Title
Safety-Critical Control of Nonlinear Systems: Input Delay Compensation and Prescribed-Time Safety

Author
Abel, Imoleayo

Publication Date
2022

Peer reviewed|Thesis/dissertation

UNIVERSITY OF CALIFORNIA SAN DIEGO

Safety-Critical Control of Nonlinear Systems: Input Delay Compensation and Prescribed-Time Safety

A dissertation submitted in partial satisfaction of the
requirements for the degree Doctor of Philosophy

in

Engineering Sciences (Mechanical Engineering)

by

Imoleayo Abel

Committee in charge:

Professor Miroslav Krstić, Chair
Professor Nikolay Atanasov
Professor Jorge Cortés
Professor Sylvia Herbert
Professor Sorin Lerner

2022

The dissertation of Imoleayo Abel is approved, and it is acceptable in quality and form for publication on microfilm and electronically.

University of California San Diego

2022

DEDICATION

This dissertation is dedicated to the memory of my father,

Olukayode Japhet Abel

who passed on the 21st day of December 2019,

after a life of immeasurable sacrifice for his family.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

ACKNOWLEDGEMENTS

I want to express my deepest gratitude to my advisor and committee chair, Professor Miroslav Krstić, for his guidance and tremendous support. His willingness and readiness to help, irrespective of time of day or day of week, and in spite of his numerous other commitments was a blessing. I am particularly grateful for his patience with me in the process of writing technical papers. He leads by example, and his work ethic is one I will continue to attempt to emulate. I cannot thank him enough.

I would also like to thank my Ph.D. committee members: Professor Nikolay Atanasov, Professor Jorge Cortés, Professor Sylvia Herbert, and Professor Sorin Lerner for their time and their helpful comments and suggestions. The knowledge I gained from their classes, and the numerous discussions I had with them laid the foundation for my dissertation topic.

Special gratitude goes to Dr. Mrdjan Janković at Ford Research and Advanced Engineering for his support and for working closely with me throughout my graduate studies. His guidance, patience, and wealth of theoretical and practical expertise were an invaluable resource for me during this academic journey.

I would like to acknowledge current and past graduate students and postdocs in my research group for their support and friendship and for making our lab feel like home: Ulf Jakob Aarsnes, Mohammad Al Suwaidan, Mostafa Bagheri, Alex Bertino, Mark Burkhardt, Leobardo Camacho-Solorio, Stephen Chen, Cenk Demir, Azad Ghaffari, Shumon Koga, Drew Steeves, Ji Wang, Nicholas Willems, Alan Williams, Tugrul Yilmaz, and Huan Yu. Special appreciation goes to Lusia Vesker for her support and for always being there to provide whatever we needed to make our lives as graduate students easier.

My journey towards a Ph.D. would not be possible without amazing teachers whose tutelage and passion for the subjects they taught spurred me to pursue a Ph.D. My gratitude goes to my professors at Swarthmore College; in particular, Professors Erik Cheever, Scott Cook, Carr Everbach, and Matt Zucker for igniting my interest in Control Theory, Differential Equations, Dynamics, and Robotics respectively.

I would also like to thank Mmes. Ifeoma Anyigbo, Ifeanyi Olagbaju, Chinenye Uwadilike, and Mr. Douglas Yebrifa at the EducationUSA center in Lagos Nigeria, for going above and beyond the requirements of their jobs to facilitate my journey from Nigeria to the USA for my university education. I cannot thank you enough and I am forever indebted to you all. Special appreciation also goes to Mmes. Modupe Okebukola, Nkem Uzowulu, Ify Anyaoku, and Esther Okparadike for believing in me and for seeing in me potential that I did not see in myself.

I have benefited immensely from a strong community of friends. Thanks in particular to Gottlieb Uahengo, Temitope Lawal, and (in loving memory) David Sannor for their selflessness and for becoming my brothers from other mothers; and to my former classmates in the "FGCL 08/09" WhatsApp group for their friendship, banter, and for making home feel not too far away.

Finally and most importantly, I would like to acknowledge the unconditional love and support of my family. To my mother Morenike Abel, my brother Dr. Olusegun Abel, and my father Olukayode Abel (in whose memory this dissertation is dedicated), thank you all for your unwavering support, your endless sacrifice, and for encouraging me whenever I doubted myself. And to my best friend turned spouse, Bulouere Princess Abel, thank you for your love and patience, and for tolerating my absent mindedness during particularly busy times during my PhD.

This dissertation contains reprints and adaptations of published and unpublished papers that are a product of work done in collaboration with several authors whose contributions I acknowledge.

Chapter 2, in part, is a reprint and adaptation of the paper: I. Abel, M. Krstić, and M. Janković, "Safety-Critical Control of Systems with Time-Varying Input Delay", IFAC Workshop on Time Delay Systems (TDS), 2021. The dissertation author was the primary investigator and author of this paper.

Chapter 3, in part, is a reprint and adaptation of the paper: I. Abel, M. Janković, and M. Krstić, "Constrained Stabilization of Multi-Input Linear Systems with Distinct Input Delays",

3rd IFAC Joint Workshop on Control of Systems Governed by PDEs and Distributed Parameter Systems (CPDE/CPDS), 2019. The dissertation author was the primary investigator and author of this paper.

Chapter 4, in part, is a reprint and adaptation of the following papers: (1) I. Abel, M. Krstić, and M. Janković, “Constrained Control of Input Delayed Systems with Partially Compensated Input Delays”, ASME Dynamic Systems and Controls Conference (DSCC), 2020. (2) I. Abel, M. Krstić, and M. Janković, “Constrained Control of Multi-Input Systems with Distinct Input Delays”, under review for publication in the IEEE Transactions on Automatic Control. The dissertation author was the primary investigator and author of both papers.

Chapter 5, in part, is a reprint and adaptation of the following papers: (1) I. Abel, D. Steeves, M. Krstić, and M. Janković, “Prescribed-Time Safety Design for a Chain of Integrators”, IEEE American Control Conference (ACC), 2022. (2) I. Abel, D. Steeves, M. Krstić, and M. Janković, “Prescribed-Time Safety Design for Strict-Feedback Nonlinear Systems”, in preparation for publication in the IEEE Transactions on Automatic Control. The dissertation author was the primary investigator and author of both papers.

VITA

2014	Bachelor of Arts in Computer Science, Swarthmore College
2014	Bachelor of Science in Engineering, Swarthmore College
2014–2016	Software Developer, Epic Systems Corporation
2018	Master of Science in Engineering Sciences (Mechanical Engineering), University of California San Diego
2019	Visiting Student Researcher, Ford Research and Advanced Engineering, Ford Motor Company
2020	Finalist for Best Student Paper Competition, ASME Dynamic Systems and Control Conference, 2020
2021	Autonomous Vehicle Control Intern, NVIDIA Corporation
2022	Doctor of Philosophy in Engineering Sciences (Mechanical Engineering), University of California San Diego

PUBLICATIONS

I. Abel, D. Steeves, M. Krstić, and M. Janković, "Prescribed-Time Safety Design for Strict-Feedback Nonlinear Systems", in preparation for submission to the *IEEE Transactions on Automatic Control*.

I. Abel, M. Janković, and M. Krstić, "Constrained Control of Multi-Input Systems with Distinct Input Delays", *IEEE Transactions on Automatic Control*, under review.

I. Abel, D. Steeves, M. Krstić, and M. Janković, "Prescribed-Time Safety Design for a Chain of Integrators", *IEEE American Control Conference (ACC)*, Atlanta, GA, 2022

I. Abel, M. Krstić, and M. Janković, "Safety-Critical Control of Systems with Time-Varying Input Delay", *IFAC Workshop on Time Delay Systems (TDS)*, Guangzhou, China, 2021

I. Abel, M. Janković, and M. Krstić, "Constrained Control of Input Delayed Systems with Partially Compensated Input Delays", *ASME Dynamic Systems and Control Conference (DSCC)*, Virtual Conference, 2020.

I. Abel, M. Janković, and M. Krstić, "Constrained Stabilization of Multi-Input Linear Systems with Distinct Input Delays", *IEEE/IFAC Workshop on Control of Systems Governed by Partial Differential Equations (CPDE)*, Oaxaca, Mexico, 2019.

A. Ghaffari, **I. Abel**, D. Ricketts, S. Lerner, and M. Krstić, “Safety Verification Using Barrier Certificates with Application to Double Integrator with Input Saturation and Zero-Order Hold”, *IEEE American Control Conference (ACC)*, Milwaukee, WI, 2018.

E. Verhasselt, C. Macfarland, **I. Abel**, R. Quevedo, and N. Macken, “Design, Construction, and Testing of a Hydrogen Fuel Cell Powered Vehicle”, *ASME International Conference on Fuel Cell Science, Engineering and Technology*, Boston, MA, 2014.

ABSTRACT OF THE DISSERTATION

**Safety-Critical Control of Nonlinear Systems:
Input Delay Compensation and Prescribed-Time Safety**

by

Imoleayo Abel

Doctor of Philosophy in Engineering Sciences (Mechanical Engineering)

University of California San Diego, 2022

Professor Miroslav Krstić, Chair

The ubiquity of dynamical systems in society has brought the subject matter of their safety to the fore. From small personal drones to autonomous vehicles and jumbo jets, the safety of dynamical systems has become critical and of utmost importance. In this dissertation, we develop control strategies for enforcing safety — characterized by state constraints — in nonlinear dynamical systems of two kinds.

First, systems with input delays are considered. This class of systems is particularly relevant because delays abound in engineering applications and their effect can be catastrophic if not systematically addressed. We develop control strategies for enforcing safety in (i) systems

with a single, *time-varying* input delay across all input channels, and (ii) systems with *distinct*, constant input delays across input channels. The control strategies developed utilize a nontrivial combination of state-predictors with a safe feedback law designed for the corresponding delay-free system. In the case of systems with distinct input delays, we introduce algorithms that enforce safety using a subset of shorter-delayed input channels whenever possible, allowing safety guarantees to be made *before* longer input delays have been compensated. This is a feature that is especially beneficial when input delays are of significantly different lengths.

Next, a prescribed-time safety (PTSf) algorithm is developed for systems where safety is required for a finite, *a priori* prescribed time. In contrast to existing algorithms that only permit an asymptotic approach to (but not arrival at) the boundary of a safe set, PTSf allows a system to reach the boundary of a safe set — at an *infinitely soft* rate — at the end of the prescribed time, irrespective of initial condition. This feature is of interest in systems where the desired nominal behavior is prescribed-time stabilization to a set point on the boundary of the safe set. In addition to enforcing safety only for a prescribed time, PTSf is developed for safety constraints of arbitrarily high relative degree. With the aid of time-varying backstepping, and an explicit initial control gain selection criteria, PTSf retains the entirety of a safe set with no restriction on a system's initial condition.

Chapter 1

Introduction

The safety of dynamical systems has always been considered important. For example, systematic safeguards exists in large chemical plants to ensure the concentration of reagents stay within safe limits. Likewise, industrial factory-floor robots have robust safety measures in place to prevent catastrophic accidents. However, as recent technological advances continue to place dynamical systems front and center in everyday life, the need for strong safety guarantees has become even more crucial. From ensuring that small vacuum-like robots at homes operate safely in close proximity to unsuspecting infants, to ensuring that a distracted teenager on a semi-autonomous e-skateboard safely navigates bustling pedestrian traffic on a university campus, the problem of safety has never been more important.

Present in many practical systems are input delays which arise largely due to physical limitations. Some common sources of delays include computationally expensive control algorithms requiring tangible execution times, lag in communication networks, transmission delays in highly interconnected systems e.t.c. In essence, the impossibility of instantaneously transferring matter or information from one point to another inherently introduces delays in physical systems. While significantly short input delays can often be ignored in practise, it is sometimes crucial to intentionally address their effects, especially in systems where safety is critical. In many scenarios, a controller which is otherwise safe in the absence of delays can lead to catastrophic outcomes in the presence of input delays. Perhaps a familiar albeit unfortunate example is the case of human driving, where many fatal accidents have been caused solely because of a distracted or inebriated driver's delayed reaction.

In this dissertation, we develop control strategies for enforcing safety in nonlinear systems with input delays. Specifically, we consider systems with a single time-varying input delay across all input channels, and systems with multiple, distinct input delays. For the latter class of systems, our design prioritizes shorter-delayed input channels for enforcing safety when the longer-delayed channels have yet to be delay-compensated. Separately, and in addition, we introduce a prescribed-time safety design for systems with high relative degree which allows a system to reach the boundary of a safe set but no sooner than a prescribed time of interest. The designs presented in this dissertation will rely on some established concepts on safety-critical control, predictor-feedback control, and prescribed-time control, all of which we briefly describe next.

1.1 Safety-critical control

System safety is typically defined in terms of constraints on the states and/or inputs of a system. For example, a measure of safety for an autonomous vehicle might be that it stays below the posted speed limit on a highway and maintains a minimum following distance from a leading vehicle. This safety measure is a constraint on the vehicle's states, specifically its velocity and its relative position. Similarly, a separate safety measure for the same vehicle might relate to passenger comfort, wherein the rate of change of the vehicle's acceleration/braking input might be required to stay below a desired magnitude to limit the vehicle's jerk. This constitutes an input constraint on the system. While many practical applications involve a combination of state and input constraints, the problems we consider in this dissertation will focus on safety characterized by state constraints but not input constraints.

Some popular approaches for enforcing state constraints in nonlinear systems include the use of control barrier functions (CBFs) [4, 6, 35, 93, 98], Hamilton-Jacobi reachability analysis [8, 16, 17], safe MPC [75], nonovershooting control [50, 52], but to mention a few. The designs we develop in this dissertation will adopt the use of CBFs, in particular the *zeroing*-type CBFs. Control barrier functions are special functions that characterize state constraints and satisfy additional rate conditions along solutions of a system. Specifically, CBFs are functions whose 0-superlevel sets represent the safe set of a system, and for which control input exists to prevent a faster than desired rate of decrease along solutions of the system. Like control Lyapunov functions for stabilization, CBFs can sometimes be difficult to find in practise but several methods have been studied for synthesizing CBFs [18, 55, 87, 97].

With a CBF available, safe controllers can be synthesized via the so-called *safety filtering* method where a baseline, safety-agnostic controller is first designed for the system, and it is filtered (i.e. overridden) when the system is at risk of being unsafe. This allows safety to be implemented in a supervisory manner as an add-on feature for already existing controllers. The safety filtering process is often achieved via the formulation of a quadratic program (QP) problem that aims to find the closest control to the baseline control (in the Euclidean norm sense) that ensures that the CBF does not decrease faster than a pre-specified rate. This QP-based safety-filtering will feature in many of our designs.

1.2 Predictor feedback control

A common approach for the control of systems with input delays is the use of predictor-feedback [9–11, 46, 47]. In comparison to state-feedback control where the current state of a system is used to determine appropriate control actions, predictor feedback control involves using a prediction of *future* states of the system to determine inputs to be applied at the current time. Particularly, because of input delays, inputs applied to a system at a particular time do not impact the system until a later time in the future; therefore, the idea of predictor-feedback is to predict the state of a system at the time when new inputs will arrive, and to use those predictions to determine inputs to be applied at the current time. As a result, the state-predictors *compensate* the effect of input delays and allow the system to behave as though the input delays were not present. Chapters 2 and 3 will cover the use of state predictors for compensating the effect of time-varying input delays and multiple, distinct input delays respectively. Despite the use of the word “prediction”, the determination of future states of an input-delayed system is not heuristic. Instead, it is an exact, (sometimes explicit) expression of the future state in terms of the current state and the history of inputs as we shall see in the aforementioned chapters.

It is important to note that a system with input delays will initially evolve uncontrolled. For example, a system with a 2-second input delay cannot be controlled in the first two seconds of operation as the inputs applied at the initial time must incur the 2-second delay. This initial period of no control is called the “dead time”. Moreover, in the case that a system has multiple input channels with delays of different lengths, the system can only be under partial control until the dead time of the input channel with the longest delay has elapsed and all input delays have been compensated. When the control objective is asymptotic regulation to a set point or reference trajectory — especially for systems that are globally asymptotically stable in the absence of delays — it is often enough to require that the system does not exhibit finite escape before all control inputs first arrive at the plant; that is, the system states do not grow unbounded before full delay compensation. This requirement is sufficient because system behavior exhibited before full delay compensation will be transient and therefore not affect the asymptotic properties of the closed-loop system. In contrast, when the control objective is safety — a property that is required for all times and not just asymptotically, extra care must be taken to ensure that a system does not go unsafe after some, but not all input delays have been compensated. We will discuss designs that achieve this in Chapter 4.

1.3 Prescribed-time control

The control of continuous time nonlinear systems traditionally involve the asymptotic regulation of the system's states to a desired set point or reference trajectory. That is, as time tends toward *infinity*, the states of the system get increasingly closer to the desired regulation target but never quite arrive at it — in theory. However, in time-constrained practical applications like missile guidance, there is the need for convergence to occur within a finite amount of time. Advances is this regard has been made generally across three fronts: finite-time stabilization (FnTS) [13, 27, 64, 67], fixed-time stabilization (FxTS) [51, 66, 68, 104], and prescribed-time stabilization (PTS) [30, 31, 45, 84]. All of these notions involve the design of controllers that regulate a system's state in a manner where convergence occurs in a *finite* amount of time. In the case of FnTS, the finite settling time depends on the initial condition of the system. With FxTS, the settling time is also finite, initial condition dependent, but is uniformly bounded by a fixed constant that is often very conservative. Lastly and more recently developed is the strongest notion of PTS, in which the finite settling time of the control objective can be arbitrarily specified beforehand, independent of the system parameters and initial conditions.

Motivated by these recent advancements in the study of prescribed-time stabilization, we introduce an analogous notion for safety, termed "Prescribed Time Safety" (PTSf), which is the enforcement of safety in a manner where a system is allowed to reach the boundary of a safe set at, *but no sooner than*, a user-specified prescribed time. A consequence of this notion is that safety is enforced only for a finite time of interest to a designer. While PTSf amounts to "less safety" than traditional notions of safety where safety is enforced indefinitely, PTSf is particularly advantageous in systems where the nominal objective is to regulate the state to a set point on the boundary of a safe set in a specified amount of time. In these cases, for the nominal system behavior to be preserved, a supervisory safety enforcing controller must be such that it does not prevent the system from reaching this set point at the desired terminal time. Traditional infinite-time safety designs do not come with such guarantees as they only allow systems to approach the boundary of a safe set asymptotically, but not in a finite amount of time. Also, as we shall later show in Chapter 5, our PTSf design will endow systems with an *infinitely soft landing* property in the case that a nominal controller attempting to violate safety constraints is overridden over a time interval that includes the terminal time.

1.4 Organization

This dissertation is organized as follows. In Chapter 2, we study the safety critical control of nonlinear systems with a single, time-varying input delay across all input channels. This is an extension of existing results in literature for system with a single, constant input delay. Chapter 3 focuses on the constrained stabilization of linear systems with distinct input input delays across all input channels via the use of recursively defined state predictors for delay compensation. In Chapter 4, we extend the results in Chapter 3 to nonlinear systems, and in particular, we introduce two additional approaches for enforcing system safety before delays in all input channels have been compensated. This is especially of practical importance in systems where the delays across input channels are of significantly different lengths. Finally in Chapter 5, we develop the notion of Prescribed-Time Safety for delay-free systems with high-relative degree safety constraints, where we use a time-varying backstepping design that ensures a system can reach the boundary of a safe set, but no sooner than an a priori prescribed time irrespective of initial condition.

Chapter 2

Safety-Critical Control of Systems with Time-Varying Input Delay

2.1 Introduction

Control Barrier Functions (CBFs) have become popular for synthesizing controllers that enforce the safety of nonlinear systems via safe-set forward invariance. First defined in [98] and later refined and popularized by [4, 6], CBFs have found use in vast application domains ranging from mobile robotics ([14, 32]) to infectious disease control ([5, 57]). Like control Lyapunov functions for stabilization, CBFs provide sufficient conditions that guarantee the forward invariance of an admissible set and can therefore be used for synthesizing safe controllers for nonlinear systems.

Recently, the utility of CBFs has been extended to systems with delays. Due to physical limitations, time delays are inherent in many practical systems and just as they can degrade the performance of stable closed-loop systems, they can also cause an otherwise safe system to become unsafe. Therefore, the need to deliberately address the effect of delays on the safety of systems is of high practical relevance. Existing studies in this regard have considered safety-critical control of nonlinear systems with state delay ([62,72]), linear systems with input delay(s) ([1,34]), and nonlinear systems with input delays ([2,57,82]). All of these results have been for systems with *constant* state or input delays, which while beneficial, is limiting because several real-life applications include models where the delays are time-varying. For example, the reaction time of a human driver varies with time and can be modeled as a time-varying input delay ([63,83]). In addition, time-varying input delays abound in networked control systems ([20,58]). In this chapter, we extend the use of CBFs to nonlinear systems with time-varying input delays, by drawing from the wealth of results for the stabilization of systems with such time-varying input delays. The stabilization problem for this class of systems has been studied extensively for linear systems ([48,60]) and nonlinear systems ([9,10,54]). In particular, we rely on the results of [9] to design state predictors that compensate time-varying input delays, and we use these predictors in combination with CBF-based controllers designed for a nominal delay-free system. We show that under certain reasonable assumptions on the time-varying input delay, a safe-feedback law designed for the nominal delay-free system achieves the same closed loop performance under predictor-feedback after the input delay has been compensated.

The remainder of this chapter is structured as follows. Section 2.2 includes the problem description, and an enumeration of assumptions made about the plant and the time-varying input delay. In Section 2.3, we describe our control design methodology and prove the main result. Section 2.4 includes two numerical examples that demonstrate the effectiveness of our control approach, and it is followed by a brief conclusion in Section 2.5.

Notation. For differentiable function $h(x)$ and vector field $f(x)$, the notation $\mathscr{L}_f h(x)$ denotes the Lie derivative of $h(x)$ with respect to $f(x)$ and is defined as $\frac{\partial h}{\partial x} f(x)$. A function $f(x)$ is said to be Lipschitz continuous at x_0 if there exists constants L and $\varepsilon > 0$ (that may depend on x_0) such that $\|f(x) - f(y)\| \leq L\|x - y\|$ wherever $\|x - x_0\| \leq \varepsilon$ and $\|y - x_0\| \leq \varepsilon$. A function $\alpha : (-b, a) \to (-\infty, \infty)$ for some $a, b > 0$ belongs to the *extended* class $\mathscr{K}$, denoted as $\mathscr{K}_e$, if it is continuous, strictly increasing, and $\alpha(0) = 0$. We shall also assume that all functions $\alpha \in \mathscr{K}_e$ used in control laws are Lipschitz continuous.

2.2 Problem description

We consider the following system with time-varying input delay

$$\dot{x}(t) = f(x(t)) + g(x(t))u\big(t - D(t)\big) \tag{2.1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, $D(t)$ is the time-varying input delay satisfying conditions we shall enumerate shortly, and the vector fields $f(x),\ g(x) : \mathbb{R}^n \to \mathbb{R}^n$ are assumed to be locally Lipschitz. We shall further assume that the system (2.1) is forward complete with respect to the input; that is, for every initial condition $x_0 \in \mathbb{R}^n$ and locally bounded input signal $u(t)$, the corresponding solution of (2.1) is well defined for all $t \geq -D(0)$. In essence, this assumption guarantees that the system does not exhibit finite escape. As done in [9, 10, 60] where the problem of stabilizing (2.1) was considered, we introduce the function

$$\phi(t) = t - D(t). \tag{2.2}$$

We call the function $\phi(t)$ the *delay time*, and it represents the time when the input arriving at the plant at time t was issued. Using the function $\phi(t)$ in place of $t - D(t)$ is notationally convenient because the state predictor that we will utilize requires the inverse function $\phi^{-1}(t)$, whose value represents the time when input command issued at time t will arrive at the plant. To that effect, we shall impose invertibility conditions on $\phi(t)$, and we will refer to $\phi^{-1}(t)$ as the *prediction time*. Notice that for the special case of constant delays $D(t) \equiv D$, the delay horizon $t - \phi(t)$ and prediction horizon $\phi^{-1}(t) - t$ are one and the same. With these definitions, the system (2.1) can be written as

$$\dot{x}(t) = f(x(t)) + g(x(t))u(\phi(t)). \tag{2.3}$$

Now, we describe conditions that the delay time $\phi(t)$ must satisfy for solutions of (2.1) to be well defined and meaningful.

Assumption 1: The function $\phi(t)$ satisfies

$$\phi(t) < t, \quad \forall t \geq 0, \tag{2.4}$$

and

$$\sup_{\theta \geq \phi^{-1}(0)} (\theta - \phi(\theta)) < \infty. \tag{2.5}$$

The condition in (2.4) guarantees causality of the plant, and can be written equivalently as $D(t) > 0$ for all $t \geq 0$; while the bound in (2.5) ensures that all inputs applied to the plant eventually arrive at the plant. Said differently, (2.5) ensures that there exists a positive constant $\overline{D}$ such that $D(t) < \overline{D}$ for all $t \geq 0$.

Assumption 2: The function $\phi(t)$ is continuously differentiable and its time-derivative $\phi'(t)$ satisfies

$$\phi'(t) > 0, \quad \forall t \geq 0 \tag{2.6}$$

and

$$\sup_{\theta \geq \phi^{-1}(0)} \phi'(\theta) < \infty. \tag{2.7}$$

Here, the inequality (2.6) which is equivalent to $\frac{dD(t)}{dt} < 1$ guarantees that the input signal direction is not reversed, that is, the plant does not feel input values older than the ones it has already felt. In addition, condition (2.6) alongside the continuous differentiability of $\phi(t)$ implies that $\phi(t)$ is invertible and therefore the prediction time is well defined for all $t \geq 0$. Lastly, the inequality (2.7) ensures that the input delay does not vanish instantaneously but only gradually; in other words, it guarantees that $\frac{dD(t)}{dt}$ is uniformly bounded from below.

The control objective is to enforce the forward invariance of an admissible set $\mathscr{C}$ defined as

$$\mathscr{C} = \{x \in \mathbb{R}^n \mid h(x) \geq 0\}, \tag{2.8}$$

where $h(x)$ is a continuously differentiable function. In other words, we desire a controller $u(t)$

that achieves $x(0) \in \mathscr{C} \implies x(t) \in \mathscr{C}$ for all $t \geq 0$. We shall call the set $\mathscr{C}$ the *safe set* and assume that it is nonempty and has no isolation points i.e. $\text{Int}(\mathscr{C}) \neq \emptyset$ and $\overline{\text{Int}(\mathscr{C})} = \mathscr{C}$. We say the system is safe if $\mathscr{C}$ is forward invariant. Now, because of the input delay, the system evolves uncontrolled in the first $\phi^{-1}(0)$ units of time before inputs applied at time $t = 0$ arrive at the plant, and we must therefore require that the system remains safe before any designed control effort arrives at the plant. We summarize this requirement by making the following assumption.

Assumption 3: The initial state $x(0)$ and initial input functional $u(\theta) : [\phi(0), 0) \to \mathbb{R}^m$ are such that the state of the system (2.3) satisfies

$$x(t) \in \mathscr{C}, \quad \forall t \in [0, \phi^{-1}(0)]. \tag{2.9}$$

With system and input delay assumptions specified, we now proceed to describe our control approach in the following section.

2.3 Control design and safety analysis

2.3.1 Delay-free systems

When the system (2.3) has no input delay, that is when $\phi(t) = t$, the problem of enforcing forward invariance of an admissible set like (2.8) can be solved with the use of control barrier functions (CBFs) defined as follows:

Definition 2.1 (Control Barrier Function (CBF)). *A continuously differentiable function $h : \mathbb{R}^n \to \mathbb{R}$ is a control barrier function for the system* (2.3) *with respect to the admissible set $\mathscr{C}$ in* (2.8) *if there exists function $\alpha \in \mathscr{K}_e$ such that for all $x \in \mathscr{C}$,*

$$\mathscr{L}_g h(x) = 0 \implies \mathscr{L}_f h(x) + \alpha(h(x)) > 0. \tag{2.10}$$

We note here that the definition of a CBF is independent of the delay time $\phi(t)$. Now, with a CBF $h(x)$ and a corresponding function $\alpha \in \mathscr{K}_e$ in hand, we define the following set for every $x \in \mathscr{C}$ and $u \in \mathbb{R}^m$

$$\Delta_u(x,u) = \{\delta_u \in \mathbb{R}^m \mid \mathscr{L}_f h(x) + \mathscr{L}_g h(x)(u+\delta_u) + \alpha(h(x)) \geq 0\}. \tag{2.11}$$

For every state $x \in \mathscr{C}$ and control $u \in \mathbb{R}^m$, the set $\Delta_u(x,u)$ is the set of permissible modifications to u required for the "barrier constraint" $\mathscr{L}_f h(x) + \mathscr{L}_g h(x)(u+\delta_u) + \alpha(h(x)) \geq 0$ to be satisfied. With (2.11) defined, the task of enforcing forward invariance of $\mathscr{C}$ when $\phi(t) = t$ boils down to ensuring that at all time t when the state $x(t) = x$, a potentially unsafe reference control $u_{\text{ref}} = u_{\text{ref}}(t)$ is modified by an element of $\Delta_u(x, u_{\text{ref}})$ as established in the following result adapted from Corollary 2 of [6].

Theorem 2.1. *If h is a CBF with an associated class $\mathscr{K}_e$ function α for the system* (2.3) *with $\phi(t) = t$, and if $u_{ref} = \kappa_{ref}(t,x) : \mathbb{R}_{\geq 0} \times \mathscr{C} \to \mathbb{R}^m$ is a locally Lipschitz reference controller, then the controller*

$$u = u_{ref} + \delta_u(u_{ref}, x), \qquad \delta_u(u_{ref}, x) \in \Delta_u(x, u_{ref}) \tag{2.12}$$

renders the set $\mathscr{C}$ in (2.8) *forward invariant.*

While this theorem is not by itself constructive, it provides a sufficient condition on δ_u (the required modification of the reference input) for guaranteeing safety, and this condition can be used for synthesizing safe controllers. One example approach for doing so involves choosing the smallest element (in the Euclidean norm sense) of the permissible set of input modifications Δ_u. In particular, given a potentially unsafe, possibly time-varying, reference controller $u_{\text{ref}} = \kappa_{\text{ref}}(t,x)$, we can synthesize a safe controller by overriding at all time t, the reference control u_{ref} with $u = u_{\text{ref}} + \delta_u(u_{\text{ref}}, x(t))$ where $\delta_u(\nu, \chi)$ is the solution of the following quadratic programming (QP) problem:

$$\begin{aligned} \delta_u(\nu,\chi) = \underset{\delta \in \mathbb{R}^m}{\arg\min}\ \|\delta\|^2 \quad \text{subject to} \\ \mathscr{L}_f h(\chi) + \mathscr{L}_g h(\chi)(\nu+\delta) + \alpha(h(\chi)) \geq 0. \end{aligned} \tag{2.13}$$

The feasibility of this QP problem follows from the definition of a CBF, and the explicit solution is given by the Karush-Kuhn-Tucker (KKT) optimality conditions as

$$\delta_u(\nu,\chi) = \begin{cases} 0 & \text{if } \zeta(\nu,\chi) \geq 0 \\ -\frac{\zeta(\nu,\chi)}{\|\mathscr{L}_g h(\chi)\|^2} \mathscr{L}_g h(\chi)^\top & \text{otherwise} \end{cases} \tag{2.14}$$

where

$$\zeta(\nu,\chi) = \mathscr{L}_f h(\chi) + \mathscr{L}_g h(\chi)\nu + \alpha(h(\chi)) \tag{2.15}$$

The feedback modification (2.14) satisfies $\delta_u(\nu,\chi) \in \Delta_u(\chi,\nu)$, and therefore the controller

$$u(t) = u_{\text{ref}}(t) + \delta_u(u_{\text{ref}}(t), x(t)) \tag{2.16}$$

is the smallest modification (pointwise) to the potentially-unsafe reference controller u_{ref} that renders the set $\mathscr{C}$ forward invariant for the system (2.3) with no input delay. We note that even though the reference control u_{ref} could potentially be a time-varying feedback law, the feedback modification δ_u in (2.14) is time-invariant.

2.3.2 Systems with time-varying input delay

As mentioned in the introduction, the described CBF approach for safe-set invariance has been extended to systems with constant input delay(s) ([1, 2, 34, 57, 82]) and these extensions utilize a state predictor in conjunction with a CBF-based feedback-law designed for the delay-free system. With a safe feedback modification $\delta_u : \mathbb{R}^m \times \mathscr{C} \to \mathbb{R}^m$ satisfying $\delta_u(\nu,\chi) \in \Delta_u(\chi,\nu)$ available, the approach for systems with constant input delay $\phi(t) = t - D$ amounts to using the controller

$$u(t) = u_{\text{ref}}(t) + \delta_u(u_{\text{ref}}(t), p(t)) \tag{2.17}$$

(see Figure 2.1) where $p(t)$ is the D time units ahead prediction of the state, which depends on the current state $x(t)$ and the input history $u_t(\theta) : [-D, 0) \to \mathbb{R}^m$ with $u_t(\theta) = u(t+\theta)$. The

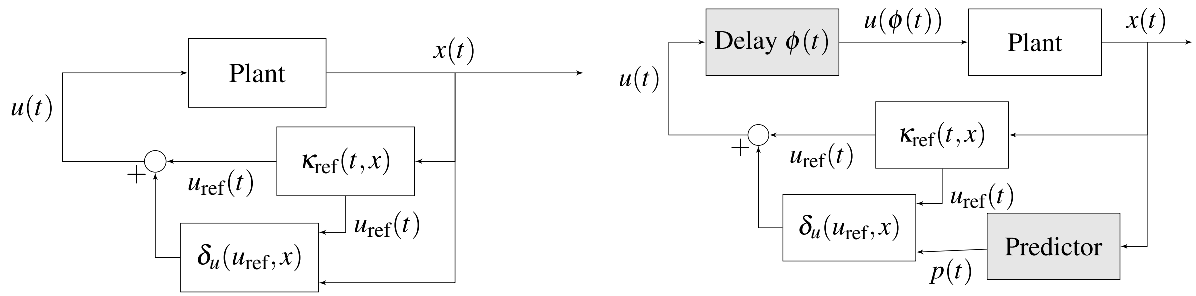

Figure 2.1. System block diagram with feedback modification of a reference controller κ_{ref}. Left: Nominal delay-free system. Right: System with input delay time $\phi(t)$.

predictor feedback approach (2.17) works because the predictor state $p(t) = x(t+D)$ satisfies the ODE

$$\frac{d}{dt}(p(t)) = f(p(t)) + g(p(t))u(t) \tag{2.18}$$

which is of the same form as the system (2.1) with no input delay. Thus, the use of predictor feedback eliminates the effect of the input delay, and the safe feedback law designed for the delay-free system achieves the same closed behavior under predictor-feedback.

Now, consider the original system (2.3) with time-varying input delay. Similar to the constant delay case, we desire at all times t, a prediction of the state at time $\phi^{-1}(t)$ when the input applied at time t arrives at the plant. We therefore use a $\phi^{-1}(t) - t$ time units ahead state predictor $p(t)$ given in [9] as

$$p(\theta) = x(t) + \int_{\phi(t)}^{\theta} \frac{f(p(\sigma)) + g(p(\sigma))u(\sigma)}{\phi'(\phi^{-1}(\sigma))}\, d\sigma, \qquad \phi(t) \leq \theta \leq t \tag{2.19}$$

for $\theta = t$, with initial condition

$$p(\theta) = x(0) + \int_{\phi(0)}^{\theta} \frac{f(p(\sigma)) + g(p(\sigma))u(\sigma)}{\phi'(\phi^{-1}(\sigma))}\, d\sigma, \qquad \forall\theta \in [\phi(0), 0]. \tag{2.20}$$

Note that the condition (2.6) in Assumption 2 guarantees that the division by $\phi'(\phi^{-1}(\sigma))$ in (2.19) and (2.20) are safe. While the predictor equation (2.19) is implicitly defined, explicit formulas exist for some classes of systems e.g. linear systems. In particular, if $f(x) = Ax$ and

$g(x) = B$ with $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$, then the predictor (2.19) can be written explicitly as

$$p(\theta) = e^{A(\phi^{-1}(\theta)-t)}x(t) + \int_{\phi(t)}^{\theta} \frac{e^{A(\phi^{-1}(\theta)-\phi^{-1}(\sigma))}Bu(\sigma)}{\phi'(\phi^{-1}(\sigma))}\,d\sigma, \qquad \phi(t) \le \theta \le t. \tag{2.21}$$

Nonetheless, even in the implicit form (2.19), the predictor state at time t is given in terms of the current state $x(t)$ as well as the predictor and input history over the time interval $[\phi(t),t)$, and can therefore be implemented numerically. The design of numerical approximations of state predictors, and its implication for theoretical results is an area of research that is beyond the scope of the subject matter of this chapter. The reader can consult [38] for details. In this chapter, we assume that predictor implementations are accurate. In the case that they aren't and a bound on the prediction error is known, one can use the robust CBF ([35]) or ISSf CBF ([43]) framework in place of the standard vanilla CBF used here. Now, returning focus to the predictor equation (2.19), we note that for the special case of constant input delay $D(t) \equiv D$, we have $\phi(t) = t - D$, $\phi'(t) = 1$, and $\phi^{-1}(t) = t + D$ and the predictor equation (2.19) for $\theta = t$ becomes

$$p(t) = x(t) + \int_{t-D}^{t} \left[f(p(\sigma)) + g(p(\sigma))u(\sigma)\right] d\sigma \tag{2.22}$$

which is easily understood as the D time units ahead prediction of the state in the case of a constant input delay. To see that $p(t)$ from (2.19) is indeed equivalent to $x(\phi^{-1}(t))$, notice that the derivative of $p(\theta)$ in (2.19) satisfies

$$\frac{d}{d\theta}p(\theta) = \frac{f(p(\theta)) + g(p(\theta))u(\theta)}{\phi'(\phi^{-1}(\theta))} \tag{2.23}$$

which is of the same form as $\frac{d}{dt}x(\phi^{-1}(t))$ which can be found using the chain rule, (2.3), and the fact that $\frac{d}{dt}\phi^{-1}(t) = \frac{1}{\phi'(\phi^{-1}(t))}$. Furthermore, we have from (2.19) that $p(\phi(t)) = x(t) = x(\phi^{-1}(\phi(t)))$ and it follows from the similarity of the ODEs (2.23) and $\frac{d}{dt}x(\phi^{-1}(t))$ that $p(\theta) = x(\phi^{-1}(\theta))$ for all $\theta \in [\phi(t),t]$, and therefore $p(t) = x(\phi^{-1}(t))$. We note here that instead of viewing $p(\theta)$ as the output of a system evolving in the same time domain as $x(t)$, $p(\theta)$ should be viewed as the output of a separate operator parameterized by t, and acting on $p(\sigma)$ and $u(\sigma)$ on the interval $\phi(t) \leq \sigma < \theta$.

Now, suppose we have a CBF h and an associated class $\mathscr{K}_e$ function α, we desire at all times t, control input $u(t)$ that achieves

$$\mathscr{L}_f h(p(t)) + \mathscr{L}_g h(p(t))u(t) + \alpha(h(p(t))) \geq 0. \tag{2.24}$$

Since $p(t) = x(\phi^{-1}(t))$, the LHS of this inequality is equivalent to $\dot{h} + \alpha(h) \geq 0$ for the system (2.3) at time $\phi^{-1}(t)$. In particular, (2.24) can be re-written as

$$\frac{\partial h}{\partial x}(p(t))\frac{dx}{dt}(\phi^{-1}(t)) + \alpha(h(p(t))) \geq 0. \tag{2.25}$$

The choice of notation $\frac{dx}{dt}(\phi^{-1}(t))$ is to emphasize that we desire $\frac{dx}{dt}$ evaluated at time $\phi^{-1}(t)$, as opposed to $\frac{d}{dt}\left[x(\phi^{-1}(t))\right]$. This distinction is important because we care about the rate of change of the barrier function along trajectories of the original system as opposed to the rate of change along the values of $p(t)$.

Theorem 2.2. *If h is a CBF with an associated class $\mathscr{K}_e$ function α for the system* (2.3) *with initial conditions satisfying Assumption 3, and if $u_{ref} = \kappa_{ref}(t,x)$ with $\kappa_{ref} : \mathbb{R}_{\geq 0} \times \mathscr{C} \to \mathbb{R}^m$ is a Lipschitz continuous reference controller, then the predictor-based feedback modified controller*

$$u(t) = u_{ref}(t) + \delta_u(u_{ref}(t), p(t)) \tag{2.26}$$

for every $\delta_u(u_{ref}, p(t)) \in \Delta_u(p(t), u_{ref})$, $t \geq 0$ with $p(t)$ defined in (2.19), (2.20) *renders the set $\mathscr{C}$ in* (2.8) *forward invariant.*

Proof. Since the system (2.3) evolves uncontrolled in the first $\phi^{-1}(0)$ units of time, we have by Assumption 3 that $x(0) \in \mathscr{C} \implies x(t) \in \mathscr{C}$ for all $t \in [0, \phi^{-1}(0)]$. It remains to show that $x(\phi^{-1}(0)) \in \mathscr{C} \implies x(t) \in \mathscr{C}$ for all $t \geq \phi^{-1}(0)$. Since $\phi(t)$ is continuous and monotonically increasing (Assumption 2), we have for every $t \geq \phi^{-1}(0)$, a unique $\tau \geq 0$ such that $t = \phi^{-1}(\tau)$. Therefore, it suffices to show that $x(\phi^{-1}(0)) \in \mathscr{C} \implies x(\phi^{-1}(\tau)) \in \mathscr{C}$ for all $\tau \geq 0$. Now, since for all $\tau \geq 0$, $u(\tau) = u_{\mathrm{ref}}(\tau) + \delta_u(u_{\mathrm{ref}}(\tau), p(\tau))$ with $\delta_u(u_{\mathrm{ref}}(\tau), p(\tau)) \in \Delta_u(p(\tau), u_{\mathrm{ref}}(\tau)) = \Delta_u(x(\phi^{-1}(\tau)), u_{\mathrm{ref}}(\tau))$, we have

$$\mathscr{L}_f h(x(\phi^{-1}(\tau))) + \mathscr{L}_g h(x(\phi^{-1}(\tau)))u(\tau) + \alpha(h(x(\phi^{-1}(\tau)))) \geq 0 \tag{2.27}$$

for all $\tau \geq 0$, or equivalently

$$\mathscr{L}_f h(x(t)) + \mathscr{L}_g h(x(t))u(\phi(t)) + \alpha(h(x(t))) \geq 0 \tag{2.28}$$

for all $t \geq \phi^{-1}(0)$. Therefore, we have

$$\frac{d}{dt}\left[h(x(t))\right] \geq -\alpha(h(x(t))) \tag{2.29}$$

for all $t \geq \phi^{-1}(0)$ along trajectories of the system (2.3). With (2.29), the rest of the proof follow directly from [62, Theorem 2]. □

As established in the delay-free case, an example feedback modification δ_u satisfying $\delta_u(\nu, \chi) \in \Delta_u(\chi, \nu)$ for all $\chi \in \mathscr{C}$ and $\nu \in \mathbb{R}^m$ is given in (2.14). Therefore, for the system (2.3) with time-varying input delay, a safeguarding controller that renders a potentially unsafe reference controller $u_{\mathrm{ref}}(t)$ safe is the controller (2.26) with δ_u defined in (2.14) and $p(t)$ defined in (2.19), (2.20). We now present some numerical examples.

2.4 Numerical simulations

2.4.1 Scalar system

Consider the system

$$\dot{x}(t) = x(t) + u(\phi(t)) \tag{2.30}$$

from Example 5.4 of [48], where the inverse of the delay time function is given as

$$\phi^{-1}(t) = t + 1 + \frac{1}{2}\cos(t). \tag{2.31}$$

While the delay-time $\phi(t)$ is not given explicitly, it can be approximated numerically to sufficiently high precision as a reflection of $\phi^{-1}(t)$ about the line $y = t$ in the (t, y)-plane. The initial condition of the input is $u(\theta) = 0$ for all $\theta \in [\phi(0), 0)$. The goal is to keep the state of the system inside the safe interval $\mathscr{C} = [-2, 2]$ for which we use the candidate CBF

$$h(x) = 4 - x^2 \tag{2.32}$$

We have

$$\mathscr{L}_f h(x) = -2x^2, \tag{2.33}$$

$$\mathscr{L}_g h(x) = -2x \tag{2.34}$$

and $\mathscr{L}_g h(x) = 0 \implies \mathscr{L}_f h(x) + \alpha(h(x)) = \alpha(h(x)) \geq 0$ for all $\alpha \in \mathscr{K}_e$ and $x \in \mathscr{C}$; therefore h is indeed a CBF. We choose $\alpha(h) = h$, and assume a reference control signal $u_{\text{ref}}(t) = 0$ for all $t \geq 0$ — a signal that renders the system unsafe if unmodified. We consider the system behavior under the uncompensated controller (2.16) and the delay-compensated controller (2.26) with the predictor state given explicitly as

$$p(t) = e^{1+\frac{1}{2}\cos(t)}x(t) + \int_{\phi(t)}^{t} \left(1 - \frac{1}{2}\sin(\sigma)\right) e^{t+\frac{1}{2}\cos(t)-\sigma-\frac{1}{2}\cos(\sigma)} u(\sigma)\, d\sigma. \tag{2.35}$$

The state $x(t)$, barrier function $h(x(t))$, and input signal $u(t)$ are shown in Figure 2.2 for initial states $x(0) = 0.15$ and $x(0) = -0.5$. In the latter case, the system exits the safe-set before the controller kicks in at time $\phi^{-1}(0) = 1.5$ but the predictor-feedback controller (2.26) forces the state back inside of the safe region. On the contrary, under the uncompensated controller (2.16), the system always exits the safe-set.

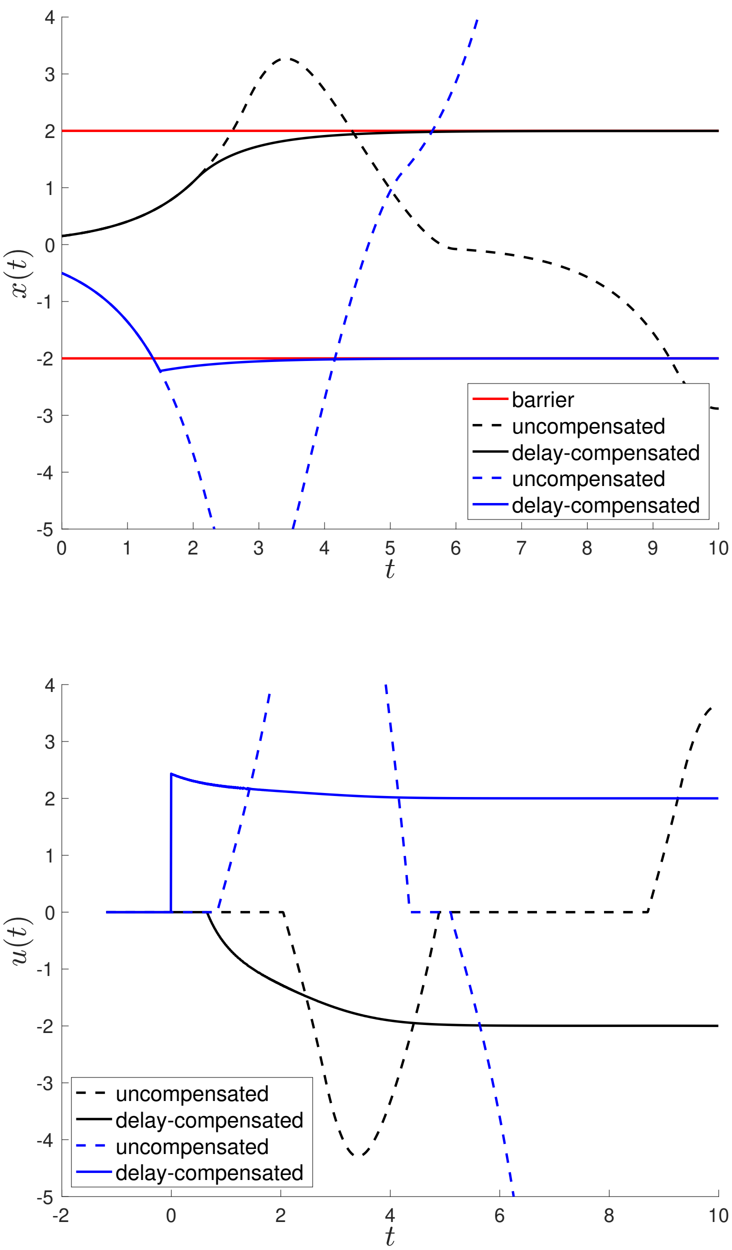

Figure 2.2. System response for Example 2.4.1 using uncompensated controller (2.16), (2.14) and delay-compensated controller (2.26), (2.14) for initial states $x(0) = 0.15$ (black) and $x(0) = -0.5$ (blue). Top: state trajectory. Bottom: input signal.

2.4.2 Nonholonomic unicycle

Consider the nonholonomic unicycle and time-varying input delay from Example 2 of [9]

$$\begin{aligned}\dot{x}(t) &= v(\phi(t))\cos(\theta(t))\\ \dot{y}(t) &= v(\phi(t))\sin(\theta(t))\\ \dot{\theta}(t) &= \omega(\phi(t))\end{aligned} \tag{2.36}$$

where the delay time function is

$$\phi(t) = t - \frac{1+t}{1+2t}, \tag{2.37}$$

with derivative and inverse given as

$$\phi'(t) = 1 + \frac{1}{(1+2t)^2}, \tag{2.38}$$

$$\phi^{-1}(t) = \frac{t+\sqrt{1+(1+t)^2}}{2}. \tag{2.39}$$

For the reference controller $(v_{\text{ref}}(t), \omega_{\text{ref}}(t))^\top$, we use the stabilizing, time-varying, predictor feedback controller from [9] given as

$$\omega_{\text{ref}}(t) = -5A(t)^2\cos(3\phi^{-1}(t)) - A(t)B(t)\left(1+25\cos^2(3\phi^{-1}(t))\right) - p_\theta(t) \tag{2.40}$$

$$v_{\text{ref}}(t) = -A(t) + 5B(t)\sin(3\phi^{-1}(t)) - 5B(t)\cos(3\phi^{-1}(t)) + B(t)\omega_{\text{ref}}(t) \tag{2.41}$$

$$A(t) = p_x(t)\cos(p_\theta(t)) + p_y(t)\sin(p_\theta(t)) \tag{2.42}$$

$$B(t) = p_x(t)\sin(p_\theta(t)) - p_y(t)\cos(p_\theta(t)) \tag{2.43}$$

$$p_x(t) = x(t) + \int_{\phi(t)}^{t} \frac{v(\sigma)\cos(p_\theta(\sigma))}{\phi'(\phi^{-1}(\sigma))}\,d\sigma \tag{2.44}$$

$$p_y(t) = y(t) + \int_{\phi(t)}^{t} \frac{v(\sigma)\sin(p_\theta(\sigma))}{\phi'(\phi^{-1}(\sigma))}\,d\sigma \tag{2.45}$$

$$p_\theta(t) = \theta(t) + \int_{\phi(t)}^{t} \frac{\omega(\sigma)}{\phi'(\phi^{-1}(\sigma))}\,d\sigma \tag{2.46}$$

Notice that all expressions on the RHS of (2.40)-(2.46) are known at every time t since they depend on the state and input history. The initial condition on the input is given as $(v(s), \omega(s))^\top = (0,0)^\top$ for all $s \in [\phi(0), 0)$. The safety objective is to keep the x-position of the unicycle to the right of a desired boundary x_{bndry} and for that we use the CBF

$$h(x) = x - x_{\text{bndry}} \tag{2.47}$$

which satisfies the CBF condition (2.10) since the system is driftless. We choose $\alpha(h) = 8h$ and show in Figure 2.3 simulation results for initial state $x(0) = y(0) = \theta(0) = 1$ and boundary $x_{\text{bndry}} = -0.1$ under uncompensated controller (2.16) and delay-compensated controller (2.26) with state predictor $p(t) = (p_x(t), p_y(t), p_\theta(t))^\top$. Despite the reference controller using predictor feedback, the uncompensated safeguarding controller (2.16) fails to keep the unicycle in the safe set. However, when the predictor-based safeguarding controller (2.26) is used, the unicycle never crosses the safe/unsafe boundary as shown in Figure 2.3.

2.5 Conclusion and acknowledgements

In this chapter, we described a predictor-based approach for designing safe controllers for control-affine nonlinear systems with time-varying input delays. We showed that in combination with a predictor that compensates the time-varying input delay, a safe feedback law designed for the nominal system achieves the same closed-loop safety guarantees for the input-delayed system after the controller kicks in. We included two numerical examples to illustrate the effectiveness of our approach. A natural extension of this work — an extension that would lend the results of this chapter to more practical applications — is the study of safety-critical control for systems where the time-varying input delay $D(t)$ is not known *a priori*.

Chapter 2, in part, is a reprint and adaptation of the paper: I. Abel, M. Krstić, and M. Janković, "Safety-Critical Control of Systems with Time-Varying Input Delay", IFAC Workshop on Time Delay Systems (TDS), 2021. The dissertation author was the primary investigator and

author of this paper.

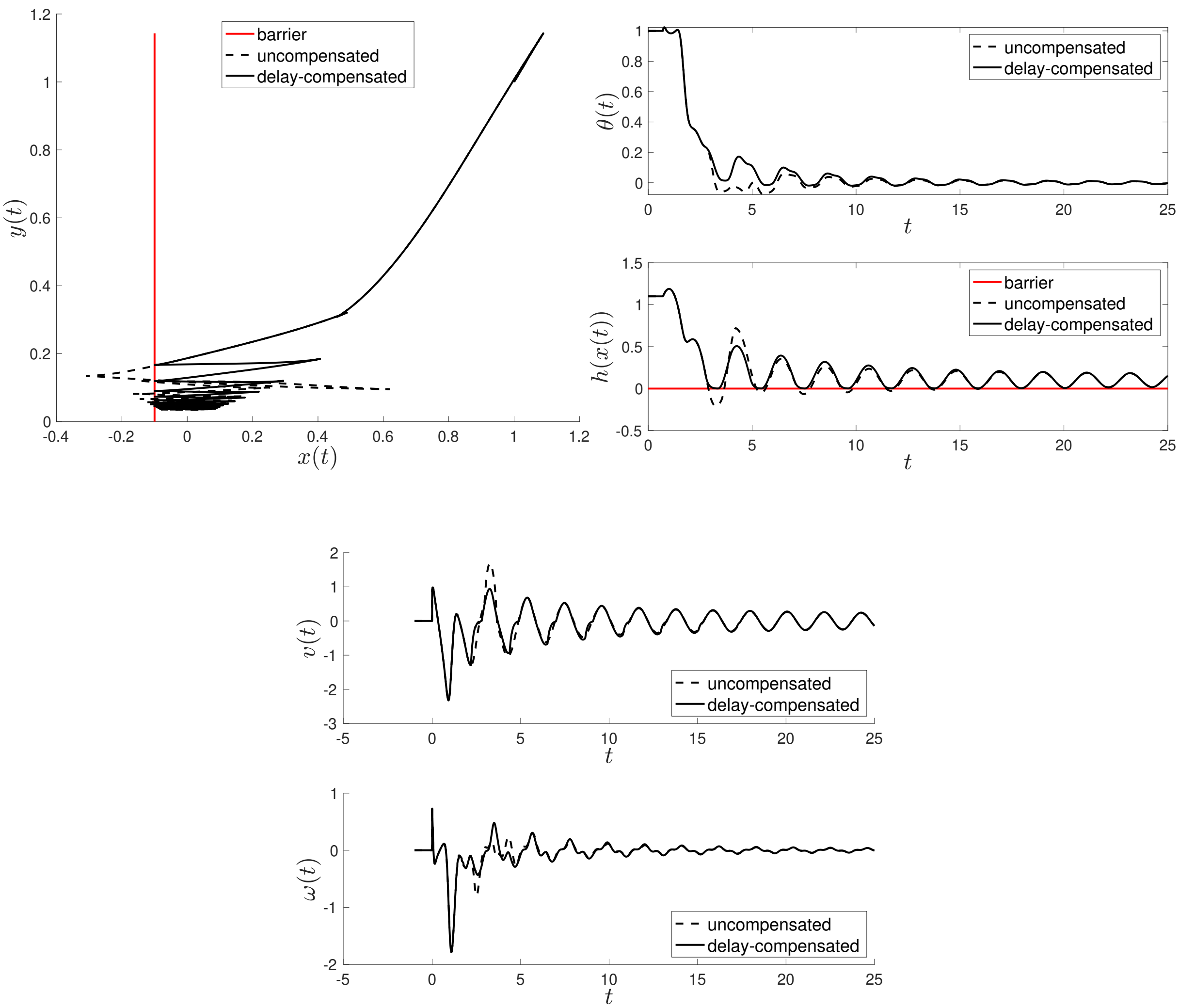

Figure 2.3. Closed loop response for nonholonomic unicycle using uncompensated controller (2.16), (2.14) and delay-compensated controller (2.26), (2.14) for initial state $x(0) = y(0) = \theta(0) = 1$. Top Left: x, y state trajectory. Top Right: angle $\theta(t)$ and barrier function $h(x(t))$. Bottom: input signal $(v(t), \omega(t))^\top$.

Chapter 3

Constrained Stabilization of Multi-Input Linear Systems with Distinct Input Delays

3.1 Introduction

Control Lyapunov functions (CLFs) introduced in [7] have been used in the design of robust stabilizing control laws for nonlinear systems. It has been shown that the existence of a CLF for a system implies the existence of a continuous stabilizing controller. Given a CLF V, universal formulas such as the pointwise minimum norm (PMN) formula [23], and Sontag's formula [86] exist for explicitly computing stabilizing control laws that provide a half-space robustness property for multi-input systems. Likewise, Control Barrier Functions (CBFs) [98] have been used for enforcing that the states of a system remain in an admissible set. When a CBF is available, the same universal formulas for CLF can be used to develop controllers that render the admissible set forward invariant. Several approaches have also emerged for combining CLFs and CBFs to achieve both stabilization and forward invariance of an admissible set ([4, 6, 35, 98]) with applications in robotics ([14]) and automotive ([4]) systems. More recently, the CLF-CBF based approach was extended to linear systems with similar delay in all input channels ([34]).

This chapter serves as an incremental extension of the results in [34] to multi-input linear systems with distinct input delays across all input channels – an extension that lends the results of this chapter to more practical applications than its predecessor. The goal here is to compensate for delays of different lengths across input channels while stabilizing the system – if possible – and keeping the admissible set invariant. As with [34], we assume that all the delays are known and remain constant, and we use predicted values of future states of the system in the baseline CLF and CBF developed for the zero-delay system.

When input delays are of equal length across all input channels, future states of a linear system can be computed explicitly using only the current state and past values of the inputs. However, when the delays are of different lengths, the process of predicting future states becomes more involved. In [95], a predictor feedback approach for multi-input linear systems with distinct input delays was developed, but with the assumption that the baseline controller for the delay-free system is linear. This result was later generalized in [11] to nonlinear systems with distinct input delays. In this chapter, because the baseline Quadratic Programming (QP)-based controller we adopt is nonlinear, we employ the predictor feedback methodology from [11]. There, a predictor is assigned to each individual input channel and the predictor states are recursively computed starting with the input channel with the shortest delay and progressing until the input channel with the longest delay. The predictor states are then fed back to a stabilizing controller designed for the delay-free system. This approach was shown to retain the properties of the nominal controller after the longest delay has been compensated, provided the delay-free system satisfies some additional conditions like strong forward completeness and input-to-state stability when in closed-loop with the nominal controller – conditions that we will show our baseline QP-based CLF-CBF controller satisfies.

We design the baseline CLF-CBF controller in this chapter using the γm-QP approach from [35] which removes the assumption that the CBF has uniform relative degree one. Unlike

[35], all results in this chapter are stated for “zeroing” Control Barrier Functions (zCBF) even though a corresponding “reciprocal” Control Barrier Function can easily be recovered from the zCBF. For the rest of this chapter, we shall refer to zCBFs simply as CBFs.

The remainder of this chapter is organized as follows: Section 2 provides an overview of the predictor-feedback framework with which a stabilizing CLF-based control law is developed. Section 3 discusses the development of a CBF-based controller to keep the admissible set invariant in the presence of delay and a known disturbance. We combine the independently designed controllers in section 4 and show that the closed-loop behavior of the system is similar to the delay-free case after the longest delay has been compensated. Section 5 includes an example to illustrate the result of this chapter.

Notation: In addition to notation from Chapter 2, a continuous function $\beta(s,t) : (\mathbb{R}^+ \times \mathbb{R}^+) \to \mathbb{R}^+$ is said to belong to class $\mathscr{K}\mathscr{L}$ if it belongs to class $\mathscr{K}$ as a function of the first argument and converges to zero as the second argument goes to infinity i.e. for all s, $\lim_{t\to\infty}\beta(s,t) = 0$.

3.2 Control Lyapunov functions for linear systems with distinct input delays

We consider the following system

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^{r} b_i u_i(t-\tau_i) \tag{3.1}$$

where $x \in \mathbb{R}^n$ is the state, $t \geq 0$ is time, $u_1, \ldots, u_r \in \mathbb{R}$ are control inputs, $\tau_i, \ldots, \tau_r$ are potentially distinct input delays satisfying (without loss of generality) $0 \leq \tau_i \leq \cdots \leq \tau_r$, and $b_i \in \mathbb{R}^n$ for all $i = 1, \ldots, r$. We adopt the approach in [11] for stabilizing general nonlinear systems with distinct input delays. This approach involves designing a stabilizing controller $\kappa(x)$ for the zero-delay system which in this case is:

$$\dot{x} = Ax + Bu \tag{3.2}$$

where $B \in \mathbb{R}^{n\times r}$ is the matrix whose i-th column is b_i and $u \in \mathbb{R}^r$ is the column vector of inputs; and then using predictions of the state as feedback to the nominal controller to stabilize the original delayed system. We present the predictor design approach here as it applies to the linear delay system in question.

The predictor feedback approach involves designing state predictors $p_1(t),\dots,p_r(t)$ corresponding to each of the r input channels with $p_i(t)$ being the τ_i-time units ahead prediction of $x(t)$ i.e. $p_i(t) = x(t+\tau_i)$, for all $i = 1,\dots,r$. Defining $p_0(t) = x(t)$, $\tau_0 = 0$, and $\tau_{ij} = \tau_i - \tau_j$, the predictor state equations are given as:

$$p_i(t) = p_{i-1}(t) + \int_{t-\tau_{i,i-1}}^{t} \left[Ap_i(s) + \sum_{j=1}^{i-1} b_j \kappa_j(p_i(s)) + \sum_{j=i}^{r} b_j u_j(s-\tau_{j,i}) \right] ds \tag{3.3}$$

with initial conditions

$$p_i(\theta) = p_{i-1}(0) + \int_{-\tau_{i,i-1}}^{\theta} \left[Ap_i(s) + \sum_{j=1}^{i-1} b_j \kappa_j(p_i(s)) + \sum_{j=i}^{r} b_j u_j(s-\tau_{j,i}) \right] ds, \tag{3.4}$$

$$\text{for } -\tau_{i,i-1} \le \theta \le 0$$

The predictor states are then used in the nominal control law as follows:

$$u_i(t) = \kappa_i(p_i(t)) \tag{3.5}$$

where κ_i is the i-th component of κ.

Note that, in the absence of model uncertainties and disturbances, at a given time $\bar{t}$, the input *reaching* the system from each individual input channel is given as $u_i(\bar{t}-\tau_i) = \kappa_i(p_i(\bar{t}-\tau_i)) = \kappa_i(x(\bar{t}))$. Thus, the complete input vector impacting the system at time $\bar{t}$ comes from the feedback law $\kappa(\cdot)$ called on the same state vector $x(\bar{t})$, even though the feedback of $x(\bar{t})$ occurs at different times $(\bar{t}-\tau_i)$ for each input channel. A consequence of this is that, after the last (longest) delay τ_r has been compensated, i.e. for $t > \tau_r$, the behavior of the closed loop system with delay would be exactly the same as the delay-free closed loop system. Secondly, we note while analytical solutions of the predictor state equations in (3.3) and (3.4) cannot always be written out explicitly – especially when the nominal feedback law is nonlinear – the predictor states can be solved for numerically and there are studies focusing on implementation and approximation issues for nonlinear predictor feedback laws ([36, 37]).

It is shown in Theorems 1 and 2 of [11] that the predictor feedback law (3.5) is Lipschitz continuous if κ_i is Lipschitz continuous, and that it stabilizes the delay system (3.1) if three assumptions on the delay-free system and the nominal feedback law κ hold. The first assumption is that the delay-free system is strongly forward complete i.e. the solution is well defined and does not exhibit finite escape – an assumption that holds automatically for linear systems. Secondly, it is assumed that the system $\dot{x} = Ax + B(\kappa(x) + \omega)$ is input-to-state stable (ISS) with respect to ω. Lastly, because of the distinct input delays in the original system (3.1), the effect of the controller reaches the system along input channels with the shortest delays first, therefore it is assumed that after the i-th controller "kicks in" and the the i-th input delay has been compensated, the resulting system $\dot{x} = Ax + \sum_{j=1}^{i} b_j \kappa_j(x) + \sum_{j=i+1}^{r} b_j \omega_j$ is strongly forward complete with respect to $\omega = (\omega_{i+1}, \ldots, \omega_r)^T$. While linear systems are by nature forward complete, this third assumption is still important because when κ is a nonlinear control law, the first i controllers could cause the evolution of the linear system to become nonlinear and as such automatic forward completeness can no longer be assumed.

For the second and third assumptions, we present the PMN controller that we show to be input-to-state stabilizing for the zero-delay system, and to keep the system strongly forward complete when only a portion of the delays has been compensated. We begin with a definition of CLFs (for linear systems) from which a PMN controller is developed.

Definition 3.1 (CLF - Linear Systems)**.** *A positive definite function $V(x) = x^T Px$ is a CLF for the system $\dot{x} = Ax + Bu$ if there exists positive definite matrix $Q > 0$ such that for all $x \neq 0$,*

$$x^T PB = 0 \implies x^T(A^T P + PA + Q)x < 0 \tag{3.6}$$

With a CLF $V = x^T Px$ for the system $\dot{x} = Ax + Bu$ known, a stabilizing controller is given by the PMN formula as:

$$\kappa_{pmn}(a_1(x), b_1(x)) = \begin{cases} -\frac{a_1(x)}{\|b_1(x)\|^2} b_1^T(x) & \text{if } a_1(x) > 0 \\ 0 & \text{if } a_1(x) \leq 0 \end{cases} \tag{3.7}$$

where

$$a_1(x) = x^T (A^T P + PA + Q)x, \tag{3.8}$$

$$b_1(x) = 2x^T PB. \tag{3.9}$$

This PMN control law achieves $\dot{V} \leq -x^T Qx$ for the zero-delay system and hence guarantees global asymptotic stability. In addition, as the name implies, the PMN formula gives the minimum control effort necessary to keep $\dot{V} \leq -x^T Qx$, and thus κ_{pmn} is the unique solution to the following quadratic programming problem:

QP-L: Find control u that satisfies

$$\begin{gathered} u = \arg\min_{v \in \mathbb{R}^r} \ v^T v \quad \text{subject to} \\ a_1(x) + b_1(x)v \leq 0 \end{gathered} \tag{3.10}$$

where $a_1(x)$ and $b_1(x)$ are as defined in (3.8), (3.9).

For the system with delay, the stabilizing control is computed component-wise as given in (3.5). For each input channel i, the corresponding predictor state $p_i(t)$ is used in the κ_{pmn} feedback law, and the i-th component of the resulting control is used as the i-th input to the system as follows,

$$u_{L_i}(t) = \kappa_{pmn_i}(a_1(p_i(t)), b_1(p_i(t))), \qquad i = 1, \ldots, r \tag{3.11}$$

While the predictor feedback law (3.11) is defined component-wise using different predicted state values for each input channel, we re-iterate here that the full input vector $u_L(\bar{t})$ arriving at the system at some time $\bar{t} > \tau_r$, would be equivalent to $\kappa_{pmn}(a_1(x(\bar{t})), b_1(x(\bar{t})))$ since $x(\bar{t}) = p_i(\bar{t} - \tau_i)$ for all i.

Proposition 3.1. *If V is a CLF for the system* (3.2)*, then the control law* (3.11) *is Lipschitz continuous and the closed loop system* (3.1)*,* (3.11) *is globally asymptotically stable in the sense that, for a class $\mathscr{K}\mathscr{L}$ function β, the closed loop trajectories satisfy*

$$\Gamma(t) \leq \beta(\Gamma(0), t), \quad \forall t \geq 0 \tag{3.12}$$

where

$$\Gamma(t) = \|x(t)\| + \sum_{i=1}^{r} \sup_{t-\tau_i \leq \theta \leq t} \|u_i(\theta)\| \tag{3.13}$$

Proof. The proofs for stabilization and Lipschitz continuity of (3.11) follow from Theorems 1 and 2 of [11] respectively, provided the aforementioned assumptions are satisfied. The first assumption of strong forward completeness of the zero-delay system is satisfied automatically for linear systems. For the second assumption, we establish input-to-state stability with the help of the Lyapunov function V. In this case,

$$\dot{V} = -x^T Q x + 2x^T P B \omega \leq -\frac{\lambda_{\min}(Q)}{2}\|x\|^2 + \frac{2\|B^T P\|}{\lambda_{\min}(Q)}|\omega|^2 \tag{3.14}$$

and the ISS property follows. For the third assumption, while the PMN feedback law $\kappa_{pmn_j}(x)$ is nonlinear in x, it is continuous and smaller in magnitude than a linear feedback that achieves

$\dot{V} = -x^T Qx$; therefore, the closed-loop system is globally Lipschitz and thus forward complete during times $\tau_1 \leq t \leq \tau_r$ when only a subset of the delays has been compensated. With all three assumptions satisfied, the proof for global asymptotic stability follows from Theorem 1 of [11] which provides the Lyapunov functional (3.12). For Lipschitz continuity of $\kappa_{pmn_i}(p_i(t))$, we need to show that $\kappa_{pmn_i}(x(t))$ is Lipschitz continuous to be able to use the result of Theorem 2 of [11]. The Lipschitz continuity of $\kappa_{pmn_i}(x(t))$ follows directly from Proposition 1 of [34] and therefore concludes this proof. □

3.3 Control barrier functions for linear systems with distinct input delays

Given the system (3.1) and an admissible set $\mathscr{C} = \{x \in \mathbb{R}^n : h(x) > 0\}$ where $h : \mathbb{R}^n \to \mathbb{R}$ is a continuously differentiable function, the second objective in addition to stabilization is to keep the admissible set $\mathscr{C}$ forward invariant i.e. we want to ensure that if $h(0) > 0$ then $h(t) > 0$ for all $t > 0$. To achieve this, we start by developing a nominal control law that keeps the delay-free system forward invariant, and then we use predicted states in this nominal feedback law to arrive at a predictor-feedback controller that keeps the original delay system (3.1) forward invariant.

Definition 3.2 (Control Barrier Function (CBF) - Linear System)**.** *A continuously differentiable function $h(x)$ is a CBF for the system $\dot{x} = Ax + Bu$ with respect to the admissible set $\mathscr{C} = \{x \in \mathbb{R}^n : h(x) > 0\}$ if there exists a class $\mathscr{K}$ function α_h such that, for $x \in \mathscr{C}$,*

$$L_B h(x) = 0 \implies L_{Ax} h(x) + \alpha_h(h(x)) > 0 \tag{3.15}$$

For the delay-free system, with a CBF $h(x)$, the goal is to find the minimal control effort that achieves $\dot{h}(x) > -\alpha_h(h(x))$. This is equivalent to solving the following quadratic programming problem:

QP-B: Find control u that satisfies

$$u = \underset{v \in \mathbb{R}^r}{\arg\min} \; v^T v \quad \text{subject to} \\ a_2(x) + b_2(x)v \leq 0 \tag{3.16}$$

where

$$a_2(x) = -L_{Ax}h(x) - \alpha_h(h(x)), \tag{3.17}$$

$$b_2(x) = -L_B h(x). \tag{3.18}$$

As with the QP-L, the optimal control law solving the QP-B problem for the delay-free system is given by the pointwise minimum norm controller $\kappa_{pmn}(a_2(x), b_2(x))$. For the original system with delays, the feedback law is defined component-wise for each input channel i by using the predictor state $p_i(t)$ in place of $x(t)$ and taking the i-th component of the resulting control as the input to the system across the i-th input channel as follows:

$$u_{B_i}(t) = \kappa_{pmn_i}(a_2(p_i(t)), b_2(p_i(t))), \qquad i = 1, \ldots, r \tag{3.19}$$

We note that with (3.19), there is no control of the system in the first τ_1 time units when none of the inputs at $t = 0$ has reached the system, therefore, we assume that the initial values of the inputs $u_i(t)$ on the interval $t \in [-\tau_i, 0]$ are such that the system remains in the safe set $\mathscr{C}$ during the first τ_1 time units. In addition, during times $t \in [\tau_i, \tau_{i+1})$ for $i = 1, \ldots, r-1$, only the first i controllers have reached the system, therefore we do not have full control of the system yet, and thus we need to impose a new condition — akin to the third assumption for predictor-feedback stabilization — that the system $\dot{x}(t) = Ax(t) + \sum_{j=1}^{i} b_j \kappa_{pmn_j}(x) + \sum_{j=i+1}^{r} b_j u_j$ remains in the safe set $\mathscr{C}$ for all times $t \in [\tau_i, \tau_{i+1}]$, or that if the system does exit the safe set in this time period, it would return to the safe set eventually. The latter assumption holds using a similar argument as in [103] where the following Lyapunov function is defined:

$$V_{\mathscr{C}}(x) = \begin{cases} 0 & \text{if } x \in \mathscr{C} \\ -h(x) & \text{if } x \in \mathscr{C} \setminus \mathscr{D} \end{cases} \tag{3.20}$$

(for some open set $\mathscr{D}$ containing $\mathscr{C}$ upon which $h(x)$ is defined) and is used to show that the set $\mathscr{C}$ is asymptotically stable i.e. trajectories starting outside the admissible set $\mathscr{C}$ asymptotically converge to the boundary $\partial\mathscr{C}$ where $h = 0$.

Proposition 3.2. *If h is a CBF for the system* (3.2)*, then the control law* (3.19) *is Lipschitz continuous in $\mathscr{C}$. If $x(0)$ and the initial conditions $u_i(t)$ for $t \in [-\tau_i, 0]$ are such that $h(t) > 0$ for $0 \leq t \leq \tau_1$, and if the systems*

$$\dot{x}(t) = Ax(t) + \sum_{j=1}^{i} b_j \kappa_{pmn_j}(x) + \sum_{j=i+1}^{r} b_j u_j, \quad i = 1, \dots, r-1 \tag{3.21}$$

are such that the system remains in the safe set at $t = \tau_r$, then the control law (3.19) *renders the set $\mathscr{C}$ forward invariant.*

Proof. The Lipschitz continuity part of the proof follows from Theorem 1 in [35]. For the forward invariance part, notice that with the control law (3.19), for all $t \geq \tau_r$, the control input *affecting* the system at the $i-$th input channel at time t, denoted $u_i^a(t)$ is given as

$$u_i^a(t) = u_{B_i}(t - \tau_i) = \kappa_{pmn_i}(a_2(x(t)), b_2(x(t))). \tag{3.22}$$

Therefore, for all $t \geq \tau_r$, the complete input vector affecting the system is $\kappa_{pmn_i}(a_2(x(t)), b_2(x(t)))$ and we have that

$$\dot{h}(x(t)) = L_{Ax}h(x(t)) + L_B h(x(t)) \kappa_{pmn}(x(t)). \tag{3.23}$$

Since $\kappa_{pmn}(x(t))$ solves (3.16), we have that $\dot{h}(x(t)) \geq -\alpha_h(h(x(t)))$ for all $t \geq \tau_r$. Thus, following from Theorem 1 in [6], the set $\mathscr{C}$ is forward invariant for $t \geq \tau_r$. With $h(x(t)) > 0$ assumed for $0 \leq t \leq \tau_r$, it follows that $h(x(t)) > 0$ for all $t > 0$. □

Proposition 3.2 assumes perfect state prediction i.e. $p_i(t) = x(t+\tau_i)$ for all $i = 1, \ldots, r$, which is the case when there are no disturbances or system modeling imperfections, and when the predictor integrals are computed exactly. These assumptions make the guarantees of Proposition 3.2 less practical. To accommodate for this, we make the forward invariance of the admissible set $\mathscr{C}$ robust by following the approach in [34]. First, it is assumed that an external disturbance $\omega(t) : \mathbb{R}^+ \to \mathbb{R}^n$ with $\|\omega(t)\| \leq \overline{\omega}$ enters the system as follows:

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^{r} b_i u(t-\tau_i) + B_\omega \omega(t). \tag{3.24}$$

and then a constant buffer η is added to the barrier and the new objective becomes keeping a subset of $\mathscr{C}$ denoted $\mathscr{C}_\eta = \{x : h(x) > \eta\}$ forward invariant. The overarching idea is that while $\omega(t)$ might cause the system to exit the subset $\mathscr{C}_\eta = \{x : h(x) > \eta\}$, the constant η can be chosen so that in the worst case, the system remains within the original admissible set $\mathscr{C} = \{x : h(x) > 0\}$. To compute the conditions on η, it is assumed that $H(x) = h(x) - \eta$ is a CBF for the disturbance free system, α_h is linear, $\partial\mathscr{C}$ is compact, and that one or more of the following are true: τ_r is small, $\overline{\omega}$ is small, $h(x)$ is approximately linear. With these set of assumptions, any constant η satisfying

$$\eta \geq \left\|\frac{\partial h}{\partial x}\right\| \left(\nu \left\|(I + \frac{1}{l_0}A)B_\omega\right\| + \frac{1}{l_0}\|B_\omega\| \right) \overline{\omega} \tag{3.25}$$

on the boundary $\partial\mathscr{C}$, where

$$\nu = \int_0^{\tau_r} e^{A\theta} d\theta, \tag{3.26}$$

will guarantee the forward invariance of $\mathscr{C}$ in the presence of $\omega(t)$.

3.4 QP design for constrained stabilization

The ultimate goal is to design a controller that simultaneously stabilizes (3.1) and keeps $\mathscr{C}$ forward invariant. To do this, we begin by combining and solving the QP-L and QP-B problems as done in [35] to arrive at a nominal feedback law $\kappa_{LB}(x(t))$ that achieves both stabilization and constraint adherence. We then use this feedback law in combination the state predictors to arrive at a control law for the original system with distinct delays.

For the delay-free system, the QP-L and QP-B problems are combined as follows:

QP-LB: Find control u and relaxation variable δ that satisfy

$$\begin{aligned}(u,\delta) = \underset{(v,\sigma)\,\in\,\mathbb{R}^r\times\mathbb{R}^r}{\arg\min}\ (v^T v + m\sigma^T\sigma) \quad \text{subject to} \\ F_1 := \bar{a}_1(x) + b_1(x)(v+\sigma) \leq 0 \\ F_2 := a_2(x) + b_2(x)v \leq 0\end{aligned} \tag{3.27}$$

where $a_i(x)$, $b_i(x)$, $i = 1,2$ are as defined in (3.8), (3.9), (3.17), (3.18), and $\bar{a}_1 = \gamma_f\big(a_1(x)\big)$ where $\gamma_f(\cdot)$ is the Lipschitz continuous function defined as

$$\gamma_f(s) = \begin{cases} \gamma s & \text{if } s \geq 0 \\ s & \text{if } s < 0, \end{cases} \tag{3.28}$$

with m and γ being design parameters.

Depending on which constraint is active, the control $u = \kappa_{LB}(x(t))$ solving (3.27) is given explicitly as:

Case A ($F_1 < 0$ or $x = 0, F_2 < 0, \lambda_1 = 0, \lambda_2 = 0$):

$$u = 0 \tag{3.29}$$

in $\Omega_1 = \{x \in \mathbb{R}^n : \bar{a}_1 < 0, a_2 < 0\}$.

Case B ($F_1 = 0, x \neq 0, F_2 < 0, \lambda_1 \geq 0, \lambda_2 = 0$):

$$u = -\frac{m}{m+1}\frac{\bar{a}_1}{\|b_1\|^2}b_1^T \tag{3.30}$$

in $\Omega_2 = \left\{x \in \mathbb{R}^n : \bar{a}_1 \geq 0, a_2 < \frac{m}{m+1}\frac{b_2 b_1^T}{\|b_1\|^2}\bar{a}_1\right\}$.

Case C ($F_1 < 0, F_2 = 0, \lambda_1 = 0, \lambda_2 \geq 0$):

$$u = -\frac{a_2}{\|b^2\|^2}b_2^T \tag{3.31}$$

in $\Omega_3 = \left\{x \in \mathbb{R}^n : a_2 \geq 0, \bar{a}_1 < \frac{b_1 b_2^T}{\|b_2\|^2}a_2\right\}$.

Case D ($F_1 = 0, F_2 = 0, \lambda_1 \geq 0, \lambda_2 \geq 0$):

$$u = \frac{-\|b_2\|^2 b_1^T \bar{a}_1 + b_1 b_2^T (b_1^T a_2 + b_2^T \bar{a}_1) - \left(1 + \frac{1}{m}\right)\|b_1\|^2 b_2^T a_2}{\left(1 + \frac{1}{m}\right)\|b_1\|^2\|b_2\|^2 - \|b_1 b_2^T\|^2} \tag{3.32}$$

in $\Omega_4 = \left\{x \in \mathscr{A} : \bar{a}_1 \geq \frac{b_1 b_2^T}{\|b_2\|^2}a_2, a_2 \geq \frac{m}{1+m}\frac{b_1 b_2^T}{\|b_1\|^2}\bar{a}_1\right\}$ where $\mathscr{A} = \mathbb{R}^n - \{x \in \mathbb{R}^n : \bar{a}_1 < 0, a_2 < 0\}$.

Now, for the system (3.1) with delay, we use the predictor states $p_i(t)$ in combination with the nominal feedback law κ_{LB} in (3.29)-(3.32) as:

$$u_i(t) = \kappa_{LB_i}(p_i(t)), \quad \forall i = 1, \ldots, r \tag{3.33}$$

and present the following main result.

Theorem 3.3. *If $V(x)$ is a CLF and $h(x)$ is a CBF for the delay-free system* (3.2)*, and if $p_i(t) = x(t + \tau_i)$ for all $t \geq 0$ and all $i = 1, \ldots, r$, then*

1. *The QP-LB problem* (3.27) *is feasible and the control law* (3.33) *is Lipschitz continuous in $\mathscr{C}$.*

2. *$\dot{h}(x(t)) \geq -\alpha_h(x(t))$ for $t \geq \tau_r$ and for all $x \in \mathscr{C}$. If $h(x(\theta)) > 0$ for all $0 \leq \theta \leq \tau_r$, the set $\mathscr{C}$ is forward invariant for all $t \geq 0$.*

3. *If the barrier constraint is inactive $(F_2 < 0)$ and if we select $\frac{\gamma m}{m+1} = 1$ ($\gamma, m \geq 1$ by assumption) the control κ_{LB} becomes identical to the PMN formula* (3.7) *and achieves $\dot{V}(x(t)) \leq -x^T(t)Qx(t)$ for $t \geq \tau_r$.*

4. *If $0 \in \mathscr{C}$ then the barrier constraint F_2 is inactive around the origin and the closed loop system is locally asymptotically stable.*

5. *If $a_1 + b_1\kappa_{LB}(p_i(t)) \leq 0$ for all $p_i(t) \in \mathscr{C}$ and $i = 1, \ldots, r$, the closed loop system is globally asymptotically stable with respect to $\mathscr{C}$ i.e every trajectory starting in $\mathscr{C}$ converges to* 0*.*

Proof. The proof of Theorem 1 follows from Theorem 1 of [34] for linear systems with delays of equal length. □

3.5 Numerical simulation

We present the following example to illustrate the result of this chapter:

$$\begin{aligned} \dot{x}_1(t) &= -x_1(t) + \frac{1}{2}x_2(t) + u_1(t - \tau_1) + \omega_1(t) \\ \dot{x}_2(t) &= -\frac{1}{2}x_1(t) - x_2(t) + u_2(t - \tau_2) + \omega_2(t) \end{aligned} \tag{3.34}$$

with $0 \leq \tau_1 \leq \tau_2$ and $\|\omega(t)\| = \|(\omega_1(t), \omega_2(t))^T\|$ is such that $\|\omega(t)\| \leq \overline{\omega}$ for a known constant $\overline{\omega}$. We note that unlike the example in [34], this system includes a drift that can force trajectories outside the admissible set, and the delays in both input channels are distinct. The control objective is to steer the system to the origin while avoiding a disk of radius $\rho > 0$ centered at $(\sigma, 0)$ where $\sigma > \rho$. Therefore, the admissible set is

$$\mathscr{C} = \{x \in \mathbb{R}^2 \mid h(x) \geq 0\} \tag{3.35}$$

where

$$h(x) = (x_1 - \sigma)^2 + x_2^2 - \rho^2 \tag{3.36}$$

We begin by verifying that $h(x)$ is an CBF for (3.34). Defining the center of the disc as

$$c = \begin{bmatrix} \sigma \\ 0 \end{bmatrix}, \tag{3.37}$$

we have

$$L_{Ax}h = 2(x-c)^T Ax, \tag{3.38}$$

$$L_B h = 2(x-c)^T, \tag{3.39}$$

and we choose

$$\alpha_h(h) = l_0 h, \qquad l_0 > 0. \tag{3.40}$$

Since $L_B h = 0 \implies x = c$ and $c \notin \mathscr{C}$, we have that h is indeed a CBF. Following the same approach, we can show that $H(x) = h(x) - \eta$ is also a CBF. Therefore, we have:

$$a_2(x) = -2(x-c)^T Ax - l_0(h(x) - \eta) \tag{3.41}$$

$$b_2(x) = -2(x-c)^T \tag{3.42}$$

Table 3.1. Table of Parameter Values for Example 3.5

Parameter	Value
τ_1	0.2
τ_2	0.8
ρ	5
σ	6
l_0	2
$\overline{\omega}$	1
p	0.4
q	$0.9p^2$
η	8.06
m	8
γ	2.25
Δt	0.005

Next, we choose the candidate CLF $V = x^T Px$ where $P > 0$ and can be chosen as $diag\{p\}$ for some $p > 0$. Any matrix Q satisfying $0 < Q < P^2$ provides $\alpha(x) = x^T Qx$. An easy choice for Q is $diag\{q\}$ for some $0 < q < p^2$. Therefore we have that

$$L_{Ax}V = x^T(A^T P + PA)x, \tag{3.43}$$

$$L_B V = 2x^T PB = 2px^T, \tag{3.44}$$

and

$$\overline{a}_1(x) = \gamma_f\left(x^T(A^T P + PA + Q)x\right) \tag{3.45}$$

$$b_1(x) = 2px^T \tag{3.46}$$

The values of (3.41)-(3.46) computed with the predictor states $p_i(t)$ are used in the QP control formulas (3.29)-(3.32). The system and constraint parameters used for simulation are listed in Table 3.1 below

The input signal was initialized as

$$u_1(\theta) = 0, \qquad \forall\theta \in [-\tau_1, 0], \tag{3.47}$$

$$u_2(\theta) = 0, \qquad \forall\theta \in [-\tau_2, 0]. \tag{3.48}$$

The predictor integrals are computed using trapezoid rule with time step Δt that is significantly smaller than the shortest input delay τ_1. For the CBF, the choices of l_0 and $\overline{\omega}$ in Table 3.1, and the inequality (3.25) lead to $\eta \geq 8.06$. For the CLF, we choose p and q as done in [34]. The QP design parameters m and γ for adjusting the aggressiveness and twitchiness of the controller are chosen as prescribed in [35].

With initial condition $x(0) = [16,\ 6]^T$, we attempt to drive the system to the origin in two different scenarios: with and without an external disturbance. In the first scenario with no disturbance – shown at the top of Figure 3.1 – we see that the system trajectory follows the drift of the system initially before the controller kicks in to steer the system tightly around the buffer of the admissible set and to the origin. Here, $\mathscr{C}_\eta$ and $\mathscr{C}$ are kept invariant. In the second scenario, we have known disturbances

$$\omega_1(t) = \frac{\sqrt{2}}{2}\sin(0.4\pi t) \tag{3.49}$$

$$\omega_2(t) = \frac{\sqrt{2}}{2}\operatorname{sgn}(\omega_1(t)) \tag{3.50}$$

entering the system. As seen in the bottom plot, the system is successfully steered to the origin without leaving the admissible set $\mathscr{C}$ even though the buffer is violated.

3.6 Conclusion and acknowledgements

We extend the QP-based CLF-CBF approach for constrained stabilization to linear systems with distinct input delays. We show that in combination with predictor feedback, the nominal QP controller renders the origin locally asymptotically stable and adheres to state

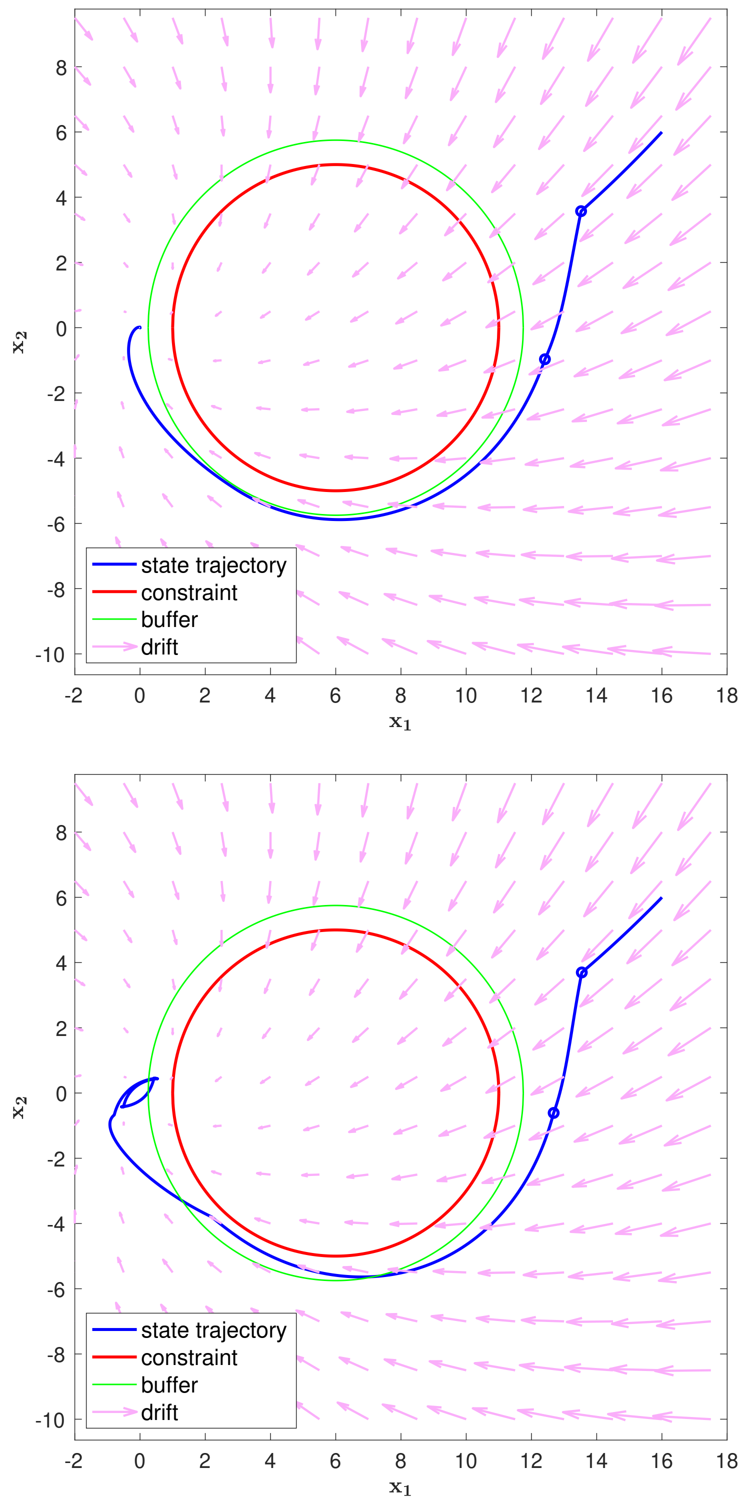

Figure 3.1. System response for Example 3.5. Top: Predictor feedback with no disturbance, Bottom: Predictor feedback with bounded disturbance. In both figures, the right and left circles on the state trajectory indicate the moments when the first and second inputs kick in respectively.

constraints even when a bounded external disturbance is present. A natural extension to the results in this chapter would be to consider the effect of predictor-feedback on constrained stabilization of nonlinear systems with distinct input delays.

Chapter 3, in part, is a reprint and adaptation of the paper: I. Abel, M. Janković, and M. Krstić, "Constrained Stabilization of Multi-Input Linear Systems with Distinct Input Delays", 3rd IFAC Joint Workshop on Control of Systems Governed by PDEs and Distributed Parameter Systems (CPDE/CPDS), 2019. The dissertation author was the primary investigator and author of this paper.

Chapter 4

Constrained Control of Multi-Input Nonlinear Systems with Distinct Input Delays

4.1 Introduction

4.1.1 Background and motivation

The task of enforcing the safe operation of nonlinear systems is often framed as a constrained control problem, where the output of a system is constrained to be within a desired set. One increasingly popular approach for this problem is the use of Control Barrier Functions (CBFs) [4, 6, 33, 71, 94, 98]. Like Lyapunov functions for stabilization, CBFs provide sufficient conditions on system input for enforcing forward invariance of a safe-set, and they have been used in various applications ranging from safe navigation [14, 44] to infectious disease control [5, 57].

As interest in CBFs has developed over time, several extensions have emerged that lend CBFs to applications of different forms. For example, [3, 25, 56] studied the use of CBFs for the safe operation of hybrid and discrete-time systems; [35, 43, 103] developed robust-CBF formulations for enforcing safety in the presence of disturbances, and [1,2,5,34,57,62,72,82] explored the usage of CBFs for systems with delays. In particular, [42,62,72,76,77] developed Lyapunov-Krasovskii-like functionals for verifying safety in systems with state delays, and in [5,57], the authors considered the problem of safety in the context of infectious disease control with a single measurement delay. Similarly in [82] the safety of sampled-data systems with a single input delay was considered and in [34], the author leverages results from [46,47] to develop controllers for safely stabilizing multi-input linear systems with same-length delays across input channels. We later extended this result in [1], to linear systems with distinct input delays using state-predictor designs from [11,95]. The result in [1] was the first to consider safety for multi-input systems with distinct input delays. However, just like all the aforementioned results for systems with input delays, [1] provides safety-guarantees only after the longest input delay has been compensated; that is, after the longest input delay time has elapsed. In essence, there is an assumption that the initial conditions of the inputs are such that the system remains safe until the designed control inputs have "kicked in" at *all* input channels. While this assumption is reasonable, it is limiting, especially in the case where the delay in some input channels is significantly longer than the delay in other input channels. In this case, the longest input delays would need to have been compensated before safety guarantees can be made. Ideally, it would be beneficial to be able to guarantee safety as soon as the delays in some shorter-delayed input channels have been compensated.

While there are several examples of real-world multi-input systems with distinct input delays, the underlying motivation for this work is collision avoidance in autonomous vehicles with different actuation delays in the steering and braking inputs. Inspired by human driving, where it is often possible – sometimes necessary – to avoid unforeseen collisions using only the more-responsive steering input, we desire a control approach that uses the shortest-delayed actuators to avoid collision whenever possible, especially in scenarios where there isn't enough time to compensate the delays in longer-delayed actuators. The continued emphasis on "whenever possible" is because it will sometimes be impossible to enforce barrier constraints without depending on all input channels, including longer-delayed ones. This will manifest as infeasibility of the CBF-based QP problems designed under the assumption that some – not all – inputs are available for safety enforcement; and we will address methods for accommodating this challenge.

4.1.2 Contributions

In this chapter, we study the safety problem for control-affine nonlinear systems with distinct input delays. Relative to the results of [34] that studies linear systems with same-length delays, we present results for nonlinear systems with distinct input delays. Specifically, we introduce a predictor-based approach for systems with distinct input delays that enforces safety before all input delays have been compensated – whenever it is possible to do so. We do this by treating inputs from longer-delayed input channels as known disturbances when determining inputs for shorter-delayed input channels. In particular, we consider the following two cases

1. First, we present results for the case where there exists a strict subset of shorter-delayed input channels sufficient for enforcing safety constraint everywhere in the safe-set. When such subset of input channels exists, they are preferentially used for enforcing safety everywhere in the safe set, therefore allowing safety guarantees to be made as soon as the longest input delay in that subset of input has been compensated.

2. Secondly, we present results for the general case where a strict subset of input channels sufficient for enforcing safety constraint everywhere in the safe set may not exist. In this case, we introduce a control approach that ensures that the subset of input channels with already-compensated delays exert their "best" effort to enforce safety constraints whether or not this effort is sufficient. When it isn't, the control approach allows the already delay-compensated inputs to minimize the violation of safety constraints until enough input delays have been compensated and system safety can be enforced.

4.1.3 Organization and notation

The remainder of this chapter is organized as follows. Section 4.2 contains the problem description and an overview of some preliminaries. In Section 4.3, we discuss our control design methodology and prove the results in this chapter with examples illustrating the effectiveness of the proposed designs. We end with a brief conclusion in Section 4.5.

Notation: In addition to notation from previous chapters we use the following notation. For the matrix-valued function $G : \mathbb{R}^n \to \mathbb{R}^{n\times r}$ with columns $g_i \in \mathbb{R}^n$, we define $\mathscr{L}_G h(x) = \left[\frac{\partial h}{\partial x} g_1, \ldots, \frac{\partial h}{\partial x} g_r\right]$. For vector $u \in \mathbb{R}^r$, and constants $1 \leq j \leq k \leq r$, we define $u_{j:k} = [u_j, u_{j+1}, \ldots, u_{k-1}, u_k]^\top$; and for matrix $G \in \mathbb{R}^{n\times r}$ with columns $g_i \in \mathbb{R}^n$ and constants $1 \leq j \leq k \leq r$, we define $G_{j:k} = [g_j, g_{j+1}, \ldots, g_{k-1}, g_k]$.

4.2 Problem description and preliminaries

4.2.1 Problem description

We consider systems of the following form:

$$\dot{x}(t) = f(x(t)) + \sum_{i=1}^{r} g_i(x(t)) u_i(t - D_i), \tag{4.1}$$

where $x \in \mathscr{X} \subset \mathbb{R}^n$ is the state, and $u_i \in \mathbb{R}$, $i = 1, \ldots, r$ are inputs with possibly distinct delays D_i satisfying (without loss of generality) $0 \leq D_1 \leq \cdots \leq D_r$, with initial conditions $u_i(\theta)$ for $\theta \in [-D_i, 0)$. The vector fields $g_1(x), \ldots, g_r(x), f(x) : \mathbb{R}^n \to \mathbb{R}^n$ are assumed to be locally Lipschitz continuous. We further assume that the delay-free system

$$\dot{x}(t) = f(x(t)) + G(x(t))u(t) \tag{4.2}$$

with $G(x) := [g_1(x), \ldots, g_r(x)]$ and $u := [u_1, \ldots, u_r]^\top$ is forward complete with respect to the input u; that is, for every initial condition $x(0) = x_0$ and locally bounded input signals $u_i(t)$, $i = 1, \ldots, r$, the corresponding solution of (4.2) is well defined for all $t \geq 0$. The importance of this assumption is that the plant in (4.1) does not exhibit finite escape before all feedback control inputs arrive.

The goal of this chapter is to design controls $u_i(t)$, $i = 1, \ldots, r$ for the system (4.1) that enforce the forward invariance of an admissible set

$$\mathscr{C} = \{x \in \mathscr{X} \mid h(x) \geq 0\} \tag{4.3}$$

characterized by a continuously differentiable scalar-valued function $h(x)$. That is, we aim to design controls $u_i(t),\ i = 1, \ldots, r$ that enforce $h(x(t)) \geq 0$ for all $t \geq 0$ provided that $h(x(0)) \geq 0$. We will refer to the set $\mathscr{C}$ as the "safe set", and we shall use the phrase "enforcing safety" to mean keeping the state $x(t)$ inside the set $\mathscr{C}$.

Remark 4.1. *We emphasize that the sole objective of our design will be to render the set $\mathscr{C}$ forward invariant, and as such we shall assume that any other objective e.g. stabilization is achieved by a baseline controller that shall be modified for safety. Yet, and irrespective of input delays, it would be sometimes impossible to enforce state constraints without the system state growing unbounded. For example, in linear systems with a non-minimum phase function $h(x)$ as output, there are initial states from which enforcing $h \geq 0$ will lead to the state growing unbounded (see [78] for details).*

Due to the presence of input delays, the system (4.1) evolves uncontrolled in the first D_1 time units and we therefore make the following assumptions.

A1. The initial state $x(0)$ and initial input histories $u_i(\theta),\ \theta \in [-D_i, 0)$ for $i = 1, \ldots, r$ are such that $x(t) \in \mathscr{C}$ for all $t \in [0, D_1]$.

This assumption ensures that the system does not exit the safe set before any feedback controller has arrived at the plant. Now, because the input signals $u_i(t)$ "kick in" at different times due to the distinct input delays, we desire a control approach that keeps the state inside of the set $\mathscr{C}$ using as few input channels as possible, whenever it is possible to do so. This is of particular interest in the case that the input delays are of significantly different lengths.

4.2.2 Control barrier functions

For systems with no input delays, the constrained control problem is often solved with the use of Control Barrier Functions (CBFs) defined as follows.

Definition 4.1 (Control Barrier Function (CBF)). *A continuously differentiable function $h(x)$ is a Control Barrier Function (CBF) for the delay-free system* (4.2) *with respect to the admissible set $\mathscr{C} = \{x \in \mathscr{X} \mid h(x) \geq 0\}$ if there exists a Lipschitz continuous, extended class $\mathscr{K}$ function α such that, for all $x \in \mathscr{X}$,*

$$\mathscr{L}_G h(x) = 0 \implies \mathscr{L}_f h(x) + \alpha(h(x)) > 0. \tag{4.4}$$

It was shown in Corollary 2 of [6] that if $h(x)$ is a CBF for (4.2), then the controller $u(x(t))$ where $u : \mathscr{X} \to \mathbb{R}^r$ is any locally Lipschitz feedback law satisfying

$$\mathscr{L}_f h(x) + \mathscr{L}_G h(x) u(x) + \alpha(h(x)) \geq 0 \tag{4.5}$$

for every $x \in \mathscr{X}$, renders the set $\mathscr{C}$ forward invariant. Moreover, it was shown in Proposition 4 of [103] that the set $\mathscr{C}$ is also rendered asymptotically stable in the case that the system starts outside of the admissible set $\mathscr{C}$. The inequality (4.5) is called the barrier constraint, and is equivalent to $\dot{h}(x) + \alpha(h(x)) \geq 0$ along the solution of (4.2) with $u(t) = u(x(t))$.

When safe set invariance is to be combined with other control objectives like stabilization or trajectory tracking, a common approach is to first design a baseline controller $u_0 : \mathscr{X} \to \mathbb{R}^r$ achieving the desired control objective, and then treat safe-set invariance as an add-on objective enforced by a supervisory controller. In essence, the baseline controller is modified by the 'smallest' (in the Euclidean norm sense) additional control $\bar{u}$ that allows the barrier constraint

(4.5) to be satisfied. More concretely, to render a baseline control u_0 safe, it is overridden with $u = u_0 + \overline{u}$ where $\overline{u}$ is the solution of the following quadratic programming (QP) problem:

$$\begin{aligned} \overline{u} = \underset{v \in \mathbb{R}^r}{\arg\min} \ \|v\|^2 \quad \text{subject to} \\ a(x) + b_1(x)(u_0 + v) \geq 0 \end{aligned} \tag{4.6}$$

where

$$a(x) = \mathscr{L}_f h(x) + \alpha(h(x)), \tag{4.7}$$

$$b_1(x) = \mathscr{L}_G h(x). \tag{4.8}$$

The feasibility of the QP problem (4.6) follows from the definition of a CBF. Using the Karush-Kuhn-Tucker (KKT) optimality conditions for the solution of (4.6), we get the following explicit solution for $\overline{u}$:

$$\overline{u}(x, u_0) = \begin{cases} 0 & \text{if } \mu(x, u_0) \geq 0 \\ -\frac{\mu(x, u_0)}{b_1(x) b_1(x)^\top} b_1(x)^\top & \text{otherwise} \end{cases} \tag{4.9}$$

where

$$\mu(x, u_0) = a(x) + b_1(x) u_0. \tag{4.10}$$

The controller

$$u(x) = u_0(x) + \overline{u}(x, u_0(x)) \tag{4.11}$$

for any baseline controller $u_0(x)$ and $\overline{u}$ given in (4.9) enforces the forward invariance of the set $\mathscr{C}$ in (4.3) for the delay-free system (4.2) as established in Corollary 2 of [6].

4.2.3 Robust control barrier functions (RCBF)

Here, we briefly describe the robust CBF formulation in [35] as it applies to the zeroing-type CBF utilized in this chapter. Consider the delay-free system (4.2), but with an external bounded disturbance $\omega(t)$

$$\dot{x}(t) = f(x) + G(x)u + Z(x)\omega, \tag{4.12}$$

where the disturbance $\omega(t) \in \mathbb{R}^\nu$ satisfies $0 \leq \|\omega(t)\| \leq \omega_{\max}$ for all t, and $Z : \mathbb{R}^n \to \mathbb{R}^{n\times\nu}$ is locally Lipschitz.

Definition 4.2 (Robust-CBF)**.** *A continuously differentiable function $h(x)$ is a robust-CBF (RCBF) for the system* (4.12) *with respect to the admissible set $\mathscr{C} = \{x \in \mathscr{X} \mid h(x) \geq 0\}$ if there exists a Lipschitz continuous, extended class $\mathscr{K}$ function α such that, for all $x \in \mathscr{X}$,*

$$\mathscr{L}_G h(x) = 0 \implies \mathscr{L}_f h(x) - \|\mathscr{L}_Z h(x)\|\omega_{max} + \alpha(h(x)) > 0. \tag{4.13}$$

The $-\|\mathscr{L}_Z h(x)\|\omega_{\max}$ term in (4.13) represents the worst case impact of the disturbance ω on $\dot{h} = \mathscr{L}_f h + \mathscr{L}_G hu + \mathscr{L}_Z h\omega$, therefore the condition in (4.13) necessitates that an RCBF satisfies $\dot{h}(x) + \alpha(h(x)) \geq 0$ in the presence of the worst-case disturbance even when the input has no effect on $\dot{h}(x)$.

As shown in [35], with an RCBF available, the task of enforcing forward invariance of $\mathscr{C}$ in the presence of disturbance signal $\omega(t)$ boils down to choosing for every $x \in \mathscr{X}$, control input u from the set of controls satisfying

$$\mathscr{L}_f h(x) + \mathscr{L}_G h(x)u - \|\mathscr{L}_Z h(x)\|\omega_{\max} + \alpha(h(x)) \geq 0. \tag{4.14}$$

Here, the input u will achieve $\dot{h} + \alpha(h) \geq 0$ in the presence of the worst case disturbance, thereby guaranteeing safe-set invariance. In the case that an accurate measurement or estimate of the disturbance $\hat{\omega}(t)$ is known *a priori*, the robust barrier constraint (4.14) can be relaxed to

$$\mathscr{L}_f h(x) + \mathscr{L}_G h(x)u + \mathscr{L}_Z h(x)\hat{\omega} + \alpha(h(x)) \geq 0 \tag{4.15}$$

in which case the input u counteracts the actual impact of ω on $\dot{h}$ when $\mathscr{L}_G h(x) \neq 0$ as opposed to accounting for the worst case impact at all times. For the rest of this chapter, we shall utilize the RCBF formulation that assumes an accurate estimate or measurement of the disturbance is available. As in the disturbance-free case, rendering a baseline control u_0 robustly safe translates to overriding u_0 with the closest control $u = u_0 + \overline{u}$ that allows the robust barrier constraint (4.15) to be satisfied, where $\overline{u}$ is the solution of the QP problem

$$\begin{aligned} \overline{u}(x, u_0, \hat{\omega}) = \underset{v \in \mathbb{R}^r}{\arg\min} \; \|v\|^2 \quad \text{subject to} \\ a(x) + b_1(x)(u_0 + v) + b_2(x)\hat{\omega} \geq 0, \end{aligned} \tag{4.16}$$

with $a(x)$, $b_1(x)$ as defined in (4.7), (4.8), and

$$b_2(x) = \mathscr{L}_Z h(x). \tag{4.17}$$

The feasibility of this QP problem follows from the definition of an RCBF. and has the explicit solution

$$\overline{u}(x, u_0, \hat{\omega}) = \begin{cases} 0 & \text{if } \mu(x, u_0, \hat{\omega}) \geq 0 \\ -\frac{\mu(x, u_0, \hat{\omega})}{b_1(x) b_1(x)^\top} b_1(x)^\top & \text{otherwise} \end{cases} \tag{4.18}$$

where

$$\mu(x, u_0, \hat{\omega}) = a(x) + b_1(x) u_0 + b_2(x)\hat{\omega}. \tag{4.19}$$

The controller

$$u(x, u_0, \hat{\omega}) = u_0(x) + \overline{u}(x, u_0, \hat{\omega}) \tag{4.20}$$

for any baseline controller $u_0(x)$ and with $\overline{u}$ given in (4.18) enforces the forward invariance of the set $\mathscr{C}$ in (4.3) for the delay-free system (4.2) with a bounded external disturbance as shown in Theorem 2 of [35]. We note that while the results of [35] were stated for the CLF-RCBF QP problem of simultaneous stabilization and safety, its adaptation in this chapter replaces the use of a Lyapunov function with the baseline control u_0 which the designer is allowed to design using any desired approach.

4.2.4 Predictor feedback for nonlinear systems with distinct input delays

Now we return to the original disturbance-free system (4.1) with distinct input delays. A predictor-feedback approach for stabilization was developed in [11] which combines state predictors with a globally asymptotically stabilizing feedback law $u = \kappa(x)$ designed for the delay-free system (4.2). Specifically, the approach involves designing individual state predictors $p_1(t), \ldots, p_r(t)$ corresponding to each of the r input channels with $p_i(t)$ being the D_i-time units ahead prediction of $x(t)$ i.e. $p_i(t) = x(t + D_i)$ for all $i = 1, \ldots, r$. These predictor states are then used as feedback to get the following control law

$$u_i(t) = \kappa_i(p_i(t)) \tag{4.21}$$

where κ_i is the i-th component of κ. Defining $p_0(t) = x(t)$, $D_0 = 0$, and

$$D_{j,i} = D_j - D_i, \tag{4.22}$$

the predictor state equations are given recursively as:

$$\begin{aligned} p_i(t) = p_{i-1}(t) + \int_{t-D_{i,i-1}}^{t} \Bigg[& f(p_i(s)) + \sum_{j=1}^{i-1} g_j(p_i(s))\kappa_j(p_i(s)) \\ & + \sum_{j=i}^{m} g_j(p_i(s))u(s - D_{j,i}) \Bigg] ds, \quad i = 1, \ldots, r \end{aligned} \tag{4.23}$$

with initial conditions

$$\begin{aligned} p_i(\theta) = p_{i-1}(0) + \int_{-D_{i,i-1}}^{\theta} \Bigg[& f(p_i(s)) + \sum_{j=1}^{i-1} g_j(p_i(s))\kappa_j(p_i(s)) \\ & + \sum_{j=i}^{m} g_j(p_i(s))u_j(s - D_{j,i}) \Bigg] ds, \quad \text{for } -D_{i,i-1} \le \theta \le 0 \end{aligned} \tag{4.24}$$

It was shown in [11] that the predictor (4.23) satisfies

$$p_i(t) = x(t + D_i) \quad \forall\ i = 1, \ldots, r. \tag{4.25}$$

The main idea of the predictor-feedback approach is that the controllers (4.21) satisfy

$$u_i(t - D_i) = \kappa_i(x(t)) \tag{4.26}$$

leading to the closed-loop dynamics of (4.1), (4.21) being

$$\dot{x}(t) = f(x) + \sum_{i=1}^{r} g_i(x)\kappa_i(x(t)) \tag{4.27}$$

which is delay-free. Therefore, a feedback controller designed for the delay free system (4.2) achieves the same closed loop behavior for the system (4.1) under predictor feedback. Importantly, since the feedback control signals $\kappa_i(p_i(t))$, $i = 1, \ldots, r$ kick in at different times, we need to make the additional assumption that the system does not exhibit finite escape during time $D_1 \leq t < D_r$ when only some, not all of the feedback controllers have kicked in. Specifically, we make the following additional assumption

A2. The systems

$$\dot{x} = f(x) + \sum_{j=1}^{i} g_j(x)\kappa_j(x) + \sum_{j=i+1}^{r} g_j(x)\vartheta_j, \quad \forall i = 1, \ldots, r-1 \tag{4.28}$$

are forward complete with respect to $[\vartheta_{i+1}, \ldots, \vartheta_r]^\top$.

Remark 4.2. *While the predictor equations* (4.23)*,* (4.24) *are implicitly defined and may not have explicit solutions, they can be implemented numerically and the implications of such numerical implementation are an area of research that is beyond the scope of this work; see [38] for details. We shall assume here that numerical predictor implementations are accurate.*

4.3 Control design and safety analysis

4.3.1 Safety after longest input delay compensation

Guided by the predictor-feedback approach in [11], a first-attempt at designing safe controllers for the system (4.1) with distinct input delays is to design a safe CBF-based controller for the delay-free system (4.2), and then use the predictor (4.23) for each $i = 1, \ldots, r$ to compensate the input delays. Specifically, using the CBF-based approach in 4.2.2, we have the following safe-guarding controller for the delay-free (DF) system (4.2)

$$u = \kappa^{\mathrm{DF}}(x) = u_0(x) + \overline{u}(x, u_0(x)) \tag{4.29}$$

where $u_0(x)$ is any safety-agnostic baseline controller for the delay-free system (4.2), and $\overline{u}$ is the CBF-based modification defined in (4.9). It follows from Corollary 2 of [6] that the controller (4.29) enforces the forward invariance of the safe set $\mathscr{C}$ for the delay-free system (4.2) and any baseline controller u_0. Now, with a safe controller for the delay-free system available, we can enforce safety for the original system (4.1) with input delays using the control law

$$u_i(t) = \kappa_i^{\mathrm{DF}}(p_i(t)), \qquad t \geq 0,\ i = 1, \ldots, r \tag{4.30}$$

where $p_i(t)$ is as defined in (4.23) with κ_j replaced with κ_j^{DF}.

Theorem 4.1. *Let $h(x)$ be a CBF for the delay free system* (4.2)*, and consider the closed loop system of* (4.1) *and controller* (4.30)*, where κ^{DF} is the feedback law in* (4.29) *and $p_i(t)$ are state predictors given in* (4.23), (4.24)*. If $p_i(t) = x(t + D_i)$ for all $t \geq 0$ and $i = 1, \ldots, r$, then for all $t \geq D_r$,*

$$\dot{h}(x(t)) + \alpha(h(x(t))) \geq 0. \tag{4.31}$$

Moreover, if $x(D_r) \in \mathscr{C}$, then the set $\mathscr{C}$ is forward invariant.

Proof. If $p_i(t) = x(t + D_i)$, then the controller (4.30) satisfies

$$u_i(t - D_i) = \kappa_i^{\mathrm{DF}}(x(t)), \quad t \geq D_i,\ i = 1, \ldots, r. \tag{4.32}$$

Thus, for all $t \geq D_r$, the closed loop system of (4.1), (4.30) becomes

$$\dot{x}(t) = f(x(t)) + G(x(t))\kappa^{\mathrm{DF}}(x(t)) \tag{4.33}$$

which has no input delays, and since κ^{DF} is the CBF QP based controller in (4.11), it follows by definition that for all $t \geq D_r$, $\dot{h}(x(t)) + \alpha(h(x(t))) \geq 0$, and therefore $h(x(D_r)) \geq 0$ implies $h(x(t)) \geq 0$ for all $t \geq D_r$. □

Remark 4.3. *The result of Theorem 4.1 was stated for times $t \geq D_r$ because only after time $t = D_r$ does the input vector $[u_1(t - D_1), \ldots, u_r(t - D_r)]^\top$ become equivalent to $\kappa^{DF}(x(t))$. Therefore, the enforcement of the barrier constraint $\dot{h} + \alpha(h) \geq 0$ for the system* (4.1) *under control law* (4.30) *can only be guaranteed for times $t \geq D_r$ when all the input delays have been compensated and the closed loop system exhibits nominal delay-free behavior. In fact, it is possible that the system exits the safe-set between time $t = D_1$ and $t = D_r$ even though some of the input delays have been compensated. This is because the controller* (4.30) *is not guaranteed to achieve $\dot{h} + \alpha(h) \geq 0$ until time $t = D_r$, even though the delays in some shorter-delayed input channels would have been compensated, and these input channels may be sufficient for enforcing the barrier constraint.*

To allow comparison with subsequent control approaches we shall present, the controller (4.30) can be written as

$$u_i(t) = u_{0_i}(p_i(t)) + \overline{u}_i(p_i(t), u_0(p_i(t))) \tag{4.34}$$

where $u_0 : \mathbb{R}^n \to \mathbb{R}^r$ is any baseline controller and $\overline{u}(p_i(t), u_0(p_i(t)))$ is the solution of the QP problem

$$\underset{v\in\mathbb{R}^r}{\arg\min}\ \|v\|^2 \quad \text{subject to} \tag{4.35}$$
$$\mathscr{L}_f h(p_i(t)) + \alpha(h(p_i(t))) + \mathscr{L}_G h(p_i(t))(u_0(p_i(t)) + v) \geq 0$$

which is given as

$$\overline{u}_i(p_i(t), u_0(p_i(t))) = \begin{cases} 0, & \mu_i \geq 0 \\ -\mu_i \frac{\mathscr{L}_{g_i} h(p_i)}{\mathscr{L}_G h(p_i)(\mathscr{L}_G h(p_i))^\top}, & \text{otherwise} \end{cases} \tag{4.36}$$

where

$$\mu_i = \mathscr{L}_f h(p_i(t)) + \alpha(h(p_i(t))) + \mathscr{L}_G h(p_i(t)) u_0(p_i(t)) \tag{4.37}$$

In the controller (4.34), (4.36), there is an implicit assumption that at every time $t \geq D_r$, the predicted states $p_i(t - D_i)$ for all input channels $i = 1, \ldots, r$ are equal and consistent with the actual state $x(t)$. The same assumption is made about the predicted values of the barrier function and its Lie derivatives. An implication of this assumption is that the controller (4.34) can only respond to perturbations in $h(x(t))$ whose effects do not render the system unsafe in under D_r units of time. One practical example of this limitation is the following: consider an autonomous robot with distinct longitudinal and lateral input delays D_{lat}, D_{lon} with $D_{\text{lat}} < D_{\text{lon}}$ navigating an obstacle-free path under controller (4.34), (4.36). If an obstacle is introduced sufficiently close to the robot so that a collision occurs in $D_{\text{lat}} < t < D_{\text{lon}}$ units of time, the predictions made by longer delayed input channels about the current status of the system will become outdated, and the controller may not enforce safety because it assumes that the predictions made for *all* control channels match the current reality. To formalize this notion of prediction consistency, we introduce the following definition

Definition 4.3. *For the system* (4.1) *and state predictors* (4.23)*, a function* $s(x)$ *is said to be*

i-consistent *at time* $t \geq D_i$ *if, for all* $j = 1, \ldots, i$

$$s(x(t)) = s(p_j(t - D_j)). \tag{4.38}$$

A function being i-consistent at time t means that the *actual* value of the function at time t is equivalent to the values predicted for the first i input channels. When a function is r-consistent at time t, we say it is *fully* consistent; if not, we say it is *partially* consistent.

Using this definition, the assumption made for the controller (4.34), (4.36) is that the functions $I(x(t)) = x(t)$, $h(x(t))$, $\mathscr{L}_f h(x(t))$ and $\mathscr{L}_G h(x(t))$ are fully consistent at all times $t \geq 0$. Notice however that by definition these functions cannot become r-consistent until time $t = D_r$, and when the r-consistency assumption does not hold at a particular time t, barrier constraint satisfaction is not guaranteed at that time t. We note here that a function $s(x(t))$ can also become $i < r$ consistent due to several factors including but not limited to prediction error, or an unforeseen perturbation of the function $s(x(t))$. Ideally, it would be preferred to have a controller that guarantees barrier constraint enforcement when predictions are i-consistent for some $i < r$. One approach for doing this is presented next.

4.3.2 Safety before longest input delay compensation using fixed number of input channels

To enable barrier constraint satisfaction when predictions are $i < r$ consistent, we utilize the robust RCBF-based QP formulation in 4.2.3 in the following way. When solving for the i-th input $u_i(t)$, we compute the state prediction $p_i(t)$ and compute the baseline control $u_0(p_i(t)) \in \mathbb{R}^r$ as before, but for safety, we assume only the first i input channels are available for enforcing the barrier constraint. The idea here is that when determining input $u_i(t)$ that would reach the plant at time $t + D_i$, all other inputs from channels $j = i+1, \ldots, r$ that would reach the plant at the same time $t + D_i$ have already been determined, and can no longer be modified. In addition to assuming only the first i inputs are available for barrier constraint enforcement, we will treat these already determined inputs $u_j(t-(D_j-D_i))$ for $j=i+1,\dots,r$ as known disturbances entering the system at time $t+D_i$ and include them in the barrier constraint inequality as is the case with the robust QP problem (4.16). In essence, when determining input $u_i(t)$, for each $i=1,\dots,r$ the modification of the baseline control $u_{0_i}(p_i(t))$ is chosen as the i-th component of the solution of the following "robust" QP problem

$$\begin{aligned} \bar{u} \; &= \; \underset{v\in\mathbb{R}^i}{\arg\min}\, \|v\|^2 \quad \text{subject to} \\ \mathscr{L}_f h(p_i(t)) + \alpha(h(p_i(t))) &+ \sum_{j=1}^{i} \mathscr{L}_{g_j} h(p_i(t))\big(u_{0_j}(p_i(t)) + v_j\big) \\ &+ \sum_{j=i+1}^{r} \mathscr{L}_{g_j} h(p_i(t)) \underbrace{u_j(t-D_{j,i})}_{\triangleq \omega_j} \geq 0 \end{aligned} \tag{4.39}$$

where $D_{j,i}=D_j-D_i$, and ω_j for $j>i$ are already-determined inputs from longer-delayed channels arriving at the same time $t+D_i$ as $u_i(t)$. While it may appear computationally expensive to solve r different QP problems at every time t, the availability of explicit solutions via (4.18) mitigates this challenge. Now, the idea behind QP problem (4.39) is to mimic the robust QP problem (4.16) and treat the already determined inputs as known disturbances. This way, the first i input channels alone can be used for enforcing safety at time $t+D_i$; especially in the case where delays in longer input channels have not been compensated.

Observe that for $i=r$, the QP problem (4.39) is equivalent to (4.35) and it is feasible by virtue of h being a CBF for the system when all r input channels are used for barrier constraint enforcement. However, for $i<r$, the feasibility of (4.39) is not guaranteed because $\mathscr{L}_{g_1}h,\dots,\mathscr{L}_{g_i}h$ could potentially all be 0 and h is not an RCBF for the system with only the first i inputs available for safety enforcement. In other words, when $\mathscr{L}_{G_{1:i}}h(p_i)=0$, (4.39) can be infeasible since the additional control v has no impact on the inequality constraint. This infeasibility is a reflection of the inability of the first i inputs alone to impact $\dot{h}(p_i(t))$. To mitigate this challenge, we introduce the set

$$\Phi = \left\{ i \in \{1, \dots, r-1\} \;\middle|\; \mathscr{L}_{G_{1:i}} h(x) \neq 0 \;\; \forall x \in \mathscr{X} \right\} \cup \left\{ r \right\} \tag{4.40}$$

and define

$$\varphi = \min \Phi. \tag{4.41}$$

Here, φ is the minimum number of input channels sufficient for enforcing the barrier constraint everywhere in the state space $\mathscr{X}$. Notice that for all $i \geq \varphi$, the QP problem (4.39) will be feasible because $\mathscr{L}_{G_{1:\varphi}} h \neq 0 \implies \mathscr{L}_{G_{1:i}} h \neq 0$. However for $i < \varphi$, (4.39) may be infeasible. To prevent infeasibility for $i < \varphi$, we will assume when solving for $u_i(t)$, $i < \varphi$ that the first φ input channels are available for barrier constraint enforcement, and use the i-th component of the resulting control modification. Thus, the QP (4.39) becomes

$$\begin{aligned} \overline{u} \;=\; \underset{v \in \mathbb{R}^s}{\arg\min}\, \|v\|^2 \quad \text{subject to} \\ \mathscr{L}_f h(p_i(t)) + \alpha(h(p_i(t))) + \sum_{j=1}^{s} \mathscr{L}_{g_j} h(p_i(t)) \big(u_{0_j}(p_i(t)) + v_j \big) \\ + \sum_{j=s+1}^{r} \mathscr{L}_{g_j} h(p_i(t)) \underbrace{u_j(t - D_{j,i})}_{\triangleq \omega_j} \geq 0 \end{aligned} \tag{4.42}$$

where

$$s = \max\{\varphi, i\}. \tag{4.43}$$

This leads to the controller

$$u_i(t) = u_{0_i}(p_i(t)) + \overline{u}_i\left(p_i(t), u_{0_{1:s}}(p_i(t)), \omega_{s+1:r}\right) \tag{4.44}$$

for all $t \geq 0$, where

$$\omega_{s+1:r} = \begin{bmatrix} u_{s+1}(t-(D_{s+1}-D_s)) \\ \vdots \\ u_r(t-(D_r-D_s)) \end{bmatrix} \in \mathbb{R}^{r-s}, \tag{4.45}$$

and

$$\overline{u}_i = \begin{cases} 0, & \mu_s \geq 0 \\ -\mu_s \frac{\mathscr{L}_{g_i} h(p_i)}{\mathscr{L}_{G_{1:s}} h(p_i)\left(\mathscr{L}_{G_{1:s}} h(p_i)\right)^\top}, & \text{otherwise} \end{cases} \tag{4.46}$$

with

$$\mu_s = \mathscr{L}_f h(p_i(t)) + \alpha(h(p_i(t))) + \sum_{j=1}^{s} \mathscr{L}_{g_j} h(p_i(t)) u_{0_j}(p_i(t)) + \sum_{j=s+1}^{r} \mathscr{L}_{g_j} h(p_i(t)) \omega_j \tag{4.47}$$

Notice the similarity of (4.44)-(4.47) to the RCBF controller in (4.20), (4.19), (4.18). If $\varphi < r$, then the predictions of the state and barrier function (and its Lie derivatives) only need be φ-consistent since the shorter-delayed inputs $u_1, \dots, u_\varphi$ are sufficient for barrier constraint enforcement.

Theorem 4.2. *Let $I : \mathbb{R}^n \to \mathbb{R}^n$ be the identity function $I(x) = x$, and let $h(x)$ be a CBF for the delay-free system* (4.2) *with φ as defined in* (4.40) *and* (4.41)*. Consider the closed-loop system of* (4.1) *and the controller* (4.44)-(4.46)*. If $I(x)$, $h(x)$, $\mathscr{L}_f h(x)$ and $\mathscr{L}_G h(x)$ are φ-consistent at time $t > D_\varphi$, then*

$$\dot{h}(x(t)) + \alpha(h(x(t))) \geq 0. \tag{4.48}$$

Proof. If $\varphi = r$, the result follow directly from Theorem 4.1. Now, consider the case where $\varphi < r$. By the φ-consistency of $I(x(t))$, $h(x)$, $\mathscr{L}_f h(x)$ and $\mathscr{L}_G h(x)$, we have that $x(t) = p_i(t - D_i)$, $h(x(t)) = h(p_i(t - D_i))$, $\mathscr{L}_f h(x(t)) = \mathscr{L}_f h(p_i(t - D_i))$ and $\mathscr{L}_G h(x(t)) = \mathscr{L}_G h(p_i(t - D_i))$ for all $i = 1, \dots, \varphi$, and therefore the controller (4.44) for $i = 1, \dots, \varphi$ satisfies

$$u_i(t-D_i) = u_{0_i}(x(t)) + \overline{u}_i\left(x(t), u_{0_{1:\varphi}}(x(t)), \omega_{\varphi+1:r}\right) \tag{4.49}$$

which can be re-written in vector form as

$$\begin{bmatrix} u_1(t-D_1) \\ \vdots \\ u_\varphi(t-D_\varphi) \end{bmatrix} = \underbrace{u_{0_{1:\varphi}}(x(t)) + \overline{u}_{1:\varphi}\left(x(t), u_{0_{1:\varphi}}(x(t)), \omega_{\varphi+1:r}\right)}_{u_{1:\varphi}}. \tag{4.50}$$

Thus, the system evolution at time t satisfies

$$\dot{x}(t) = f(x(t)) + G_{1:\varphi}(x(t))u_{1:\varphi} + G_{\varphi+1:r}(x(t))\omega_{\varphi+1:r}. \tag{4.51}$$

By the definition of φ, we have that the CBF $h(x)$ is an RCBF for the system (4.51) with respect to the "disturbance" $\omega_{\varphi+1:r}$, and since $\overline{u}_{1:\varphi}$ solves the robust QP problem (4.42), it follows that $\dot{h}(x(t)) + \alpha(h(x(t)) \geq 0$. □

Remark 4.4. *The result of Theorem 4.2, i.e. the satisfaction of the barrier constraint* $\dot{h}(x(t)) + \alpha(h(x(t))) \geq 0$ *holds for all* $t \geq D_\varphi$ *where predictions are* φ*-consistent. If the state starts inside of the admissible set* $\mathscr{C}$ *at some time* $t^* \geq D_\varphi$ *and* φ*-consistency holds for all* $t \geq t^*$*, then the forward invariance of the set* $\mathscr{C}$ *from time* t^* *follows.*

Example 1: Consider the following nonlinear system motivated by the example in [35]

$$\begin{aligned} \dot{x}_1(t) &= -x_1(t) - x_2(t) + u_1(t-D_1) \\ \dot{x}_2(t) &= x_1^3(t) + x_2(t)u_2(t-D_2) \end{aligned} \tag{4.52}$$

with $D_1 = 0.2$, $D_2 = 1$, $u_1(\theta) = 0$ for $\theta \in [-0.2, 0)$, and $u_2(\theta) = 0$ for $\theta \in [-1, 0)$. The control objective is to stabilize the system to the origin while keeping the set

$$\mathscr{C} = \{x : h(x) = x_2^2 - x_1 + 1 \geq 0\} \tag{4.53}$$

forward invariant.

First, we verify that when $D_1 = D_2 = 0$, the candidate barrier function $h(x)$ in (4.53) is indeed a CBF for (4.52). Specifically, we verify that for some class $\mathscr{K}$ function $\alpha(h)$, $\mathscr{L}_g h(x) = 0 \implies \mathscr{L}_f h(x) + \alpha(h(x)) \geq 0$. From (4.52) and (4.53) we have

$$\mathscr{L}_f h(x) = x_1 + x_2 + 2x_1^3 x_2, \tag{4.54}$$

$$\mathscr{L}_g h(x) = \begin{bmatrix} -1, & 2x_2^2 \end{bmatrix}, \tag{4.55}$$

and it follows that $\mathscr{L}_g h(x) \neq 0$ for all $x \in \mathscr{X}$ which implies that $h(x)$ is indeed a CBF for (4.52) with no delays. For control design, we choose $\alpha(h) = h$. Next, we determine the set Φ and its minimum φ as defined in (4.40) and (4.41). For the system (4.52) and control barrier function (4.53), we have

$$\Phi = \{1\} \cup \{2\}, \implies \varphi = 1. \tag{4.56}$$

Thus, we only need 1-consistent predictions to enforce safe set invariance and safety can be enforced earlier at time $t = D_1$ before both input delays have been fully compensated at time $t = D_2$.

For the baseline controller $u_0(x)$, we use a standard control Lyapunov function (CLF) based controller to stabilize the system to origin. Specifically, we use the Lyapunov function

$$V(x) = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2 \tag{4.57}$$

and baseline control

$$u_0(x) = \begin{cases} 0, & \mathscr{L}_g V = 0 \\ -\frac{\left(\mathscr{L}_f V + \|x\|^2\right)\mathscr{L}_g V^\top}{\mathscr{L}_g V (\mathscr{L}_g V^\top)}, & \mathscr{L}_g V \neq 0 \end{cases} \tag{4.58}$$

which achieves

$$\dot{V} \leq -\|x\|^2 \quad \forall x \in \mathbb{R}^2 \setminus \{0\} \tag{4.59}$$

when the barrier constraint is inactive.

We run numerical simulations for two separate cases: the first using control law (4.34), (4.36) where 2-consistency is assumed, and the second using the updated control law (4.44)-(4.46) where 1-consistency is assumed. To capture the significance of the new approach, we choose initial conditions close enough to the boundary of the admissible set so that the drift of the system causes a constraint violation in $D_1 \leq t \leq D_2$ units of time. This way, only the delay in the first input channel would have been compensated by the time the state trajectory exits the admissible set. This is reflected in Figure 4.1 where the system evolves uncontrolled for 0.2 time units due to the input delays, and arrives very close to the boundary of the admissible set.

At time $t = 0.2$, the delay in the first input channel has been compensated and the designed control u_1 begins to impact the system. As shown in Figure 4.2, under control law (4.34), (4.36), the system violates the barrier constraint because the control inputs were designed under the assumption that both inputs are always available for enforcing safety and have consistent

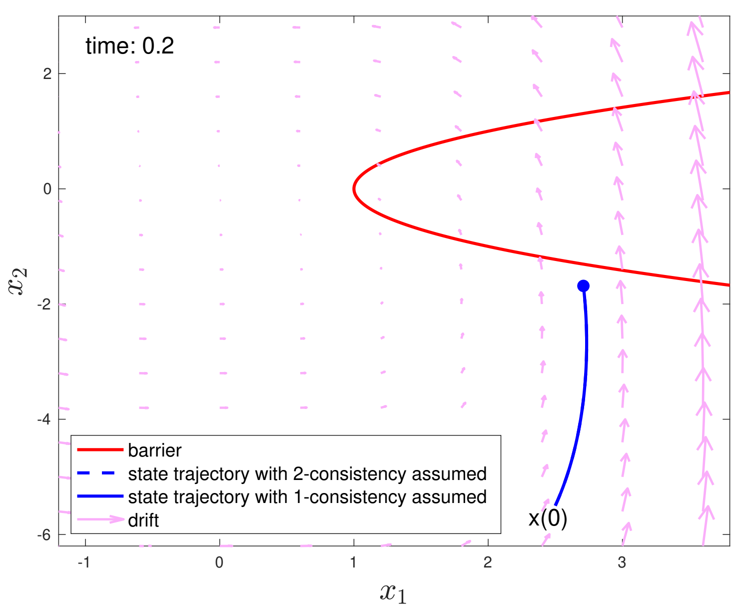

Figure 4.1. System response for $0 \leq t \leq 0.2$ when none of the designed control inputs have arrived at the system. The system evolution is driven by the drift and the initial conditions of the inputs, therefore the state trajectory in this time interval is as shown irrespective of the control law used.

predictions of state, which is not the case in the time interval $0.2 \le t < 1$. However, under the new control law (4.44), (4.46) where at all times $t \ge 0$, we incorporate the impact of the already-determined input signal $u_2(t-0.8)$ when determining $u_1(t)$, the system remains in the

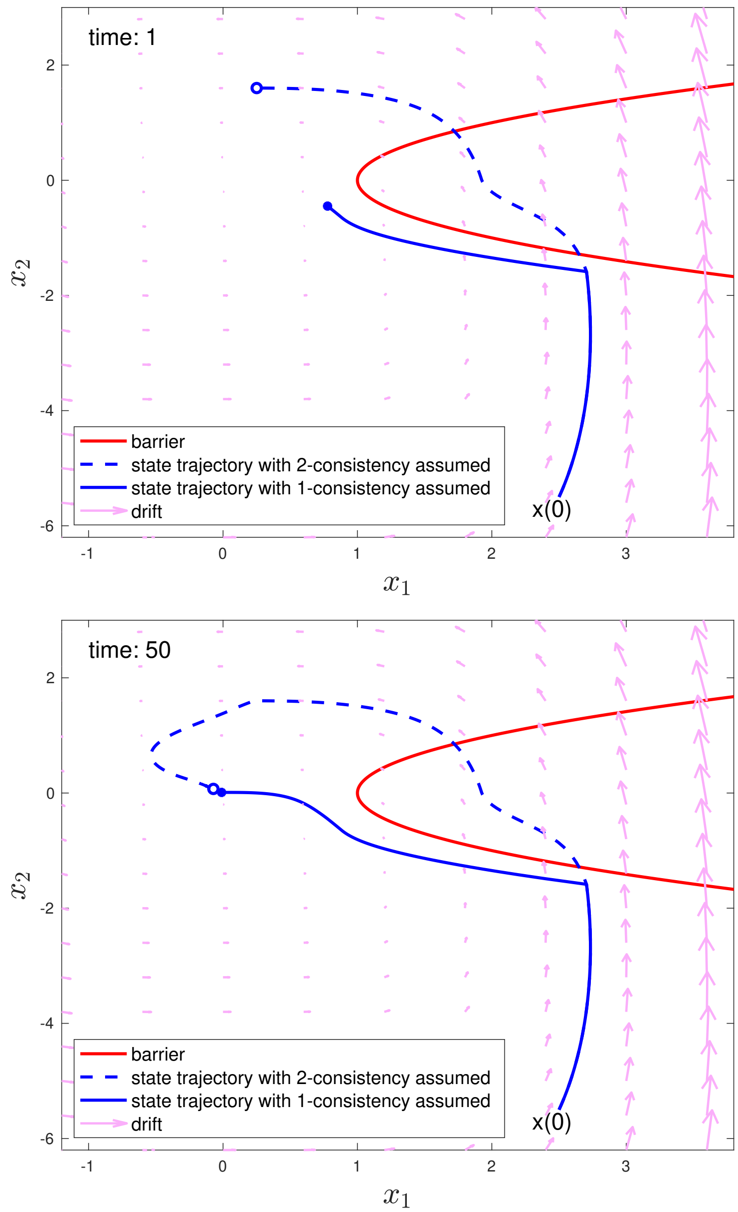

Figure 4.2. Top: System response for $0 \le t \le 1$ when only the shorter-delayed designed input u_1 has been delay-compensated. Under controller (4.44), (4.46) (solid blue) the system remains in the admissible set using only u_1, while for controller (4.34), (4.36) (dashed blue), the system exits the admissible set. Bottom: With both controllers, the system eventually gets stabilized to the origin once the constraint has been cleared.

admissible set even though only the delay in first input channel has been compensated. The control signals under both control approaches are shown in Figure 4.3

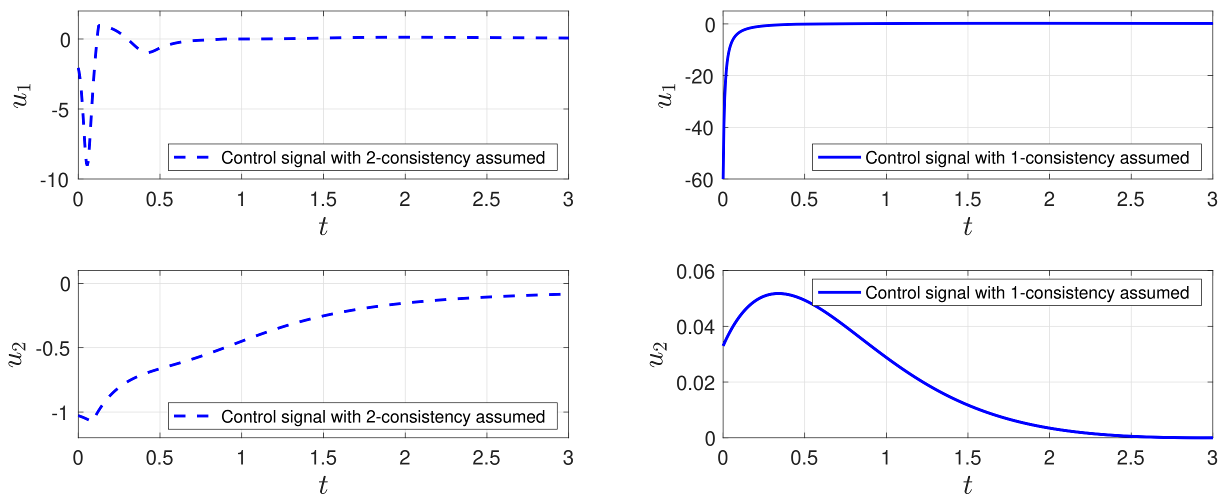

Figure 4.3. Control signals $u_1(t)$ and $u_2(t)$ under controllers (4.44), (4.46) (solid blue) and (4.34), (4.36) (dashed blue).

4.3.3 Safety before longest input delay compensation using variable number of input channels

While the previous approach alleviates the need for r-consistent predictions for safety to only needing φ-consistent predictions, it is only beneficial when the constant φ is less than r i.e. when there exists a strict subset of input channels sufficient to impact $\dot{h}$ everywhere in the set $\mathscr{X}$. When there isn't a strict subset of input channels, i.e. when $\varphi = r$, the controller (4.44)-(4.46) would be equivalent to controller (4.34)-(4.36) and fully consistent predictions would be required to guarantee safety. Ideally, we want the smallest subset of input channels to be utilized whenever they can enforce constraint adherence, even if they cannot be used everywhere in the safe set. To this effect, we introduce a control approach that uses the following modified QP problem.

Consider the following QP problem denoted QP_i^t for input channel $i \in \{1, \ldots, r\}$ at time $t \geq 0$:

$$\underline{\mathrm{QP}_i^t}: \qquad (\overline{u},\sigma^*) = \underset{\substack{v\in\mathbb{R}^i\\ \sigma\in\mathbb{R}^r}}{\arg\min}\ \|v\|^2 + m\|\sigma\|^2 \quad \text{subject to}$$

$$\gamma_f\left(\mathscr{L}_f h + \alpha(h) + \sum_{j=1}^{i}\mathscr{L}_{g_j} h u_{0_j}\right) + \sum_{j=1}^{i}\mathscr{L}_{g_j} h\big(v_j+\sigma_j\big) + \sum_{j=i+1}^{r}\mathscr{L}_{g_j} h(\omega_j+\sigma_j) \geq 0 \tag{4.60}$$

where σ_j are slack variables and all functions of state $h, \mathscr{L}_f h$, $\mathscr{L}_{g_j} h$, and u_{0_i} with suppressed arguments are evaluated at $p_i(t)$ defined in (4.23), $\omega_j = u_j(t-D_{j,i})$, $m \geq 1$ is a design parameter, and

$$\gamma_f(s) = \begin{cases} \gamma s & \text{if } s \leq 0 \\ s & \text{if } s > 0 \end{cases} \tag{4.61}$$

for some $\gamma \geq 1$ (another design parameter).

The notation QP_i^t indicates that this QP problem is "solved" for each $i \in \{1,\dots,r\}$ at all times $t \geq 0$, and will be useful for clarity in the analysis that follow. As previously mentioned, an explicit solution for (4.60) exists and no online optimization is required. The QP problem (4.60) attempts to find the control modifications $\overline{u}_j$, $j = 1,\dots,i$ for the first i input channels, that allows the barrier function to satisfy the inequality in (4.60) at time $t+D_i$. Notice that the inequality in (4.60) is not exactly the barrier constraint $\dot{h}(p_i(t)) + \alpha(h(p_i(t))) \geq 0$ since it includes the additional term $\mathscr{L}_G h(p_i(t))\sigma$ and the function γ_f. The terms $\sigma_{i+1},\dots,\sigma_r$ in (4.60) are fictitious control inputs included to make QP_i^t always feasible, while the terms $\sigma_1,\dots,\sigma_i$, also fictitious, are included to ensure that when all predictions are consistent, the resulting $\overline{u} \in \mathbb{R}^j$ from QP_j^t, is equivalent to the first j components of $\overline{u} \in \mathbb{R}^i$ from $\mathrm{QP}_i^{t-D_{i,j}}$. This equivalence will be shown later in Lemma 4.7. Since the terms $\sigma_1,\dots,\sigma_r$ are fictitious, we include the penalty $m \geq 1$ in the cost function of the QP problem. Lastly, the function γ_f is used to overcome the impact of the fictitious control inputs by exaggerating the negative effect of the baseline control on $\dot{h}(p_i(t))$. Thus, for a fixed penalty m, a larger value of γ leads to the baseline control being considered more unsafe than it actually is. This "γm" approach is similar to and motivated by the γm

CLF-CBF QP problem in [35]. While it allows the benefit of using *any* subset of input channels for enforcing safety before full delay compensation, it does lead to a "safer than necessary" controller relative to the design (4.42), (4.43) where a priori knowledge of the minimum number of input channels sufficient for safety is available. Subsequent analysis will reveal that a suitable choice of γ is $\frac{m+1}{m}$.

The QP problem (4.60) has solution

$$\overline{u}_j^{\mathrm{QP}_i^t} = \begin{cases} 0, & \text{if } \zeta_i(t) \geq 0 \\ -\frac{\zeta_i(t)}{\xi_i(t)} \mathscr{L}_{g_j} h(p_i(t)), & \text{otherwise} \end{cases} \tag{4.62}$$

for $j = 1, \ldots, i$ and

$$\sigma_j^{*\mathrm{QP}_i^t} = \begin{cases} 0, & \text{if } \zeta_i(t) \geq 0 \\ -\frac{1}{m} \frac{\zeta_i(t)}{\xi_i(t)} \mathscr{L}_{g_j} h(p_i(t)), & \text{otherwise} \end{cases} \tag{4.63}$$

for $j = 1, \ldots, r$ where

$$\zeta_i(t) = \gamma_f(z_i(t)) + \sum_{k=i+1}^{r} \mathscr{L}_{g_k} h(p_i(t)) \omega_k(t - D_{k,i}) \tag{4.64}$$

$$\xi_i(t) = \|\mathscr{L}_{G_{1:i}} h(p_i(t))\|^2 + \frac{1}{m} \|\mathscr{L}_G h(p_i(t))\|^2 \tag{4.65}$$

and

$$z_i(t) := \mathscr{L}_f h(p_i(t)) + \alpha(h(p_i(t))) + \sum_{j=1}^{i} \mathscr{L}_{g_j} h(p_i(t)) u_{0_j}(p_i(t)) \tag{4.66}$$

The proposed control law is given as

$$u_i(t) = u_{0_i}(p_i(t)) + \overline{u}_i^{\mathrm{QP}_k^t}, \quad t \geq 0,\ i = 1, \ldots, r \tag{4.67}$$

where

$$k = \max\{j \in \{i, i+1, \ldots, r\} \mid D_j = D_i\} \tag{4.68}$$

For the rest of this chapter, we will assume for the sake of clarity and without loss of generality that the input delays are all distinct i.e. $0 \le D_1 < D_2 < \cdots < D_r$. With this assumption, we have $k = i$ in (4.68). The results and proofs that follow apply with minor modifications when this assumption does not hold.

Proposition 4.3. *If the baseline controller $u_0(x)$ is continuous over a compact set $\mathscr{Y} \subset \mathscr{X}$ with $\mathscr{Y} \cap \mathscr{C} \neq \emptyset$, then the QP solution $\overline{u}_j^{QP_i^t}$ in* (4.62) *is bounded if $p_i(t) \in \mathscr{Y}$. Furthermore, the controller* (4.67) *maps $\mathscr{Y}$ to a compact set $\mathscr{U} \subset \mathbb{R}^r$.*

Proof. Since $\overline{u}_j^{\mathrm{QP}_i^t}$ is the solution of a γm QP problem, its Lipschitz continuity follows from Theorem 1, Part 1 of [35]. The continuity of $u_0(x)$ and $\overline{u}_j^{\mathrm{QP}_i^t}$ implies that the control law (4.67) is continuous and therefore maps a compact set $\mathscr{Y} \subset \mathscr{X}$ to some compact subset $\mathscr{U}$ of $\mathbb{R}^r$. □

Theorem 4.4 (Main Result)**.** *For the closed loop system of* (4.1) *and controller* (4.67)*,* (4.62) *we have the following result. Suppose trajectories of the system stay inside of a compact subset $\mathscr{Y} \subset \mathscr{X}$ where the parameter $m \ge 1$ satisfies*

$$m \ge \frac{1}{\varepsilon \delta^2} \max_{\substack{y \in \mathscr{Y} \subset \mathscr{X}, \\ v \in \mathscr{U}(\mathscr{Y})}} \Big\{ -\|\mathscr{L}_G h(y)\|^2 \Big(\mathscr{L}_f h(y) + \alpha(h(y)) + \mathscr{L}_G h(y) v + \varepsilon \Big) \Big\}, \tag{4.69}$$

for some $\delta > 0$, $\varepsilon > 0$, then the i-consistency of the identity function $I(x) = x$, CBF $h(x)$, and Lie derivatives $\mathscr{L}_f h(x)$ and $\mathscr{L}_G h(x)$ at time $t > D_i$ for some $i \in \{1, \ldots, r\}$ implies that

$$\|\mathscr{L}_{G_{1:i}} h(x(t))\| > \delta \implies \dot{h}(x(t)) + \alpha(h(x(t))) > -\varepsilon. \tag{4.70}$$

for all $x(t) \in \mathscr{Y}$. Moreover, if $I(x)$, $h(x)$, $\mathscr{L}_f h(x)$ and $\mathscr{L}_G h(x)$ are r-consistent at $t > D_r$ and if $\gamma \ge \frac{m+1}{m}$, then

$$\dot{h}(x(t)) + \alpha(h(x(t))) \ge 0 \tag{4.71}$$

for all $x(t) \in \mathscr{Y}$.

Remark 4.5. *The second part of Theorem 4.4 states that after time $t > D_r$, whenever all input delays are fully compensated as reflected by the r-consistency of $I(x)$ and $h(x)$ (and its Lie derivatives), the controller* (4.67) *achieves the standard barrier constraint inequality* (4.71) *as long as γ is chosen to satisfy $\gamma \geq \frac{m+1}{m}$. In fact, since $\frac{m+1}{m} \leq 2$, choosing $\gamma \geq 2$ guarantees the satisfaction of* (4.71) *everywhere in the set $\mathscr{X}$. The first part of the theorem however states that at any given time $t > D_i$, if predictions are only i-consistent, then the first i input channels would enforce the barrier constraint as best as possible, with a possible violation dependent on the magnitude of $\|\mathscr{L}_{G_{1:i}} h(x(t))\|$, provided that trajectories remain inside of some compact set $\mathscr{Y}$ where the inequality* (4.70) *holds. Notice that the inverse relationship of the parameter m and $\varepsilon\delta^2$ in* (4.69) *suggests the choice of a sufficiently large penalty m to keep ε small for a fixed δ. In addition, for a fixed choice of parameter m, the magnitude ε of the violation of the barrier constraint in* (4.70) *decreases as δ – the lower bound on $\|\mathscr{L}_{G_{1:i}} h(x(t))\|$ increases.*

Remark 4.6. *The maximum on the RHS of* (4.69) *exists because it is taken over continuous functions over compact sets $\mathscr{Y} \subset \mathscr{X} \subset \mathbb{R}^n$ and $\mathscr{U} \subset \mathbb{R}^r$. In addition, if one chooses a sufficiently large m, then there exists a non-empty compact set $\mathscr{Y}$ inside of which the inequality holds.*

Proposition 4.5. *For all $t^* \geq D_1$, if $h(x(t^*)) \geq 0$ and if the conditions of Theorem 4.4 are satisfied for all $t \in [t^*, t^* + T]$, $T > 0$, then*

$$h(x(t)) > -\alpha^{-1}(-\varepsilon), \qquad \forall t \in [t^*, t^* + T] \tag{4.72}$$

We prove Theorem 4.4 by using the following lemmas whose proofs are included at the end of this chapter. The proof of Proposition 4.5 is also included at the end of the chapter.

Lemma 4.6. *Let $\overline{u} \in \mathbb{R}^b$ be the unique solution of*

$$\begin{aligned} \overline{u} = \arg\min_{w \in \mathbb{R}^b} \|w\|^2 \quad & \textit{subject to} \\ \beta^\top w \geq c & \end{aligned} \tag{4.73}$$

for some $\beta \in \mathbb{R}^b$, $\beta \neq 0$ and $c \in \mathbb{R}$, then the vector $\overline{v} \in \mathbb{R}^a$, $a < b$ that uniquely solves

$$\begin{aligned} \overline{v} = \underset{w \in \mathbb{R}^a}{\arg\min} \ \|w\|^2 \quad & \textit{subject to} \\ \beta_{1:a}^\top w + \beta_{a+1:b}^\top \overline{u}_{a+1:b} \geq c & \end{aligned} \tag{4.74}$$

satisfies

$$\overline{v} = \overline{u}_{1:a} \tag{4.75}$$

Lemma 4.7. *If the identity function $I(x) = x$, CBF $h(x)$, and the Lie derivatives $\mathscr{L}_f h(x)$ and $\mathscr{L}_G h(x)$ are i-consistent at time t, then for all $j = 1, \ldots, i$,*

$$\overline{u}_j^{QP_j^{t-D_j}} = \overline{u}_j^{QP_i^{t-D_i}} \tag{4.76}$$

Remark 4.7. *Lemma 4.7 states that the i-consistency of $I(x)$, $h(x)$, $\mathscr{L}_f h(x)$ and $\mathscr{L}_G h(x)$ at time t implies that the j-th component of the solution of $QP_j^{t-D_j}$ is equivalent to the j-th component of the solution of $QP_i^{t-D_i}$ for $j < i$. This is intuitive since i-consistency implies that the predicted states, barrier function values, and Lie derivatives used in $QP_i^{t-D_i}$ and $QP_j^{t-D_j}$ are equivalent.*

Lemma 4.8. *For all $t > D_r$ and $\gamma \geq \frac{m+1}{m}$, if $I(x)$, $h(x)$, $\mathscr{L}_f h(x)$ and $\mathscr{L}_G h(x)$ are r-consistent at time t, then the controller* (4.67), (4.62) *achieves*

$$\dot{h}(x(t)) + \alpha(h(x(t))) \geq 0. \tag{4.77}$$

Lemma 4.9. *If m satisfies* (4.69) *for some $\delta > 0$, $\varepsilon > 0$, then the i-consistency of $I(x)$, $h(x)$, $\mathscr{L}_f h(x)$ and $\mathscr{L}_G h(x)$ at $t > D_i$ for some $i \in \{1, \ldots, r\}$ implies* (4.70).

Proof of Theorem 4.4. The proof follows from Lemmas 4.8 and 4.9. The proofs for these Lemmas are included at the end of this chapter. □

Example 2: Consider the following kinematic bicycle model of a vehicle for a circular robot (the host)

$$\begin{aligned}
\dot{x}_1(t) &= x_4(t)\cos(x_3(t)) \\
\dot{x}_2(t) &= x_4(t)\sin(x_3(t)) \\
\dot{x}_3(t) &= \frac{x_4}{L}u_1(t-D_1), \quad \delta(t) = \arctan(u_1(t)) \\
\dot{x}_4(t) &= u_2(t-D_2),
\end{aligned} \tag{4.78}$$

where $(x_1, x_2) \in \mathbb{R}^2$ is the position of the host, $x_3 \in [0, 2\pi)$ is its orientation, $x_4 \in \mathbb{R}$ is the longitudinal velocity, and L is the distance between its front and rear axles. The input u_1 is the tangent of the steering wheel angle δ, and u_2 is the acceleration/deceleration input. The input channels have delays D_1 and D_2 respectively with $0 < D_1 < D_2$ and have initial values

$$u_1(\theta) = 0, \qquad \text{for } \theta \in [-D_1, 0), \tag{4.79}$$

$$u_2(\theta) = 0, \qquad \text{for } \theta \in [-D_2, 0). \tag{4.80}$$

The control objective is to keep the host traveling at a reference velocity $\bar{x}_4$ along a prescribed path in the (x_1, x_2)-plane while avoiding obstacles introduced in its path. For collision avoidance, we use the following candidate barrier function

$$h(x) = \left(x_1 - x_1^{\text{obs}}\right)^2 + \left(x_2 - x_2^{\text{obs}}\right)^2 - r^2 \tag{4.81}$$

where $(x_1^{\text{obs}}, x_2^{\text{obs}})$ is the position of the obstacle and r is the sum of the radii of the host and the obstacle. Keeping $h(x) \geq 0$ therefore ensures that a collision does not occur.

We note here that the candidate barrier function (4.81) has a relative degree of 2 from the inputs and as such the inputs do not appear in the barrier constraint $\dot{h} + \alpha(h) \geq 0$. To circumvent this, we use the barrier constraint formulation from [59, 101] for CBFs with relative degree greater than one, which in the relative degree 2 case is given as:

$$\ddot{h} + l_1 \dot{h} + l_0 h \geq 0, \tag{4.82}$$

$$\mathscr{L}_f^2 h(x) + \mathscr{L}_G \mathscr{L}_f h(x) u + l_1 \mathscr{L}_f h(x) + l_0 h(x) \geq 0 \tag{4.83}$$

where l_0 and l_1 are constants chosen so that the roots of the polynomial $s^2 + l_1 s + l_0 = 0$ are negative real. The idea being that this allows $h(x)$ to decrease towards zero but not oscillate around zero as the case would be for a polynomial with complex roots. The consequences of using this higher relative degree barrier constraint are the following

- In addition to ensuring forward invariance of the set $\mathscr{C} = \{x \mid h(x) \geq 0\}$, satisfying the barrier constraint (4.83) also ensures $h(x) \geq \frac{1}{|\lambda|_{\max}} \dot{h}(x)$ where $|\lambda|_{\max}$ is the magnitude of the more negative root of the characteristic equation $s^2 + l_1 s + l_0 = 0$. Therefore, satisfying (4.83) guarantees the forward invariance of

$$\mathscr{C}_2 = \left\{ x \,\middle|\, h(x) \geq 0,\ h(x) + \frac{1}{|\lambda|_{\max}} \dot{h}(x) \geq 0 \right\}, \tag{4.84}$$

  and since $\mathscr{C}_2 \subseteq \mathscr{C}$, the original set $\mathscr{C}$ is also kept forward invariant.

With the barrier constraint (4.83), the results in this chapter apply with the following substitutions made in the constraint of all QP problems utilized:

$$\mathscr{L}_f^2 h + l_1 \mathscr{L}_f h + l_0 h \to \mathscr{L}_f h + \alpha(h), \text{ and} \tag{4.85}$$

$$\mathscr{L}_G \mathscr{L}_f h \to \mathscr{L}_G h. \tag{4.86}$$

Next, we verify that the candidate barrier function (4.81) is indeed a CBF for the system (4.78) i.e. we want to show that

$$\mathscr{L}_G \mathscr{L}_f h = 0 \implies \mathscr{L}_f^2 h + l_1 \mathscr{L}_f h + l_0 h \geq 0. \tag{4.87}$$

We have

$$\mathscr{L}_f h(x) = 2(x_1 - x_1^{\text{obs}})x_4 \cos(x_3) + 2(x_2 - x_2^{\text{obs}})x_4 \sin(x_3) \tag{4.88}$$

$$\mathscr{L}_f^2 h(x) = 2x_4^2 \tag{4.89}$$

$$\mathscr{L}_G \mathscr{L}_f h(x) = \begin{bmatrix} -\frac{2}{L}(x_1 - x_1^{\text{obs}})x_4^2 \sin(x_3) + \frac{2}{L}(x_2 - x_2^{\text{obs}})x_4^2 \cos(x_3) \\ 2(x_1 - x_1^{\text{obs}}) \cos(x_3) + 2(x_2 - x_2^{\text{obs}}) \sin(x_3) \end{bmatrix}^\top \tag{4.90}$$

It follows from (4.88) and (4.90) that

$$\mathscr{L}_f h(x) = x_4 \mathscr{L}_{g_2} \mathscr{L}_f h(x), \tag{4.91}$$

and therefore,

$$\mathscr{L}_G \mathscr{L}_f h = 0 \implies \mathscr{L}_{g_2} \mathscr{L}_f h = 0 \tag{4.92}$$

$$\implies \mathscr{L}_f h = 0 \tag{4.93}$$

$$\implies \mathscr{L}_f^2 h + l_1 \mathscr{L}_f h + l_0 h = 2x_4^2 + l_0 h \tag{4.94}$$

$$\geq 0 \tag{4.95}$$

for any $l_0 > 0$, and thus (4.81) is indeed a control barrier function for (4.78) with no delays.

For the baseline controller u_0, we linearize the dynamics about a desired nominal trajectory and use an LQR controller for trajectory tracking. Once a baseline control is computed, it is applied unmodified if no obstacle is present in the host's path. If an obstacle is present, we modify the baseline control using two predictor-based safety approaches for comparison. In the first case, we use the controller (4.67), (4.62) where the longer delayed acceleration input u_2 is considered unavailable for safety-enforcement when determining the shorter delayed steering input u_1. In the second case, we use controller (4.34), (4.36) where 2-consistency is implicitly assumed and both inputs are always considered available for safety enforcement. Notice that

Table 4.1. Table of Parameter Values for Example 4.3.3

Parameter	Value
D_1	0.2
D_2	1.5
L	6
r	6
l_0	4
l_1	4
m	50
γ	$\frac{m+1}{m}$
$\bar{x}_4$	3
$(x_1^{\text{obs}}, x_2^{\text{obs}})$	$(-3, 0.5)$

for this example, $\varphi = 2$ (as defined in (4.41)) since $\mathscr{L}_{g_1}\mathscr{L}_f h(x) = 0$ if $x_4 = 0$. This is supported by intuition because the steering input alone cannot affect the distance of a circular agent to a circular obstacle if the velocity x_4 is zero. Thus, the controller (4.44), (4.46) is equivalent to (4.34), (4.36) and is therefore not considered separately. The simulation parameter values are given in table 4.1.

Figure 4.4 shows the initial configuration of the host and the reference path to be followed. The host travels at a constant velocity for the first 2.75 time units before an obstacle gets introduced in its path (Figure 4.5). For the purpose of safety enforcement, the time $t = 2.75$ is the "initial time", and any steering input u_1 applied at this time *with knowledge of the obstacle* will not arrive at the plant until time $t = 2.75 + D_1 = 2.95$. Similarly for the acceleration/breaking input u_2, any new control signal issued will not arrive at the plant until $t = 2.75 + D_2 = 4.25$. Thus, for the purpose of safety, the delay is not fully compensated until time $t = 4.25$ and only becomes partially compensated at time $t = 2.95$. As shown in the Figure 4.6, under controller (4.67), (4.62), the host clears the obstacle without collision using the shorter-delayed control input preferentially. Under controller (4.34), (4.36) however, a collision occurs as shown in Figure 4.7. For both controllers, we include a plot of the input signals in Figure 4.8, and a plot of barrier function h, and the function $h + \frac{1}{|\lambda|_{\max}}\dot{h}$ (c.f. (4.84)) in Figure 4.9.

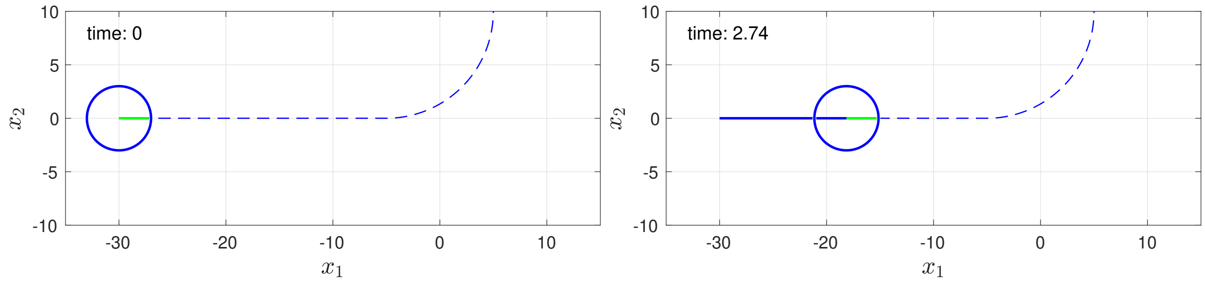

Figure 4.4. Left: Host (solid blue) initial condition with heading (green) along prescribed path (dotted blue). Right: Host (solid) blue accelerates to reference velocity. Since path is obstacle-free, the control effort under controllers (4.67), (4.62) and controller (4.34), (4.36) are equivalent.

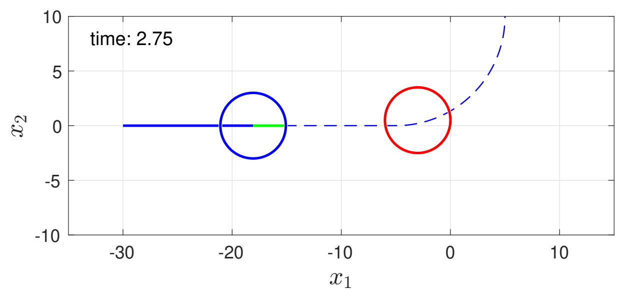

Figure 4.5. Obstacle (solid red) introduced in path of the host (solid blue)

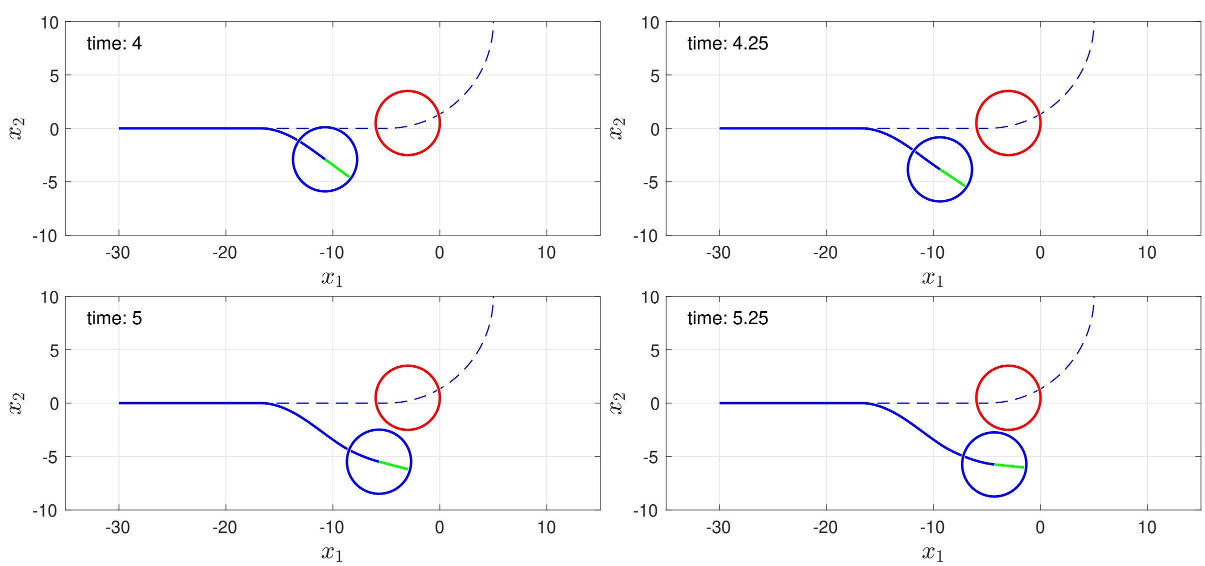

Figure 4.6. Obstacle avoidance under controller (4.67), (4.62). Here, the host does not collide with the obstacle (see barrier function plot in Figure 4.9).

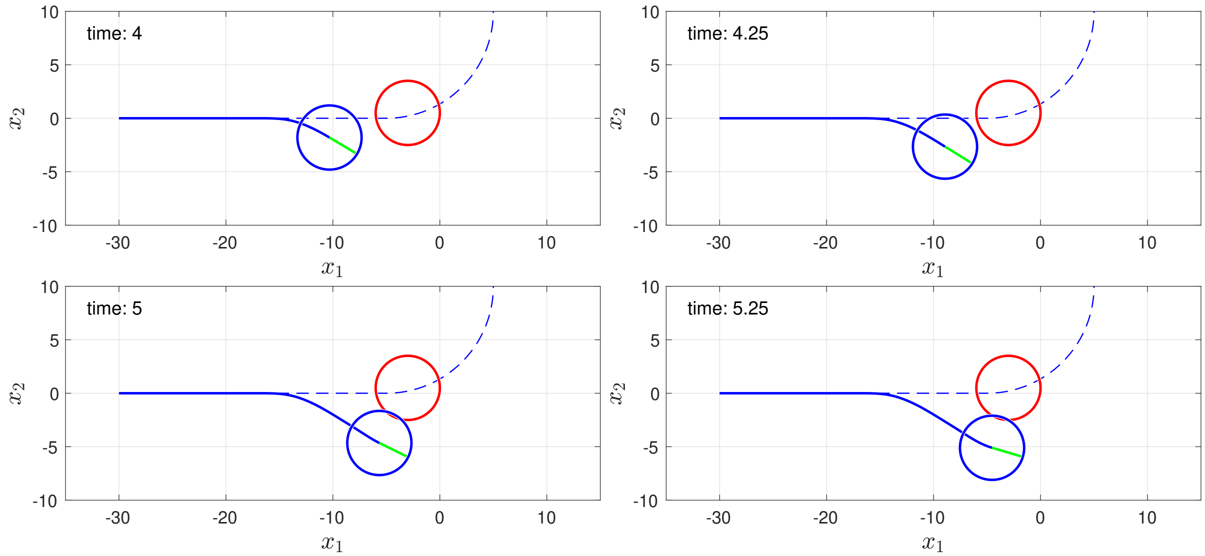

Figure 4.7. Obstacle avoidance under controller (4.34), (4.36) where collision occurs.

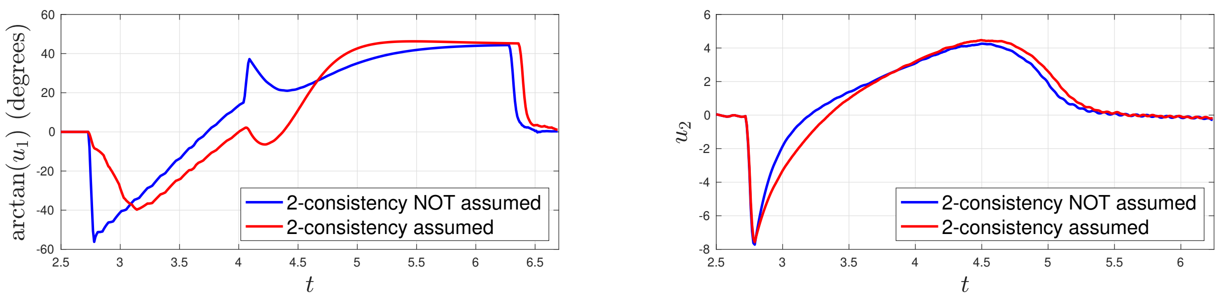

Figure 4.8. Input signal under controller (4.67), (4.62) (blue) and controller (4.34), (4.36) (red).

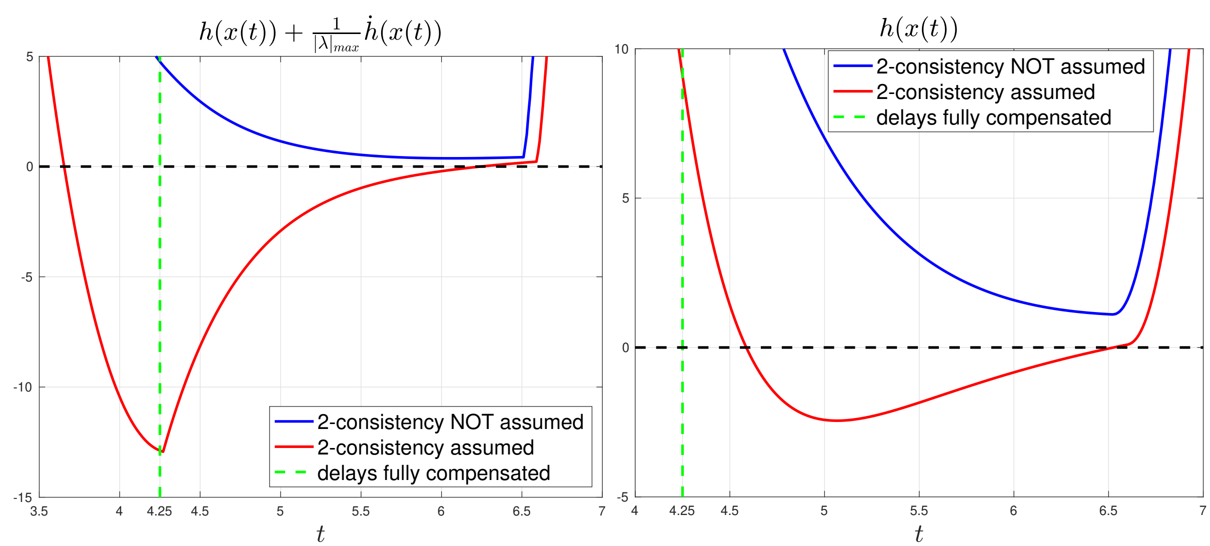

Figure 4.9. CBF value under controller (4.67), (4.62) (blue) and controller (4.34), (4.36) (red).

4.4 Proofs of Lemmas 4.6 – 4.9 and Proposition 4.5

Proof of Lemma 4.6 (by contradiction). Assume $\overline{v} \neq \overline{u}_{1:a}$. Since $\overline{v}$ solves (4.74), we have

$$\beta_{1:a}^{\top}\overline{v} + \beta_{a+1:b}^{\top}\overline{u}_{a+1:b} \geq c \tag{4.96}$$

$$\beta^{\top}\underbrace{\begin{bmatrix} \overline{v} \\ \overline{u}_{a+1:b} \end{bmatrix}}_{\triangleq v^{\dagger}} \geq c, \tag{4.97}$$

therefore $v^{\dagger}$ is in the feasible set of (4.73). Since $\overline{u}$ solves (4.73), we have

$$\beta^{\top}\overline{u} \geq c \tag{4.98}$$

$$\beta_{1:a}^{\top}\overline{u}_{1:a} + \beta_{a+1:b}^{\top}\overline{u}_{a+1:b} \geq c, \tag{4.99}$$

therefore $\overline{u}_{1:a}$ is in the feasible set of (4.74). Since $\overline{v}$ uniquely solves (4.74) and $\overline{v} \neq \overline{u}_{1:a}$, we have

$$\|\overline{v}\|^2 < \|\overline{u}_{1:a}\|^2. \tag{4.100}$$

Now consider $\|v^{\dagger}\|^2$

$$\|v^{\dagger}\|^2 = \|\overline{v}\|^2 + \|\overline{u}_{a+1:b}\|^2 \tag{4.101}$$

$$< \|\overline{u}_{1:a}\|^2 + \|\overline{u}_{a+1:b}\|^2 \tag{4.102}$$

$$= \|\overline{u}\|^2 \tag{4.103}$$

Since $v^{\dagger} \neq \overline{u}$ is in the feasible set of (4.73) and $\|v^{\dagger}\|^2 < \|\overline{u}\|^2$, it implies that $\overline{u}$ does not solve the QP problem (4.73), which is a contradiction. □

Proof of Lemma 4.7. Define

$$\phi := \begin{bmatrix} \frac{\sigma_1^*}{\sqrt{m}} \\ \vdots \\ \frac{\sigma_r^*}{\sqrt{m}} \\ \bar{u}_1 \\ \vdots \\ \bar{u}_i \end{bmatrix}, \qquad \nu := \begin{bmatrix} \frac{\sigma_1^*}{\sqrt{m}} \\ \vdots \\ \frac{\sigma_r^*}{\sqrt{m}} \\ \bar{u}_1 \\ \vdots \\ \bar{u}_j \end{bmatrix}, \tag{4.104}$$

and let

$$c = -\gamma_f \left(\mathscr{L}_f h + \alpha(h) + \sum_{k=1}^{i} \mathscr{L}_{g_k} h u_{0_k} \right) - \sum_{k=i+1}^{r} \mathscr{L}_{g_k} h \omega_k, \tag{4.105}$$

$$\beta = \begin{bmatrix} \sqrt{m}\mathscr{L}_{g_1} h \\ \vdots \\ \sqrt{m}\mathscr{L}_{g_r} h \\ \mathscr{L}_{g_1} h \\ \vdots \\ \mathscr{L}_{g_i} h \end{bmatrix}, \tag{4.106}$$

where functions $h, \mathscr{L}_f h, \mathscr{L}_{g_k} h, u_{0_k}$ have suppressed argument $x(t)$, and $\omega_k = \omega_k(t - D_{k,i})$; then $\mathrm{QP}_i^{t-D_i}$ in (4.60) can be written as

$$\begin{aligned} &\arg\min_{\phi} \|\phi\|^2 \quad \text{subject to} \\ &\beta^\top \phi \geq c. \end{aligned} \tag{4.107}$$

Also, $\mathrm{QP}_j^{t-D_j}$ can be written as

$$\begin{aligned} &\arg\min_{\nu} \|\nu\|^2 \quad \text{subject to} \\ &\beta_{1:r+j}^\top \nu + \beta_{r+j+1:r+i}^\top \phi_{r+j+1:r+i}^* \geq c \end{aligned} \tag{4.108}$$

where ϕ^* is the solution of (4.107). With the representations (4.107), (4.108), the rest of the proof follow from Lemma 4.6. □

Proof of Lemma 4.8. In what follows, we suppress the argument of $h(x(t))$ for brevity. When h appears without an argument, it refers to $h(x(t))$. Now, for all $t > D_r$, it follows from Lemma 4.7 that

$$\dot{h}(x(t)) + \alpha(h(x(t))) = z_r(t-D_r) + \sum_{j=1}^{r} \mathscr{L}_{g_j} h \overline{u}_j^{\mathrm{QP}_r^{t-D_r}} \tag{4.109}$$

where $z_r(\cdot)$ is as defined in (4.66). Now we consider the two cases for $\overline{u}_j^{\mathrm{QP}_r^{t-D_r}}$ in (4.62).

<u>Case 1</u>: $\zeta_r(t-D_r) = \gamma_f(z_r(t-D_r)) \geq 0$.
Here, $\overline{u}_j^{\mathrm{QP}_r^{t-D_r}} = 0$ for all $j = 1, \ldots, r$. In addition,

$$\gamma_f(z_r(t-D_r)) \geq 0 \implies z_r(t-D_r) \geq 0. \tag{4.110}$$

Therefore, (4.109) becomes

$$\dot{h}(x(t)) + \alpha(h(x(t))) = z_r(t-D_r) \tag{4.111}$$

$$\geq 0 \tag{4.112}$$

<u>Case 2</u>: $\zeta_r(t-D_r) = \gamma_f(z_r(t-D_r)) < 0$.
Here,

$$\gamma_f(z_r(t-D_r)) < 0 \implies z_r(t-D_r) < 0, \tag{4.113}$$

and

$$\gamma_f(z_r(t-D_r)) = \gamma z_r(t-D_r). \tag{4.114}$$

Also, by r-consistency of $I(x)$, $h(x)$, $\mathscr{L}_f h(x)$ and $\mathscr{L}_G h(x)$ (4.109) becomes

$$\begin{aligned}
\dot{h}(x(t)) + \alpha(h(x(t))) &= z_r(t-D_r) - \sum_{j=1}^{r} \mathscr{L}_{g_j} h \left(\frac{\gamma z_r(t-D_r)}{\frac{m+1}{m}\|\mathscr{L}_G h\|^2} \right) \mathscr{L}_{g_j} h && (4.115)\\
&= z_r(t-D_r) - \frac{\gamma z_r(t-D_r)}{\frac{m+1}{m}\|\mathscr{L}_G h\|^2} \sum_{j=1}^{r} \left(\mathscr{L}_{g_j} h\right)^2 && (4.116)\\
&= z_r(t-D_r)\left(1 - \frac{\gamma}{\frac{m+1}{m}}\right) && (4.117)\\
&\geq 0, \qquad \forall \gamma \geq \frac{m+1}{m} && (4.118)
\end{aligned}$$

□

Proof of Lemma 4.9. For all $t > D_i$

$$\begin{aligned}
\dot{h}(x(t)) + \alpha(h(x(t))) &= \mathscr{L}_f h + \alpha(h) + \sum_{j=1}^{r} \mathscr{L}_{g_j} h u_j(t-D_j) && (4.119)\\
&= \mathscr{L}_f h + \alpha(h) + \sum_{j=1}^{i} \mathscr{L}_{g_j} h \left(u_{0_j}(p_j(t-D_j)) + \overline{u}_j^{\mathrm{QP}_j^{t-D_j}} \right) \\
&\quad + \sum_{j=i+1}^{r} \mathscr{L}_{g_j} h u_j(t-D_j) && (4.120)\\
&= \mathscr{L}_f h + \alpha(h) + \sum_{j=1}^{i} \mathscr{L}_{g_j} h \left(u_{0_j}(p_i(t-D_i)) + \overline{u}_j^{\mathrm{QP}_i^{t-D_i}} \right) \\
&\quad + \sum_{j=i+1}^{r} \mathscr{L}_{g_j} h \omega_j(t-D_j) && (4.121)\\
&= z_i(t-D_i) + \sum_{j=1}^{i} \mathscr{L}_{g_j} h \overline{u}_j^{\mathrm{QP}_i^{t-D_i}} + \sum_{j=i+1}^{r} \mathscr{L}_{g_j} h \omega_j(t-D_i-D_{j,i}) && (4.122)
\end{aligned}$$

The equality in (4.121) uses Lemma 4.7, and the equality in (4.122) uses the definition of $z_i(\cdot)$ in (4.66). Now, consider the two cases for $\overline{u}_j^{\mathrm{QP}_i^{t-D_i}}$ in (4.62).

Case 1: $\zeta_i(t-D_i) = \gamma_f(z_i(t-D_i)) + \sum_{k=i+1}^{r} \mathscr{L}_{g_k} h(p_i(t-D_i))\omega_k(t-D_i-D_{k,i}) \geq 0$.

Here, $\overline{u}_j^{\mathrm{QP}_i^{t-D_i}} = 0$ for all j and (4.122) becomes

$$\dot{h}(x(t)) + \alpha(h(x(t))) = z_i(t-D_i) + \sum_{j=i+1}^{r} \mathscr{L}_{g_j} h\omega_j(t-D_i-D_{j,i}) \tag{4.123}$$

$$\geq \gamma_f(z_i(t-D_i)) + \sum_{j=i+1}^{r} \mathscr{L}_{g_j} h\omega_j(t-D_i-D_{j,i}) \tag{4.124}$$

$$= \zeta_i(t-D_i) \tag{4.125}$$

$$\geq 0 \tag{4.126}$$

where the equality in (4.125) uses the the i-consistency of $I(x(t))$, $h(x(t))$, $\mathscr{L}_f h(x(t))$ and $\mathscr{L}_G h(x(t))$.

Case 2: $\zeta_i(t-D_i) = \gamma_f(z_i(t-D_i)) + \sum_{k=i+1}^{r} \mathscr{L}_{g_k} h(p_i(t-D_i))\omega_k(t-D_i-D_{k,i}) < 0$.

Here, (4.122) becomes

$$\begin{aligned}
\dot{h}(x(t)) + \alpha(h(x(t))) &= z_i(t-D_i) - \frac{\zeta_i(t-D_i)}{\xi_i(t-D_i)} \sum_{j=1}^{i} \left(\mathscr{L}_{g_j} h\right)^2 \\
&\quad + \sum_{j=i+1}^{r} \mathscr{L}_{g_j} h(x(t))\omega_j(t-D_i-D_{j,i}) \qquad (4.127) \\
&= z_i(t-D_i) - \underbrace{\frac{\|\mathscr{L}_{G_{1:i}} h(p_i(t-D_i))\|^2}{\xi_i(t-D_i)}}_{\triangleq \rho_i(t-D_i)} \zeta_i(t-D_i) \\
&\quad + \underbrace{\sum_{j=i+1}^{r} \mathscr{L}_{g_j} h(p_i(t-D_i))\omega_j(t-D_i-D_{j,i})}_{\triangleq \eta_i(t-D_i)} \qquad (4.128) \\
&= z_i(t-D_i) + \eta_i(t-D_i) - \rho_i(t-D_i)\zeta_i(t-D_i) \qquad (4.129)
\end{aligned}$$

Now, consider the inequality in (4.69). It follows that

$$m \geq -\frac{\|\mathscr{L}_G h(p_i(t-D_i))\|^2}{\varepsilon\delta^2}(z_i(t-D_i)+\eta_i(t-D_i)+\varepsilon) \tag{4.130}$$
$$\geq -\frac{\|\mathscr{L}_G h(p_i(t-D_i))\|^2}{\delta^2}\frac{(z_i(t-D_i)+\eta_i(t-D_i)+\varepsilon)}{\varepsilon+z_i(t-D_i)-\gamma_f(z_i(t-D_i))}, \tag{4.131}$$

since $m>0$ and $s-\gamma_f(s)\geq 0$ for all s. For brevity, we suppress the arguments of $\mathscr{L}_G h$, z_i, and η_i. We have

$$\frac{m\delta^2}{\|\mathscr{L}_G h\|^2}(\varepsilon+z_i-\gamma_f(z_i)) \geq -(z_i+\eta_i+\varepsilon) \tag{4.132}$$
$$\implies \frac{m\delta^2}{\|\mathscr{L}_G h\|^2}\frac{(\gamma_f(z_i)-z_i-\varepsilon)}{\gamma(z_i)+\eta_i} \geq \frac{z_i+\eta_i+\varepsilon}{\gamma_f(z_i)+\eta_i}, \tag{4.133}$$

since $\gamma_f(z_i(t-D_i))+\eta_i(t-D_i)=\zeta_i(t-D_i)<0$. Simplifying further, we get

$$\delta^2\left(1-\frac{z_i+\eta_i+\varepsilon}{\gamma_f(z_i)+\eta_i}\right) \geq \frac{\|\mathscr{L}_G h\|^2}{m}\frac{z_i+\eta_i+\varepsilon}{\gamma_f(z_i)+\eta_i} \tag{4.134}$$
$$\frac{\delta^2}{\delta^2+\frac{\|\mathscr{L}_G h\|^2}{m}} \geq \frac{z_i+\eta_i+\varepsilon}{\gamma_f(z_i)+\eta_i} \tag{4.135}$$

Note from (4.128) that $\rho_i = \frac{\|\mathscr{L}_{G_{1:i}}h\|^2}{\|\mathscr{L}_{G_{1:i}}h\|^2+\frac{1}{m}\|\mathscr{L}_G h\|^2} > \frac{\delta^2}{\delta^2+\frac{\|\mathscr{L}_G h\|^2}{m}}$ if $\|\mathscr{L}_{G_{1:i}}h\|>\delta>0$. Therefore, we have from (4.135) that

$$\rho_i(t-D_i) > \frac{z_i(t-D_i)+\eta_i(t-D_i)+\varepsilon}{\gamma_f(z_i(t-D_i))+\eta_i(t-D_i)} \tag{4.136}$$
$$= \frac{z_i(t-D_i)+\eta_i(t-D_i)+\varepsilon}{\zeta_i(t-D_i)} \tag{4.137}$$

Substituting (4.137) into (4.129) gives

$$\dot{h}(x(t))+\alpha(h(x(t))) > z_i(t-D_i)+\eta_i(t-D_i)-\frac{z_i+\eta_i+\varepsilon}{\zeta_i(t-D_i)}\Big(\zeta_i(t-D_i)\Big) \tag{4.138}$$
$$> -\varepsilon \tag{4.139}$$

□

Proof of Proposition 4.5. From Theorem 4.4, we have for all $t \in [t^*, t^* + T]$

$$\dot{h}(x(t)) + \alpha(h(x(t))) > -\varepsilon. \tag{4.140}$$

We apply the comparison lemma [39] to the initial value problem

$$\dot{y}(t) = -\alpha(y(t)) - \varepsilon, \quad t \in [t^*, t^* + T], \tag{4.141}$$

$$y(t^*) = h(x(t^*)). \tag{4.142}$$

Since $y(t^*) > 0$ and $\varepsilon > 0$, we have that y decreases monotonically until $\dot{y} = 0$. Therefore, it follows that

$$y(t) \geq \alpha^{-1}(-\varepsilon), \quad \forall t \in [t^*, t^* + T]. \tag{4.143}$$

By the comparison lemma, we have

$$h(x(t)) > \alpha^{-1}(-\varepsilon), \quad \forall t \in [t^*, t^* + T]. \tag{4.144}$$

□

4.5 Conclusion and acknowledgements

In this chapter, we studied the problem of enforcing the safety in multi-input nonlinear systems with distinct input delays using predictor feedback. In particular, we presented two approaches that preferentially use shorter-delayed inputs for enforcing safety before all input delays have been compensated. Compared to a naive combination of state-predictors and a nominal safe-guarding controller for the delay free system where safety-guarantees cannot be made until after the longest input delay has been compensated, we showed that the two introduced methods enforce safety before the longest input delay has been compensated whenever it is possible to do so. We included illustrative examples to demonstrate the performance of the two introduced control methods. A possible direction for future work is the study of safety in the presence of distinct input delays that are non-constant and/or unknown a priori, a problem that is relevant in many practical systems.

Chapter 4, in part, is a reprint and adaptation of the following papers: (1) I. Abel, M. Krstić, and M. Janković, “Constrained Control of Input Delayed Systems with Partially Compensated Input Delays”, ASME Dynamic Systems and Controls Conference (DSCC), 2020. (2) I. Abel, M. Krstić, and M. Janković, “Constrained Control of Multi-Input Systems with Distinct Input Delays”, under review for publication in the IEEE Transactions on Automatic Control. The dissertation author was the primary investigator and author of both papers.

Chapter 5

Prescribed-Time Safety for High Relative Degree Systems

5.1 Introduction

5.1.1 High relative degree control barrier functions

Control barrier functions (CBFs) have become a popular tool for synthesizing safe controllers for dynamical systems and have been used in a wide range of problem domains: multi-agent robotics [26, 79, 96], robust safety [35, 43, 103], automotive systems [4, 74, 102], delay systems [2, 34, 57, 72], and stochastic systems [19, 73, 80] to mention a few. First defined in [98] and later refined and popularized by the seminal papers [4, 6], CBFs are often employed in a "safety filter" framework where they are used for generating safe control overrides for a potentially unsafe nominal controller. In essence, a nominal controller is designed to achieve a desired performance objective, and a CBF-based control override is used whenever the nominal controller is at risk of making the system unsafe.

In its initial conception [4, 98], CBFs were specified for safety constraints of relative degree one i.e. constraints whose time derivative depend explicitly on the input. The extension of CBFs to constraints with high-relative degree was first studied independently in [32, 99] with much progress following in [15, 59, 100, 101]. In [99], the extension was limited to the relative degree two case, and in [32] where CBFs of arbitrarily high relative degree was introduced, its usage for a relative degree r case involves choosing $r-1$ bounded, positive definite functions that satisfy additional derivative conditions [32, Eq. (26)] – a requirement that limits the utility of [32] for significantly high relative degree constraints. A similar limitation applies to more recent treatment [100] which requires choosing and tuning r class-$\mathscr{K}$ functions, whose choice determines the subset of the original safe set that is kept forward invariant. Building off [99], exponential CBFs were reported in [59] and allowed the use of simple linear control tools to design CBFs for high relative degree constraints.

5.1.2 The "non-overshooting control" roots of high relative degree CBFs

A year before [98], eight years before [4], and ten years before [59], a design for stabilization to an equilibrium point *at* the barrier was introduced, under the name "non-overshooting control," in the 2006 paper [50] (following its conference version in 2005). This design possesses all the attributes of a safety design with a CBF of a *uniform* and high relative degree[1] (sans the CBF terminology) with only the QP step absent since, for stabilization at the barrier, QP is subsumed in the stabilization design (the nominal feedback and the safety filtered feedback are the same).

The interest in non-overshooting control in the 1990s came from applications—spacecraft docking, aerial refueling, machining, etc., with no margin for error in downward setpoint regulation. The non-overshooting control problem for linear systems, albeit mostly for zero initial conditions and nonzero setpoints, was solved in [12, 65, 81].

The paper [50] introduced the following two ideas (translated to the current CBF terminology). First, for a system with a high relative degree CBF, a transformation, by backstepping,

[1]Uniformity of the CBF's relative degree gives the equivalence of a general control-affine system with the strict-feedback class and the convertibility of the safe set, given by the CBF positivity constraint, into a semi-infinite interval constraint for the first state of the strict-feedback system.

into a particular target system in the form of a chain of first-order CBF subsystems (resulting in all real poles in the linear case) is performed. Second, in order to ensure that all the CBF "states" of this chain begin and remain positive, the positivity of their initial values is ensured by choosing the backstepping gains in accordance with the initial conditions so the entire CBF chain is initialized positively.

This chain structure and the gain selection of [50, Eq. (12), (13)], regarded through the lens of pole placement, were independently discovered in the 2016 paper [59, Cor. 2]. Likewise, the nonlinear damping choices in the CBF chain in [50, Eq. (53), (54)] was independently proposed in the 2019 paper [100]. Additionally pursued in [50], but not in [59, 100], was a form of input-to-state safety (ISSf) in the presence of disturbances. This notion, though not explored for high relative degree CBFs, is rigorously conceived in [43].

Inspired by non-overshooting control under disturbances in [50], i.e., by stabilization to an equilibrium at the barrier along with ISSf, mean-square stabilization of stochastic nonlinear systems to an equilibrium at the barrier, along with a guarantee of non-violation of the barrier in the mean sense, is solved in [52].

With a nearly negligible QP modification, the stabilizing feedbacks in [50] can be used in safety filters. Hence, backstepping generates a safety filter with explicit tuning variables that dictate the exponential approach to the barrier.

5.1.3 Prescribed-time safety (PTSf)

Recent advances in prescribed-time stabilization (PTS) [85] have resulted in time-varying backstepping controllers that guarantee settling times independent of initial conditions. Extensions have been developed to stochastic nonlinear systems [53], infinite-dimensional systems [21, 90, 91], and even coupled systems with finite/infinite-dimensional subsystems [22, 88, 89]. PTS is a subset of both the finite-time [28] and fixed-time [69] notions (i.e., stronger than both).

The success in achieving stabilization in prescribed time, independent of initial conditions, inevitably raises the question of pursuing the safety counterpart of the same notion. The "translation" from stability to safety may be a bit counterintuitive: while PTS guarantees that the state *reaches the equilibrium no later* than a prescribed time T, PTSf guarantees that the state *cannot reach the barrier sooner* than a prescribed time T.

In what context is such a PTSf property useful? First, it should be noted that PTSf is "less safe" than exponential safety (ESf). Less safe is useful by not being restrictive for longer than necessary. If the desired operation is in the barrier's proximity, and especially if the desired operation is beyond the barrier, where it is safe to be after time T, PTSf offers obvious performance (or liveliness) advantages over ESf. It is even known in the automotive area that "too safe" may mean possibly unsafe: a follower vehicle that keeps a large distance 'invites' vehicles from other lanes to cut in [29].

We distinguish our notion of PTSf from recently defined notions of fixed- and finite-time safety (FxTSf/FnTSf) in [70]. In PTSf, safety is enforced only for a fixed time duration T, after which the system is allowed to enter the unsafe set as dictated by the nominal system behavior. In contrast, FxTSf enforces safety *indefinitely* and acts in a manner where, whenever the safety filter kicks in and the nominal system behavior is overridden for a duration of T time units, the system is necessary brought to the boundary of the safe-set at the end of that time duration. FnTSf acts similarly as FxTSf, but the time duration T is dependent on the state and nominal control input at the time the safety override kicks in. Furthermore, PTSf is also different from the notions of limited duration safety [61] and periodic safety using fixed-time CBFs [24]. Specifically, [24] introduced the notion of periodic safety where the objective is to keep a system safe for *all times* while enforcing that it periodically (with time period T) visits a goal set *inside* the safe set. In [61] the notion of limited duration safety was studied, and like PTSf it implies that a system is kept safe only for a limited duration T. While [61] restricts the set of initial conditions—a set that shrinks as T increases [61, Rk. 2]—to be a strict subset of the safe set [61, Eq. (3)], our notion of PTSf places no restriction on the initial conditions of the system.

Finally, two distinct features of the time-varying backstepping technique make it quite attractive for use in safety filter design. The first is that, compared to ESf designs, the PTSf filters designed with time-varying backstepping do not exhibit large transients when the safety filter overrides the nominal controller. This is not the case for ESf filters with rapid decay rates: the so-called "peaking" phenomenon [40, 41, 92] is exhibited, which can cause some of the states to become very large near the initialization time, before rapidly converging to the equilibrium. This behavior can cause large state-derivatives, which, e.g., is undesirable in vehicle systems where maneuvers causing large acceleration and "jerk" can be dangerous. PTSf safety filter designs avoid peaking by using small gains near initialization time that only grow large as the state grows "small". In essence, PTSf behaves like a smooth, automatic transition from slow ESf at the initialization time to fast ESf as time approaches the terminal time. The second feature making time-varying backstepping attractive is the behavior of the convergence it achieves near the terminal time. PTSf achieves convergence with "infinitely-soft" landing, that is, the state and *all* of its derivatives converge to the equilibrium by the terminal time. This feature occurring in finite time is unique to PTSf, and is desirable because it can ensure, e.g., "jerk-free" safety maneuvers by the terminal time.

5.2 Problem description

We study linear systems in the following chain-of-integrator form

$$\begin{aligned} \dot{x}_i(t) &= x_{i+1}(t), \quad i = 1, \ldots, n-1, \\ \dot{x}_n(t) &= u(t), \\ y(t) &= x_1(t), \qquad t \geq t_0, \end{aligned} \tag{5.1}$$

with relative degree n, where $t_0 \geq 0$ is the initialization time, $x = [x_1, \ldots, x_n]^\top \in \mathbb{R}^n$ is the state with initial condition $x(t_0)$, and $u \in \mathbb{R}$ is the control input. Our objective is to enforce non-positivity of the output $y(t)$ over the finite time horizon $[t_0, t_0 + T)$, where T is a terminal time that can be a priori prescribed. As is the case in many safety-critical systems, the control input will be filtered through a so-called safety-filter which overrides a nominal control input u_{nom}

whenever it violates conditions that lead to $y(t) > 0$. Distinct to our safety-filter design is that it only enforces safety for times $t \in [t_0, t_0 + T)$, and our methodology generates an explicit safety-filter expression which depends on $x(t_0)$ and parameters that can be readily tuned for system performance objectives.

5.2.1 Preliminaries

Our PTSf designs will be generated by the following "blow-up" function:

$$\mu_m(t - t_0, T) = \frac{1}{\nu^m(t - t_0, T)}, \quad t \in [t_0, t_0 + T) \tag{5.2}$$

for $m \in \mathbb{N}_{\geq 2}$ and the *terminal time* $T > 0$, where

$$\nu(t - t_0, T) := \frac{T + t_0 - t}{T} \tag{5.3}$$

decays linearly from one to zero by the terminal time. We denote by $m^{\bar{k}}$ the *rising factorial* for $m, k \in \mathbb{N}$, that is,

$$m^{\bar{k}} := m(m+1)\cdots(m+k-1); \tag{5.4}$$

the derivatives of μ_m are

$$\mu_m^{(i)}(t - t_0, T) = \frac{m^{\bar{i}}}{T^i}\mu_{m+i}(t - t_0, T). \tag{5.5}$$

For the rest of this chapter, we shall use μ_m and $\mu_m(t)$ to denote $\mu_m(t - t_0, T)$ for brevity when there is no confusion. We denote my $\mathscr{P}^n(x)$ an nth-order polynomial in x.

5.3 A second-order design demonstration

We motivate our design with the double integrator

$$
\begin{aligned}
\dot{x}_1(t) &= x_2(t), \\
\dot{x}_2(t) &= u(t), \qquad (5.6)\\
y(t) &= x_1(t), \quad t \geq t_0.
\end{aligned}
$$

which can be used to model the longitudinal dynamics of a vehicle, where the states x_1, x_2 represent the position and velocity of the vehicle respectively, and the input u represents the longitudinal acceleration/deceleration command. The safety objective is to enforce that the position x_1 remains negative for a prescribed time duration T during which the system is driven by a nominal control u_{nom} that can be overridden as needed to enforce safety. One application of this is the problem of ensuring that an ego vehicle does not violate a red traffic light that is on for a fixed duration T, irrespective of driver input. We begin our design by performing a time-varying backstepping transformation defined as follows

$$h_1(t) := -x_1(t) \qquad (5.7)$$

$$h_2(t) := -x_2(t) + c_1\mu_2(t-t_0,T)h_1(t) \qquad (5.8)$$

where c_1 is a design parameter to be determined. We call the transformed states h_i's barrier functions to connote the desire to keep their values positive, provided that the initial values $h_i(t_0)$ are positive as is typical with CBFs. The positivity of $h_1(t_0)$ follows from requiring that the system is initially safe i.e. $x_1(t_0) < 0$. For $h_2(t_0)$, we achieve positivity by choosing $c_1 > \max\left\{0, -\frac{x_2(t_0)}{x_1(t_0)}\right\}$ where we have used $\mu_2(t_0-t_0,T) = 1$. The importance of the restriction of c_1 to positive values will become apparent from the dynamics of the target h-system. Under the transformation (5.7), (5.8) and the choice of control

$$u_{\text{safe}} = c_2\mu_2 h_2 + \frac{d}{dt}(c_1\mu_2 h_1) \tag{5.9}$$

$$= -(c_1+c_2)\mu_2 x_2 - c_1\left(c_2\mu_2 + \frac{2\mu_1}{T}\right)\mu_2 x_1 \tag{5.10}$$

where $c_2 > 0$, the h-dynamics satisfy

$$\begin{aligned} \frac{d}{dt}h_1 &= -c_1\mu_2 h_1 + h_2 \\ \frac{d}{dt}h_2 &= -c_2\mu_2 h_2 \end{aligned} \tag{5.11}$$

which we will later show converges to the origin in T time units for positive constants c_1, c_2 chosen to satisfy the aforementioned conditions. Thus, to enforce that the nominal controller u_{nom} does not make the state x_1 positive *before* time t_0+T, we apply the safety filter

$$u = \begin{cases} \min\{u_{\text{nom}}, u_{\text{safe}}\}, & \text{if } t_0 \le t < t_0+T, \\ u_{\text{nom}}, & \text{if } t \ge t_0+T. \end{cases} \tag{5.12}$$

We shall address the continuity of the controller at t_0+T when we consider the general n-dimensional problem in Section 5.5. Now, consider the case where the nominal controller u_{nom} is unsafe in the interval $[t_0, t_0+T)$ so that we apply the overriding controller u_{safe} in (5.10), the solution of the system (5.11) is given as

$$h_1(t) = e^{-c_1 T[\mu_1(t)-1]}\left[h_1(t_0) + h_2(t_0)\int_{t_0}^{t} e^{(c_1-c_2)T[\mu_1(\tau)-1]}\,d\tau\right], \tag{5.13}$$

$$h_2(t) = e^{-c_1 T[\mu_1(t)-1]} h_2(t_0) \tag{5.14}$$

which satisfies $h_1(t) > 0$, $h_2(t) > 0$ for $t \in [t_0, t_0+T)$ and in particular $\lim_{t\to(t_0+T)^-} h_1(t) = \lim_{t\to(t_0+T)^-} h_2(t) = 0$. Thus, in the case of a potentially unsafe nominal control u_{nom}, we have

$x_1(t) < 0$, $t \in [t_0, t_0 + T)$ and $\lim_{t \to (t_0+T)^-} x_1(t) = 0$.

In the next section, we present a simulation and interpret the results of our second-order safety-filter design.

5.4 Simulation and interpretations of design

5.4.1 PTSf versus ESf: The peaking phenomenon

We return to the double integrator (5.6). Suppose that the nominal control input u_{nom} is at risk of making our system unsafe, and we wish to design a *time-invariant* safety filter that overrides the nominal controller and takes the system to the origin. This problem was studied in [59, Sec. 3.B] for input-output linearized systems via pole-placement (which inherently relies on the backstepping method—see [59, Rk. 5]), which achieves exponential convergence to the origin with arbitrary decay rate. We define the barrier functions

$$h_1 := -y, \tag{5.15}$$

$$h_2 := -x_2 + \rho h_1, \quad \rho > 0, \tag{5.16}$$

with the goal of keeping $h_1 \geq 0$ uniformly. Consider the following time-invariant safety filter designed as in [59]:

$$u = \min\left\{u_{\text{nom}}, -\left(2\rho^2 \;\; 3\rho\right) x\right\}, \qquad t_0 \leq t < \infty, \tag{5.17}$$

with

$$\rho \geq \max\left\{0, -\frac{x_2(t_0)}{x_1(t_0)}\right\}. \tag{5.18}$$

Suppose the safety-filter overrides the nominal controller at $t = t_0 + \bar{t} < \infty$ and continues to enforce safety thereafter (i.e., $u(t) = -\left(2\rho^2 \;\; 3\rho\right) x(t)$ for all $t \geq t_0 + \bar{t}$, placing the closed-loop poles for the x-system at $\{-\rho,-2\rho\}$). Then the closed-loop system (5.6)–(5.17) is given by

$$x(t)=e^{-\rho(t-t_0-\bar{t})}\times\begin{pmatrix} 2-e^{-\rho(t-t_0)} & \frac{1}{\rho}-\frac{e^{-\rho(t-t_0-\bar{t})}}{\rho} \\ 2\rho\left(e^{-\rho(t-t_0-\bar{t})}-1\right) & 2e^{-\rho(t-t_0-\bar{t})}-1 \end{pmatrix}x(t_0+\bar{t}). \tag{5.19}$$

If we wish to achieve large exponential decay when the system is unsafe, we can select $\rho\gg\max\left\{0,-\frac{x_2(t_0)}{x_1(t_0)}\right\}$ as large as desired. However, for small $t-t_0-\bar{t}$, the righthand side of (5.19) can be very large depending on the size of ρ (in particular, x_2 grows with ρ). This illustrates the "peaking" phenomenon, which was studied for ODE control systems in [40,41,92]. The celebrated work of Sussmann and Kokotovic [92] has exposed the possibility of disastrous outcomes in input-output feedback linearization, where rapid regulation of the output can have catastrophic consequences on the zero dynamics. Moreover, due to the structure of the feedback (5.17), even for systems with a full relative degree (systems without zero dynamics, as above), the control input becomes extremely large near time $t_0+\bar{t}$.

In the context of safety, if (x_1,x_2) represent position and velocity, seeking time-invariant safety filters with large exponential decay ($\rho\gg\max\left\{0,-\frac{x_2(t_0)}{x_1(t_0)}\right\}$) results in a very large and rapid transient response in the velocity, which is undesirable as it causes a large "jerk" to the system.

For our time-varying safety-filter (5.17), the control gains are chosen to initially start quite small and depending on the initial conditions (see (5.10)), and only grow very large simultaneously as the states grow very small and as time approaches the prescribed terminal time. This eliminates the possibility of peaking.

We now compare these results graphically to demonstrate the advantages of time-varying backstepping. We perform numerical simulations for the double-integrator system under the nominal controller

$$u_{\text{nom}} = -k_1 z_1 - k_2 z_2, \tag{5.20}$$

$$z_1 = x_1 + a\sin(\omega t) + b, \tag{5.21}$$

$$z_2 = x_2 + a\omega\cos(\omega t), \tag{5.22}$$

with $k_1 = k_2 = 4$, $a = 1$, $b = 0.8$ and $\omega = 2\pi/T$ where $T = 4$ is the prescribed time. The initial condition is chosen as $x(0) = (-4,2)^\top$. For safety, we use our time-varying PTSf safety-filter (5.12), (5.10) with choice of gains $c_2 = c_1 = \max\left\{0, -\frac{x_2(0)}{x_1(0)}\right\} + 0.1 = 0.6$ and use ramp function (5.28) with $m = 2$ and $\bar{T} = 0.5$ for controller continuity at $t_0 + T$ (see Section 5.5, Eq. (5.28) for details). For comparison, we use the time-invariant ESf safety-filter (5.17) with $\rho = 0.6$ and $\rho = 3.2$. The choice $\rho = 0.6$ was made to allow a gain equivalent to the initial gains $c_1\mu_2(0)$, $c_1\mu_2(0)$ of PTSf, and the choice $\rho = 3.2$ was tuned to make ESf less conservative and to react at around the same instant as PTSf. For numerical stability near the origin, we clip the blow-up function μ_2 at a maximum value $\mu_{2,\max} = 1000$ — which still allows the PTSf gains grow to several orders of magnitudes larger than $\rho = 3.2$. The system trajectories under PTSf and ESf are shown in Figure 5.1 where we observe the ESf filter with $\rho = 0.6$ being overly conservative, overriding the nominal trajectory much sooner, despite being significantly far from the barrier. With ρ increased to 3.2, the ESf filter becomes less conservative like PTSf but at the cost of a significantly higher jerk as evident in Figure 5.2. In essence, while ESf can be tuned to be less conservative like PTSf by choosing larger gains, it comes at the expense of a significant jerk. Lastly, as evident in both figures, the PTSf filter eventually allows the system evolve freely after the prescribed time $T = 4$ has elapsed.

We now present our prescribed-time safety-filter design for the nth order chain-of-integrator systems. While the time-varying backstepping technique is applicable to a wider class of systems (*nonlinear* strict feedback ones), we opt for increased clarity by limiting our exposition to linear systems.

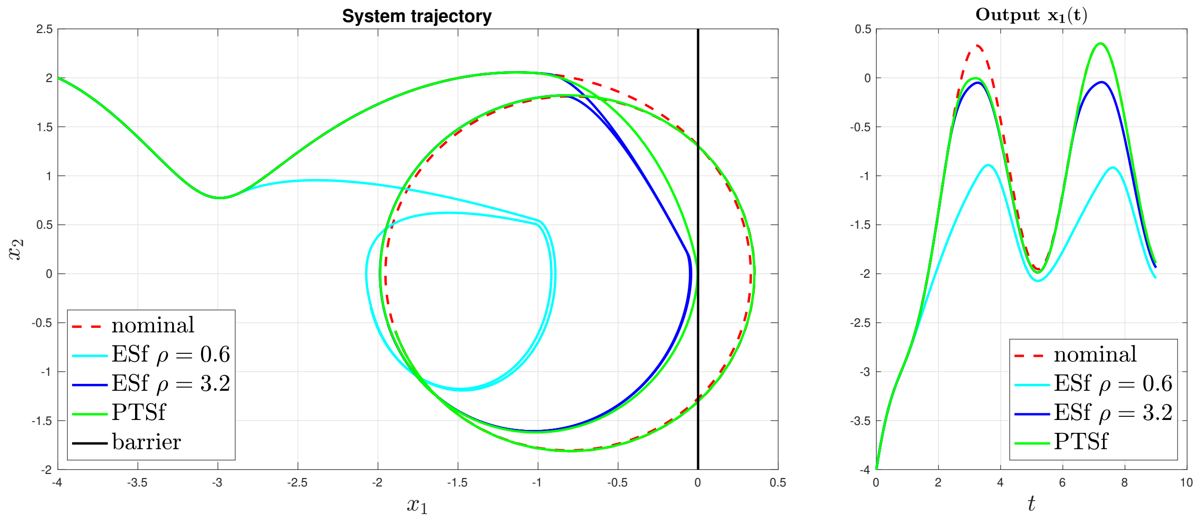

Figure 5.1. System trajectories (left) and outputs (right) for double integrator under nominal controller (5.20) with terminal time $T = 4$, and initial condition $(x_1(0), x_2(0)) = (-4, 2)$. The PTSf safety-filter uses (5.12), (5.10) with $c_2 = c_1 = 0.6$ while the ESf safety-filter uses (5.17) with $\rho = 0.6$ and $\rho = 3.2$ – the latter value tuned to make ESf react at the same instant as PTSf.

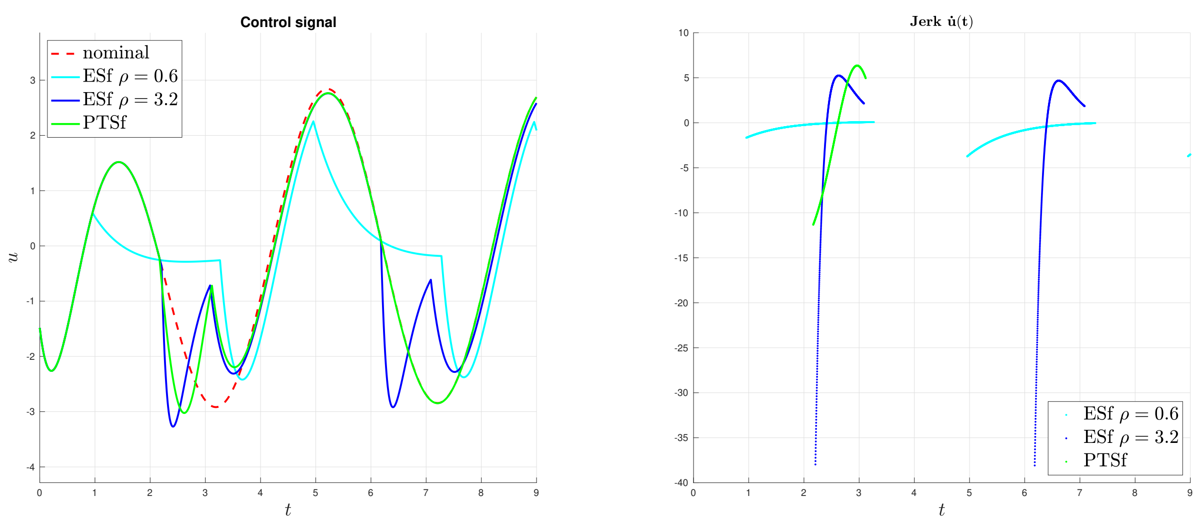

Figure 5.2. Left: Control signal. Right: Jerk during intervals when nominal command is overridden. When ESf is tuned ($\rho = 3.2$ case) to be less conservative like PTSf, the magnitude of the jerk increases significantly.

5.5 *N*th-order chain-of-integrators design

We recall the system of interest:

$$\begin{aligned} \dot{x}_i(t) &= x_{i+1}(t), \quad i = 1, \ldots, n-1, \\ \dot{x}_n(t) &= u(t), \\ y(t) &= x_1(t), \qquad t \geq t_0. \end{aligned} \tag{5.23}$$

We aim to design a safety-filter such that $y(t) \leq 0$ uniformly over the finite time horizon $[t_0, t_0 + T)$; that is, we wish to only enforce safety while it may be needed. We reproduce the barrier function treatment in Section 5.3 by defining the one-dimensional barrier functions, for $t \in [t_0, t_0 + T)$:

$$h_i := -x_i + \alpha_{i-1}(\underline{x}_{i-1}, t), \quad i = 1, \ldots, n, \tag{5.24}$$

$$\alpha_0(\underline{x}_0, t) \equiv 0, \tag{5.25}$$

$$\alpha_i(\underline{x}_i, t) := c_i \mu_2 h_i + \frac{d}{dt} \alpha_{i-1}(\underline{x}_{i-1}, t), \quad i = 1, \ldots, n, \tag{5.26}$$

with the safety-filter

$$u = \begin{cases} \min\left\{u_{\mathrm{nom}},\ \alpha_n(\underline{x}_n, t)\right\}, & \text{if } t_0 \leq t < t_0 + T, \\ u_{\mathrm{nom}} g(t, x_1(t_0 + T)), & \text{if } t \geq t_0 + T, \end{cases} \tag{5.27}$$

for the "ramp" function

$$g(t, x_1(t_0 + T)) := \begin{cases} 1 - \nu^m(t - t_0 - T, \bar{T}), & \text{if } x_1(t_0 + T) = 0, \\ & t_0 + T \leq t \leq t_0 + T + \bar{T} \\ 1, & \text{otherwise.} \end{cases} \tag{5.28}$$

where $m \in \mathbb{N}$ and $\bar{T} > 0$ are design parameters. The role of g in the product $u_{\text{nom}} g(t, x_1(t_0 + T))$ in (5.27) is to ensure that the control law is continuous at $t = t_0 + T$, since we will show that the feedback law $\alpha_n(\underline{x}_n, t_0 + T) = 0$ (cf. Section 5.6). Hence, (5.27) overrides a potentially "dangerous" nominal controller only for T time units.

Strictly speaking, the safety-filter $\min\{u_{\text{nom}},\ \alpha_n(\underline{x}_n, t)\}$ from (5.27) during times $t_0 \leq t < t_0 + T$ is the solution of the QP problem

$$\begin{aligned} u = \arg\min_{v \in \mathbb{R}} \ & |v - u_{\text{nom}}|^2 \ \text{ subject to} \\ & v \leq \alpha_n \end{aligned} \tag{5.29}$$

where constraint $v \leq \alpha_n$ is equivalent to $\frac{d}{dt} h_n + c_n \mu_2 h_n \geq 0$ under input v. Therefore, we refer to our design as a QP-backstepping PT-CBF design.

With the safety-filter (5.27), the CBFs satisfy

$$\frac{d}{dt} h_i = -c_i \mu_2 h_i + h_{i+1}, \quad i = 1, \ldots, n-1, \tag{5.30}$$

$$\frac{d}{dt} h_n \geq -c_n \mu_2 h_n, \tag{5.31}$$

for $t \in [t_0, t_0 + T)$. We can now state our main result.

Theorem 5.1. *If the system* (5.23) *is initially safe and away from the barrier, that is* $y(t_0) < -\varepsilon$, $\varepsilon > 0$, *then the controller* (5.24)–(5.27) *ensures that* $y(t) < 0$ *for all* $t \in [t_0, t_0 + T)$ *for the* initial *control gains*

$$c_i > \max\{c_{i-1}, \underline{c}_i\}, \quad i = 1, \ldots, n-1, \tag{5.32}$$

$$c_n > 0 \tag{5.33}$$

where

$$\underline{c}_i = \frac{x_{i+1}(t_0) - \frac{d}{dt}\alpha_{i-1}(\underline{x}_{i-1}(t_0), t_0)}{\alpha_{i-1}(\underline{x}_{i-1}(t_0), t_0) - x_i(t_0)}. \tag{5.34}$$

Moreover, the control law (5.27) *is uniformly bounded provided that* u_{nom} *is continuous*[2] *in the interval* $[t_0, t_0+T]$.

Remark 5.1. *While not characterized in Theorem 5.1, if the safety filter overrides the nominal controller over the time interval* $[t_0+\bar{t}, t_0+T)$ *for some* $\bar{t} < T$*, then the convergence of the CBFs to zero with be "infinitely-soft": in other words, all of the derivatives* $\frac{d^k h_i(t)}{dt^k}$, $k \in \mathbb{N}$, *will also converge to zero by the terminal time* t_0+T. *This also holds true for the x-system states. This is advantageous in many applications, e.g., when performing overriding safety maneuvers for vehicles, this feature ensures the maneuver to be "jerk-free" at the terminal time. See Section 5.6, and in particular,* (5.76) *and* (5.78) *for the mathematical treatment of this "infinitely-soft" convergence.*

We now pursue a proof of Theorem 5.1.

5.6 Proof of theorem 5.1

The structure of our proof comes in two parts: one to establish non-positivity of $y(t)$ for $t \in [t_0, t_0+T)$; and another to establish uniform boundedness of the control law which filters the nominal controller to enforce safety. While these two parts are intimately connected, each part requires different treatments. To ensure positivity of $h_1(t)$ for $t \in [t_0, t_0+T)$, it is enough to invoke the control barrier *constraint* (5.31) and our choice of control gains (5.32). On the other hand, due to the possible switches in control input between u_{nom} and α_n in (5.27), one must take care to ensure uniform boundedness of the control input $u = \alpha_n$, whose gains increase with time according to (5.26) *even when* $u = u_{\mathrm{nom}}$. To this end, we first present the following commutativity property of the "blow-up" function (5.2) which we will leverage to show controller boundedness. To simplify our presentation, we take $t_0 = 0$ henceforth.

[2]The result of Theorem 5.1 hold if u_{nom} is only bounded so long as the nominal system is well defined.

Lemma 5.2. *For $m \in \mathbb{N}$ and $0 \le \bar{t} \le t < T$, the "blow-up" function* (5.2) *satisfies*

$$\mu_m(t,T) = \mu_m(\bar{t},T)\mu_m(t-\bar{t},T-\bar{t}). \tag{5.35}$$

Proof. It follows directly from the definition (5.2):

$$\begin{aligned} \mu_m(t,T) &:= \frac{1}{\left(1-\frac{t}{T}\right)^m} && (5.36)\\ &= \frac{1}{\left(1-\frac{\bar{t}}{T}\right)^m}\frac{1}{\left(\frac{T-t}{T-\bar{t}}\right)^m} && (5.37)\\ &= \mu_m(\bar{t},T)\mu_m(t-\bar{t},T-\bar{t}). && (5.38) \end{aligned}$$

□

To demonstrate controller uniform boundedness, we must leverage the fact that our feedback law invokes PTSf whose convergence dominates the rate of divergence of the time-varying control gains in (5.26). To accomplish this, we characterize the following property of our closed-loop system.

Lemma 5.3. *For $c > 0$, the ith derivative of the function*

$$\xi(t) := e^{-cT(\mu_1(t,T)-1)} \tag{5.39}$$

satisfies

$$\lim_{t\to T^-} \frac{d^i\xi(t)}{dt^i} = \lim_{t\to T^-} \mathscr{P}^{2i}\left(\mu_1(t,T)\right)\xi(t) = 0, \quad i \in \mathbb{N}. \tag{5.40}$$

Proof. We compute the first derivative according to (5.5):

$$\frac{d\xi(t)}{dt} = -2c\mu_2(t,T)e^{-cT(\mu_1(t,T)-1)} \tag{5.41}$$

An application of l'Hôpital's rule to (5.41) twice verifies (5.40) for $i = 1$, since

$$\lim_{t \to T^-} \frac{d\xi(t)}{dt} = -2ce^{cT} \lim_{t \to T^-} \mu_2(t,T) e^{-cT\mu_1(t,T)} \tag{5.42}$$

$$= -2ce^{cT} \lim_{\tau \to +\infty} \frac{\tau^2}{e^{cT\tau}} = 0. \tag{5.43}$$

For successive derivatives, we rely on the general Leibniz rule to study the time-varying structure of the expression:

$$\frac{d^i\xi(t)}{dt^i} = -2c\frac{d^{i-1}}{dt^{i-1}}\left(\mu_2(t,T)e^{-cT(\mu_2(t,T)-1)}\right) \tag{5.44}$$

$$= -2c\sum_{k=0}^{i-1}\binom{i-1}{k}\mu_2^{(k)}(t,T)\frac{d^{i-k-1}\xi(t)}{dt^{i-k-1}} \tag{5.45}$$

$$= -2c\sum_{k=0}^{i-1}\frac{2^{\bar{k}}}{T^k}\binom{i-1}{k}\mu_{2+k}(t,T)\frac{d^{i-k-1}\xi(t)}{dt^{i-k-1}}. \tag{5.46}$$

We assume by induction that (5.40) holds for the $(i-1)$th derivative such that

$$\frac{d^{i-1}\xi(t)}{dt^{i-1}} = \mathscr{P}^{2(i-1)}\left(\mu_1(t,T)\right)\xi(t); \tag{5.47}$$

it follows from applying l'Hôpital's rule to (5.44) with (5.47) $i+2$ times that (5.40) holds for all $i \in \mathbb{N}$. $\square$

We can now proceed with the proof of Theorem 5.1, where we select $t_0 = 0$ for clarity.

Proof. We first pursue non-positivity of $y(t)$ under (5.27), (5.32), (5.34). The system beginning from safety, that is, $y(t_0) = x_1(t_0) < 0$, implies that $h_1(t_0) > 0$. We proceed by induction: suppose $h_i(t_0) > 0$ for some $i = 1, \ldots, n-1$; it follows from (5.30) and differentiating (5.24) along (5.23) that

$$h_{i+1}(t_0) = c_i h_i(t_0) + \frac{d}{dt} h_i(t_0) \tag{5.48}$$

$$= c_i h_i(t_0) - x_{i+1}(t_0) + \frac{d}{dt} \alpha_{i-1}(\underline{x}_{i-1}(t_0), t_0). \tag{5.49}$$

Our *initial* control gains (5.32), (5.34) are designed so that

$$c_i h_i(t_0) - x_{i+1}(t_0) + \frac{d}{dt} \alpha_{i-1}(\underline{x}_{i-1}(t_0), t_0) > 0, \tag{5.50}$$

where we've used (5.24). We now show that $h_i(t_0) > 0$ for all $i = 1, \ldots, n$ is a sufficient condition for non-negativity of $h_i(t)$ for $t \in [t_0, t_0 + T)$. Applying the Comparison lemma and variation of constants formula to (5.30), (5.31) gives

$$h_i(t) = h_i(t_0) e^{-c_i \int_{t_0}^{t} \mu_2(s)\, ds} + \int_{t_0}^{t} e^{-c_i \int_{\tau}^{t} \mu_2(s)\, ds} h_{i+1}(\tau)\, d\tau, \tag{5.51}$$

$$h_n(t) \geq h_n(t_0) e^{-c_n \int_{t_0}^{t} \mu_2(s)\, ds} \tag{5.52}$$

$$> 0, \tag{5.53}$$

for $t \in [t_0, t_0 + T)$. Substituting (5.53) into (5.51) for $i = n-1$ yields

$$h_{n-1}(t) \geq h_{n-1}(t_0) e^{-c_{n-1} \int_{t_0}^{t} \mu_2(s)\, ds} + h_n(t_0) \int_{t_0}^{t} \left[e^{-c_{n-1} \int_{\tau}^{t} \mu_2(s)\, ds} e^{-c_n \int_{t_0}^{\tau} \mu_2(s)\, ds} \right] d\tau \tag{5.54}$$

$$\geq h_{n-1}(t_0) e^{-c_{n-1} \int_{t_0}^{t} \mu_2(s)\, ds} \tag{5.55}$$

$$> 0. \tag{5.56}$$

By using (5.53), (5.56) and by proceeding by backwards strong induction, it follows that

$$h_1(t) \geq h_1(t_0) e^{-c_1 \int_{t_0}^{t} \mu_2(s)\, ds} \tag{5.57}$$

which is equivalent to

$$y(t) = x_1(t) \le x_1(t_0)e^{-c_1 \int_{t_0}^{t} \mu_2(s)\,ds} \tag{5.58}$$

$$< 0. \tag{5.59}$$

for all $t \in [t_0, t_0 + T)$.

We now pursue uniform boundedness of the the control law (5.27). We partition the time horizon $[0, T)$ into intervals for which the system is either deemed safe or unsafe according to our safety filter by defining

$$t_k := \begin{cases} \min\{t_{k-1} < t \le T : u_{\text{nom}}(t) = \alpha_n(\underline{x}_n, t)\}, & \text{if it exists,} \\ T, & \text{otherwise,} \end{cases} \tag{5.60}$$

for $k \in \mathbb{N}$ with $t_0 = 0$, where

$$[0, T) = \bigcup_{\substack{k \in \mathbb{N} \cup \{0\} \\ t_{k+1} \le T}} [t_k, t_{k+1}). \tag{5.61}$$

We have constructed this partition such that the control law (5.27) remains continuous at t_k, precluding Zeno behavior of the closed-loop system. Since the system is initially safe, $t_1 \neq T$ represents the first time that safety is enforced by (5.27). For $t \in [t_{2k}, t_{2k+1})$, $k \in \mathbb{N} \cup \{0\}$ and $t_{2k+1} \le T$, we define the CBFs

$$h_i^{2k} := -x_i + \alpha_{i-1}^{2k}(\underline{x}_{i-1}, t - t_{2k}), \tag{5.62}$$

$$\alpha_i^{2k}(\underline{x}_i, t - t_{2k}) := c_i^{2k} \mu_2(t - t_{2k}, T - t_{2k}) h_i^{2k} + \frac{d}{dt} \alpha_{i-1}^{2k}(\underline{x}_{i-1}, t - t_{2k}) \tag{5.63}$$

for $i = 1, \ldots, n$, with

$$\alpha_0^{2k}(\underline{x}_0, t - t_{2k}) \equiv 0. \tag{5.64}$$

It follows from (5.27) that during these intervals, the CBFs satisfy

$$\frac{d}{dt}h_i^{2k} = -c_i^{2k}\mu_2(t-t_{2k}, T-t_{2k})h_i^{2k} + h_{i+1}^{2k}, \tag{5.65}$$
$$\frac{d}{dt}h_n^{2k} = -u_{\text{nom}} + \frac{d}{dt}\alpha_{n-1}^{2k}(\underline{x}_{n-1}, t-t_{2k}), \tag{5.66}$$

for $i = 1, \ldots, n-1$. Similarly, for $t \in [t_{2k-1}, t_{2k})$, $k \in \mathbb{N}$ and $t_{2k} \leq T$, we define the CBFs

$$h_i^{2k-1} := -x_i + \alpha_{i-1}^{2k-1}(\underline{x}_{i-1}, t-t_{2k-1}), \tag{5.67}$$
$$\alpha_i^{2k-1}(\underline{x}_i, t-t_{2k-1}) := \frac{d}{dt}\alpha_{i-1}^{2k-1}(\underline{x}_{i-1}, t-t_{2k-1}) + c_i^{2k-1}\mu_2(t-t_{2k-1}, T-t_{2k-1})h_i^{2k-1} \tag{5.68}$$

for $i = 1, \ldots, n$, with

$$\alpha_0^{2k-1}(\underline{x}_0, t-t_{2k-1}) \equiv 0. \tag{5.69}$$

It follows from (5.27) adapted as $u = \alpha_n^{2k-1}(\underline{x}_n, t-t_{2k-1})$ that during these intervals, the CBFs satisfy

$$\frac{d}{dt}h_i^{2k-1} = -c_i^{2k-1}\mu_2(t-t_{2k-1}, T-t_{2k-1})h_i^{2k-1} + h_{i+1}^{2k-1}, \tag{5.70}$$
$$\frac{d}{dt}h_n^{2k-1} = -c_n^{2k-1}\mu_2(t-t_{2k-1}, T-t_{2k-1})h_n^{2k-1}. \tag{5.71}$$

for $i = 1, \ldots, n-1$. We select $c_i^0 = c_i$ according to (5.32), (5.34), and we select

$$c_i^k = c_i^{k-1}\mu_2(t_k - t_{k-1}, T - t_{k-1}), \quad k \in \mathbb{N}. \tag{5.72}$$

Since $\alpha_0^{2k}(\underline{x}_0, t-t_{2k}) = \alpha_0^{2k-1}(\underline{x}_0, t-t_{2k-1}) \equiv 0$, it follows that $h_1^{2k}(t_{2k-1}) = h_1^{2k-1}(t_{2k-1})$ for $k \in \mathbb{N}$. Furthermore, by applying our initial gain selection (5.72) to (5.67), (5.68) and comparing them to (5.62), (5.63) at $t = t_{2k-1}$, we deduce that $h_i^{2k}(t_{2k-1}) = h_i^{2k-1}(t_{2k-1})$ for $i = 2, \ldots, n$. The same treatment leads to the equalities $h_i^{2k}(t_{2k}) = h_i^{2k-1}(t_{2k})$ for $i = 1, \ldots, n$. Hence, our initial gain selection (5.72) for each time partition in (5.61) ensures that the system dynamics remain continuous at every time. In fact, it simply tracks the growth of the "blow-up" function μ_2 over the time intervals.

Furthermore, we can leverage Lemma 5.2 and our initial gain selection (5.72) to show that

$$\prod_{k\in\mathbb{N}} c_i^k \mu_2(t-t_k, T-t_k) = \mu_2(t,T); \tag{5.73}$$

in other words, the CBF design over the partitioned set (5.61) is consistent with the design (5.24)–(5.34).

For $t \in [t_{2k}, t_{2k+1})$, $k \in \mathbb{N}\cup\{0\}$ and $t_{2k+1} \leq T$, the system is safe and the nominal control—which we assume to be uniformly bounded (continuous over a compact time interval)—is being applied. For $t \in [t_{2k-1}, t_{2k})$, $k \in \mathbb{N}$ and $t_{2k} \leq T$, we must estimate the size of the time-varying input to verify that it is bounded.

To this end, we first study the stability of (5.70), (5.71). We can solve (5.71) explicitly to obtain

$$h_n^{2k-1}(t) = e^{-c_n^{2k-1}(T-t_{2k-1})(\mu_1(t-t_{2k-1},T-t_{2k-1})-1)} h_n^{2k-1}(t_{2k-1}), \tag{5.74}$$

whereas for $i = 1, \dots, n-1$, we have the relationship

$$\begin{aligned} h_i^{2k-1}(t) = {}& e^{-c_i^{2k-1}(T-t_{2k-1})(\mu_1(t-t_{2k-1},T-t_{2k-1})-1)} h_i^{2k-1}(t_{2k-1}) \\ & + \int_{t_{2k-1}}^{t} e^{-c_i^{2k-1}\int_\tau^t \mu_2(z-t_{2k-1},T-t_{2k-1})dz} h_{i+1}(\tau) d\tau. \end{aligned} \tag{5.75}$$

We apply Lemma 5.3 to (5.74) to establish that successive derivatives of (5.74) will converge to zero by the terminal time:

$$\lim_{t\to T^-} \frac{d^r h_n^{2k-1}(t)}{dt^r} = 0, \quad t \in [t_{2k-1}, T^-], \tag{5.76}$$

for $r \in \mathbb{N} \cup \{0\}$. For $i = 1, \ldots, n-1$, we compute

$$\frac{dh_i^{2k-1}(t)}{dt} = h_{i+1}(t) + h_i^{2k-1}(t_{2k-1}) \frac{d}{dt} \left(e^{-c_i^{2k-1}(T-t_{2k-1})(\mu_1(t-t_{2k-1}, T-t_{2k-1})-1)} \right). \tag{5.77}$$

By applying Lemma 5.3 to the second term within (5.77), and by backward strong induction on (5.76), we get for $r \in \mathbb{N} \cup \{0\}$

$$\lim_{t \to T^-} \frac{d^r h_i^{2k-1}(t)}{dt^r} = 0, \quad i = 1, \ldots, n-1. \tag{5.78}$$

Hence, we have shown that when the nominal controller is overridden by our safety filter to enforce safety during $t \in [t_{2k-1}, t_{2k})$, $k \in \mathbb{N}$ and $t_{2k} \leq T$, our time-varying backstepping design ensures that the CBF converge very smoothly to zero by the terminal time—indeed, all of their derivatives also converge to zero by the terminal time.

By using (5.70) within the derivative term of (5.68) in an iterative fashion, we can verify by induction that (5.68) for $i = n$ is equivalent to

$$\begin{aligned} \alpha_n^{2k-1}(\underline{x}_n, t - t_{2k-1}) = \sum_{r=1}^{n-1} \frac{d^r}{dt^r} \left(h_{n-r}^{2k-1} - \frac{d}{dt} h_{n-r-1}^{2k-1} \right) \\ + c_n^{2k-1} \mu_2(t - t_{2k-1}, T - t_{2k-1}) h_n^{2k-1}. \end{aligned} \tag{5.79}$$

We conclude from (5.76), (5.78) and applying l'Hôpital's rule to (5.79) as before that

$$\left| \alpha_n^{2k-1}(\underline{x}_n, t - t_{2k-1}) \right| < +\infty, \quad \forall\, t \in [t_{2k-1}, t_{2k}), \tag{5.80}$$

for $k \in \mathbb{N}$ and $t_{2k} \leq T$. This concludes our proof of controller uniform boundedness. □

Corollary 5.3.1 (Rescue to Safety in Prescribed Time)**.** *If the system* (5.23) *is initially unsafe, that is, if* $y(t_0) > \varepsilon$, $\varepsilon > 0$, *then the controller* (5.24)–(5.27) *ensures that* $y(t_0 + T) = 0$ *for the initial control gains selected as in* (5.32), (5.33). *Moreover the control law* (5.27) *is uniformly bounded provided that* u_{nom} *is continuous in the interval* $[t_0, t_0 + T]$.

Proof. The proof follows by noticing that the barrier function definitions (5.24), choice of initial gains (5.32)–(5.33), and control law (5.27) lead to the same target system (5.30)–(5.31), but with initial condition $h_1(t_0) < 0$, $h_i(t_0) > 0$, for $i = 2, \ldots, n$. Using a similar argument as in Theorem 5.1, we arrive at the same lower bound on $h_1(t)$ in (5.57) which leads to

$$y(t_0 + T) = h_1(t_0 + T) = 0. \tag{5.81}$$

□

5.7 Preservation of nominal prescribed-time stabilization performance

One advantage of PTSf safety filters over infinite-time ESf filters is the ability to enforce safety without destroying finite-time performance of the underlying nominal system. In particular, for systems where the nominal control objective is to be achieved in finite time as is the case in prescribed-time stabilization (PTS), the ephemeral nature of PTSf makes it possible to enforce safety, and retain the finite properties of the nominal controller. To illustrate this advantage, we consider the following example. Consider the following nonlinear strict-feedback system

$$\begin{aligned} \dot{x}_1(t) &= x_1^2(t) + x_2(t) \\ \dot{x}_2(t) &= u(t) \\ y(t) &= x_1(t), \quad t \geq t_0, \end{aligned} \tag{5.82}$$

where the safety-objective is to keep the system output $y(t)$ negative for all times $t \in [0, T)$ with $T = 1$. The nominal controller is designed to stabilize the system to the origin in the same prescribed time $T = 1$ time units. We first transform the system into double integrator form using state transformation

$$z_1(t) = x_1(t), \tag{5.83}$$

$$z_2(t) = x_1^2(t) + x_2(t), \tag{5.84}$$

leading to

$$\begin{aligned}\dot{z}_1(t) &= z_2(t) \\ \dot{z}_2(t) &= v(t) \\ y(t) &= z_1(t), \quad t \geq t_0.\end{aligned} \tag{5.85}$$

with

$$v(t) = 2z_1(t)z_2(t) + u(t). \tag{5.86}$$

For the nominal controller, we use a prescribed-time stabilizing (PTS) controller as in [45] for the transformed system. Specifically, we use

$$v_{\text{nom}}(t) = -K\mu_2(t)z(t) \tag{5.87}$$

$$K = \begin{bmatrix}\omega^2 & 2\zeta\omega\end{bmatrix} \tag{5.88}$$

with $\omega = 4$ and $\zeta = 0.1$. Starting from initial state $(x_1(0), x_2(0)) = (-1, -1)$ which corresponds to $(z_1(0), z_2(0)) = (-1, 0)$, we apply the ESf filter

$$v(t) = \min\left\{v_{\text{nom}}(t),\ -3\rho z_2 - 2\rho^2 z_1\right\} \tag{5.89}$$

with $\rho = 1.2$ (low-gain ESf) and $\rho = 10$ (high-gain ESf); and the PTSf filter

$$v(t) = \min\left\{v_{\text{nom}}(t),\ -(c_1 + c_2)\mu_2 z_2 - c_1\left(c_2\mu_2 + \frac{2\mu_1}{T}\right)\mu_2 z_1\right\} \tag{5.90}$$

with initial gains $c_1 = c_2 = 1$. In all three cases, the resulting control v for the transformed z-system is used for the original x-system via

$$u(t) = v(t) - 2x_1(x_1^2 + x_2). \tag{5.91}$$

The low-gain ESf filter ($\rho = 1.2$) is tuned to be initially as restrictive as the PTSf controller, overriding the nominal control by the same amount, while the high-gain ESf filter ($\rho = 10$) is tuned so that the output arrives sufficiently close to the boundary of the safe-region by the terminal time, thereby retaining the PTS property as much as possible. The resulting plots are shown in Figure 5.3 and Figure 5.4. While the PTSf filter initially limits the rate of approach to the $y(t) = 0$ boundary to enforce safety, it still preserves the prescribed-time stabilization property of the nominal PTS controller. For the high-gain ESf filter, the system gets sufficiently close to the boundary of the safe-set at the terminal time; however, choosing the appropriate ESf

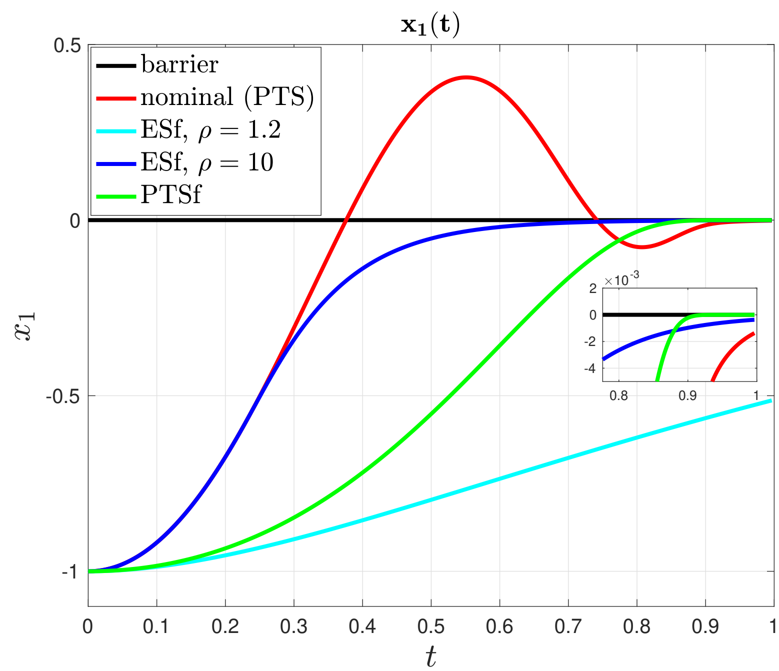

Figure 5.3. Output for strict-feedback system (5.82) under nominal PTS controller (5.87) with terminal time $T = 1$, a PTSf filter with terminal time $T = 1$, an ESf filter with $\rho = 1.2$, and an ESf filter with $\rho = 10$.

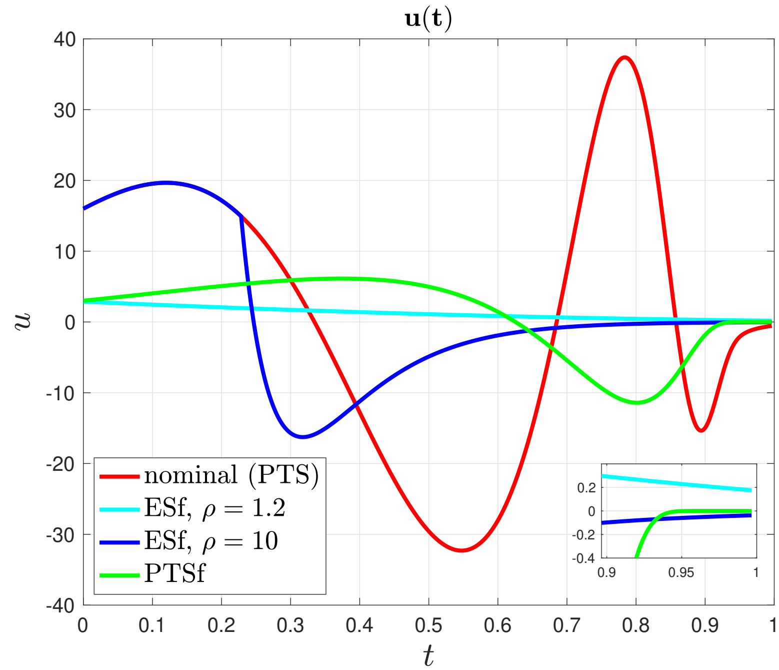

Figure 5.4. Input signal for strict-feedback system (5.82) under nominal PTS controller (5.87) with terminal time $T = 1$, a PTSf filter with terminal time $T = 1$, an ESf filter with $\rho = 1.2$, and an ESf filter with $\rho = 10$.

gain ρ requires more effort from the designer, unlike in PTSf where the designer only chooses the terminal time to match the terminal time of the nominal PTS controller, and gains c_1, c_2 according to the explicit prescription in (5.32). Also evident in the figures is the infinitely soft convergence of the output, internal state x_2 (Figure 5.5), and input under the PTSf filter — a property that is of importance in practical applications.

5.8 Conclusion and acknowledgements

In this chapter, we present a safety filter design for strict-feedback systems that enforces state constraints for an a priori prescribed time. Our design uses a time-varying backstepping transformation with gains that grow as time approaches the terminal time. Despite the use of gains that grow towards infinity, we show that our safegaurding controller remains uniformly bounded provided that the nominal controller is uniformly bounded.

Compelling future research directions include: studying predictor-based safety filter

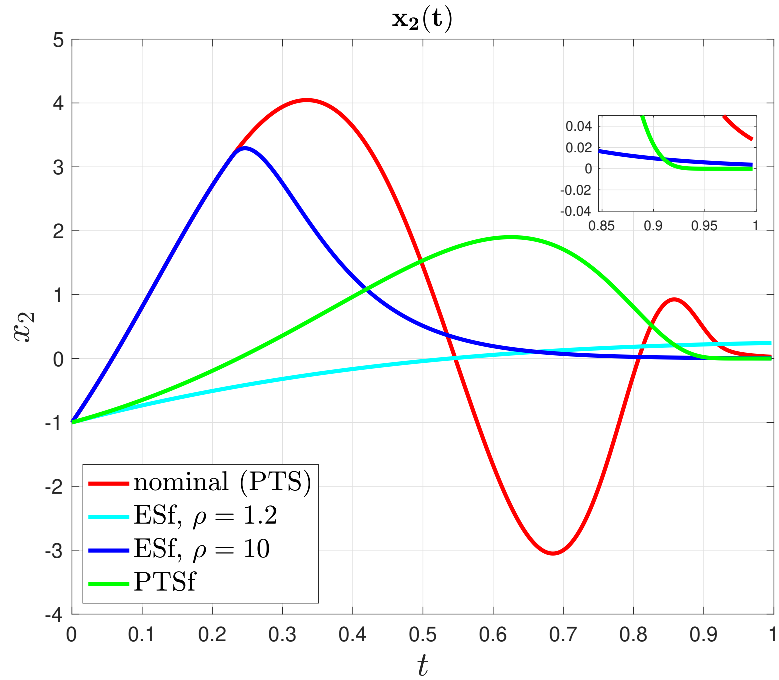

Figure 5.5. Internal state x_2 for strict-feedback system (5.82) under nominal PTS controller (5.87) with terminal time $T = 1$, a PTSf filter with terminal time $T = 1$, an ESf filter with $\rho = 1.2$, and an ESf filter with $\rho = 10$.

designs to compensate for input delays which are omnipresent in applications, characterizing and compensating for the effect of persistent disturbances appearing on the right-hand side of the dynamics, and developing a discretization algorithm that preserves the properties of our prescribed-time safety filter. Lastly, QP-based safety filters are only pointwise optimal and do not minimize a meaningful cost function over the duration of safety enforcement. However, recent results on the design of inverse optimal (infinite-time) safety filters [49], and inverse optimal prescribed-time stabilizing controllers [53] have made promising the prospect of studying the design of inverse-optimal prescribed-time safety filters.

Chapter 5, in part, is a reprint and adaptation of the following papers: (1) I. Abel, D. Steeves, M. Krstić, and M. Janković, "Prescribed-Time Safety Design for a Chain of Integrators", IEEE American Control Conference (ACC), 2022. (2) I. Abel, D. Steeves, M. Krstić, and M. Janković, "Prescribed-Time Safety Design for Strict-Feedback Nonlinear Systems", in preparation for publication in the IEEE Transactions on Automatic Control. The dissertation author was the primary investigator and author of both papers.

Bibliography

[1] I. Abel, M. Janković, and M. Krstić. Constrained stabilization of multi-input linear systems with distinct input delays. *IFAC Workshop on Control of Systems Governed by Partial Differential Equations (CPDE)*, 2019.

[2] I. Abel, M. Janković, and M. Krstić. Constrained control of input delayed systems with partially compensated input delays. *ASME Dynamic Systems and Controls Conference (DSCC)*, 2020.

[3] M. Ahmadi, A. Singletary, J. W. Burdick, and A. D. Ames. Safe policy synthesis in multi-agent POMDPs via discrete-time barrier functions. In *arXiv, https://arxiv.org/abs/1903.07823*, 2019.

[4] A. D. Ames, J. W. Grizzle, and P. Tabuada. Control barrier function based quadratic programs with application to adaptive cruise control. In *53rd IEEE Conference on Decision and Control*, pages 6271–6278, 2014.

[5] A. D. Ames, T. G. Molnár, A. W. Singletary, and G. Orosz. Safety-critical control of active interventions for covid-19 mitigation. *medRxiv*, 2020.

[6] A. D. Ames, X. Xu, J. W. Grizzle, and P. Tabuada. Control barrier function based quadratic programs for safety critical systems. *IEEE Transactions on Automatic Control*, 62(8):3861–3876, 2017.

[7] Z. Artstein. Stabilization with relaxed controls. *Nonlinear Analysis*, 7(11):1163 – 1173, 1983.

[8] S. Bansal, M. Chen, S. Herbert, and C. J. Tomlin. Hamilton-jacobi reachability: A brief overview and recent advances. In *2017 IEEE 56th Annual Conference on Decision and Control (CDC)*, pages 2242–2253, 2017.

[9] N. Bekiaris-Liberis and M. Krstić. Compensation of Time-Varying Input and State Delays for Nonlinear Systems. *Journal of Dynamic Systems, Measurement, and Control*, 134(1), 12 2011.

[10] N. Bekiaris-Liberis and M. Krstić. *Nonlinear Control Under Nonconstant Delays*. Society for Industrial and Applied Mathematics, Philadelphia, PA, 2013.

[11] N. Bekiaris-Liberis and M. Krstić. Predictor-feedback stabilization of multi-input nonlinear systems. *IEEE Transactions on Automatic Control*, 62(2):516–531, Feb 2017.

[12] M. Bement and S. Jayasuriya. Construction of a set of nonovershooting tracking controllers. *J. Dyn. Sys., Meas., Control*, 126(3):558–567, 2004.

[13] S. P. Bhat and D. S. Bernstein. Finite-time stability of continuous autonomous systems. *SIAM Journal on Control and Optimization*, 38(3):751–766, 2000.

[14] U. Borrmann, L. Wang, A. D. Ames, and M. Egerstedt. Control barrier certificates for safe swarm behavior. *IFAC-PapersOnLine*, 48(27):68 – 73, 2015.

[15] J. Breeden and D. Panagou. High relative degree control barrier functions under input constraints. In *60th IEEE Conference on Decision and Control (CDC)*, pages 6119–6124, 2021.

[16] M. Chen, S. Herbert, and C. J. Tomlin. Exact and efficient hamilton-jacobi guaranteed safety analysis via system decomposition. In *2017 IEEE International Conference on Robotics and Automation (ICRA)*, pages 87–92, 2017.

[17] M. Chen, S. L. Herbert, H. Hu, Y. Pu, J.F. Fisac, S. Bansal, S. Han, and C. J. Tomlin. Fastrack: A modular framework for real-time motion planning and guaranteed safe tracking. *IEEE Transactions on Automatic Control*, 66(12):5861–5876, 2021.

[18] J. J. Choi, D. Lee, K. Sreenath, C. J. Tomlin, and S. L. Herbert. Robust control barrier–value functions for safety-critical control. In *2021 60th IEEE Conference on Decision and Control (CDC)*, pages 6814–6821, 2021.

[19] A. Clark. Control barrier functions for stochastic systems. *Automatica*, 130:109688, 2021.

[20] M. B. G. Cloosterman, N. van de Wouw, W. P. M. H. Heemels, and H. Nijmeijer. Stability of networked control systems with uncertain time-varying delays. *IEEE Transactions on Automatic Control*, 54(7):1575–1580, 2009.

[21] N. Espitia, A. Polyakov, D. Efimov, and W. Perruquetti. Boundary time-varying feedbacks for fixed-time stabilization of constant-parameter reaction-diffusion systems. *Automatica*, 103:398 – 407, 2019.

[22] N. Espitia, D. Steeves, W. Perruquetti, and M. Krstić. Sensor delay-compensated prescribed-time observer for lti systems. *Automatica*, 135:110005, 2022.

[23] R. Freeman and P. Kokotovic. *Robust Control of Nonlinear Systems*. Birkhauser, Boston, 1996.

[24] K. Garg, R. K. Cosner, U. Rosolia, A. D. Ames, and D. Panagou. Multi-rate control design under input constraints via fixed-time barrier functions. *IEEE Control Systems Letters*, 6:608–613, 2022.

[25] A. Ghaffari, I. Abel, D. Ricketts, S. Lerner, and M. Krstić. Safety verification using barrier certificates with application to double integrator with input saturation and zero-order hold. In *2018 Annual American Control Conference (ACC)*, pages 4664–4669, June 2018.

[26] P. Glotfelter, J. Cortés, and M. Egerstedt. Nonsmooth barrier functions with applications to multi-robot systems. *IEEE Control Systems Letters*, 1(2):310–315, 2017.

[27] V. T. Haimo. Finite time controllers. *SIAM Journal on Control and Optimization*, 24(4):760–770, 1986.

[28] V. T. Haimo. Finite time controllers. *SIAM Journal on Control and Optimization*, 24(4):760–770, 1986.

[29] C. R. He and G. Orosz. Safety guaranteed connected cruise control. In *2018 21st International Conference on Intelligent Transportation Systems (ITSC)*, pages 549–554, 2018.

[30] J. Holloway and M. Krstić. Prescribed-time observers for linear systems in observer canonical form. *IEEE Transactions on Automatic Control*, 64(9):3905–3912, 2019.

[31] J. Holloway and M. Krstić. Prescribed-time output feedback for linear systems in controllable canonical form. *Automatica*, 107:77–85, 2019.

[32] S. Hsu, X. Xu, and A. D. Ames. Control barrier function based quadratic programs with application to bipedal robotic walking. In *2015 American Control Conference (ACC)*, pages 4542–4548, 2015.

[33] M. Janković. Combining control Lyapunov and barrier functions for constrained stabilization of nonlinear systems. In *American Control Conference*, pages 1916–1922, 2017.

[34] M. Janković. Control barrier functions for constrained control of linear systems with input delay. In *2018 Annual American Control Conference (ACC)*, pages 3316–3321, June 2018.

[35] M. Janković. Robust control barrier functions for constrained stabilization of nonlinear systems. *Automatica*, 96:359 – 367, 2018.

[36] I. Karafyllis. Stabilization by means of approximate predictors for systems with delayed input. *IFAC Proceedings Volumes*, 43(2):9 – 14, 2010.

[37] I. Karafyllis and M. Krstić. Numerical schemes for nonlinear predictor feedback. *Mathematics of Control, Signals, and Systems*, 26(4):519–546, 2014.

[38] I. Karafyllis and M. Krstić. *Predictor Feedback for Delay Systems: Implementations and Approximations*. Birkhauser, Cambridge, MA, 2017.

[39] H. K. Khalil. *Nonlinear systems; 3rd ed.* Prentice-Hall, Upper Saddle River, NJ, 2002.

[40] H. K. Khalil and F. Esfandiari. Semiglobal stabilization of a class of nonlinear systems using output feedback. In *Proceedings of the 31st IEEE Conference on Decision and Control*, pages 3423–3428. IEEE, 1992.

[41] M. Kimura. A new approach to the perfect regulation and the bounded peaking in linear multivariable control systems. *IEEE Transactions on Automatic Control*, 26(1):253–270, 1981.

[42] A. K. Kiss, T. G. Molnar, A. D. Ames, and G. Orosz. Control barrier functionals: Safety-critical control for time delay systems. In *arXiv, https://arxiv.org/abs/2206.08409*, 2021.

[43] S. Kolathaya and A. D. Ames. Input-to-state safety with control barrier functions. *IEEE Control Systems Letters*, 3(1):108–113, Jan 2019.

[44] R. Konda, E. Squires, P. Pierpaoli, M. Egerstedt, and S. Coogan. Provably-safe autonomous navigation of traffic circles. In *2019 IEEE Conference on Control Technology and Applications (CCTA)*, pages 876–881, 2019.

[45] P. Krishnamurthy, F. Khorrami, and M. Krstić. Robust adaptive prescribed-time stabilization via output feedback for uncertain nonlinear strict-feedback-like systems. *European Journal of Control*, 55:14–23, 2020.

[46] M. Krstić. *Delay Compensation for Nonlinear, Adaptive, and PDE Systems*. Birkhauser, Cambridge, MA, 2009.

[47] M. Krstić. Input delay compensation for forward complete and strict-feedforward nonlinear systems. *IEEE Transactions on Automatic Control*, 55(2):287–303, 2010.

[48] M. Krstić. Lyapunov stability of linear predictor feedback for time-varying input delay. *IEEE Transactions on Automatic Control*, 55(2):554–559, 2010.

[49] M. Krstić. Inverse optimal safety filters. In *arXiv, https://arxiv.org/abs/2112.08225*, 2021. submitted to *IEEE Transactions on Automatic Control*.

[50] M. Krstić and M. Bement. Nonovershooting control of strict-feedback nonlinear systems. *IEEE Transactions on Automatic Control*, 51(12):1938–1943, 2006.

[51] S. Li, H. Du, and X. Lin. Finite-time consensus algorithm for multi-agent systems with double-integrator dynamics. *Automatica*, 47(8):1706–1712, 2011.

[52] W. Li and M. Krstić. Mean-nonovershooting control of stochastic nonlinear systems. *IEEE Transactions on Automatic Control*, 2020.

[53] W. Li and M. Krstić. Stochastic nonlinear prescribed-time stabilization and inverse optimality. *IEEE Transactions on Automatic Control*, 2021.

[54] K. Liu and X. Sun. Stabilization of nonlinear systems with time-varying input and state delays by approximate predictor. *IEEE Transactions on Automatic Control*, 63(10):3449–3455, 2018.

[55] K. Long, C. Qian, J. Cortés, and N. Atanasov. Learning barrier functions with memory for robust safe navigation. *IEEE Robotics and Automation Letters*, 6(3):4931–4938, 2021.

[56] M. Maghenem and R. G. Sanfelice. Characterizations of safety in hybrid inclusions via barrier functions. In *Proceedings of the 22Nd ACM International Conference on Hybrid Systems: Computation and Control*, HSCC '19, pages 109–118, New York, NY, USA, 2019. ACM.

[57] T. G. Molnár, A. W. Singletary, G. Orosz, and A. D. Ames. Safety-critical control of compartmental epidemiological models with measurement delays. *IEEE Control Systems Letters*, 5(5):1537–1542, 2021.

[58] L. A. Montestruque and P. Antsaklis. Stability of model-based networked control systems with time-varying transmission times. *IEEE Transactions on Automatic Control*, 49(9):1562–1572, 2004.

[59] Q. Nguyen and K. Sreenath. Exponential control barrier functions for enforcing high relative-degree safety-critical constraints. In *American Control Conference*, pages 322–328, 2016.

[60] M. T. Nihtila. Finite pole assignment for systems with time-varying input delays. In *Proceedings of the 30th IEEE Conference on Decision and Control*, volume 1, pages 927–928, 1991.

[61] M. Ohnishi, G. Notomista, M. Sugiyama, and M. Egerstedt. Constraint learning for control tasks with limited duration barrier functions. *Automatica*, 127:109504, 2021.

[62] G. Orosz and A. D. Ames. Safety functionals for time delay systems. In *2019 American Control Conference (ACC)*, pages 4374–4379, July 2019.

[63] G. Orosz and G. Stepan. Subcritical hopf bifurcations in a car-following model with reaction-time delay. *Proceedings of the Royal Society London*, 462:2643–2670, 2006.

[64] W. Perruquetti, T. Floquet, and E. Moulay. Finite-time observers: Application to secure communication. *IEEE Transactions on Automatic Control*, 53(1):356–360, 2008.

[65] S. F. Phillips and D. E. Seborg. Conditions that guarantee no overshoot for linear systems. *International Journal of Control*, 47(4):1043–1059, 1988.

[66] A. Polyakov. Nonlinear feedback design for fixed-time stabilization of linear control systems. *IEEE Transactions on Automatic Control*, 57(8):2106–2110, 2012.

[67] A. Polyakov, J.-M. Coron, and L. Rosier. On homogeneous finite-time control for linear evolution equation in hilbert space. *IEEE Transactions on Automatic Control*, 63(9):3143–3150, 2018.

[68] A. Polyakov, D. Efimov, and W. Perruquetti. Finite-time and fixed-time stabilization: Implicit lyapunov function approach. *Automatica*, 51:332–340, 2015.

[69] A. Polyakov, D. Efimov, and W. Perruquetti. Finite-time and fixed-time stabilization: Implicit lyapunov function approach. *Automatica*, 51:332–340, 2015.

[70] A. Polyakov and M. Krstić. Finite-and Fixed-Time Nonovershooting Stabilizers and Safety Filters by Homogeneous Feedback. submitted to *IEEE Transactions on Automatic Control.*, January 2022.

[71] S. Prajna and A. Jadbabaie. *Safety Verification of Hybrid Systems Using Barrier Certificates*, pages 477–492. Springer Berlin Heidelberg, Berlin, Heidelberg, 2004.

[72] S. Prajna and A. Jadbabaie. Methods for safety verification of time-delay systems. In *Proceedings of the 44th IEEE Conference on Decision and Control*, pages 4348–4353, 2005.

[73] S. Prajna, A. Jadbabaie, and G. J. Pappas. A framework for worst-case and stochastic safety verification using barrier certificates. *IEEE Transactions on Automatic Control*, 52(8):1415–1428, 2007.

[74] Y. Rahman, M. Janković, and M. Santillo. Driver intent prediction with barrier functions. In *2021 American Control Conference (ACC)*, pages 224–230, 2021.

[75] S. V. Raković, S. Zhang, Y. Hao, L. Dai, and Y. Xia. Convex mpc for exclusion constraints. *Automatica*, 127:109502, 2021.

[76] W. Ren. Razumikhin-type control lyapunov and barrier functions for time-delay systems. In *2021 60th IEEE Conference on Decision and Control (CDC)*, pages 5471–5476, 2021.

[77] W. Ren, R. M. Jungers, and D. V. Dimarogonas. Razumikhin and krasovskii approaches for safe stabilization. In *arXiv, https://arxiv.org/abs/2204.12106*, 2022.

[78] A. Saberi, J. Han, and A. A. Stoorvogel. Constrained stabilization problems for linear plants. *Automatica*, 38(4):639–654, 2002.

[79] M. Santillo and M. Janković. Collision free navigation with interacting, non-communicating obstacles. In *2021 American Control Conference (ACC)*, pages 1637–1643, 2021.

[80] C. Santoyo, M. Dutreix, and S. Coogan. A barrier function approach to finite-time stochastic system verification and control. *Automatica*, 125:109439, 2021.

[81] R. Schmid and L. Ntogramatzidis. A unified method for the design of nonovershooting linear multivariable state-feedback tracking controllers. *Automatica*, 46(2):312–321, 2010.

[82] A. Singletary, Y. Chen, and A. D. Ames. Control barrier functions for sampled-data systems with input delays. In *2020 59th IEEE Conference on Decision and Control (CDC)*, pages 804–809, 2020.

[83] R. Sipahi, F. M. Atay, and S. Niculescu. Stability of traffic flow behavior with distributed delays modeling the memory effects of the drivers. *SIAM Journal on Applied Mathematics*, 68(3):738–759, 2008.

[84] Y. Song, Y. Wang, J. Holloway, and M. Krstić. Time-varying feedback for regulation of normal-form nonlinear systems in prescribed finite time. *Automatica*, 83:243–251, 2017.

[85] Y. Song, Y. Wang, J. Holloway, and M. Krstić. Time-varying feedback for regulation of normal-form nonlinear systems in prescribed finite time. *Automatica*, 83:243–251, 2017.

[86] E. D. Sontag. A 'universal' construction of Artstein's theorem on nonlinear stabilization. *Systems & Control Letters*, 13(2):117 – 123, 1989.

[87] E. Squires, P. Pierpaoli, and M. Egerstedt. Constructive barrier certificates with applications to fixed-wing aircraft collision avoidance. In *2018 IEEE Conference on Control Technology and Applications (CCTA)*, pages 1656–1661, 2018.

[88] D. Steeves, N. Espitia, M. Krstić, and W. Perruquetti. Input delay compensation in prescribed-time of boundary-actuated reaction-diffusion pdes. In *2021 American Control Conference (ACC)*, pages 274–279, 2021.

[89] D. Steeves and M. Krstić. Prescribed-time stabilization of odes with diffusive actuator dynamics. *IFAC-PapersOnLine*, 54(9):434–439, 2021.

[90] D. Steeves, M. Krstić, and R. Vazquez. Prescribed-time h^1-stabilization of reaction-diffusion equations by means of output feedback. In *2019 18th European Control Conference (ECC)*, pages 1932–1937. IEEE, 2019.

[91] D. Steeves, M. Krstić, and R. Vazquez. Prescribed–time estimation and output regulation of the linearized schrödinger equation by backstepping. *European Journal of Control*, 2020.

[92] H. J. Sussmann and P. V. Kokotovic. The peaking phenomenon and the global stabilization of nonlinear systems. *IEEE Transactions on Automatic Control*, 36(4):424–440, 1991.

[93] A. J. Taylor, P. Ong, J. Cortés, and A. D. Ames. Safety-critical event triggered control via input-to-state safe barrier functions. *IEEE Control Systems Letters*, 5(3):749–754, 2021.

[94] K. P. Tee, S. S. Ge, and E. H. Tay. Barrier Lyapunov functions for the control of output-constrained nonlinear systems. *Automatica*, 45:918–927, 2009.

[95] D. Tsubakino, M. Krstić, and T. R. Oliveira. Exact predictor feedbacks for multi-input LTI systems with distinct input delays. *Automatica*, 71:143–150, 2016.

[96] L. Wang, A. D. Ames, and M. Egerstedt. Safety barrier certificates for collisions-free multirobot systems. *IEEE Transactions on Robotics*, 33(3):661–674, 2017.

[97] L. Wang, D. Han, and M. Egerstedt. Permissive barrier certificates for safe stabilization using sum-of-squares. In *2018 Annual American Control Conference (ACC)*, pages 585–590, 2018.

[98] P. Wieland and F. Allgöwer. Constructive safety using control barrier functions. *IFAC Proceedings Volumes*, 40(12):462 – 467, 2007. 7th IFAC Symposium on Nonlinear Control Systems.

[99] G. Wu and K. Sreenath. Safety-critical and constrained geometric control synthesis using control lyapunov and control barrier functions for systems evolving on manifolds. In *2015 American Control Conference (ACC)*, pages 2038–2044, 2015.

[100] W. Xiao and C. Belta. Control barrier functions for systems with high relative degree. In *2019 IEEE 58th Conference on Decision and Control (CDC)*, pages 474–479. IEEE, 2019.

[101] X. Xu. Constrained control of input-output linearizable systems using control sharing barrier functions. *Automatica*, 87:195 – 201, 2018.

[102] X. Xu, J. W. Grizzle, P. Tabuada, and A. D. Ames. Correctness guarantees for the composition of lane keeping and adaptive cruise control. *IEEE Transactions on Automation Science and Engineering*, 15(3):1216–1229, 2018.

[103] X. Xu, P. Tabuada, J. W. Grizzle, and A. D. Ames. Robustness of control barrier functions for safety critical control. *IFAC-PapersOnLine*, 48(27):54 – 61, 2015.

[104] Z. Zuo. Nonsingular fixed-time consensus tracking for second-order multi-agent networks. *Automatica*, 54:305–309, 2015.