]). It is an open subsehaf of X which, roughly speaking, is given as the locus in X of points that specialize to x.

[13] In fact, we expect moduli spaces of p-adic shtukas to be modeled by kimberlites, but for our purposes this difference is minor, as the specialization map is defined for both kimberlites and prekimberlites.

3.3.2 B^+_{dR}-Grassmannians and local models

Let Gr_G be the B^+_{dR}-Grassmannian attached to G [SW20, §19, 20]. This is an ind-diamond over $\mathrm{Spd}\,\breve{\mathbb{Q}}_p$. We omit G from the notation from now on, and denote by Gr_μ the Schubert variety [SW20, Definition 20.1.3] attached to G and μ. This is a spatial diamond over $\mathrm{Spd}\,\breve{E}$ where $\breve{E} = E \cdot \breve{\mathbb{Q}}_p$ and E is the field of definition of $\boldsymbol{\mu}$. Now, Gr_μ contains the Schubert cell attached to μ, which we denote by Gr°_μ. This is an open dense subdiamond of Gr_μ.

Let $\mathrm{Gr}_{\mathcal{K}_p}$ be the Beilinson–Drinfeld Grassmannian attached to $\mathcal{K}_p$. This is a v-sheaf that is ind-representable in diamonds over $\mathrm{Spd}\,\breve{\mathbb{Z}}_p$, whose generic fiber is Gr_G, and whose reduced special fiber is $\mathcal{F}\ell_{\breve{\mathcal{K}}_p}$. Let $\mathcal{M}_{\mathcal{K}_p,\mu}$ be the local models first introduced in [SW20, Definition 25.1.1] for minuscule μ and later extended to non-minuscule μ in [AGLR22, Definition 4.11].

A priori, these local models are defined only as v-sheaves over $\mathrm{Spd}\,O_{\breve{E}}$, but when μ is minuscule, $\mathcal{M}_{\mathcal{K}_p,\mu}$ is representable by a normal scheme flat over $\mathrm{Spec}\,O_{\breve{E}}$ by [AGLR22, Theorem 1.1] and [GL22b, Corollary 1.4][14]. Moreover, in the general case, i.e. μ not necessarily minuscule, $\mathcal{M}_{\mathcal{K}_p,\mu}$ is a kimberlite by [AGLR22, Proposition 4.14], and it is unibranch by [GL22b, Theorem 1.2]. Let $\mathcal{A}_{\mathcal{K}_p,\mu}$ denote the μ-admissible locus inside $\mathcal{F}\ell_{\breve{\mathcal{K}}_p}$ (see for example [AGLR22, Definition 3.11]). This is a perfect scheme whose $\bar{\mathbb{F}}_p$-valued points agree with $\breve{\mathcal{K}}_p\,\mathrm{Adm}(\mu)\breve{\mathcal{K}}_p/\breve{\mathcal{K}}_p$. The generic fiber of $\mathcal{M}_{\mathcal{K}_p,\mu}$ is Gr_μ and the reduced special fiber is $\mathcal{A}_{\mathcal{K}_p,\mu}$ by [AGLR22, Theorem 1.5].

3.3.3 Functoriality of affine Deligne–Lusztig varieties

The formation of affine Deligne–Lusztig varieties is functorial with respect to morphisms of tuples $(G_1, b_1, \mu_1, \mathcal{K}_{1,p}) \to (G_2, b_2, \mu_2, \mathcal{K}_{2,p})$. More precisely, we have the following lemma.

Lemma 3.3.2. *Let $f : G_1 \to G_2$ be a group homomorphism such that $b_2 = f(b_1)$, $\mu_2 = f \circ \mu_1$ and $f(\mathcal{K}_{1,p}) \subseteq \mathcal{K}_{2,p}$. Then we have a map $X^{\mathcal{K}_{1,p}}_{\mu_1}(b_1) \to X^{\mathcal{K}_{2,p}}_{\mu_2}(b_2)$ that fits in the following commutative diagram:*

$$\begin{tikzcd} X^{\mathcal{K}_{1,p}}_{\mu_1}(b_1) \arrow[r] \arrow[d] & X^{\mathcal{K}_{2,p}}_{\mu_2}(b_2) \arrow[d] \\ \mathcal{F}\ell_{\breve{K}_{1,p}} \arrow[r] & \mathcal{F}\ell_{\breve{K}_{2,p}} \end{tikzcd} \tag{3.25}$$

Proof. This follows directly from the definitions and from Lemma 3.3.3. □

Lemma 3.3.3. $f(\breve{\mathcal{K}}_{1,p}\,\mathrm{Adm}(\mu_1)\breve{\mathcal{K}}_{1,p}) \subseteq \breve{\mathcal{K}}_{2,p}\,\mathrm{Adm}(\mu_2)\breve{\mathcal{K}}_{2,p}$.

Proof. We give a geometric argument. Let $\mathcal{M}_{\mathcal{K}_{1,p},\mu_1}$ and $\mathcal{M}_{\mathcal{K}_{2,p},\mu_2}$ denote the v-sheaf local models in [AGLR22, Definition 4.11]. Since $f(\mathcal{K}_{1,p}) \subseteq \mathcal{K}_{2,p}$, we have a morphism of parahoric group schemes $\mathcal{K}_{1,p} \to \mathcal{K}_{2,p}$. By the functoriality result of v-sheaf local models [AGLR22,

[14] Representability is proved in full generality in [AGLR22] and normality is proven when $p \geq 5$. In [GL22b] normality is proved even when $p < 5$.

Proposition 4.16], we obtain a morphism $\mathcal{M}_{\mathcal{K}_{1,p},\mu_1} \to \mathcal{M}_{\mathcal{K}_{2,p},\mu_2}$ of v-sheaves. Moreover, by [AGLR22, Theorem 6.16], we know that $\mathcal{M}_{\mathcal{K}_{i,p},\mu_i,\bar{\mathbb{F}}_p} \subseteq \mathcal{F}\ell_{\breve{K}_{i,p}}$ consists of Schubert cells parametrized by $\mathrm{Adm}(\mu_i)$. More precisely, $\mathcal{M}_{\mathcal{K}_{i,p},\mu_i,\bar{\mathbb{F}}_p}(\bar{\mathbb{F}}_p) = \breve{\mathcal{K}}_{i,p}\,\mathrm{Adm}(\mu_i)\breve{\mathcal{K}}_{i,p}/\breve{\mathcal{K}}_{i,p}$. Therefore, the existence of the map of perfect schemes $\mathcal{M}_{\mathcal{K}_{1,p},\mu_1,\bar{\mathbb{F}}_p} \to \mathcal{M}_{\mathcal{K}_{2,p},\mu_2,\bar{\mathbb{F}}_p}$ immediately implies that $f(\breve{\mathcal{K}}_{1,p}\,\mathrm{Adm}(\mu_1)\breve{\mathcal{K}}_{1,p}) \subseteq \breve{\mathcal{K}}_{2,p}\,\mathrm{Adm}(\mu_2)\breve{\mathcal{K}}_{2,p}$. ☐

Lemma 3.3.2 is most relevant in the following situations:

1. When $G_1 = G_2$, $f = \mathrm{id}$, and $\mathcal{K}_{1,p} \subseteq \mathcal{K}_{2,p}$.
2. When $G_2 = G_1^{\mathrm{ab}}$ and $\mathcal{K}_{2,p}$ is the only parahoric of the torus G_1^{ab}.
3. When $G_2 = G_1/Z$, where Z a central subgroup of G_1 and $\mathcal{K}_{2,p} = f(\mathcal{K}_{1,p})$.

To simplify certain proofs, we will also need the following statement.

Lemma 3.3.4. *Suppose $G = G_1 \times G_2$, $b = (b_1, b_2)$, $\mu = (\mu_1, \mu_2)$ and $\mathcal{K}_p = \mathcal{K}_p^1 \times \mathcal{K}_p^2$. Then $X_\mu^{\mathcal{K}_p}(b) = X_{\mu_1}^{\mathcal{K}_p^1}(b_1) \times X_{\mu_2}^{\mathcal{K}_p^2}(b_2)$.*

Proof. This follows directly from the definition. ☐

3.3.4 Moduli spaces of p-adic shtukas

9. *Recall from [SW20, §23] that to each (G, b, μ) and a closed subgroup $K \subseteq G(\mathbb{Q}_p)$, one can attach a locally spatial diamond $\mathrm{Sht}_{(G,b,\mu,K)}$ over $\mathrm{Spd}\,\breve{E}$, where $\breve{E} = \breve{\mathbb{Q}}_p \cdot E$ and E is the reflex field of $\boldsymbol{\mu}$, i.e. $\mathrm{Sht}_{(G,b,\mu,K)}$ is the moduli space of p-adic shtukas with level K.*

This association is functorial in the tuple (G, b, μ, K), i.e. if $f : G \to H$ is a morphism of groups, we let $b_H := f(b)$, $\mu_H := f \circ \mu$ and we assume $f(K) \subseteq K_H$, then we have a morphism of diamonds

$$\mathrm{Sht}_{(G,b,\mu,K)} \to \mathrm{Sht}_{(H,b_H,\mu_H,K_H)}. \tag{3.26}$$

1. *When $H = G^{\mathrm{ab}}$, $f = \det : G \to G^{\mathrm{ab}}$ is the natural quotient map, and $K_H = \det(K) =: K^{\mathrm{ab}}$, we let $b^{\mathrm{ab}} := \det(b)$, $\mu^{\mathrm{ab}} := \det \circ \mu$, and the morphism (3.26) in this case is called the "determinant map"*

$$\det : \mathrm{Sht}_{(G,b,\mu,K)} \to \mathrm{Sht}_{(G^{\mathrm{ab}},b^{\mathrm{ab}},\mu^{\mathrm{ab}},K^{\mathrm{ab}})}. \tag{3.27}$$

2. *When $H = G$, $f = \mathrm{id}$, and the inclusion $K_1 \subseteq K_2$ is proper, we have a change-of-level-structures map:*

$$\mathrm{Sht}_{(G,b,\mu,K_1)} \to \mathrm{Sht}_{(G,b,\mu,K_2)} \tag{3.28}$$

10. *For parahoric levels K_p, $\mathrm{Sht}_{(G,b,\mu,K_p)}$ is the generic fiber of a canonical*[15] *integral model, which is a v-sheaf $\mathrm{Sht}_\mu^{\mathcal{K}_p}(b)$ over $\mathrm{Spd}\,\mathcal{O}_{\breve{E}}$ defined in [SW20, §25]. In [Gle22a, Theorem 2], the*

[15] canonical in the sense that $\mathrm{Sht}_\mu^{\mathcal{K}_p}(b)$ represents a moduli problem.

first author proved that $\mathrm{Sht}_\mu^{\mathcal{K}_p}(b)$ is a prekimberlite (see [Gle22b, Definition 4.15]). Moreover, by [Gle22a, Proposition 2.30], its reduction (or its reduced special fiber in the sense of [Gle22b, §3.2]) can be identified with $X_\mu^{\mathcal{K}_p}(b)$. Furthermore, the formalism of kimberlites developed in [Gle22b] gives a continuous specialization map which turns out to be surjective (on the underlying topological spaces).

Theorem 3.3.5. *[Gle22a, Theorem 2] The pair $(\mathrm{Sht}_\mu^{\mathcal{K}_p}(b), \mathrm{Sht}_{(G,b,\mu,K_p)})$ is a rich smelted kimberlite[16]. Moreover, $\mathrm{Sht}_\mu^{\mathcal{K}_p}(b)^{\mathrm{red}} = X_\mu^{\mathcal{K}_p}(b)$. In particular, we have a surjective and continuous specialization map.*

$$\mathrm{sp} : |\mathrm{Sht}_{(G,b,\mu,K_p)}| \to |X_\mu^{\mathcal{K}_p}(b)|. \tag{3.29}$$

11. *Now we recall the infinite-dimensional local model diagram of [Gle22a, Theorem 3][17]. It has the form*

$$\begin{tikzcd} & X \arrow[dl, "g"'] \arrow[dr, "f"] & \\ (\mathrm{Sht}_\mu^{\mathcal{K}_p}(b) \times \mathrm{Spd}\,\mathcal{O}_F)^{\circledcirc}_x & & (\mathcal{M}_{\mathcal{K}_p,\mu} \times \mathrm{Spd}\,\mathcal{O}_F)^{\circledcirc}_y \end{tikzcd} \tag{3.30}$$

where $x \in X_\mu^{\mathcal{K}_p}(b)(k_F)$, $y \in \mathcal{A}_{\mathcal{K}_p,\mu}(k_F)$, and the maps f and g are $\widehat{L_W^+G}$-torsors for a certain infinite-dimensional connected group v-sheaf $\widehat{L_W^+G}$.

Theorem 3.3.6. *[GL22b, Theorem 1.3] For any parahoric $K_p \subseteq G(\mathbb{Q}_p)$ and any field extension $\breve{E} \subseteq F \subseteq \mathbb{C}_p$, the tubular neighborhoods of $(\mathcal{M}_{\mathcal{K}_p,\mu} \times \mathrm{Spd}\,\mathcal{O}_F, \mathrm{Gr}_\mu \times \mathrm{Spd}\,F)$ are connected.*

Using (3.30) and Theorem 3.3.6, one can show that the specialization map (3.29) induces a map $\pi_0(\mathrm{sp})$ on connected components.

Proposition 3.3.7. *For any parahoric $K_p \subseteq G(\mathbb{Q}_p)$ and any field extension $\breve{E} \subseteq F \subseteq \mathbb{C}_p$, the map*

$$\pi_0(\mathrm{sp}) : \pi_0(\mathrm{Sht}_{(G,b,\mu,K_p)} \times \mathrm{Spd}\,F) \xrightarrow{\sim} \pi_0(X_\mu^{\mathcal{K}_p}(b)) \tag{3.31}$$

is bijective.

Proof. Recall that by [Gle22b, Lemma 4.55], whenever $(\mathfrak{X}, X)$ is a rich smelted kimberlite, to prove that

$$\pi_0(\mathrm{sp}) : \pi_0(X) \to \pi_0(\mathfrak{X}^{\mathrm{red}}) \tag{3.32}$$

[16] The term “rich” refers to some technical finiteness assumption that ensures that the specialization map can be controlled by understanding the preimage of the closed points in the reduced special fiber.

[17] We warn the reader that this local model correspondence does not agree with the more classical local model diagrams considered in the literature.

is bijective, it suffices to prove that $(\mathfrak{X}, X)$ is unibranch[18] (in the sense of [Gle22b, Definition 4.52]), i.e. tubular neighborhoods are connected. By Theorem 3.3.5, $(\mathrm{Sht}_\mu^{\mathcal{K}_p}(b), \mathrm{Sht}_{(G,b,\mu,K_p)})$ is a rich smelted kimberlite, and thus it suffices to prove that $(\mathrm{Sht}_\mu^{\mathcal{K}_p}(b) \times \mathrm{Spd}\,\mathcal{O}_F, \mathrm{Sht}_{(G,b,\mu,K_p)} \times \mathrm{Spd}\,F)$ is unibranch, i.e. their tubular neighborhoods are connected.

By (3.30), it suffices to prove that the tubular neighborhoods of $(\mathcal{M}_{\mathcal{K}_p,\mu} \times \mathrm{Spd}\,\mathcal{O}_F, \mathrm{Gr}_\mu \times \mathrm{Spd}\,F)$ are connected, which follows from Theorem 3.3.6. ☐

With a similar argument as in Lemma 3.3.3, one can prove that the formation of $\mathrm{Sht}_\mu^{\mathcal{K}_p}(b)$ *is also functorial in tuples* $(G, b, \mu, \mathcal{K}_p)$.

Lemma 3.3.8. *Let* $f : G_1 \to G_2$ *be a group homomorphism such that* $b_2 = f(b_1)$*,* $\mu_2 = f \circ \mu_1$ *and* $f(\mathcal{K}_{1,p}) \subseteq \mathcal{K}_{2,p}$*. Then we have a map*

$$\mathrm{Sht}_{\mu_1}^{\mathcal{K}_{1,p}}(b_1) \to \mathrm{Sht}_{\mu_2}^{\mathcal{K}_{2,p}}(b_2) \tag{3.33}$$

of v-sheaves. Moreover, taking the reduction functor [Gle22b, §3.2] of map (3.33) *induces the map* $X_{\mu_1}^{\mathcal{K}_{1,p}}(b_1) \to X_{\mu_2}^{\mathcal{K}_{2,p}}(b_2)$ *of Lemma 3.3.2.*

Proof. The first statement follows from the definition of $\mathrm{Sht}_{\mu_1}^{\mathcal{K}_{1,p}}(b_1)$ (see for example [Gle22a, Definition 2.26]) and from Lemma 3.3.3. By [Gle22a, Proposition 2.30], we have the identity $X_{\mu_i}^{\mathcal{K}_{i,p}}(b_i) = \mathrm{Sht}_{\mu_i}^{\mathcal{K}_{i,p}}(b_i)^{\mathrm{red}}$, with $i \in \{1, 2\}$, where the right-hand side is the reduced special fiber (more precisely, it is the image under the reduction functor defined *loc.cit.*). ☐

As a special case, if we fix a datum (G, b, μ) *and two parahorics* $K_1 \subseteq K_2$ *of* $G(\mathbb{Q}_p)$*, we have a map*

$$\mathrm{Sht}_\mu^{\mathcal{K}_1}(b) \to \mathrm{Sht}_\mu^{\mathcal{K}_2}(b) \tag{3.34}$$

of v-sheaves. On the generic fiber, the map (3.34) *gives the change-of-level-structures map of* (3.28)*. After applying the reduction functor to the map* (3.34)*, we recover the map* $X_\mu^{\mathcal{K}_1}(b) \to X_\mu^{\mathcal{K}_2}(b)$ *from Lemma 3.3.2 applied to scenario (1).*

3.3.5 The Grothendieck–Messing period map

Recall that given a triple (G, b, μ), there is a quasi-pro-étale *Grothendieck–Messing period morphism* (see for example [SW20, §23]):

$$\pi_{\mathrm{GM}} : \mathrm{Sht}_{(G,b,\mu,\infty)} \times \mathrm{Spd}\,\mathbb{C}_p \to \mathrm{Gr}_\mu \times \mathrm{Spd}\,\mathbb{C}_p. \tag{3.35}$$

Now, the b-admissible locus $\mathrm{Gr}_\mu^b \subseteq \mathrm{Gr}_\mu$, can be defined as the image of π_{GM}. Note that $\mathrm{Gr}_\mu^b \times \mathrm{Spd}\,\mathbb{C}_p \subseteq \mathrm{Gr}_\mu \times \mathrm{Spd}\,\mathbb{C}_p$ is a dense open subset. Moreover, there is a (universal) $\underline{G(\mathbb{Q}_p)}$-torsor $\mathbb{L}_b$ over Gr_μ^b, such that for each finite extension K over $\breve{E}$ and $x \in \mathrm{Gr}_\mu^b(K)$, $x^*\mathbb{L}_b$ is a crystalline representation associated to the isocrystal with G-structure defined

[18] The definition of unibranchness, or alternatively *topological normality*, for smelted kimberlites is inspired by a useful criterion for the unibranchness of a scheme (see [AGLR22, Proposition 2.38]).

by b (for more details see for example [Gle21, §2.2-2.4]). The map in (3.35) can be then constructed as the geometric $\underline{G(\mathbb{Q}_p)}$-torsor attached to $\mathbb{L}_b$, i.e. $\mathrm{Sht}_{(G,b,\mu,\infty)}$ is the moduli space of trivializations of $\mathbb{L}_b$. The first author together with Lourenço prove the following theorem using diamond-theoretic techniques.

Theorem 3.3.9 ([GL22a]). *Let (G, b, μ) be a p-adic shtuka datum with $\mathbf{b} \in B(G, \boldsymbol{\mu})$. The b-admissible locus $\mathrm{Gr}^b_\mu \times \mathrm{Spd}\,\mathbb{C}_p$ is connected and dense within $\mathrm{Gr}_\mu \times \mathrm{Spd}\,\mathbb{C}_p$.*

From this we deduce the following.

Proposition 3.3.10. *The $G(\mathbb{Q}_p)$-action on $\pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)} \times \mathrm{Spd}\,\mathbb{C}_p)$ is transitive.*

Proof. Note that π_0 commutes with colimits. This gives an identification $\pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)} \times \mathrm{Spd}\,\mathbb{C}_p)/G(\mathbb{Q}_p) = \pi_0(\mathrm{Gr}^b_\mu \times \mathrm{Spd}\,\mathbb{C}_p)$. From which we deduce that $G(\mathbb{Q}_p)$ acts transitively on $\pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)} \times \mathrm{Spd}\,\mathbb{C}_p)$. □

We also get the following result which from a different perspective is much harder to obtain.

Corollary 3.3.11. *For any point $x \in \mathrm{Sht}_{(G,b,\mu,\mathcal{K}_p)}(\mathbb{C}_p)$ the map:*

$$G(\mathbb{Q}_p)/\mathcal{K}_p \xrightarrow{g \mapsto g\cdot x} \mathrm{Sht}_{(G,b,\mu,\mathcal{K}_p)}(\mathbb{C}_p) \xrightarrow{\mathrm{sp}} X^{\mathcal{K}_p}_\mu(b)(\bar{\mathbb{F}}_p)$$

induces a surjection $G(\mathbb{Q}_p)/\mathcal{K}_p \to \pi_0(X^{\mathcal{K}_p}_\mu(b))$.

Proof. Let $\tilde{x} \in \mathrm{Sht}_{(G,b,\mu,\infty)}(\mathbb{C}_p)$ be any lift of x and consider the following diagram:

$$\begin{tikzcd} G(\mathbb{Q}_p) \arrow[r, "g\mapsto g\cdot\tilde{x}"] \arrow[d] & \mathrm{Sht}_{(G,b,\mu,\infty)}(\mathbb{C}_p) \arrow[r] \arrow[d] & \pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)} \times \mathrm{Spd}\,\mathbb{C}_p) \arrow[d] \\ G(\mathbb{Q}_p)/\mathcal{K}_p \arrow[r, "g\cdot\mathcal{K}_p\mapsto g\cdot x"] & \mathrm{Sht}_{(G,b,\mu,\mathcal{K}_p}(\mathbb{C}_p) \arrow[r] & \pi_0(\mathrm{Sht}_{(G,b,\mu,\mathcal{K}_p} \times \mathrm{Spd}\,\mathbb{C}_p) \end{tikzcd}$$

the map $G(\mathbb{Q}_p) \to \pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)} \times \mathrm{Spd}\,\mathbb{C}_p)$ is $G(\mathbb{Q}_p)$-equivariant and by Proposition 3.3.10 surjective. The right arrow is also surjective since the map of spaces $\mathrm{Sht}_{(G,b,\mu,\infty)} \times \mathrm{Spd}\,\mathbb{C}_p \to \mathrm{Sht}_{(G,b,\mu,\mathcal{K}_p)} \times \mathrm{Spd}\,\mathbb{C}_p$ is a $\underline{\mathcal{K}_p}$-torsor. Finally, by Proposition 3.3.7 the map $G(\mathbb{Q}_p)/\mathcal{K}_p \to \pi_0(X^{\mathcal{K}_p}_\mu(b))$ is surjective since it is the composition of two surjective maps. □

3.3.6 The Bialynicki-Birula map

Recall the Bialynicki-Birula map (see [SW20, Proposition 19.4.2]) from the Schubert cell Gr°_μ to the generalized flag variety $\mathrm{Fl}_\mu := G/P_\mu$

$$\mathrm{BB} : \mathrm{Gr}^\circ_\mu \to \mathrm{Fl}_\mu\,. \tag{3.36}$$

In general, the map (3.36) is not an isomorphism (it is an isomorphism only when μ is minuscule), but it always induces a bijection on classical points, i.e. finite extensions F of $\breve{E}$ (see for example [Vie21, Theorem 5.2]).

Let $\mathrm{Fl}_\mu^{\mathrm{adm}} \subseteq \mathrm{Fl}_\mu$ denote the weakly admissible (or equivalently, semistable) locus inside the flag variety [DOR10, §5], and let $\mathrm{Gr}_\mu^{\circ,b} := \mathrm{Gr}_\mu^\circ \cap \mathrm{Gr}_\mu^b$. By [CF00], we have a bijection $\mathrm{BB} : \mathrm{Gr}_\mu^{\circ,b}(F) \cong \mathrm{Fl}_\mu^{\mathrm{adm}}(F)$ for all finite extensions F of $\breve{E}$. Moreover, (3.36) fits in the following commutative diagram:

$$\begin{tikzcd} \mathrm{Gr}_\mu^{\circ,b} \arrow[r] \arrow[d, "\mathrm{BB}"'] & \mathrm{Gr}_\mu^\circ \arrow[d, "\mathrm{BB}"] \\ \mathrm{Fl}_\mu^{\mathrm{adm}} \arrow[r] & \mathrm{Fl}_\mu. \end{tikzcd} \tag{3.37}$$

3.3.7 Ad-isomorphisms and z-extensions

Definition 3.3.12. *[Kot97a, §4.8] A morphism $f : G \to H$ is called an ad-isomorphism if f sends the center of G to the center of H and induces an isomorphism of adjoint groups.*

An important example of ad-isomorphisms are z-extensions.

Definition 3.3.13. *[Kot82, §1] A map of connected reductive groups $f : G' \to G$ is a z-extension if: f is surjective, $Z = \mathrm{Ker}(f)$ is central in G', Z is isomorphic to a product of tori of the form $\mathrm{Res}_{F_i/\mathbb{Q}_p} \mathbb{G}_m$ for some finite extensions $F_i \subseteq \overline{\mathbb{Q}}_p$, and G' has simply connected derived subgroup.*

Lemma 3.3.14. *Let $f : \tilde{G} \to G$ be a z-extension and $\mathbf{b} \in B(G, \boldsymbol{\mu})$.*
(1) There exist a conjugacy class of cocharacters $\tilde{\boldsymbol{\mu}}$ and an element $\tilde{\mathbf{b}} \in B(\tilde{G}, \tilde{\boldsymbol{\mu}})$ which, under the map $B(\tilde{G}, \tilde{\boldsymbol{\mu}}) \to B(G, \boldsymbol{\mu})$, map to $\boldsymbol{\mu}$ and $\mathbf{b}$, respectively.
(2) $c_{\tilde{b},\tilde{\mu}}\pi_1(\tilde{G})_I^\varphi \to c_{b,\mu}\pi_1(G)_I^\varphi$ is surjective.

Proof. (1) Let $T \subseteq G$ be a maximal torus and $\tilde{T} \subseteq \tilde{G}$ its preimage under f. Let $Z = \mathrm{Ker}(f)$. We have an exact sequence

$$0 \to Z \to \tilde{T} \to T \to 0 \tag{3.38}$$

Since Z is a torus, we have an exact sequence:

$$0 \to X_*(Z) \to X_*(\tilde{T}) \to X_*(T) \to 0 \tag{3.39}$$

In particular, we can lift $\boldsymbol{\mu}$ to an arbitrary $\tilde{\boldsymbol{\mu}} \in X_*(\tilde{T})$. To lift $\tilde{\mathbf{b}}$ compatibly, it suffices to recall from [Kot97b, (6.5.1)] that

$$B(G, \boldsymbol{\mu}) \cong B(G^{\mathrm{ad}}, \boldsymbol{\mu}^{\mathrm{ad}}) \cong B(\tilde{G}, \tilde{\boldsymbol{\mu}}). \tag{3.40}$$

(2) Recall that the map $G(\mathbb{Q}_p) \to \pi_1(G)_I^{\varphi}$ is surjective (see for example [Zho20, Lemma 5.18]). Indeed, this follows from the exact sequence

$$0 \to \mathcal{T}(\breve{\mathbb{Z}}_p) \to T(\breve{\mathbb{Q}}_p) \to \pi_1(G)_I \to 0 \tag{3.41}$$

and the group cohomology vanishing $H^1(\mathbb{Z}, \mathcal{T}(\breve{\mathbb{Z}}_p)) = 0$, where $\mathcal{T}$ is the unique parahoric of T and the $\mathbb{Z}$-action on $\mathcal{T}(\breve{\mathbb{Z}}_p)$ is given by the Frobenius φ. Consider the following commutative diagram:

$$\begin{tikzcd} \tilde{G}(\mathbb{Q}_p) \arrow[r] \arrow[d] & \pi_1(\tilde{G})_I^{\varphi} \arrow[d] \\ G(\mathbb{Q}_p) \arrow[r] & \pi_1(G)_I^{\varphi} \end{tikzcd} \tag{3.42}$$

The horizontal arrows in (3.42) are surjective. Since Z is an induced torus, $H^1_{\text{ét}}(\operatorname{Spec}\mathbb{Q}_p, Z) = 0$. Thus by the exact sequence of pointed sets that

$$0 \to Z \to \tilde{G} \to G \to 0, \tag{3.43}$$

induces, the map $\tilde{G}(\mathbb{Q}_p) \to G(\mathbb{Q}_p)$ is surjective. Therefore $\pi_1(\tilde{G})_I^{\varphi} \to \pi_1(G)_I^{\varphi}$ is surjective. Finally, since $\tilde{\mathbf{b}}$ and $\tilde{\boldsymbol{\mu}}$ map to $\mathbf{b}$ and $\boldsymbol{\mu}$, the coset $c_{\tilde{b},\tilde{\mu}}\pi_1(\tilde{G})_I^{\varphi}$ also maps to the coset $c_{b,\mu}\pi_1(\tilde{G})_I^{\varphi}$. □

Assume that f is an ad-isomorphism for the rest of this subsection. Let $b_H := f(b)$ and $\mu_H := f \circ \mu$. Let $\mathcal{K}_p^H$ denote the unique parahoric of H that corresponds to the same point in the Bruhat–Tits building as $\mathcal{K}_p$.

Proposition 3.3.15. *The following diagram is Cartesian:*

$$\begin{tikzcd} \pi_0(X_\mu^{\mathcal{K}_p}(b)) \arrow[r] \arrow[d] & c_{b,\mu}\pi_1(G)_I^{\varphi} \arrow[d] \\ \pi_0(X_\mu^{\mathcal{K}_p^H}(b)) \arrow[r] & c_{b_H,\mu_H}\pi_1(H)_I^{\varphi} \end{tikzcd} \tag{3.44}$$

Proof. This is a consequence of [PR22, Lemma 5.4.2], which is a generalization of [CKV15, Corollary 2.4.2] for arbitrary parahorics. □

3.4 Hodge–Newton decomposition

We can classify elements in $B(G, \boldsymbol{\mu})$ into two kinds: Hodge-Newton decomposable or indecomposable.

Definition 3.4.1 (Hodge-Newton Decomposability)**.** *Assume* $\mathbf{b} \in B(G, \boldsymbol{\mu})$*. We say* $\mathbf{b}$ *is Hodge-Newton decomposable (with respect to* M*) in* $B(G, \boldsymbol{\mu})$ *if there exists a* φ_0*-stable standard Levi subgroup* M *containing* $M_{\boldsymbol{\nu}_{\mathbf{b}}}$*, and*

$$\mu^\diamond - \boldsymbol{\nu}_{\mathbf{b}} \in \mathbb{Q}_{\geq 0}\Delta_M^\vee. \tag{3.45}$$

If no such M *exists,* $\mathbf{b}$ *is said to be Hodge-Newton indecomposable in* $B(G, \boldsymbol{\mu})$*.*

Example 3.4.2. *A basic element* $\mathbf{b}$ *is always HN-indecomposable in* $B(G, \boldsymbol{\mu})$ *since* $M_{\boldsymbol{\nu}_{\mathbf{b}}} = G$*.*

For a HN-decomposable $\mathbf{b}$ in $B(G, \boldsymbol{\mu})$, affine Deligne–Lusztig varieties admit a decomposition theorem (Theorem 3.4.3). More precisely, suppose $\mathbf{b}$ is HN-decomposable with respect to a Levi subgroup M. Let P be the standard parabolic subgroup containing M and B. As in [GHN19, 4.4], let $\mathfrak{P}^\varphi$ be the set of φ-stable parabolic subgroups containing the maximal torus T and conjugate to P. Given $P' \in \mathfrak{P}^\varphi$, let N' be the unipotent radical, and M' the Levi subgroup containing T such that $P' = M'N'$. We let $\mathcal{K}_p^{M'}$ denote the parahoric group scheme of M' such that $\mathcal{K}_p^{M'}(\breve{\mathbb{Q}}_p) = K_p \cap M'(\breve{\mathbb{Q}}_p)$. Let W_K be the subgroup of W_0 generated by the set of simple reflections corresponding to $\mathcal{K}_p$. Let W_K^φ be the φ-invariant elements of W_K. We have the following.

Theorem 3.4.3 ([GHN19, Theorem A])**.** *Let* $\mathbf{b} \in B(G, \boldsymbol{\mu})$ *be HN-decomposable with respect to* $M \subset G$*. Then there is an isomorphism*

$$X_\mu^{\mathcal{K}_p}(b) \simeq \bigsqcup_{P' = M'N'} X_{\mu_{P'}}^{\mathcal{K}_p^{M'}}(b_{P'}), \tag{3.46}$$

where P' *ranges over the set* $\mathfrak{P}^\varphi / W_K^\varphi$*.*

Note that the natural embedding

$$\phi_{P'} : X_{\mu_{P'}}^{\mathcal{K}_p^{M'}}(b_{P'}) \hookrightarrow X_\mu^{\mathcal{K}_p}(b) \tag{3.47}$$

is the composite of the closed immersion $\mathcal{F}\ell_{\mathcal{K}_p^{M'}} \hookrightarrow \mathcal{F}\ell_{\mathcal{K}_p}$ of affine flag varieties and the map $g\breve{\mathcal{K}}_p \mapsto h_{P'} g \breve{\mathcal{K}}_p$, where $h_{P'} \in G(\breve{\mathbb{Q}}_p)$ satisfies $b_{P'} = h_{P'}^{-1} b \sigma(h_{P'})$ ([GHN19, 4.5]).

By the following lemma, we may assume–without loss of generality in the proof of Proposition 3.4.10–that each $(b_{P'}, \mu_{P'})$ is HN-indecomposable.

Lemma 3.4.4 ([Zho20, Lemma 5.7])**.** *There exists a unique* φ_0*-stable* $M \subset G$ *such that, for each* P' *appearing in decomposition* (3.46)*,* $b_{P'}$ *is HN-indecomposable in* $B(M', \boldsymbol{\mu}_{P'})$*.*

Example 3.4.5. *When* G^{ad} *is simple,* $\mathbf{b}$ *is basic and* $\boldsymbol{\mu}$ *is not central, then* $\mathbf{b}$ *is Hodge-Newton irreducible (Definition 3.2.1) in* $B(G, \boldsymbol{\mu})$ *because if a linear combination of coroots is dominant then all the coefficients are positive.*

Example 3.4.5 shows that, except for the "central cocharacter" case, HN-indecomposability is the same as HN-irreducibility whenever $\mathbf{b}$ is basic. The general version of this phenomena is Proposition 3.4.6 below, which asserts that the gap between HN-indecomposable and HN-irreducible elements consists only of central elements.

Proposition 3.4.6 (cf. [Zho20, Lemma 5.3])**.** *Suppose that $G = G^{\mathrm{ad}}$ and that G is $\mathbb{Q}_p$-simple. Let $b \in G(\breve{\mathbb{Q}}_p)$ and μ a dominant cocharacter, such that $\mathbf{b} \in B(G, \boldsymbol{\mu})$. Suppose $(\mathbf{b}, \boldsymbol{\mu})$ is HN-indecomposable. Then either $(\mathbf{b}, \boldsymbol{\mu})$ is HN-irreducible or b is φ-conjugate to some $\dot{t}_{\bar{\mu}}$ with $\bar{\mu} \in X_*(T)_I$ central.*

Moreover, when b is φ-conjugate to $\dot{t}_{\bar{\mu}}$ for a central μ, the connected components of affine Deligne–Lusztig varieties have been computed in Proposition 3.4.7 below. Note that if μ is central, there is a unique $\mathbf{b} \in B(G, \boldsymbol{\mu})$. Moreover, this $\mathbf{b}$ is basic and represented by $\dot{t}_{\bar{\mu}}$, which is a lift of $t_{\bar{\mu}}$ to $N(\breve{\mathbb{Q}}_p)$. We can then apply the following result.

Proposition 3.4.7 ([HZ20a, Theorem 0.1 (1)])**.** *Suppose that G^{ad} is $\mathbb{Q}_p$-simple. Let $b \in G(\breve{\mathbb{Q}}_p)$ be a representative for a basic element $\mathbf{b} \in B(G)$. If μ is central and $\mathbf{b} \in B(G, \boldsymbol{\mu})$, then $X_{\mu}^{\mathcal{K}_p}(b)$ is discrete and*

$$X_{\mu}^{\mathcal{K}_p}(b) \simeq G(\mathbb{Q}_p)/\mathcal{K}_p(\mathbb{Z}_p). \tag{3.48}$$

12. *Next we show that HN-irreducibility is preserved under ad-isomorphisms and taking projection onto direct factors. Let $f : G \to H$ be an ad-isomorphism. Let $b_H := f(b)$ and $\mu_H = \mu \circ f$. Let T_H denote a maximal torus containing $f(T)$. By functoriality, we have commutative diagrams*

$$\begin{tikzcd} X_*(T) \arrow[r, "f_*"] \arrow[d] & X_*(T_H) \arrow[d] \\ \pi_1(G)_\Gamma \arrow[r] & \pi_1(H)_\Gamma \end{tikzcd} \tag{3.49}$$

and

$$\begin{tikzcd} B(G) \arrow[r] \arrow[d] & B(H) \arrow[d] \\ \pi_1(G)_\Gamma \arrow[r] & \pi_1(H)_\Gamma \end{tikzcd} \qquad \begin{tikzcd} B(G) \arrow[r] \arrow[d] & B(H) \arrow[d] \\ X_*(T)_{\mathbb{Q}}^+ \arrow[r] & X_*(T)_{\mathbb{Q}}^+ \end{tikzcd} \tag{3.50}$$

We have the following.

Proposition 3.4.8. *Let $\mathbf{b} \in B(G, \boldsymbol{\mu})$ and let f be an ad-isomorphism. Then $(\mathbf{b}_H, \boldsymbol{\mu}_H)$ is HN-irreducible if and only if $(\mathbf{b}, \boldsymbol{\mu})$ is HN-irreducible.*

Proof. Since $\mathbf{b} \in B(G, \boldsymbol{\mu})$, we have $\kappa_G(\mathbf{b}) = \mu^\natural$ (see 2). By (3.49) and (3.50), we have $\kappa_H(\mathbf{b}_H) = \mu_H^\natural$. Moreover, we can write

$$\mu^\diamond - \boldsymbol{\nu}_{\mathbf{b}} = \sum_{\alpha \in \Phi^+} c_\alpha \alpha^\vee, \tag{3.51}$$

where $c_\alpha \geq 0$. On the other hand, note that $f_*(\mu^\diamond - \boldsymbol{\nu}_{\mathbf{b}}) = \mu_H^\diamond - \boldsymbol{\nu}_{\mathbf{b}_H}$. Since f is an ad-isomorphism, $f_*(\alpha^\vee) = \alpha^\vee$. Thus we have $\mu_H^\diamond - \boldsymbol{\nu}_{\mathbf{b}_H} = \sum_{\alpha\in\Phi^+} c_\alpha \alpha^\vee$, and hence $\mathbf{b}_H \in B(H, \boldsymbol{\mu}_H)$. Now, each $(\mathbf{b}_H, \boldsymbol{\mu}_H)$ is HN-irreducible if and only if $(\mathbf{b}, \boldsymbol{\mu})$ is, since this is in turn equivalent to $c_\alpha > 0$ for all $\alpha \in \Phi_G^+$. ☐

13. *Let $G = G_1 \times G_2$, then $T = T_1 \times T_2$, $B(G) = B(G_1) \times B(G_2)$, $\pi_1(G)_\Gamma = \pi_1(G_1)_\Gamma \times \pi_1(G_2)_\Gamma$ and $X_*(T) = X_*(T_1) \times X_*(T_2)$. In this case, the Kottwitz and Newton maps 3 can be computed coordinatewise.*

Proposition 3.4.9. *The following hold:*

1. $\mathbf{b} \in B(G, \boldsymbol{\mu})$ *if and only if each* $\mathbf{b}_i \in B(G_i, \boldsymbol{\mu}_i)$ *for* $i \in \{1, 2\}$.
2. $(\mathbf{b}, \boldsymbol{\mu})$ *is HN-irreducible if and only if each* $(\mathbf{b}_i, \mu_i)$ *is HN-irreducible for* $i \in \{1, 2\}$.

Proof. The condition $\kappa_G(\mathbf{b}) = \mu^\natural$ can be checked component-wise. Moreover, since $\mu^\diamond - \boldsymbol{\nu}_{\mathbf{b}} = (\mu_1^\diamond - \boldsymbol{\nu}_{\mathbf{b}1}, \mu_2^\diamond - \boldsymbol{\nu}_{\mathbf{b}2})$, verifying whether it is a non-negative (resp. positive) sum of positive coroots (see Definition 3.2.1) can also be done component-wise. ☐

Proposition 3.4.10. *Assume G^{ad} has only isotropic factors. If the Kottwitz map $\omega : \pi_0(X_\mu(b)) \to c_{b,\mu}\pi_1(G)_I^\varphi$ is a bijection, then $(\mathbf{b}, \boldsymbol{\mu})$ is HN-irreducible.*

Proof. By Proposition 3.4.8, Proposition 3.4.9, Lemma 3.3.4 and Proposition 3.3.15, we may assume without loss of generality that G is adjoint and $\mathbb{Q}_p$-simple. We prove by contradiction and assume that $(\mathbf{b}, \boldsymbol{\mu})$ is not HN-irreducible.

(I) If $\mathbf{b}$ is HN-decomposable in $B(G, \boldsymbol{\mu})$, then by Theorem 3.4.3, we have

$$\pi_0(X_\mu(b)) = \bigsqcup_{P' \in \mathfrak{P}^\varphi / W_K^\varphi} \pi_0(X_{\mu_{P'}}^{M'}(b_{P'})). \tag{3.52}$$

Thus by Lemma 3.4.4, we may assume that each $b_{P'}$ is HN-indecomposable in $B(M', \mu_{P'})$. Recall from (3.47) that for each $P' \in \mathfrak{P}^\varphi / W_K^\varphi$ we have an embedding $\phi_{P'} : X_{\mu_{P'}}^{M'}(b_{P'}) \hookrightarrow X_\mu(b)$, which induces a map

$$\pi_0(\phi_{P'}) : \pi_0(X_{\mu_{P'}}^{M'}(b_{P'})) \hookrightarrow \pi_0(X_\mu(b)). \tag{3.53}$$

The disjoint union over $P' \in \mathfrak{P}^\varphi / W_K^\varphi$ in (3.53) gives the bijection (3.52).

Consider $\iota : M'(\breve{F}) \to G(\breve{F})$. Let $\iota_I : \pi_1(M')_I \to \pi_1(G)_I$ be the induced map, which then induces a map $\iota_I^\varphi : \pi_1(M')_I^\varphi \to \pi_1(G)_I^\varphi$. By [Kot97b, 7.4], the following diagram commutes:

$$\begin{array}{ccc} M'(\breve{\mathbb{Q}}_p) & \xrightarrow{\kappa_{M'}} & \pi_1(M')_I \\ {\scriptstyle\iota}\downarrow & & \downarrow{\scriptstyle\iota_I} \\ G(\breve{\mathbb{Q}}_p) & \xrightarrow{\kappa_G} & \pi_1(G)_I \end{array} \tag{3.54}$$

Denote by $+_{h_{P'}} : \pi_1(G)_I \to \pi_1(G)_I$ the addition-by-$\kappa_G(h_{P'})$ map. Then (3.54) shows that $+_{h_{P'}} \circ \iota_I$ sends $c_{b_{P'},\mu_{P'}}\pi_1(M')_I^{\varphi}$ to $c_{b,\mu}\pi_1(G)_I^{\varphi}$. Moreover, we have the following commutative diagram

$$\begin{tikzcd} \pi_0(X^{M'}_{\mu_{P'}}(b_{P'})) \arrow[r, two heads, "\omega_{M'}"] \arrow[d, hook, "\pi_0(\phi_{P'})"'] & c_{b_{P'},\mu_{P'}}\pi_1(M')_I^{\varphi} \arrow[d, "+_{h_{P'}}\circ\iota_I"] \\ \pi_0(X_{\mu}(b)) \arrow[r, "\omega_G", "\cong"'] & c_{b,\mu}\pi_1(G)_I^{\varphi} \end{tikzcd} \tag{3.55}$$

Here the surjectivity of $\omega_{M'}$ follows from [HZ20a, Lemma 6.1]. Now, if the lower horizontal arrow ω_G is a bijection, then the upper horizontal arrow $\omega_{M'}$ should also be a bijection. Moreover, this implies that $+_{h_{P'}} \circ \iota_I$ is injective, which then implies that $\iota_I^{\varphi} : \pi_1(M')_I^{\varphi} \to \pi_1(G)_I^{\varphi}$ is injective. This contradicts Lemma 3.4.11.

(II) If $\mathbf{b}$ is HN-indecomposable in $B(G,\boldsymbol{\mu})$, by Proposition 3.4.6, we may assume that μ is central and $b = \dot{t}_{\bar{\mu}}$. We now show that $\pi_0(X_{\mu}(b)) \to \pi_1(G)_I^{\varphi}$ is not bijective.

By Proposition 3.4.7, there is a bijection $\pi_0(X_{\mu}(b)) \simeq G(\mathbb{Q}_p)/\mathcal{K}_p(\mathbb{Z}_p)$. Since G is not anisotropic, there exists a non-trivial $\mathbb{Q}_p$-split torus S, and we can consider the composition of maps

$$S(\mathbb{Q}_p) \hookrightarrow G(\mathbb{Q}_p) \twoheadrightarrow G(\mathbb{Q}_p)/\mathcal{K}_p(\mathbb{Z}_p). \tag{3.56}$$

Since $S(\mathbb{Q}_p) \cap \mathcal{K}_p(\mathbb{Z}_p)$ is compact, we have $S(\mathbb{Q}_p) \cap \mathcal{K}_p(\mathbb{Z}_p) \subseteq S(\mathbb{Z}_p)$. Therefore, we obtain an injective homomorphism

$$X_*(S) \cong S(\mathbb{Q}_p)/S(\mathbb{Z}_p) \hookrightarrow G(\mathbb{Q}_p)/\mathcal{K}_p(\mathbb{Z}_p). \tag{3.57}$$

Since G is adjoint, $\pi_1(G)_I^{\varphi}$ is finite. However, $X_*(S)$ is infinite, thus the map $\omega_G : \pi_0(X_{\mu}(b)) \to \pi_1(G)_I^{\varphi}$ cannot be bijective. We have a contradiction. $\square$

Now we finish the proof of Proposition 3.4.10 by proving the following lemma.

Lemma 3.4.11. *Let G be adjoint and $\mathbb{Q}_p$-simple. Let $P \subseteq G$ be a proper parabolic defined over $\mathbb{Q}_p$ with Levi factor M. The natural map $\iota_I^{\varphi} : \pi_1(M)_I^{\varphi} \to \pi_1(G)_I^{\varphi}$ is not injective.*

Proof. Recall that $(\pi_1(G)_I)_{\hat{\mathbb{Z}}} \simeq \pi_1(G)_{\Gamma}$. We prove by contradiction and assume that the natural map $\iota_I^{\varphi} : \pi_1(M)_I^{\varphi} \to \pi_1(G)_I^{\varphi}$ is injective. In particular, $\pi_1(M)_I^{\varphi} \otimes \mathbb{Q} \hookrightarrow \pi_1(G)_I^{\varphi} \otimes \mathbb{Q}$ is also injective. Via the "average map" under φ-action, we have

$$\pi_1(-)_I^{\varphi} \otimes \mathbb{Q} \simeq (\pi_1(-)_I)_{\langle\varphi\rangle} \otimes \mathbb{Q} \simeq \pi_1(-)_{\Gamma} \otimes \mathbb{Q} \simeq \pi_1(-)^{\Gamma} \otimes \mathbb{Q}. \tag{3.58}$$

If ι_I^{φ} is injective, we deduce that the natural map

$$\pi_1(M)^{\Gamma} \otimes \mathbb{Q} \to \pi_1(G)^{\Gamma} \otimes \mathbb{Q} \tag{3.59}$$

is injective. Let $M \subseteq P \subseteq G$ be the corresponding parabolic subgroup. Let $\theta_P = \sum_{\alpha\in\Phi_P} \alpha^{\vee} \in X_*(T)$ denote the sum of coroots of P. Now, θ_P is Γ-stable since P is defined over $\mathbb{Q}_p$. Moreover, its image under the natural projection map $q_M : X_*(T) \to \pi_1(M)$ is Γ-stable. One can check that $q_M(\theta) \neq 0$ in $\pi_1(M)^{\Gamma} \otimes \mathbb{Q}$. Since $q_G(\theta_P) = 0$ in $\pi_1(G)$, the map in (3.59) is not injective. We have a contradiction, this proves that ι_I^{φ} is not injective. $\square$

3.5 Generic Mumford–Tate groups

3.5.1 Mumford–Tate groups of crystalline representations

We will use the theory of *crystalline representations with G-structures* (see for example [DOR10]). Let Rep_G be the category of algebraic representations of G in $\mathbb{Q}_p$-vector spaces. Let Isoc be the category of isocrystals over $\bar{\mathbb{F}}_p$.

Fix a finite extension K of $\breve{\mathbb{Q}}_p$. Let $\mathrm{Rep}^{\mathrm{cris}}_{\Gamma_K}$ be the category of crystalline representations of Γ_K on finite-dimensional $\mathbb{Q}_p$-vector spaces. Let $\omega : \mathrm{Rep}^{\mathrm{cris}}_{\Gamma_K} \to \mathrm{Vec}_{\mathbb{Q}_p}$ be the forgetful fibre functor. Let $\mathrm{IsocFil}_{K/\breve{\mathbb{Q}}_p}$ be the category of filtered isocrystals whose objects are pairs of an isocrystal N and a decreasing filtration of $N \otimes K$. Furthermore, let $\mathrm{IsocFil}^{\mathrm{ad}}_{K/\breve{\mathbb{Q}}_p}$ be Fontaine's subcategory of weakly admissible filtered isocrystals [Fon94]. This is a $\mathbb{Q}_p$-linear Tannakian category, which is equivalent to $\mathrm{Rep}^{\mathrm{cris}}_{\Gamma_K}$ through Fontaine's functor V_{cris} [CF00].

14. *Fix a pair (b, μ^η) with $b \in G(\breve{\mathbb{Q}}_p)$ and $\mu^\eta : \mathbb{G}_m \to G_K$ a group homomorphism over K. This defines a $\otimes$-functor*

$$\mathcal{G}_{(b,\mu^\eta)} : \mathrm{Rep}_G \to \mathrm{IsocFil}_{K/\breve{\mathbb{Q}}_p} \tag{3.60}$$

sending $\rho : G \to \mathrm{GL}(V)$ to the filtered isocrystal $(V \otimes \breve{\mathbb{Q}}_p, \rho(b)\sigma, \mathrm{Fil}^\bullet_{\mu^\eta} V \otimes K)$, where the filtration on $V \otimes K$ is the one induced by μ^η. The pair is called admissible [RZ96, Definition 1.18], if the image of $\mathcal{G}_{(b,\mu^\eta)}$ lies in $\mathrm{IsocFil}^{\mathrm{ad}}_{K/\breve{\mathbb{Q}}_p}$. Moreover, when $\mathbf{b} \in B(G, \boldsymbol{\mu})$, $V_{\mathrm{cris}} \circ \mathcal{G}_{(b,\mu^\eta)}$ defines a conjugacy class of crystalline representations $\xi_{(b,\mu^\eta)} : \Gamma_K \to G(\mathbb{Q}_p)$ [DOR10, Proposition 11.4.3].

Definition 3.5.1. *With notation as above, let $\mathrm{MT}_{(b,\mu^\eta)}$ denote the identity component of the Zariski closure of $\xi_{(b,\mu^\eta)}(\Gamma_K)$ in $G(\mathbb{Q}_p)$. This is the Mumford–Tate group attached to (b, μ^η).*

Theorem 3.5.2. *([Ser79, Théorème 1],[Sen73, §4, Théorème 1], [Che14, Proposition 3.2.1]) The image of $\xi_{(b,\mu^\eta)}$ contains an open subgroup of $\mathrm{MT}_{(b,\mu^\eta)}$.*

15. *As in [Che14, §3], we let $\mathscr{T}^{\mathrm{cris}}_{(b,\mu^\eta)} := \mathcal{G}_{(b,\mu^\eta)}(\mathrm{Rep}_G)$ and $\mathscr{T}_{(b,\mu^\eta)} := V_{\mathrm{cris}} \circ \mathcal{G}_{(b,\mu^\eta)}(\mathrm{Rep}_G)$ be the images of Rep_G. Then $\mathrm{MT}_{(b,\mu^\eta)}$ is the Tannakian group for the fiber functor $\omega : \mathscr{T}_{(b,\mu^\eta)} \to \mathrm{Vec}_{\mathbb{Q}_p}$ by [Che14, Proposition 3.2.3].*

In [Che14, §3], there is a fiber functor $\omega_s : \mathscr{T}^{\mathrm{cris}}_{(b,\mu^\eta)} \to \mathrm{Vec}_{\mathbb{Q}_{p^s}}$ for s sufficiently large[19]*, with Tannakian group $\mathrm{MT}^{\mathrm{cris},s}_{(b,\mu^\eta)} := \mathrm{Aut}^\otimes \omega_s$ as in [Che14, Définition 3.3.1]. When b is s-decent (see Definition 3.2.2), there is a canonical embedding $\mathrm{MT}^{\mathrm{cris},s}_{(b,\mu^\eta)} \subseteq G_{\mathbb{Q}_{p^s}}$ [Che14, Lemme 3.3.2]. Moreover, $\mathrm{MT}^{\mathrm{cris},s}_{(b,\mu^\eta)}$ and $\mathrm{MT}_{(b,\mu^\eta)} \otimes_{\mathbb{Q}_p} \mathbb{Q}_{p^s}$ are pure inner forms of each other [Che14, Proposition 3.3.3]. Both claims follow immediately using Tannakian formalism. In particular, to prove that $\mathrm{MT}_{(b,\mu^\eta)}$ contains G^{der}, it suffices to prove $\mathrm{MT}^{\mathrm{cris},s}_{(b,\mu^\eta)}$ contains $G^{\mathrm{der}}_{\mathbb{Q}_{p^s}}$ (since G^{der} is normal).*

[19] Note that our notation ω_s differs from the notations *loc.cit.*, where the notation $\omega^{\mathrm{cris},s}_{b,\mu}$ is used instead.

16. *In fact, there is a more concrete description of* $\mathrm{MT}^{\mathrm{cris},s}_{(b,\mu^\eta)}$ *given as follows.*

Let $(V,\rho) \in \mathrm{Rep}_G$. *The* μ^η*-filtration of* V_K *induces a degree function*

$$\deg_{\mu^\eta} : V \setminus \{0\} \to \mathbb{Z}, \tag{3.61}$$

where $\deg_{\mu^\eta}(v) = i$ *if* $v \in \mathrm{Fil}^i_{\mu^\eta} V \setminus \mathrm{Fil}^{i+1}_{\mu^\eta} V$. *We shall consider a subset* $V^{s,k}_{(b,\mu^\eta)} \subseteq V \otimes \mathbb{Q}_{p^s}$ *of elements that satisfy a certain "Newton equation" (3.62) and a certain "Hodge equation" (3.63) with respect to* k.

Let $T^{s\cdot\nu_b}_\rho : V \otimes \mathbb{Q}_{p^s} \to V \otimes \mathbb{Q}_{p^s}$ *be the operator with formula*

$$T^{s\cdot\nu_b}_\rho := \rho \circ [s \cdot \nu_b](p). \tag{3.62}$$

Consider also the function $d^s_{\rho,\mu^\eta} : V \otimes \mathbb{Q}_{p^s} \setminus \{0\} \to \mathbb{Z}$ *where*

$$d^s_{\rho,\mu^\eta}(v) = \sum_{i=0}^{s-1} \deg_{\mu^\eta}([\rho(b)\varphi]^i(v)). \tag{3.63}$$

We consider the following subset of $V \otimes \mathbb{Q}_{p^s}$ *given by*

$$V^{s,k}_{(b,\mu^\eta)} := \{v \in V \otimes \mathbb{Q}_{p^s} \mid T^{s\cdot\nu_b}_\rho(v) = p^k v,\, d^s_{\rho,\mu^\eta}(v) = k\} \tag{3.64}$$

By [Che14, Proposition 3.3.6], $\mathrm{MT}^{\mathrm{cris},s}_{(b,\mu^\eta)}$ *consists of those elements* $g \in G_{\mathbb{Q}_{p^s}}$ *such that: for any* $(V,\rho) \in \mathrm{Rep}_G$ *and any* $k \in \mathbb{Z}$, *all of the elements* $v \in V^{s,k}_{(b,\mu^\eta)}$ *are eigenvectors of* $\rho(g)$. *In particular, to prove* $G^{\mathrm{der}}_{\mathbb{Q}_{p^s}} \subseteq \mathrm{MT}^{\mathrm{cris},s}_{(b,\mu^\eta)}$, *it suffices to prove that* $G^{\mathrm{der}}_{\mathbb{Q}_{p^s}}$ *acts trivially on* $V^{s,k}_{(b,\mu^\eta)}$ *for all* V *and* k.

3.5.2 Generic filtrations

17. *As in [Che14, §4], we give a representation-theoretic formula for* d^s_{ρ,μ^η} *when* μ^η *is generic. In our case,* G *is not assumed to be neither unramified nor quasisplit.*

We first recall some generalities, which we will apply later to $G_{\mathbb{Q}_{p^s}}$ *for s-sufficiently large such that* $G_{\mathbb{Q}_{p^s}}$ *is quasisplit. Until further notice,* K *will denote an arbitrary field of characteristic* 0, G *a quasisplit reductive group over* K, *and* $\boldsymbol{\mu}$ *a conjugacy class of group homomorphisms* $\mu : \mathbb{G}_m \to G_{\bar{K}}$. *Let* E/K *be the reflex field of* $\boldsymbol{\mu}$. *Since* G *is quasisplit, we can choose a representative* $\mu \in \boldsymbol{\mu}$ *defined over* E *such that it is dominant for a choice of* K*-rational Borel* $B \subseteq G$. *To this data, we can associate a flag variety* $\mathrm{Fl}_\mu := G_E/P_\mu$ *over* $\mathrm{Spec}(E)$ *as in (3.36). It parametrizes filtrations of* Rep_G *of type* $\boldsymbol{\mu}$. *Given a field extension* K'/K, $x \in \mathrm{Fl}_\mu(K')$ *and* $(V,\rho) \in \mathrm{Rep}_G$, *we obtain a filtration* $\mathrm{Fil}^\bullet_x V_{K'}$ *as in [Che14, Définition 4.1.1].*

Definition 3.5.3 ([Che14, Définition 4.2.1])**.** *With the setup as above, let*

$$\overline{\mathrm{Fil}}^{\bullet}_{\mu} V_E := \left(\bigcap_{x \in \mathrm{Fl}_\mu(E)} \mathrm{Fil}^{\bullet}_x V_E \right) \tag{3.65}$$

$$\overline{\mathrm{Fil}}^{\bullet}_{\boldsymbol{\mu}} V := V \cap \left(\bigcap_{x \in \mathrm{Fl}_\mu((E)} \mathrm{Fil}^{\bullet}_x V_E \right). \tag{3.66}$$

We refer to (3.65) *(resp.* (3.66)*) as the generic filtration of V_E (resp. V) attached to μ (resp. $\boldsymbol{\mu}$).*

18. *Each step of $\overline{\mathrm{Fil}}^{i}_{\boldsymbol{\mu}} V$ is a subrepresentation of V. Moreover,*

$$\overline{\mathrm{Fil}}^{\bullet}_{\boldsymbol{\mu}} V = V \cap \left(\bigcap_{g \in G(E)} \rho(g) \mathrm{Fil}^{\bullet}_{\mu} V_E \right). \tag{3.67}$$

This filtration $\overline{\mathrm{Fil}}^{i}_{\boldsymbol{\mu}} V$ gives rise to a degree function $\overline{\deg}_{\boldsymbol{\mu}} : V \setminus \{0\} \to \mathbb{Z}$ which can be computed as:

$$\overline{\deg}_{\boldsymbol{\mu}}(v) = \inf_{g \in G(E)} \deg_\mu(\rho(g) \cdot v). \tag{3.68}$$

Let K' be an arbitrary extension of K.

Definition 3.5.4. *We say that a map $\mathrm{Spec}(K') \to \mathrm{Fl}_\mu$ is generic if, at the level of topological spaces $|\mathrm{Spec}(K')| \to |\mathrm{Fl}_\mu|$, the image of the unique point on the left is the generic point of Fl_μ.*

The following statement relates $\overline{\mathrm{Fil}}^{\bullet}_{\boldsymbol{\mu}} V$ (see (3.66) or (3.67)) to the generic points of Fl_μ in the sense of Definition 3.5.4.

Proposition 3.5.5 ([Che14, Lemme 4.2.2])**.** *Let $\mu^\eta : \mathrm{Spec}(K') \to \mathrm{Fl}_\mu$ be generic (in the sense of Definition 3.5.4). Then for all $i \in \mathbb{Z}$, we have*

$$\overline{\mathrm{Fil}}^{i}_{\boldsymbol{\mu}} V = V \cap \mathrm{Fil}^i_{\mu^\eta} V_{K'}, \tag{3.69}$$

where the inclusion $V \subseteq V \otimes_K E \subseteq V \otimes_K K'$ is the natural one.

Proof. The following proof is in [Che14, 4.2.2]. We recall the argument for the convenience of the reader. Note that we do not assume G split over K, which is the running assumptions in *loc.cit.*

Let $\widetilde{\mathcal{Y}}_\mu$ be the the universal P_μ-bundle over $\mathrm{Fl}_\mu = G_E/P_\mu$ coming from the natural map to $[*/P_\mu]$. Consider the vector bundle $\mathcal{E} := \widetilde{\mathcal{Y}}_\mu \times_{P_{\mu,\rho}} V$, with a filtration

$$\cdots \supseteq \mathrm{Fil}^0\mathcal{E} \supseteq \mathrm{Fil}^1\mathcal{E} \supseteq \cdots \supseteq \mathrm{Fil}^n\mathcal{E} \supseteq \dots$$

of locally free locally direct factors of the form $\widetilde{\mathcal{Y}}_\mu \times_{P_\mu,\rho} \mathrm{Fil}^\bullet V$, where $\mathrm{Fil}^\bullet V$ is the natural filtration of V by subrepresentations of P_μ.

We may regard elements $v \in V$ as global sections of $\mathcal{E}$, and we have that

$$v \in \mathrm{Fil}^i_x V \Leftrightarrow v \in \ker\left(\Gamma(\mathrm{Fl}_\mu, \mathcal{E}/\mathrm{Fil}^i\mathcal{E}) \to \Gamma(\mathrm{Spec}\,\kappa(x), \mathcal{E}/\mathrm{Fil}^i\mathcal{E})\right).$$

The vanishing locus of such an element is a Zariski closed subset, and it contains the generic point if and only if it contains all the E-rational points. Thus $\overline{\mathrm{Fil}}^i_{\boldsymbol{\mu}} V = V \cap \mathrm{Fil}^i_{\mu^\eta} V_{K'}$. □

19. *We need a more easily computable description of* $\overline{\mathrm{Fil}}^\bullet_{\boldsymbol{\mu}} V$*. In [Che14, Proposition 4.3.2], there is such a description assuming that G is split over K. We now prove a generalization in the quasisplit case.*

Let Γ_K denote the Galois group of K. We fix K-rational tori $S \subseteq T \subseteq B \subseteq G$ where S is maximally split and T is the centralizer of S. Recall that, by combining the theory of highest weights and Galois theory, one can classify all irreducible representations of a quasisplit group by the Galois orbits $\mathcal{O} \subseteq X_*(T)^+$ of dominant weights. Given $\lambda \in X_*(T)^+$, let $\mathcal{O}_\lambda := \Gamma_K \cdot \lambda$ denote its Galois orbit. We also consider $\mathcal{O}^E_\lambda := \Gamma_E \cdot \lambda$. Given a Γ_K-Galois orbit (resp. Γ_E-Galois orbit) $\mathcal{O} \subseteq X_*(T)^+$ (resp. $\mathcal{O}^E \subseteq X_*(T)^+$), let $V_\mathcal{O}$ (resp. $V_{\mathcal{O}^E}$) denote the $\mathcal{O}$-isotypic (resp. $\mathcal{O}^E$-isotypic) direct summand of V (resp. V_E). We have

$$V_\mathcal{O} \otimes_K \bar{K} = \bigoplus_{\lambda \in \mathcal{O}} V^\lambda_{\bar{K}}. \tag{3.70}$$

$$V_{\mathcal{O}^E} \otimes_E \bar{K} = \bigoplus_{\lambda \in \mathcal{O}^E} V^\lambda_{\bar{K}}. \tag{3.71}$$

Where $V^\lambda_{\bar{K}}$ is the λ-isotypic part of $V_{\bar{K}}$. Let $\underline{\mathcal{O}} \in (X_*(T)^+_\mathbb{Q})^{\Gamma_K}$ (resp. $(X_*(T)^+_\mathbb{Q})^{\Gamma_E}$) be given by $\underline{\mathcal{O}} := \frac{1}{|\mathcal{O}|}\sum_{\lambda\in\mathcal{O}} \lambda$. When $\mathcal{O}_\lambda = \Gamma_K \cdot \lambda$, we have

$$\underline{\mathcal{O}_\lambda} = \frac{1}{[\Gamma_K : \Gamma_\lambda]} \sum_{\gamma \in \Gamma_K/\Gamma_\lambda} \gamma(\lambda) \tag{3.72}$$

Analogously, we have $\underline{\mathcal{O}^E_\lambda} = \frac{1}{[\Gamma_E:\Gamma_\lambda]} \sum_{\gamma\in\Gamma_E/\Gamma_\lambda} \gamma(\lambda))$. Let $\mathcal{W}$ denote the absolute Weyl group of G. Let $w_0 \in \mathcal{W}$ be the longest element, which is Γ_K-invariant.

Proposition 3.5.6. *Let the setup be as above. For any $(V,\rho) \in \mathrm{Rep}_G$, the generic filtration attached to μ is given by the formula:*

$$\overline{\mathrm{Fil}}^k_{\boldsymbol{\mu}} V = \bigoplus_{\substack{\lambda \in X_*(T)^+ \\ \langle \underline{\mathcal{O}^E_{\tau(\lambda)}}, w_0.\mu\rangle \geq k \\ \tau \in \mathrm{Gal}(E/K)}} V_{\mathcal{O}_\lambda} \tag{3.73}$$

Proof. Since $\overline{\mathrm{Fil}}^k_{\boldsymbol{\mu}} V$ consists of subrepresentations, it suffices to show that

$$V_{\mathcal{O}_\lambda} \subseteq \overline{\mathrm{Fil}}^k_{\boldsymbol{\mu}} V \iff k \leq \langle \underline{\mathcal{O}}^E_{\tau(\lambda)}, w_0.\mu \rangle \quad \forall \tau \in \mathrm{Gal}(E/K). \tag{3.74}$$

Let us first prove "$\Longrightarrow$". Let $V = \bigoplus_{\sigma \in \mathrm{Irrep}(T)} V_\sigma$ be the decomposition of $\rho|_T$. Over an algebraic closure, each V_σ decomposes as $V_\sigma = \bigoplus_{\chi' \in \mathcal{O}_\chi} V_{\chi'}$ for some $\chi \in X^*(T)$.

Observe that $\overline{\mathrm{Fil}}^k_{\boldsymbol{\mu}} V \subseteq \mathrm{Fil}^k_{\tau(\mu)} V_E$ for all $\tau \in \mathrm{Gal}(E/K)$, and by definition we have

$$\mathrm{Fil}^k_{\tau(\mu)} V_{\bar{K}} = \bigoplus_{\substack{\langle \chi, \tau(\mu) \rangle \geq k, \\ \chi \in X^*(T)}} V_{\bar{K},\chi}. \tag{3.75}$$

In particular, the anti-dominant weights appearing in $V_{\mathcal{O}_\lambda}$ pair with $\tau(\mu)$ to a number greater than or equal to k. In other words, $k \leq \langle w_0.\xi, \tau(\mu) \rangle$ for $w_0.\xi \in \mathcal{O}_{w_0.\lambda}$, but then pairing $\tau(\mu)$ with their Γ_E-average $w_0.\underline{\mathcal{O}}^E_\lambda$ will still be greater than or equal to k, i.e. $k \leq \langle w_0.\underline{\mathcal{O}}^E_\lambda, \tau(\mu) \rangle = \langle \underline{\mathcal{O}}^E_{\tau(\lambda)}, w_0.\mu \rangle$.

Thus

$$\overline{\mathrm{Fil}}^k_{\boldsymbol{\mu}} V \subset \bigoplus_{\substack{\lambda \in X^*(T)^+ \\ \langle \underline{\mathcal{O}}^E_{\tau(\lambda)}, w_0.\mu \rangle \geq k \\ \tau \in \mathrm{Gal}(E/K)}} V_{\mathcal{O}_\lambda}. \tag{3.76}$$

Let us now prove "$\Longleftarrow$". Suppose $k \leq \langle w_0.\underline{\mathcal{O}}^E_\lambda, \tau(\mu) \rangle$ for all $\tau \in \mathrm{Gal}(E/K)$, this implies that for at least one $w_0.\xi \in \mathcal{O}^E_{w_0.\lambda}$, we have $k \leq \langle w_0.\xi, \tau(\mu) \rangle$. Since $\langle \cdot, \cdot \rangle$ is Γ_E-equivariant, $k \leq \langle w_0.\xi, \tau(\mu) \rangle$ for all $w_0.\xi \in \mathcal{O}^E_{w_0.\lambda}$. We can view $w_0.\xi$ as a cocharacter of T and $V_{w_0.\xi} \subseteq \mathrm{Fil}^k_{\tau(\mu)} V$. Consider $W_\xi := V^\xi_{\bar{K}}$ the isotypic part of $V_{\bar{K}}$ associated to the highest weight representation of ξ on an algebraic closure of K. If χ is a weight appearing in W_ξ, then $k \leq \langle w_0.\xi, \tau(\mu) \rangle \leq \langle \chi, \tau(\mu) \rangle$, and thus $W_\xi \subseteq \mathrm{Fil}^k_{\tau(\mu)} V_{\bar{K}}$. In particular,

$$W_E := \left(\bigoplus_{w_0.\xi \in \mathcal{O}^E_{w_0\lambda}} W_\xi \right)^{\Gamma_E} \tag{3.77}$$

is a subrepresentation of $\mathrm{Fil}^k_{\tau(\mu)} V_E$ defined over E. Thus $W_E \subseteq \overline{\mathrm{Fil}}^k_{\tau(\mu)} V_E$ for all $\tau \in \mathrm{Gal}(E/K)$. Then $W := \bigoplus_{\tau \in \mathrm{Gal}(E/K)} \tau(W_E)$ is contained in

$$\bigcap_{\tau \in \mathrm{Gal}(E/K)} \overline{\mathrm{Fil}}^k_{\tau(\mu)} V_E, \tag{3.78}$$

and the $\mathrm{Gal}(E/K)$-fixed points of W are contained in

$$\overline{\mathrm{Fil}}^k_{\mu} V = V \cap \bigcap_{\tau \in \mathrm{Gal}(E/K)} \overline{\mathrm{Fil}}^k_{\tau(\mu)} V_E.$$

But $W^{\mathrm{Gal}(E/K)} = V_{\mathcal{O}_\lambda}$, proving the claim. $\square$

3.5.3 Mumford–Tate group computations

The goal of this section is to prove Theorem 3.5.7 (or Theorem 3.1.11 in the introduction).

Let G be a reductive group over $\mathbb{Q}_p$. Let K be a finite extension of $\breve{\mathbb{Q}}_p$. Let $b \in G(\breve{\mathbb{Q}}_p)$ be decent (Definition 3.2.2) and $\mu^\eta : \mathbb{G}_m \to G_K$ be generic (Definition 3.5.4) with $\mu^\eta \in \boldsymbol{\mu}$. As before, let $\mu \in X_*(T)^+$ be the unique B-dominant cocharacter of $\boldsymbol{\mu}$.

Theorem 3.5.7. *Suppose that b is decent, μ^η is generic and $\mathbf{b} \in B(G, \boldsymbol{\mu})$. The following hold:*

1. *(b, μ^η) is admissible.*
2. *If $(\mathbf{b}, \boldsymbol{\mu})$ is HN-irreducible, then $\mathrm{MT}_{(b,\mu^\eta)}$ contains G^{der}.*

Proof. We fix s large enough so that b is s-decent, G is quasisplit over $\mathbb{Q}_{p^s}$ and splits over a totally ramified extension of $\mathbb{Q}_{p^s}$ that we denote by L. Recall that replacing b by $g^{-1}b\varphi(g)$ and μ^η by $g^{-1}\mu^\eta g$ gives isomorphic fiber functors $\mathcal{G}_{(b,\mu^\eta)}$ (see (3.60)). Moreover, via this kind of replacement, we can arrange that $\nu_b = \boldsymbol{\nu}_{\mathbf{b}}$ as in 4. Note that this replacement preserves genericity of μ^η.

(1) The argument in [Che14, Théorème 5.0.6.(1)] goes through in our setting. Indeed, the only part in the proof *loc.cit.* using that G is unramified is to justify that $\mathrm{Fl}^{\mathrm{ad}}_\mu \neq \emptyset$ whenever $\mathbf{b} \in B(G, \boldsymbol{\mu})$, but this is true by [DOR10, Theorem 9.5.10] in full generality.

(2) Let $(V, \rho) \in \mathrm{Rep}_G$ and let $v \in V^{s,k}_{(b,\mu^\eta)}$ as in (3.64). By 16, it suffices to show that $\rho(g)v = v$ for all $g \in G^{\mathrm{der}}_{\mathbb{Q}_{p^s}}$. Over L, we can write $v = \sum_{\lambda \in \Lambda_v} v_\lambda$ where $\Lambda_v \subseteq X^*(T)^+$, $v_\lambda \in V^\lambda$ and $v_\lambda \neq 0$. Since v is defined over $\mathbb{Q}_{p^s}$, we have $\gamma(v_\lambda) = v_{\gamma(\lambda)}$ for $\gamma \in \Gamma_{\mathbb{Q}_{p^s}}$.

Given $\mathcal{O} \in \mathrm{Irrep}_{G_{\mathbb{Q}_{p^s}}}$, let $v_\mathcal{O} = \sum_{\lambda \in \mathcal{O} \subseteq \Lambda_v} v_\lambda$. We have $v_\mathcal{O} \in V_{\mathbb{Q}_{p^s}}$. By Proposition 3.5.6 and Proposition 3.5.5, we can write

$$\begin{aligned}
\deg_{\mu^\eta}(v) &= \overline{\deg}_{\boldsymbol{\mu}}(v) && (3.79)\\
&= \inf_{\lambda \in \Lambda_v} \overline{\deg}_{\boldsymbol{\mu}}(v_{\mathcal{O}_\lambda}) && (3.80)\\
&\leq \overline{\deg}_{\boldsymbol{\mu}}(v_{\mathcal{O}_\lambda}) && (3.81)\\
&= \inf_{\tau \in \mathrm{Gal}(E/K)} \langle \underline{\mathcal{O}^E_{\tau(\lambda)}}, w_0 \cdot \mu \rangle && (3.82)\\
&\leq \langle \underline{\mathcal{O}_\lambda}, w_0 \cdot \mu \rangle && (3.83)\\
&= \langle w_0 \cdot \lambda, \underline{\mu} \rangle, && (3.84)
\end{aligned}$$

Here (3.79) follows from Proposition 3.5.5. Since each step of $\overline{\mathrm{Fil}}^\bullet_{\boldsymbol{\mu}} V$ is a subrepresentation of V, in order for $v \in \overline{\mathrm{Fil}}^k_{\boldsymbol{\mu}} V$, each $v_{\mathcal{O}_\lambda}$ has to be in $\overline{\mathrm{Fil}}^\bullet_{\boldsymbol{\mu}} V$, and hence (3.80). Inequality (3.81) follows from the definition of infimum. (3.82) follows from Proposition 3.5.6, and the fact that

$$v_{\mathcal{O}_\lambda} = \sum_{\tau \in \mathrm{Gal}(E/K)} v_{\mathcal{O}^E_{\tau(\lambda)}}. \tag{3.85}$$

Since the average is smaller than the infimum, (3.83) follows. Finally, (3.84) follows from equivariance of the pairing $\langle \cdot, \cdot \rangle$ with respect to the Γ_K-action, and invariance of the pairing under the w_0-action.

Write $v^i = (\rho(b)\varphi)^i v$. Therefore, we have the following formula

$$
\begin{aligned}
d^s_{\rho,\mu^\eta}(v) &= \sum_{i=0}^{s-1} \deg_{\mu^\eta}((\rho(b)\varphi)^i v) && (3.86)\\
&= \sum_{i=0}^{s-1} \inf_{\lambda \in \Lambda_{v^i}} \overline{\deg}_{\boldsymbol{\mu}}(v^i_{\mathcal{O}_\lambda}) && (3.87)\\
&\leq \sum_{i=0}^{s-1} \langle \underline{\mathcal{O}_{\varphi^i(\lambda)^{\mathrm{dom}}}}, w_0 \cdot \mu \rangle && (3.88)\\
&= \sum_{i=0}^{s-1} \langle w_0 \cdot \varphi^i(\lambda)^{\mathrm{dom}}, \underline{\mu} \rangle && (3.89)\\
&= \sum_{i=0}^{s-1} \langle w_0 \cdot \varphi_0^i(\lambda), \underline{\mu} \rangle && (3.90)\\
&= \sum_{i=0}^{s-1} \langle w_0 \cdot \lambda, \varphi_0^i(\underline{\mu}) \rangle && (3.91)\\
&= s \cdot \langle w_0 \cdot \lambda, \mu^\diamond \rangle && (3.92)
\end{aligned}
$$

Equality (3.86) follows from the definition in (3.63). By Proposition 3.5.6, we obtain (3.87). Inequality (3.88) follows from the inequalites (3.79) through (3.84) above. Since $\lambda \in \Lambda_v$, we have $\varphi^i(\lambda)^{\mathrm{dom}} \in \Lambda_{v^i}$. Equality (3.89) follows from equivariance of $\langle , \rangle$ under the Galois action and w_0-action. Equality (3.90) follows from the definition of φ_0 in (1). Since T is φ_0-stable, (3.91) follows from equivariance of $\langle , \rangle$ under the φ_0-action. Equality (3.92) follows from the definition of $\mu^\diamond$ (see (3.17)).

Since $v \in V^{s,k}_{(b,\mu^\eta)}$, by (3.86) through (3.92), we have $\frac{k}{s} \leq \langle w_0 \cdot \lambda, \mu^\diamond \rangle$ for all $\lambda \in \Lambda_v$. On the other hand, over L, we have a decomposition $v = \sum\limits_{\chi \in X^*(T)} v_\chi$. Since we have arranged that $\nu_b = \boldsymbol{\nu}_{\mathbf{b}}$, by (3.62) and (3.64) we have

$$
T_\rho^{s\cdot\nu_b}(v) = \sum_{\chi \in X^*(T)} T_\rho^{s\cdot\nu_b}(v_\chi) = \sum_{\chi \in X^*(T)} p^{\langle \chi, s\cdot \boldsymbol{\nu}_{\mathbf{b}} \rangle} v_\chi. \tag{3.93}
$$

The assumption $v \in V^{s,k}_{(b,\mu^\eta)}$ forces χ to satisfy $\langle \chi, s \cdot \boldsymbol{\nu}_{\mathbf{b}} \rangle = k$ for all χ where $v_\chi \neq 0$. In particular, since $w_0 \cdot \lambda \leq \chi$ when $V_L^\chi \subseteq V_L^\lambda$, we have $\langle w_0 \cdot \lambda, \boldsymbol{\nu}_{\mathbf{b}} \rangle \leq \frac{k}{s}$ for all $\lambda \in \Lambda_v$. Therefore $\langle w_0 \cdot \lambda, \mu^\diamond - \boldsymbol{\nu}_{\mathbf{b}} \rangle \leq 0$. Since $(\mathbf{b}, \boldsymbol{\mu})$ is HN-irreducible, we have $\langle w_0 \cdot \lambda, \alpha^\vee \rangle = 0$ for all $\alpha \in \Delta$. Therefore, the action of G_L^{der} on V^λ is trivial for all $\lambda \in \Lambda_v$. Thus we are done with the proof of (2) in Theorem 3.1.11. □

Proposition 3.5.8. *Let (G, b, μ) be a local shtuka datum over $\mathbb{Q}_p$ with $(\mathbf{b}, \boldsymbol{\mu})$ HN-irreducible. There exists a finite extension K over $\breve{\mathbb{Q}}_p$ containing the reflex field of $\boldsymbol{\mu}$, and a point $x \in \mathrm{Gr}_\mu^b(K)$ whose induced (conjugacy class of) crystalline representation(s)*

$$\rho_x : \Gamma_K \to G(\mathbb{Q}_p)$$

satisfies that $\rho_x(\Gamma_K) \cap G^{\mathrm{der}}(\mathbb{Q}_p)$ is open in $G^{\mathrm{der}}(\mathbb{Q}_p)$.

Proof. Recall from [Gle22a, Proposition 2.12] (see also [Vie21, Theorem 5.2]) that the Bialynicki-Birula map BB in (3.36) induces a bijection of classical points. Therefore it suffices to construct the image $\mathrm{BB}(x) \in \mathrm{Fl}_\mu$, which corresponds to constructing a weakly admissible filtered isocrystal with G-structure.

By Lemma 3.5.9, we can take $\mathrm{BB}(x) = \mu^\eta$ to be generic (Definition 3.5.4). By Theorem 3.5.7(2), $\mathrm{MT}_{(b,\mu^\eta)}$ contains G^{der}. By Theorem 3.5.2, the image of the generic crystalline representation $\xi_{(b,\mu^\eta)}$ contains an open subgroup of $\mathrm{MT}_{(b,\mu^\eta)}$, thus containing an open subgroup of G^{der}. □

Lemma 3.5.9. *There exist a finite extension K over $\breve{\mathbb{Q}}_p$ and a map $\mu^\eta : \mathrm{Spec}(K) \to \mathrm{Fl}_\mu$ such that $|\mu^\eta| : \{*\} \to |\,\mathrm{Fl}_\mu\,|$ maps to the generic point.*

Proof. Recall from [Che14, Proposition 2.0.3] that the transcendence degree of $\breve{\mathbb{Q}}_p$ over $\mathbb{Q}_p$ is infinite. By the structure theorem of smooth morphisms [Sta18, Tag 054L], one can find an open neighborhood $U \to \mathrm{Fl}_\mu$ that is étale over $\mathbb{A}^n_{\breve{\mathbb{Q}}_p}$. On the other hand, one can always find a map $\mathrm{Spec}(\breve{\mathbb{Q}}_p) \to \mathbb{A}^n_{\mathbb{Q}_p}$ mapping to the generic point by choosing n trascendentally independent elements of $\breve{\mathbb{Q}}_p$. Its pullback to U is an étale neighborhood of $\mathrm{Spec}(\breve{\mathbb{Q}}_p)$ that consists of a finite disjoint union of finite extensions K of $\breve{\mathbb{Q}}_p$. Any of these components will give a map to the generic point of Fl_μ. □

The following is a partial converse to Theorem 3.1.11, and it follows directly from Theorem 3.6.1.

Proposition 3.5.10 (Proposition 3.5.10)**.** *Assume that G^{ad} has only isotropic factors. If $\mathrm{MT}_{(b,\mu^\eta)}$ contains G^{der}, then $(\mathbf{b}, \boldsymbol{\mu})$ is HN-irreducible.*

Proof. If $G^{\mathrm{der}} \subseteq \mathrm{MT}_{(b,\mu^\eta)}$, then by Theorem 3.5.2 there exists a finite field extension K over $\breve{\mathbb{Q}}_p$, and a crystalline representation $\xi : \Gamma_K \to G(\mathbb{Q}_p)$ with invariants $(\mathbf{b}, \boldsymbol{\mu})$ whose image in $G^{\mathrm{der}}(\mathbb{Q}_p)$ is open. Indeed, we can let K be generic as in Definition 3.5.4. The result follows from the equivalence (3) $\iff$ (4) in Theorem 3.6.1. □

3.6 Proof of main theorems

The first goal in this section is to prove the following main theorem:

Theorem 3.6.1. *Suppose that $\mathbf{b} \in B(G, \boldsymbol{\mu})$ and that $G^{\mathrm{ad}} \neq \{e\}$ does not have anisotropic factors. The following statements are equivalent:*

1. *The map $\omega_G : \pi_0(X_\mu(b)) \to c_{b,\mu}\pi_1(G)_I^\varphi$ is bijective.*
2. *The map $\omega_G : \pi_0(X_\mu^{\mathcal{K}_p}(b)) \to c_{b,\mu}\pi_1(G)_I^\varphi$ is bijective.*
3. *The pair $(\mathbf{b}, \boldsymbol{\mu})$ is HN-irreducible.*
4. *There exists a finite field extension K over $\breve{\mathbb{Q}}_p$, and a crystalline representation $\xi : \Gamma_K \to G(\mathbb{Q}_p)$ with invariants $(\mathbf{b}, \boldsymbol{\mu})$ whose image in $G^{\mathrm{der}}(\mathbb{Q}_p)$ is open.*
5. *The action of $G(\mathbb{Q}_p)$ on $\mathrm{Sht}_{(G,b,\mu,\infty)}$ makes $\pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)} \times \mathrm{Spd}\,\mathbb{C}_p)$ into a G°-torsor.*

The second goal in this section is to prove the following corollary of Theorem 3.6.1.

Theorem 3.6.2. *Let G be arbitrary. Suppose that $(\mathbf{b}, \boldsymbol{\mu})$ is HN-irreducible, then the Kottwitz map $\omega_G : \pi_0(X_\mu^{\mathcal{K}_p}(b)) \to c_{b,\mu}\pi_1(G)_I^\varphi$ is bijective.*

The proof of the above main theorems proceeds as follows and will occupy the rest of section 3.6. We first prove a modified version of the statement in the case of tori (see §3.6.1, Proposition 3.6.4, Lemma 3.6.5). We then use z-extensions and ad-isomorphisms to reduce the proof of Theorem 3.6.1 and Theorem 3.6.2 to the case where $G^{\mathrm{der}} = G^{\mathrm{sc}}$ (see Proposition 3.6.7). We prove the circle of implications of Theorem 3.6.1 in this case. Then, we deduce Theorem 3.6.2 from Theorem 3.6.1 whenever G has no anisotropic factors. Finally, we deduce Theorem 3.6.2 in the anisotropic case.

Before we dive into the proofs of Theorem 3.1.1, we deduce Corollary 3.6.3 below. Let $(p, \mathbf{G}, X, \mathbf{K})$ be a tuple of global Hodge type [PR21, §1.3], let $\mathscr{S}_{\mathbf{K}}$ denote the integral model of [PR21, Theorem 1.3.2], let k an algebraically closed field in characteristic p and let $x_0 \in \mathscr{S}_{\mathbf{K}}(k)$. Pappas and Rapoport consider a map of v-sheaves $c : \mathrm{RZ}^\diamondsuit_{\mathcal{G},\mu,x_0} \to \mathcal{M}^{\mathrm{int}}_{\mathcal{G},b,\mu}$ [PR21, Lemma 4.1.0.2], where the source is a Rapoport–Zink space and the target is another name for $\mathrm{Sht}^{\mathcal{G}}_\mu(b)$ i.e. $\mathcal{M}^{\mathrm{int}}_{\mathcal{G},b,\mu} = \mathrm{Sht}^{\mathcal{G}}_\mu(b)$. Let the notations be as in [PR21, Theorem 4.10.6, §4.10.2].

Corollary 3.6.3. *The map $c : \mathrm{RZ}^\diamondsuit_{\mathcal{G},\mu,x_0} \to \mathcal{M}^{\mathrm{int}}_{\mathcal{G},b,\mu}$ is an isomorphism. Thus, $\mathcal{M}^{\mathrm{int}}_{\mathcal{G},b,\mu}$ is representable by a formal scheme $\mathscr{M}_{\mathcal{G},b,\mu}$, and we obtain a p-adic uniformization isomorphism of $O_{\breve{E}}$-formal schemes*

$$I_x(\mathbb{Q})\backslash(\mathscr{M}_{\mathcal{G},b,\mu} \times \mathbf{G}(\mathbb{A}_f^p)/\mathbf{K}^p) \to (\mathscr{S}_{\mathbf{K}} \widehat{\otimes_{O_E}} O_{\breve{E}})_{/\mathcal{I}(x)}. \tag{3.94}$$

Proof. It suffices to verify condition (U_x) in [PR21, §4.10.2]. Throughout the argument, we let the notations be as in [PR21, §4.8]. By [PR21, Proposition 4.10.3 and Lemma 4.10.2.b)], $c : \mathrm{RZ}^\diamondsuit_{\mathcal{G},\mu,x_0} \subseteq \mathcal{M}^{\mathrm{int}}_{\mathcal{G},b,\mu}$ is an open and closed immersion, and it suffices to check that for every $x \in \pi_0(\mathcal{M}^{\mathrm{int}}_{\mathcal{G},b,\mu})$ there is $y \in \pi_0(\mathrm{RZ}^\diamondsuit_{\mathcal{G},\mu,x_0})$ with $c(y) = x$. Let $\tilde{x}_0 \in \mathscr{S}_{\mathbf{K}}(\breve{E}_{\tilde{x}_0})$ be a $\breve{E}_{\tilde{x}_0}$-valued point of $\mathscr{S}_{\mathbf{K}}$ specializing to x_0 with $[\breve{E}_{\tilde{x}_0} : \breve{E}] < \infty$. Such a point exists by flatness of $\mathscr{S}_{\mathbf{K}}$ over

$\mathbb{Z}_{(p)}$. By Serre–Tate theory, $\tilde{x}_0$ induces a canonical point in $\tilde{y}_0 \in \mathrm{RZ}_{\mathcal{H},\iota(x_0)}(\breve{E}_{\tilde{x}_0})$, which overall induces a point $\tilde{z}_0 \in \mathrm{RZ}^{\diamondsuit}_{\mathcal{G},\mu,x_0}(\breve{E}_{\tilde{x}_0})$. Recall that to any element of $g \cdot \mathcal{G}(\mathbb{Z}_p) \in G(\mathbb{Q}_p)/\mathcal{G}(\mathbb{Z}_p)$ we may attach a point in $g \cdot \tilde{x}_0 \in \mathscr{S}_{\mathbf{K}}(\mathbb{C}_p)$ by acting through at-p G-isogenies. Analogously, for every $h \cdot \mathcal{H}(\mathbb{Z}_p) \in \mathcal{H}(\mathbb{Q}_p)/\mathcal{H}(\mathbb{Z}_p)$ we get an element $h \cdot \tilde{y}_0 \in \mathrm{RZ}_{\mathcal{H},\iota(x_0)}(\mathbb{C}_p)$, and we get a commutative diagram:

$$\begin{tikzcd}
G(\mathbb{Q}_p)/\mathcal{G}(\mathbb{Z}_p) \arrow[r, "g\cdot\tilde{z}_0"] \arrow[d] \arrow[rr, bend left, "g\cdot\tilde{x}_0"] & \mathrm{RZ}_{\mathcal{G},\mu,x_0}(\mathbb{C}_p) \arrow[r] \arrow[d] & \mathscr{S}_{\mathbf{K}}(\mathbb{C}_p) \arrow[d] \\
H(\mathbb{Q}_p)/\mathcal{H}(\mathbb{Z}_p) \arrow[r, "h\cdot\tilde{y}_0"] & \mathrm{RZ}_{\mathcal{H},\iota(x_0)}(\mathbb{C}_p) \arrow[r] & \mathcal{A}_{K^\flat}(\mathbb{C}_p)
\end{tikzcd}$$

Moreover, we get a further compatibility

$$\begin{tikzcd}
G(\mathbb{Q}_p)/\mathcal{G}(\mathbb{Z}_p) \arrow[r, "\mathrm{GM}_{\tilde{z}_0}"] \arrow[d, "g\cdot\tilde{z}_0"] & \mathcal{M}^{\mathrm{int}}_{\mathcal{G},b,\mu} \\
\mathrm{RZ}^{\diamondsuit}_{\mathcal{G},\mu,x_0} \arrow[ur, "c"] &
\end{tikzcd}$$

where $\mathrm{GM}_{\tilde{z}_0}$ is the map $G(\mathbb{Q}_p)/\mathcal{G}(\mathbb{Z}_p) \to \mathcal{M}^{\mathrm{int}}_{\mathcal{G},b,\mu}(\mathbb{C}_p) = \mathrm{Sht}_{(G,b,\mu,\mathcal{G}(\mathbb{Z}_p))}(\mathbb{C}_p)$ induced from choosing an identification of the fibers of the Grothendieck–Messing period morphism $G(\mathbb{Q}_p) = \pi^{-1}_{GM}(\pi_{GM}(\tilde{z}_0)) \subseteq \mathrm{Sht}_{(G,b,\mu,\infty)}(\mathbb{C}_p)$ (see §3.3.5). It suffices to prove that $\mathrm{GM}_{\tilde{z}_0} : G(\mathbb{Q}_p)/\mathcal{G}(\mathbb{Z}_p) \to \pi_0(\mathcal{M}^{\mathrm{int}}_{\mathcal{G},b,\mu})$ is surjective, but this is precisely the content of Corollary 3.3.11. □

3.6.1 The tori case

When $G = T$ is a torus, there is only one parahoric model that we denote by $\mathcal{T}$. The tori analogue of Theorem 3.6.1 is as follows.

Proposition 3.6.4. *Suppose that* $\mathbf{b} \in B(T, \boldsymbol{\mu})$. *The following hold:*

1. *The map* $\omega_T : \pi_0(X^{\mathcal{T}}_\mu(b)) \to c_{b,\mu}\pi_1(T)^{\varphi}_I$ *is bijective.*

2. *The action of* $T(\mathbb{Q}_p)$ *on* $\mathrm{Sht}_{(T,b,\mu,\infty)}$ *makes* $\pi_0(\mathrm{Sht}_{(T,b,\mu,\infty)} \times \mathrm{Spd}\,\mathbb{C}_p)$ *into a* T°*-torsor.*

In this case, both $X^{\mathcal{T}}_\mu(b)$ and $\mathrm{Sht}_{(T,b,\mu,\mathcal{T})} \times \mathrm{Spd}\,\mathbb{C}_p$ are zero-dimensional. Since we are working over algebraically closed fields, they are of the form $\coprod_J \mathrm{Spec}\,\bar{\mathbb{F}}_p$ and $\coprod_I \mathrm{Spd}\,\mathbb{C}_p$ for some index sets I and J, respectively. Moreover, by Proposition 3.3.7, the specialization map (3.8) induces a bijection $\pi_0(\mathrm{sp}) : I \cong J$. Also, $T^\circ = T(\mathbb{Q}_p)$ and $\mathrm{Sht}_{(T,b,\mu,\infty)} \times \mathrm{Spd}\,\mathbb{C}_p$ is a $\underline{T(\mathbb{Q}_p)}$-torsor over $\mathrm{Spd}\,\mathbb{C}_p$ (see for example [Gle22a, Theorem 1.24]). In particular, $\pi_0(\mathrm{Sht}_{(T,b,\mu,\infty)} \times \mathrm{Spd}\,\mathbb{C}_p)$ is a $T(\mathbb{Q}_p)$-torsor and Proposition 3.6.4 (5) holds. The content of Proposition 3.6.4 (1) becomes the following lemma.

Lemma 3.6.5. *Let T be a torus. We have a $T(\mathbb{Q}_p)$-equivariant commutative diagram, where the horizontal arrows are isomorphisms:*

$$\begin{tikzcd} \pi_0(\mathrm{Sht}_{(T,b,\mu,\mathcal{T})} \times \mathrm{Spd}\,\mathbb{C}_p) \arrow[r, "\cong"] & \pi_0(X_\mu^{\mathcal{T}}(b)) \arrow[r, "\cong"] \arrow[d] & c_{b,\mu}\pi_1(T)_I^{\varphi} \arrow[d] \\ & \pi_0(\mathcal{F}\ell_{\mathcal{T}}) \arrow[r, "\cong"] & \pi_1(T)_I \end{tikzcd}$$

Proof. Upon fixing an element of $\mathrm{Sht}_{(T,b,\mu,\infty)} \times \mathrm{Spd}\,\mathbb{C}_p$, we can identify $\pi_0(\mathrm{Sht}_{(T,b,\mu,\infty)} \times \mathrm{Spd}\,\mathbb{C}_p) \cong T(\mathbb{Q}_p)$ (see for example [Gle22a, Theorem 1.24]), which then gives an identification $\mathrm{Sht}_{(T,b,\mu,\mathcal{T})} \times \mathrm{Spd}\,\mathbb{C}_p \cong T(\mathbb{Q}_p)/\mathcal{T}(\mathbb{Z}_p) \cong T(\breve{\mathbb{Q}}_p)^{\varphi=\mathrm{id}}/\mathcal{T}(\breve{\mathbb{Z}}_p)^{\varphi=\mathrm{id}}$. Since $H^1_{\acute{e}t}(\mathrm{Spec}\,\mathbb{Z}_p, \mathcal{T})$ vanishes, we can write

$$T(\breve{\mathbb{Q}}_p)^{\varphi=\mathrm{id}}/\mathcal{T}(\breve{\mathbb{Z}}_p)^{\varphi=\mathrm{id}} \cong (T(\breve{\mathbb{Q}}_p)/\mathcal{T}(\breve{\mathbb{Z}}_p))^{\varphi=\mathrm{id}}, \tag{3.95}$$

where the right-hand side is $X_*(T)_I^{\varphi} = \pi_1(T)_I^{\varphi}$. Therefore the $T(\mathbb{Q}_p)$-action makes $\pi_0(X_\mu^{\mathcal{T}}(b))$ and $\pi_0(\mathrm{Sht}_{(T,b,\mu,\mathcal{T})} \times \mathrm{Spd}\,\mathbb{C}_p)$ into $\pi_1(T)_I^{\varphi}$-torsors (via the specialization map (3.8)). Thus by equivariance of $\pi_1(T)_I^{\varphi}$-action, $\pi_0(\mathrm{Sht}_{(T,b,\mu,\mathcal{T})} \times \mathrm{Spd}\,\mathbb{C}_p)$ and $\pi_0(X_\mu^{\mathcal{T}}(b))$ can be identified with a unique coset $c_{b,\mu}\pi_1(T)_I^{\varphi} \subseteq \pi_1(T)_I$ (by the definition of $c_{b,\mu}$). □

3.6.2 Reduction to the $G^{\mathrm{der}} = G^{\mathrm{sc}}$ case

For the rest of this subsection, assume that f is an ad-isomorphism. Let $b_H := f(b)$ and $\mu_H := f \circ \mu$. Let $\mathcal{K}_p^H$ denote the unique parahoric of H that corresponds to the same point in the Bruhat–Tits building as $\mathcal{K}_p$.

Proposition 3.6.6. *(1) We have a canonical identification of diamonds*

$$\mathrm{Sht}_{(H,b_H,\mu_H,\infty)} \cong \mathrm{Sht}_{(G,b,\mu,\infty)} \times^{\underline{G(\mathbb{Q}_p)}} \underline{H(\mathbb{Q}_p)}. \tag{3.96}$$

(2) In particular, if $\pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)} \times \mathrm{Spd}\,\mathbb{C}_p)$ is a G°-torsor, then

$$\pi_0(\mathrm{Sht}_{(H,b_H,\mu_H,\infty)} \times \mathrm{Spd}\,\mathbb{C}_p) \tag{3.97}$$

is a H°-torsor.

Proof. (1) A version of (3.96) was proven in [Gle21, Proposition 4.15], where the result is phrased in terms of the torsor $\mathbb{L}_b$ from §3.3.5. [20] We sketch the proof for the reader's convenience:

[20] Although [Gle21, Proposition 4.15] only considers unramified groups G (since this was the ongoing assumption in *loc.cit.*), the proof goes through without this assumption.

A more detailed proof of Proposition 3.6.6 (1) can also be found in [PR22, Proposition 5.2.1], which was obtained independently as *loc.cit.*

Step 1. $\mathrm{Gr}_\mu = \mathrm{Gr}_{\mu_H}$: there is an obvious proper map $\mathrm{Gr}_\mu \to \mathrm{Gr}_{\mu_H}$ of spatial diamonds. Therefore, to prove that it is an isomorphism, it suffices to prove bijectivity on points, which can be done as in the classical Grassmannian case (see [AGLR22, Proposition 4.16] for a stronger statement).

Step 2. $\mathrm{Gr}^b_\mu = \mathrm{Gr}^{b_H}_{\mu_H}$: the b-admissible and b_H-admissible loci are open subsets of $\mathrm{Gr}_\mu = \mathrm{Gr}_{\mu_H}$. To prove that they agree, we can prove it on geometric points. This ultimately boils down to the fact that an element $e \in B(G)$ is basic if and only if $f(e) \in B(H)$ is basic, which holds because centrality of the Newton point ν_e can be checked after applying an ad-isomorphism.

Step 3. $\mathrm{Sht}_{(H,b_H,\mu_H,\infty)} \cong \mathrm{Sht}_{(G,b,\mu,\infty)} \times^{\underline{G(\mathbb{Q}_p)}} \underline{H(\mathbb{Q}_p)}$: recall that the Grothendieck–Messing period map (3.35) from §3.3.5 realizes $\mathrm{Sht}_{(G,b,\mu,\infty)}$ (respectively $\mathrm{Sht}_{(H,b_H,\mu_H,\infty)}$) as a $\underline{G(\mathbb{Q}_p)}$-torsor (respectively an $\underline{H(\mathbb{Q}_p)}$-torsor) over $\mathrm{Gr}^b_\mu = \mathrm{Gr}^{b_H}_{\mu_H}$. Since the $\underline{G(\mathbb{Q}_p)}$-equivariant map $\mathrm{Sht}_{(G,b,\mu,\infty)} \to \mathrm{Sht}_{(H,b_H,\mu_H,\infty)}$ extends to a map of $\underline{H(\mathbb{Q}_p)}$-torsors

$$\mathrm{Sht}_{(G,b,\mu,\infty)} \times^{\underline{G(\mathbb{Q}_p)}} \underline{H(\mathbb{Q}_p)} \to \mathrm{Sht}_{(H,b_H,\mu_H,\infty)}, \tag{3.98}$$

and any map of torsors is an isomorphism, the conclusion follows.

(2) Recall that since $G \to H$ is an ad-isomorphism, we have an isomorphism $G^{\mathrm{sc}} \to H^{\mathrm{sc}}$. Recall $G^\circ := G(\mathbb{Q}_p)/\operatorname{Im}(G^{\mathrm{sc}}(\mathbb{Q}_p))$ and $H^\circ := H(\mathbb{Q}_p)/\operatorname{Im}(G^{\mathrm{sc}}(\mathbb{Q}_p))$. By (3.96), we have a canonical isomorphism

$$\pi_0(\mathrm{Sht}_{(H,b_H,\mu_H,\infty)} \times \operatorname{Spd}\mathbb{C}_p) \cong \pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)} \times \operatorname{Spd}\mathbb{C}_p) \times^{G(\mathbb{Q}_p)} H(\mathbb{Q}_p). \tag{3.99}$$

The right-hand side of (3.99) is by definition

$$\left(\pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)} \times \operatorname{Spd}\mathbb{C}_p) \times H(\mathbb{Q}_p)\right)/G(\mathbb{Q}_p), \tag{3.100}$$

where the quotient is via the diagonal action. Since $G^{\mathrm{sc}}(\mathbb{Q}_p)$ acts trivially on $\pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)} \times \operatorname{Spd}\mathbb{C}_p)$, quotienting (3.100) by $G^{\mathrm{sc}}(\mathbb{Q}_p)$ first gives

$$\pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)} \times \operatorname{Spd}\mathbb{C}_p) \times^{G(\mathbb{Q}_p)} H(\mathbb{Q}_p) \tag{3.101}$$

$$\cong \left(\pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)} \times \operatorname{Spd}\mathbb{C}_p) \times (H(\mathbb{Q}_p)/\operatorname{Im} G^{\mathrm{sc}}(\mathbb{Q}_p))\right)/G^\circ, \tag{3.102}$$

which simplifies, via (3.99) and since $G^{\mathrm{sc}}(\mathbb{Q}_p) = H^{\mathrm{sc}}(\mathbb{Q}_p)$, to

$$\pi_0(\mathrm{Sht}_{(H,b_H,\mu_H,\infty)} \times \operatorname{Spd}\mathbb{C}_p) = \pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)} \times \operatorname{Spd}\mathbb{C}_p) \times^{G^\circ} H^\circ. \tag{3.103}$$

The right-hand side of (3.103) is clearly an H°-torsor. □

Proposition 3.6.7. *If Theorem 3.6.1 holds for $G^{\mathrm{der}} = G^{\mathrm{sc}}$, then it holds in general as well.*

Proof. For each item $i \in \{1, \dots, 5\}$, we show that if (i) holds for $G^{\mathrm{der}} = G^{\mathrm{sc}}$, then (i) also holds for general G. Consider an arbitrary z-extension $\tilde{G} \to G$ (see Definition 3.3.13). By definition of z-extensions, $\tilde{G}^{\mathrm{der}} = \tilde{G}^{\mathrm{sc}}$. By Lemma 3.3.14 (1), we may choose a conjugacy class of cocharacters $\tilde{\boldsymbol{\mu}}$ and an element $\tilde{\mathbf{b}} \in B(\tilde{G}, \tilde{\boldsymbol{\mu}})$ that map to $\boldsymbol{\mu}$ and $\mathbf{b}$, respectively, under the map $B(\tilde{G}, \tilde{\boldsymbol{\mu}}) \to B(G, \boldsymbol{\mu})$.

We first justify (1) and (2) of Theorem 3.6.1. Recall that by Lemma 3.3.14 (2), $c_{\tilde{b},\tilde{\mu}}\pi_1(\tilde{G})^{\varphi}_I \to c_{b,\mu}\pi_1(G)^{\varphi}_I$ is surjective. We apply Proposition 3.3.15 to the ad-isomorphism $\tilde{G} \to G$. Since the top horizontal arrow in (3.44) is a bijection (of sets), the bottom horizontal arrow in (3.44) is also a bijection of sets, as it is the pullback of the top horizontal arrow under a surjective map. Now, (3) of Theorem 3.6.1 is a direct consequence of Proposition 3.4.8.

For (4) recall that the map $\tilde{G}^{\mathrm{der}} \to G^{\mathrm{der}}$ is surjective with finite kernel. In particular, it is an open map. Finally for (5) we use Proposition 3.6.6 (2). □

3.6.3 Argument for (1) $\implies$ (2)

We start by giving a new proof to [He18, Theorem 7.1].

Theorem 3.6.8 (He)**.** *The map $X^{\mathcal{I}}_\mu(b) \to X^{\mathcal{K}_p}_\mu(b)$ is surjective.*

Proof. By functoriality of the specialization map [Gle22b, Proposition 4.14] applied to $\mathrm{Sht}^{\mathcal{I}}_\mu(b) \to \mathrm{Sht}^{\mathcal{K}}_\mu(b)$ from (3.34), we get a commutative diagram:

$$\begin{tikzcd} \mid \mathrm{Sht}_{(G,b,\mu,\mathcal{I}(\mathbb{Z}_p))} \mid \arrow[r] \arrow[d, "\mathrm{sp}"'] & \mid \mathrm{Sht}_{(G,b,\mu,\mathcal{K}_p)} \mid \arrow[d, "\mathrm{sp}"] \\ \mid X^{\mathcal{I}}_\mu(b) \mid \arrow[r] & \mid X^{\mathcal{K}_p}_\mu(b) \mid \end{tikzcd}$$

The top arrow is given by (3.28). By [SW20, Proposition 23.3.1], it is a $\mathcal{K}_p/\mathcal{I}(\mathbb{Z}_p)$-torsor and thus surjective. It then suffices to prove that the specialization map is surjective, which follows directly from [Gle22a, Theorem 2 b)]. □

Now, Theorem 3.6.8 implies the (1) $\implies$ (2) part of Theorem 3.6.1: by Lemma 3.3.2, we have the following commutative diagram:

$$\begin{tikzcd} \pi_0(X^{\mathcal{I}}_\mu(b)) \arrow[r] \arrow[d] & \pi_0(X^{\mathcal{K}_p}_\mu(b)) \arrow[d] \\ \pi_0(\mathcal{F}\ell_{\breve{\mathcal{I}}}) \arrow[r, "\cong"] & \pi_0(\mathcal{F}\ell_{\breve{\mathcal{K}}_p}) \end{tikzcd} \tag{3.104}$$

For the bijection of the lower horizontal arrow, see for example [AGLR22, Lemma 4.17]. The left downward arrow is injective by assumption (1), and the top arrow is surjective by Theorem 3.6.8. Thus the right downward arrow is also injective.

3.6.4 Argument for (2) $\implies$ (3)

This is the content of Proposition 3.4.10.

3.6.5 Argument for (3) $\implies$ (4)

This is the content of Proposition 3.5.8.

3.6.6 Argument for (5) $\implies$ (1)

Proposition 3.6.9. (5) $\implies$ (1) *in Theorem 3.6.1.*

Proof. Consider the map $\det : G \to G^{\mathrm{ab}}$ where $G^{\mathrm{ab}} = G/G^{\mathrm{der}}$. Let $\mathcal{I}^{\mathrm{der}}$ denote the Iwahori subgroup of G^{der} attached to our alcove $\mathbf{a}$ (see §3.2). Let $\mathcal{G}^{\mathrm{ab}}$ be the unique parahoric group scheme of G^{ab}. We have an exact sequence:

$$e \to \mathcal{I}^{\mathrm{der}} \to \mathcal{I} \to \mathcal{G}^{\mathrm{ab}} \to e, \tag{3.105}$$

which induces maps $\mathrm{Sht}^{\mathcal{I}}_{\mu}(b) \to \mathrm{Sht}^{\mathcal{G}^{\mathrm{ab}}}_{\mu^{\mathrm{ab}}}(b^{\mathrm{ab}})$ and $X_\mu(b) \to X_{\mu^{\mathrm{ab}}}(b^{\mathrm{ab}})$ by (3.33) and Lemma 3.3.2, respectively. Recall that by Proposition 3.6.7, it suffices to assume $G^{\mathrm{der}} = G^{\mathrm{sc}}$. When $G^{\mathrm{der}} = G^{\mathrm{sc}}$, we automatically have $G^\circ = G^{\mathrm{ab}}(\mathbb{Q}_p)$ and $\pi_1(G) = X_*(G^{\mathrm{ab}})$, which induces an isomorphism $\pi_1(G)_I = X_*(G^{\mathrm{ab}})_I$. In this case, by functoriality of the Kottwitz map κ, we have the following commutative diagram

$$\begin{tikzcd} \pi_0(X_\mu(b)) \arrow[r, "\omega_G"] \arrow[d, "\pi_0(\det)"'] & \pi_1(G)_I \arrow[d, "\cong"] \\ X_{\mu^{\mathrm{ab}}}(b^{\mathrm{ab}}) \arrow[r, "\omega_{G^{\mathrm{ab}}}"] & X_*(G^{\mathrm{ab}})_I. \end{tikzcd} \tag{3.106}$$

Which fits into the following diagram.

$$\begin{tikzcd} \pi_0(X_\mu(b)) \arrow[r, "\omega_G"] \arrow[d, "\pi_0(\det)"'] & c_{b,\mu}\pi_1(G)_I^{\varphi} \arrow[r, hook] \arrow[d, "\cong"] & \pi_1(G)_I \arrow[d, "\cong"] \\ X_{\mu^{\mathrm{ab}}}(b^{\mathrm{ab}}) \arrow[r, "\omega_{G^{\mathrm{ab}}}", "\cong"'] & c_{b^{\mathrm{ab}},\mu^{\mathrm{ab}}}X_*(G^{\mathrm{ab}})_I^{\varphi} \arrow[r, hook] & X_*(G^{\mathrm{ab}})_I. \end{tikzcd} \tag{3.107}$$

In particular, it suffices to prove that left-hand side arrow is a bijection. By functoriality of the specialization map [Gle22b, Proposition 4.14] and Proposition 3.3.7, we have

$$\begin{tikzcd} \pi_0(\mathrm{Sht}_{(G,b,\mu,\mathcal{I}(\mathbb{Z}_p))} \times \mathrm{Spd}\,\mathbb{C}_p) \arrow[r, "\pi_0(\mathrm{sp})", "\cong"'] \arrow[d, "\pi_0(\det)"'] & \pi_0(X_\mu(b)) \arrow[d, "\pi_0(\det)"] \\ \mathrm{Sht}_{(G^{\mathrm{ab}},b^{\mathrm{ab}},\mu^{\mathrm{ab}},\mathcal{G}^{\mathrm{ab}})} \times \mathrm{Spd}\,\mathbb{C}_p \arrow[r, "\pi_0(\mathrm{sp})", "\cong"'] & X_{\mu^{\mathrm{ab}}}(b^{\mathrm{ab}}) \end{tikzcd} \tag{3.108}$$

Note that we have the following identification

$$\begin{aligned} \pi_0(\mathrm{Sht}_{(G,b,\mu,\mathcal{I}(\mathbb{Z}_p))} \times \mathrm{Spd}\,\mathbb{C}_p) &= \pi_0\left(\mathrm{Sht}_{(G,b,\mu,\infty)} \times \mathrm{Spd}\,\mathbb{C}_p / \underline{\mathcal{I}(\mathbb{Z}_p)}\right) \\ &= \pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)} \times \mathrm{Spd}\,\mathbb{C}_p)/\mathcal{I}(\mathbb{Z}_p) \end{aligned}$$

Since $\pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)} \times \mathrm{Spd}\,\mathbb{C}_p)$ is a G^{ab}-torsor (i.e. assumption (5) of Theorem 3.6.1), up to choosing an $x \in \pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)} \times \mathrm{Spd}\,\mathbb{C}_p)$, we have compatible identifications $\pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)} \times \mathrm{Spd}\,\mathbb{C}_p) \cong G^{\mathrm{ab}}(\mathbb{Q}_p)$ and

$$\pi_0(\mathrm{Sht}_{(G,b,\mu,\mathcal{I}(\mathbb{Z}_p))} \times \mathrm{Spd}\,\mathbb{C}_p) \cong G^{\mathrm{ab}}(\mathbb{Q}_p)/\det(\mathcal{I}(\mathbb{Z}_p)). \tag{3.109}$$

Analogously, taking $x^{\mathrm{ab}} \in \pi_0(\mathrm{Sht}_{(G^{\mathrm{ab}},b^{\mathrm{ab}},\mu^{\mathrm{ab}},\infty)} \times \mathrm{Spd}\,\mathbb{C}_p)$ as $x^{\mathrm{ab}} = \pi_0(\det(x))$, we obtain a compatible identification $\pi_0(\mathrm{Sht}_{(G^{\mathrm{ab}},b^{\mathrm{ab}},\mu^{\mathrm{ab}},\mathcal{G}^{\mathrm{ab}})} \times \mathrm{Spd}\,\mathbb{C}_p) = G^{\mathrm{ab}}(\mathbb{Q}_p)/\mathcal{G}^{\mathrm{ab}}(\mathbb{Z}_p)$ by Lemma 3.6.5. Moreover, the map $\det : \pi_0(\mathrm{Sht}_{G,b,\mu,\infty} \times \mathrm{Spd}\,\mathbb{C}_p) \to \pi_0(\mathrm{Sht}_{(G^{\mathrm{ab}},b^{\mathrm{ab}},\mu^{\mathrm{ab}},\infty)} \times \mathrm{Spd}\,\mathbb{C}_p)$ is equivariant with respect to the $G(\mathbb{Q}_p)$-action on the left and the $G^{\mathrm{ab}}(\mathbb{Q}_p)$-action on the right. Thus we have the following commutative diagram:

$$\begin{tikzcd}
G^{\mathrm{ab}}(\mathbb{Q}_p)/\det(\mathcal{I}(\mathbb{Z}_p)) \arrow[r, "\cong"] \arrow[d, "\det"'] & \pi_0(\mathrm{Sht}_{(G,b,\mu,\mathcal{I}(\mathbb{Z}_p))} \times \mathrm{Spd}\,\mathbb{C}_p) \arrow[r, "\cong", "\pi_0(\mathrm{sp})"'] \arrow[d, "\det"] & \pi_0(X_\mu(b)) \arrow[d, "\det"] \\
G^{\mathrm{ab}}(\mathbb{Q}_p)/\mathcal{G}^{\mathrm{ab}}(\mathbb{Z}_p) \arrow[r, "\cong"] & \mathrm{Sht}_{(G^{\mathrm{ab}},b^{\mathrm{ab}},\mu^{\mathrm{ab}},\mathcal{G}^{\mathrm{ab}})} \times \mathrm{Spd}\,\mathbb{C}_p \arrow[r, "\cong", "\pi_0(\mathrm{sp})"'] & X_{\mu^{\mathrm{ab}}}(b^{\mathrm{ab}})
\end{tikzcd}$$

Thus in order to prove that the vertical arrow on the left-hand side is a bijection, it suffices to show that $\mathcal{I} \to \mathcal{G}^{\mathrm{ab}}$ is surjective on the level of $\mathbb{Z}_p$-points. But this follows from Lang's theorem. □

3.6.7 Argument for (4) $\implies$ (5)

Proposition 3.6.10. (4) $\implies$ (5) *in Theorem 3.6.1.*

Proof. As seen earlier (for example in §3.6.6), the map

$$\det : \pi_0(\mathrm{Sht}_{G,b,\mu,\infty} \times \mathrm{Spd}\,\mathbb{C}_p) \to \pi_0(\mathrm{Sht}_{(G^{\mathrm{ab}},b^{\mathrm{ab}},\mu^{\mathrm{ab}},\infty)} \times \mathrm{Spd}\,\mathbb{C}_p) \tag{3.110}$$

is equivariant with respect to the $G(\mathbb{Q}_p)$-action on the source and the $G^{\mathrm{ab}}(\mathbb{Q}_p)$-action on the target. By the assumption that $G^{\mathrm{der}} = G^{\mathrm{sc}}$ (in particular $G^\circ = G^{\mathrm{ab}}(\mathbb{Q}_p)$), it suffices to show that

$$\pi_0(\det) : \pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)} \times \mathrm{Spd}\,\mathbb{C}_p) \to \mathrm{Sht}_{(G^{\mathrm{ab}},b^{\mathrm{ab}},\mu^{\mathrm{ab}},\infty)} \times \mathrm{Spd}\,\mathbb{C}_p \tag{3.111}$$

is bijective. Since the map $G(\mathbb{Q}_p) \to G^{\mathrm{ab}}(\mathbb{Q}_p)$ is surjective, by equivariance of the respective group actions, the map (3.111) is always surjective. By Proposition 3.3.10, $G(\mathbb{Q}_p)$ acts transitively on $\pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)} \times \mathrm{Spd}\,\mathbb{C}_p)$, thus up to picking an $x \in \pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)} \times \mathrm{Spd}\,\mathbb{C}_p)$ we have an identification of sets $\pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)} \times \mathrm{Spd}\,\mathbb{C}_p) \cong G(\mathbb{Q}_p)/H_x$ for some subgroup $H_x := \mathrm{Stab}(x)$. To prove (4), it suffices to show that $H_x = G^{\mathrm{der}}(\mathbb{Q}_p)$. Firstly, it is easy to see that $H_x \subseteq G^{\mathrm{der}}(\mathbb{Q}_p)$: take any $g \in H_x$, we have $g \cdot x = x$; thus $\deg(g) \cdot \det(x) = \det(g \cdot x) = \det(x)$; by the tori case (see §3.6.1), $\det(g)$ is trivial, thus $g \in G^{\mathrm{der}}(\mathbb{Q}_p)$.

We now prove the other inclusion, i.e. that $G^{\mathrm{der}}(\mathbb{Q}_p)$ acts trivially on $\pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)} \times \mathrm{Spd}\,\mathbb{C}_p)$. We may argue over finite extensions of $\breve{\mathbb{Q}}_p$.

Indeed, recall from [Sch17, Lemma 12.17] that, the underlying topological space of a cofiltered inverse limit of locally spatial diamonds along qcqs[21] transitions maps is the

[21] i.e. quasi-compact quasi-separated

limit of the underlying topological spaces. Thus it suffices to prove that $G^{\mathrm{der}}(\mathbb{Q}_p)$ acts trivially on $\pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)} \times \mathrm{Spd}\, K)$ for all finite degree extensions K over $\breve{\mathbb{Q}}_p$. For any fixed $x \in \pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)} \times \mathrm{Spd}\, K)$, we denote by $G_x \subseteq G(\mathbb{Q}_p)$ the stabilizer of x. Let $G_x^{\mathrm{der}} := G_x \cap G^{\mathrm{der}}(\mathbb{Q}_p)$. It suffices to prove that $G_x^{\mathrm{der}} = G^{\mathrm{der}}(\mathbb{Q}_p)$, which is shown in Lemma 3.6.13. □

Lemma 3.6.11. *G_x^{der} is open in $G^{\mathrm{der}}(\mathbb{Q}_p)$.*

Proof. For any $y \in \mathrm{Gr}_\mu^b(K)$, let $\mathcal{T}_y := \mathrm{Sht}_{(G,b,\mu,\infty)} \times_{\mathrm{Gr}_\mu^b} \mathrm{Spd}\, K$ be the fiber of y under the Grothendieck–Messing period morphism. Take an arbitrary $w \in \pi_0(\mathcal{T}_y)$, by Proposition 3.3.10, we assume without loss of generality that $w \mapsto x$ under the surjection $\pi_0(\mathcal{T}_y) \to \pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)} \times \mathrm{Spd}\, K)$. Consider $G_w^{\mathrm{der}} := G_w \cap G^{\mathrm{der}}(\mathbb{Q}_p)$, and the inclusion of groups $G_w^{\mathrm{der}} \subseteq G_x^{\mathrm{der}} \subseteq G^{\mathrm{der}}(\mathbb{Q}_p)$. It suffices to find a $y \in \mathrm{Gr}_\mu^b(K)$, such that

$$(*) \quad \text{there exists a } w \in \pi_0(\mathcal{T}_y) \text{ with } G_w^{\mathrm{der}} \text{ open in } G^{\mathrm{der}}(\mathbb{Q}_p).$$

Recall the $\underline{G(\mathbb{Q}_p)}$-torsor $\mathbb{L}_b$ over Gr_μ^b from § 3.3.5. Let $y^*\mathbb{L}_b$ be the corresponding torsor over $\mathrm{Spd}\, K$, which induces a crystalline representation $\rho_y : \Gamma_K \to G(\mathbb{Q}_p)$, well-defined up to conjugacy. We claim that G_w is equal to $\rho_y(\Gamma_K)$ up to $G(\mathbb{Q}_p)$-conjugacy. We now justify the claim. Consider the pullback $\mathcal{T}_t$ of $\mathcal{T}_y$ under the geometric point $t : \mathrm{Spd}\, \mathbb{C}_p \to \mathrm{Spd}\, K$. Thus $\mathcal{T}_t$ is a trivial torsor that gives a section $s : \mathrm{Spd}\, \mathbb{C}_p \to \mathcal{T}_t$. The Galois action of Γ_K on $\mathcal{T}_t$ defines a representative of the crystalline representation ρ_y. The orbit $\Gamma_K \cdot s$ descends to a unique component $w_s \in \pi_0(\mathcal{T}_y)$. Therefore, for any $g \in G(\mathbb{Q}_p)$ such that $g \cdot s \in \Gamma_K \cdot s$, we have $g \cdot w_s = w_s$. This gives us the desired claim. By Proposition 3.5.8 (which is a consequence of our Theorem 3.1.11), any generic y satisfies property $(*)$. □

Lemma 3.6.12. *Assuming hypothesis (4) in Theorem 3.6.1. Let N_x denote the normalizer of G_x in $G(\mathbb{Q}_p)$. Then N_x has finite index in $G(\mathbb{Q}_p)$. In particular, N_x contains $G^{\mathrm{der}}(\mathbb{Q}_p)$.*

Proof. Let S be the set of orbits of $\pi_0(\mathrm{Sht}_{(G,b,\mu,\mathcal{I}(\mathbb{Z}_p))} \times \mathrm{Spd}\, K)$ under the $J_b(\mathbb{Q}_p)$-action. By [HV20, Theorem 1.2], S is finite. For each $s \in S$, we choose a representative $x_s \in \pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)} \times \mathrm{Spd}\, K)$ that maps to s under the map

$$\pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)} \times \mathrm{Spd}\, K) \to \pi_0(\mathrm{Sht}_{(G,b,\mu,\mathcal{I}(\mathbb{Z}_p))} \times \mathrm{Spd}\, K) \to S.$$

We can always arrange that x is in this set of representatives for some s. By Proposition 3.3.10, we can find an element $h_s \in G(\mathbb{Q}_p)$ such that $x_s \cdot h_s = x$, for each $s \in S$. We construct a surjection $\coprod_{s \in S} \mathcal{I}(\mathbb{Z}_p) \cdot h_s \twoheadrightarrow G(\mathbb{Q}_p)/N_x$. We do this in two steps. The first step is to construct, for any $g \in G(\mathbb{Q}_p)$, a triple (i, j, s) where $i \in \mathcal{I}(\mathbb{Z}_p)$, $j \in J_b(\mathbb{Q}_p)$ and $s \in S$ such that

$$j \cdot (x \cdot g) \cdot i = x_s \tag{3.112}$$

(note that s is uniquely determined by x and g). We do this by choosing j so that $j \cdot (x \cdot g)$ and x_s map to the same element in $\pi_0(\mathrm{Sht}_{(G,b,\mu,\mathcal{I}(\mathbb{Z}_p))} \times \mathrm{Spd}\, K)$. Since $\mathcal{I}(\mathbb{Z}_p)$ acts transitively on the fibers of the map $\pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)} \times \mathrm{Spd}\, K) \to \pi_0(\mathrm{Sht}_{(G,b,\mu,\mathcal{I}(\mathbb{Z}_p))} \times \mathrm{Spd}\, K)$, there exists an i satisfying (3.112). Thus we have

$$j \cdot x \cdot (gih_s) = x \tag{3.113}$$

The second step is to eliminate j from (3.113). By Proposition 3.3.10, there exists an $n \in G(\mathbb{Q}_p)$ such that $x \cdot n = j \cdot x$. We now show that $n \in N_x$. Indeed, $n^{-1}G_x n = G_{x\cdot n} = G_{j\cdot x} = G_x$ since the actions of $J_b(\mathbb{Q}_p)$ and $G(\mathbb{Q}_p)$ commute. Thus we have $(x \cdot n) \cdot (gih_s) = x$. Since $G_x \subseteq N_x$, in particular $n \cdot (gih_s) \in G_x \subseteq N_x$. Thus $g \cdot i \cdot h_s \in N_x$, and we have a surjection:

$$\coprod_{s\in S} \mathcal{I}(\mathbb{Z}_p) \cdot h_s \twoheadrightarrow G(\mathbb{Q}_p)/N_x. \tag{3.114}$$

The target of (3.114) is discrete, and the source is compact. Thus the index of N_x in $G(\mathbb{Q}_p)$ is finite.

Recall that $G^{\mathrm{der}} = G^{\mathrm{sc}}$. Since G^{der} only has $\mathbb{Q}_p$-simple isotropic factors, and $N_x \cap G^{\mathrm{der}}(\mathbb{Q}_p)$ has finite index in $G^{\mathrm{der}}(\mathbb{Q}_p)$, we have $N_x \cap G^{\mathrm{der}}(\mathbb{Q}_p) = G^{\mathrm{der}}(\mathbb{Q}_p)$. Indeed, it is a standard fact that $G^{\mathrm{der}}(\mathbb{Q}_p)$ has no open subgroups of finite index [Mar91, Chapter II, Theorem 5.1], thus we are done. □

Lemma 3.6.13. $G_x^{\mathrm{der}} = G^{\mathrm{der}}(\mathbb{Q}_p)$.

Proof. By Lemma 3.6.11 and Lemma 3.6.12, $G_x^{\mathrm{der}} \subseteq G^{\mathrm{der}}(\mathbb{Q}_p)$ is open and normal. This already implies $G_x^{\mathrm{der}} = G^{\mathrm{der}}(\mathbb{Q}_p)$, since $G^{\mathrm{der}}(\mathbb{Q}_p)$ does not have open normal subgroups (recall that $G^{\mathrm{der}} = G^{\mathrm{sc}}$). □

This finishes the argument for (4) $\implies$ (5).

Proof of Theorem 3.6.1. We have now justified the circle of implications (1) $\implies$ (2) $\implies$ (3) $\implies$ (4) $\implies$ (5) $\implies$ (1), i.e. Theorem 3.6.1 holds in the case $G^{\mathrm{der}} = G^{\mathrm{sc}}$. But by Proposition 3.6.7 the general case follows. □

Proof of Theorem 3.6.2. (and of Theorem 3.1.1) Using z-extension ad-isomorphisms and decomposition into products (Proposition 3.4.8, Proposition 3.4.9, Proposition 3.3.15 and Lemma 3.3.4), we may assume without loss of generality that $G^{\mathrm{der}} = G^{\mathrm{sc}}$ and that G^{ad} is $\mathbb{Q}_p$-simple. We split into two cases: (1) when G^{ad} is isotropic, and (2) when G^{ad} is anisotropic. The first case holds from the equivalence (2) $\iff$ (3) of Theorem 3.6.1.

We now consider the case where G^{ad} is anisotropic. Recall that

$$\mathrm{Gr}_\mu^b \times \mathrm{Spd}\, \mathbb{C}_p = \mathrm{Sht}_{(G,b,\mu,\infty)} \times \mathrm{Spd}\, \mathbb{C}_p / G(\mathbb{Q}_p). \tag{3.115}$$

When G^{ad} is $\mathbb{Q}_p$-simple and anisotropic, we have that $\mathcal{I}(\mathbb{Z}_p)$ is normal in $G(\mathbb{Q}_p)$, contains $G^{\mathrm{sc}}(\mathbb{Q}_p)$, and $G(\mathbb{Q}_p)/\mathcal{I}(\mathbb{Z}_p) = G^{\mathrm{ab}}(\mathbb{Q}_p)/\mathcal{G}^{\mathrm{ab}}(\mathbb{Z}_p) = \pi_1(G)_I^\varphi$ (see § 3.6.1 for the last identification). Since $\mathcal{I}(\mathbb{Z}_p)$ is normal in $G(\mathbb{Q}_p)$, $\mathrm{Sht}_{(G,b,\mu,\mathcal{I}(\mathbb{Z}_p))} \times \mathrm{Spd}\, \mathbb{C}_p$ becomes a $\pi_1(G)_I^\varphi$-torsor over $\mathrm{Gr}_\mu^b \times \mathrm{Spd}\, \mathbb{C}_p$. Moreover, the map

$$\det : \mathrm{Sht}_{(G,b,\mu,\mathcal{I}(\mathbb{Z}_p))} \times \mathrm{Spd}\, \mathbb{C}_p \to \mathrm{Sht}_{(G,b^{\mathrm{ab}},\mu^{\mathrm{ab}},\mathcal{G}^{\mathrm{ab}}(\mathbb{Z}_p))} \times \mathrm{Spd}\, \mathbb{C}_p \tag{3.116}$$

is $\pi_1(G)_I^\varphi$-equivariant. Since $\mathrm{Sht}_{(G,b^{\mathrm{ab}},\mu^{\mathrm{ab}},\mathcal{G}^{\mathrm{ab}}(\mathbb{Z}_p))} \times \mathrm{Spd}\,\mathbb{C}_p$ is a $\pi_1(G)_I^\varphi$-torsor over $\mathrm{Spd}\,\mathbb{C}_p$, $\mathrm{Sht}_{(G,b,\mu,\mathcal{I}(\mathbb{Z}_p))} \times \mathrm{Spd}\,\mathbb{C}_p$ is the trivial $\pi_1(G)_I^\varphi$-torsor over $\mathrm{Gr}_\mu^b \times \mathrm{Spd}\,\mathbb{C}_p$. That is

$$\mathrm{Sht}_{(G,b,\mu,\mathcal{I}(\mathbb{Z}_p))} \times \mathrm{Spd}\,\mathbb{C}_p \cong (\mathrm{Gr}_\mu^b \times \mathrm{Spd}\,\mathbb{C}_p) \times \underline{\pi_1(G)_I^\varphi}. \tag{3.117}$$

Taking π_0 in (3.117), we have

$$\pi_0(\mathrm{Sht}_{(G,b,\mu,\mathcal{I}(\mathbb{Z}_p))} \times \mathrm{Spd}\,\mathbb{C}_p) \cong \pi_0(\mathrm{Gr}_\mu^b \times \mathrm{Spd}\,\mathbb{C}_p) \times \pi_1(G)_I^\varphi. \tag{3.118}$$

By Theorem 3.3.9, $\pi_0(\mathrm{Sht}_{(G,b,\mu,\mathcal{I}(\mathbb{Z}_p))} \times \mathrm{Spd}\,\mathbb{C}_p) \cong \pi_1(G)_I^\varphi$ and the map

$$\omega_G \circ \pi_0(\mathrm{sp}) : \pi_0(\mathrm{Sht}_{(G,b,\mu,\mathcal{I}(\mathbb{Z}_p))}) \times \mathrm{Spd}\,\mathbb{C}_p) \to c_{b,\mu}\pi_1(G)_I^\varphi$$

is an isomorphism as we needed to show. We can finish by recalling that the map of (3.9) is bijective. $\square$

Chapter 4

Hodge-Newton indecomposability and the identity of He-Nie-Yu

4.1 Background

In [HNY22], He-Nie-Yu studies the affine Deligne-Lusztig varieties with finite Coxeter parts. They study such types of varieties using the Deligne-Lusztig reduction method from [DL76] and carefully investigating the reduction path. In the approach, they establish the "multiplicity one" result which is, roughly speaking, for any σ-conjugacy class $[b] \in B(G)$, there is at most one path in the reduction tree that corresponds to $[b]$. The proof of the "multiplicity one" result is obtained by showing that a certain combinatorial identity (of two $\mathbf{q}$-polynomials, or more precisely, of the class polynomials) of the following form holds. Here, $\mathbf{q}$ is an indeterminate.

$$\sum_{[b] \in B(G,\mu)_{\text{indec}}} (\mathbf{q}-1)^{?}\mathbf{q}^{-??} = 1.$$

They first reduce this to the case when G is split and simply-laced and μ is a fundamental coweight ([HNY22, 6.5 and 6.6]). Then, for type A, they check the identity using some geometric properties of affine Deligne-Lusztig varieties, such as dimension formulae and injectiveness of the projection map from the affine flag variety to the affine Grassmannian ([HNY22, 5.4]). Also, for example, the type E case is proven by computer. In this chapter, we prove it all at once in the quasi-split case via a combinatorial proof.

4.2 An essential case

The motivation for our combinatorial approach is originated from the following case from type A.

Theorem 4.2.1 ([HNY22, 1.3]). *For natural numbers $i < n$, the following holds:*

$$\sum_{\substack{k\geq 1, 1>\frac{a_1}{b_1}>\cdots>\frac{a_k}{b_k}>0 \\ a_1+\cdots+a_k=i, b_1+\cdots+b_k=n}} (\mathbf{q}-1)^{k-1}\mathbf{q}^{1-k+\frac{\sum_{1\leq l_1<l_2\leq k}(a_{l_1}b_{l_2}-a_{l_2}b_{l_1})+\sum_{1\leq l\leq k}\gcd(a_l,b_l)}{2}} = \mathbf{q}^{\frac{i(n-i)-n}{2}+1}.$$

Our strategy is to use a coordinate plane to understand the (index set of the) identity. A polygon always means a polygon *whose vertices are all lattice points (i.e., the coordinates are integers)* and we count segments as 2-gons. We denote by $A(P)$ the area of P, by $i(P)$ the number of lattice points interior to P, and by $b(P)$ the number of lattice points on the boundary of P. Finally, we will use $\mathbb{N}$ for the set of natural numbers.

Let D_k be the index set of $((a_l, b_l))_{l\leq k}$ in Theorem 4.2.1 and let j be defined as $n-i$ for simplicity.

Lemma 4.2.2. *Let C_k be the set of $(a_l, b_l) \in \mathbb{N}^2$'s for $l \leq k$ such that $a_1 + \cdots + a_k = i$, $b_1 + \cdots + b_k = j$, and $\frac{b_1}{a_1} < \cdots < \frac{b_k}{a_k}$. Then,*

$$\sum_{\substack{k\geq 1 \\ ((a_l,b_l))_{l\leq k}\in ?_k}} (\mathbf{q}-1)^{k-1}\mathbf{q}^{1-k+\frac{\sum_{1\leq l_1<l_2\leq k}(a_{l_1}b_{l_2}-a_{l_2}b_{l_1})+\sum_{1\leq l\leq k}\gcd(a_l,b_l)}{2}}$$

for $? = C$ and $? = D$ are equal.

Proof. The one-to-one correspondence from C_k to D_k is given by $(x_l, y_l) \mapsto (x_l, x_l + y_l)$. It is easy to check that the conditions on the sums and the slopes are all equivalent. The one that needs justification is the exponent part. However, $x_{l_1}(x_{l_2}+y_{l_2})-x_{l_2}(x_{l_1}+y_{l_1}) = x_{l_1}y_{l_2}-x_{l_2}y_{l_1}$ and $\gcd(x_l, x_l + y_l) = \gcd(x_l, y_l)$ obviously. □

Next, the idea to interpret this summation is by making a one-to-one correspondence between an element $((a_l, b_l))_{l\leq k} \in C_k$ and a convex polygon satisfying some condition. Let us denote by Δ the triangle whose vertices are $(0, 0)$, $(i, 0)$, and (i, j). For simplicity, let L denote the segment(=2-gon) connecting $(0, 0)$ and (i, j).

Lemma 4.2.3. *There is a one-to-one correspondence between C_k and the set of convex $(k+1)$-gons lying in Δ but not touching the horizontal and vertical edges of Δ such that L is an edge. Under this correspondence, we have*

$$\frac{1}{2}\left(\sum_{1\leq l_1<l_2\leq k}(a_{l_1}b_{l_2}-a_{l_2}b_{l_1}) + \sum_{1\leq l\leq k}\gcd(a_l,b_l)\right) = i(P) + b(P) - 1 - \frac{1}{2}\gcd(i,j),$$

where P is the corresponding $(k+1)$-gon.

We will denote the set of such $(k+1)$-gons by C_k abusing notation.

Proof. Given $((x_l, y_l))_{l\leq k} \in C_k$, consider the convex polygon P with vertices $(0,0)$, (x_1, y_1), $(x_1 + x_2, y_1 + y_2)$, $\cdots$, $(x_1 + \cdots + x_k, y_1 + \cdots + y_k) = (i, j)$. As $(\frac{x_i}{y_i})_i$ is increasing, the polygon P is convex. Now, $y_1, x_k > 0$ implies that P does not touch the horizontal and vertical edges of Δ. The inverse map from the set of polygons to the set of pairs is the obvious one.

Regarding the formula, it is easy to see that, using induction,

$$\frac{1}{2}\sum_{1\leq l_1<l_2\leq k}(x_{l_1}y_{l_2} - x_{l_2}y_{l_1}) = A(P).$$

Noting that $\gcd(a, b) + 1$ is the number of lattice points on the segment connecting (m, n) and $(m + a, n + b)$ for any integers m and n, we get

$$\sum_{1\leq l\leq k}\gcd(x_l, y_l) = b(P) - \gcd(i, n - i).$$

Applying the following well-known Pick's Theorem, we get the conclusion.

Pick's Theorem. *Let P be a polygon in a coordinate plane. Then,*

$$A(P) = i(P) + \frac{b(P)}{2} - 1.$$

□

The main lemma in our proof is the following:

Lemma 4.2.4. *Let C be the set of all convex polygons which lies in Δ not touching the horizontal and vertical edges and contains L as an edge. Then, the following identity holds:*

$$\sum_{P\in C} x^{u(P)}(1 - x)^{v(P)-2} = 1,$$

where $u(P)$ is the number of lattice points interior to $\Delta \setminus P$ and $v(P)$ is the number of vertices of P.

Proof of Lemma 4.2.4. Both sides are polynomials in x, so we only need to prove it for all $0 < x < 1$. Let us consider the following probabilistic process:

For each point interior to Δ, choose it with the probability x and abandon it with the probability $1 - x$. Then, we form the convex hull containing $(0, 0)$, (i, j), and the chosen points. It is easy to see that the resulting convex hull is an element of C. For example, if all interior points are abandoned, we end up getting L and so the probability of obtaining L is $(1 - x)^{u(L)}$.

For $P \in C$, let $\mathrm{prob}(P)$ be the probability of obtaining P as a result of the aforementioned process. Obviously, $\sum_{P\in C}\mathrm{prob}(P) = 1$. So, it is enough to show that $\mathrm{prob}(P) = (1 - x)^{u(P)}x^{v(P)-2}$ for all $P \in C$. However, this holds because the case when the resulting convex hull is P is exactly when

1) the vertices of P (except $(0,0)$ and (i,j)) are chosen $(= x^{v(P)-2})$ and
2) the vertices outside of P are abandoned $(= (1-x)^{u(P)})$
with no conditions on the other remaining points. $\square$

Now, Theorem 4.2.1 is simply the result of the previous lemmas.

Proof of Theorem 4.2.1. By Lemmas 4.2.2 and 4.2.3, it is enough to show that

$$\sum_{k\geq 1,\ P\in C_k} (\mathbf{q}-1)^{k-1}\mathbf{q}^{-(k-1)+i(P)+b(P)} = q^{\frac{ij-n+\gcd(i,j)}{2}+2}.$$

Now, we observe that the exponent part on the right-hand side can be written as $i(\Delta)+b(\Delta)-(n-1)$ simply using the facts that $A(\Delta)=\frac{ij}{2}$ and $b(\Delta)=n+\gcd(i,j)$ and applying Pick's Theorem.

As $C=\cup_{k\geq 1}C_k$, the index set of the left-hand side summation is C. Now, observing that $u(P)=i(\Delta)+b(\Delta)-(n-1)-(i(P)+b(P))$, we are reduced to show

$$\sum_{C}\left(\frac{\mathbf{q}-1}{\mathbf{q}}\right)^{k-1}\mathbf{q}^{-u(P)} = 1.$$

This is nothing but the resulting identity of Lemma 4.2.4 by letting $x=\mathbf{q}^{-1}$ because, for any $P\in C_k$, we have $v(P)-2=k+1-2=k-1$. $\square$

Remark 13. We do not know if the identity of Lemma 4.2.4 is well-known. It looks interesting to us because the left-hand side is not homogeneous in the sense that $u(P)+v(P)-2$ is not constant but this gives a way to generate 1 using polynomials of the form $x^a(1-x)^b$.

We wonder if $\{(u(P),v(P)-2) : P\in C\}$ parametrizes all such pairs $\{(a_i,b_i)\in\mathbb{N}^2 : i\in I\}$ such that $\sum_{i\in I}x^{a_i}(1-x)^{b_i}=1$. More precisely, let $S=\{(a_i,b_i)\in\mathbb{N}^2 : 1\leq i\leq k\}$ be a set satisfying

$$\sum_{i=1}^{k}x^{a_i}(1-x)^{b_i}=1,$$

and $(0,1)\in S$. Then we would like to ask if there exist $m,n\in\mathbb{N}$ such that $S=\{(u(P),v(P)-2) : P\in C_{m,n}\}$ where $C_{m,n}$ is the set defined in Lemma 4.2.4 corresponding to the triangle $\Delta_{m,n}$ whose vertices are $(0,0)$, $(m,0)$, and (m,n).

4.3 The general case

As in [HNY22, 2.1], let G be a quasi-split reductive group over a local field F and T be a maximal torus constructed in *loc.cit.*. Denote by W the relative Weyl group and by $\mathbb{S}$ the set of simple reflections, equipped with a Frobenius action σ.

Moreover, denote by Γ_0 the inertia group of F and define $V := X_*(T)_{\Gamma_0,\mathbb{Q}}$. The subset of dominant vectors will be denoted by V^+. Now, for each $i\in\mathbb{S}$, we denote the corresponding root, coroot, fundamental weight, and fundamental coweight by α_i, $\alpha_i^\vee$, ϖ_i, and $\varpi_i^\vee$. Moreover, we denote by $\mathcal{O}_i$ the σ-orbit of i and define $\alpha_{\mathcal{O}_i}^\vee := \frac{1}{\#\mathcal{O}_i}\sum_{j\in\mathcal{O}_i}\alpha_j^\vee$ and $\varpi_{\mathcal{O}_i} := \sum_{j\in\mathcal{O}_i}\varpi_j$.

Definition 4.3.1. *Given $v \in V$, define $I(v)$ as $\{i \in \mathbb{S} : \langle v, \alpha_i\rangle = 0\}$ and define*

$$B(G,\mu)_{indec} := \{v \in (V^+)^\sigma : v \leq \mu \text{ and } \langle \mu - v, \varpi_{\mathcal{O}_i}\rangle \in \mathbb{Z}_{\geq 0}\ \forall i \in \mathbb{S}\setminus I(v)\}.$$

Note that $I(v)$ is σ-stable for any $v \in (V^+)^\sigma$. We also note that our $B(G,\mu)_{\mathrm{indec}}$ is defined in this way for the sake of simplicity of the proof, but it is the (bijective) image of the original one via the dominant Newton map (see [HNY22, 6.2]).

Our main theorem is the following.

Theorem 4.3.2. *Let $\mu \in V^+$. Then, as polynomials of a variable* $\mathbf{p}$,

$$\sum_{v\in B(G,\mu)_{indec}} (1-\mathbf{p})^{\sum_{i\in\mathbb{S}/\langle\sigma\rangle}\lceil\langle\mu-v,\varpi_{\mathcal{O}_i}\rangle\rceil - \#(\mathbb{S}/\langle\sigma\rangle)}\mathbf{p}^{\#(\mathbb{S}/\langle\sigma\rangle)-\#(I(v)/\langle\sigma\rangle)} = 1.$$

4.3.1 The setup and the proof

We explain the proof straight, but, in order to see the main idea more clearly, we suggest seeing Section 4.2.

In $(\mathbb{S}/\langle\sigma\rangle)\times\mathbb{R}_{>0}$, consider the following set T of points

$$\{([i], \langle\mu,\varpi_{\mathcal{O}_i}\rangle - m) : i \in \mathbb{S} \text{ and } m \in \mathbb{Z}_{\geq 0} \text{ such that } m < \langle\mu,\varpi_{\mathcal{O}_i}\rangle\}.$$

Definition 4.3.3. *Fix a subset $C \subset T$. For each $i \in \mathbb{S}$, let us define $C([i])$ as $\max\{r \in \mathbb{R}_{>0} : ([i], r) \in C\}$ with the convention that $\max\emptyset = 0$.*

We also define the rough envelope of C by

$$\text{R-env}(C) := \{v \in (V^+)^\sigma : v \leq \mu \text{ and } \langle v, \varpi_{\mathcal{O}_i}\rangle \geq C([i])\ \forall i \in \mathbb{S}\}.$$

We will denote by $\mathrm{env}(C)$, the envelope of C, the set $\text{R-env}(C) \cap B(G,\mu)_{indec}$.[1]

The main proposition of this chapter is as follows.

Proposition 4.3.4. *Let $v \in \mathrm{env}(C)$. TFAE:*

1. *v is the smallest among the ones in $\mathrm{env}(C)$.*
2. *v is minimal among the ones in $\mathrm{env}(C)$.*
3. *For all $i \in \mathbb{S}\setminus I(v)$, we have $\langle v, \varpi_{\mathcal{O}_i}\rangle = C([i])$.*

By taking any minimal element of $\mathrm{env}(C)$, we get the following corollary.

[1] In particular, we are not assuming that $\langle \mu - v, \varpi_{\mathcal{O}_i}\rangle \in \mathbb{Z}_{\geq 0}$ for all $i \in \mathbb{S}\setminus I(v)$ when considering $\text{R-env}(C)$.

Corollary 4.3.5. *For any $C \subset T$, the smallest element of* $\mathrm{env}(C)$ *exists.*

Now, we can prove the main theorem as follows.

Proof of Theorem 4.3.2. We may assume that $0 < \mathbf{p} < 1$. Now, consider the following probabilistic process: For each $t \in T$, we **select** it with the probability $\mathbf{p}$. Then, for the set of the selected points, say C, we take the smallest element v in $\mathrm{env}(C)$ whose existence is guaranteed by Corollary 4.3.5.

Given $v \in B(G,\mu)_{\mathrm{indec}}$, let $\mathrm{prob}(v)$ be the probability that this process results in v. Then, it is obvious that $\sum_{v\in B(G,\mu)_{\mathrm{indec}}} \mathrm{prob}(v) = 1$ by the definition of $\mathrm{env}(C)$. Now, it is enough to show that

$$\mathrm{prob}(v) = (1-\mathbf{p})^{\sum_{i\in\mathbb{S}/\langle\sigma\rangle}\lceil\langle\mu-v,\varpi_{\mathcal{O}_i}\rangle\rceil - \#(\mathbb{S}/\langle\sigma\rangle)}\mathbf{p}^{\#(\mathbb{S}/\langle\sigma\rangle)-\#(I(v)/\langle\sigma\rangle)}.$$

By Proposition 4.3.4, the element v is the smallest in $\mathrm{env}(C)$ if and only if $C([i]) \le \langle v, \varpi_{\mathcal{O}_i}\rangle$ for all $i \in \mathbb{S}$ with the equality holding for each $i \in \mathbb{S}\setminus I(v)$. The inequality part is equivalent to that the points above $([i], \langle v, \varpi_{\mathcal{O}_i}\rangle)$ are not selected. There are $\lceil\langle\mu - v, \varpi_{\mathcal{O}_i}\rangle\rceil - 1$ number of such points so that we get the $(1-\mathbf{p})$-part. The equality part means that the point $([i], \langle v, \varpi_{\mathcal{O}_i}\rangle)$ must be selected for all $i \in \mathbb{S}\backslash I(v)$, giving $\mathbf{p}^{\#(\mathbb{S}\backslash I(v))/\langle\sigma\rangle}$ which is $\mathbf{p}^{\#(\mathbb{S}/\langle\sigma\rangle)-\#(I(v)/\langle\sigma\rangle)}$ as $\mathbb{S}$ and $I(v)$ are σ-stable. □

4.3.2 The identity ♠′ of [HNY22, 6.1]

By applying $\mathbf{p} = \frac{\mathbf{q}-1}{\mathbf{q}}$ to Theorem 4.3.2, we get

$$\sum_{[b]\in B(G,\mu)_{\mathrm{indec}}} (\mathbf{q}-1)^{\#(S/\langle\sigma\rangle)-\#(I(\nu_b)/\langle\sigma\rangle)}\mathbf{q}^{-\sum_{i\in\mathbb{S}/\langle\sigma\rangle}\lceil\langle\mu-\nu_b,\varpi_{\mathcal{O}_i}\rangle\rceil+\#(I(\nu_b)/\langle\sigma\rangle)} = 1,$$

where $B(G,\mu)_{\mathrm{indec}}$ here is the original one defined in [HNY22, 2.3].

4.4 Proof of the main proposition

4.4.1 Two lemmas

Given a subset $J \subset \mathbb{S}$, we say that J is connected if the corresponding graph in the Dynkin diagram is connected. We denote by ∂J the set of vertices of $\mathbb{S}$ that is adjacent to J but not in J. For example, when $J = \mathbb{S}$, we have $\partial J = \emptyset$. Note that if J is σ-stable, then so is ∂J.

The following is crucial in the proofs of Proposition 4.3.4 and Corollary 4.3.5.

Lemma 4.4.1. *Let $J \subset \mathbb{S}$ be connected and j be a vertex of J. Then, there exist $c_i \in \mathbb{Q}_{>0}$ for each $i \in J$ and $d_k \in \mathbb{Q}_{>0}$ for each $k \in \partial J$ such that*

$$\varpi_j^\vee = \sum_{i\in J} c_i\alpha_i^\vee + \sum_{k\in\partial J} d_k\varpi_k^\vee.$$

The dual version also holds (with possibly different c_i's and d_k's).

Proof. The main idea is that J is a Dynkin diagram again. We now set c_i's to be the coefficients, when expressing fundamental coweights of J, of the linear combination by coroots of J. Then, they are all positive (cf. [Lim23, Lemma 2.18]). Now, we need to compute what $\varpi_j^\vee - \sum_{i\in J} c_i\alpha_i^\vee$ looks like. We do this by computing $\langle -, \alpha_l\rangle$ for each α_l. By construction, it is nonzero if and only if $l \in \partial J$. In that case, it is not just nonzero, but it is always negative as $\langle \alpha_i^\vee, \alpha_l\rangle < 0$ when i and l are adjacent, and c_i is positive. □

Lemma 4.4.2. *For a subset $C \subset T$ and $v \in \mathrm{R\text{-}env}(C)$, suppose that there exists $i_0 \in \mathbb{S} \setminus I(v)$ such that $\langle v, \varpi_{\mathcal{O}_{i_0}}\rangle \neq C([i_0])$. Then, one can find $v' \in \mathrm{R\text{-}env}(C)$ such that either (S) or (I^c) holds.*

$$\#\{i \in \mathbb{S}\colon \langle v', \varpi_{\mathcal{O}_i}\rangle = C([i])\} > \#\{i \in \mathbb{S}\colon \langle v, \varpi_{\mathcal{O}_i}\rangle = C([i])\} \tag{S}$$

$$\#\{i \in \mathbb{S} \setminus I(v')\colon \langle v', \varpi_{\mathcal{O}_i}\rangle \neq C([i])\} < \#\{i \in \mathbb{S} \setminus I(v)\colon \langle v, \varpi_{\mathcal{O}_i}\rangle \neq C([i])\} \tag{I^c}$$

Proof. Definition of $u^\diamond$: Define $A(v) := \{i \in \mathbb{S} : \langle v, \varpi_{\mathcal{O}_i}\rangle \neq C([i])\}$ and J be its connected component containing i_0. It is σ-stable. Applying lemma 4.4.1 to those J and i_0, we denote by u the resulting $\sum_{i\in J} c_i\alpha_i^\vee$ part and by $u^\diamond$ its σ-average. We want to set $v' = v - \epsilon u^\diamond$ for some $\epsilon \in \mathbb{Q}_{>0}$.

Construction of v': Note that $\langle v, \varpi_{\mathcal{O}_i}\rangle - C([i]) > 0$ for all $i \in J$ by definition of $A(v)$. As $\langle u^\diamond, \varpi_{\mathcal{O}_i}\rangle = \sum_{j\in\mathcal{O}_i} c_j > 0$ for all $i \in J$ by lemma 4.4.1, we can find the maximal one ϵ_1 such that $\langle v - \epsilon_1 u^\diamond, \varpi_{\mathcal{O}_i}\rangle - C([i]) \geq 0$ for all $i \in J$. We take $\epsilon = \max\{\epsilon_1, \#\mathcal{O}_{i_0}\langle v, \alpha_{i_0}\rangle\}$ and let $v' := v - \epsilon u^\diamond$.

Verification of $v' \in \mathrm{R\text{-}env}(C)$: Observe that $\langle u^\diamond, \alpha_i\rangle > 0$ only for $i \in \mathcal{O}_{i_0}$ and $\langle u^\diamond, \alpha_i\rangle < 0$ only for $i \in \partial J$. Other cases give $\langle u^\diamond, \alpha_i\rangle = 0$. Hence, we only need to consider $i \in \mathcal{O}_{i_0}$. However, $\langle v - \epsilon u^\diamond, \alpha_i\rangle = \langle v, \alpha_{i_0}\rangle - \frac{1}{\#\mathcal{O}_{i_0}}\epsilon$ as v is σ-invariant. It is ≥ 0 since $\epsilon \geq$. As $u^\diamond$ is σ-invariant, so is v'.

Verification of "(S) or (I^c)": As $J \subset A(v)$ and $\langle u^\diamond, \varpi_{\mathcal{O}_i}\rangle = 0\ \forall i \in \mathbb{S} \setminus J$,

$$\begin{aligned}\{i \in \mathbb{S} : \langle v, \varpi_{\mathcal{O}_i}\rangle = C([i])\} &= \{i \in \mathbb{S} \setminus J : \langle v, \varpi_{\mathcal{O}_i}\rangle = C([i])\} \\ &= \{i \in \mathbb{S} \setminus J : \langle v', \varpi_{\mathcal{O}_i}\rangle = C([i])\}.\end{aligned}$$

So, (S) is equivalent to $\{i \in J : \langle v - \epsilon u^\diamond, \varpi_{\mathcal{O}_i}\rangle = C([i])\} \neq \emptyset$ which is equivalent to that $\epsilon = \epsilon_1$. Next, from the previous paragraph, we know that $\mathbb{S} \setminus I(v') = (\mathbb{S} \setminus I(v) \setminus \mathcal{O}_{i_0}) \cup \partial J$ if and only if $\epsilon = \#\mathcal{O}_{i_0}\langle v, \alpha_{i_0}\rangle$.[2] However, for $i \in \partial J$, we have $\langle v, \varpi_{\mathcal{O}_i}\rangle = \langle v', \varpi_{\mathcal{O}_i}\rangle = C([i])$ but i_0 belongs to only the right-hand side set in (I^c). This proves the claim. □

4.4.2 The proposition

Proof of Proposition 4.3.4. (1)⇒(2): Trivial.

(2)⇒(3): Assume that (3) does not hold. As $v \in \mathrm{env}(C) \subset \mathrm{R\text{-}env}(C)$, we can apply lemma 4.4.2. If the new v' satisfies the assumption again, we keep repeating this process. As the number in (S) is $\leq \#S$ and that in (I^c) is ≥ 0, the process terminates in finite

[2] If $\epsilon_1 = \epsilon < \#\mathcal{O}_{i_0}\langle v, \alpha_{i_0}\rangle$, then $\mathbb{S} \setminus I(v') = (\mathbb{S} \setminus I(v)) \cup \partial J$.

steps. The final v_{fin} then satisfies that $\langle v_{\mathrm{fin}}, \varpi_{\mathcal{O}_i}\rangle = C([i])$ for all $i \in \mathbb{S} \setminus I(v_{\mathrm{fin}})$. Noting that $C([i]) = \langle \mu, \varpi_{\mathcal{O}_i}\rangle - m$ for some $m \in \mathbb{Z}_{\geq 0}$, we get $\langle \mu - v_{\mathrm{fin}}, \varpi_{\mathcal{O}_i}\rangle \in \mathbb{Z}_{\geq 0}$ for all $i \in \mathbb{S} \setminus I(v_{\mathrm{fin}})$ meaning that $v_{\mathrm{fin}} \in \mathrm{env}(C)$. It contradicts to (2).

(3)⇒(1): Suppose that $w \in \mathrm{env}(C)$ and let us show that $w \geq v$. This is equivalent to that $\langle w - v, \varpi_i\rangle \geq 0$ for all $i \in \mathbb{S}$. Now recall that $\langle w, \varpi_{\mathcal{O}_i}\rangle \geq C([i])$ for all $i \in \mathbb{S}$ as $w \in \mathrm{env}(C)$. Moreover, we have $\langle v, \varpi_{\mathcal{O}_i}\rangle = C([i])$ for all $i \in \mathbb{S} \setminus I(v)$ by assumption. Hence, $\langle w - v, \varpi_{\mathcal{O}_i}\rangle \geq 0$ for all $i \in \mathbb{S} \setminus I(v)$. We get $\langle w - v, \varpi_i\rangle \geq 0$ for all $i \in \mathbb{S} \setminus I(v)$ as w and v are σ-invariant.

Let $j \in I(v)$. The dual version of lemma 4.4.1 applied to the connected component of $I(v)$ containing j tells us that ϖ_j is a nonnegative linear combination of α_i ($i \in I(v)$) and ϖ_k ($k \in \mathbb{S} \setminus I(v)$). However, $\langle w, \alpha_i\rangle \geq 0$ as $w \in V^+$ and $\langle v, \alpha_i\rangle = 0$ for all $i \in I(v)$, so we have $\langle w - v, \alpha_i\rangle \geq 0$ for all $i \in I(v)$. Therefore, $\langle w - v, \varpi_j\rangle \geq 0$ for all $j \in I(v)$. $\square$

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