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On the Extrinsic Geometry of Conformally Embedded Hypersurfaces

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On the Extrinsic Geometry of Conformally Embedded Hypersurfaces

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To my wife, Vivian.

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On the Extrinsic Geometry of Conformally Embedded Hypersurfaces

Abstract

The relationship between the boundary of a manifold and its interior is important for studying many problems in science, as it allows us to predict the behavior of certain problems that can be modeled by partial differential equations. We study bulk-boundary relationships for conformal manifolds. A key tool for analyzing conformal manifolds is tractor calculus. By comparing the conformal structure in the interior with that of the boundary, we provide a complete hypersurface tractor calculus and develop a conformally-invariant characterization of the extrinsic curvature of the embedded hypersurface. These tools provide a characterization of families of conformal manifolds with boundaries that are of particular interest to physicists: so-called Poincarè–Einstein manifolds and Willmore manifolds. Furthermore, we produce a series of conformally-invariant hypersurface operators and curvatures in boundary dimension four and discuss generalizations of these objects to arbitrary dimension.

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CHAPTER 1

Introduction

1.1. Motivation

Central to science is the ability to specify data at a given time or place and make predictions about other times or places where that data was not specified. Indeed, the scientific endeavor's success relies on universal laws that relate quantities *now* or *here* to quantities *then* or *there*. Often, such relationships are described by differential equations, and the input data is given in the form of *initial values* or *boundary values*. We will refer to all such problems as boundary value problems (BVPs). To describe such a problem generally, we treat BVPs as the specification of an object (*i.e.* a function, a tensor, or an operator) on the boundary of some smooth manifold. That specification, along with some prescribed relationship, is then used to predict the behavior of the same object away from the boundary.

One such particularly interesting class of BVPs are those where the *conformal infinity* of a (pseudo-)Riemannian manifold is specified (see Section 5.1 for details). When the underlying manifold has a Lorentzian signature, then one can use these BVPs to study scattering of fundamental fields or to study the causal structure of the manifold itself. In Riemannian signatures, we can use these BVPs to better understand systems that are invariant under local changes in unit systems: those that have local scale symmetries. Such systems are called conformal structures and will be our primary focus.

In the general relativity setting, the study of smooth and complete Cauchy slices as initial data is important for understanding the causal structure of solutions to Einstein's field equations on a manifold. Thus, it is natural to examine the geometric constraints on these Cauchy slices. In the specific case where the cosmological constant is negative, we can view the conformal compactification of the Cauchy slice as a conformal manifold with Euclidean signature. In that case, the geometric constraints on a smooth Cauchy slice specify a conformal BVP which has been well-studied; a seminal result of Andersson, Chrusciel, and Friedrich [**2**] characterized the allowed boundary data sets so that the Cauchy slice is smooth and complete. This result (and others) suggest a deeper connection between the causal structure of spacetime and conformal BVPs.

Another motivation for studying conformal BVPs is a proposed relationship between type IIB string theory in an asymptotically $\mathrm{AdS}_5 \times S^5$ and $\mathcal{N} = 4$ supersymmetric Yang–Mills on its four-dimensional boundary—called the AdS/CFT correspondence— proposed by Maldacena [**60**]. In particular, one can view this correspondence as a prescription for the relationship between boundary data and the interior, effectively specifying a BVP where the boundary data is some matter content and a conformal structure. Since this proposal, many explicit relationships have been found between properties of the conformal theory on the boundary and the gravitating theory in the AdS spacetime. One early such result by Henninson and Skenderis [**55**] is the relationship between the Weyl anomaly on the boundary CFT and a particular conformally-invariant geometric quantity found in the Laurent series expansion of the infinite volume of the AdS bulk (see Section 6.4 for more details).

Later, the Bekenstein–Hawking entropy [**8**, **54**] (reproduced in another setting by Strominger and Vafa [**76**]) was explicitly reproduced by Ryu and Takayanagi in 2006 [**70**, **71**] in the classical limit of the AdS/CFT correspondence. A consequence of their result is a proposed generalization of the Bekenstein–Hawking entropy: that the entanglement entropy of a given state on the CFT side of the AdS/CFT correspondence is given (in natural units) by one quarter of the surface area of a particular spacelike codimension-2 submanifold attached to the boundary and extending into the bulk. Again, this suggests that there is a fundamental relationship between a geometric quantity (the area of a particular codimension-2 submanifold attached to the boundary) and a quantum quantity (the entanglement entropy of the state in the CFT). Various other "dictionary entries," relating geometric quantities in the bulk to observables of the boundary CFT, have since been proposed. Notably, in 2014, Susskind [**74**, **78**, **79**] proposed a correspondence between the quantum complexity of a system on the boundary CFT and the volume of a particular associated region in the AdS bulk.

Thus, we study the geometric properties of BVPs on conformal manifolds in a general setting.

1.2. Organization

The next section provides a brief overview of some important aspects of Riemannian differential geometry and lays out the notation that will be used throughout this dissertation.

Chapter 2 begins by contrasting conformal geometry with Riemannian geometry. From there, a conformally-invariant tractor calculus on conformal manifolds is built, much like how one builds the tensor calculus on Riemannian manifolds. For those who are familiar with the standard tractor construction, we recommend skipping all but Section 2.5, where somewhat arcane (but useful) tractor operator results are provided.

In Chapter 3, we provide a comparison between the standard treatment of hypersurfaces in Riemannian geometry and its analog in conformal geometry. Using well-known results in the conformal setting, we provide new tractor relationships that mimic classical results for Riemannian geometries. At the end of the chapter, we provide an explicit description of geometric holography, the notion that we can learn about the boundary of a system by studying solutions to a prescribed differential equation in the holographic bulk.

In Chapter 4, we conclude our exposition of new tools developed for conformal hypersurface calculus by introducing conformal fundamental forms—conformally-invariant tensors that characterize extrinsic curvatures of an embedding. These are generalizations of the well-known second fundamental form and provide a basis for understanding extrinsic conformal hypersurface geometry. Due to complications in constructing conformally-invariant tensors on a conformal manifold, we provide two distinct constructions for these tensors: one which does not generate the entire family of conformal fundamental forms but is applicable in all dimensions and another which does generate the entire family but only applies when the bulk dimension is even.

In Chapter 5, we give a first application of these conformal hypersurface geometry tools. In particular, given a conformally-compact manifold, we prove that there is a one-to-one correspondence between conformal fundamental forms vanishing on its conformal infinity and the manifold being asymptotically Einstein. This result suggests that these conformal fundamental forms are obstructions to such a manifold being Einstein. In this chapter we also investigate a generalization of the celebrated Willmore invariant and find that it can be almost entirely characterized in terms of the conformal fundamental forms.

Chapter 6 contains a series of lengthy computations where we explicitly use the conformal hypersurface geometry described in Chapters 3 and 4 to compute certain conformal hypersurface invariants of particular interest, including the Weyl anomaly for a four-dimensional CFT as well as the Willmore energy and corresponding invariant for a hypersurface embedded in a five-manifold.

We conclude with proposed future work in Chapter 7, including investigations of higher codimension conformally-embedded submanifolds, global phenomena using the calculus developed, and new families of conformal manifolds. Appendix A gives an introduction to the software used to generate many of the results in this dissertation, FORM [**83**].

1.3. Riemannian geometry and notations

The geometry of smooth manifolds with well-defined notions of both distance and angle takes place on *Riemannian manifolds*. Throughout, we will restrict our considerations to manifolds of dimension 3 or greater. A Riemannian manifold can be described by a pair (M^d, g) consisting of a d-dimensional differentiable manifold M^d and a rank-2 positive-definite (and thus invertible) symmetric tensor field g known as the *metric tensor*. Given a point $x \in M$, the metric tensor assigns to each tangent space T_xM an inner product, so that for $u, v \in T_xM$, we can write $\langle u, v\rangle_{g_x} := g_x(u, v)$. Then, the magnitude of the vector u can be defined in the usual way via $|u|_{g_x} := \sqrt{g_x(u, u)}$, and the angle θ between vectors u and v is given by

$$\cos\theta = \tfrac{g_x(u,v)}{|u|_{g_x}|v|_{g_x}}\,.$$

Importantly, it is the existence of the inner product that allows for an invariant notion of length and angle to be defined. In pseudo-Riemannian settings (like Lorentzian manifolds, which of are particular physics interest), the requirement that the metric tensor be positive-definite is relaxed—although it must still be invertible. Much of what will be said in this dissertation can be generalized to pseudo-Riemannian manifolds, but we restrict our discussion to those with positive-definite metric tensors.

Because the metric tensor is non-degenerate, it also has a well-defined inverse g^{-1}, which similarly assigns to each point $x \in M$ an inner product on the cotangent bundle T^*_xM. Thus, lengths of and angles between covectors can just as well be defined in similar ways to those given above.

In a coordinate patch of M with coordinates $(x^1, \ldots, x^d)$, we can write a vector $u \in T_xM$ as $u = u^a \frac{\partial}{\partial x^a}$ and a covector $\alpha \in T_x^*M$ as $\alpha = \alpha_a dx^a$, where indices take values between 1 and d and repeated indices are implicitly summed over via the Einstein summation convention. Because g defines an inner product on pairs of vectors in T_xM, we can write $g = g_{ab}dx^a dx^b$. Further, because $dx^a(\frac{\partial}{\partial x^b}) = \delta^a_b$ where δ^a_b is the Kronecker delta, we can thus write

$$g_x(u, v) = g_{ab}u^a v^b$$

and

$$\cos\theta = \frac{g_{ab}u^a v^b}{\sqrt{g_{cd}g_{ef}u^c u^d v^e v^f}} \, ,$$

where we have dropped the subscript x when it is clear from context. In this notation, we can write $|u|_g^2 = g_{ab}u^a u^b$.

Observe from the above computations that vectors can be represented by symbols with "upper" indices like u^a and similarly covectors can be represented by symbols with "lower" indices like α_a. In a similar fashion, the metric tensor g can be represented by g_{ab} and its inverse g^{-1} can be represented by g^{ab}. For the tangent and cotangent bundle of M (and their tensor products), we will explicitly use letters from the first part of the Latin alphabet for abstract indices. In general, more complex tensor structures can be represented by combinations of upper and lower indices, *i.e.* one might represent a type (m, n) tensor T by $T^{a_1\cdots a_m}{}_{b_1\cdots b_n}$. This notation, where tensor structures are represented by their index structure, is often called Penrose's abstract index notation [**65**] and will be used throughout this dissertation. Note that in this notation, these indices do not take on numerical values, nor do the symbols with indices represent sets of scalar fields—this is merely a tool for representing tensor operations. Moreover, no choice of coordinates is made. Alternatively, sometimes we will place (co)vectors in the place of the abstract indices to represent the appropriate contraction so that, for example, for vectors $u, v \in TM$ one might write $g_{uv} = g_{ab}u^a v^b$.

To denote tensor symmetries, we will use round brackets for the symmetric part of a given tensor structure so that, for example, $v_{a(bc)d} := \frac{1}{2}(v_{abcd} + v_{acbd})$, and square brackets for the antisymmetric part, *i.e.* $v_{a[bc]d} := \frac{1}{2}(v_{abcd} - v_{acbd})$. The trace of a rank-2 tensor is defined to be the contraction of its indices via the metric or its inverse, so that $\mathrm{tr}(v_{ab}) := v_a{}^a$. When we are only interested in the trace-free part of some symmetric part of a tensor, we will follow the round brackets with a $\circ$

so that $v_{(ab)\circ} := v_{(ab)} - \frac{1}{d} g_{ab} v^c{}_c$. Alternatively, when the tensor is fully symmetric and trace-free, we will sometimes place a $\circ$ over the tensor so that $v_{(ab)\circ} \equiv \mathring{v}_{(ab)}$. Symmetric and antisymmetric tensor products of vector bundles will be represented by $\odot$ and $\wedge$, respectively, and the trace-free symmetric tensor products of vector bundles will be represented by $\odot_\circ$. For example, at a point $x \in M$, we have that $g_x \in \odot^2 T^*_x M$. We will often refer to an arbitrary but fixed tensor product of (co)tangent bundles by $T^\phi M$, where in general we will use lower-case Greek letters. Thus, if $t \in T^\phi M$, we will say that t has tensor type ϕ. Furthermore, given a tensor $v^{abc\cdots e}$, we will use the shorthand $\mathcal{E}(v)$ to denote $v^{abc\cdots e} t$ where t is an unspecified tensor or tensor-valued operator and $a, b, c, \ldots, e$ are any open indices. While the above discussion was given in the context of the tangent and cotangent bundles of M, the same language will be used (with different index names) for different vector bundles. In particular, when referring to arbitrary but fixed tensor products of a generic vector bundle (that is not the tangent or cotangent bundles) then we will use capital Greek letters for the same.

The metric and its inverse allow us to prescribe a natural isomorphism between $T_x M$ and $T^*_x M$, and thus we can use g and g^{-1} to "raise" and "lower" indices so that $u_a = g_{ab} u^b$. Thus, for $u, v \in T_x M$ we can write $g_{ab} u^a v^b \equiv u_a v^a \equiv u^a v_a$. Mimicking the standard meaning of the Euclidean dot product, we will often write such an inner product as $u \cdot v$ and similarly for covectors. Abusing this notation, we use the same binary operation to represent the action of covectors on vectors, so that for $\alpha \in T^*_x M$ and $u \in T_x M$, we have that $\alpha \cdot u \equiv \alpha_a u^a$. Furthermore, when the index contraction is clear, we will also use this notation, so that $s \cdot t \equiv s_{abc\cdots} t^{abc\cdots}$.

One of the key properties of a Riemannian manifold (M, g) is that there exists a (unique) canonical affine connection ∇, known as the *Levi-Civita connection*, on (M^d, g) that preserves g and is torsion-free. An affine connection ∇ is a bilinear map

$$\Gamma(TM) \times \Gamma(TM) \ni (u, v) \mapsto \nabla_u v \in \Gamma(TM) \,,$$

where $\Gamma(\mathcal{V}M)$ is the section space of any bundle $\mathcal{V}M$ over M. In this notation, if $u \in \Gamma(TM)$, then u is a vector field on M. Given any $u, v, w \in \Gamma(TM)$, an affine connection ∇ preserves the metric g when

$$\nabla_u(g(v, w)) = g(\nabla_u v, w) + g(v, \nabla_u w) \,.$$

Observe that the connection ∇ can be extended to act on sections of the tensor product of the tangent bundle (and its dual) in a natural way. Thus, we can write the above metric-preserving property as, for any $u \in \Gamma(TM)$, we have that $\nabla_u g = 0$. Furthermore, if such an affine connection is torsion-free, then

$$[u, v] = \nabla_u v - \nabla_v u \,,$$

where $[\cdot, \cdot]$ is the Lie bracket of vector fields (defined as the commutator of vector fields viewed as derivations).

We conclude this section by providing definitions of a useful set of *Riemannian invariants*, which are objects which do not change under generic differentiable coordinate transformations. Because a Riemannian manifold can be characterized entirely without reference to a choice of coordinates, those scalars and tensors that are independent of the choice of coordinates can be viewed as properties of (M^d, g). These objects are useful to characterize a given Riemannian manifold.

The most fundamental invariant of a Riemannian manifold is the *Riemann curvature*, defined by

$$R(u, v)w := (\nabla_u \nabla_v - \nabla_v \nabla_u)w - \nabla_{[u,v]} w \,,$$

where $u, v, w \in \Gamma(TM)$. This invariant measures the failure of the vector w to be unchanged when parallel transported along an infinitesimal parallelogram spaned by the vectors u, v. In the abstract index notation above, we can write

$$u^a v^b R_{ab}{}^c{}_d w^d = u^a \nabla_a v^b \nabla_b w^c - v^a \nabla_a u^b \nabla_b w^c - (u^a (\nabla_a v^b) - v^a (\nabla_a u^b)) \nabla_b w^c \,.$$

In this formula (and all formulas in this dissertation) we will assume that differential operators act on everything to their right unless they are enclosed by brackets. When acting on a generic tensor $t \in \Gamma(T^\phi M)$, observe that the commutator of two covariant derivatives act as a derivation on each factor in the tensor bundle, so that

$$\begin{aligned}[\nabla_a, \nabla_b] T^{c_1 c_2 \cdots}{}_{d_1 d_2 \cdots} = R_{ab}{}^{c_1}{}_e T^{e c_2 \cdots}{}_{d_1 d_2 \cdots} + R_{ab}{}^{c_2}{}_e T^{c_1 e \cdots}{}_{d_1 d_2 \cdots} + \cdots \\ + R_{abd_1}{}^e T^{c_1 c_2 \cdots}{}_{e d_2 \cdots} + R_{abd_2}{}^e T^{c_1 c_2 \cdots}{}_{d_1 e \cdots} + \cdots \,.\end{aligned}$$

We simplify this action on arbitrary tensor types with the notation in which we write the above display as

$$[\nabla_a, \nabla_b]T = R_{ab}{}^{\sharp}T\,.$$

This notation will be generalized to the action of any endomorphism on any vector bundle over M. Sometimes, we will use dots $\cdot$ to indicate which indices of an antisymmetric tensor will be treated as the endomorphism indices, *i.e.* we might write

$$(u_{ab[\cdot}v_{\cdot]cd})^{\sharp}(w^e) = u_{ab}{}^{e}v_{fcd}w^f\,,$$

and then extend as above to higher-rank tensors.

The Riemann curvature (often denoted by R_{abcd}) has the following symmetry properties:

$$R_{abcd} + R_{bacd} = 0$$

$$R_{abcd} - R_{cdab} = 0$$

$$R_{abcd} + R_{cabd} + R_{bcad} = 0\,.$$

The Riemann curvature also satisfies a Bianchi identity, given by $\nabla_{[a}R_{bc]de} = 0$. The trace of the Riemann curvature is called the *Ricci tensor*, which is given by $Ric_{ab} := R_{ca}{}^{c}{}_{b}$ and is symmetric. Finally, the trace of the Ricci tensor is called the *Ricci scalar* and is given by $Sc := Ric_a{}^a$.

A particularly useful trace-correction of the Ricci tensor is known as the *Schouten tensor* and is given, in dimensions $d \geq 3$, by

$$P := \frac{1}{d-2}\left(Ric - \frac{Sc}{2(d-1)}g\right)\,.$$

The trace of the Schouten tensor is denoted by $J := P_a{}^a$. The Ricci tensor can be expressed in terms of the Schouten tensor and its trace via

$$Ric = (d-2)P + gJ\,.$$

The Riemann curvature can be decomposed into its trace-free part, the *Weyl tensor* W, and a combination of the metric and Schouten tensors:

$$R_{abcd} = W_{abcd} + g_{ac}P_{bd} - g_{bc}P_{ad} - g_{ad}P_{bc} + g_{bd}P_{ac}\,.$$

Observe that the Weyl tensor has all of the symmetries of the Riemann curvature but has the additional property that a trace over any pair of its indices vanishes. A surprising feature of the Weyl tensor is that it vanishes in dimension $d = 3$.

The *Cotton tensor* (sometimes called the Cotton-York tensor) is the covariant curl of the Schouten tensor, so that

$$C_{abc} := 2\nabla_{[a}P_{b]c}\,.$$

In dimensions $d > 3$, the Cotton tensor can also be written in terms of the divergence of the Weyl tensor, which follows as a result of the Bianchi identity described above. In particular,

$$(d-3)C_{abc} = \nabla^d W_{dcab}\,.$$

From these expressions for the Cotton tensor it is clear that it is antisymmetric in its first two indices and is trace-free under contraciton of any of its pair of indices. Further, by the symmetry properties of the Weyl tensor, it satisfies

$$C_{abc} + C_{cab} + C_{bca} = 0\,.$$

The final named invariant that we will need for this dissertation is the *Bach tensor*, which is formed by the action of a particular second order differential operator on the Weyl tensor. Specifically, for $d \geq 4$, we define the symmetric trace-free tensor

$$B_{ab} := (\tfrac{1}{d-3}\nabla^c\nabla^d + P^{cd})W_{dbca}\,.$$

Using the above identities, we can also write

$$B_{ab} = \Delta P_{ab} - \nabla^c\nabla_a P_{cb} + P^{cd}W_{dbca} = \nabla^c C_{cab} + P^{cd}W_{dbca}\,.$$

A straightforward computation using the definition of the tensors described above yields the following useful identity for the divergence of the Bach tensor:

$$\nabla^b B_{ab} = (d-4) P^{bc} C_{abc} \, .$$

Indeed, this identity establishes that in four dimensions, the divergence of the Bach tensor vanishes.

The unifying idea behind the above definitions is that we can proliferate diffeomorphism invariant tensors (or scalars) by applying covariant derivatives to tensors (or scalars) that are diffeomorphism invariant. This is because the Levi-Civita connection is a diffeomorphism invariant operator, so it maps diffeomorphism invariants to diffeomorphism invariants.

CHAPTER 2

Conformal Geometry

Roughly speaking, conformal geometry is the geometry of smooth manifolds with a well-defined notion of angle but no invariant notion of length. Such manifolds are called *conformal manifolds*. Indeed, a conformal manifold can be viewed as an equivalence class of Riemannian manifolds where two Riemannian manifolds (M, g) and $(\tilde{M}, \tilde{g})$ are equivalent when $M = \tilde{M}$ and the angles between any two vectors $u, v \in \Gamma(TM)$ are equal. From the above description of Riemannian manifolds, we observe that this holds so long as there exists a positive function $\Omega \in C^\infty_+ M$ such that $\tilde{g} = \Omega^2 g$. Note that this does not necessarily imply that the lengths of such vectors are preserved: indeed, $|u|^2_{\tilde{g}} = \Omega^2 |u|_g$. Thus, we consider an equivalence class of metrics on a manifold M such that $[\tilde{g}] = [g]$ when there exists some $\Omega \in C^\infty_+ M$ such that $\tilde{g} = \Omega^2 g$. We call such an equivalence class the *conformal class of metrics* and denote it by $\boldsymbol{c} := [g]$. Then, we can describe a conformal manifold by a pair $(M^d, \boldsymbol{c})$, similar to the description of a Riemannian manifold.

Like in the Riemannian case, a useful class of objects belonging to a conformal manifold are those that depend only on the structure $(M, \boldsymbol{c})$. However, these are, in general, much harder to generate because there is not a canonical Levi-Civita connection (or corresponding curvature) on $(M, \boldsymbol{c})$ because the Levi-Civita connection for one representative metric in general will not preserve another metric representative. Nonetheless, objects in a given Riemannian manifold do obey specific transformation laws when $g \mapsto \Omega^2 g$, and hence one might expect that a subset of Riemannian invariants might transform in a "covariant" way even under rescaling of the metric. Specifically, we will call Riemannian objects (scalars, tensors, or operators) *conformally invariant* when such an object O^g transforms according to

$$g \mapsto \tilde{g} := \Omega^2 g$$

$$O^g \mapsto \tilde{O}^{\tilde{g}} := \Omega^w O^g ,$$

for any positive function $\Omega \in C^\infty_+ M$ and some real-valued *weight* w.

One such conformal invariant is the Weyl tensor W_{abcd} described in the previous section which transforms with weight 2—this is somewhat non-obvious for $d \geq 4$, but is trivially true in three dimensions, where the Weyl tensor vanishes identically. By a similar calculation in three dimensions, the Cotton tensor is conformally invariant. A distinguished family of conformal manifolds are those manifolds whose conformal classes $\boldsymbol{c}$ contain a metric g that has a vanishing Riemann curvature. Because this is a property of the conformal manifold $(M, \boldsymbol{c})$, we say that such manifolds are *conformally flat*. Given a metric $\tilde{g} \in \boldsymbol{c}$, one can show **[30]** that $(M, \boldsymbol{c})$ is conformally flat if and only if $W^{\tilde{g}} = 0$ and $C^{\tilde{g}} = 0$. In three dimensions, this condition reduces to the vanishing of the Cotton tensor for any choice of representative. In dimensions $d \geq 4$, we need only that the Weyl tensor vanishes for any choice of representative.

2.1. Conformal Densities

That we can discuss the Weyl tensor of a conformal manifold $(M, \boldsymbol{c})$ without reference to a choice of metric representative suggests that, rather than Riemannian invariants themselves, instead we ought be interested in *classes* of Riemannian invariants. Such classes are called conformal densities (or simply "densities" for brevity) and are one of the core objects in discussion of conformal geometry.

Naively, a *conformal density of weight* w is a double equivalence class $\varphi = [g; f] = [\Omega^2 g; \Omega^w f]$ where $f \in C^\infty M$. We denote the bundle of weight w densities by $\mathcal{E}M[w]$ and its section space by $\Gamma(\mathcal{E}M[w])$. It is also useful to consider tensor-valued densities: a tensor-valued (of tensor type ϕ) weight w density $\theta = [g; t]$ belongs to the section space of the product bundle $T^\phi M \otimes \mathcal{E}M[w] =: T^\phi M[w]$; this notation is generically used to refer to tensor bundle-valued densities. A fundamental example of such a density is the *conformal metric*: $\gamma \in \Gamma(\odot^2 T^* M[2])$. Going forward, rather than describing a conformal manifold by the pair $(M, \boldsymbol{c})$, we often (and equivalently) use the pair (M, γ) instead. Of special interest are scalar densities of weight 1; these are referred to as *scales*. In particular, a nowhere vanishing scale $\tau \in \Gamma(\mathcal{E}M[1])$ is called a *true scale* and canonically determines a metric representative from the conformal class $g_\tau \in \boldsymbol{c}$ by trivializing the conformal metric via

$$g_\tau := \gamma/\tau^2 \,.$$

Thus, there is an isomorphism between the conformal manifold-scale triple (M, γ, τ) and the Riemannian manifold $(M, \gamma/\tau^2)$. We call such an isomorphism a *choice of scale*.

More geometrically, one can understand the density bundle $\mathcal{E}M[w]$ as follows. First, observe that we can view a conformal manifold $(M, \boldsymbol{c})$ as a ray subbundle $\pi : \mathcal{Q} \to M$ where $\mathcal{Q} \subset \odot^2 T^*M$. The natural $\mathbb{R}_+$ action (denoted by ρ and parametrized by t) on an element $(x, g_x) \in \mathcal{Q}$ is given by $\rho_t(x, g_x) = (x, t^2 g_x)$ so that $\mathcal{Q} \to M$ is a principal $\mathbb{R}_+$ bundle. Then, for some $w \in \mathbb{R}$, there exists a representation of $\mathbb{R}_+$ given by $\rho_w : \mathbb{R}_+ \to \mathrm{End}(\mathbb{R})$ (viewing $\mathbb{R}$ as a vector space) where

$$\mathbb{R} \ni z \mapsto \rho_w(t)(z) := t^{-w} z \in \mathbb{R}\,.$$

So, for $t \in \mathbb{R}_+$, there exists a right $\mathbb{R}_+$ action on $\mathcal{Q} \times \mathbb{R}$ according to

$$((x, g_x), z) \cdot t = ((x, t^2 g_x), \rho_w(t^{-1})(z)) = ((x, t^2 g_x), t^w z)\,.$$

Quotienting out by this $\mathbb{R}_+$ action, we have equivalence classes

$$[(x, g_x); z] = [(x, t^2 g_x); t^w z] \in \mathcal{E}M[w]\,.$$

With the projection map $\pi_w : \mathcal{E}M[w] \to M$ given by $\pi_w([(x, g_x); z]) = \pi((x, g_x)) = x$, then $\pi_w : \mathcal{E}M[w] \to M$ is a line bundle with structure group $\mathbb{R}_+$ as desired. Then, smooth sections of this bundle are the double equivalence classes $[g; f] = [\Omega^2 g; \Omega^w f]$ for $f \in C^\infty M$. Tensor-valued density bundles are then formed in the usual way. For more detail, see [**15**].

For later use, note that the conformal structure $(M, \boldsymbol{c})$ also determines log-density bundles [**40**], $\mathcal{F}M[w]$. Similar to the discussion above, we can consider the log representations of $\mathbb{R}_+$ to construct equivalence classes $[g; \ell] = [\Omega^2 g; \ell + w \log \Omega] \in \Gamma(\mathcal{F}M[w])$. In particular, for a strictly positive density $\tau = [g; t] \in \Gamma(\mathcal{E}M[w])$, we have that $\log \tau = [g; \log t] \in \Gamma(\mathcal{F}M[w])$.

While there does not exist a uniquely determined Levi-Civita connection on $(M, \boldsymbol{c})$, we can define a density-coupled Levi-Civita connection in a uniform way. Specifically, given a fixed $\tau \in \Gamma(\mathcal{E}M[1])$, there exists a unique Levi-Civita connection for $g_\tau \in \boldsymbol{c}$, so we can define $\nabla^\tau := \tau^w \circ \nabla^{g_\tau} \circ \tau^{-w}$ on weight w densities, where ∇^{g_τ} is the usual Levi-Civita connection on (M, g_τ). For brevity, we will often drop the g_τ superscript. Additionally, the density-coupled Levi-Civita connection can also be defined to act on log densities, so that for $\lambda \in \Gamma(\mathcal{F}M[w])$, we write in a choice of scale τ,

$$\nabla\lambda := d(\lambda - w\log\tau) \in \Gamma(T^*M)\,.$$

Observe that the combination $\lambda - w\log\tau$ is an element of $C^\infty M$.

While this definition of a density-coupled Levi-Civita connection is conformally invariant, it depends on a choice of scale τ and hence picks out a special metric representative $g_\tau \in \boldsymbol{c}$. To truly capture the conformal geometry of $(M, \boldsymbol{c})$, our analysis should be agnostic toward any particular metric representative in $\boldsymbol{c}$.

2.2. The Tractor Bundle

In order to systematically construct conformal invariants as we would diffeomorphism invariants in Riemannian geometry, we must look further than tensor products of the tangent and cotangent bundles over M. Instead, we consider a rank $d+2$ vector bundle $\mathcal{T}M$ known as the *standard tractor bundle* determined by the conformal structure (M^d, γ). This vector bundle can be canonically constructed in one of several ways: via the Cartan conformal connection [**17**], via Thomas' associated bundle [**6**, **81**], or via the ambient construction [**15**]. A brief summary of the ambient construction is given in Section 2.4. Note that all of these constructions yield the same structure, $\mathcal{T}M$. This vector bundle comes equipped with several canonical objects: a *tractor metric* h, a null vector field called the *canonical tractor* X, and a *tractor connection* $\nabla^{\mathcal{T}}$.

First, note that given a metric representative of the conformal structure $g \in \boldsymbol{c}$, there exists an isomorphism between the tractor bundle and a triple of density bundles:

$$\mathcal{T}M \overset{g}{\cong} \mathcal{E}M[1] \oplus TM[-1] \oplus \mathcal{E}M[-1]\,. \tag{2.1}$$

We call such an isomorphism a *choice of splitting*. So, given a section of the standard tractor bundle $T \in \Gamma(\mathcal{T}M)$, we can apply the isomorphism to write

$$T^A \overset{g}{=} (\tau^+, \tau^a, \tau^-)\,,$$

where $\overset{g}{=}$ will be used to indicate that an equality holds in a choice of splitting specified by g. Sometimes we will use a column vector notation for the same decomposition. To represent tensors of this bundle, we will use capital Latin letters for our abstract indices. Further, because for each representative $g \in \boldsymbol{c}$ there exists such a choice of splitting, the relationship between two choices of splitting is implied by the relationships

$$
\begin{aligned}
T^A &\overset{g}{=} (\tau^+, \tau^a, \tau^-)\,, \\
T^A &\overset{\Omega^2 g}{=} (\tau^+, \tau^a + \Upsilon^a \tau^+, \tau^- - \Upsilon \cdot \tau - \tfrac{1}{2}|\Upsilon|_g^2 \tau^+)\,,
\end{aligned}
\tag{2.2}
$$

where $\Upsilon = d(\log \Omega)$. Often we will use the language of "slots" to refer to specific entries in a tractor viewed as a vector or matrix. In particular, in the above display, the entry containing τ^+ might be referred to as the "top slot" (as motivated by a column vector notation). For the square of the tractor bundle (a choice of splitting of which is given in Equation (2.1)), we will refer to the entry corresponding to the term $TM[-1] \otimes TM[-1]$ in this decomposition as the "middle slot." Such language will be used sparingly.

Just as (co)tangent tensor bundles can be given weights (by taking the product with a density bundle) so too can tractor bundles: for this we will write $\mathcal{T}M[w] := \mathcal{T}M \otimes \mathcal{E}M[w]$ and also call these bundles *tractor bundles*. Just as for Riemannian tensors, arbitrary but fixed tensor products of tractor bundles will be specified with capital Greek letters, so that if $T \in \Gamma(\mathcal{T}^\Phi M[w])$, the tractor T is of tensor type Φ.

From Equation (2.2), we can observe a key feature: the first non-zero entry in a tractor defines a density independent of the choice of splitting and hence is conformally-invariant. In any given tractor, this first non-zero slot is referred to as the *projecting part* of the tractor; the map q^* extracts this term and is called the *extraction map*. Additionally, observe that the section $X \overset{g}{=} (0, 0, 1) \in \Gamma(\mathcal{T}M[1])$ is also canonically defined independently of the choice of splitting: this is the *canonical tractor* mentioned above.

The conformal structure also canonically determines a natural symmetric, non-degenerate (but non-positive) inner product between tractors that is independent of the choice of splitting, given by

$$
h(U, V) \overset{g}{=} u^+ v^- + \gamma_{ab} u^a v^b + u^- v^+\,,
$$

for two tractors $U, V \in \Gamma(\mathcal{T}M)$ specified in a choice of splitting in the obvious way. Associated to this inner product is a canonical isomorphism between the tractor bundle and its dual

$$\mathcal{T}^*M \overset{g}{\cong} \mathcal{E}M[1] \oplus T^*M[1] \oplus \mathcal{E}M[-1]\,.$$

This inner product defines the *tractor metric* $h_{AB} \in \Gamma(\odot^2 \mathcal{T}^*M)$, given, in a choice of splitting, by

$$h_{AB} \overset{g}{=} \begin{pmatrix} 0 & 0 & 1 \\ 0 & \gamma_{ab} & 0 \\ 1 & 0 & 0 \end{pmatrix}\,,$$

with its inverse denoted by h^{AB}.

Given the above structure, one can show that the tractor bundle, while not irreducible itself, has a composition series given by

$$\mathcal{T}M = \mathcal{E}M[1] \oplus\!\!\!\!\leftarrow TM[-1] \oplus\!\!\!\!\leftarrow \mathcal{E}M[-1]\,.$$

Indeed, note that multiplication by the canonical tractor X maps sections of $\mathcal{E}M[-1]$ to the tractor bundle. Furthermore, action by $h(X,\cdot)$ maps sections of $\mathcal{T}$ to sections of $\mathcal{E}M[1]$.

Finally, there exists a canonically determined *tractor connection* acting on tractor bundles

$$\nabla^{\mathcal{T}} : \Gamma(\mathcal{T}M) \to \Gamma(T^*M \otimes \mathcal{T}M)$$

given in a choice of splitting by

$$\nabla_a^{\mathcal{T}} T^B \overset{g}{=} \begin{pmatrix} \nabla_a \tau^+ - \tau_a \\ \nabla_a \tau^b + \delta_a^b \tau^- + (P^g)_a^b \tau^+ \\ \nabla_a \tau^- - P^g_{ab} \tau^b \end{pmatrix}\,, \tag{2.3}$$

where ∇ is the density-coupled Levi-Civita connection and P^g is the Schouten tensor associated with the splitting metric g. This connection extends as usual to arbitrary tensor products of the tractor bundle as well as products of the standard tractor bundle with density bundles. Just as for the density-coupled Levi-Civita connection, we will usually drop the superscript $\mathcal{T}$ in $\nabla^{\mathcal{T}}$ when the connection is clear from context.

With these canonical structures defined, we can usefully define *injecting operators* (or *injectors*) that, roughly speaking, play the roles of basis tractors for the tractor bundle. In particular, observe that for $\phi \in \Gamma(\mathcal{E}M[w])$, we have that $\phi X \overset{g}{=} (0, 0, \phi) \in \Gamma(\mathcal{E}M[w+1])$. Further, given a choice of splitting specified by the metric representative $g \in \boldsymbol{c}$ (and hence its corresponding scale $\tau \in \Gamma(\mathcal{E}M[1])$) and a vector-valued density $\mu^a \in \Gamma(TM[w])$, we can write

$$\mu^a \tau \nabla_a^{\mathcal{T}}(\tau^{-1} X^B) \overset{g}{=} (0, \mu^b, 0) \in \Gamma(\mathcal{T}M[w+1])\,.$$

In fact, we can define the injector

$$Z_a^B := \tau \nabla_a^{\mathcal{T}}(\tau^{-1} X^B) \overset{g}{=} (0, \delta_a^b, 0) \in \Gamma(T^*M \otimes \mathcal{T}M[1])\,.$$

This one-form valued tractor acts as a set of d basis tractors for the tractor bundle, but unlike X it depends on a choice of splitting given by g. Finally, given X and Z and a choice of splitting specified by $g \in \boldsymbol{c}$ and its corresponding scale $\tau \in \Gamma(\mathcal{E}M[1])$, we can uniquely define the weight -1 tractor Y^A by the decomposition

$$h^{AB} \overset{g}{=} X^A Y^B + \gamma^{ab} Z_a^A Z_b^B + X^B Y^A\,.$$

Hence, we can write $Y^A \overset{g}{=} (1, 0, 0) \in \Gamma(\mathcal{E}M[-1])$, where Y^A also depends on the choice of splitting. That is, there exists a unique pair of injecting operators Z and Y for each choice of metric $g \in \boldsymbol{c}$. So, for a tractor $T \in \Gamma(\mathcal{T}M[w])$, in a choice of splitting specified by $g \in \boldsymbol{c}$, we have

$$T^A \overset{g}{=} Y^A \tau^+ + Z_a^A \tau^a + X^A \tau^-\,.$$

Often we drop the implied dependence of injectors on a choice of splitting and the corresponding scale. The action of the tractor connection on the injectors is then given by

$$\begin{aligned}
&\nabla_a^g X^A \overset{g}{=} Z_b^A\,,\\
&\nabla_b^g Z_a^A \overset{g}{=} -P_{ab}^g X^A - \gamma_{ab} Y^A\,,\\
&\nabla_b^g Y^A \overset{g}{=} (P^g)_b^a Z_a^A\,.
\end{aligned}$$

Note that this is just a rewriting of Equation (2.3).

For notational purposes, we will occasionally write contractions of the injectors with tractors. For example, given a tractor $T^{ABC\cdots}$, we will write $Z^a_A T^{ABC\cdots} = T^{aBC\cdots}$. Similarly, we write $X_A T^{ABC\cdots} = T^{+BC\cdots}$ and $Y_A T^{ABC\cdots} = T^{-BC\cdots}$. This notation mirrors the notion of the "slots" of a given tractor.

2.3. The Thomas-D Operator

Observe that the action of the tractor connection on a tractor is not unique but depends on a choice of scale (used to determine the density-coupled connection). However, there exists an invariant second order differential operator containing the tractor connection that is independent of this choice: the *Thomas-D operator* **[80]** mapping

$$D : \Gamma(\mathcal{T}^\Phi M[w]) \to \Gamma(\mathcal{T}M \otimes \mathcal{T}^\Phi M[w-1])\,.$$

Given a tractor $T \in \Gamma(\mathcal{T}^\Phi M[w])$, we write, in a choice of splitting,

$$D^A T \stackrel{g}{=} (d+2w-2)wY^A T + (d+2w-2)\gamma^{ab} Z^A_a \nabla_b T - X^A(\Delta T + wJ^g T) \in \Gamma(\mathcal{T}M \otimes \mathcal{T}^\Phi M[w-1])\,, \tag{2.4}$$

where $\Delta := \gamma^{ab}\nabla^{\mathcal{T}}_a \nabla^{\mathcal{T}}_b$ is the tractor-coupled rough Laplacian and J^g is the trace of the Schouten tensor associated to the splitting metric g. We can also define such an operator on weight w log densities, so that

$$D : \Gamma(\mathcal{F}M[w]) \to \Gamma(\mathcal{T}M[-1])\,.$$

In particular, acting on $\lambda \in \Gamma(\mathcal{F}M[w])$, we have that

$$D^A \lambda \stackrel{g}{=} (d-2)wY^A + (d-2)\gamma^{ab} Z^A_a \nabla_b \lambda - X^A(\Delta\lambda + wJ^g)\,.$$

Even though this operator is expressed in a choice of splitting, one can show that it does not depend on the underlying choice of scale. Thus, this operator can be used to proliferate conformal invariants, much like how the Levi-Civita connection is used to proliferate diffeomorphism invariants. Importantly, the Thomas-D operator is not a derivation because it is a second-order differential operator; however, it does have the properties

$$D^A \circ D_A = 0 = D^A h_{BC}\,.$$

A useful modification of the D^A operator is the "hatted" Thomas-D operator, $\hat{D}$. For any tractor $T \in \Gamma(\mathcal{T}^\Phi M[w])$ where $w \neq 1 - \frac{d}{2}$, this is defined by

$$\hat{D}^A T^\Phi := (d + 2w - 2)^{-1} D^A T^\Phi \,.$$

As a consequence of this definition, observe that

$$\hat{D}_A X_B = h_{AB} \qquad \text{and} \qquad \hat{D}_A h_{BC} = 0 \,.$$

Further, the operator $\hat{D}$ acting on a product of tractors satisfies a relationship known as the *Leibniz failure* [**45**, **57**].

Proposition 2.3.1 (Leibniz failure). *Let $T_i \in \Gamma(\mathcal{T}^{\Phi_i} M[w_i])$ for $i = 1, 2$, $h_i := d + 2w_i$, and $h_{12} := d + 2w_1 + 2w_2$ such that $h_i \neq 2 \neq h_{12}$. Then,*

$$\hat{D}^A (T_1 T_2) - (\hat{D}^A T_1) T_2 - T_1 (\hat{D}^A T_2) = -\tfrac{2}{h_{12}} X^A (\hat{D}^B T_1)(\hat{D}_B T_2) \,.$$

Because we have defined the Thomas-D operator on log densities, it is also useful to have a result similar to Proposition 2.3.1 for log densities. This requires some preliminary work.

First, we define the *weight operator* $\underline{w}$ on sections of tractor bundles and on sections of log density bundles. In particular, for $T \in \Gamma(\mathcal{T}^\Phi M[w])$ and $\lambda \in \Gamma(\mathcal{F} M[w])$, we have that

$$\underline{w} T = w T \qquad \text{and} \qquad \underline{w} \lambda = w \,.$$

Acting on tractors, it is evident that the weight operator is a derivation, but a tensor product of log-density bundles is not also a log-density bundle, so the notion of a derivation on a product of log-densities is tricky to define and unneeded here.

We can extend the weight operator $\underline{w}$ (and the tractor connection ∇) to act on tensor products of a log-density bundle with tractor bundles as a derivation. For $\lambda T \in \Gamma(\mathcal{T}^\Phi M[w] \otimes \mathcal{F} M[w'])$, we write

$$\underline{w}(\lambda T) := \lambda \underline{w} T + T \underline{w} \lambda = w \lambda T + w' T \in \Gamma(\mathcal{T}^\Phi M[w] \ \oplus\ \mathcal{T}^\Phi M[w] \otimes \mathcal{F} M[w']) \,,$$

and similarly for the tractor connection:

$$\nabla(\lambda T) := \lambda \nabla T + T \nabla \lambda \in \Gamma(T^*M \otimes \mathcal{T}^\Phi M[w] \ \oplus \ T^*M \otimes \mathcal{T}^\Phi M[w] \otimes \mathcal{F}M[w'])\,,$$

Thus, we can define the Thomas-D operator on the tensor product of some tractor bundle with a log-density bundle via

$$D^A :\stackrel{g}{=} Y^A(d+2\underline{w}-2)\underline{w} + \gamma^{ab} Z_a^A \nabla_b (d+2\underline{w}-2) - X^A(\Delta + J^g \underline{w})\,.$$

Written in terms of the weight operator $\underline{w}$, we can reexpress the hatted Thomas-D operator as

$$\hat{D}^A = D \circ \tfrac{1}{d+2\underline{w}-2} = \tfrac{1}{d+2\underline{w}} \circ D\,,$$

where, acting on $\lambda \in \Gamma(\mathcal{F}M[w'])$, we have that (for $k \neq d$),

$$\frac{1}{d+2\underline{w}-k}\lambda := \frac{\lambda}{d-k} - \frac{2w'}{(d-k)^2} \in \Gamma(\mathcal{F}M[w'] \oplus \mathcal{E}M[0])\,.$$

Similarly, we have, for $T \in \Gamma(\mathcal{T}^\Phi M[w])$ and $d+2w-k \neq 0$, we have that

$$\frac{1}{d+2\underline{w}-k}(\lambda T) := T \frac{1}{d+2\underline{w}+2w-k}\lambda \in \Gamma(\mathcal{T}^\Phi M[w] \ \oplus \ \mathcal{T}^\Phi M[w] \otimes \mathcal{F}M[w'])\,.$$

A consequence of the definition of $\hat{D}$ implies that

$$X^A \hat{D}_A = \underline{w}\,. \tag{2.5}$$

In much of the above discussion, sections of Whitney sum bundles of the form $\Gamma(\mathcal{V}M \ \oplus \ \mathcal{V}'M)$ appeared, and these are particularly challenging to work with. However, unlike either the operator D or $\frac{1}{\alpha\underline{w}+\beta}$ alone, their composition lies in a weighted tractor bundle when acting on a log-density. This is captured in the following lemma, which is easily proved by computing in a choice of scale.

Lemma 2.3.2. *For any $0 \neq \beta \in \mathbb{R}$,*

$$\Big(D \circ \frac{1}{\alpha\underline{w}+\beta}\Big)\lambda = \frac{1}{\beta} D\lambda \in \Gamma(\mathcal{T}M[-1])\,.$$

In particular, this lemma allows us to directly verify that $\hat{D}$ has a Leibniz failure property, even on log-densities, as desired. The following result can also be easily proved by computing in a choice of scale.

LEMMA 2.3.3 (Log Leibniz failure). *Let λ be any log density and let $w \neq 1 - \frac{d}{2}$. Then,*

$$\hat{D} \circ \lambda - \lambda \circ \hat{D} : \Gamma(\mathcal{T}^{\Phi} M[w]) \to \Gamma(\mathcal{T} M \otimes \mathcal{T}^{\Phi} M[w-1])\,,$$

and moreover

$$\hat{D} \circ \lambda - \lambda \circ \hat{D} = (\hat{D}\lambda) - \tfrac{2}{d+2\underline{w}} X\,(\hat{D}\lambda) \cdot \hat{D}\,.$$

While in principle, this is enough to use the tractor calculus, there exists one more canonical tractor that roughly plays the role of curvature for the Thomas-D operator: the W-tractor. The projecting part of the W tractor is the Weyl curvature, and hence we use the same symbol for both this tensor and the W tractor. In dimensions $d \geq 5$ and acting on a weight w tractor $T \in \Gamma(\mathcal{T}^{\Phi} M[w])$ with $w \neq 1 - \frac{d}{2}, 2 - \frac{d}{2}$, the commutator of (hatted) Thomas-D operators obeys **[9, 39]**

$$[\hat{D}_A, \hat{D}_B] T = W_{AB}{}^{\sharp} T^E + \tfrac{4}{d+2w-4} X_{[A} W_{B]C}{}^{\sharp} \hat{D}^C T^E\,. \tag{2.6}$$

The tractor content of the W tractor is given in the following lemma:

LEMMA 2.3.4. *Let $d \geq 5$. The W tractor has the symmetries of the Weyl curvature, so that*

$$W^{[ABC]D} = W^{ABCD} + W^{ABDC} = W^{ABCD} - W^{CDAB} = 0 = h_{AC} W^{ABCD}\,.$$

Further, $X \cdot W = 0$. Finally, in any scale $g \in \boldsymbol{c}$, the W-tractor is given by

$$\begin{aligned}
Z_a^A Z_b^B Z_c^C Z_d^D W_{ABCD} &= W^g_{abcd}\,,\\
Z_a^A Z_b^B Z_c^C Y^D W_{ABCD} &= C^g_{abc}\,,\\
Z_a^A Y^B Z_b^C Y^D W_{ABCD} &= \frac{B^g_{ab}}{d-4}\,.
\end{aligned}$$

All other components are either zero or determined by the symmetry of the W-tractor.

PROOF. This lemma follows by direct computation from Equation 2.6 and the curvature identities at the end of Section 1.3. □

REMARK 2.3.5. A consequence of Lemma 2.3.4 is that, away from $d = 6$, the W-tractor is D-free, *i.e.*

$$\hat{D}_A W^{ABCD} = 0\,.$$

Further, this tractor contains our first example of a residue of a dimensionful pole yielding a conformally invariant tensor. Indeed, the tractor $(d-4)W$ is well-defined in four dimensions and has the Bach tensor as its projecting part in four dimensions, which implies that the Bach tensor is a conformally-invariant tensor when $d = 4$. Further, observe that in three dimensions, the Weyl tensor vanishes, so the projecting part of the W tractor becomes the Cotton tensor—which is another proof of the conformal invariance of the Cotton tensor in three dimensions. ■

2.4. The Ambient Construction and Consequences

As mentioned in the previous section, one of the ways to produce the tractor bundle is via the ambient construction. In this section we provide a brief summary of that construction and some useful results that generalize those found in [**37**] for later use. For more details and other expositions, see [**15**, **25**, **26**, **34**, **40**].

As in Section 2.1, we begin by viewing a conformal manifold $(M^d, \boldsymbol{c})$ as a ray subbundle $\pi : \mathcal{Q} \to M$ where $\mathcal{Q} \subset \odot^2 T^*M$, with the natural $\mathbb{R}_+$ action denoted by ρ and parametrized by t. Observe that this subbundle carries with it a tautological symmetric 2-tensor defined at a point $(x, g_x) \in \mathcal{Q}$ by $g_0 := \pi^* g$ which obeys $\rho_t^*(g_0) = t^2 g_0$. This is the conformal metric $\boldsymbol{c}$. The associated *ambient manifold* to the conformal manifold (represented by $\mathcal{Q}$ here) is then defined to be a $(d+2)$-dimensional manifold $\widetilde{M}$ with signature $(d+1, 1)$ in which $\mathcal{Q}$ is embedded as a null hypersurface. Importantly, $\widetilde{M}$ must have an $\mathbb{R}_+$ action extended naturally from the $\mathbb{R}_+$ action on $\mathcal{Q}$. Locally near $\mathcal{Q} \subset \widetilde{M}$, we can write $\widetilde{M} = \mathcal{Q} \times (-1, 1)$, so because we are only interested in local behavior, we will assume that $\widetilde{M} = \mathcal{Q} \times (-1, 1)$. We denote by $\boldsymbol{X}$ the infinitesimal generator of the $\mathbb{R}_+$ action on $\widetilde{M}$ and we denote by Q the defining function for $\mathcal{Q}$.

The metric $\boldsymbol{h}$ on $\widetilde{M}$ is defined such that $\boldsymbol{h}$ pulls back to g_0 on $\mathcal{Q}$ and has the same homogeneity property it, *i.e.* $\rho_t^* \boldsymbol{h} = t^2 \boldsymbol{h}$. Further, we construct $(\widetilde{M}, \boldsymbol{h})$ such that the associated Ricci curvature

$\boldsymbol{Ric}$ vanishes to the maximal possible order in the defining function Q. In particular, when d is odd, a result of Fefferman and Graham [**25**] shows that $(\widetilde{M}, \boldsymbol{h})$ can be constructed so that $\boldsymbol{Ric}$ vanishes to arbitrary order; when d is even, $(\widetilde{M}, \boldsymbol{h})$ can only be constructed asymptotically so that $\boldsymbol{Ric}$ vanishes to order $\frac{d}{2}-2$. Indeed, this implies that in the even-dimensional case, we can at most uniquely determine $\boldsymbol{h}$ to order $\mathcal{O}(Q^{d/2})$. Note that this fixes $\boldsymbol{R}$ (the ambient Riemann curvature) and $\boldsymbol{Ric}$ uniquely to order $\mathcal{O}(Q^{d/2-2})$ in the even-dimensional case and to arbitrary order in the odd-dimensional case. Then, associated to metric $\boldsymbol{h}$ is the unique Levi-Civita connection $\boldsymbol{\nabla}$, constructed in the usual way.

Now, the standard tractor bundle can be defined as $\mathcal{T}M \equiv T\widetilde{M}|_{\mathcal{Q}}/\sim_{\mathcal{T}}$. Specifically, we say that for $U, V \in T\widetilde{M}$, we have that $U \sim_{\mathcal{T}} V$ when $\pi(U) = \pi(V) \in M$ and U is $\boldsymbol{\nabla}$-parallel to V. Then, we can say that certain operators and tensors acting and living on $\widetilde{M}|_{\mathcal{Q}}$ "descend" to tractors and tractor-valued operators on M. Such operators are precisely those operators whose actions on tensors on $\mathcal{Q}$ do not depend on the extension of those tensors off of $\mathcal{Q}$. For a homogeneous weight w tensor field T of type Φ on $\widetilde{M}|_{\mathcal{Q}}$, we will denote the section space by $\Gamma(\mathcal{T}^\Phi\widetilde{M}(w))|_{\mathcal{Q}}$, using similar notation as that used for tractors. In particular, the ambient operator given by

$$\boldsymbol{D}_A := \boldsymbol{\nabla}_A(d + 2\boldsymbol{\nabla}_{\boldsymbol{X}} - 2) - \boldsymbol{X}_A\boldsymbol{\Delta}$$

descends to the Thomas-D operator when restricted to $\mathcal{Q}$ and

$$\boldsymbol{\Delta}|_{\mathcal{Q}} : \Gamma(\mathcal{T}^\Phi\widetilde{M}(1-d/2))|_{\mathcal{Q}} \to \Gamma(\mathcal{T}^\Phi\widetilde{M}(-1-d/2))|_{\mathcal{Q}}$$

descends to the *Yamabe operator* $\square$ in the same way, defined according to $D_A = -X_A\square$ when acting on tractors with weight $1-d/2$. The fundamental vector field $\boldsymbol{X}$ descends to X, $\boldsymbol{h}$ descends to the tractor metric h, and $\boldsymbol{R}$ descends to the W-tractor.

A key result using this ambient construction of the tractor bundle is the following straight forward generalization of [**37**, Proposition 4.1] (which in turn is a rewriting of [**48**, Proposition 2.3]). Note that the result directly generalizes with no changes.

PROPOSITION 2.4.1 (Generalization of Proposition 4.1 of [**37**]). *For d even and k an integer satisfying $1 \le k \le d/2$ or for d odd and $k \in \mathbb{Z}_{\ge 1}$, let $T \in \Gamma(\mathcal{T}^\Phi\widetilde{M}(k-d/2))|_{\mathcal{Q}}$ have a homogeneous* *extension* $\tilde{T}$ *to* $\widetilde{M}$. *Then* $\mathbf{\Delta}^k \tilde{T}|_{\mathcal{Q}}$ *depends only on* T *and the conformal structure on* M *but not on the choice of extension* $\tilde{T}$ *nor on any choices in the ambient metric. Thus the operator*

$$\mathbf{\Delta}^k : \Gamma(\mathcal{T}^\Phi \widetilde{M}\,(k - d/2))|_{\mathcal{Q}} \to \Gamma(\mathcal{T}^\Phi \widetilde{M}\,(-k - d/2))|_{\mathcal{Q}}$$

is conformally invariant and descends to a natural conformally invariant differential operator on tractors

$$P^\Phi_{2k} : \Gamma(\mathcal{T}^\Phi M[k - d/2]) \to \Gamma(\mathcal{T}^\Phi M[-k - d/2])\,.$$

The operators P^Φ_{2k} are known as GJMS operators, named after Graham, Jenne, Mason, and Sparling, who demonstrated their existence using this construction. This result completes the injection that we require to relate ambient calculus to the tractor calculus.

Another useful result is [**37**, Lemma 4.4], recorded here for later use:

LEMMA 2.4.2 (Lemma 4.4 of [**37**]). *Suppose* d *is odd or* $t + u \leq d/2 - 3$ *for* d *even. Then on* $\mathcal{Q}$ *there is an expression for* $\boldsymbol{\nabla}^t \mathbf{\Delta}^u \boldsymbol{R}$ *as a partial contraction polynomial in* $\boldsymbol{D}$, $\boldsymbol{R}$, $\boldsymbol{X}$, $\boldsymbol{h}$, *and* $\boldsymbol{h}^{-1}$. *This expression is rational in* d *and each term is of degree at least* 1 *in* $\boldsymbol{R}$.

For later purposes, we will only be interested in the even-dimensional case, so we provide a generalization of another result from [**37**] in that dimension parity. The proof follows their proof closely.

PROPOSITION 2.4.3 (Generalization of Proposition 4.5 of [**37**]). *Let* $k < d/2$ *with* d *even. For any tractor* $T \in \Gamma(\mathcal{T}^\Phi M[k - d/2])$,

$$X_{A_1} \cdots X_{A_{k-1}} P^\Phi_{2k} T = (-1)^{k-1} \Box D_{A_1} \cdots D_{A_{k-1}} T + \Psi_{A_1 \cdots A_{k-1}}{}^C D_C T\,,$$

where Ψ *is some tractor opperator operator*

$$\Psi_{A_1 \cdots A_{k-1}}{}^C : \Gamma(\mathcal{T}^\Phi_C M[k - 1 - d/2]) \to \Gamma(\mathcal{T}^\Phi_{A_1 \cdots A_{k-1}} M[-1 - d/2])\,,$$

that can be written as a partial contraction polynomial in D_A, W_{ABCD}, X_A, h_{AB}, *and* h^{AB}, *with each term being of order at least* 1 *in* W_{ABCD}.

Proof. The proof of this result follows the discussion (beginning on page 36) in [**37**]. The strategy will be to find a relationship between ambient operators and then descend to their corresponding tractor counterparts.

Given the ambient construction described above for an even dimensional base manifold M^d, we fix an integer $0 < k < d/2$ and a homogeneous ambient tensor T, fixed along $\mathcal{Q}$, of weight $k - d/2$ with an arbitrary extension, also labeled by T. Then, we consider the ambient expression $\boldsymbol{\Delta}\boldsymbol{D}_{A_1}\cdots\boldsymbol{D}_{A_{k-1}}T$. Our goal will be to rearrange terms so that we have

$$\boldsymbol{\Delta}\boldsymbol{D}_{A_1}\cdots\boldsymbol{D}_{A_{k-1}}T = (-1)^{k-1}\boldsymbol{X}_{A_1}\cdots\boldsymbol{X}_{A_{k-1}}\boldsymbol{\Delta}^k T + \text{curvature terms.}$$

Observe that $\boldsymbol{\nabla}_{\boldsymbol{X}}$ acting on a tensor with homogeneity w returns wT, so we can write

$$\boldsymbol{\Delta}\boldsymbol{D}_{A_1}\cdots\boldsymbol{D}_{A_{k-1}}T = \boldsymbol{\Delta}(2\boldsymbol{\nabla}_{A_{k-1}} - \boldsymbol{X}_{A_{k-1}}\boldsymbol{\Delta})\cdots(2(k-1)\boldsymbol{\nabla}_{A_1} - \boldsymbol{X}_{A_1}\boldsymbol{\Delta})T\,. \tag{2.7}$$

Our first step is to expand this display and then commute all of the $\boldsymbol{X}$s to the left of any $\boldsymbol{\nabla}$s and $\boldsymbol{\Delta}$s using the identities $[\boldsymbol{\nabla}_A, \boldsymbol{X}_B] = \boldsymbol{h}_{AB}$ and $[\boldsymbol{\Delta}, \boldsymbol{X}_A] = 2\boldsymbol{\nabla}_A$. Our next step will be to commute all $\boldsymbol{\Delta}$s to the right of any $\boldsymbol{\nabla}$s; doing so requires the following operator identity:

$$\begin{aligned}\boldsymbol{\Delta}\boldsymbol{\nabla}_{A_1}\cdots\boldsymbol{\nabla}_{A_\ell} =& \boldsymbol{\nabla}_{A_1}\cdots\boldsymbol{\nabla}_{A_\ell}\boldsymbol{\Delta}\\ &+ 2\boldsymbol{R}^C{}_{A_1}{}^\sharp\boldsymbol{\nabla}_C\boldsymbol{\nabla}_{A_2}\cdots\boldsymbol{\nabla}_{A_\ell}\\ &+ 2\boldsymbol{\nabla}_{A_1}\boldsymbol{R}^C{}_{A_2}{}^\sharp\boldsymbol{\nabla}_C\boldsymbol{\nabla}_{A_3}\cdots\boldsymbol{\nabla}_{A_\ell}\\ &+\cdots\\ &+ 2\boldsymbol{\nabla}_{A_1}\cdots\boldsymbol{\nabla}_{A_{\ell-1}}\boldsymbol{R}^C{}_{A_\ell}{}^\sharp\boldsymbol{\nabla}_C + \mathcal{O}(Q^{d/2-\ell-1})\,.\end{aligned} \tag{2.8}$$

Note that this identity holds to order $\mathcal{O}(Q^{d/2-\ell-1})$ because $\boldsymbol{R}$ is only determined uniquely to order $\mathcal{O}(Q^{d/2-2})$. Going forward, we will use the simplifying notation of [**37**], which omits the details of contractions and coefficients in exchange for symbolic brevity. For example, applying the Leibniz rule, we can write the result in Equation (2.8) as

$$\boldsymbol{\Delta}\boldsymbol{\nabla}^\ell = \boldsymbol{\nabla}^\ell\boldsymbol{\Delta} + \sum(\boldsymbol{\nabla}^p\boldsymbol{R})\boldsymbol{\nabla}^q\,,$$

where each term on the right-hand side of the above equation has $q \geq 1$ and $p+q=\ell$. Observe from simple counting that any term of the form $\boldsymbol{\Delta}\boldsymbol{\nabla}^{\ell}$ appearing in Equation (2.7) has $\ell < k$, so because $k < d/2$, we have that using Equation (2.8) is valid in all cases here. In order to further simplify our results, we need another identity, which applies for any expression E:

$$\boldsymbol{\Delta}(\boldsymbol{\nabla}^t\boldsymbol{\Delta}^u\boldsymbol{R})E = (\boldsymbol{\Delta}\boldsymbol{\nabla}^t\boldsymbol{\Delta}^u\boldsymbol{R})E + (\boldsymbol{\nabla}^t\boldsymbol{\Delta}^u\boldsymbol{R})\boldsymbol{\Delta}E + (\boldsymbol{\nabla}^{t+1}\boldsymbol{\Delta}^u\boldsymbol{R})\boldsymbol{\nabla}E\,. \tag{2.9}$$

A result from [**37**, Proposition 4.3] shows that for a conformally flat structure, when $T \in \Gamma(\mathcal{T}^{\Phi}\widetilde{M}\,(k-d/2))$, we have that

$$\boldsymbol{\Delta}\boldsymbol{D}_{A_{k-1}}\cdots\boldsymbol{D}_{A_1}T = (-1)^{k-1}\boldsymbol{X}_{A_1}\cdots\boldsymbol{X}_{A_{k-1}}\boldsymbol{\Delta}^k T\,.$$

Therefore, except the term $(-1)^{k-1}\boldsymbol{X}_{A_1}\cdots\boldsymbol{X}_{A_{k-1}}\boldsymbol{\Delta}^k$, all of the terms remaining after commuting $\boldsymbol{X}$s to the left except must contain at least one curvature $\boldsymbol{R}$ (because $\boldsymbol{R}$ vanishes for conformally flat structures). Thus, we can write

$$\boldsymbol{\Delta}\boldsymbol{D}_{A_1}\cdots\boldsymbol{D}_{A_{k-1}}T = (-1)^{k-1}\boldsymbol{X}^{k-1}\boldsymbol{\Delta}^k T + \sum \boldsymbol{h}^s\boldsymbol{X}^x(\boldsymbol{\nabla}^{p_1}\boldsymbol{\Delta}^{r_1}\boldsymbol{R})\cdots(\boldsymbol{\nabla}^{p_n}\boldsymbol{\Delta}^{r_n}\boldsymbol{R})\boldsymbol{\nabla}^q\boldsymbol{\Delta}^r T\,, \tag{2.10}$$

where $n \geq 1$ for each term on the right-hand side.

We now apply some counting arguments to check that our expression is uniquely determined. With the exception of the application of Equation (2.9), all of the identities we have considered cannot increase the sum of the number of $\boldsymbol{\nabla}$s, $\boldsymbol{\Delta}$s, and $\boldsymbol{R}$s on the right-hand side of Equation (2.10), represented by $n+q+r$. On the other hand, Equation (2.9) increases $q+r$ by at most one at the cost of one $\boldsymbol{\Delta}$ acting from the left. However, we had at most k such symbols initially, so we observe that $n+q+r \leq k$ and hence $k-q-r \geq 1$. Thus, because $k < d/2$, we have that $q+r < d/2-1$. Because both $\boldsymbol{\nabla}$ and $\boldsymbol{\Delta}$ are determined modulo terms of order $\mathcal{O}(Q^{d/2-1})$ and because $[\boldsymbol{\Delta}, Q] = 2(d+2\boldsymbol{\nabla}_{\boldsymbol{X}}+2)$, we have that the operator $\boldsymbol{\nabla}^q\boldsymbol{\Delta}^r$ is uniquely determined modulo terms of order $\mathcal{O}(Q^{d/2-q-r})$. Thus, $\boldsymbol{\nabla}^q\boldsymbol{\Delta}^r$ as an operator on T is uniquely determined along $\mathcal{Q}$.

Further, observe that because there is always at least one $\boldsymbol{\nabla}$ on the right in Equation (2.8), we have that $q \geq 1$, and so at most $q-1$ $\boldsymbol{\nabla}$s on T can come from Equation (2.9) or from the application of the typical Leibniz rule. Thus, $p_i + r_i \leq q-1$, because the Leibniz rule and Equation (2.9) are the only way to get more derivatives on $\boldsymbol{R}$. But because $k-q-r \geq 1$ and $r \geq 0$, we have that

$p_i + r_i + 2 \leq k$ and hence $p_i + r_i < d/2 - 2$. But by a similar argument as above, $\boldsymbol{\nabla}^{p_i}\boldsymbol{\Delta}^{r_i}\boldsymbol{R}$ is determined uniquely up to order $\mathcal{O}(Q^{d/2-2-p_i-r_i})$ and so we have that all of the terms containing curvature are uniquely determined.

A straightforward extension of [**48**, Proposition 2.2] to general tensor structures implies that, given some homogeneous tensor T of weight $k - d/2$ along Q, $T|_{\mathcal{Q}}$ uniquely (to order $\mathcal{O}(Q^k)$) determines a canonical extension $\tilde{T}$ by requiring that $\boldsymbol{\Delta}\tilde{T} = \mathcal{O}(Q^{k-1})$. Thus, to make our calculations easier, we choose $T := \tilde{T}$ that is harmonic in this sense. With this prescription in place, we have that $\boldsymbol{\nabla}^q\boldsymbol{\Delta}^r T = \mathcal{O}(Q^{k-r-q})$ and thus vanishes along Q. So we will drop all terms in Equation (2.10) with $r > 0$.

A useful identity that holds regardless of tensor type is [**37**, Equation 42]; acting on $T \in \Gamma(\mathcal{T}^\Phi\widetilde{M}\,(k - d/2))$, we have that

$$2(k-\ell-1)\boldsymbol{\nabla}^{\ell+1}T = \boldsymbol{D}\boldsymbol{\nabla}^\ell\tilde{T} + \boldsymbol{X}\sum(\boldsymbol{\nabla}^p\boldsymbol{R})\boldsymbol{\nabla}^q T + \boldsymbol{X}\boldsymbol{\nabla}^\ell\boldsymbol{\Delta}T\,, \tag{2.11}$$

where $q \geq 1$ and $p + q = \ell$. Note that this identity follows from Equation (2.8). But because $\boldsymbol{\Delta}T = \mathcal{O}(Q^{k-1})$, we can drop the last term on the right-hand side when using this identity. Because $p_i + r_i + 2 \leq k$ and $k < d/2$, we have that $p_i + r_i \leq d/2 - 3$, so we can safely apply Lemma 2.4.2 and Equation (2.11) to substitute all occurrences of $\boldsymbol{\nabla}$ and $\boldsymbol{\Delta}$ in Equation (2.10) with $\boldsymbol{D}$, $\boldsymbol{X}$, and $\boldsymbol{R}$. Therefore, we can write

$$(-1)^{k-1}\boldsymbol{X}^{k-1}\boldsymbol{\Delta}^k\tilde{T} = \boldsymbol{\Delta}\boldsymbol{D}_{A_1}\cdots\boldsymbol{D}_{A_{k-1}}\tilde{T} + \Psi(\tilde{T})\,,$$

where Ψ is an operator that is polynomial in $\boldsymbol{h}, \boldsymbol{X}, \boldsymbol{D}, \boldsymbol{R}$ and rational in d. Note that Ψ must end in $\boldsymbol{D}$ because $q \geq 1$ everywhere. Then note that each of these operators on the right-hand side descend to their tractor counterparts. From Proposition 2.4.1, the left-hand side descends to the desired tractor operator P_{2k}^Φ which completes the proof.

For a more detailed exposition of the scalar case, see [**37**, Section 4]. □

A series of results that will be useful later follow from the above proposition. These results are tractor generalizations of results from [**38**]. We first rewrite the operator Ψ in Proposition 2.4.3 in the following proposition:

Proposition 2.4.4 (Generalization of Proposition 5.10 of [**38**]). *Let $k < d/2$ and d even. For any tractor $T \in \Gamma(\mathcal{T}^\Phi M[k-d/2])$, there exists a conformally invariant operator*

$$\mathcal{P}^{\Phi,k}_{A_1\cdots A_{k-1}} : \Gamma(\mathcal{T}^\Phi M[k-d/2]) \to \Gamma(\mathcal{T}^\Phi_{A_1\cdots A_{k-1}} M[-1-d/2])$$

such that

$$(-1)^{k-1} X_{A_1} \cdots X_{A_{k-1}} P^\Phi_{2k} T = \Box D_{A_1} \cdots D_{A_{k-1}} T + \mathcal{P}^{\Phi,k}_{A_1\cdots A_{k-1}} T\,,$$

where $\mathcal{P}^{\Phi,k}_{A_1\cdots A_{k-1}}$ has a tractor formula, is at least order 1 in W, and has weight $-k-1$.

Proof. By identifying $\mathcal{P}^{\Phi,k}_{A_1\cdots A_{k-1}}$ with $(-1)^{k-1}\Psi_{A_1\cdots A_{k-1}}{}^C D_C$, the proposition follows. $\square$

Proposition 2.4.4 directly implies the following proposition:

Proposition 2.4.5 (Generalization of Proposition 5.14 of [**38**]). *There exists a family of operators*

$$P^\Phi_{A_1\cdots A_k} : \Gamma(\mathcal{T}^\Phi M[w]) \to \Gamma(\mathcal{T}^\Phi_{A_1\cdots A_k} M[w-k])$$

defined by

$$P^\Phi_{A_1\cdots A_k} T = D_{A_1} \cdots D_{A_k} T - X_{A_1} \mathcal{P}^{\Phi,k}_{A_2\cdots A_k} T$$

for all $T \in \Gamma(\mathcal{T}^\Phi M[w])$. Then, we have that when $w = k - d/2$,

$$P^\Phi_{A_1\cdots A_k} T = (-1)^k X_{A_1} \cdots X_{A_k} P^\Phi_{2k} T\,.$$

Proof. Observe that there exists a tractor formula for $\mathcal{P}^{\Phi,k}_{A_1\cdots A_{k-1}}$ in terms of h, X, D, and W, so combining that tractor formula with $D_{A_1} \cdots D_{A_k}$ allows us to construct a well-defined operator $P^\Phi_{A_1\cdots A_k}$ as in the proposition statement. Then, observe that acting on tractors of weight $1 - d/2$, $D_A = -X_A \Box$, so the remainder of the proposition follows from Proposition 2.4.3 and Proposition 2.4.4. $\square$

These generalizations will be key in Section 4.1.

2.5. Insertion and Construction of Conformal Invariants

Given a tractor expression, one can extract a conformally-invariant Riemannian expression by computing the tractor expression in a choice of scale and applying the extraction operator q^* if necessary. However, the problem of constructing new conformal invariants given a conformally-invariant Riemannian expression, *i.e.* a density-valued tensor (or scalar), is more challenging. To do so, we introduce the *insertion operator* q, which is a right-inverse of the extraction map q^*. The insertion operator is a canonical map that inserts a tensor $t \in \Gamma(T^\phi M[w])$ into a tractor $T \in \Gamma(\mathcal{T}^\Phi M[w'])$ where ϕ and Φ have the same tensor structures such that the projecting part of the tractor T is given by t. While this operator is defined for (almost) all tensor-valued densities, we provide the explicit map for three such operators.

LEMMA 2.5.1. *Let $g \in \boldsymbol{c}$.*

(i) Given $v_a \in \Gamma(T^*M[w+1])$ where $w \neq 1-d$, then

$$q(v_a) =: V^A \in \Gamma(\mathcal{T}M[w])$$

$$\stackrel{g}{=} \begin{pmatrix} 0 \\ v^a \\ -\frac{\nabla \cdot v}{d+w-1} \end{pmatrix},$$

and

$$D_A V^A = X_A V^A = 0\,.$$

(ii) Given $t_{ab} \in \Gamma(\odot^2_\circ T^*M[w+2])$ where $w \neq -d, 1-d$, then

$$q(t_{ab}) =: T^{AB} \in \Gamma(\odot^2_\circ \mathcal{T}M[w])$$

$$\stackrel{g}{=} \begin{pmatrix} 0 & 0 & 0 \\ 0 & t^{ab} & -\frac{\nabla \cdot t^a}{d+w} \\ 0 & -\frac{\nabla \cdot t^b}{d+w} & \frac{\nabla \cdot \nabla \cdot t + (d+w) P_{ab} t^{ab}}{(d+w)(d+w-1)} \end{pmatrix},$$

and

$$D_A T^{AB} = 0 = X_A T^{AB}\,.$$

(iii) Given $t_{abcd} \in \Gamma(\otimes^4 T^*M[w+4])$, where $w \neq 1-d, 2-d$, such that t has the algebraic symmetries of the Riemann tensor and is trace-free, then

$$q(t_{abcd}) =: T^{ABCD} \in \Gamma(\otimes^4 \mathcal{T}M[w])$$

where

$$
\begin{aligned}
T^{abcd} &\stackrel{g}{=} t^{abcd} \\
T^{abc-} &\stackrel{g}{=} -\frac{\nabla_d t^{dabc}}{d+w-1} \\
T^{a-b-} &\stackrel{g}{=} \frac{\nabla_a \nabla_c t^{abcd} + (d+w-1)P_{ac}t^{abcd}}{(d+w-1)(d+w-2)}\,.
\end{aligned}
$$

Further, T also has the algebraic symmetries of the Riemann tensor and

$$
D_A T^{ABCD} = X_A T^{ABCD} = 0 = h_{AC} T^{ABCD}\,.
$$

PROOF. The proofs of the above three results can either be given in a much more general setting (see [**33**]), or by explicit computation whose intricacy increases in concordance with tensor rank. We give the lowest rank case, and the rest follow by similar arguments.

The "bottom slot" V^- of $q(v_a) := V^A \stackrel{g}{=} (0, v^a, v^-)$ can be computed by writing out the constraint $D_A V^A \stackrel{g}{=} 0$ for some $g \in \boldsymbol{c}$:

$$
\begin{aligned}
0 &= (d+2w-2)(wv^- + \nabla\!\cdot\! v + Jv^+ + dv^-) - (\Delta + (w-1)J)v^+ + 2\nabla\!\cdot\! v + dv^- \\
&= (d+2w)(\nabla\!\cdot\! v + (d+w-1)v^-)\,,
\end{aligned}
$$

where the second equality comes from the requirement that $X_A V^A = 0$. When $w \neq -\frac{d}{2}$, this yields the quoted result. If $w = -\frac{d}{2}$, we need to verify that the result for V^A given for a choice of $g \in \boldsymbol{c}$ defines a section of $\Gamma(\mathcal{T}M[w])$. This is easily established by transforming the quoted result to a conformally related metric. $\square$

REMARK 2.5.2. Observe that the W tractor can be constructed by the insertion of the Weyl tensor into a tractor with the same symmetries:

$$
W^{ABCD} = q(W_{abcd})\,.
$$

$\blacksquare$

While the insertion operator is a right-inverse of the extraction map so that $q^* \circ q = \mathrm{Id}$, it is not a left-inverse. Instead, we can compute the difference between the identity operator and $q \circ q^*$; we do so on a particularly useful tractor bundle, the result of which is given in the following lemma.

LEMMA 2.5.3. *Let the tractor $T \in \Gamma(\odot_\circ^2 \mathcal{T}M[w])$, where $w \neq 1 - \frac{d}{2}, -\frac{d}{2}, -d, 1-d$, obey*

$$X_A T^{AB} = 0 \ \ \text{and} \ \ q^*(T) \in \Gamma(\odot_\circ^2 TM[w-2])\,.$$

Then

$$(q \circ q^*)(T) = \tilde{T}\,,$$

where

$$\tilde{T}_{AB} := T_{AB} - \tfrac{2}{(d+w)(d+2w)} X_{(A} D^C T_{B)C} + \tfrac{1}{(d+w)(d+w-1)(d+2w)} X_A X_B D^C \hat{D}^D T_{CD}\,.$$

PROOF. We will establish that there is a unique $\tilde{T}$ that satisfies

$$\hat{D}^A \tilde{T}_{AB} = X^A \tilde{T}_{AB} = 0 = h^{AB} \tilde{T}_{AB}\,,$$

and obeys $q^*(\tilde{T}) = q^*(T)$ whenever $q^*(T) \in \Gamma(\odot_\circ^2 TM[w-2])$. This ensures that $(q \circ q^*)(T) = \tilde{T}$. For that, we use the operator version of Proposition 2.3.1, valid acting on tractors of weight $w \neq 1 - d/2, -d/2$:

$$\hat{D}^A \circ X^B = X^B \hat{D}^A + h^{AB} - \tfrac{2}{d+2w} X^A \hat{D}^B\,. \tag{2.12}$$

We first verify that $X \cdot \tilde{T} = 0$. Because $X^2 = 0 = X^A T_{AB}$, we simply need to check that $X^A \hat{D}^B T_{AB}$ vanishes. Applying Equation (2.12) we have that

$$\begin{aligned} X^A \hat{D}^B T_{AB} &= \left(\hat{D}^B X^A - h^{AB} + \tfrac{2}{d+2w} X^B \hat{D}^A\right) T_{AB} \\ &= \tfrac{2}{d+2w} X^B \hat{D}^A T_{AB}\,, \end{aligned}$$

where we have used that $h^{AB} T_{AB} = 0 = X^A T_{AB}$. Because T is symmetric, we are left with the identity

$$\tfrac{d+2w-2}{d+2w} X^A \hat{D}^B T_{AB} = 0\,.$$

Thus, thanks to the weight assumptions, it follows that $X^A \hat{D}^B T_{AB} = 0$, and hence $X^A \tilde{T}_{AB} = 0$. Similarly, we have that $h^{AB} \tilde{T}_{AB} = 0$.

Finally, we check that $\hat{D}^A \tilde{T}_{AB} = 0$. We do this in stages. First, we evaluate $\hat{D}^A(X_A \hat{D}^C T_{BC})$:

$$\begin{aligned}\hat{D}^A(X_A \hat{D}^C T_{BC}) &= \left[(w-1) + d + 2 - \tfrac{2(w-1)}{d+2(w-1)}\right] \hat{D}^C T_{BC} \\ &= \tfrac{(d+w-1)(d+2w)}{d+2w-2} \hat{D}^C T_{BC} \,.\end{aligned}$$

Next, we evaluate the term $\hat{D}^A(X_B \hat{D}^C T_{AC})$:

$$\begin{aligned}\hat{D}^A(X_B \hat{D}^C T_{AC}) &= X_B \hat{D}^A \hat{D}^C T_{AC} + \hat{D}^C T_{BC} - \tfrac{2}{d+2(w-1)} X^A \hat{D}_B \hat{D}^C T_{AC} \\ &= X_B \hat{D}^A \hat{D}^C T_{AC} + \hat{D}^C T_{BC} \\ &\qquad - \tfrac{2}{d+2(w-1)} \left[\hat{D}_B(X^A \hat{D}^C T_{AC}) - h^A_B \hat{D}^C T_{AC} + \tfrac{2}{d+2(w-1)} X_B \hat{D}^A \hat{D}^C T_{AC}\right] \\ &= \tfrac{(d+2w)(d+2w-4)}{(d+2w-2)^2} X_B \hat{D}^A \hat{D}^C T_{AC} + \tfrac{d+2w}{d+2w-2} \hat{D}^C T_{BC} \,.\end{aligned}$$

Last, we evaluate the term $\hat{D}^A(X_A X_B \hat{D}^C \hat{D}^D T_{CD})$:

$$\hat{D}^A(X_A X_B \hat{D}^C \hat{D}^D T_{CD}) = \tfrac{(d+w-1)(d+2w)}{d+2w-2} X_B \hat{D}^C \hat{D}^D T_{CD} \,.$$

Combining these terms, we find that $\hat{D}^A \tilde{T}_{AB} = 0$, thus completing the proof. □

REMARK 2.5.4. Note that if a weight $w \neq 1 - \frac{d}{2}, -\frac{d}{2}$ tractor $\tilde{T}^{AB\cdots}$ obeys

$$X_A \tilde{T}^{AB\cdots} = 0 = \hat{D}_A \tilde{T}^{AB\cdots} \,,$$

then it follows directly from Equation (2.12) that

$$X_A \hat{D}^C \tilde{T}^{AB\cdots} = -\tilde{T}^{CB\cdots} \,.$$

■

While the insertion operator allows us to apply conformally-invariant tractor operators (such as contraction with W or application of $\hat{D}$) to Riemannian tensor-valued densities, more conformal invariants can be extracted by manipulating the projecting part of a tractor. Given a tractor $T \in \Gamma(\mathcal{T}^\Phi M[w])$, this can be achieved by applying specific tractor-valued differential operators to

T so that the original projecting part of T vanishes. We name such operators *removal operators*, denoted by r; two examples of such removal operators are given in the following lemma.

LEMMA 2.5.5. *Let $V^A \in \Gamma(\mathcal{T}M[w])$ and $T \in \Gamma(\odot^2_\circ \mathcal{T}M[w])$. Then if $w \neq -1, -1-\frac{d}{2}$,*

$$r^A(V) := V^A - \frac{1}{w+1}\hat{D}^A\big(X_B V^B\big)$$

obeys

$$X_A\, r^A(V) = 0\,,$$

while if $w \neq 0, -1, -\frac{d}{2}, -1-\frac{d}{2}, -2-\frac{d}{2}$,

$$\begin{aligned} r^{AB}(T) := T^{(AB)\circ} - \tfrac{2}{w}\hat{D}^{(A}\big(X_C T^{|C|B)\circ}\big) + \tfrac{1}{w(w+1)}\hat{D}^{(A}\hat{D}^{B)\circ}\big(X_C X_D T^{CD}\big) \\ - \tfrac{8}{wd(d+2w+2)} X^{(A}\hat{D}^{B)\circ}\big(\hat{D}_C(X_D T^{CD})\big)\,, \end{aligned}$$

obeys

$$X_A\, r^{AB}(T) = 0 = h_{AB}\, r^{AB}(T)\,.$$

PROOF. The proof is an elementary application of the identity

$$X_A \hat{D}^A T = wT\,,$$

valid for any weight $w \neq 1-\frac{d}{2}$ tractor T, the fact that X and D are null, and Equation (2.12). $\square$

Acting on a trace-free rank-2 tractor, the operator r adds tractor-valued terms to a trace-free rank-2 symmetric tractor T so that the projecting part is the middle slot rather than the top slot. It can then be composed with the extraction operator to extract a conformally invariant rank-2 symmetric trace-free tensor. That is, for a generic $T \in \Gamma(\odot^2_\circ \mathcal{T}M[w])$, we have that

$$(q^* \circ r)(T^{AB}) \in \Gamma(\odot^2_\circ TM[w-2]) \cong \Gamma(\odot^2_\circ T^*M[w+2])\,,$$

where the isomorphism is obtained in the usual way with the conformal metric $\boldsymbol{\gamma}$.

CHAPTER 3

Conformal Hypersurface Geometry

Growing out of work on the AdS/CFT conjecture, including studies in entanglement entropy [**56**, **71**] and quantum complexity [**1**, **13**], there has been increased interest in conformal hypersurface geometry. In the mathematics literature, conformal hypersurface geometry has shown to be useful for analyzing properties of formal eigenfunctions of the Laplace equation on hyperbolic manifolds [**11**, **21**, **40**, **50**]. In this chapter, we provide an overview of Riemannian hypersurface geometry, a summary of known results in conformal hypersurface geometry, and new results in the development of the hypersurface tractor calculus.

3.1. Riemannian Hypersurface Calculus

Let (M^d, g) be a d-dimensional Riemannian manifold with a hypersurface Σ smoothly embedded in M; we will denote this by $\Sigma \hookrightarrow (M, g)$. In this regard, we treat Σ as a codimension 1 submanifold of M and use the same coordinates and indices for Σ as for M, and we will assume that Σ is closed, orientable, and that its embedding in M is separating so that $M = M^+ \sqcup \Sigma \sqcup M^-$. Because Σ is codimension 1, it is particularly easy to describe the embedding via a defining function. A *defining function* for Σ is a function $s \in C^\infty M$ such that, for every $p \in M$, $ds|_p \neq 0$ and $s(p) = 0$ if and only if $p \in \Sigma$. We can therefore study Riemannian geometry by studying the triple (M, g, s). Observe, however, that s is not uniquely determined by the hypersurface Σ because for any positive function $f \in C^\infty_+ M$, the function fs is also defining for Σ.

To resolve this ambiguity, we can demand that s solves a canonical problem on M. In general, one can always solve [**86**] the problem where $|ds|_g = 1$ in a neighborhood of Σ to arbitrarily high order in s; another proof of this result was given in [**45**] via induction. Thus, given $\Sigma \hookrightarrow (M, g)$ we can always uniquely determine a triple (M, g, s) such that $|ds|_g = 1$, and we label each such triple with a subscript u, *i.e.* $(M, g, s)_u$. We now provide a summary of well-known results for the geometry of (M, g, s). Note that these results are simplified when $(M, g, s) = (M, g, s)_u$.

Given such a triple (M, g, s), we define the *conormal* $n := ds|_\Sigma \in \Gamma(T^*M)|_\Sigma$. When $|ds|_g \neq 1$, we define the *unit conormal* by $\hat{n} := (n/|n|_g)|_\Sigma \in \Gamma(T^*M)|_\Sigma$. Using this covector, we can decompose the ambient tangent bundle into a hypersurface tangent bundle and the normal bundle via the hypersurface projector (also sometimes called the *first fundamental form*)

$$\bar{g} : \Gamma(TM)|_\Sigma \to \Gamma(T\Sigma)\,,$$

specified by

$$\bar{g}_a^b := \delta_a^b - \hat{n}_a \hat{n}^b\,.$$

As mentioned above, this isomorphism allows us to use the same abstract index notation to represent tensors on the tangent and normal bundles on Σ as on the tangent bundle on M. Using this projector, we can compute the metric on Σ induced by g. For notational simplicity, we denote this induced metric with the same symbol as the projector, so that

$$\bar{g}_{ab} := \bar{g}_a^{a'} \bar{g}_b^{b'} g_{a'b'} = g_{ab} - \hat{n}_a \hat{n}_b \in \Gamma(\odot^2 T^*\Sigma)\,.$$

We will use the notation $\top$ to represent tensors that have been projected to the hypersurface, so that for $v \in \Gamma(TM)|_\Sigma$, we might write $(v^\top)^a := v^a - \hat{n}^a \hat{n}_b v^b = \bar{g}_b^a v^b$. Often, we will extend the operator $\top$ to act on tensors (or operators) in M via restriction to Σ, so that for $v \in \Gamma(TM)$, we use $v^\top$ to denote $(v|_\Sigma)^\top$. Sometimes, the symbol $\top$ will be used to represent an operator, so that $\top(v) = v^\top$. Using that notation, it can be useful to project to the trace-free part of the hypersurface tensor bundle, and we represent this operation with $\mathring{\top}$. Thus, we might write $\mathring{\top}(t_{ab}) = t_{ab}^\top - \frac{1}{d-1}\bar{g}_{ab}(t^\top)_c^c$. Further, as first exemplified by the induced metric $\bar{g}$, when a scalar, tensor, or operator is intrinsic to the hypersurface, we will decorate that object with an overbar. Finally, we will use the symbol "$\stackrel{\Sigma}{=}$" to refer to equalities that only hold (or make sense) along Σ.

Given the induced metric, there exists a canonical induced Levi-Civita connection on Σ denoted by $\bar{\nabla}$. A fundamental result attributable to Gauß is the relationship between the induced Levi-Civita connection and the projected Levi-Civita connection. Acting on a hypersurface vector $\bar{v} \in \Gamma(T\Sigma)$, we have that

$$\bar{\nabla}_a \bar{v}^b = \nabla_a^\top v^b\big|_\Sigma + \hat{n}^b \mathrm{I\!I}_{ac} \bar{v}^c\,, \tag{3.1}$$

where v^b is any extension of $\bar{v}$ to M and II_{ab} is the *second fundamental form*, defined by $\mathrm{II}_{ab} := (\nabla \hat{n}^{\mathrm{e}})^{\top}_{ab} \in \Gamma(\odot^2 T^*\Sigma)$ where $\hat{n}^{\mathrm{e}}_a$ is any extension of the unit conormal. The trace of the second fundamental form yields the *mean curvature* given by $H := \frac{1}{d-1}\mathrm{II}_a^a$ and, and its trace-free part is

$$\mathring{\mathrm{II}}_{ab} = \mathrm{II}_{ab} - H\bar{g}_{ab}\,.$$

Of note is that the operator $\nabla^{\top}$ is called *tangential* because $\nabla^{\top} \circ s \stackrel{\Sigma}{=} 0$. Specifically, we say that an operator O is tangential when the action of O on a section of a vector bundle over M evaluated along the boundary depends only on the restriction of that section to the boundary. We say that such an operator acts *tangentially along* Σ when there exists a smooth operator O' such that $\mathsf{O} \circ s = s \circ \mathsf{O}'$. Such operators are particularly important in the study of embedded hypersurface geometry.

A consequence of this relationship is the Gauß equation, relating the intrinsic Riemann curvature of $\bar{\nabla}$ to the projected Riemann curvature of ∇:

$$R^{\top}_{abcd} = \bar{R}_{abcd} - \mathrm{II}_{ac}\mathrm{II}_{bd} + \mathrm{II}_{ad}\mathrm{II}_{bc}\,. \tag{3.2}$$

Similarly, we can decompose the various projections of the bulk Weyl curvature and Schouten tensors in terms of hypersurface tensors. For $d \geq 3$,

$$2\bar{\nabla}_{[a}\mathring{\mathrm{II}}_{b]c} - \tfrac{2}{d-2}\bar{g}_{c[a}\bar{\nabla}\cdot\mathring{\mathrm{II}}_{b]} = W^{\top}_{abc\hat{n}}\,, \tag{3.3}$$

$$\mathring{\mathrm{II}}^2_{ab} - \tfrac{1}{2(d-2)}\mathring{\mathrm{II}}^2\bar{g}_{ab} + (d-3)\left(\bar{P}_{ab} - H\mathring{\mathrm{II}}_{ab} - \tfrac{1}{2}\bar{g}_{ab}H^2\right) = W_{\hat{n}ab\hat{n}} + (d-3)P^{\top}_{ab}\,, \tag{3.4}$$

$$\bar{\nabla}\cdot\mathring{\mathrm{II}}_a - (d-2)\bar{\nabla}_a H = P^{\top}_{a\hat{n}}\,, \tag{3.5}$$

$$\bar{J} - \tfrac{d-1}{2}H^2 + \tfrac{\mathring{\mathrm{II}}^2}{2(d-2)} = J - P_{\hat{n}\hat{n}}\,, \tag{3.6}$$

where in general we write $T^{\top}_{\hat{n}ab\cdots} \equiv (T_{\hat{n}ab\cdots})^{\top}$ and we will often write $\bar{\nabla}\cdot\mathring{\mathrm{II}}_a \equiv \bar{\nabla}^b\mathring{\mathrm{II}}_{ab}$. Equations (3.3) and (3.5) are usually referred to as the Codazzi–Mainardi equations, Equation (3.4) is usually known as the Fialkow–Gauß equation, and Equation (3.6) is usually known as Gauß' *Theorema Egregium.*

3.2. Hypersurface Density Calculus

We now consider the geometry of conformally embedded hypersurfaces. Analogous to the above section, we let $(M^d, \boldsymbol{\gamma})$ be a d-dimensional (with $d \geq 3$) conformal manifold with a hypersurface Σ

smoothly embedded in M so that $\Sigma \hookrightarrow (M, \gamma)$, where Σ is closed, orientable, and the embedding of Σ in M is separating, as above. Rather than using a defining function, we instead describe $\Sigma \hookrightarrow (M, \gamma)$ with a *defining density.* A defining density is a scale $\sigma \in \Gamma(\mathcal{E}M[1])$ such that for each $g \in \boldsymbol{c}$, we have that s is defining, where s is determined by $\sigma = [g; s]$. Then, a conformally embedded hypersurface $\Sigma \hookrightarrow (M, \gamma)$ can be specified by the triple (M, γ, σ). Note that many of the names and symbols in what follow overload the names and symbols in the Riemannian setting, but their meaning should be clear from context.

Given such a triple, we can begin to characterize the conformal invariants by studying σ. First, observe that for a defining function σ for Σ, we have that in the scale specified by $[\Omega^2 g; \Omega s]$, the exterior derivative of σ along Σ satisfies $d(\Omega s) \stackrel{\Sigma}{=} \Omega ds$. Because each choice of scale induces a Riemannian metric on Σ, the conformal structure $\boldsymbol{c}$ on M induces a conformal structure $\bar{\boldsymbol{c}}$ on Σ and hence we can consider density bundles over Σ. In particular, we observe that $d\sigma|_\Sigma$ is a covector-valued density, and so we define the *conormal* by $n := d\sigma|_\Sigma = [g; ds] \in \Gamma(T^*M[1])|_\Sigma$. While the conormal is often useful, because n has weight 1, we cannot demand that it has some canonical length like in the Riemannian case. Instead, only the *unit conormal* $\hat{n} := n/|n|_\gamma \in \Gamma(T^*M[0])|_\Sigma$ is canonically determined along Σ. But then the induced metric is a representative of the induced conformal metric so that $\bar{\gamma}_{ab} := \gamma_{ab} - \hat{n}_a \hat{n}_b \in \Gamma(\odot^2 T^*M[2])$.

Given $g \in \boldsymbol{c}$, the corresponding second fundamental form is not conformally invariant because its trace $(d-1)H^g$ (where H^g is the mean curvature of $\Sigma \hookrightarrow (M, g)$) cannot be viewed as a representative of a density. Indeed, under the transformation $g \mapsto \Omega^2 g$, we have that $H^g \mapsto H^{\Omega^2 g} = \Omega^{-1}(H^g + \hat{n} \cdot \Upsilon)$. However, the trace-free part of the second fundamental form is conformally invariant; that is, $\mathring{\mathrm{II}}_{ab} := [g; \mathring{\mathrm{II}}_{ab}] \in \Gamma(\odot^2_\circ T^*\Sigma[1])$. As is done here, we will typically use the same symbol for a density as its representative. Observe that the tensor-valued density $\mathring{\mathrm{II}}$ can also be defined in the same way as the Riemannian second fundamental form via a tangential density-coupled Levi-Civita connection $\nabla^\top$ so that the density-valued definition holds: $\mathring{\mathrm{II}} = \nabla^\top \hat{n}^{\mathrm{e}}$. Note that tangentiality extends to operators acting on densities: specifically, an operator O acting on weight w densities can be said to act tangentially when there exists a smooth operator O' acting on weight $w-1$ densities such that $\mathsf{O} \circ \sigma = \sigma \circ \mathsf{O}'$.

The Weyl curvature, the unit conormal, and the second fundamental form are critical components of conformal hypersurface geometry. Indeed, from Equation (3.3) we see that the the covariant trace-free curl of the second fundamental form is conformally invariant. This is because the tensor $W^{\top}_{abc\hat{n}}$ is constructed solely from conformally invariant tensors. Thus, we find that there exists a conformally-invariant operator, known as the Codazzi operator [**45**] denoted by Cod that acts on weight 1 trace-free rank two symmetric tensors via

$$\mathring{K}_{ab} \stackrel{\mathrm{Cod}}{\mapsto} 2\bar{\nabla}_{[a}\mathring{K}_{b]c} - \tfrac{2}{d-2}\bar{g}_{c[a}\bar{\nabla}\cdot\mathring{K}_{b]}\,.$$

Furthermore, we can rearrange Equation (3.4) to obtain the Fialkow-Gauß equation:

$$\mathring{\mathrm{I\!I}}^2_{ab} - \tfrac{1}{2(d-2)}\mathring{\mathrm{I\!I}}^2\bar{g}_{ab} - W_{\hat{n}ab\hat{n}} = (d-3)\left(P^{\top}_{ab} - \bar{P}_{ab} + H\,\mathring{\mathrm{I\!I}}_{ab} + \tfrac{1}{2}\bar{g}_{ab}H^2\right)\,, \tag{3.7}$$

which shows that the right-hand of the above display is conformally invariant. Defining this tensor as $(d-3)F_{ab}$, where F is the *Fialkow tensor*, we have that

$$F := \left[g; P^{\top} - \bar{P} + H\,\mathring{\mathrm{I\!I}} + \tfrac{1}{2}\bar{g}H^2\right] \in \Gamma(\odot^2 T^*\Sigma[0])\,. \tag{3.8}$$

Often it is more useful to consider the trace-free part of this tensor so a simple computation yields

$$F^a_a = \frac{K}{2(d-2)}\,,$$

where $K := \mathring{\mathrm{I\!I}}^2 \in \Gamma(\mathcal{E}\Sigma[-2])$ is known as the *ridigity density*. Then, we have that

$$\mathring{F}_{ab} := \left[g; \tfrac{1}{d-3}\left(\mathring{\mathrm{I\!I}}^2_{(ab)\circ} - W_{\hat{n}ab\hat{n}}\right)\right] \in \Gamma(\odot^2_\circ T^*\Sigma[0])\,.$$

3.3. Hypersurface Tractor Calculus

Because Σ has an induced conformal structure $\bar{\boldsymbol{c}}$ from $(M^d, \boldsymbol{c})$, all of the tractor constructions in the Chapter 2 also exist for the hypersurface Σ (so long as $d \geq 4$). Thus, just as the relationship between $\nabla^{\top}$ and $\bar{\nabla}$ (and hence Equations 3.3-3.6) enabled the construction of new extrinsic conformal invariants, it is useful to find relationships between the tractor analogs. The key idea behind what follows is that the Thomas-D operator fills the role of the covariant derivative in the tractor setting, so we proceed as in the above discussion. In this section, we consider the triple (M, γ, σ) which specifies the hypersurface embedding of interest.

Rather than beginning with the defining density as above, we begin with the tractor equivalent of the conormal. (We will backtrack later to fill in this initial gap.) Observe that [**6, 23**], in a representative $g \in \boldsymbol{c}$, we can pair the unit normal (which transforms as a weight -1 density) with the mean curvature H^g to form the triple

$$N^A :\overset{g}{=} (0, \hat{n}^a, -H^g)\,.$$

A short calculation shows that this triple transforms as a standard tractor of the boundary of M, so $N^A \in \Gamma(\mathcal{T}M)|_\Sigma$. Observe that, like the unit conormal, we have that $h(N, N) = 1$, so we can define a projection operator on bulk tractors along the hypersurface to a space isomorphic to the sum of the boundary tractor bundle. Specifically, we write

$$I^A_B := \delta^A_B - N^A N_B\,.$$

REMARK 3.3.1. The above language suggests that we must be more careful when discussing the isomorphism between the boundary tractor bundle $\mathcal{T}\Sigma$ and the bulk tractor bundle orthogonal to N denoted by $\mathcal{T}M_{||}$. Indeed, the isomorphism is given by [**34**]

$$\mathcal{T}M_{||} \ni \begin{pmatrix} \sigma \\ \mu^b \\ \rho \end{pmatrix} \mapsto \begin{pmatrix} \sigma \\ \mu^b - H\hat{n}^b\sigma \\ \rho + \frac{1}{2}H^2\sigma \end{pmatrix} \in \mathcal{T}\Sigma$$

From this isomorphism, we can write the hypersurface injectors $\bar{X}$, $\bar{Z}$, and $\bar{Y}$ in terms of the bulk injectors X, Z, and Y according to [**73**]

$$\begin{aligned}
\bar{X} &\overset{\Sigma}{=} X\,,\\
\bar{Z}^A_a &\overset{\Sigma}{=} \bar{g}^b_a Z^A_b\,,\\
\bar{Y}^A &\overset{\Sigma}{=} Y^A + \hat{n}^a Z^A_a H - \tfrac{1}{2}H^2 X^A\,.
\end{aligned}$$

Because the projecting isomorphism is the identity on the canonical tractor, we usually drop the bar on $\bar{X}$ and use the same symbol for X both on Σ and M. For many of our calculations, we can avoid a choice of splitting and instead just rely on the existence of the isomorphism $\mathcal{T}M|_\Sigma \cong \mathcal{T}\Sigma \oplus \mathcal{N}\Sigma$ where $\mathcal{N}\Sigma = \operatorname{span}(N^A)$ is the orthogonal complement of $\mathcal{T}\Sigma$ as a submanifold of $\mathcal{T}M$. Thus, when writing tractor expressions we can use the same indices to denote hypersurface tractors as bulk tractors. Often, the notation $\stackrel{\Sigma}{=}$ will imply this isomorphism. ■

Just as for the induced conformal metric on Σ, we can use the projection operator to write the induced tractor metric:

$$\bar{h}_{AB} := h_{AB} - N_A N_B \in \Gamma(\odot^2 \mathcal{T}^*\Sigma)\,.$$

Going forward, we will sometimes write $I^A_B \equiv \bar{h}^A_B$. We will also use the same $\top$ notation for tractors to indicate tractors that are orthogonal to N and the same overbar notation will be used to denote tractors that belong to the hypersurface Σ.

The relationship between the induced and projected Levi-Civita connections in the Riemannian setting suggests that there exists a similar relationship between the induced tractor-coupled Levi-Civita connection on Σ to the projected tractor-coupled Levi-Civita. Just as Equation (3.1) requires the second fundamental form, the tractor-coupled analog requires a *tractor second fundamental form*. We define this tractor by insertion into the hypersurface tractor bundle:

$$L^{AB} := \bar{q}(\mathring{\mathrm{I\!I}}_{ab}) \stackrel{g}{=} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \mathring{\mathrm{I\!I}}^{ab} & -\frac{1}{d-2}\bar{\nabla}\cdot\mathring{\mathrm{I\!I}}^a \\ 0 & -\frac{1}{d-2}\bar{\nabla}\cdot\mathring{\mathrm{I\!I}}^b & \frac{\bar{\nabla}\cdot\bar{\nabla}\cdot\mathring{\mathrm{I\!I}} + (d-2)\bar{P}^{ab}\mathring{\mathrm{I\!I}}_{ab}}{(d-2)(d-3)} \end{pmatrix} \in \Gamma(\odot^2_\circ \mathcal{T}\Sigma[-1])\,.$$

Observe from Equation (2.3) that the tractor-coupled Levi-Civita connection also depends on the Schouten tensor. The relationship between the Schouten tensor induced on Σ and the projected Schouten tensor is described by the Fialkow tensor, see Equation (3.8). Thus, it is reasonable to infer that the Fialkow tensor (and its trace) will play a role in this relationship. So for uniformity, we also define the *Fialkow tractor* by insertion of the (trace-free) Fialkow tensor into a tractor bundle (when $d \geq 4$):

$$F^{AB} := \bar{q}(\mathring{F}_{ab}) \stackrel{g}{=} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \mathring{F}^{ab} & -\frac{1}{d-3}\bar{\nabla}\cdot\mathring{F}^{a} \\ 0 & -\frac{1}{d-3}\bar{\nabla}\cdot\mathring{F}^{b} & \frac{\bar{\nabla}\cdot\bar{\nabla}\cdot\mathring{F}+(d-3)\bar{P}^{ab}\mathring{F}_{ab}}{(d-3)(d-4)} \end{pmatrix} \in \Gamma(\odot^2_\circ \mathcal{T}\Sigma[-2])\,.$$

These two tractors, together with the rigidity density K, come together to form a useful rank-3 tractor that plays the role of $\hat{n} \otimes \mathring{\mathrm{I\!I}}$ in the Riemannian case:

$$\Gamma_{ABC} := 2N_{[C}L_{B]A} + 2X_{[C}F_{B]A} + \tfrac{K}{(d-1)(d-2)} X_{[C}\bar{h}_{B]A} \in \Gamma(\otimes^3(\mathcal{T}^* \oplus \mathcal{N}^*)\Sigma[-1])\,.$$

Note that the bundle to which this tractor belongs implicitly uses the isomorphism $\mathcal{T}M|_\Sigma \cong \mathcal{T}\Sigma \oplus \mathcal{N}\Sigma$.

With these definitions in place, we can describe a relationship between the induced tractor-coupled Levi-Civita on Σ and its projected counterpart. This result was first hinted at in [**52**] and expanded upon in [**73**, **85**]. A more concise formulation was given in [**44**]; we provide a more compact result using Γ that is equivalent to their work for later use.

Proposition 3.3.2 (Fialkow-Gauß formula). *Let $\bar{V}^A \in \Gamma(\mathcal{T}\Sigma)$ be any standard hypersurface tractor. Then, for $d \geq 4$,*

$$\nabla_a \bar{V}^B \stackrel{\Sigma}{=} \nabla^\top_a V^B|_\Sigma - \Gamma_a{}^B{}_C \bar{V}|_\Sigma\,, \tag{3.9}$$

where the isomorphism $\mathcal{T}M|_\Sigma \cong \mathcal{T}\Sigma \oplus \mathcal{N}\Sigma$ is used to equate tractors in distinct bundles and $V \in \Gamma(\mathcal{T}M)$ is any extension of $\bar{V}$ off Σ.

Using the above relationship, we can relate the tangential Thomas-D operator to the induced Thomas-D operator along Σ. To that end, we first define the tangential analog of the Thomas-D operator [**9**, **36**]:

Proposition 3.3.3. *Let $w + \frac{d}{2} \neq 1, \frac{3}{2}, 2$ and let N^{e} be any extension of the normal tractor. Then, the operator*

$$\hat{D}^T : \Gamma(\mathcal{T}^\Phi M[w]) \to \Gamma(\mathcal{T}M \otimes \mathcal{T}^\Phi M[w-1])\,,$$

given by

$$\hat{D}^T_A := \hat{D}_A N^{\rm e}_A N^{\rm e} \cdot \hat{D} + \frac{X_A}{d+2w-3}\left(N^{\rm e}_B N^{\rm e}_C \hat{D}^B \hat{D}^C + \frac{wK_{\rm e}}{d-2} \right),$$

is tangential and is called the tangential Thomas-D operator. Here, $K_{\rm e}$ is any extension of the rigidity density. Moreoever, for any operator O^A acting on tractors of weight $\frac{1-d}{2}$ that obeys $\mathsf{O}^A \circ X_A = 0$, the operator

$$\mathsf{O}^A \circ (\hat{D}_A - N^{\rm e}_A N^{\rm e} \cdot \hat{D}) =: \mathsf{O}^A \circ \tilde{D}_A$$

is also tangential.

The proof of this proposition is given in [**44**] but can be verified by direct computation. We can then obtain the following result.

THEOREM 3.3.4. *Let $w + \frac{d}{2} \neq 1, \frac{3}{2}, 2$. Acting on weight w tractors, the bulk tangential and hypersurface Thomas-D operators obey*

$$\hat{D}^T_A \overset{\Sigma}{=} \hat{\bar{D}}_A + \Gamma_A{}^\sharp - \tfrac{X_A}{d+2w-3}\left\{ 2\Gamma_B{}^\sharp \circ \hat{\bar{D}}^B + \Gamma^{B\sharp} \circ \Gamma_B{}^\sharp + \tfrac{1}{(d-1)(d-2)}\left[\left(\hat{\bar{D}} K \right) \wedge X \right]^\sharp - \tfrac{(3d-1)wK}{2(d-1)(d-2)} \right\}, \tag{3.10}$$

where the isomorphism $\mathcal{T}M \cong \mathcal{T}\Sigma \oplus \mathcal{N}\Sigma$ is implicitly used.

PROOF. We first prove the special case of Theorem 3.3.4 where the operators act on a tractor vector $V \in \Gamma(\mathcal{T}\Sigma[w])$. We rely heavily on [**44**, Lemma 4.9], which states that

$$\left(\hat{D}^T_A \right)_\Sigma = \begin{pmatrix} w \\ \nabla^\top_a \\ -\frac{\Delta^\top + w\bar{J}}{\bar{d}+2w-2} + \frac{wK}{2(\bar{d}-1)(\bar{d}+2w-2)} \end{pmatrix}, \tag{3.11}$$

where the subscript Σ indicates that the domain of the denoted operator is restricted to products of $\mathcal{T}M|_\Sigma$ and its image is mapped via the isomorphism to the bundle $\mathcal{T}\Sigma \oplus \mathcal{N}\Sigma$.

To prove the theorem, we check Equation (3.10) slot by slot by contracting with the three possible injectors. First note that $w = \bar{w}$ and $X_A \Gamma^A{}_{BC} = 0$. Thus, the theorem holds upon contraction with X^A.

Next, we check that Equation (3.10) holds upon contraction with $\bar{Z}_a^A$. Note that, according to Equation (3.11), $\bar{Z}_a^A \hat{D}^T{}_A = \left(\nabla_a^\top\right)_\Sigma$. Because $X_A \bar{Z}_a^A = 0$, we have that

$$\left(\nabla_a^\top\right)_\Sigma V^B = \bar{\nabla}_a V^B + \Gamma_a{}^B{}_C \bar{V}^C ,$$

which is the Fialkow-Gauss equation given in Equation (3.9). Thus, the equation holds upon contraction with $\bar{Z}_a^A$.

Finally, we check that the identity holds upon contraction with $\bar{Y}^A$. From Equation. (3.11), we have that

$$\begin{aligned}
\bar{Y}^A \left[\hat{D}_A^T - \hat{\bar{D}}_A\right] &= \left(-\frac{\Delta^\top + w\bar{J}}{\bar{d}+2w-2} + \frac{wK}{2(\bar{d}-1)(\bar{d}+2w-2)}\right) + \frac{1}{\bar{d}+2w-2}\left(\bar{\Delta} + w\bar{J}\right) \\
&= -\frac{1}{\bar{d}+2w-2}\left(\Delta^\top - \bar{\Delta} - \frac{wK}{2(\bar{d}-1)}\right).
\end{aligned}$$

We now explicitly compute the tractor Laplacian difference $\Delta^\top - \bar{\Delta}$.

From the definition of the tractor Laplacian and defining $\Gamma_\perp^{ABC} := \Gamma^{ABC} - 2N^{[C}L^{B]A} \in \Gamma(\mathcal{T}^3\Sigma[-1])$, we have

$$\begin{aligned}
\Delta^\top V^B &= \nabla_a^\top \left(\bar{\nabla}^a V^B + \Gamma^{aBC} V_C\right) \\
&= \nabla_a^\top \left(\bar{\nabla}^a V^B + \Gamma_\perp^{aBC} V_C + 2N^{[C}L^{B]a} V_C\right) \\
&= \bar{\Delta} V^B + \Gamma^{aB}{}_C \bar{\nabla}_a V^C + \bar{\nabla}_a(\Gamma_\perp^{aBC} V_C) + \Gamma_a{}^B{}_E \Gamma_\perp^{aEC} V_C + \nabla_a^\top \left(2N^{[C}L^{B]a} V_C\right) \\
&= \bar{\Delta} V^B + \Gamma^{aB}{}_C \bar{\nabla}_a V^C + \left(\bar{\nabla}_a \Gamma_\perp^{aBC}\right) V_C + \left(\Gamma^{aBC} - 2N^{[C}L^{B]a}\right) \bar{\nabla}_a V_C \\
&\quad + \Gamma_a{}^B{}_E \Gamma^{aEC} V_C - 2\Gamma_a{}^B{}_E N^{[C}L^{E]a} V_C + 2\left(\nabla_a^\top (N^{[C}L^{B]a})\right) V_C + 2N^{[C}L^{B]a} \nabla_a^\top V_C \\
&= \bar{\Delta} V^B + 2\Gamma^{aB}{}_C \bar{\nabla}_a V^C + \left(\bar{\nabla}_a \Gamma_\perp^{aBC}\right) V_C \\
&\quad + \Gamma_a{}^B{}_E \Gamma^{aEC} V_C - 2\Gamma_a{}^B{}_E N^{[C}L^{E]a} V_C + 2\left(\nabla_a^\top (N^{[C}L^{B]a})\right) V_C + 2N^{[C}L^{B]a} \Gamma_{aC}{}^E V_E .
\end{aligned}$$

Writing $\Gamma = \Gamma_\perp + 2NL$ allowed us to use the Fialkow–Gauß equation (3.9) for the above simplifications.

We now break up this calculation into smaller parts.

$$\begin{aligned}\bar{\nabla}_a \Gamma_\perp^{aBC} &= \bar{\nabla}_a \left(\bar{Z}_A^a \Gamma_\perp^{ABC}\right) \\ &= \bar{Z}_A^a \bar{\nabla}_a \Gamma_\perp^{ABC} + \Gamma_\perp^{ABC}\left(\bar{J}\bar{X}_A - \bar{d}\bar{Y}_A\right) \\ &= \bar{Z}_A^a \bar{\nabla}_a \Gamma_\perp^{ABC} - \bar{d}\Gamma_\perp^{-BC}.\end{aligned}$$

Here, the second equality comes from the fact that $\bar{\nabla}_a \bar{Z}_A^b = -\bar{P}_a^b \bar{X}_A - \bar{g}_a^b \bar{Y}_A$ and the last equality holds because $X_A \Gamma^{ABC} = 0$. Similarly,

$$\begin{aligned}\nabla_a^\top N^{[C} L^{B]a} &= \left(\nabla_a^\top N^{[C}\right) L^{B]a} + N^{[C}\left[\bar{\nabla}_a\left(L^{B]A}\bar{Z}_A^a\right) + \Gamma_a{}^{B]}{}_E L^{Ea}\right] \\ &= \left(\nabla_a^\top N^{[C}\right) L^{B]a} + N^{[C}\left[\left(\bar{\nabla}_a L^{B]A}\right)\bar{Z}_A^a - \bar{d}L^{B]-} + \Gamma_a{}^{B]}{}_E L^{Ea}\right]\end{aligned}$$

In order to simplify the above two displays, we need results that follow from Equation (2.3):

$$\begin{aligned}\bar{Z}_A^a \bar{\nabla}_a L^{AC} &= 2L^{-C}\,, \\ \bar{Z}_A^a \bar{\nabla}_a F^{AC} &= 3F^{-C}\,, \\ \nabla_a^\top N^B &= L_a^B.\end{aligned}$$

Using the above identities, we can write

$$\begin{aligned}\bar{\nabla}_a \Gamma_\perp^{aBC} &= 2\left(\bar{\nabla}_a X^{[C}\right) F^{B]a} + 6X^{[C}F^{B]-} + \frac{1}{\bar{d}(\bar{d}-1)}\left(\nabla_a K\right) X^{[C}\bar{h}^{B]a} - \bar{d}\Gamma_\perp^{-BC} \\ &= 4X^{[C}F^{B]-} + \frac{1}{\bar{d}(\bar{d}-1)}\left(X^{[C}\hat{\bar{D}}^{B]}K + 2KX^{[C}\bar{h}^{B]-}\right) - \bar{d}\Gamma_\perp^{-BC} \\ &= (2-\bar{d})\Gamma_\perp^{-BC} + \frac{1}{\bar{d}(\bar{d}-1)} X^{[C}\hat{\bar{D}}^{B]}K\,,\end{aligned}$$

and

$$2\nabla_a^\top N^{[C}L^{B]a} = 2(2-d)N^{[C}L^{B]-} + N^C\Gamma^{ABE}L_{AE} - N^B\Gamma^{ACE}L_{AE}\,.$$

We can now use these formulæ to write

$$\begin{aligned}
\Delta^{\top}\bar{V}^B &= \bar{\Delta}V^B + 2\Gamma^{aBC}\bar{\nabla}_a V_C + (2-\bar{d})\Gamma_{\perp}^{-BC}V_C + \frac{V_C}{\bar{d}(\bar{d}-1)}X^{[C}\hat{\bar{D}}^{B]}K + \Gamma_a{}^B{}_E\Gamma^{aEC}V_C \\
&\quad - 2\Gamma_a{}^B{}_E N^{[C}L^{E]a}V_C + 2N^{[C}L^{B]a}\Gamma_{aC}{}^E V_E \\
&\quad + \left(2(2-d)N^{[C}L^{B]-} + N^C\Gamma^{ABE}L_{AE} - N^B\Gamma^{ACE}L_{AE}\right)V_C \\
&= \bar{\Delta}\bar{V}^B + 2\Gamma^{aBC}\bar{\nabla}_a V_C + (2-\bar{d})\Gamma^{-BC}V_C + \frac{V_C}{\bar{d}(\bar{d}-1)}X^{[C}\hat{\bar{D}}^{B]}K + \Gamma_a{}^B{}_E\Gamma^{aEC}V_C \\
&\quad - 2\Gamma_a{}^B{}_E N^{[C}L^{E]a}V_C + 2N^{[C}L^{B]a}\Gamma_{aC}{}^E V_E + \left(N^C\Gamma^{ABE}L_{AE} - N^B\Gamma^{ACE}L_{AE}\right)V_C \\
&= \bar{\Delta}V^B + 2\Gamma^{aBC}\bar{\nabla}_a V_C + (2-\bar{d})\Gamma^{-BC}V_C + \frac{V_C}{\bar{d}(\bar{d}-1)}X^{[C}\hat{\bar{D}}^{B]}K + \Gamma_a{}^B{}_E\Gamma^{aEC}V_C \\
&= \bar{\Delta}V^B + 2\Gamma^{ABC}\hat{\bar{D}}_A V_C - (\bar{d}+2w-2)\Gamma^{-BC}V_C + \frac{V_C}{\bar{d}(\bar{d}-1)}X^{[C}\hat{\bar{D}}^{B]}K + \Gamma_A{}^B{}_E\Gamma^{AEC}V_C \\
&= \bar{\Delta}V^B + 2\Gamma^A{}_A{}^C\hat{\bar{D}}_C V^B - \frac{(\bar{d}+1)wK}{\bar{d}(\bar{d}-1)}V^B + 2\Gamma^{ABC}\hat{\bar{D}}_A V_C - (\bar{d}+2w-2)\Gamma^{-BC}V_C \\
&\quad + \frac{V_C}{\bar{d}(\bar{d}-1)}X^{[C}\hat{\bar{D}}^{B]}K + \Gamma_A{}^B{}_E\Gamma^{AEC}V_C \\
&= \bar{\Delta}V^B + 2\Gamma^{A\sharp}\circ\hat{\bar{D}}_A V^B - (\bar{d}+2w-2)\Gamma^{-BC}V_C + \frac{V_C}{\bar{d}(\bar{d}-1)}X^{[C}\hat{\bar{D}}^{B]}K \\
&\quad + \Gamma^{A\sharp}\circ\Gamma_A{}^{\sharp}V^B - \frac{(\bar{d}+1)wK}{\bar{d}(\bar{d}-1)}V^B\,.
\end{aligned}$$

In the display above, the second equality comes from the definition of $\Gamma_{\perp}$. The third equality is a result of the last four terms canceling, and the fourth equality comes from the fact that $X_A\Gamma^{ABC} = 0$. The last inequality follows from $\Gamma_A{}^A{}_E\Gamma^{EBC}V_C = 0$.

But,

$$\left(\hat{D}^T_A - \hat{\bar{D}}_A\right)V^B = \bar{Z}^A_a\Gamma_A{}^B{}_C V^C - \frac{X_A}{\bar{d}+2w-2}\left(\Delta^{\top} - \bar{\Delta} - \frac{wK}{2(\bar{d}-1)}\right)V^B,$$

so the proof of the special case for a tractor vector is completed by combining terms involving wK. The proof of the full theorem follows by the same calculations but accounting for the possible action by derivation on tensor products of the tractor bundle. $\square$

REMARK 3.3.5. A useful consequence of Theorem 3.3.4 is that, for a scalar density $\mu \in \Gamma(\mathcal{E}M[w])$ where $w \neq \frac{3-d}{2}$, the following identity holds:

$$\hat{D}^T_A \mu \overset{\Sigma}{=} \hat{\bar{D}}_A \mu + \tfrac{wK}{2(d-2)(d+2w-3)} X_A \mu \,.$$

■

As observed in Equation (2.6), the commutator of two Thomas-D operators produces the W tractor. The commutator of two hypersurface Thomas-D operators produces the induced $\bar{W}$ tractor on Σ. One can thus infer that there is a relationship between W and $\bar{W}$. Indeed, a consequence of Theorem 3.3.4 is the following corollary.

COROLLARY 3.3.6 (Gauß-Thomas Equation). *Let $d > 5$. Then the bulk and hypersurface W-tractors are related by*

$$\begin{aligned} W^{\top}_{ABCD}\big|_\Sigma = \bar{W}_{ABCD} &- 2L_{A[C}L_{D]B} - 2\bar{h}_{A[C}F_{D]B} + 2\bar{h}_{B[C}F_{D]A} - \tfrac{2}{(d-1)(d-2)}\bar{h}_{A[C}\bar{h}_{D]B}K \\ &+ 2X_{[A}T_{B]CD} + 2X_{[C}T_{D]AB} - 2X_AX_{[C}V_{D]B} + 2X_BX_{[C}V_{D]A} \\ &+ \tfrac{1}{3(d-1)(d-2)}X_AX_{[C}\hat{\bar{D}}_{D]}\hat{\bar{D}}_BK - \tfrac{1}{3(d-1)(d-2)}X_BX_{[C}\hat{\bar{D}}_{D]}\hat{\bar{D}}_AK, \end{aligned} \tag{3.12}$$

where

$$T_{ABC} := 2\hat{\bar{D}}_{[C}F_{B]A} + \tfrac{1}{(d-1)(d-2)}\bar{h}_{A[B}\hat{\bar{D}}_{C]}K \ \in\ \Gamma(\mathcal{T}\Sigma \otimes \wedge^2\mathcal{T}\Sigma[-3])\,,$$

and $V_{AB} \in \Gamma(\odot^2\mathcal{T}\Sigma[-4])$ is a symmetric tractor built from curvatures such that $X^AV_{AB} = X_BV$ for some $V \in \Gamma(\mathcal{E}M[-4])$.

PROOF OF COROLLARY 3.3.6. Recall that the Gauß equation is a corollary of the Gauß formula, in the sense that it is obtained by applying the latter to $[\nabla^{\top}_a, \nabla^{\top}_b]v_c$ where v_c is an extension of a hypersurface tangent vector. Similarly, the present proof could be completed by applying the Gauß–Thomas formula to $[\hat{D}^T_A, \hat{D}^T_B]V_C$. But, because $\hat{D}$ is not a derivation, that computation is rather involved. Instead, we approach the proof via equality of all possible contractions (in some scale $g \in \boldsymbol{c}$) by hypersurface injectors $(X^A, \bar{Z}^A_a, \bar{Y}^A)$ on both sides of the lemma's result. Note that it is unnecessary to check contractions with more than one $\bar{Y}$—this only probes V_{AB}. Also, without loss of generality, we may choose g to be a scale in which the mean curvature H^g of the embedding $\Sigma \hookrightarrow (M, g)$ vanishes.

We begin by contracting with a single X. For that, we first use Proposition 2.3.1 and the Fialkow tractor identities $\hat{\bar{D}}^A F_{AB} = 0 = X^A F_{AB}$, $0 = F_A{}^A$ as well as the ansatz $X^A V_{AB} = X_B V$, to obtain

$$\begin{aligned}
X^A T_{ABC} &= \tfrac{1}{(d-1)(d-2)} X_{[B} \hat{\bar{D}}_{C]} K\,,\\
X^C T_{ABC} &= -F_{AB} - \tfrac{K}{(d-1)(d-2)} \bar{h}_{AB} - \tfrac{1}{2(d-1)(d-2)} X_A \hat{\bar{D}}_B K\,,\\
X^A X_{[B} V_{C]A} &= 0\,.
\end{aligned}$$

Now $X^A W^{\top}_{ABCD} = 0$, so we need to show contraction of the right-hand side of Equation (3.12) with X^A vanishes. Clearly $X^A \bar{W}_{ABCD} = 0$ and the contraction of X with the second term also vanishes because $X^A L_{AB} = 0$. Using $X^A F_{AB} = 0$ along with the identities of the above display, the remaining terms are

$$\begin{aligned}
&-2X_{[C} F_{D]B} + \tfrac{2}{(d-1)(d-2)} X_{[D} \bar{h}_{C]B} K\\
+2X_{[C} F_{D]B} - \tfrac{2}{(d-1)(d-2)} X_{[D} \bar{h}_{C]B} K &- \tfrac{1}{(d-1)(d-2)} X_B X_{[C} \hat{\bar{D}}_{D]} K\\
&+ \tfrac{1}{(d-1)(d-2)} X_B X_{[C} \hat{\bar{D}}_{D]} K = 0\,.
\end{aligned}$$

Because the W-tractor has Weyl curvatures symmetries this establishes consistency of the identity when any index is contracted with a canonical tractor.

Next, note that $\bar{Z}^A_a \bar{Z}^B_b \bar{Z}^C_c \bar{Z}^D_d W^{\top}_{ABCD} = W^{\top}_{abcd}$ and that the trace-free Gauß equation says

$$W^{\top}_{abcd} = \bar{W}_{abcd} - 2\mathring{\mathrm{I\!I}}_{a[c} \mathring{\mathrm{I\!I}}_{d]b} - 2\bar{g}_{a[c} \mathring{F}_{d]b} + 2\bar{g}_{b[c} \mathring{F}_{d]a} - \tfrac{2}{(d-1)(d-2)} \bar{g}_{a[c} \bar{g}_{d]b} K.$$

It is easy to check, using $X_A \bar{Z}^A_a = 0$, that this is the right hand side of Equation (3.12) when contracted with this combination of injectors.

The last case to check is contraction of Equation (3.12) by $\bar{Z}^A_a \bar{Z}^B_b \bar{Z}^C_c \bar{Y}^D$. By directly applying the definitions of L_{AB}, F_{AB}, $\bar{W}_{ABCD}$, and the hatted hypersurface Thomas-D operator, after some computation, we find for the right-hand side

$$\begin{aligned}
&\bar{Z}_a^A \bar{Z}_b^B \bar{Z}_c^C \bar{Y}^D \Big[\bar{W}_{ABCD} - 2L_{A[C}L_{D]B} - 2\overline{h}_{A[C}F_{D]B} + 2\overline{h}_{B[C}F_{D]A} - \tfrac{2}{(d-1)(d-2)}\overline{h}_{A[C}\overline{h}_{D]B}K \\
&\qquad\qquad + 2X_{[A}T_{B]CD} + 2X_{[C}T_{D]AB}\Big] \\
&= \bar{C}_{abc} + \tfrac{2}{d-2}\,\mathring{\mathrm{II}}_{c[a}\bar{\nabla}\cdot\mathring{\mathrm{II}}_{b]} + 2\bar{\nabla}_{[a}\mathring{F}_{b]c} - \tfrac{1}{(d-1)(d-2)}\bar{g}_{c[a}\bar{\nabla}_{b]}K.
\end{aligned}$$

We must then contract the left-hand side with the same injector product. Because we use a scale where $H^g = 0$,

$$\bar{Z}_a^A \bar{Z}_b^B \bar{Z}_c^C \bar{Y}^D W^{\top}_{ABCD} = C^{\top}_{abc}\big|_{\Sigma}\,.$$

Showing that this contraction yields equality in Equation (3.12) is now equivalent to showing that, when $H^g = 0$, the projected Cotton tensor is related to the hypersurface Cotton tensor by

$$C^{\top}_{abc}\big|_{\Sigma} = \bar{C}_{abc} + 2\bar{\nabla}_{[a}\mathring{F}_{b]c} + \tfrac{2}{d-2}\,\mathring{\mathrm{II}}_{c[a}\bar{\nabla}\cdot\mathring{\mathrm{II}}_{b]} - \tfrac{1}{(d-1)(d-2)}\bar{g}_{c[a}\bar{\nabla}_{b]}K.$$

For that, first observe that the projected covariant derivative of (any extension of) the induced metric form obeys

$$\nabla^{\top}_a \bar{g}^{\mathrm{e}}_{bc}\big|_{\Sigma} = -\mathrm{II}_{ab}\hat{n}_c - \mathrm{II}_{ac}\hat{n}_b \overset{H^g=0}{=} -\mathring{\mathrm{II}}_{ab}\hat{n}_c - \mathring{\mathrm{II}}_{ac}\hat{n}_b.$$

Applying this identity, Equation (3.8), and the traced Codazzi–Mainardi equation, the projected Cotton tensor can be written in terms of the hypersurface Cotton tensor:

$$\begin{aligned}
C^{\top}_{abc}\big|_{\Sigma} &= (\nabla_a P_{bc})^{\top} - (a \leftrightarrow b) \\
&= \nabla^{\top}_a P^{\top}_{bc} + \mathring{\mathrm{II}}_{ab}P^{\top}_{\hat{n}c} + \mathring{\mathrm{II}}_{ac}P^{\top}_{\hat{n}b} + \hat{n}_b\,\mathring{\mathrm{II}}_a^d P^{\top}_{dc} + \hat{n}_c\,\mathring{\mathrm{II}}_a^d P^{\top}_{bd} - (a \leftrightarrow b) \\
&= \nabla^{\top}_a P^{\top}_{bc} + \hat{n}_b\,\mathring{\mathrm{II}}_a^d P^{\top}_{dc} + \hat{n}_c\,\mathring{\mathrm{II}}_a^d P^{\top}_{bd} - (a \leftrightarrow b) + \tfrac{2}{d-2}\,\mathring{\mathrm{II}}_{c[a}\bar{\nabla}\cdot\mathring{\mathrm{II}}_{b]} \\
&= \bar{\nabla}_a P^{\top}_{bc} - (a \leftrightarrow b) + \tfrac{2}{d-2}\,\mathring{\mathrm{II}}_{c[a}\bar{\nabla}\cdot\mathring{\mathrm{II}}_{b]} \\
&= \bar{\nabla}_a \bar{P}_{bc} + \bar{\nabla}_a \mathring{F}_{bc} + \tfrac{\bar{g}_{bc}}{2(d-1)(d-2)}\bar{\nabla}_a K - (a \leftrightarrow b) + \tfrac{2}{d-2}\,\mathring{\mathrm{II}}_{c[a}\bar{\nabla}\cdot\mathring{\mathrm{II}}_{b]} \\
&= \bar{C}_{abc} + 2\bar{\nabla}_{[a}\mathring{F}_{b]c} + \tfrac{2}{d-2}\,\mathring{\mathrm{II}}_{c[a}\bar{\nabla}\cdot\mathring{\mathrm{II}}_{b]} - \tfrac{1}{(d-1)(d-2)}\bar{g}_{c[a}\bar{\nabla}_{b]}K.
\end{aligned}$$

Here $X_{ab} - (a \leftrightarrow b)$ denotes $X_{ab} - X_{ba} = 2X_{[ab]}$ and refers to everything in the expression to the left of it. The second line above relies on the previous display, while the third relies on the trace of the Codazzi–Mainardi equation. The penultimate line uses the Fialkow–Gauß equation. This completes the proof.

□

REMARK 3.3.7. The corollary does not contain an explicit formula for the tractor V_{AB} for reasons of brevity only. It measures the difference between hypersurface and bulk Bach tensors. While explicit knowledge of the tensor content of V_{AB} is unnecessary for the computations that follow, it is nonetheless interesting. A computer-aided computation gives

$$V_{AB} = \bar{q}(U_{ab}) + \tfrac{1}{(d-1)(d-4)(d-5)}\, \bar{h}_{AB} U,$$

where, for $d \neq 5, 7$,

$$\begin{aligned} \Gamma(\mathcal{E}\Sigma[-4]) \ni U = \;& \tfrac{d-3}{d-1} K^2 + 2\mathring{\mathrm{I\!I}}^{ad}\mathring{\mathrm{I\!I}}^{bc}\overline{W}_{abcd} - 2(d-3)\mathring{\mathrm{I\!I}}\cdot\mathring{F}\cdot\mathring{\mathrm{I\!I}} + (d-3)(d-5)\mathring{F}^2 \\ & + \tfrac{1}{d-7}\left(\bar{D}_A L_{BC}\right)\left(\hat{\bar{D}}^A L^{BC}\right) - L^{BC} N^A N^D \delta_R W_{ABCD}\,, \end{aligned} \tag{3.13}$$

where $\delta_R = N\cdot\hat{D} = \nabla_{\hat{n}} - H\underline{w}$, and

$$\begin{aligned} \Gamma(\odot^2 T^*_\circ\Sigma[-2]) \ni U_{ab} = \;& \tfrac{1}{d-5}\overline{B}_{ab} - \tfrac{1}{d-4}B^\top_{(ab)\circ} + \tfrac{2}{d-7}E(\mathring{F})_{ab} \\ & + \tfrac{1}{6(d-1)(d-2)}\bar{\nabla}_{(a}\bar{\nabla}_{b)\circ}K - \tfrac{1}{(d-2)(d-3)}\mathring{\mathrm{I\!I}}_{ab}\bar{\nabla}\cdot\bar{\nabla}\cdot\mathring{\mathrm{I\!I}} + \tfrac{1}{(d-2)^2}\bar{\nabla}\cdot\mathring{\mathrm{I\!I}}_{(a}\bar{\nabla}\cdot\mathring{\mathrm{I\!I}}_{b)\circ} \\ & + 2HC^\top_{\hat{n}(ab)} + H^2 W_{\hat{n}ab\hat{n}} - \tfrac{1}{3(d-1)(d-2)}K\overline{P}_{(ab)\circ} - \tfrac{1}{d-3}\mathring{\mathrm{I\!I}}_{ab}\mathring{\mathrm{I\!I}}\cdot\overline{P}\,. \end{aligned}$$

Here the operator $E \in \mathrm{End}\left(\Gamma(\odot^2_\circ T^*\Sigma)\right)$ is defined by

$$E(\mathring{X})_{ab} := \bar{\Delta}\mathring{X}_{ab} - \bar{\nabla}_{(a}\nabla\cdot\mathring{X}_{b)\circ} - (d-3)\bar{P}_{c(a}\mathring{X}^c_{b)\circ} - 2\bar{J}\mathring{X}_{ab}\,.$$

When $\bar{d} = 6$, the operator E defines a conformally invariant map $\Gamma(\odot^2_\circ T^*\Sigma) \to \Gamma(\odot^2_\circ T^*\Sigma[-2])$. ■

A further corollary of the Gauß–Thomas equation in Theorem 3.3.4 characterizes the Fialkow tractor in terms of the W-tractor and tractor second fundamental form and generalizes Equation (3.7).

Corollary 3.3.8. *Let $7 \neq d > 5$. Then the Fialkow tractor obeys*

$$(d-3)F_{AB} = \Big(L_A^C L_{CB} - \tfrac{1}{d-1} K \bar{h}_{AB} - W_{NABN}\Big) - \tfrac{1}{d-1} X_{(A} \hat{\bar{D}}_{B)} K - \tfrac{1}{(d-4)(d-5)} X_A X_B U\,,$$

where $U \in \Gamma(\mathcal{E}\Sigma[-4])$ is the density built from curvatures given in Equation (3.13).

Proof. The proof amounts to tracing Equation (3.12) with the hypersurface tractor metric. Note that U is given in the previous remark. □

Observe that the above corollary mirrors the Riemannian relationship between the Fialkow tensor, the square of the second fundamental form, and the Weyl tensor. Further paralleling the Riemannian case, one might expect that the tractor second fundamental form arises as the action of a connection-like tractor operator on the normal tractor. This expectation is captured by the following lemma.

Lemma 3.3.9. *Let $d \geq 4$ and $N^{\rm e}$ be any extension of the normal tractor. Then the tractor second fundamental form obeys*

$$L_{AB} = \hat{D}^T_{(A} N^{\rm e}_{B)}\big|_\Sigma - \tfrac{1}{d-3}\left(X_{(A} N_{B)} K + X_A X_B M\right),$$

with $M = L_{AB} F^{AB} = F^{AB} (\hat{D}_A N^{\rm e}_B)\big|_\Sigma$.

While this result can be proved directly, a holographic approach to the proof is more instructive; thus we wait to provide the proof until we have discussed holography.

3.4. Geometric Holography

A particularly useful way to study both Riemannian and conformal hypersurface invariants is via the notion of *preinvariants*. The notion of a preinvariant was described in [**9**,**45**] and is recorded here for our use.

Definition 3.4.1. For hypersurfaces, a Riemannian preinvariant is a diffeomorphism-invariant (possibly tensor-valued) function P which assigns to each Riemannian d-manifold (M, g) and hypersurface defining function s for Σ pair the function $P(s; g)$ such that:

(1) $P(s; g)|_\Sigma$ is independent of the choice of defining function s for Σ;

(2) $P(s;g)$ is given by a universal polynomial expression such that, in a local coordinate system (x^a) on (M,g), $P(s;g)$ can be expressed as a polynomial in g, its inverse, the defining function s, their partial derivatives, the inverse of the length of the conormal $|ds|_g^{-1}$, the inverse of the metric determinant, and the volume form on M.

■

Observe that action via the Levi-Civita connection can also appear in preinvariants, if only because this action can be expressed in terms of partial derivatives and derivatives of the metric. The simplest example of a preinvariant is the preinvariant for the unit conormal: $(|ds|_g^{-1}\partial s)|_\Sigma = \hat{n}$. Another example of a useful preinvariant is $\mathscr{P}_{ab}$, the preinvariant for the second fundamental form:

$$\mathscr{P}_{ab} := \left(\nabla_a - |ds|_g^{-1}(\nabla_a s)\nabla_{|ds|_g^{-1}\text{grad } s}\right)\frac{\nabla_b s}{|ds|_g}\,.$$

It is straightforward to check that this reduces to the standard definition of the second fundamental form when restricted to Σ, so that $\mathscr{P}_{ab}|_\Sigma = \mathrm{II}_{ab}$.

The notion of a preinvariant allows us to define the *transverse order* of a Riemannian (or conformal) preinvariant $P(s;g)$. Given such a preinvariant, we can represent it in a choice of coordinates (s, y^i) in a neighborhood of Σ such that vector fields $\partial/\partial y^i$ are tangent to Σ. To each coordinate representation we associate the non-negative integer that is the highest order of $\partial/\partial s$ derivatives of g upon restriction to Σ. Then, the transverse order of the preinvariant $P(s;g)$ is the minimum such integer, minimizing over all coordinate representations of $P(s;g)$. A consequence of this definition is that, because to each preinvariant there is an associated hypersurface invariant, there is a unique transverse order associated to each hypersurface invariant. As simple examples, the transverse order of the unit conormal is 0, the transverse order of the second fundamental form is 1, and the transverse order of the Fialkow tensor is 2.

Another useful notion is the transverse order of an operator. In particular, we say that an operator O has transverse order $k \in \mathbb{Z}_{\geq 0}$ when there exists v in the domain of O such that $\mathsf{O}(s^k v)|_\Sigma \neq 0$, but $\mathsf{O}(s^{k+1}v')|_\Sigma = 0$ for all v' in the domain of O. Given an operator O with transverse order k and a hypersurface invariant with transverse order ℓ and an associated preinvariant $P(s;g)$, the transverse order of the hypersurface invariant $\mathsf{O}(P(s;g))|_\Sigma$ is less than or equal to $k+\ell$. Indeed,

$\mathsf{O}(P(s;g))|_\Sigma$ in general depends on more than the underlying hypersurface invariant defined by $P(s;g)$.

With this notion of preinvariants given, we have the tools to discuss *geometric holography*. Among other things, geometric holography is a method of studying geometric embedding data of a hypersurface by studying particular extensions of hypersurface quantities into a bulk space. Specifically, if $\Sigma \hookrightarrow (M, \mathcal{S})$ is a hypersurface embedding where $(M, \mathcal{S})$ is a smooth manifold with some structure $\mathcal{S}$, we can study hypersurface invariants of Σ by finding and studying corresponding canonical preinvariants (*i.e.* finding canonical extensions) that are determined by and preserve the structure $\mathcal{S}$. We call such canonical preinvariants *holographic formulæ* for their hypersurface invariants. In some cases, these holographic formulæ can only be determined up to some order in a defining function—nonetheless, they can still be useful.

Viewing the hypersurface invariants as more fundamental, we can view preinvariants instead as extensions of the hypersurface invariant. Typically, canonical extensions are found by demanding that an object solves a particular natural partial differential equation. In Section 3.1, we demanded that $|ds|_g = 1$ in a neighborhood of Σ. Indeed, this is a partial differential equation on the defining function s which can be used to canonically choose a triple $(M, g, s)_u$. With such a canonical choice made, we can study the hypersurface by studying the specific function s and the preinvariants associated to it rather than an entire family of equivalently valid defining functions. While this was not strictly necessary for the discussion that followed, it indeed can be useful for simplifying challenging calculations. This type of simplification is the power of holography: it allows for the dramatic simplification of otherwise enormous computations.

We are primarily interested in the case where $\mathcal{S}$ is a conformal structure $\boldsymbol{c}$ on M. As in the Riemannian case, we are interested in a canonical partial differential equation on σ that, given some hypersurface embedding $\Sigma \hookrightarrow (M, \boldsymbol{\gamma})$, will fix the triple $(M, \boldsymbol{\gamma}, \sigma)$. Indeed, as suggested above, for any positive function $f \in C^\infty M$, the triple $(M, \boldsymbol{\gamma}, f\sigma)$ specifies the same hypersurface embedding $\Sigma \hookrightarrow (M, \boldsymbol{\gamma})$, so we seek out a partial differential equation to fix f. To that end, we specify the *singular Yamabe problem* [**88**]:

Problem 3.4.2. Given a conformal hypersurface embedding $(M, \boldsymbol{\gamma}, \sigma)$, find a positive function $f \in C^\infty M$ such that the singular metric g^o associated with the triple $(M, \boldsymbol{\gamma}, f\sigma)$ has a constant scalar curvature:

$$Sc^{g^o} = -d(d-1)\,.$$

An analogous problem was solved by Loewner and Nirenberg on round structures [**59**]. Recall that for the hypersurfaces we are interested in, Σ is separating so that $M = M^- \sqcup \Sigma \sqcup M^+$. A one-sided global solution to this problem always exists [**2**, **5**, **62**] but in general relies on global data of M^+; however, a solution f that depends only on local data of the embedding can always be found such that $g^o = \gamma/(f\sigma)^2$ asymptotically solves the singular Yamabe problem [**2**, **34**]:

$$Sc^{g^o} = -d(d-1) + \mathcal{O}(s^d)\,,$$

where s here is any defining function for $\Sigma \hookrightarrow M$. In this dissertation, we will use the notation $\mathcal{O}(s^m)$ (or, for densities, equivalently $\mathcal{O}(\sigma^m)$) to indicate that the remaining terms in an expression can be written as $s^m f$ where f is some function (or density) that is regular in the limit where s (or σ) approaches zero. Indeed, note that for σ a defining density for $\Sigma \hookrightarrow (M,\gamma)$, the above display is thus equivalent to

$$Sc^{g^o} = -d(d-1) + \mathcal{O}(\sigma^d)\,. \tag{3.14}$$

because for two representative $s, \tilde{s}$ of σ, we have that, for some $\Omega \in C^\infty_+ M$ relating s and $\tilde{s}$,

$$\tilde{s}^d f = (\Omega s)^d f = s^d(\Omega^d f)\,.$$

Thus, Equation (3.14) can be viewed as a conformally-invariant equation.

Given a hypersurface embedding specified by (M,γ,σ) that has $f=1$ as the solution to the above asymptotic singular Yamabe problem, we will say that $(M,\gamma,\sigma) \in \mathrm{ASY}$ and we will often denote such embeddings by a subscript $\mathcal{Y}$, *i.e.* we will write $(M,\gamma,\sigma)_{\mathcal{Y}}$. Given a triple (M,γ,σ'), we can simply define $\sigma := f\sigma'$, where f solves the singular Yamabe problem for (M,γ,σ'), so that $(M,\gamma,f\sigma') = (M,\gamma,\sigma)_{\mathcal{Y}}$.

A priori the above canonical problem is very different from the canonical PDE imposed by requiring that the extension of the conormal has unit length. However, we can reframe the singular

Yamabe problem as a similar unit-length type problem using the framing of tractors [**34**, **40**]. Suppose that (M,γ,σ) specifies the hypersurface embedding $\Sigma \hookrightarrow (M,\gamma)$. Then, we define the *scale tractor* by

$$I^A_\sigma := \hat{D}^A\sigma \in \Gamma(\mathcal{T}M[0])\,.$$

In a choice of splitting given by $g \in \boldsymbol{c}$, we can write

$$I^A_\sigma = (\sigma, \nabla^a\sigma, \rho)\,,$$

where $\rho := -\frac{1}{d}(\Delta\sigma + J^g\sigma)$. Observe that, using the tractor metric, we have that

$$I^2_\sigma = (\nabla\sigma)^2 + 2\sigma\rho\,.$$

In a choice of scale $\sigma = [g; s]$, we have that

$$I^2_\sigma = |ds|^2_g - \tfrac{2}{d}(\Delta s + Js)\,.$$

Because this equality holds everywhere in M, and in particular is valid in M^+, we choose the scale (valid only away from Σ) given by $\sigma = [g^o; 1]$. In that case, we have that $I^2_\sigma = -\frac{2}{d}J^{g^o}$. However, $J^{g^o} = \frac{1}{2(d-1)}Sc^{g^o}$, so we have that

$$I^2_\sigma = -\frac{Sc^{g^o}}{d(d-1)}\,.$$

From this equation we conclude that if a hypersurface specified by (M,γ,σ) exactly solves the singular Yamabe problem, then we must have that $I^2_\sigma = 1$. We call such a defining densities σ that solve this equation a *unit defining density*. Similarly, given a conformal hypersurface embedding specified by $(M,\gamma,\sigma)_\mathcal{Y}$, we must have that

$$I^2_\sigma = 1 + \sigma^d B\,,$$

for some density $B \in \Gamma(\mathcal{E}M[-d])$ that extends smoothly to the boundary. Note that the density $B|_\Sigma \in \Gamma(\mathcal{E}\Sigma[-d])$, called the *obstruction density* because it obstructs solving the singular Yamabe problem smoothly [**2**, **45**], is a local invariant and plays a special role in conformal geometry. It will be the subject of further discussion in Chapters 5 and 6. For holographic purposes, then, we will demand that $I^2_\sigma = 1 + \mathcal{O}(\sigma^d)$ going forward so that the hypersurface is specified by $(M,\gamma,\sigma)_\mathcal{Y}$. This uniquely determines the defining density σ to order σ^{d+1}. In the literature, such a σ that solves the asymptotic singular Yamabe problem is sometimes called a *asymptotic unit defining density*. When it is clear that we are discussing asymptotic unit defining densities, we will typically drop the subscript σ decorating I—because given a conformal hypersurface embedding $(M, \boldsymbol{\gamma}, \sigma)_{\mathcal{Y}}$, this tractor is (asymptotically) uniquely determined.

Given the above discussion, the relationship between the defining density σ and the normal tractor becomes clear. A result of [**34**] showed that so long as $I^2_\sigma = 1 + \mathcal{O}(\sigma^2)$, we have that $I^A_\sigma \stackrel{\Sigma}{=} N^A$. (Note that this implies that in a choice of scale $\sigma = [g; s]$ we have that $\rho \stackrel{\Sigma}{=} -H^g$.) But given $(M, \boldsymbol{\gamma}, \sigma)_{\mathcal{Y}}$, this is always satisfied for $d \geq 4$, which is our regime of interest—so the hypersurface tractor calculus developed in Section 3.3 can directly parallel the Riemannian developments in Section 3.1. Thus, we use the scale tractor as a holographic formula for the normal tractor, and furthermore refer to any formula using this canonical extension as a holographic formula.

The operator $I \cdot D$ (and its hatted counterpart, $I \cdot \hat{D}$, which is often called the *Laplace–Robin operator*) play an important role in holography, in part because this operator satisfies an $\mathfrak{sl}(2)$ algebra. This fact is recorded in the following lemma from Gover and Waldron:

Proposition 3.4.3 ([**40**]). *Suppose $\sigma \in \Gamma(\mathcal{E}M[1])$ obeys $I^2_\sigma \neq 0$, and denote by $\mathsf{h} : \Gamma(\mathcal{T}^\Phi M[w]) \to \Gamma(\mathcal{T}^\Phi M[w])$ the operator defined by $\mathsf{h} = d + 2\underline{w}$. Then, viewing $x := \sigma : \Gamma(\mathcal{T}^\Phi M[w]) \to \Gamma(\mathcal{T}^\Phi M[w+1])$ as a multiplicative operator and $y := -\frac{1}{I^2} I \cdot D : \Gamma(\mathcal{T}^\Phi M[w]) \to \Gamma(\mathcal{T}^\Phi M[w-1])$ as a differential operator, commutators of the operators (x, h, y) satisfy the $\mathfrak{sl}(2)$ defining relations,*

$$[\mathsf{h}, x] = 2x\,, \quad [x, y] = \mathsf{h}\,, \quad [\mathsf{h}, y] = -2y\,.$$

Remark 3.4.4. Because we will often assume that σ is an asymptotic unit defining density, we may neglect the $\frac{1}{I^2}$ coefficient when using the above-displayed relations because it equals unity to sufficiently high order for the situations discussed. ■

We now finally have the tools to prove Lemma 3.3.9 via holography. The method requires a holographic formula for the tangential Thomas-D operator; this is captured in the following lemma.

LEMMA 3.4.5. *Let $w+\frac{d}{2} \neq 1, \frac{3}{2}, d$ and let $(M,\gamma,\sigma)_{\mathcal{Y}}$ specify a conformal hypersurface embedding. Then, acting on tractors of weight w, we have that*

$$\hat{D}^T_A \stackrel{\Sigma}{=} \hat{D}_A - I_A I \cdot \hat{D} + \tfrac{1}{d+2w-3} X_A I \cdot \hat{D}^2 \,. \tag{3.15}$$

PROOF. For this proof we will assume that σ is asymptotic unit defining. Observe from the Leibniz failure (Lemma 2.3.1) that, for ϕ, ψ tractors of weight $w_1, w_2 \neq 1 - \frac{d}{2}$ respectively,

$$I \cdot \hat{D}(\phi\psi) \stackrel{\Sigma}{=} \phi(I \cdot \hat{D}\phi) + (I \cdot \hat{D}\phi)\psi \,.$$

Indeed, along Σ, the operator $I \cdot \hat{D}$ is a derivation. Thus, we can write the operator equation

$$(I \cdot \hat{D})^2 \stackrel{\Sigma}{=} (I \cdot \hat{D} I^B)\hat{D}_B + I^A I^B \hat{D}_A \hat{D}_B \,. \tag{3.16}$$

Now observe that

$$\begin{aligned}
I \cdot \hat{D} I^B &= I_A \hat{D}^A \hat{D}^B \sigma \\
&= I_A \hat{D}^B \hat{D}^A \sigma \\
&= I_A \hat{D}^B I^A \\
&= \tfrac{1}{2} \hat{D}^B (I^2) - \tfrac{1}{d-2} X^B (\hat{D}^C I^A)(\hat{D}_C I_C) \\
&= \tfrac{1}{2} \hat{D}^B (1 + \sigma^d B) - \tfrac{1}{d-2} X^B (\hat{D}^C I^A)(\hat{D}_C I_C) \\
&\stackrel{\Sigma}{=} -\tfrac{1}{d-2} X^B (\hat{D}^C I^A)(\hat{D}_C I_A) \,.
\end{aligned} \tag{3.17}$$

Above, the second equality holds because $[\hat{D}_A, \hat{D}_B]$ on scalars vanishes, the fourth equality holds via the Leibniz failure, and the fifth equality holds because σ is asymptotic unit defining. By substituting Equation (3.16) into Equation (3.15), the proof then amounts to showing that $(\hat{D}^C I^A)(\hat{D}_C I_A)$ is an extension of the rigidity density. To do so, observe from [**9**, Lemma 3.15] that $q^*(\nabla I) \stackrel{\Sigma}{=} \mathring{\mathrm{II}}$, and thus $q^*(\hat{D} I) \stackrel{\Sigma}{=} \mathring{\mathrm{II}}$. That is, we can write

$$\hat{D}_A I_B \stackrel{\Sigma}{=} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \mathring{\mathrm{II}}_{ab} & * \\ 0 & * & * \end{pmatrix} .$$

Hence it is clear that $(\hat{D}^C I^A)(\hat{D}_C I_A) \overset{\Sigma}{=} K$, and thus is an extension of K. $\square$

The above proof included the first use of the tractor

$$P_{AB} := \hat{D}_A I_B \in \Gamma(\odot_\circ^2 \mathcal{T}^* M[-1])\,.$$

This tractor will play a major role going forward in our analyses of conformal hypersurface invariants. Indeed, we define a holographic formula for the rigidity density K by

$$K_{\mathrm{e}} := P_{AB} P^{AB}\,.$$

Observe that this tractor is symmetric because $[\hat{D}_A, \hat{D}_B]$ vanishes on scalars, is trace-free because $\hat{D}^A \circ \hat{D}_A = 0$, and is "top-slot free," *i.e.* $X^A P_{AB} = 0$, as a consequence of Equation (2.5) and that $\underline{w} I = 0$. Another useful property of the tractor P is that $\hat{D}^A P_{AB} = 0$. This directly implies that $(q \circ q^*)(P_{AB}) = P_{AB}$, so knowing the projecting part of P_{AB} is sufficient to determine the entire tractor. Because $q^*(P_{AB}) \overset{\Sigma}{=} \mathring{\mathrm{I\!I}}$, we define the canonical extension of the trace-free second fundamental form by

$$\mathring{\mathrm{I\!I}}^{\mathrm{e}} := q^*(P_{AB}) = [g; \nabla_{(a} n_{b)\circ} + s \mathring{P}^g_{ab}] \in \Gamma(\odot_\circ^2 T^* M[1])\,, \tag{3.18}$$

so that $\mathring{\mathrm{I\!I}}^{\mathrm{e}}|_\Sigma = \mathring{\mathrm{I\!I}}$. By inserting the above tensor-valued density into a tractor, we find that $q(\mathring{\mathrm{I\!I}}^{\mathrm{e}}) = P_{AB}$. Hence, we can write that

$$P_{AB} \overset{\Sigma}{=} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \mathring{\mathrm{I\!I}}^{\mathrm{e}}_{ab} & -\frac{\nabla \cdot \mathring{\mathrm{I\!I}}^{\mathrm{e}}_a}{d-1} \\ 0 & -\frac{\nabla \cdot \mathring{\mathrm{I\!I}}^{\mathrm{e}}_b}{d-1} & \frac{\nabla \nabla \cdot \mathring{\mathrm{I\!I}}^{\mathrm{e}} + (d-1) P \cdot \mathring{\mathrm{I\!I}}^{\mathrm{e}}}{(d-1)(d-2)} \end{pmatrix}. \tag{3.19}$$

A useful consequence of this extension of $\mathring{\mathrm{I\!I}}$ and the tractor P_{AB} is the following lemma.

LEMMA 3.4.6. *Let $(M, \gamma, \sigma)_{\mathcal{Y}}$ represent a conformally embedded hypersurface and let $\ell \in \mathbb{Z}_{\leq d-2}$ be non-negative. If $\mathring{\mathrm{I\!I}}^{\mathrm{e}} = \mathcal{O}(\sigma^\ell)$, then $n^a \mathring{\mathrm{I\!I}}^{\mathrm{e}}_{ab} = \mathcal{O}(\sigma^{\ell+1})$ and $\nabla \cdot \mathring{\mathrm{I\!I}}^{\mathrm{e}} = \mathcal{O}(\sigma^\ell)$.*

PROOF. First observe that, when $\mathring{\mathrm{I\!I}}^{\mathrm{e}} = \mathcal{O}(\sigma^\ell)$, an extension of the tangential divergence of $\mathring{\mathrm{I\!I}}^{\mathrm{e}}$ satisfies $\nabla_{\mathrm{e}}^{\top a} \mathring{\mathrm{I\!I}}^{\mathrm{e}}_{ab} := (g^{ac} - n^a n^c) \nabla_c \mathring{\mathrm{I\!I}}^{\mathrm{e}}_{ab} = \mathcal{O}(\sigma^\ell)$. This follows because

$$[\nabla_n, \nabla^\top_{\mathrm{e}\,a}] = 2\rho n_a \nabla_n + \mathrm{ltots}_0 \,,$$

where $\rho := -\frac{1}{d}(\Delta s + Js)$. Note that by ltots_k we mean any (possibly differential) operator with transverse order less than or equal to k. That is, we can write

$$\nabla \cdot \mathring{\mathrm{I\!I}}^{\mathrm{e}} - n \cdot \nabla_n \mathring{\mathrm{I\!I}}^{\mathrm{e}} = \nabla^\top_{\mathrm{e}} \cdot \mathring{\mathrm{I\!I}}^{\mathrm{e}} = \mathcal{O}(\sigma^\ell) \,. \tag{3.20}$$

Further, from Equation (3.17), we have that

$$q^*(2I^B P_{AB}) = d\sigma^{d-1} n_a B + \sigma^d \nabla_a B \,.$$

From Equation (3.19), we also have that $q^*(I \cdot P) = -\frac{\sigma}{d-1} \nabla \cdot \mathring{\mathrm{I\!I}}^{\mathrm{e}} + n \cdot \mathring{\mathrm{I\!I}}^{\mathrm{e}}$. Therefore,

$$\sigma \nabla \cdot \mathring{\mathrm{I\!I}}^{\mathrm{e}} - (d-1) n \cdot \mathring{\mathrm{I\!I}}^{\mathrm{e}}_b = \mathcal{O}(\sigma^{d-1}) \,. \tag{3.21}$$

Now choose an integer $0 \le k \le d-2$ and observe that $[\nabla_n^m, n_a] = \mathrm{ltots}_{m-1}$ for any positive integer m. Combining Equations (3.20) and (3.21) and taking k transverse derivatives, we find that

$$\nabla_n^k n \cdot \mathring{\mathrm{I\!I}}^{\mathrm{e}} \stackrel{\Sigma}{=} \frac{k}{d-1-k} \nabla_n^{k-1} \nabla^\top_{\mathrm{e}} \cdot \mathring{\mathrm{I\!I}}^{\mathrm{e}} + \mathrm{ltots}_{k-1}(n \cdot \mathring{\mathrm{I\!I}}^{\mathrm{e}}) + \mathrm{ltots}_{k-2}(\nabla^\top_{\mathrm{e}} \cdot \mathring{\mathrm{I\!I}}^{\mathrm{e}}) \,, \tag{3.22}$$

where for $m < 0$, ltots_m is the zero operator. Observe that for any $0 \le k \le \ell$, the right hand side vanishes to order $\mathcal{O}(\sigma^{\ell-k+1})$. Thus, if $\mathring{\mathrm{I\!I}}^{\mathrm{e}} = \mathcal{O}(\sigma^\ell)$, we have that $\nabla_n^k n \cdot \mathring{\mathrm{I\!I}}^{\mathrm{e}} \stackrel{\Sigma}{=} 0$ and hence $n \cdot \mathring{\mathrm{I\!I}}^{\mathrm{e}} = \mathcal{O}(\sigma^{\ell+1})$. The remainder of the lemma follows from Equation (3.20). □

REMARK 3.4.7. Because $[\nabla_n, \nabla^\top_{\mathrm{e}\,a}] = 2\rho n_a \nabla_n + \mathrm{ltots}_0$, it follows from Lemma 3.4.6 that

$$\nabla_n^{k-1} \nabla^\top_{\mathrm{e}} \cdot \mathring{\mathrm{I\!I}}^{\mathrm{e}} \stackrel{\Sigma}{=} \nabla^\top \cdot \nabla_n^{k-1} \mathring{\mathrm{I\!I}}^{\mathrm{e}} + \mathrm{ltots}_{k-2}(\mathring{\mathrm{I\!I}}^{\mathrm{e}}) \tag{3.23}$$

for $k \le d-2$. ■

A corollary about the tractor content of P_{AB} follows directly from the above lemma and Equation (3.19).

COROLLARY 3.4.8. *Let* $(M, \gamma, \sigma)_{\mathcal{Y}}$ *specify a conformally embedded hypersurface, let* $\ell \in \mathbb{Z}_{\leq d-2}$ *be non-negative, and let* $\mathring{\mathrm{I\!I}}^{\mathrm{e}} = \mathcal{O}(\sigma^\ell)$*. Then,*

$$P_{AB} = \mathcal{O}(\sigma^\ell) + X_A X_B \mathcal{O}(\sigma^{\ell-1})\,.$$

As another application of the tractor P_{AB}, we can prove Lemma 3.3.9 using Equation (3.15) and the tractor P_{AB}.

PROOF OF LEMMA 3.3.9. Let $(M, \gamma, \sigma)_{\mathcal{Y}}$ specify a conformally embedded hypersurface $\Sigma \hookrightarrow (M, \gamma)$. Because σ is asymptotic unit defining and $d \geq 4$, we have that $I^2 = 1 + \mathcal{O}(\sigma^4)$. As noted above in the proof of Lemma 3.4.5, we have that $I \cdot \hat{D} I_A = \frac{K_{\mathrm{e}} X_A}{d-2} + \mathcal{O}(\sigma^2)$. Next, we choose the holographic extension I of N so that, using Equation (3.17), we have that

$$\begin{aligned}
\hat{D}^T_{(A} I_{B)} &\stackrel{\Sigma}{=} \hat{D}_A I_B - I_{(A} I \cdot \hat{D} I_{B)} + \tfrac{1}{d-3} X_{(A} I \cdot \hat{D}\left(\tfrac{1}{d-2} X_{B)} K_{\mathrm{e}}\right) \\
&\stackrel{\Sigma}{=} P_{AB} - \tfrac{1}{d-2} I_{(A} X_{B)} K + \tfrac{1}{(d-2)(d-3)}\left(I_{(A} X_{B)} K_{\mathrm{e}} + X_A X_B I \cdot \hat{D} K_{\mathrm{e}}\right) \\
&\stackrel{\Sigma}{=} P_{AB} - \tfrac{d-4}{(d-2)(d-3)} I_{(A} X_{B)} K + \tfrac{1}{(d-2)(d-3)} X_A X_B I \cdot \hat{D} K_{\mathrm{e}}\,,
\end{aligned} \tag{3.24}$$

where the second equality holds because $I \cdot \hat{D}$ is a derivation along Σ. A result of [**44**] showed that the holographic formula for L_{AB} is given by

$$P_{AB} \stackrel{\Sigma}{=} L_{AB} + \tfrac{2}{d-2} I_{(A} X_{B)} K|_\Sigma + \tfrac{3d-8}{(d-2)(d-3)} X_A X_B M\,, \tag{3.25}$$

where $M = F^{AB} P_{AB}|_\Sigma$ as in the lemma statement. Combining Equation (3.24) with the above display, proving the lemma amounts to showing that $I \cdot \hat{D} K \stackrel{\Sigma}{=} -2(d-3)M$. This identity can be checked explicitly. By the Leibniz failure, we have that $I \cdot \hat{D} K_{\mathrm{e}} \stackrel{\Sigma}{=} 2P^{AB} I \cdot \hat{D} P_{AB}$. Because $I^A \stackrel{g}{=} (\sigma, n^a, \rho)$, we have that in a choice of scale $\sigma = [g; s]$, $I \cdot \hat{D} \stackrel{\Sigma}{=} \nabla_n - H^g \underline{w}$. Then, we have that

$$I \cdot \hat{D} K_{\mathrm{e}} \stackrel{\Sigma}{=} 2P^{AB}(\nabla_n + H) P_{AB}\,.$$

Using Equation (3.19) and (2.3), we have that

$$\nabla_n P_{AB} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \nabla_n \mathring{\mathrm{I\!I}}^{\mathrm{e}}_{ab} - \frac{2}{d-1} n_{(a} \nabla \cdot \mathring{\mathrm{I\!I}}^{\mathrm{e}}_{b)} & * \\ 0 & * & * \end{pmatrix} .$$

Hence, we have that

$$I \cdot \hat{D} K_{\mathrm{e}} \stackrel{\Sigma}{=} 2HK + 2 \mathring{\mathrm{I\!I}}^{ab} (\nabla_n \mathring{\mathrm{I\!I}}^{\mathrm{e}}_{ab} - \tfrac{2}{d-1} n_a \nabla \cdot \mathring{\mathrm{I\!I}}^{\mathrm{e}}_{b)})|_{\Sigma} .$$

Performing an explicit Riemannian computation, we find precisely that $I \cdot \hat{D} K_{\mathrm{e}} \stackrel{\Sigma}{=} -2(d-3) \mathring{\mathrm{I\!I}} \cdot \mathring{F} = -2(d-3) L_{AB} F^{AB}$. This completes the proof. □

CHAPTER 4

Conformal Fundamental Forms

A particularly useful application of hypersurface differential geometry is in the Arnowitt–Deser–Misner (ADM) formalism [**4**], which is one of the fundamental tools of numerical relativity. The core idea of that formalism is to take a spacelike hypersurface in $3+1$ dimensional spacetime and evolve that hypersurface forward in time using Einstein's equations to foliate all of spacetime, thereby obtaining a solution by construction with some given initial data. Given such an initial data slice, evolving towards a full solution requires two pieces of data because Einstein's equations are second order PDEs. Indeed, the ADM formalism requires both the induced metric on and the second fundamental form of the initial Cauchy slice.

In the same vein, if the problem we are interested in is conformally invariant (such as the Bach-flat problem in four dimensions), a conformally-invariant set of initial data would be required to evolve to solutions. Furthermore, in the example given of the Bach-flat problem, the PDE of interest is a fourth-order PDE rather than second order, so we need more than just two pieces of initial data. Indeed, to solve such problems, one might look toward producing a family of conformally-invariant extrinsic curvatures that generalize the second fundamental form. In this section, we examine such a family.

Using holography, one way we can further probe conformal hypersurface invariants of the embedding $\Sigma \hookrightarrow (M, \gamma)$ is by directly studying the canonically-determined unit defining density. Indeed, because the unit defining density σ for the embedding is determined uniquely (to sufficiently high order) by solving the singular Yamabe problem, we can extract conformal hypersurface invariants by studying a family of conformally-invariant jet coefficients of σ. These jet coefficients encode extrinsic embedding data so we study those tensors. Note that by construction, these jet coefficients are tensor-valued densities on tensor products of the cotangent bundle of M along Σ.

In Section 3.2, we introduced the first two conformally-invariant jet coefficients of σ. The first such coefficient is the conormal $\nabla\sigma|_\Sigma \in \Gamma(T^*M[1])|_\Sigma$. The second such jet is the trace-free second fundamental form, given by $\mathring{\mathrm{I\!I}}_{ab} := \nabla_{(a}\nabla_{b)\circ}\sigma|_\Sigma \in \Gamma(\odot^2_\circ T^*\Sigma[1])$. Unfortunately, additional gradients of this tensor do not fall into the cotangent bundle along Σ. There is thus a fork in our path: we can consider longitudinal jet coefficients of σ by considering hypersurface gradients of $\mathring{\mathrm{I\!I}}$ or we can consider transverse jet coefficients by considering normal derivatives of an extension of $\mathring{\mathrm{I\!I}}$. Because conformally-invariant operators on a conformal manifold $(\Sigma, \bar{\gamma})$ are well-studied [**7**,**16**,**48**], we do not investigate those jet coefficients in depth here. Furthermore, we are particularly interested in finding higher-order analogs of the second fundamental form. Thus, we probe transverse jet coefficients of σ. To better navigate our path forward, we analogize with the Riemannian case.

Consider a surface Σ embedded in a flat $(\mathbb{R}^3, \delta_{ab})$ specified by a defining function s. In this case, the square of the second fundamental form is sometimes termed the *third fundamental form*. We are interesting in examining how the third fundamental form is related to the jet coefficients of the defining function s. In this case, the conormal is given by the gradient of s, so that $n := \nabla s$, and the second fundamental form is simply the Hessian of s, *i.e.* $\mathrm{I\!I} = \nabla\nabla s$. The third fundamental form then obeys

$$\mathrm{I\!I\!I}_{ab} := -(\nabla_n \nabla_a \nabla_b s)|_\Sigma = \mathrm{I\!I}^2_{ab}\,.$$

We then proceed to define higher (Riemannian) fundamental forms by taking successive normal derivatives, so that

$$\overline{\underline{\mathrm{k+2}}} := \tfrac{(-1)^k}{k!} \nabla_n^k \nabla_a \nabla_b s|_\Sigma = \mathrm{I\!I}^{k+1}_{ab}\,,$$

where $\overline{\underline{\mathrm{m}}}$ denotes the (Riemannian) mth fundamental form.

While the fundamental forms described above naively do not encode any more information of the embedding than the second fundamental form, we see that for hypersurfaces embedded in a generally curved manifold $\Sigma \hookrightarrow (M, g)$, these tensors contain additional embedding information that is not captured by the second fundamental form. For example, if s is unit defining, then

$$\mathrm{I\!I\!I}_{ab} \stackrel{\Sigma}{=} \mathrm{I\!I}^2_{ab} - R_{nabn}\,.$$

A hint of the path forward is given by this formula: recall that the Fialkow tensor is given by

$$\mathring{F}_{ab} = \tfrac{1}{d-3}\left(\mathring{\mathrm{I\!I}}^2_{(ab)\circ} - W_{\hat{n}ab\hat{n}}\right) .$$

The similarity between $\mathring{F}$ and the formula for $\mathrm{I\!I\!I}$ above suggests that the Fialkow tensor is (proportional to) the next conformally-invariant transverse jet coefficient of σ beyond $\mathring{\mathrm{I\!I}}$—a conformal third fundamental form. Furthermore, observe that the second fundamental form and the Fialkow tensor have transverse order and weight pairs $(1,1)$ and $(2,0)$, respectively. Also note that under constant rescalings of the metric, derivatives with respect to the defining density ∂_σ have weight -1. This suggests that the transverse jet coefficients of σ probe, in a natural, conformally-invariant way, derivatives of the conformal metric with respect to the defining density. Here, natural tensors are those that are diffeomorphism invariant and are built from the metric, its derivatives, and the conormal. We are thus led to the following definition.

DEFINITION 4.0.1. Let $m \in \mathbb{Z}_{\geq 2}$. An mth conformal fundamental form is any natural section of $\odot^2_\circ T^*\Sigma[3-m]$ with transverse order $m-1$. ■

Often when discussing such tensor-valued densities, we drop the adjective conformal. Using this definition, we have that the Fialkow tensor is indeed a third conformal fundmental form. Of interest is that the definition above does not uniquely fix fundamental forms. Indeed, both the tensors

$$\mathring{F} \qquad \text{and} \qquad \mathring{F} + \alpha\,\mathring{\mathrm{I\!I}}^2_{(ab)\circ}$$

are third fundamental forms. However, their leading transverse derivative structure is (essentially) unique. This fact is made explicit in the following lemma.

LEMMA 4.0.2. *Suppose $2 \leq n \leq d-1$. Then if the conformal embedding $\Sigma \hookrightarrow (M, \boldsymbol{c})$ is such that at least one ℓth conformal fundamental form vanishes for every $2 \leq \ell < n$, then up to an overall non-zero coefficient, there is a unique nth conformal fundamental form.*

PROOF. The proof is based on an inductive argument where we show that the leading normal derivative terms of the metric are unique. Thus we begin by considering the leading transverse order term in the preinvariant expression for an nth fundamental form. As before, using coordinates $\{s, y^i\}$ in a collar neighborhood $I \times \Sigma \subset M$, we can express any natural preinvariant in terms of a defining function s, partial derivatives ∂_s and $\partial_i = \partial/\partial y^i$, the metric components g_{ab}, the components of its inverse g^{ab}. Because fundamental forms are conformally invariant, it is useful to view part of the preinvariant alphabet—the metric g and the defining function s—as representatives of weighted densities: the metric is a weight 2 representative of the conformal class of metrics $\boldsymbol{c}$ and the defining function is a representative of a defining density $\sigma = [g; s]$ with weight 1. We next determine the leading transverse order term in the preinvariant expression for an nth fundamental form using Definition 4.0.1.

From the definition of transverse order, we must be able to choose coordinates such that the leading derivative term in the preinvariant for the nth fundamental form is of the form

$$\mathrm{O}_{(ab)}{}^{cd}\,\partial_s^{n-1} g_{cd}|_\Sigma\,,$$

where, as an operator, $\mathrm{O}_{(ab)}{}^{cd}$ has transverse order 0. Moreover the above is annihilated by the normal vector and the hypersurface trace. Note that the weight of an nth fundamental form is $3-n$ and its transverse order is $n-1$. By considering only conformal transformations by a constant we may still analyze expressions such as that displayed above in terms of weights. Because the weight of the operator ∂_s is -1, the weight of the above display is $3-n+w_{\mathrm{O}}$ where the operator $\mathrm{O}_{(ab)}{}^{cd}$ has weight w_{O}. Hence we must have that $w_{\mathrm{O}} = 0$. By an elementary weight argument, ones sees that this operator is algebraic and therefore made only from the metric, its inverse, and a preinvariant for the conormal. Together with elementary $O(d)$ and $O(d-1)$ representation theory, this implies that (along Σ) this operator must be a non-zero multiple of the trace-free hypersurface projector, and hence proportional to

$$\mathring{\top}_{\mathrm{e}}\left(\partial_s^{n-1} g_{ab}\right)\big|_\Sigma\,,$$

where $\mathring{\top}_{\mathrm{e}}$ is any preinvariant expression for the operator $\mathring{\top}$.

Now suppose that $L^{(n)}$ and $L^{(n)\prime}$ are two nth fundamental forms with the same coefficient for the above-displayed term and the conformal embedding $\Sigma \hookrightarrow (M, \boldsymbol{c})$ is such that an ℓth fundamental form vanishes for every $\ell < n$. We then seek to show that $L^{(n)} - L^{(n)\prime} \stackrel{\Sigma}{=} 0$. Clearly, because $L^{(n)}$ and $L^{(n)\prime}$ have the same leading term, their difference must have transverse order at most $n-2$. Put another way,

$$L^{(n)}_{ab} - L^{(n)\prime}_{ab} = \mathrm{P}_{(ab)}{}^{cd}\partial_s^{n-2} g_{cd}\big|_\Sigma + \mathrm{ltots}_{n-3}(g_{ab})\,,$$

where $\mathrm{P}_{(ab)}{}^{cd}$ is a preinvariant operator with weight -1 and transverse operator order 0. But then $\partial_s^{n-2} g_{ab}|_\Sigma$ can be rewritten as an $(n-1)$th fundamental form plus lower-order terms. Thus, the induction only requires that we check that the second fundamental form is unique up to an overall non-zero coefficient, which is again easily verified by an elementary weight and representation theoretic argument. □

A consequence of this lemma is that the leading derivative structure of a conformal fundamental form is unique up to multiplication by a nonzero constant. By this uniqueness property, there is *a priori* no preferred method for constructing these fundamental forms. Arguing by analogy with the Riemannian case suggests that one way to form these tensors is by computing conformally-invariant transverse derivatives of an extension of the second fundamental form. In the following section, we provide several methods to construct such tensors.

4.1. Conformally-Invariant Transverse Derivative Operators

In general, the derivative operator ∇_n is not conformally invariant except when evaluated along Σ and acting on weight 0 functions. However, we are interested in the development of conformally-invariant operators with transverse order k that can act on sections of $\odot^2_\circ T^*M[1]$ (because that is the section space in which an extension of $\mathring{\mathrm{I\!I}}$ resides). While in general this is no easy task, holography makes our work easier. In this section, we provide two distinct methods of constructing such operators: one is iterative and one is non-holographic.

4.1.1. Iterative Construction. An important transverse derivative operator in conformal hypersurface geometry is the *tractor Robin operator* δ_{R}, first mentioned here in the proof Lemma 3.3.9, defined as the map

$$\Gamma(\mathcal{T}^\Phi M[w]) \ni T \overset{g}{\mapsto} (\nabla_{\hat{n}} - H^g \underline{w})T|_\Sigma \in \Gamma(\mathcal{T}^\Phi M[w-1])|_\Sigma \,.$$

When $w \neq 1 - \frac{d}{2}$, this operator can be written in a more compact form, *i.e.* $\delta_{\mathrm{R}} T = N^A \hat{D}_A T|_\Sigma$.

Remark 4.1.1. When restricted to densities, the tractor Robin operator is the conformally-invariant Robin combination of Neumann and Dirichlet operators first constructed by Cherrier [**22**]. ■

Because we wish to develop transverse derivative operators not on tractors or scalars but on rank-2 trace-free symmetric (Riemannian) tensors, the Robin operator as described above is not appropriate. However, given such a tensor, we can canonically construct a tractor on which δ_{R} can operate via the insertion operator, and we can then extract a hypersurface tensor with the same tensor structure by composing the hypersurface extraction map, the hypersurface removal operator, and the trace-free projection operator. This leads us to the following lemma.

LEMMA 4.1.2. *Let $\Sigma \hookrightarrow (M, \boldsymbol{c})$ be a conformal hypesurface embedding and let $t \in \Gamma(\odot^2_\circ T^*M[w])$, with $w \neq 3$. Then, given $g \in \boldsymbol{c}$,*

$$\delta^{(1)}\, t_{ab} \stackrel{g}{\triangleq} \mathring{\top}\left[(\nabla_{\hat{n}} - (w-2)H)\, t_{ab} + \tfrac{2}{w-3}\bar{\nabla}_{(a} t^{\top}_{\hat{n}b)\circ}\right].$$

PROOF. We proceed by working in a generic dimension d and with generic weight w rank-2 trace-free symmetric tensors $t \in \Gamma(\odot^2_\circ T^*M[w])$. For generic weights and dimensions, we may compute the composition of maps $\bar{q}^* \circ \bar{r} \circ \mathring{\top} \circ \delta_R \circ q$. We must compute $\bar{q}^* \circ \bar{r} \circ \mathring{\top}$, so we first apply this operator to a general rank-2 tractor $T \in \Gamma(\odot^2_\circ \mathcal{T}M[w])$. Note that $\mathring{\top}$ maps $T \mapsto \mathring{\bar{T}}$ where $\mathring{\bar{T}} := (I_A^{A'} I_B^{B'} - \frac{1}{d+1} I_{AB} I^{A'B'}) T_{A'B'}|_\Sigma$. Because $\bar{r}$ achieves $X \cdot \bar{r}(\mathring{\bar{T}}) = 0$, we have that $(\bar{q}^* \circ \bar{r} \circ \mathring{\top})(T)_{ab} = \bar{Z}_{Aa}\bar{Z}_{Bb}\bar{r}(\mathring{\bar{T}})^{AB}$ for an arbitrary tractor T. Thus, for generic dimensions and weights, we have that

$$\begin{aligned}
(\bar{q}^* \circ \bar{r} \circ \mathring{\top})(T)_{ab} = \; & \bar{Z}_{Aa}\bar{Z}_{Bb}\mathring{\bar{T}}^{AB} \\
& - \tfrac{2}{w}\bar{Z}_{B(a}\bar{\nabla}_{b)}(X_C\mathring{\bar{T}}^{CB}) + \tfrac{2}{w(d+1)}\bar{\gamma}_{ab}\hat{\bar{D}}_C(X_D\mathring{\bar{T}}^{CD}) \\
& + \tfrac{1}{w(w+1)}\bar{Z}_{Bb}\bar{\nabla}_a\hat{\bar{D}}^B(X_C X_D\mathring{\bar{T}}^{CD}) \\
& + \tfrac{8\bar{\gamma}_{ab}}{(d-1)(d+1)(d+2w+1)}\hat{\bar{D}}_C(X_D\mathring{\bar{T}}^{CD}) .
\end{aligned}$$

We now compute each of the terms above. First, using Equation (2.3), we obtain

$$\bar{Z}_{Bb}\bar{\nabla}_a(X_C\mathring{\bar{T}}^{CB}) = \bar{\nabla}_a\mathring{\bar{T}}^+_b + \bar{\gamma}_{ab}\mathring{\bar{T}}^{+-} + \bar{P}_{ab}\mathring{\bar{T}}^{++} ,$$

$$\bar{Z}_{Bb}\bar{\nabla}_a\hat{\bar{D}}^B(X_C X_D\mathring{\bar{T}}^{CD}) = \big[\bar{\nabla}_a\bar{\nabla}_b - \tfrac{\bar{\gamma}_{ab}}{d+2w+1}(\bar{\Delta} + (w+2)\bar{J}) + (w+2)\bar{P}_{ab}\big]\mathring{\bar{T}}^{++} .$$

Next (see for example the Appendix B of [**72**]), we have

$$\hat{\bar{D}}_C(X_D\mathring{\bar{T}}^{CD}) = \tfrac{1}{d+2w-1}\big(-[\bar{\Delta}-(d+w-1)\bar{J}]\mathring{\bar{T}}^{++}+(d+2w+1)\bar{\nabla}_a\mathring{\bar{T}}^{+a}+(d+w-1)(d+2w+1)\mathring{\bar{T}}^{+-}\big)\,.$$

Finally, because $\mathring{\bar{T}}^{AB}$ is hypersurface tractor trace-free, we have that $0=\mathring{\bar{T}}_a^a+2\mathring{\bar{T}}^{+-}$. Thus,

$$\bar{Z}_{Aa}\bar{Z}_{Bb}\mathring{\bar{T}}^{AB}+\tfrac{2\bar{\gamma}_{ab}}{d-1}T^{+-}=\mathring{\bar{T}}_{ab}-\tfrac{1}{d-1}\bar{\gamma}_{ab}\mathring{\bar{T}}_c^c=:\mathring{\bar{T}}_{(ab)\circ}\,.$$

Substituting these identities into the above display for $(\bar{q}^*\circ\bar{r}\circ\mathring{\top})(T)$ gives

$$(\bar{q}^*\circ\bar{r}\circ\mathring{\top})(T)_{ab}=\tfrac{1}{w(w+1)}\bar{\nabla}_{(a}\bar{\nabla}_{b)\circ}\mathring{\bar{T}}^{++}-\tfrac{2}{w}\bar{\nabla}_{(a}\mathring{\bar{T}}^{+}_{b)\circ}-\tfrac{1}{w+1}\bar{P}_{(ab)\circ}\mathring{\bar{T}}^{++}+\mathring{\bar{T}}_{(ab)\circ}\,.$$

Proving the lemma now amounts to computing the components of $\mathring{\bar{T}}$ when $T=\delta_{\rm R}\circ q(t)$. Note that by construction,

$$\begin{aligned}
\mathring{\bar{T}}^{++} &\stackrel{\Sigma}{=} T^{++}\,,\\
\mathring{\bar{T}}^{+}_b &\stackrel{\Sigma}{=} \bar{\gamma}_b^c T_c^+\,,\\
\mathring{\bar{T}}_{(ab)\circ} &\stackrel{\Sigma}{=} \bar{\gamma}_a^c\bar{\gamma}_b^d T_{cd}+\tfrac{1}{d-1}\bar{\gamma}_{ab}(2T^{+-}+T_{\hat{n}\hat{n}})\,.
\end{aligned}$$

Thus, we can simplify our calculations by only computing the components of T appearing on the right hand side above.

We can use Equation (2.12) and the definition of $\delta_{\rm R}$ to show that

$$\begin{aligned}
X_A T^{AB} = X_A\delta_{\rm R}q(t)^{AB} &= \delta_{\rm R}X_A q(t)^{AB}-N_A q(t)^{AB}\\
&= -N_A q(t)^{AB}\,,
\end{aligned}$$

where the second equality holds because $X\cdot q(t_{ab})=0$ by definition. Thus, using Lemma 2.5.1, we have

$$T^{++}=0\,,\qquad T_a^+=-t_{\hat{n}a}\,,\qquad T^{+-}=\frac{\hat{n}\cdot\nabla\cdot t}{d+w-2}\,.$$

Using again Lemma 2.5.1 as well as Equation (2.3) and the fact that $q(t)$ has weight $w-2$, we have that

$$T_{ab}=\nabla_{\hat{n}}t_{ab}-(w-2)Ht_{ab}-\frac{2\hat{n}_{(a}\nabla\cdot t_{b)}}{d+w-2}\,,$$

so that in a choice of scale $\sigma = [g;s]$,

$$\begin{aligned}\mathring{\bar{T}}_{(ab)\circ} &\stackrel{\Sigma}{=} \top[\nabla_{\hat{n}} t_{ab} - (w-2)Ht_{ab}] + \tfrac{1}{d-1}\bar{g}_{ab}(\hat{n}^a\hat{n}^b\nabla_{\hat{n}} t_{ab} - (w-2)Ht_{\hat{n}\hat{n}}) \\ &\stackrel{\Sigma}{=} \mathring{\top}[\nabla_{\hat{n}} t_{ab} - (w-2)Ht_{ab}]\,.\end{aligned}$$

Combining the above and noting that T has weight $w-3$, we have

$$\bar{q}^* \circ \bar{r} \circ \mathring{\top} \circ \delta_{\mathrm{R}} \circ q(t) = \mathring{\top}\Big[\nabla_{\hat{n}} t_{ab} - (w-2)Ht_{ab} + \tfrac{2}{w-3}\bar{\nabla}_{(a} t^{\top}_{\hat{n}b)\circ}\Big]\,.$$

This completes the proof. $\square$

In principle, one could extend this technique to the operator $(I\cdot\hat{D})^k$, but in doing so it becomes much harder to explicitly compute such operators. Instead, we will define an operator that we can iterate to produce such higher derivative operators. Indeed, by constructing a similar composition of operators with the tractor operator $I\cdot D$, we can compute powers of

$$\mathbb{D}_\sigma = q^* \circ r \circ I\cdot D \circ q \tag{4.1}$$

on generic weights to construct the desired operators. Observe that such an operator can be well-defined even if σ is not a defining density, let alone an asymptotic unit defining density. The following results regarding the operator $\mathbb{D}_\sigma$ hold in this most general case. We now need a technical lemma.

LEMMA 4.1.3. *Let $\theta = [g;t] \in \Gamma(\otimes^r T^*M[w])$ and $\sigma = [g;s] \in \Gamma(\mathcal{E}M[1])$. Then*

$$\nabla^\sigma\theta := [g; s\nabla t - (w-r)\mathrm{d}s \otimes t + (\mathrm{d}s \circledast g)^\sharp t] \in \Gamma(\otimes^{r+1}T^*M[w+1])\,,$$

where for a general covector ω, we denote $(\omega \circledast g)^\sharp_a t_b := \omega_b t_a - g_{ba}\omega^c t_c$ (when $r=1$) and extends, in the standard Leibniz way, to higher rank r tensors.

PROOF. First note that, using the notation provided in the lemma, the Levi-Civita connection acting on $\theta := [g,t] \in \Gamma(\otimes^r T^*M[w])$ obeys

$$\nabla^{\Omega^2 g} t^{\Omega^2 g} = \nabla^{\Omega^2 g}(\Omega^w t^g) = \Omega^w\left(\nabla^g t^g + (w-r)\Upsilon \otimes t^g - (\Upsilon \circledast g)^\sharp t^g\right).$$

Here $\Upsilon := \mathrm{d}\log\Omega$. Further, denoting $n := \mathrm{d}s$, $n^{\Omega^2 g} = \mathrm{d}(\Omega s) = \Omega(n^g + s\Upsilon)$. Therefore, $n^{\Omega^2 g} \otimes t^{\Omega^2 g} = \Omega^{w+1}(n^g + s\Upsilon) \otimes t^g$ and $(n^{\Omega^2 g} \circledast \Omega^2 g)^\sharp = \Omega([n^g + s\Upsilon] \circledast g)^\sharp$. Combining these conformal transformations with the above display completes the proof. □

Note that if σ is an asymptotic unit defining density for a hypersurface Σ, then, along Σ, the operator ∇^σ is just a particular tensor multiplication by the unit conormal $\hat{n}$.

The operator $\mathbb{D}_\sigma$ of Equation (4.1) is not defined for several weights, so we instead make the following definition.

Definition 4.1.4. Let $\sigma \in \Gamma(\mathcal{E}M[1])$ be any weight one density, let $I_\sigma = \hat{D}\sigma$, and let $\tau_{ab} \in \Gamma(\odot^2_\circ T^*M[w])$ where $w \neq 3, 2-d$. Also let $\widehat{M} := M \setminus \mathcal{Z}(\sigma)$. Then we define the map

$$\mathbb{D}_\sigma : \Gamma(\odot^2_\circ T^*\widehat{M}[w]) \to \Gamma(\odot^2_\circ T^*\widehat{M}[w-1])$$

by the following formula:

$$\begin{aligned}\sigma \mathbb{D}_\sigma \tau_{ab} := -\gamma^{cd}\Big(\nabla^\sigma_c \nabla^\sigma_d \tau_{ab} + \tfrac{2d}{(w-3)(d+w-2)}\, \nabla^\sigma_{(a} \nabla^\sigma_{|c} \tau_{d|b)\circ} - \tfrac{4}{d}[\nabla^\sigma_{(a}, \nabla^\sigma_{|c|}]\tau_{d|b)\circ}\Big) \\ - \tfrac{4}{d}\sigma^2 W^c{}_{ab}{}^d \tau_{cd} + \Big[\Big(w - 2 + \tfrac{d-1}{2}\Big)^2 - \Big(\tfrac{d-1}{2}\Big)^2 + 2\Big] I^2_\sigma\, \tau_{ab}\,.\end{aligned}$$

■

Note that the definition for $\mathbb{D}_\sigma$ given above is manifestly conformally invariant and of the appropriate weight—all of the terms that make up the operator are manifestly conformally invariant, as per their definitions and Lemma 4.1.3. The combination of terms making up $\mathbb{D}_\sigma$ is distinguished because they in fact allow the operator $\mathbb{D}_\sigma$ to be defined along $\mathcal{Z}(\sigma)$. This result is described in the following lemma.

Lemma 4.1.5. *Let $\sigma \in \Gamma(\mathcal{E}M[1])$ be any weight one density and $\tau_{ab} \in \Gamma(\odot^2_\circ T^*M[w])$ with $w \neq 3, 2-d$. Let $g \in \boldsymbol{c}$ for which $\sigma = [g; s]$, $\tau_{ab} = [g; t_{ab}]$, and $\mathbb{D}_\sigma \tau_{ab} = [g; \mathbb{D}_s t_{ab}]$. Then*

$$\begin{aligned}\mathbb{D}_s t_{ab} = (d+2w-6)\Big(\big[\nabla_n + (w-2)\rho\big]\, t_{ab} - \tfrac{2(w-2)}{(w-3)(d+w-2)} n_{(a} \nabla \cdot t_{b)\circ} + \tfrac{2}{w-3}\Big[n_c \nabla_{(a} t^c_{b)\circ} + (\nabla n)_{(a} \cdot t_{b)\circ}\Big]\Big) \\ - s\left(\Delta t_{ab} + (w-2) J t_{ab} + \tfrac{2d}{(w-3)(d+w-2)} \nabla_{(a} \nabla \cdot t_{b)\circ} - 4 P_{(a} \cdot t_{b)\circ}\right),\end{aligned}$$

where $n := \mathrm{d}s$ *and* $\rho = -\frac{1}{d}(\Delta s + Js)$*, and* $\mathbb{D}_\sigma$ *is a well-defined map*

$$\mathbb{D}_\sigma : \Gamma(\odot^2_\circ T^*M[w]) \to \Gamma(\odot^2_\circ T^*M[w-1])\,.$$

Proof. The proof amounts to a computation of $\mathbb{D}_\sigma \tau_{ab}$ in a choice of scale away from $\mathcal{Z}(\sigma)$, applying Lemma 4.1.3 twice, and the identity $I^2_\sigma \stackrel{g}{=} n^2 + 2s\rho$. The resulting tensor is proportional to s and hence the apparent singularity of $\mathbb{D}_\sigma$ along $\mathcal{Z}(\sigma)$ in Definition 4.1.4 is removable. □

For weights at which the composition of maps $q^* \circ r \circ I \cdot \hat{D} \circ q : \Gamma(\odot^2_\circ T^*M[w]) \to \Gamma(\odot^2_\circ T^*M[w-1])$ is defined, a tedious but straightforward computation shows that $\mathbb{D}_\sigma = q^* \circ r \circ I \cdot D \circ q$. Thus, we have constructed a means by which to compute the action of a conformally-invariant transverse derivative on symmetric trace-free rank-2 tensors in a way that can be iterated, but is well-defined on more weights than the composition operator just mentioned.

Furthermore, we can relate the operator $\mathbb{D}_\sigma$ to $\delta^{(1)}$ quite easily by first defining the operator $\hat{\mathbb{D}}_\sigma := \frac{1}{d+2w-6}\mathbb{D}_\sigma$ acting on tensors with weight $w \neq 3 - \frac{d}{2}$. As above, when the composition operator is well-defined, we have that $\hat{\mathbb{D}}_\sigma = q^* \circ r \circ I \cdot \hat{D} \circ q$. Furthermore, because $\delta^{(1)}$ is constructed from a partially hypersurface composition operator containing $I \cdot \hat{D}$, we can find a relationship between $\delta^{(1)}$ and $\hat{\mathbb{D}}_\sigma$. Indeed, given a defining density σ subject to $I^2_\sigma \stackrel{\Sigma}{=} 1$, when $w \neq 3, 3 - \frac{d}{2}$, we have that

$$\delta^{(1)} = \mathring{\top} \circ \hat{\mathbb{D}}_\sigma - \tfrac{2}{w-3}\,\mathring{\mathrm{II}}(\hat{n}\cdot\,)^2\,.$$

With these operators $\mathbb{D}_\sigma$ and $\delta^{(1)}$, one might naively suggest that a transverse derivative operator of any order could be constructed by writing $\delta^{(1)} \circ \mathbb{D}^k_\sigma$ for any $k \in \mathbb{Z}_{\geq 0}$. However, as suggested by the work of Gover and Peterson [**38**], certain values of k yield operators with transverse order lower than $k+1$. Indeed, we can prove this result explicitly—but first we provide a definition of this iterated operator.

Definition 4.1.6. Let $(M, \gamma, \sigma) \in \mathrm{ASY}$ specify a conformal hypersurface embedding. Then, define $\delta^{(0)}_{d,w} := \mathring{\top} : \Gamma(\odot^2_\circ T^*M[w]) \to \Gamma(\odot^2_\circ T^*\Sigma[w])$, and acting on weight $w \neq 3$ tensors, define $\delta^{(1)}_{d,w} := \delta^{(1)}$. For $k \in \mathbb{Z}_{\geq 2}$ and $w \notin \{2-d, \ldots, k-d\} \cup \{3, \ldots, k+2\}$, let

$$\delta^{(k)}_{d,w} := \delta^{(1)} \circ \mathbb{D}^{k-1}_\sigma : \Gamma(\odot^2_\circ T^*M[w]) \to \Gamma(\odot^2_\circ T^*\Sigma[w-k])\,.$$

$\blacksquare$

Acting on sections of $\odot^2_\circ T^*M[w]$ for $w \neq 3$, the transverse order of $\delta^{(1)}_{d,w}$ is 1. When $w \notin \mathbb{Z}$, the operator $\delta^{(k)}_{d,w}$ is always defined. Moreover, when $2w$ is not an integer the operators $\delta^{(k)}_{d,w}$ for $k \geq 2$ have transverse order k (see Equation (4.2) below). However, when $k \in \mathbb{Z}_{\geq 2}$ and $2w \in \mathbb{Z}$, the operator $\delta^{(k)}_{d,w}$ may not be defined or it could fail to have transverse order k. In particular, for w an integer and $k \geq 2$, the operators $\delta^{(k)}_{d,w}$ are only defined in the three regions where w obeys $w < 2-d$, $k-d < w < 3$, or $k+2 < w$ (the second of these could be empty). The following lemma characterizes the transverse order of $\delta^{(k)}_{d,w}$ in these cases.

LEMMA 4.1.7. *Fix $d \geq 3$ and $k \geq 2$, and let $w \in \mathbb{Z}$ be such that*

$$w < 2-d\,,\ k-d < w < 3\,,\quad \textit{or}\ \ k+2 < w\,.$$

Then, the transverse order of $\delta^{(k)}_{d,w}$ is strictly less than k if and only if

$$\frac{7-d}{2} \leq w < 3 \ \ \textit{and}\ \ \frac{d+2w-3}{2} \leq k \leq d+2w-5\,.$$

PROOF. To evaluate the transverse order of $\delta^{(k)}_{d,w}$, we compute the coefficient of ∇^k_n. For that we first examine the leading derivative structure of the operator $\mathrm{I\!D}_\sigma$. From Lemma 4.1.5, acting on a weight w tensor $t_{ab} \in \Gamma(\odot^2_\circ T^*M[w])$ the only terms with non-zero transverse order are $\nabla_n t_{ab}$, $n_{(a}\nabla \cdot t_{b)\circ}$, $n_c \nabla_{(a} t^c_{b)\circ}$, $s\Delta t_{ab}$, and $s\nabla_{(a}\nabla \cdot t_{b)\circ}$ (here $\sigma = [g;s]$ and we work in the scale g). In a choice of coordinates $(s, y^1, \dots, y^{n-1})$ with $\{\partial_{y^i}\}$ tangential to Σ, we can write

$$\nabla_a = n_a \partial_s + \mathrm{ltots}_0\,,$$

where $n = \nabla s$ and ltots_k denotes an operator that can be expressed in these coordinates with leading derivative term ∂^ℓ_s where $\ell \leq k$. Observe that $\mathrm{ltots}_\ell \circ \mathrm{ltots}_k = \mathrm{ltots}_{\ell+k}$. Thus, we have that

$$n_c \nabla_{(a} t^c_{b)\circ} = n_c n_{(a}\partial_s t^c_{b)\circ} + \mathrm{ltots}_0(t) \ \text{ and }\ s\nabla_{(a}\nabla \cdot t_{b)\circ} = s n_{(a}\partial_s \nabla \cdot t_{b)\circ} + \mathrm{ltots}_1(t)\,.$$

Because σ is an asymptotic unit defining density, we have that $\nabla_n \circ n \stackrel{\Sigma}{=} n(\nabla_n + H)$ and so the operator $\delta^{(1)}$ composed with the conormal n has transverse order zero. Therefore, only the terms ∇_n and $s\Delta$ in $\mathrm{I\!D}_\sigma$ can contribute to the leading transverse derivatives in the operator $\delta^{(k)}_{d,w}$.

From the above, and again consulting Lemma 4.1.5, we conclude that

$$\delta^{(k)}_{d,w} = \mathring{\top} \circ \nabla_n \circ \prod_{i=0}^{k-2} [(d+2w-2i-6)\nabla_n - s\Delta] + \mathrm{ltots}_{k-1}\,.$$

Because $\Delta = \nabla_n^2 + \mathrm{ltots}_1$, we have that

$$\delta^{(2)}_{d,w} \stackrel{\Sigma}{=} \mathring{\top} \circ (d+2w-7)\nabla_n^2 + \mathrm{ltots}_1\,.$$

We now proceed inductively to find the general coefficient of the leading transverse derivative term. Suppose that for $k \geq 3$,

$$\delta^{(k-1)}_{d,w} = \mathring{\top} \circ \left[\prod_{i=1}^{k-2}(d+2w-k-i-3)\right] \nabla_n^{k-1} + \mathrm{ltots}_{k-2}\,.$$

Then, because

$$\delta^{(k)}_{d,w} = \delta^{(k-1)}_{d,w-1} \circ \mathrm{ID}_\sigma\,,$$

we have that

$$\delta^{(k)}_{d,w} = \mathring{\top} \circ \left[\prod_{i=1}^{k-2}(d+2w-k-i-5)\right] \nabla_n^{k-1} \circ [(d+2w-6)\nabla_n - s\Delta] + \mathrm{ltots}_{k-1}\,.$$

Hence, as required, we find

$$\delta^{(k)}_{d,w} = \mathring{\top} \circ \left[\prod_{i=1}^{k-1}(d+2w-k-i-4)\right] \nabla_n^{k} + \mathrm{ltots}_{k-1}\,. \tag{4.2}$$

To find when $\delta^{(k)}_{d,w}$ has transverse order strictly less than k, we study when the leading coefficient in the above display vanishes. That is, we wish to find k and w that obey

$$d+2w-2k-3 \leq 0 \leq d+2w-k-5\,. \tag{4.3}$$

Because $k \geq 2$, we find that the right inequality implies that $w \geq \frac{7-d}{2}$. Further, the inequality above can be rewritten in terms of k to obtain $\frac{d+2w-3}{2} \leq k \leq d+2w-5$. Equation (4.3) has no solutions for w in the range $w < 2-d$ since that would require $d < -3$. Therefore, when $w < 2-d$, the transverse order of $\delta^{(k)}_{d,w}$ is k. The left inequality of (4.3) rules out $w > k+2$ so the only remaining case is $k - d < w < 3$, which, in combination with Equation (4.3), gives the ranges of k and w quoted in the Lemma. $\square$

Observe that, acting on integer weight tensors, these operators $\delta^{(k)}_{d,w}$ fail to have the expected transverse order in certain cases in all dimension parities. The next provided construction has the interesting property that it has the expected transverse order in certain cases in a particular dimensional parity. We now proceed to that construction.

4.1.2. Non-Holographic Construction. For this construction, we rely on the generalizations of results from [**38**] and [**37**] found in Section 2.4.

While the operators $\delta^{(k)}_{d,w}$ constructed above typically have a dimensionful coefficient on the leading derivative term that vanishes for particular weights and dimensions, it is not necessarily the case that operators with the expected transverse order in those cases do not exist. Indeed, if every term in the formula for said operator had the same dimensionful coefficient, one could instead define an operator with the expected transverse order by dividing out by that particular coefficient. Work of Gover and Peterson [**38**] showed that this phenomenon occurs in a particular subset of similarly constructed operators on scalar-valued densities. In this section, we summarize a nearly identical construction for the action of such operators on tractors, most of which is contained in that work.

First, we record a result directly from [**38**, Proposition 5.8]:

LEMMA 4.1.8 (Proposition 5.8 of [**38**]). *Let $k \in \mathbb{Z}_+$ be given. Then, there exists a family of conformally invariant operators $\delta_k : \Gamma(\mathcal{T}^\Phi[w]) \to \Gamma(\mathcal{T}^\Phi[w-k])|_\Sigma$ defined by*

$$\delta_k := N^{A_2} \cdots N^{A_k} \delta_R D_{A_2} \cdots D_{A_k} \,,$$

with transverse order k so long as

$$w \notin \left\{ \tfrac{2k-1-d}{2}, \tfrac{2k-2-d}{2}, \ldots, \tfrac{k+1-d}{2} \right\} .$$

Here $\delta_R : \Gamma(\mathcal{T}^\Phi M[w]) \to \Gamma(\mathcal{T}^\Phi M[w-1])|_\Sigma$ and is defined by $\delta_R := \nabla_{\hat{n}} - wH$.

We now provide an analog of [**38**, Theorem 5.16] in even dimensions.

Theorem 4.1.9. *Let $J, k \in \mathbb{Z}_+$ such that $0 < J$, $0 < k < d/2$ and let d be even. Then, there exists a family of conformally invariant differential operators*

$$\delta^{\Phi}_{J,k} : \Gamma(\mathcal{T}^{\Phi}M[w]) \to \Gamma(\mathcal{T}^{\Phi}M[w-k-J])|_{\Sigma}$$

determined as follows. For $k \leq J$,

$$\delta^{\Phi}_{J,k} = N^{A_1} \cdots N^{A_k} \delta_J P^{\Phi}_{A_1 \cdots A_k} .$$

For $k > J$, then $\delta_{J,k}$ is determined by the equation

$$(d + 2w - 2k)\delta^{\Phi}_{J,k} = N^{A_1} \cdots N^{A_k} \delta_J P^{\Phi}_{A_1 \cdots A_k} .$$

When $w = k - d/2$, $\delta^{\Phi}_{J,k}$ has transverse order $J + k$.

Proof. We follow the proof of [**38**, Theorem 5.16]. That proof relies on several results that require generalization to arbitrary tractor tensor structures; specifically, it directly requires [**38**, Lemma 5.7, Proposition 5.14, and Lemma 5.15]. Of these results, [**38**, Lemmas 5.7 and 5.15] already apply to arbitrary tractors, whereas [**38**, Proposition 5.14] only applies to scalars and directly relies on [**38**, Proposition 5.10], which also only applies to scalars. [**38**, Proposition 5.10], in turn, relies on [**37**, Proposition 4.5], which is a result on scalars. Thus, the proof this theorem effectively amounts to generalizing [**37**, Proposition 4.5] to arbitrary tensor structures and then following this result through the aforementioned steps. The generalization of [**38**, Proposition 4.5] is given in Proposition 2.4.3, and the generalizations of [**38**, Propositions 5.10, 5.14] are given in Propositions 2.4.4 and 2.4.5, respectively. See Section 2.4 for these results. □

Such an operator $\delta^{\Phi}_{J,k}$ is defined on any particular tractor bundle $\mathcal{T}^{\Phi}M[w]$, so we will drop the superscript Φ when the tensor structure is clear from context. Given such operators, we can compose them with the insertion, projection, removal, and extraction operators as before to produce operators on tensors of the appropriate type. Then, such operators can be proven to have the correct transverse order, even in cases when the iteratively-constructed operators do not.

As an example, suppose $(M^6, \gamma, \sigma)_{\mathcal{Y}}$ represents a conformal hypersurface embedding. We compare the operators $\delta^{(3)}_{6,1}$ and $(\bar{q}^* \circ \bar{r} \circ \mathring{\top} \circ \delta_{1,2} \circ q)$ on weight 1 symmetric trace-free rank-2 tensor-valued densities. From Lemma 4.1.7, we see that $\delta^{(3)}_{6,1}$ has transverse order strictly less than 3, whereas we will expect (and later prove) that $(\bar{q}^* \circ \bar{r} \circ \mathring{\top} \circ \delta_{1,2} \circ q)$ has transverse order 3, as per Theorem 4.1.9. Because we are most interested in constructing such operators that act on this weight (because any extension of $\mathring{\mathrm{II}}$ is weight 1), we will find that when d is even, this second construction can be more useful.

4.2. Conformal Fundamental Forms—General Dimensions

We now proceed to explicitly construct fundamental forms in a way that is agnostic toward the dimension parity. Because the iterative construction is more straightforward, we implement that construction to explicitly compute several of the fundamental forms, as well as providing formulæ for computing them in general.

As mentioned earlier, the method we will use to compute these fundamental forms is by application of transverse derivative operators to an extension of $\mathring{\mathrm{II}}$. Observe from Equation (3.18) that we already have a prescribed holographic formula for $\mathring{\mathrm{II}}$, so we will use this as our canonical extension. Furthermore, when necessary we will use the tractor P_{AB} as the tractor equivalent of this canonical extension, as $P_{AB} = q(\mathring{\mathrm{II}}^{\mathrm{e}}_{ab})$. Because these holographic formulæ rely on a conformal hypersurface embedding specified by $(M, \gamma, \sigma)_{\mathcal{Y}}$, we will assume the conformal hypersurface embedding asymptotically solves the singular Yamabe problem going forward.

We now define a formula for a particular subset of fundamental forms. This set is distinguished because the contained fundamental forms are always defined.

Definition 4.2.1. Let $d \geq 3$ and let $2 \leq n < \frac{d+3}{2}$. An *nth fundamental form* $\mathring{\underline{\mathrm{n}}}$ is defined by

$$\mathring{\underline{\mathrm{n}}} := \delta^{(n-2)}_{d,1} \, \mathring{\mathrm{II}}^{\mathrm{e}} \, .$$

■

The following proposition confirms that the above definition is valid.

Proposition 4.2.2. *The nth fundamental form defined in Definition 4.2.1 is a fundamental form.*

PROOF. From Equation (3.18), in a choice of scale $\sigma = [g; s]$, we have $\mathring{\mathrm{II}}^{\mathrm{e}}_{ab} = \nabla_{(a} n_{b)\circ} + s\mathring{P}_{ab}$. Therefore, we have that $\nabla_n^k \mathring{\mathrm{II}}^{\mathrm{e}}_{ab} \stackrel{\Sigma}{=} \nabla_n^k \nabla_{(a} n_{b)\circ} + k\nabla_n^{k-1}\mathring{P}_{ab}$ for any positive integer k. There exists a scale g for which s is a unit defining function for Σ (see, for example [**49**]); thus in what follows, we can assume that $|ds|^2 = 1$, so that $\nabla_n n_a = 0$. Therefore, we can write that

$$\begin{aligned}\nabla_n \nabla_a n_b &= R_{nabn} - (\nabla_a n^c)(\nabla_c n_b) \\ &= W_{nabn} - P_{ab} + 2n_{(a}P_{b)n} - g_{ab}P_{nn} + \mathrm{ltots}_1(g_{ab})\,.\end{aligned}$$

Applying the above display to $\nabla_n^k \mathring{\mathrm{II}}^{\mathrm{e}}_{ab}$, we have

$$\nabla_n^k \mathring{\mathrm{II}}^{\mathrm{e}}_{ab} \stackrel{\Sigma}{=} \nabla_n^{k-1} W_{nabn} + (k-1)\nabla_n^{k-1}\mathring{P}_{ab} + 2n_{(a}\nabla_n^{k-1}P_{nb)\circ} + \mathrm{ltots}_{k-1}(\mathring{\mathrm{II}}^{\mathrm{e}}_{ab})\,.$$

Then a straightforward computation in a choice of coordinates $(s, y^1, \ldots, y^{d-1})$ with $\{\partial_{y^i}\}$ tangential to Σ shows that

$$\mathring{\top} \circ \nabla_n^k \mathring{\mathrm{II}}^{\mathrm{e}}_{ab} \stackrel{\Sigma}{=} \tfrac{d-k-2}{2(d-2)} \mathring{\top}(\partial_s^{k+1} g_{ab}) + \mathrm{ltots}_k(g_{ab}). \tag{4.4}$$

Using Lemma 4.1.7, if $n \in \mathbb{Z}$ satisfies $2 \le n < \frac{d+3}{2}$, then

$$\delta^{(n-2)}_{d,1} = \alpha\, \mathring{\top} \circ \nabla_n^{n-2} + \mathrm{ltots}_{n-3},$$

for some non-zero coefficient α. Thus, in a choice of scale and coordinates used above, we have

$$\mathring{\underline{\underline{\mathrm{n}}}}_{ab} = \alpha\, \mathring{\top} \circ \nabla_n^{n-2} \mathring{\mathrm{II}}^{\mathrm{e}}_{ab} + \mathrm{ltots}_{n-3}(\mathring{\mathrm{II}}^{\mathrm{e}}_{ab}) = \alpha \tfrac{d-n}{2(d-2)} \mathring{\top}(\partial_s^{n-1} g_{ab}) + \mathrm{ltots}_{n-2}(g_{ab}).$$

and so $\mathring{\underline{\underline{\mathrm{n}}}}$ has transverse order $n-1$. Further, by construction, $\mathring{\underline{\underline{\mathrm{n}}}}$ is a conformal tensor density of weight $3-n$. The proposition follows. □

REMARK 4.2.3. The above proof method—writing some tensor as partial derivatives with respect to s of the metric—will be utilized heavily in later results. ■

REMARK 4.2.4. In dimensions $d = 3, 4$, an nth fundamental form is defined for all $2 \le n \le d-1$. When $d \ge 4$, the operator $\delta^{(1)}_{d,1}$ can be used to compute

$$\mathring{\mathrm{III}}_{ab} := \quad \delta^{(1)}_{d,1} \mathring{\mathrm{II}}^{\mathrm{e}}_{ab} \quad = -\mathring{\mathrm{II}}^2_{(ab)\circ} + W_{\hat{n}ab\hat{n}}\,.$$

For $d \geq 6$, applying the operator $\delta^{(2)}_{d,1}$ gives

$$
\begin{aligned}
\mathring{\mathrm{IV}}_{ab} := \delta^{(2)}_{d,1} \mathring{\mathrm{II}}^{\mathrm{e}}_{ab} \ &= -(d-4)(d-5) C^{\top}_{\hat{n}(ab)} - (d-4)(d-5) H W_{\hat{n}ab\hat{n}} - (d-4) \bar{\nabla}^c W^{\top}_{c(ab)\hat{n}} \\
&\quad + 2 W_{c\hat{n}\hat{n}(a} \mathring{\mathrm{II}}^c_{b)\circ} + (d^2 - 7d + 18) \mathring{F}_{(a} \cdot \mathring{\mathrm{II}}_{b)\circ} + (d-6) \bar{W}^c{}_{ab}{}^d \mathring{\mathrm{II}}_{cd} \\
&\quad + \tfrac{d^3 - 10d^2 + 25d - 10}{(d-1)(d-2)} K \mathring{\mathrm{II}}_{ab} \, .
\end{aligned} \tag{4.5}
$$

The above computation was performed using the symbolic manipulation program FORM [**83**]. Documentation of our FORM code can be found in Appendix A. Also note that, in dimensions $d > 3$, the third fundamental form $\mathring{\mathrm{III}}$ recovers the trace-free Fialkow tensor:

$$
\mathring{\mathrm{III}} = -(d-3) \mathring{F} \, . \tag{4.6}
$$

■

A useful consequence of the uniqueness result in Lemma 4.0.2 and the proof of Proposition 4.2.2 is that if the first $m-1$ fundamental forms vanish, we must have that the canonical extension of the second fundamental form must vanish to a certain order. This fact is captured in the following lemma which will be used extensively.

Lemma 4.2.5. *Let $(M^d, \gamma, \sigma)_{\mathcal{Y}}$ with $d \geq 3$ specify a conformal hypersurface embedding such that*

$$
\mathring{\mathrm{II}} = \ldots = \underline{\mathring{\mathrm{m}}} = 0
$$

for some integer $2 \leq m < \frac{d+3}{2}$. Then, $\mathring{\mathrm{II}}^{\mathrm{e}} = \mathcal{O}(\sigma^{m-1})$.

Proof. For this proof, we will work in a scale $\sigma = [g; s]$ where $|ds|_g = 1$ and where $(s, y^1, \ldots, y^{d-1})$ are the typical choice of coordinates. Our strategy for this proof will be to show that if all such fundamental forms vanish up to the mth fundamental form, then we will have that $\mathring{\top} \circ \nabla^{\ell}_n \mathring{\mathrm{II}}^{\mathrm{e}} \overset{\Sigma}{=} 0$ for every integer $0 \leq \ell \leq m-2$. Then we will use Lemma 3.4.6 to show that this implies that $\nabla^{\ell}_n \mathring{\mathrm{II}}^{\mathrm{e}}$ vanishes along Σ for the same ℓ.

The general definition of an nth fundamental form suggests that it is a conformally invariant tensor that can be written in the form

$$
\underline{\mathring{\mathrm{n}}}_{ab} = \alpha \mathring{\top} \partial^{n-1}(g_{ab}) + \mathrm{ltots}_{n-2}(g_{ab}) \, ,
$$

for some nonzero coefficient α. If $n \leq d-1$, from the uniqueness result of Lemma 4.0.2, we can see from the existence result of Lemma 4.2.2 and the explicit formula in Equation (4.4) that this tensor can thus be expressed in the form

$$\mathring{\underline{\mathrm{n}}}_{ab} = \beta \mathring{\top} \circ \nabla_n^{n-2} \mathring{\mathrm{I\!I}}^{\mathrm{e}} + \mathrm{ltots}_{n-3}(\mathring{\mathrm{I\!I}}^{\mathrm{e}}) \,,$$

where β is some nonzero coefficient. Thus, for any integer $2 \leq m \leq \frac{d+3}{2}$, we have that if $\mathring{\mathrm{I\!I}} = \ldots = \mathring{\underline{\mathrm{m}}} = 0$, then $\mathring{\top} \circ \nabla_n^{\ell} \mathring{\mathrm{I\!I}}^{\mathrm{e}} \stackrel{\Sigma}{=} 0$ for every integer $0 \leq \ell \leq m-2$.

We now proceed by induction on $\ell \leq m$ to show that $\nabla_n^{\ell} \mathring{\mathrm{I\!I}}^{\mathrm{e}} \stackrel{\Sigma}{=} 0$ for all $\ell \leq m-2$. Because $m \geq 2$, we have that $\mathring{\mathrm{I\!I}} = 0$ so clearly $\nabla_n^{0} \mathring{\mathrm{I\!I}}^{\mathrm{e}} \stackrel{\Sigma}{=} 0$. Now, suppose that $\nabla_n^{\ell-1} \mathring{\mathrm{I\!I}}^{\mathrm{e}} \stackrel{\Sigma}{=} 0$ for all $1 \leq \ell \leq m-2$. Thus, we have that $\mathring{\mathrm{I\!I}}^{\mathrm{e}} = \mathcal{O}(\sigma^{m-2})$. We then compute directly:

$$\begin{aligned}
\nabla_n^{\ell} \mathring{\mathrm{I\!I}}^{\mathrm{e}}_{ab} \stackrel{\Sigma}{=}& (\bar{g}_a^c + n_a n^c)(\bar{g}_b^d + n_b n^d) \nabla_n^{\ell} \mathring{\mathrm{I\!I}}^{\mathrm{e}}_{cd} \\
\stackrel{\Sigma}{=}& \mathring{\top} \circ \nabla_n^{\ell} \mathring{\mathrm{I\!I}}^{\mathrm{e}}_{ab} + \frac{\bar{g}_{ab}}{d-1} \bar{g}^{cd} \nabla_n^{\ell} \mathring{\mathrm{I\!I}}^{\mathrm{e}}_{cd} + 2 n_{(a} (n \cdot \nabla_n^{\ell} \mathring{\mathrm{I\!I}}^{\mathrm{e}}_{b)})^{\top} + n_a n_b \, n \cdot (\nabla_n^{\ell} \mathring{\mathrm{I\!I}}^{\mathrm{e}}) \cdot n \\
\stackrel{\Sigma}{=}& \mathring{\top} \circ \nabla_n^{\ell} \mathring{\mathrm{I\!I}}^{\mathrm{e}}_{ab} + 2 n_{(a} (n \cdot \nabla_n^{\ell} \mathring{\mathrm{I\!I}}^{\mathrm{e}}_{b)})^{\top} + (n_a n_b - \tfrac{\bar{g}_{ab}}{d-1}) n \cdot (\nabla_n^{\ell} \mathring{\mathrm{I\!I}}^{\mathrm{e}}) \cdot n \,,
\end{aligned}$$

where the third line holds because $g^{ab} \mathring{\mathrm{I\!I}}^{\mathrm{e}}_{ab} = 0$. Because $\mathring{\mathrm{I\!I}}^{\mathrm{e}} = \mathcal{O}(\sigma^{m-2})$ and $m-2 \leq d-2$, Lemma 3.4.6 implies that $n \cdot \mathring{\mathrm{I\!I}}^{\mathrm{e}} = \mathcal{O}(\sigma^{m-1})$ and hence, for any integer $1 \leq \ell \leq m-2$, we have that $n \cdot \nabla_n^{\ell} \mathring{\mathrm{I\!I}}^{\mathrm{e}} \stackrel{\Sigma}{=} 0$. Thus, the above expression is zero along Σ, completing the induction. Therefore, for $2 \leq m < \frac{d+3}{2}$, we have that $\mathring{\mathrm{I\!I}} = \ldots = \mathring{\underline{\mathrm{m}}} = 0$ implies that $\mathring{\mathrm{I\!I}}^{\mathrm{e}} = \mathcal{O}(\sigma^{m-1})$. □

We now define a family of conformal hypersurface embeddings that are useful in defining higher fundamental forms.

Definition 4.2.6. We say that a conformal hypersurface embedding $\Sigma \hookrightarrow (M^d, \boldsymbol{c})$ is *hyperumbilic* if, for each and every $n \in \{2, \ldots, \lceil \frac{d+1}{2} \rceil\}$, an nth fundamental form vanishes. ■

Remark 4.2.7. Recall that an embedding is called umbilic when $\mathring{\mathrm{I\!I}} = 0$. The name hyperumbilic was chosen to evoke a stricter condition than mere umbilicity. Furthermore, in Definition 4.2.6, the article "an" was used specifically to suggest that there is no canonical set of fundamental forms that must vanish for such a hypersurface embedding to be hyperumbilic. Because of the uniqueness result of Lemma 4.0.2, the indefinite article is sufficient. Going forward, we can assume that hyperumbilicity implies vanishing of the above fundamental forms and vice versa. ■

We can now define a set of higher fundamental forms that are only invariant so long as the embedding is hyperumbilic. Such tensors can only conditionally be called fundamental forms. Thus, we call a transverse order $n-1$ hypersurface invariant a *conditional fundamental form* if and only if it is an nth conformal fundamental form on embeddings for which some lower transverse order fundamental form vanishes. We now provide one possible construction of such conditional fundamental forms.

Definition 4.2.8. Let $d \geq 5$ and let $(M, \gamma, \sigma)_{\mathcal{Y}}$ specify a conformal hypersurface embedding. Further, let $\tau \in \Gamma(\mathcal{E}M[1])$ be any true scale. Then, for $\frac{d+3}{2} \leq n \leq d-1$, define the *nth conditional fundamental form* by

$$\mathring{\underline{\bar{\mathrm{n}}}} := \bar{q}^* \circ \bar{r} \circ \mathring{\top} \circ \left(I \cdot D^{n-2}(P \log \tau) - \log \tau I \cdot D^{n-2} P\right) \in \Gamma(\odot^2_\circ T^*\Sigma[3-n])\,.$$

■

Remark 4.2.9. It follows that the expression $I \cdot D^{n-2} P \log \tau - \log \tau I \cdot D^{n-2} P$ in the above definition is a tractor by repeated application of Lemma 2.3.3. ■

We must now verify that the conditional fundamental forms defined above are indeed fundamental forms so long as the embedding is hyperumbilic.

Proposition 4.2.10. *Let $\Sigma \hookrightarrow (M^d, \boldsymbol{c})$ with $d \geq 5$ be a hyperumbilic conformal embedding and let $k := \lceil \frac{d+1}{2} \rceil$. Then, for all $k+1 \leq n \leq d-1$, the nth conditional fundamental form is a fundamental form.*

Proof. We begin by showing that the transverse order of the nth canonical conditional fundamental form is $n-1$. Recycling the computation in the proof of Lemma 4.1.7, but promoting the weight w to an operator (as necessary to act on log densities), we find that

$$I \cdot D^{n-2} \stackrel{\Sigma}{=} \nabla_n^{n-2} \circ \left[\prod_{i=1}^{n-2} (d + 2\underline{w} - n - i + 2)\right] + \mathrm{ltots}_{n-3}\,,$$

and in turn

$$I\cdot D^{n-2}\circ P_{AB}\stackrel{\Sigma}{=}\nabla_n^{n-2}\circ P_{AB}\circ\left[\prod_{i=1}^{n-2}(d+2\underline{w}-n-i)\right]+\mathrm{ltots}_{n-3}(P_{AB})\,.$$

Because $k+1\leq n\leq d-1$, we have that $d-n-(n-2)=d-2n+2\leq 0$ and $d-n-1\geq 0$. Therefore the product above has $\underline{w}$ as one of its factors, so we have that

$$\prod_{i=1}^{n-2}(d+2\underline{w}-n-i)=\alpha\underline{w}+\mathcal{O}(\underline{w}^2)\,,$$

for some $\alpha\neq 0$. So, remembering that $\mathring{\top}$ includes restriction to Σ and $\underline{w}\log\tau=1$ while $\underline{w}^2\log\tau=0$, we can write

$$\mathring{\top}\circ I\cdot D^{n-2}(P_{AB}\log\tau)=\alpha\mathring{\top}(\nabla_n^{n-2}P_{AB})+\mathrm{ltots}_{n-3}(P_{AB}\log\tau)\,.$$

Further, again using $\underline{w}(1)=0$, we have that $\log\tau\, I\cdot D^{n-2}P_{AB}=\log\tau\,\mathrm{ltots}_{d-3}(P_{AB})$.

We now need to verify that $(\bar{q}^*\circ\bar{r}\circ\mathring{\top})(\nabla_n^{n-2}P_{AB})$ has transverse order $n-1$. To check this, suppose that $\mathring{\mathrm{II}}^{\mathrm{e}}=\mathcal{O}(\sigma^{n-2})$. Then, because $n-2\leq d-2$, from Lemma 3.4.6 and Equation (3.23), we have that $\nabla\cdot\mathring{\mathrm{II}}^{\mathrm{e}}=\mathcal{O}(\sigma^{n-2})$. However, observe heuristically from Equation (2.3) that the action of the tractor connection on a lower slot either moves the slot "up" or applies a derivative. Indeed, for $\mathring{\mathrm{II}}^{\mathrm{e}}=\mathcal{O}(\sigma^{n-2})$, one can verify by direct computation and Lemma 3.4.6 that the projecting part of $\nabla_n^{n-2}P_{AB}$ is indeed $\nabla_n^{n-2}\mathring{\mathrm{II}}^{\mathrm{e}}$. Therefore, for arbitrary $\mathring{\mathrm{II}}^{\mathrm{e}}$, all of the terms correcting $(\bar{q}^*\circ\bar{r}\circ\mathring{\top})(\nabla_n^{n-2}P_{AB})$ must be of lower transverse order than $\nabla_n^{n-2}\mathring{\mathrm{II}}^{\mathrm{e}}$, *i.e.*

$$(\bar{q}^*\circ\bar{r}\circ\mathring{\top})(\nabla_n^{n-2}P_{AB})=\alpha\mathring{\top}(\nabla_n^{n-2}\mathring{\mathrm{II}}^{\mathrm{e}}_{ab})+\mathrm{ltots}_{n-3}(\mathring{\mathrm{II}}^{\mathrm{e}})\,,$$

for some non-zero α. From Equation (4.4), we have that (in the typical choice of coordinates and scale)

$$\mathring{\top}(\nabla_n^{n-2}\mathring{\mathrm{II}}^{\mathrm{e}})=\tfrac{d-n}{2(d-2)}\mathring{\top}(\partial_s^{n-1}g_{ab})+\mathrm{ltots}_{n-2}(g_{ab})\,,$$

and hence has transverse order $n-1$ for $n\leq d-1$.

Finally we need to show that the nth conditional fundamental form is independent of τ. Suppose that Σ is embedded hyperumbilically and let $\ell:=n-k-1$. Note that $0\leq\ell\leq d-k-2$. Because $(M,\gamma,\sigma)_{\mathcal{Y}}$ is hyperumbilic, we have that $\mathring{\mathrm{II}}=\ldots=\mathring{\underline{\mathrm{k}}}=0$, so Lemma 4.2.5 implies that

$\mathring{\mathbb{I}}^{\mathrm{e}} = \mathcal{O}(\sigma^{k-1})$. Then, from Lemma 3.4.8, we have that

$$\underline{\mathring{\mathrm{n}}} := (\bar{q}^* \circ \bar{r} \circ \mathring{\top})(\Pi_n)\,,$$

where

$$\Pi_n := I\cdot D^{\ell+(k-1)}\big([\sigma^{k-1}Q_{AB} + \sigma^{k-2}X_AX_BU]\log\tau\big) \\ - \log\tau\, I\cdot D^{\ell+(k-1)}\big(\sigma^{k-1}Q_{AB} + \sigma^{k-2}X_AX_BU\big)\,,$$

for some $Q \in \Gamma(\odot^2_\circ \mathcal{T}^*M[-k]) \cap \ker \iota_X$ and $U \in \Gamma(\mathcal{E}M[-k+1])$. We will show that this tractor is independent of τ.

Employing a quadratic Casimir of the $\mathfrak{sl}(2)$ algebra,

$$4yx + 2\mathsf{h} + \mathsf{h}^2 = 4xy - 2\mathsf{h} + \mathsf{h}^2 \stackrel{\Sigma}{=} \mathsf{h}(\mathsf{h}-2)\,,$$

we find the enveloping algebra recursion relation

$$y^{\ell+m+1}x^{m+1} \stackrel{\Sigma}{=} -y^{\ell+m}x^m(\ell+m+1)(\mathsf{h}+m-\ell)\,,$$

which can be solved to yield (for any non-negative integer m)

$$y^{\ell+m}x^m \stackrel{\Sigma}{=} (-1)^m y^\ell \prod_{i=1}^{m}(\ell+i)(\mathsf{h}-\ell+i-1)\,. \tag{4.7}$$

Note that when $m = 0$ in the above display, our convention is to define the product to be 1. In the $\mathfrak{sl}(2)$ notations of Proposition 3.4.3 we now have

$$\Pi_n = -(-1)^n \log\tau\Big[y^{\ell+(k-1)}x^{k-1}Q_{AB} + y^{\ell+1+(k-2)}x^{k-2}(X_AX_BU)\Big] \\ +(-1)^n\Big[y^{\ell+(k-1)}x^{k-1}(Q_{AB}\log\tau) + y^{\ell+1+(k-2)}x^{k-2}(X_AX_BU\log\tau)\Big]\,.$$

We define the polynomials $F_{\ell,w,i}(u) := (\ell+i)(u+2w-\ell+i-1)$ which obey $F_{\ell+1,w+1,i}(u) = F_{\ell,w,i+1}(u)$. Then, using Equation (4.7) and the fact that Q and X^2U have weights $-k$ and $1-k$, respectively, we have that:

$$y^{\ell+(k-1)}x^{k-1}\circ Q_{AB}\overset{\Sigma}{=}(-1)^{k-1}y^{\ell}\circ Q_{AB}\circ\prod_{i=1}^{k-1}F_{\ell,-k,i}(\mathsf{h})\,,$$

$$\begin{aligned}y^{\ell+1+(k-2)}x^{k-2}\circ X_AX_BU &\overset{\Sigma}{=}(-1)^{k-2}y^{\ell+1}\circ X_AX_BU\circ\prod_{i=1}^{k-2}F_{\ell+1,1-k,i}(\mathsf{h})\\ &\overset{\Sigma}{=}(-1)^{k-2}y^{\ell+1}\circ X_AX_BU\circ\prod_{i=2}^{k-1}F_{\ell,-k,i}(\mathsf{h})\,.\end{aligned}$$

Defining the polynomial

$$f_j(u):=\prod_{i=j}^{k-1}(\ell+i)(u-2k-\ell+i-1)=\prod_{i=j}^{k-1}F_{\ell,-k,i}(u)\,,$$

and remembering that $\mathsf{h}(1)=d$, we may rewrite Π_n:

$$\begin{aligned}\Pi_n\overset{\Sigma}{=}&-(-1)^{n+k-1}\log\tau\Big\{f_1(d)y^{\ell}Q_{AB}-f_2(d)y^{\ell+1}(X_AX_BU)\Big\}\\ &+(-1)^{n+k-1}\Big\{y^{\ell}\big[Q_{AB}f_1(\mathsf{h})(\log\tau)\big]-y^{\ell+1}\big[X_AX_BUf_2(\mathsf{h})(\log\tau)\big]\Big\}\,.\end{aligned}\tag{4.8}$$

Note that $f_1(u)=F_{\ell,-k,1}(u)f_2(u)$. Now, for any polynomial f for which d is a root, the operator $f(\mathsf{h})=f(d+2\underline{w})$ obeys

$$f(\mathsf{h})=2f'(d)\underline{w}+\mathcal{O}(\underline{w}^2)\,,$$

and so

$$f(\mathsf{h})\log\tau=2f'(d)\,.$$

Thus, if $f_2(d)=0$ then $f_1(d)=0$ and moreover $f_1(\mathsf{h})\log\tau$ and $f_2(\mathsf{h})\log\tau$ are independent of τ and hence Π_n would be independent of τ.

To establish τ independence, we study which values of n force $f_2(d)=0$. By examining the product formula for f_2, demanding that $f_2(d)=0$ is equivalent to the inequality

$$d-2k-\ell+1\le 0\le d-k-\ell-2\,.$$

Rearranging this inequality and using that $\ell:=n-k-1$, this suggests that

$$d-k+2\le n\le d-1\,.$$

Now suppose that d is even. In that case, $d = 2k-2$, so that the above inequality reads $k \leq n \leq d-1$. Thus we have that for $k+1 \leq n \leq d-1$ and d is even, we have that Π_n is independent of τ. Now consider the case where d is odd. In that case, $d = 2k-1$, so the above inequality reads $k+1 \leq n \leq d-1$, so again for $k+1 \leq n \leq d-1$ we have τ independence. Thus we have that for all n satisfying $k+1 \leq n \leq d-1$, the nth conditional fundamental form $\mathring{\underline{\mathrm{n}}}$ is independent of τ. This completes the proof. $\square$

Example 4.2.11. We can use Proposition 4.2.10 to compute $\mathring{\mathrm{IV}}$ in $d = 5$ for hyperumbilic embeddings (so $\mathring{\mathrm{II}} = \mathring{\mathrm{III}} = 0$) and find

$$\mathring{\mathrm{IV}}_{ab} \stackrel{\Sigma_{\mathrm{hyp}}}{=} 2C^{\top}_{\hat{n}(ab)} \,.$$

This computation relies on the expression for $I \cdot D^2$ in terms of the tractor connection and weight operators, which can be found in [**44**]. ■

With this construction given, we can now define another generalization of umbilicity.

Definition 4.2.12. We say that a conformal hypersurface embedding $\Sigma \hookrightarrow (M^d, \gamma)$ is *m-umbilic* if the embedding has that, for each $n \in \{2, \ldots, m\}$, an n (possibly conditional) fundamental form vanishes. ■

Remark 4.2.13. Given the above definition, a hyperumbilic conformal hypersurface embedding is $\lceil \frac{d+1}{2} \rceil$-umbilic. Furthermore, observe that if $m > \frac{d+1}{2}$, the conditional fundamental forms become fundamental forms which can invariantly vanish. ■

Now that the conditional fundamental forms have been defined, we can slightly generalize Lemma 4.2.5.

Lemma 4.2.14. *Let $(M^d, \gamma, \sigma)_{\mathcal{Y}}$ with $d \geq 3$ specify an m-umbilic conformal hypersurface embedding for some integer $2 \leq m \leq d-2$. Then, $\mathring{\mathrm{II}}^{\mathrm{e}} = \mathcal{O}(\sigma^{m-1})$. When $m = d-1$, we have that $\mathring{\mathrm{II}}^{\mathrm{e}} = \mathcal{O}(\sigma^{d-1})$.*

Proof. First, observe that if $m \geq \frac{d+3}{2}$, these fundamental forms are still sensibly defined because the hypersurface embedding is hyperumbilic. Then, proving this result is nearly identical to that for Lemma 4.2.5. When $m = d-1$, we can easily show that $\mathring{\mathrm{II}}^{\mathrm{e}} = \mathcal{O}(\sigma^{d-2})$ as in the case for $m < d-1$. Thus, we can write that $\mathring{\mathrm{I\!I}}^{\mathrm{e}} = \sigma^{d-2} T$ for some tensor-valued density T. However from Equation (4.4), we have that $\nabla_n^{d-2} \mathring{\mathrm{I\!I}}^{\mathrm{e}} \stackrel{\Sigma}{=} \operatorname{ltots}_{d-3}(\mathring{\mathrm{I\!I}}^{\mathrm{e}})$, and thus we have that $\nabla_n^{d-2} \mathring{\mathrm{I\!I}}^{\mathrm{e}} \stackrel{\Sigma}{=} \operatorname{ltots}_{d-3}(\sigma^{d-2} T) \stackrel{\Sigma}{=} 0$. Therefore, we have that $\mathring{\mathrm{I\!I}}^{\mathrm{e}} = \mathcal{O}(\sigma^{d-1})$, proving the generalization. □

4.3. Conformal Fundamental Forms—Even Dimensions

As mentioned at the end of Subsection 4.1.2, there are cases where the iterative operator construction fails to produce operators of the appropriate transverse order but the non-holographic construction can still produce such operators. Indeed, we find that the family of operators $\delta_{J,k}$ can produce fundamental forms in the even-dimensional case where the iterative approach could not. Thus we define another set of fundamental forms, defined using the non-holographic construction.

DEFINITION 4.3.1. Let d be even and m be an integer satisfying $3 \leq m \leq d-1$. Then the fundamental form $\underline{\mathring{\mathrm{m}}}'$ is defined by

$$\underline{\mathring{\mathrm{m}}}'_{ab} := \begin{cases} \big(\bar{q}^* \circ \bar{r} \circ \mathring{\top} \circ \delta_{m-2} \circ q\big)(\mathring{\mathrm{I\!I}}^{\mathrm{e}}) & m \leq \frac{d+2}{2} \\ \\ \big(\bar{q}^* \circ \bar{r} \circ \mathring{\top} \circ \delta_{m - \frac{d+2}{2}, \frac{d-2}{2}} \circ q\big)(\mathring{\mathrm{I\!I}}^{\mathrm{e}}) & m > \frac{d+2}{2}\,. \end{cases}$$

■

We need to check that this definition is correct, given that an mth fundamental form is defined to be a symmetric trace-free rank-2 tensor with weight $3-m$ and transverse order $m-1$.

PROPOSITION 4.3.2. *Let d be even. Then for each $m \in \mathbb{Z}_{\geq 2}$ with $m \leq d-1$, we have that $\underline{\mathring{\mathrm{m}}}'$ is a conformal fundamental form.*

PROOF. First we must check that $\underline{\mathring{\mathrm{m}}}'$ is well-defined. Note that the operator $\bar{r}$ is only ill-defined when $w \in \{0, -1, -\frac{d-1}{2}, -1-\frac{d-1}{2}, -2-\frac{d-1}{2}\}$. Because $\frac{d-1}{2}$ is not an integer and the weights of $\delta_k(P_{AB})$ and $\delta_{J,k}(P_{AB})$ are integers (because the weight of P_{AB} is -1), it is clear that application of $\bar{r}$ is permitted in Definition 4.3.1.

We next check that $\underline{\mathring{\mathrm{m}}}'$ has weight $3-m$ as required. Observe that, because $\operatorname{Im} \bar{r} \subset \ker \iota_X$, for generic $T \in \Gamma(\odot^2_\circ \mathcal{T}\Sigma[w])$, we have that $(\bar{q}^* \circ \bar{r})(T_{AB}) \in \Gamma(\odot^2_\circ T^*\Sigma[w+2])$. So, by construction, $\underline{\mathring{\mathrm{m}}}$ has weight $3-m$.

Finally, we must verify that $\underline{\mathring{\mathrm{m}}}'$ has transverse order $m-1$. Because the weight of P_{AB} is -1, for $m \leq \frac{d+2}{2}$, we can easily check (for even d) that δ_{m-2} has transverse order $m-2$ as per Lemma 4.1.8. Similarly, we can verify from Theorem 4.1.9 that the transverse order of $\delta_{m-\frac{d+2}{2},\frac{d-2}{2}}$ is $m-2$. Finally, to verify that $\bar{q}^* \circ \bar{r} \circ \mathring{\top}$ does not change the transverse order, we can apply the same argument found in the proof of Proposition 4.2.10. This completes the proof. □

The existence of these fundamental forms in even dimensions in particularly useful for describing conformal hypersurface invariants in more detail than otherwise possible in odd dimensions. An example of such a computation is carried out in Section 5.2. Furthermore, that there exists a difference in constructability of fundamental forms on even- and odd-dimensional conformal manifolds suggests a deeper relationship that is yet not fully understood.

4.4. Fundamental Forms Beyond Transverse Order $d-1$

Observe that beyond the $d-1$th fundamental form, some of the transverse derivative operators have the desired transverse order again. Indeed, for $k \geq d-2$, the operator δ_k described in Lemma 4.1.8 has transverse order k when acting on tractors with weight -1. Furthermore, when d is even and $J+k \geq d-2$, we have that if $k = \frac{d-2}{2}$, then the operator $\delta_{J,k}$ also has transverse order $d-2$. However, even though such operators exist, a consequence of Equation (4.4) suggests that a dth fundamental form may still not exist. One might note that this is a consequence of the method (differentiating an extension of $\mathring{\mathrm{II}}$) that was chosen, so perhaps it may be possible to construct fundamental forms by acting on another conformally-invariant tensor.

One such alternative construction uses a conformally-invariant extension of $\mathring{\mathrm{III}}$ given by

$$\mathring{\mathrm{III}}^{\mathrm{e}}_{ab} := W_{nabn} + 2\sigma C_{n(ab)} - \tfrac{\sigma^2}{d-4} B_{ab}\,.$$

Again, acting on $q(\mathring{\mathrm{III}}^{\mathrm{e}}) \in \Gamma(\odot^2_\circ \mathcal{T}^* M[-2])$, we see that δ_{d-3} again has transverse order $d-3$. But, in a choice of scale $|ds|^2 = 1$ and a choice of coordinates $(s, y^1, \ldots, y^{d-1})$, we have that

$$\mathring{\top} \circ \nabla_n^m \mathring{\mathrm{III}}^{\mathrm{e}}_{ab} = \tfrac{(d-m-3)(d-m-4)}{2(d-2)(d-4)} \partial_s^{m+2}(g_{ab}) + \mathrm{ltots}_{m+1}(g_{ab})\,, \tag{4.9}$$

so that for $d \geq 6$,

$$\nabla_n^{d-3} \mathring{\mathrm{III}}^{\mathrm{e}}_{ab} = \mathrm{ltots}_{d-2}(g_{ab})\,.$$

However, in $d=4$, the fourth fundamental form $\mathring{\rm IV}$ is constructible using Equation (4.9) because the leading derivative has a removable singularity at $d=4$, and thus we can define

$$\mathring{\rm IV} := (\bar{q}^* \circ \bar{r} \circ \mathring{\top} \circ \delta_R \circ q)(\mathring{\rm III}^{\rm e})\,.$$

This suggests that there may be no way to produce a dth fundamental form except when $d=4$. At present we have no evidence for or against the existence of dth fundamental forms and leave this as an open question.

Nonetheless, we can define mth fundamental forms with $m \geq d+1$:

Definition 4.4.1. Let $(M^d, \gamma, \sigma)_{\mathcal{Y}}$ specify a conformally embedded hypersurface with $d \geq 4$. Then, for every $n \geq d+1$, define an nth fundamental form by

$$\underline{\mathring{\rm n}} := (\bar{q}^* \circ \bar{r} \circ \mathring{\top} \circ \delta_{n-2} \circ q)(\mathring{\rm II}^{\rm e})\,.$$

■

Verifying that the tensors defined above are indeed fundamental forms requires more machinery than developed so far. Thus, we hold off on proving that result until Section 5.2.

Example 4.4.2. As an example, consider a fifth fundamental form in $d=4$ dimensions,

$$\mathring{\rm V} := B^{\top}_{(ab)\circ}\,.$$

Clearly $\mathring{\rm V}$ is conformally invariant because the Bach tensor is conformally invariant in four dimensions. A simple calculation shows that $\mathring{\rm V}$ has leading derivative term $\alpha \mathring{\top} \partial_s^4 (g_{ab})$ in the typical coordinate system with $\alpha \neq 0$. While this is clearly not proof, it is evidence for the hypothesis that $\underline{\overline{\mathring{\rm d+1}}}$ can always be written as a trace-free hypersurface projection of the Fefferman-Graham obstruction tensor. ■

4.5. Tractor Fundamental Forms

Just as the tractor second fundamental form and the Fialkow tractor were defined in Section 3.3, we can insert other fundamental forms into tractors to find manifestly conformally invariant relationships between more hypersurface tractors. To that end, in this section we provide various identities relating tractor fundamental forms and bulk tractors. These identities will be useful later when utilizing holographic methods to compute more complicated conformal hypersurface invariants, such as those implemented in Chapter 6.

A canonical tractor nth fundamental form is given by $\bar{q}(\mathring{\underline{\mathrm{n}}})$ in dimensions $d > n+1$. The tractor second fundamental form defined in Section 3.3 is given by $L := \bar{q}(\mathring{\mathrm{II}})$ for $d > 3$, and its holographic formula was given in Equation (3.25). A holographic formula for the canonical tractor third fundamental form, valid when $7 \neq d > 5$, is

$$\bar{q}(\mathring{\mathrm{III}}) = \dot{P}^t_{AB} - \tfrac{2}{(d-3)(d-5)} X_{(A} \bar{D} \cdot \dot{P}^t_{B)} + \tfrac{1}{(d-3)(d-4)(d-5)} X_A X_B \bar{D} \cdot \hat{\bar{D}} \cdot \dot{P}^t \,. \tag{4.10}$$

Here $\dot{P}_{AB} := I \cdot \hat{D} P_{AB}$ and $\dot{P}^t_{AB} := \bar{r} \circ \mathring{\top}(\dot{P}_{AB})$; the above result is a direct application of Lemma 2.5.3 and the definition of $\mathring{\mathrm{III}}$ in Definition 4.2.1. Just as the Fialkow tensor is related to the canonical third fundamental form by a factor $-(d-3)$, see Equation (4.6), the Fialkow *tractor* and the canonical tractor third fundamental form obey

$$\bar{q}(\mathring{\mathrm{III}})_{AB} = -(d-3) F_{AB} \,.$$

The above relationship between F and $\bar{q}(\mathring{\mathrm{III}})$ and Equation (4.10) yield a corollary to Corollary 3.3.8 that gives an analog of the Fialkow–Gauß equation (3.7):

Corollary 4.5.1 (Fialkow–Gauß–Thomas Equation). *Let $7 \neq d > 5$ and σ be an asymptotic unit conformal defining density for $\Sigma \hookrightarrow (M, \gamma)$. Then,*

$$\begin{aligned}\Big(L_A^C L_{CB} - \tfrac{1}{d-1} K \bar{h}_{AB} - W_{NABN}\Big) - \tfrac{1}{d-1} X_{(A} \hat{\bar{D}}_{B)} K - \tfrac{1}{(d-4)(d-5)} X_A X_B U \\ = -\dot{P}^t_{AB} + \tfrac{2}{(d-3)(d-5)} X_{(A} \bar{D} \cdot \dot{P}^t_{B)} - \tfrac{1}{(d-3)(d-4)(d-5)} X_A X_B \bar{D} \cdot \hat{\bar{D}} \cdot \dot{P}^t = (d-3) F_{AB} \,,\end{aligned}$$

where $U \in \Gamma(\mathcal{E}\Sigma[-4])$ is the density given in Equation (3.13).

For later use, in dimensions $d \geq 6$, we define $J := \bar{q}(\mathring{\mathrm{IV}}) \in \Gamma(\odot^2_\circ \mathcal{T}\Sigma[-3])$.

As mentioned in Section 3.4, the canonical extension of K is given by $K_{\mathrm{e}} := P_{AB} P^{AB} = (\mathring{\mathrm{II}}^{\mathrm{e}})^2$. When computing extrinsic hypersurface embedding curvatures, there are many situations in which normal derivatives of K_{e} are needed. Note that often, we will use K and K_{e} interchangeably when it is clear that we are not referring strictly to an identity that holds along Σ.

DEFINITION 4.5.2. We define the transverse derivatives of $K_{\rm e}$ according to the following:

$$\dot{K} := \delta_R K^{\rm e}\,, \quad \ddot{K} := \delta_R I\cdot \hat{D} K^{\rm e}\,,\ d \neq 6\,, \quad \dddot{K} := \delta_R I\cdot \hat{D}^2 K^{\rm e}\,,\ d\neq 6,8\,, \quad \cdots\ ,$$

where $\dot{K}\in\Gamma(\mathcal{E}\Sigma[-3])$, $\ddot{K}\in\Gamma(\mathcal{E}\Sigma[-4]$, $\dddot{K}\in\Gamma(\mathcal{E}\Sigma[-5])$, *etc.*

■

In particular, because $P_{AB}P^{AB} = (\mathring{\rm I\!I}^{\rm e})^2$, there are (rather useful) formulæ for $\dot{K}, \ddot{K}$, and $\dddot{K}$ in terms of fundamental forms and hypersurface derivatives thereof. For this we introduce the following notational device.

DEFINITION 4.5.3. Let

$$\Pi_{(2)} := \mathring{\top}\circ q(\mathring{\rm I\!I}^{\rm e}) \in \Gamma(\odot^2_\circ \mathcal{T}\Sigma[-1])\,,$$

and, for $3\leq m<d$ such that $m\notin\{\frac{d+1}{2},\frac{d+3}{2},\frac{d+5}{2}\}$, let

$$\Pi_{(m)} := (\bar{r}\circ\mathring{\top}\circ\delta_{\rm R}\circ q)\circ {\rm I\!D}_\sigma^{m-3}(\mathring{\rm I\!I}^{\rm e}) \in \Gamma(\odot^2_\circ\mathcal{T}\Sigma[1-m])\,.$$

When $3\leq m<d$ we define

$$\tilde{\Pi}_{(m)} := \bar{q}(\mathring{\underline{\overline{m}}})\,.$$

■

REMARK 4.5.4. The values $\{\frac{d+1}{2},\frac{d+3}{2},\frac{d+5}{2}\}$ are treated on a separate footing in Definition 4.5.3 for reasons of definedness of the operator $\bar{r}$; see Lemma 2.5.5. Also, in dimensions d such that $m\notin\{\frac{d+1}{2},\frac{d+3}{2},\frac{d+5}{2}\}$, by construction we have that $\tilde{\Pi}_{(m)}=\Pi_{(m)}+\mathcal{E}(X)$, since $\tilde{\Pi}_{(m)}=(\bar{q}\circ\bar{q}^*)(\Pi_{(m)})$ and Lemma 2.5.3 says $\bar{q}\circ\bar{q}^*={\rm Id}+\mathcal{E}(X)$. So, when $m\in\{\frac{d+1}{2},\frac{d+3}{2},\frac{d+5}{2}\}$, if $T\in\Gamma(\odot^2\mathcal{T}M[w])\cap\ker\iota_X$, we may define $\Pi^{AB}_{(m)}T_{AB} := \tilde{\Pi}^{AB}_{(m)}T_{AB}$. ■

By construction, the rank two, trace-free hypersurface tractors $\Pi_{(m)}$ produce the corresponding fundamental form $\mathring{\underline{\overline{m}}}$ upon acting by the extraction map $\bar{q}^*$. In general, if $\bar{q}^*(T^{AB})=t_{ab}$ and $\bar{q}^*(U^{AB})=u_{ab}$, we have that $t_{ab}u^{ab}=T_{AB}U^{AB}$. For this reason, the tractors $\Pi_{(m)}$ can be used to compute holographic formulæ for scalars built from contractions of fundamental forms. These formulæ are simpler than those for their constituent fundamental forms and are therefore particularly useful for computations of scalar densities, such as integrands for Willmore-like energies (see for example [**44**, **68**] and Section 6.4). We now give two such results.

LEMMA 4.5.5. *Let $d > 4$, then the square of the third fundamental form has a holographic formula given by*

$$\mathring{\mathrm{III}}^2 = \dot{P}_{AB}\dot{P}^{AB}\big|_\Sigma - \tfrac{3d-2}{(d-1)(d-2)^2}K^2\,.$$

PROOF. It follows from Definition 4.5.3 that $\mathring{\mathrm{III}}^2 = \Pi_{(3)\,AB}\Pi_{(3)}^{AB}$. Because the only appearance of $\Pi_{(3)}$ in this proof is when it is squared, any instances where $\tilde{\Pi}_{(3)}$ would be required can be replaced with $\Pi_{(3)}$. So, the proof amounts to relating $\dot{P}$ to $\Pi_{(3)}$. As previously noted, $q(\mathring{\mathrm{II}}^{\mathrm{e}}) = P$, so $\Pi_{(3)}^{AB} = \bar{r} \circ \mathring{\top}(\dot{P})$. In order to relate $\Pi_{(3)}$ to $\dot{P}|_\Sigma$ explicitly, following Lemma 2.5.5 specialized to hypersurface tractors, we need to rewrite $X_A\dot{P}^{AB}$:

$$\begin{aligned} X_A\dot{P}^{AB} = {}& \;I\cdot\hat{D}X_AP^{AB} - I_AP^{AB} + \tfrac{2\sigma}{d-2}\hat{D}_AP^{AB} \\ ={}& -I_A\hat{D}_BI^A \\ ={}& -\big(\tfrac{1}{2}\hat{D}_BI^2 + \tfrac{X_B}{d-2}(\hat{D}I)^2\big) \\ ={}& -\frac{K_{\mathrm{e}}X_B}{d-2} + \mathcal{O}(\sigma^{d-2})\,, \end{aligned} \tag{4.11}$$

where the first and third lines are results of the Leibniz failure (Proposition 2.3.1), the second line results from the properties of P, and the last line uses the definition of K_{e}. Further, observe that via the Leibniz failure, we have

(4.12)
$$I\cdot\dot{P}^B = \tfrac{1}{d-2}\big(\dot{K}X^B + KI^B\big) - \tfrac{2\sigma}{d-4}\big(\tfrac{1}{d-2}\hat{D}^BK - P^{AC}\hat{D}_CP_A^B\big) + \mathcal{O}(\sigma^{d-3}) \overset{\Sigma}{=} \tfrac{1}{d-2}(\dot{K}X^B + KN^B)\,.$$

Using the above identities, the definition of $\bar{r}$ and $\mathring{\top}$, as well as the standard operator identity for $\hat{\bar{D}} \circ X$, a tedious calculation along Σ yields

$$\Pi_{(3)\,AB} \overset{\Sigma}{=} \dot{P}_{AB} - \tfrac{d}{(d-1)(d-2)}X_{(A}\hat{\bar{D}}_{B)}K - \tfrac{2}{d-2}\dot{K}I_{(A}X_{B)} - \tfrac{1}{d-2}KI_AI_B - \tfrac{1}{(d-1)(d-2)}KI_{AB}\,.$$

Squaring this identity gives the quoted result. $\square$

LEMMA 4.5.6. *Let $d > 6$, then the product of the second and fourth fundamental forms has a holographic formula given by*

$$\mathring{\mathrm{II}} \cdot \mathring{\mathrm{IV}} \stackrel{\Sigma}{=} (d-4)\Big(P_{AB}\ddot{P}^{AB} + \tfrac{4}{(d-2)^2}K^2\Big)\,.$$

PROOF. First, because $d > 5$, we see that $\mathring{\mathrm{IV}}$ is a canonical fundamental form (so not conditional). As in the previous lemma, we note that $\mathring{\mathrm{II}} \cdot \mathring{\mathrm{IV}} = \Pi_{(2)\,AB}\Pi_{(4)}^{AB}$. Moreover, Equation (3.17) implies that $N^A P_{AB} \stackrel{\Sigma}{=} \frac{1}{d-2}KX_B$ so

$$\Pi_{(2)} = P|_\Sigma - \tfrac{2K}{d-2} N \odot X\,.$$

Thus we are tasked with computing $P_{AB}\Pi_{(4)}^{AB}$ along Σ. Remembering that $X \cdot P = 0$, it is sufficient to compute $\Pi_{(4)}^{AB}$ modulo terms proportional to X. Using Definition 4.5.3, we compute $\Pi_{(4)}$ in steps. Recall, from Equation (4.1), that

$$\Pi_{(4)} = (d-4)\,(\bar{r} \circ \mathring{\top} \circ \delta_R \circ q \circ q^* \circ r \circ I \cdot \hat{D} \circ q)(\mathring{\mathrm{II}}^{\mathrm{e}})\,.$$

We outline this calculation proceeding from right to left in this sequence of operators.

First, as shown previously, $q(\mathring{\mathrm{II}}^{\mathrm{e}}) = P$, so $I \cdot \hat{D} q(\mathring{\mathrm{II}}^{\mathrm{e}}) = \dot{P}$. Therefore, using Lemma 2.5.5, we next compute $r(\dot{P})$:

$$r(\dot{P})_{AB} = \dot{P}_{AB} - \tfrac{d+2}{4d(d-2)} X_{(A}\hat{D}_{B)}K - \tfrac{2}{d(d-2)} h_{AB}K + \mathcal{O}(\sigma^{d-2}) + \mathcal{O}(\sigma^{d-4})X_{(A}T_{B)}\,,$$

for some tractor T_B. Here we used Equation (4.11) and the oft-used operator identity for $\hat{D} \circ X$ given in Equation (2.12).

Next, we need to compute $(q \circ q^* \circ r)(\dot{P})$. Before we continue, we consider the operators that come next: We are only interested in the $\bar{Z}_A\bar{Z}_B$ component of the tractor $\Pi_{(4)}$, so we can ignore terms proportional to X in $(\mathring{\top} \circ \delta_R \circ q \circ q^* \circ r)(\dot{P})$ when doing this computation. Further, we can ignore terms proportional to I_A when computing $(\delta_R \circ q \circ q^* \circ r)(\dot{P})$ because these terms are projected out by $\mathring{\top}$. The various projections, therefore, amount to ignoring terms proportional to X or I when computing $(q \circ q^* \circ r)(\dot{P})$, because $\delta_R \circ I \stackrel{\Sigma}{=} \mathcal{E}(X) + \mathcal{E}(I)$ and $\delta_R \circ X \stackrel{\Sigma}{=} \mathcal{E}(X) + \mathcal{E}(I)$. From

Lemma 2.5.3, $(q \circ q^* \circ r)(\dot{P}) - r(\dot{P}) = \mathcal{E}(X)$, so

$$(q \circ q^* \circ r)(\dot{P}) = \dot{P} - \tfrac{2}{d(d-2)} Kh + \mathcal{E}(X) + \mathcal{E}(I) + \mathcal{O}(\sigma^{d-2})\,.$$

Next, note that $(\delta_R \circ q \circ q^* \circ r)(\dot{P}) = \ddot{P} - \frac{2}{d(d-2)} \dot{K} h + \mathcal{E}(X) + \mathcal{E}(I) + \mathcal{O}(\sigma^{d-3})$, and we can apply the operator $\mathring{\top}$ to obtain

$$(\mathring{\top} \circ \delta_R \circ q \circ q^* \circ r)(\dot{P}) = \mathring{\top}(\ddot{P}) + \mathcal{E}(X)\,.$$

Next, it is useful to note that $\mathring{\top}(\ddot{P}_{AB}) \stackrel{\Sigma}{=} I_A^{A'} I_B^{B'} \ddot{P}_{A'B'} + I_{AB} U$ for some $U \in \Gamma(\mathcal{E}\Sigma[-3])$, and also that $I_{AB} P^{AB} \stackrel{\Sigma}{=} 0$. Thus, finishing the calculation amounts to computing

$$P^{AB} \bar{r}(I_A^{A'} I_B^{B'} \ddot{P}_{A'B'} + I_{AB} U)\,. \tag{4.13}$$

For this, we need the identity

$$X^A \ddot{P}_{AB} \stackrel{\Sigma}{=} -\tfrac{2}{d-2}\big(\dot{K} X_B + K N_B\big)\,,$$

which is derived from the Leibniz failure, Equation (2.12), and Equation (4.12). Because we are contracting on P, any terms proportional to X or the tractor first fundamental form produced by $\bar{r}$ in Equation (**??**)display-before-last can be discarded. Hence, again consulting Lemma 2.5.5, we find that

$$\begin{aligned}
\Pi_{(2)AB} \Pi_{(4)}^{AB} &\stackrel{\Sigma}{=} (d-4) P_{AB} I_{A'}^{A} I_{B'}^{B} \ddot{P}_{A'B'} \\
&\stackrel{\Sigma}{=} (d-4)\big(P_{AB} - \tfrac{2}{d-2} K N_{(A} X_{B)}\big) \ddot{P}^{AB} \\
&\stackrel{\Sigma}{=} (d-4)\big(P_{AB} \ddot{P}^{AB} + \tfrac{4}{(d-2)^2} K^2\big)\,,
\end{aligned}$$

where the first equality is a result of the identity $I_{AB} P^{AB} \stackrel{\Sigma}{=} 0$, the second equality is an application of Equation (4.11) to yield an identity for $I \cdot P$, and the last equality is a consequence of the display above expressing $X \cdot \ddot{P}$. □

One more technical lemma is necessary in order to produce formulæ for $\dot{K}$ and $\ddot{K}$ in terms of the canonical fundamental forms.

Lemma 4.5.7. *Let $d > 5$. Then,*

$$
\begin{aligned}
(\hat{D}P)^2 \stackrel{\Sigma}{=} & (\hat{\bar{D}}L)^2 + \mathring{\mathrm{III}}^2 + \tfrac{2}{(d-4)(d-5)}\,\mathring{\mathrm{II}}\cdot\mathring{\mathrm{IV}} - \tfrac{4(d-7)}{(d-3)(d-5)}\,\mathring{\mathrm{II}}\cdot\mathring{\mathrm{III}}\cdot\mathring{\mathrm{II}} + \tfrac{2(d-7)}{d-5}\,\mathring{\mathrm{II}}^4 \\
& - \tfrac{2(3d^3-34d^2+100d-73)}{(d-1)(d-2)^2(d-5)} K^2 \,.
\end{aligned}
$$

Proof. The proof is a tedious but straightforward application of Equation (3.25), Lemmas 3.3.4, 4.5.5, 4.5.6 and standard reorderings of tractor operators based on the Leibniz failure (Proposition 2.3.1). □

Employing these lemmas, we have formulæ for $\dot{K}$ and $\ddot{K}$. These formulæ will be useful for later applications of these fundamental forms.

Proposition 4.5.8. *Let $d > 4$. Then,*

$$
\dot{K} = 2\,\mathring{\mathrm{II}}\cdot\mathring{\mathrm{III}}\,. \tag{4.14}
$$

If $d > 6$,

$$
\begin{aligned}
\ddot{K} \stackrel{\Sigma}{=} & -\tfrac{2}{d-6}(\hat{\bar{D}}L)^2 + \tfrac{2(d-7)}{d-6}\mathring{\mathrm{III}}^2 + \tfrac{2(d-7)}{(d-5)(d-6)}\,\mathring{\mathrm{II}}\cdot\mathring{\mathrm{IV}} + \tfrac{8(d-7)}{(d-3)(d-5)(d-6)}\,\mathring{\mathrm{II}}\cdot\mathring{\mathrm{III}}\cdot\mathring{\mathrm{II}} \\
& - \tfrac{4(d-7)}{(d-5)(d-6)}\,\mathring{\mathrm{II}}^4 + \tfrac{10d^3-110d^2+296d-172}{(d-1)(d-2)^2(d-5)(d-6)} K^2 \,.
\end{aligned} \tag{4.15}
$$

Moreover, if $d = 5$ then

$$
\begin{aligned}
\ddot{K} \stackrel{\Sigma}{=} & -4\,\mathring{\mathrm{II}}\cdot\bar{\Delta}\,\mathring{\mathrm{II}} + \tfrac{20}{3}\,\mathring{\mathrm{II}}\cdot\bar{\nabla}\bar{\nabla}\cdot\mathring{\mathrm{II}} + \tfrac{8}{9}\big(\bar{\nabla}\cdot\mathring{\mathrm{II}}\big)^2 + \bar{\Delta}K \\
& - 4\,\mathring{\mathrm{II}}\cdot C_n^{\top} + 20\,\mathring{\mathrm{II}}^2\cdot\bar{P} + 2\bar{J}K - 4H\,\mathring{\mathrm{II}}^3 - 4H\,\mathring{\mathrm{II}}\cdot\mathring{\mathrm{III}} \\
& + 4\,\mathring{\mathrm{III}}\cdot\mathring{\mathrm{III}} - 2\,\mathring{\mathrm{II}}^2\cdot\mathring{\mathrm{III}} + \tfrac{31}{18}K^2 + 8\,\mathring{\mathrm{II}}^{ad}\,\mathring{\mathrm{II}}^{bc}\bar{W}_{abcd}\,.
\end{aligned} \tag{4.16}
$$

Proof. We first prove that $\dot{K} = 2\,\mathring{\mathrm{II}}\cdot\mathring{\mathrm{III}}$ (note that a proof was already given in [**44**]). To do so, consider the product $\mathring{\mathrm{II}}\cdot\mathring{\mathrm{III}}$. By inserting these fundamental forms into tractors, we find that $\mathring{\mathrm{II}}\cdot\mathring{\mathrm{III}} = L\cdot\bar{q}(\mathring{\mathrm{III}})$. From Equation (4.10) and the fact that $X\cdot L = 0$, we have that $\mathring{\mathrm{II}}\cdot\mathring{\mathrm{III}} = L\cdot\dot{P}^t$. Further, $L \stackrel{\Sigma}{=} P + \mathcal{E}(X) + \mathcal{E}(N)$ and $X\cdot\dot{P}^t = 0 = N\cdot\dot{P}^t$, so $\mathring{\mathrm{II}}\cdot\mathring{\mathrm{III}} \stackrel{\Sigma}{=} P\cdot\dot{P}^t$. By definition, $\dot{P}^t = \bar{r}(I_A^{A'}I_B^{B'}\dot{P}_{A'B'} + I_{AB}U)$ for some $U \in \Gamma(\mathcal{E}\Sigma[-2])$. Thus, because $X\cdot\dot{P} = \mathcal{E}(X)$ (see Equation (4.11)), using Equation (2.5.5) we have that $P \cdot \dot{P}^t \stackrel{\Sigma}{=} P \cdot \dot{P}$ and in turn $\mathring{\mathrm{II}} \cdot \mathring{\mathrm{III}} \stackrel{\Sigma}{=} P \cdot \dot{P}$. But from Proposition 2.3.1, we have that $\dot{K} \stackrel{\Sigma}{=} 2P \cdot \dot{P}$, so the first claim of the lemma follows.

To prove the second claim, for which $d > 6$, we first apply Proposition 2.3.1 twice to $P_{AB}P^{AB}$ and find that

$$\ddot{K} \stackrel{\Sigma}{=} 2\dot{P}^2 + 2P \cdot \ddot{P} - \tfrac{2}{d-6}(\hat{D}P)^2 \,.$$

Applying Lemmas 4.5.5, 4.5.6, and 4.5.7, we obtain the second claim.

Finally, we turn to the third claim with $d = 5$. Because Lemmas 4.5.6 and 4.5.7 do not hold when $d = 5$, we need to use a different method. Also if $\Sigma \hookrightarrow (M^5, \boldsymbol{c})$ is a generic conformally embedded hypersurface, the tensor $\mathring{\mathrm{IV}}$ is only a conditional fundamental form, so in particular it cannot appear in an otherwise conformally-invariant expression for $\ddot{K}$. Thus, to compute $\ddot{K}$ in this case, we resort to a Riemannian computation, and use that when $d = 5$ (see [**44**]),

$$I \cdot \hat{D}^2 K_{\mathrm{e}} \stackrel{\Sigma}{=} \Big[\Delta^{\top} - 2\bar{J} + \tfrac{1}{3}K + 2\nabla_n^2 + 4\big(2H\nabla_n - P_{nn} - \tfrac{1}{3}K + \tfrac{5}{2}H^2\big)\Big](\nabla n + sP + g\rho)^2 \,.$$

The expression for $\nabla_n^2 \rho$ along Σ may also be found in [**44**]. The remaining terms were handled by using the computer algebra system FORM [**83**]; this computation is documented in detail in Appendix A.3. $\square$

CHAPTER 5

Asymptotically Poincaré–Einstein and Willmore Structures

5.1. Asymptotic Poincaré–Einstein Structures

The notion of geometric holography introduced in Section 3.4 was described simply as a tool for calculating. However, one can also view these relationships as equivalences in physical theories: the *holographic principle* was first introduced by 't Hooft [**75**] and developed by Susskind [**77**] (building off of work from Thorn [**82**]). The most well-known example of the holographic principle was first proposed by Maldacena [**60**] in the AdS/CFT correspondence: viewed one way, this is a correspondence that identifies a conformal manifold $(\Sigma^{d-1}, \bar{\boldsymbol{c}})$ with the conformal infinity of an asymptotically hyperbolic manifold (M^d, g^o). To be precise, we say that a manifold $(M \backslash \partial M, g^o)$ is *conformally compact* when $g = s^2 g^o$ is a non-degenerate metric on $M \backslash \partial M$ that extends smoothly to ∂M for some defining function s for ∂M—then, ∂M is the the conformal infinity of $M \backslash \partial M$. Observe that for any conformally compact manifold and any function $\Omega \in C^\infty_+ M$, both $s^2 g^o$ and $(\Omega s)^2 g^o$ extend smoothly to Σ and thus induce metrics $\bar{g}$ and $\Omega^2 \bar{g}$, respectively, on Σ. Indeed, a conformally compact manifold induces a conformal class of metrics $\bar{\boldsymbol{c}}$ on its boundary Σ. Then, the AdS/CFT correspondence suggests that there is a correspondence between a conformal field theory living on $(\Sigma, \bar{\boldsymbol{c}})$ and a gravitating theory living on $(M \backslash \partial M, g^o)$. In this sense, the holographic principle is a prescription of a bulk-boundary correspondence: that we can learn about conformal field theories by studying bulk gravitating theories and vice versa.

Perhaps the first application of this type of holography in the mathematics literature was by Fefferman and Graham [**25, 26**]. In their work, they sought to study conformal invariants of a manifold $(\Sigma^{d-1}, \bar{\boldsymbol{c}})$ by constructing a conformally-compact manifold $(M \backslash \partial M, g^o)$ whose conformal infinity $\Sigma = \partial M$ has induced conformal class $\bar{\boldsymbol{c}}$ and whose associated singular metric g^o is Poincaré–Einstein. A conformally compact manifold $(M \backslash \partial M, g^o)$ is said to be *Poincaré–Einstein* (or is said to satisfy the *Poincaré–Einstein condition*) when the Ricci curvature of the singular metric is pure trace, *i.e.* for some negative constant k, the singular metric sastisfies

$$Ric^{g^o} = kg^o \text{ on } M\backslash\partial M\,.$$

Fefferman and Graham showed that, given such a conformal manifold $(\Sigma^{d-1}, \bar{\boldsymbol{c}})$, there is an essentially unique (depending on dimension parity) Poincaré–Einstein manifold with Σ as its conformal infinity. Specifically, when d is even, such a manifold can be formally constructed to arbitrarily high order; when d is odd, such a manifold can be constructed to asymptotically satisfy the Poincaré–Einstein condition, so that for some defining function s for Σ, we have that

$$Ric^{g^o} = kg^o + \mathcal{O}(s^{d-2}) \text{ on } M\backslash\partial M\,.$$

Then, by studying Riemannian invariants of $(M\backslash\partial M, g^o)$ that extend smoothly to the boundary ∂M, we can generate conformal invariants of $(\Sigma^{d-1}, \bar{\boldsymbol{c}})$.

This method to study the intrinsic conformal structure of $(\Sigma^{d-1}, \bar{\boldsymbol{c}})$ can also be used to study the extrinsic data of a conformal hypersurface embedding specified by (M^d, γ, σ). To see this, observe that the extrinsic embedding data of $\Sigma \hookrightarrow (M, \boldsymbol{c})$ can be (locally) captured by $\bar{\boldsymbol{c}}$ and its jets. Given this data, we can uniquely (up to a certain order) determine a conformally compact manifold $(M\backslash\partial M, g^o)$ by demanding that this conformally compact manifold solves the asymptotic singular Yamabe problem. Then, extrinsic invariants of the conformal hypersurface embedding (M, γ, σ) can be generated by studying Riemannian invariants of $(M\backslash\partial M, g^o)$ that extend smoothly to its conformal infinity $\Sigma = \partial M$.

The above discussion implies a bijection between conformally compact manifolds $(M\backslash\partial M, g^o)$ and one-sided conformal hypersurface embeddings specified by (M, γ, σ), where $M = \Sigma \sqcup M^+$ and $\Sigma = \partial M$, given by $(M, \gamma, \sigma) \leftrightarrow (M^+, \gamma/\sigma^2)$. Thus, we denote by PE the family of triples (M, γ, σ) that correspond to conformally compact manifolds satisfying the Poincaré–Einstein condition. Further, we denote by APE_k the family of triples (M, γ, σ) that correspond to conformally compact manifolds satisfying the asymptotic Poincaré–Einstein condition to order k, so that

$$Ric^{g^o} = kg^o + \mathcal{O}(s^k) \text{ on } M\backslash\partial M\,. \tag{5.1}$$

In this notation, we have that $\text{PE} \subset \text{APE}_k$ for every positive k.

A key observation relating Poincaré–Einstein structures to the tractor calculus made by Gover [**34**] is that if $(M, \gamma, \sigma) \in \mathrm{PE}$, then there exists a standard tractor $I \neq 0$ parallel with respect to the tractor connection. That is, given a hypersurface embedding specified by (M, γ, σ), then $\nabla I_\sigma = 0$ if and only if $(M, \gamma, \sigma) \in \mathrm{PE}$. But a consequence of this relationship is that $\mathring{\mathrm{I\!I}}^{\mathrm{e}} = 0$ if and only if $(M, \gamma, \sigma) \in \mathrm{PE}$. Observe now that $\mathring{P}^{g^o} = 0$ if and only if $(M, \gamma, \sigma) \in \mathrm{PE}$, and so we must have that $\mathring{\mathrm{I\!I}}^{\mathrm{e}} = 0$ if and only if $\mathring{P}^{g^o} = 0$. Furthermore, note that on M^+ in the scale $\sigma = [g^o; 1]$, we have that $\mathring{\mathrm{I\!I}}^{\mathrm{e}} = \mathring{P}^{g^o}$. This suggests that there exists some ℓ such that $\sigma^\ell \mathring{P}^{g^o} = \mathring{\mathrm{I\!I}}^{\mathrm{e}}$. Because $\mathring{P}^{g^o}$ is a conformal invariant of the data (M, γ, σ), by weight we see that $\ell = 1$. That is, we have that

$$\sigma \mathring{P}^{g^o} = \mathring{\mathrm{I\!I}}^{\mathrm{e}} \,. \tag{5.2}$$

That $\mathring{\mathrm{I\!I}}^{\mathrm{e}}$ extends smoothly to the boundary further implies that $\sigma \mathring{P}^{g^o}$ also extends smoothly to the boundary.

Equation (5.2) directly implies that there is a strong relationship between asymptotically Poincaré–Einstein structures and the conformal fundamental forms. Indeed, we we have the following result.

THEOREM 5.1.1. *Let (M, γ, σ) specify a conformally embedded hypersurface and let $k \in \mathbb{Z}_+$ such that $2 \leq k \leq d-1$. Then, (M, γ, σ) is k-umbilic if and only if $(M, \gamma, \sigma) \in \mathrm{APE}_{k-2}$.*

PROOF. The proof is a straightforward application of several results developed so far. First, suppose that (M, γ, σ) is k-umbilic with $2 \leq k \leq d-1$ an integer. Without loss of generality, we can assume that $(M, \gamma, \sigma) \in \mathrm{ASY}$ (the k-umbilicity condition depends only on the embedding and not on the choice of defining density σ). By definition, we have that $\mathring{\mathrm{I\!I}} = \cdots = \underline{\mathring{\mathrm{k}}} = 0$. Then, direct application of Lemma 4.2.14 shows that $\mathring{\mathrm{I\!I}}^{\mathrm{e}} = \mathcal{O}(\sigma^{k-1})$. From Equation (5.2), we thus have that $\mathring{P}^{g^o} = \mathcal{O}(\sigma^{k-2})$.

The reverse direction is nearly identical: because $(M, \gamma, \sigma) \in \mathrm{APE}_{k-2}$, we have that $\mathring{\mathrm{I\!I}}^{\mathrm{e}} = \mathcal{O}(\sigma^{k-1})$. But $\underline{\mathring{\mathrm{m}}} = \mathring{\top} \nabla_n^{m-2} \mathring{\mathrm{I\!I}}^{\mathrm{e}} + \mathrm{ltots}_{m-3}(\mathring{\mathrm{I\!I}}^{\mathrm{e}})$, so we have that $\mathring{\mathrm{I\!I}} = \cdots = \underline{\mathring{\mathrm{k}}} = 0$ and hence (M, γ, σ) is k-umbilic. This completes the proof. $\square$

This result suggests that we can draw connections between APE_k embeddings and other families of embeddings by passing through the language of the fundamental forms (or the extension $\mathring{\mathrm{I\!I}}^{\mathrm{e}}$). An example of this phenomenon is the family of Willmore hypersurface embeddings.

5.2. The Willmore Invariant

A particularly well-studied equation describing a surface S embedded in flat $\mathbb{R}^3$ is known as the Willmore [**87**] equation:

$$\mathcal{B}_3 := \bar{\Delta} H + 2H(H^2 - k) = 0\,,$$

where k is the Gauss curvature of S, H is its mean curvature, $\bar{\Delta}$ is the Laplacian on S, and the curvature quantity $\mathcal{B}_3$ is known as the *Willmore invariant*. In particular, one can view the equation $\mathcal{B} = 0$ as the energy minimizing equation (found by functional differentiation) for a surface S with energy given by the *Willmore energy*,

$$E = \int_S (H^2 - k)\,,$$

first described in Willmore's celebrated conjecture, recently proven by Marques and Neves [**61**].

The Willmore energy and invariant can be extended to generally curved ambient manifolds, whereby we observe two critical facts: the Willmore energy and invariant are global and local conformal invariants, respectively, of the embedded surface, and the Willmore invariant is linear in its leading term. Their conformal invariance and linearity are likely major driving factors in the frequent appearance of this pair in both the mathematics and physics literature. Early examples of this include Polyakov [**67**], who found the Willmore energy when studying strings (subsequently calling it the "rigid string action") and Graham and Witten [**69**], who found the Willmore energy when studying anomalies in the AdS/CFT correspondence. Recently, there has been increased interest in higher-dimensional analogs of the Willmore energy and invariant [**3**, **45**, **53**, **68**, **85**], in part to better understand the invariants of hypersurfaces embedded in conformal manifolds.

To capture the notion of the Willmore energy and functional described above, we provide the following definition of a higher Willmore energy.

Definition 5.2.1. Let the dimension d of M be odd. Then any functional of conformal embeddings given by

$$E\big[\Sigma \hookrightarrow (M, \boldsymbol{c})\big] = \int_\Sigma \mathrm{dVol}(\bar{g})\, \mathcal{E}(g, \Sigma)\,,$$

where $g \in \boldsymbol{c}$ and $\bar{g} \in \bar{\boldsymbol{c}}$ is the corresponding induced metric, is called a *higher Willmore energy* if

(i) the energy functional E only depends on the conformal embedding $\Sigma \hookrightarrow (M, \boldsymbol{c})$, and

(ii) the functional gradient of E with respect to variations of the embedding $\Sigma \hookrightarrow M$ is a local conformal invariant and has leading linear term (in any scale with non-vanishing mean curvature) proportional to $\bar{\Delta}^{\frac{d-1}{2}} H^g$.

■

This definition precisely captures the desired properties when the dimension of the bulk manifold is odd: that the energy is a global conformal invariant, that the functional gradient of the energy is a local conformal invariant, and the leading derivative term of the functional gradient is linear in the mean curvature. Furthermore, this definition can be extended to even dimensional bulk manifolds, so long as we relax the linearity condition. Such Willmore energies have also been studied; see [**31**, **45**, **58**].

A result of [**41**] shows that the obstruction density $B|_\Sigma$ in three dimensions reproduces the standard Willmore invariant. This motivates the following definition of the generalized Willmore invariant:

Definition 5.2.2. Let $(M, \gamma, \sigma)_{\mathcal{Y}}$ specify a conformal hypersurface embedding into a d-dimensional conformal manifold $(M, \boldsymbol{c})$ such that

$$I^2 = 1 + \sigma^d B \,.$$

Then the *canonical generalized Willmore invariant* $\mathcal{B}_d$ is defined by

$$\mathcal{B}_d := B|_\Sigma \in \Gamma(\mathcal{E}\Sigma[-d]) \,.$$

In general, we define a generalized Willmore invariant as any linear combination of $\mathcal{B}_d$ and scalar-valued extrinsic curvatures that are sections of $\mathcal{E}\Sigma[-d]$. ■

Often, we will refer to the canonical generalized Willmore invariant as *the* generalized Willmore invariant. That the obstruction density is a compatible generalized Willmore invariant for a higher Willmore energy was first shown in [**47**]. In particular, the canonical generalized Willmore invariant is associated to a particular global conformal invariant (the integrated extrinsic Q-curvature), which will be discussed in detail in Section 6.4. A straightforward way to see how Definition 5.2.2 is compatible with Definition 5.2.1 is via a key result from Gover and Waldron in [**45**].

Theorem 5.2.3 (Theorem 5.1 of [**45**]). *Given a conformal hypersurface embedding specified by $(M, \gamma, \sigma)_{\mathcal{Y}}$, up to a non-zero constant multiple, the obstruction density $B|_{\Sigma}$ takes the form*

$$\bar{\Delta}^{\frac{d-1}{2}} H + \textit{lower order terms}$$

for d odd and is fully non-linear for d even.

Thus, we have that the canonical generalized Willmore invariant indeed has the correct leading derivative structure to be associated to a higher Willmore energy. Another straightforward compatibility check can be made by observing that the functional variation of any higher Willmore energy must be a linear combination of $\mathcal{B}_d$ and other independent conformally-invariant terms.

Corollary 5.2.4. *Let $(M, \gamma\sigma)_{\mathcal{Y}}$ be a conformal hypersurface embedding and let E, E' be two distinct higher Willmore energies of this embedding with generalized Willmore invariants B, B'. Furthermore, suppose without loss of generality that E and E' are normalized so that the leading derivative structure of B and B' are both equal to $\bar{\Delta}^{\frac{d-1}{2}} H^g$ in some scale with non-vanishing mean curvature. Then, we can write*

$$B = \alpha \mathcal{B}_d + I \qquad \textit{and } B' = \alpha \mathcal{B}_d + I' \,,$$

where α is some non-zero coefficient and I, I' are sections of $\mathcal{E}\Sigma[-d]$ that are not generalized Willmore invariants.

Proof. First, observe that in a choice of scale g with non-vanishing mean curvature, a consequence of Theorem 5.2.3 is that there exists a non-zero coefficient β such that

$$\mathcal{B}_d \stackrel{g}{=} \beta \bar{\Delta}^{\frac{d-1}{2}} H^g + \text{lower order terms},$$

so we must be able to write

$$B = \tfrac{1}{\beta} \mathcal{B}_d + \text{lower order terms},$$

and similarly with B'. However, because $B, B', \mathcal{B}_d$ are all conformally invariant, the lower order terms must be themselves conformally invariant. Furthermore, because they cannot contain terms that are themselves of the form $\bar{\Delta}^{\frac{d-1}{2}}$, such invariants are not generalized Willmore invariants. This completes the proof. □

The above corollary implies that for any choice of higher Willmore energy, its associated generalized Willmore invariant has the form $\alpha\mathcal{B}_d$ plus lower order invariant terms for some nonzero α.

While $\mathcal{B}_4$ was explicitly computed in [**31**] and $\mathcal{B}_5$ is explicitly computed in Chapter 6, only some features of $\mathcal{B}_d$ are known for arbitrary d. In this section we seek to make statements about the generalized Willmore invariant in arbitrary dimensions.

We call hypersurface embeddings that have a vanishing Willmore invariant $\mathcal{B}_d$ *Willmore* and use the notation $(M^d, \gamma, \sigma) \in \mathcal{W}$ to denote such a hypersurface embedding. A result of [**34**] shows that $\mathrm{PE} \subset \mathcal{W}$. Because $\mathrm{PE} \subset \mathrm{APE}_k$, it is of interest to investigate the inclusion properties of APE_k in $\mathcal{W}$. Indeed, we can sharply characterize this inclusion in the following theorem.

Theorem 5.2.5. *For conformal hypersurface embeddings specified by* (M^d, γ, σ) *with* $d \geq 4$, *we have that* $\mathrm{APE}_{d-3} \subset \mathcal{W}$ *but* $\subset \mathrm{APE}_{d-4} \not\subset \mathcal{W}$.

Proof. To prove this theorem we first show for some conformal hypersurface embedding specified by $(M^d, \gamma, \sigma) \in \mathrm{APE}_{d-3}$, we have that $(M^d, \gamma, \sigma) \in \mathcal{W}$. From the definition of APE_k, we have that $\mathring{P}^{g^o} = \mathcal{O}(\sigma^{d-3})$. Further, from Equation (5.2) we have that $\mathring{\mathrm{I\!I}}^{\mathrm{e}} = \mathring{P}^{g^o}$. So it suffices to check that $\mathcal{B}_d = 0$ when $\mathring{\mathrm{I\!I}}^{\mathrm{e}} = \mathcal{O}(\sigma^{d-2})$.

First, observe that $I^2 = 1 + \sigma^d B$, so in particular $I^B \nabla_n I_B = \frac{d}{2}\sigma^{d-1}B + \mathcal{O}(\sigma^d)$. Using Equation (2.3) in a choice of scale $\sigma = [g; s]$, we have that

$$\frac{d}{2}s^{d-1}B + \mathcal{O}(s^d) \stackrel{g}{=} n^a n^b \mathring{\mathrm{I\!I}}^{\mathrm{e}}_{ab} - \tfrac{1}{d-1} s n^a \nabla^b \mathring{\mathrm{I\!I}}^{\mathrm{e}}_{ab}\,.$$

Taking $d-1$ normal derivatives and evaluating along Σ, we have that

$$\begin{aligned}
\frac{d!}{2}B &\overset{\Sigma}{=} \nabla_n^{d-1} n^a n^b \mathring{\mathrm{I\!I}}^{\mathrm{e}}_{ab} - \nabla_n^{d-2} n^a \nabla^b \mathring{\mathrm{I\!I}}^{\mathrm{e}}_{ab} \\
&\overset{\Sigma}{=} \nabla_n^{d-1} n^a n^b \mathring{\mathrm{I\!I}}^{\mathrm{e}}_{ab} - \nabla_n^{d-2} n^a \nabla_{\mathrm{e}}^{\top b} \mathring{\mathrm{I\!I}}^{\mathrm{e}}_{ab} - \nabla_n^{d-2} n^a n^b \nabla_n \mathring{\mathrm{I\!I}}^{\mathrm{e}} \\
&\overset{\Sigma}{=} \nabla_n^{d-2} \left[2n^{(a} (\nabla_n n^{b)}) \mathring{\mathrm{I\!I}}^{\mathrm{e}}_{ab} - n^a \nabla_{\mathrm{e}}^{\top b} \mathring{\mathrm{I\!I}}^{\mathrm{e}}_{ab} \right] \\
&\overset{\Sigma}{=} \nabla_n^{d-2} \left[n^{(a} \mathring{\mathrm{I\!I}}^{\mathrm{e}}_{ab} \nabla^{b)} (1 - 2s\rho + s^d B) - \nabla_{\mathrm{e}}^{\top b} n^a \mathring{\mathrm{I\!I}}^{\mathrm{e}}_{ab} + (\nabla_{\mathrm{e}}^{\top b} n^a) \mathring{\mathrm{I\!I}}^{\mathrm{e}}_{ab} \right] \\
&\overset{\Sigma}{=} \nabla_n^{d-2} \left[K_{\mathrm{e}} - n \cdot (\mathring{\mathrm{I\!I}}^{\mathrm{e}})^2 \cdot n - \rho n \cdot \mathring{\mathrm{I\!I}}^{\mathrm{e}} \cdot n - \nabla_{\mathrm{e}}^{\top} \cdot n \cdot \mathring{\mathrm{I\!I}}^{\mathrm{e}} + s \mathring{\mathrm{I\!I}}^{\mathrm{e}}_{ab} \left(n^a n^c P_c^b - 2n^a (\nabla^b \rho) - P^{ab} \right) \right] .
\end{aligned}$$

From Lemma 3.4.6, we have that if $\mathring{\mathrm{I\!I}}^{\mathrm{e}} = \mathcal{O}(\sigma^{d-2})$, then $n \cdot \mathring{\mathrm{I\!I}}^{\mathrm{e}} \overset{g}{=} \mathcal{O}(s^{d-1})$ and $\nabla_{\mathrm{e}}^{\top} \cdot n \cdot \mathring{\mathrm{I\!I}}^{\mathrm{e}} = \mathcal{O}(s^{d-1})$, and thus $B \overset{\Sigma}{=} 0$.

To complete the proof, we must show that there exists some $(M^d, \gamma, \sigma) \in \mathrm{APE}_{d-4}$ but $(M^d, \gamma, \sigma) \notin \mathcal{W}$. In particular, we consider hypersurface embedding with $\mathring{\mathrm{I\!I}}^{\mathrm{e}} = \mathcal{O}(\sigma^{d-3})$ but $\mathring{\mathrm{I\!I}}^{\mathrm{e}} \neq \mathcal{O}(\sigma^{d-2})$. Observe from Equation (3.22) and Remark 3.4.7 that

$$\nabla_n^{d-2} n \cdot \mathring{\mathrm{I\!I}}^{\mathrm{e}} \overset{\Sigma}{=} (d-2) \nabla_{\mathrm{e}}^{\top} \cdot \nabla_n^{d-3} \mathring{\mathrm{I\!I}}^{\mathrm{e}} + \mathrm{ltots}_{d-4}(\mathring{\mathrm{I\!I}}^{\mathrm{e}}) .$$

Hence because $\mathring{\mathrm{I\!I}}^{\mathrm{e}} = \mathcal{O}(\sigma^{d-3})$, we have that

$$\nabla_n^{d-2} \nabla_{\mathrm{e}}^{\top} \cdot n \cdot \mathring{\mathrm{I\!I}}^{\mathrm{e}} \overset{\Sigma}{=} (d-2) \nabla^{\top} \cdot \nabla^{\top} \cdot \nabla_n^{d-3} \mathring{\mathrm{I\!I}}^{\mathrm{e}} ,$$

and hence

$$\frac{d!}{2} B \overset{\Sigma}{=} -(d-2) \left(\nabla^{\top a} \nabla^{\top b} + P^{ab} \right) \nabla_n^{d-3} \mathring{\mathrm{I\!I}}^{\mathrm{e}}_{ab} .$$

Using the Fialkow–Gauß Equation (3.7) and noting that the Fialkow tensor vanishes when $d \geq 5$ because $\mathring{\mathrm{I\!I}}^{\mathrm{e}} = \mathcal{O}(\sigma^{d-3})$, it then follows that

$$\frac{d!}{2} B \overset{\Sigma}{=} -(d-2) \left(\bar{\nabla}^a \bar{\nabla}^b + \bar{P}^{ab} \right) (\mathring{\top} \nabla_n^{d-3} \mathring{\mathrm{I\!I}}^{\mathrm{e}}_{ab}) .$$

We now construct an explicit example to show that a hypersurface embedding with $\mathring{\mathrm{I\!I}}^{\mathrm{e}} = \mathcal{O}(\sigma^{d-3})$ exists that has $\mathcal{B}_d \neq 0$. To do so, let $U \subset M$ be a neighborhood of Σ and work in a choice of scale $\sigma = [g; s]$ such that in a set of local coordinates $(s, y, x^2, \ldots, x^{d-1})$, we have that on U the metric representative takes the form

$$g = ds^2 + 2s^{d-3} f(y) ds \odot dy + dy^2 + dx^i dx_i \,,$$

for $i = 1, \dots, d-2$. In this choice of scale, the hypersurface is flat and a tedious but straightforward calculation shows that $\mathring{\mathrm{I\!I}}^{\mathrm{e}} = \mathcal{O}(s^{d-3})$ but $\mathring{\mathrm{I\!I}}^{\mathrm{e}} \neq \mathcal{O}(s^{d-2})$, and further that $\mathcal{B}_d \propto \partial_y^3 f(y) \neq 0$.

In the $d = 4$ case, we can also check using the same example. Using the same embedding described above, we have that $\mathring{\mathrm{I\!I}}^{\mathrm{e}} = \mathcal{O}(s)$ but $\mathring{\mathrm{I\!I}}^{\mathrm{e}} \neq \mathcal{O}(s^2)$. Furthermore, using the result for $\mathcal{B}_4$ in [**31**], we can directly compute the Willmore invariant and find that

$$\mathcal{B}_4 = \tfrac{1}{18} \left[(\partial_y f)^2 + \partial_y^3 f \right] \,.$$

And hence is nonzero for some function $f(y)$. This completes the proof. □

REMARK 5.2.6. When the singular Yamabe obstruction density vanishes, a result of Gover and Waldron [**45**, Remark 4.14] states that in general the singular Yamabe problem can be solved to arbitrarily high order. While the σ that solves the asymptotic singular Yamabe problem is asymptotically unique, to solve the (one-sided) Yamabe problem requires global information (such as the topology of the interior of the ambient manifold M). Thus, even though the obstruction density may vanish, the asymptotic unit defining density σ does not necessarily solve the singular Yamabe problem globally. However, again as a result of [**45**], we do have that I^2 is smooth to arbitrarily high order when the obstruction density vanishes. ■

The proof of Theorem 5.2.5 suggests that, if one can find fundamental forms that correspond to each normal derivative of $\mathring{\mathrm{I\!I}}^{\mathrm{e}}$, one could reframe the obstruction density (and hence the Willmore invariant) in terms of these fundamental forms. In particular, when d is even, we can construct fundamental forms up to $\mathring{\overline{\underline{\mathrm{d}-1}}}$, which we can use to express the Willmore invariant (in addition to one more non-invariant tensor). Hence we have the following corollary.

COROLLARY 5.2.7. *Let $(M^d, \gamma, \sigma)_{\mathcal{Y}}$ specify a conformally embedded hypersurface with $d \geq 4$ even. Then, the Willmore invariant $\mathcal{B}_d$ has transverse order $d-1$ and can be expressed in such a way that each term contains at least one element of the set $\{\mathring{\mathrm{I\!I}}, \dots, \mathring{\overline{\underline{\mathrm{d}-1}}}, \mathring{\top}\nabla_n^{d-3} P\}$ with a non-zero coefficient and $\mathcal{B}_d$ can be written to explicitly depend polynomially on each element of the set.*

Proof. We first check that Corollary 5.2.7 holds for $d = 4$. From an explicit formula in [**31**], we see that $\mathcal{B}_4$ contains exactly one term that contains $C^{\top}_{\hat{n}ab}$, which has transverse order 3. Further we note that $\mathcal{B}_4$ can be written in a way such that each term contains at least one of $\mathring{\mathrm{I\!I}}, \mathring{\mathrm{I\!I\!I}}, C^{\top}_{\hat{n}ab}$. However note that $C^{\top}_{\hat{n}ab} = \mathring{\top}\nabla_n P_{ab} + \mathrm{ltots}_0(P_{ab})$, so the corollary holds.

Now suppose that $d \geq 5$. We check that the transverse order of $\mathcal{B}_d$ is $d-1$ first. From the proof of Theorem 5.2.5, the terms with leading normal derivatives on $\mathring{\mathrm{I\!I}}^{\mathrm{e}}$ are $\nabla_n^{d-2}(K_{\mathrm{e}} - s\,\mathring{\mathrm{I\!I}}^{\mathrm{e}}\cdot P)$. Observe that $\nabla_n^{d-2}K = \mathring{\mathrm{I\!I}}\cdot\nabla_n^{d-2}\mathring{\mathrm{I\!I}}^{\mathrm{e}} + \mathrm{ltots}_{d-3}(\mathring{\mathrm{I\!I}}^{\mathrm{e}})$. However, from Equation (4.4) we see that the leading transverse order term of this expression vanishes. So we need only consider $\nabla_n^{d-2}(s\,\mathring{\mathrm{I\!I}}^{\mathrm{e}}\cdot P)$. A direct calculation shows that this expression has transverse order $d-1$. But because this is the only term in $\mathcal{B}_d$ with that transverse order, we have that $\mathcal{B}_d$ has transverse order $d-1$.

From the proof of Theorem 5.2.5, it is clear that each term can be expressed in terms of some operator acting on $\mathring{\mathrm{I\!I}}^{\mathrm{e}}$ and hence from Proposition 4.3.2 and Lemma 3.4.6 can be expressed in terms of a canonical fundamental form—with the exception of terms with transverse order $d-1$. In particular, for any integer ℓ satisfying $0 \leq \ell \leq d-3$, we have that

$$\nabla_n^{\ell}\,\mathring{\mathrm{I\!I}}^{\mathrm{e}} \stackrel{\Sigma}{=} \overset{\circ}{\overline{\underline{\ell+2}}} + \mathrm{ltots}_{\ell-1}(\mathring{\mathrm{I\!I}}^{\mathrm{e}})\,. \tag{5.3}$$

As noted above, the term with transverse order $d-1$ only arises in the product $\mathring{\mathrm{I\!I}}\cdot\nabla_n^{d-3}P$, and hence we can characterize that term by $\mathring{\top}\nabla_n^{d-3}P$. Thus, we have that each summand in $\mathcal{B}_d$ contains at least one element of the set $\{\mathring{\mathrm{I\!I}}, \ldots, \overset{\circ}{\overline{\underline{d-1}}}, \mathring{\top}\nabla_n^{d-3}P\}$.

It now remains to show that $\mathcal{B}_d$ can be written in such a way that it explicitly depends on each element of the set $\{\mathring{\mathrm{I\!I}}, \ldots, \overset{\circ}{\overline{\underline{d-1}}}, \mathring{\top}\nabla_n^{d-3}P\}$ with a non-zero coefficient. A key observation is that only the terms in $\mathcal{B}_d$ of the form $\nabla_n^{d-2}(K_{\mathrm{e}} - s\,\mathring{\mathrm{I\!I}}^{\mathrm{e}}\cdot P)$ can contain terms that are a product of exactly two elements from this set. One way to see this is that $n\cdot\nabla_n\mathring{\mathrm{I\!I}}^{\mathrm{e}}$ can be written in terms of $\nabla^{\top}_{\mathrm{e}}\cdot\mathring{\mathrm{I\!I}}^{\mathrm{e}}$, and that derivative cannot be eliminated by any manipulations. So we now consider an expansion of this difference:

$$\begin{aligned}\nabla_n^{d-2}(K_{\mathrm{e}} - s\,\mathring{\mathrm{I\!I}}^{\mathrm{e}}\cdot P) \stackrel{\Sigma}{=} &\sum_{k=0}^{d-2}\binom{d-2}{k}(\nabla_n^k\mathring{\mathrm{I\!I}}^{\mathrm{e}})\cdot(\nabla_n^{d-k-2}\mathring{\mathrm{I\!I}}^{\mathrm{e}})\\ &- (d-2)\sum_{k=0}^{d-3}\binom{d-3}{k}(\nabla_n^k\mathring{\mathrm{I\!I}}^{\mathrm{e}})\cdot(\nabla_n^{d-k-3}P) + \text{more},\end{aligned} \tag{5.4}$$

where more contains no terms quadratic in elements of the set $\{\nabla_n^m \mathring{\mathrm{I\!I}}^{\mathrm{e}}, \nabla_n^{\ell} P\}_{m,\ell}$ for $0 \le m \le d-2$ and $0 \le \ell \le d-3$. Before we proceed, note that the term of the form $P \cdot \nabla_n^{d-3} \mathring{\mathrm{I\!I}}^{\mathrm{e}}$ in the display above can be paired with $\nabla^\top \cdot \nabla^\top \cdot \nabla_n^{d-3} \mathring{\mathrm{I\!I}}^{\mathrm{e}}$ (resulting from manipulating the term $\nabla_n^{d-2} \nabla_{\mathrm{e}}^\top \cdot n \cdot \mathring{\mathrm{I\!I}}^{\mathrm{e}}$) to form a conformal invariant. In particular, observe that the operator L defined in [**31**] by

$$\Gamma(\odot_\circ^2 T^* \Sigma[-d+4]) \ni T_{ab} \mapsto (\bar{\nabla}^a \bar{\nabla}^b + \bar{P}^{ab}) T_{ab} \in \Gamma(\mathcal{E}\Sigma[-d]) \tag{5.5}$$

is conformally invariant on weight $4-d$ densities, which is precisely the weight of $\overset{\circ}{\overline{\underline{\mathrm{d}-1}}}$. But the leading term of $\nabla_n^{d-3} \mathring{\mathrm{I\!I}}^{\mathrm{e}}$ is indeed $\overset{\circ}{\overline{\underline{\mathrm{d}-1}}}$ as per Equation (5.3), so the leading term of

$$(\nabla^{\top a} \nabla^{\top b} + P^{ab}) \nabla_n^{d-3} \mathring{\mathrm{I\!I}}^{\mathrm{e}}_{ab}$$

is precisely $L(\overset{\circ}{\overline{\underline{\mathrm{d}-1}}})$. Thus, $\mathcal{B}_d$ contains a term of the form $L(\overset{\circ}{\overline{\underline{\mathrm{d}-1}}})$ which absorbs the term of the form $P \cdot \nabla_n^{d-3} \mathring{\mathrm{I\!I}}^{\mathrm{e}}$. Thus, from Equation (5.4), we are only interested in the terms

$$\sum_{k=0}^{d-2} \tbinom{d-2}{k} (\nabla_n^k \mathring{\mathrm{I\!I}}^{\mathrm{e}}) \cdot (\nabla_n^{d-k-2} \mathring{\mathrm{I\!I}}^{\mathrm{e}}) - (d-2) \sum_{k=0}^{d-4} \tbinom{d-3}{k} (\nabla_n^k \mathring{\mathrm{I\!I}}^{\mathrm{e}}) \cdot (\nabla_n^{d-k-3} P). \tag{5.6}$$

We now work in the scale where $|ds|_g = 1$ and choose a set of coordinates $(s, y^1, \dots, y^{d-1})$, so that

$$\begin{aligned} \nabla_n^m \mathring{\mathrm{I\!I}}^{\mathrm{e}} &= \tfrac{d-m-2}{2(d-2)} \partial_s^{m+1} g_{ab} + \mathrm{ltots}_{m-2}(g)_{ab} \\ \nabla_n^m P_{ab} &= -\tfrac{1}{2(d-2)} \partial_s^{m+2} g_{ab} + \mathrm{ltots}_{m+1}(g)_{ab} \,. \end{aligned}$$

Applying the above to Display (5.6) and keeping only terms quadratic in $\partial_s^m g$, we have

$$\tfrac{1}{4} (\partial_s g) \cdot (\partial_s^{d-1} g) + \tfrac{(d-1)(d-3)}{4(d-2)} (\partial_s^2 g) \cdot (\partial_s^{d-2} g) + \tfrac{1}{4} \sum_{k=2}^{d-4} \tbinom{d-3}{k} (\partial_s^{k+1} g) \cdot (\partial_s^{d-k-1} g) \,.$$

The definition of a fundamental form is a conformally-invariant rank-2 tensor of the appropriate weight and transverse order. Because $\mathcal{B}$ is conformally invariant, a consequence of Equation (5.3) is that we must be able to express the above display in terms of quadratic products of fundamental forms of the form $\overset{\circ}{\overline{\underline{\mathrm{k}+2}}} \cdot \overset{\circ}{\overline{\underline{\mathrm{d}-\mathrm{k}}}}$ plus subleading terms—with the exception of the term with transverse order $d-1$. But because none of the coefficients vanish for $d \ge 4$ in the above display, the corresponding coefficients for $\overset{\circ}{\overline{\underline{\mathrm{k}+2}}} \cdot \overset{\circ}{\overline{\underline{\mathrm{d}-\mathrm{k}}}}$ must also not vanish for each product for $1 \le k \le d-3$.

For the same reason as the terms quadratic in fundamental forms, the one remaining term, of the form $\frac{1}{4}(\partial_s g)\cdot(\partial_s^{d-1} g)$ can be written in the form $\mathring{\mathrm{I\!I}}\cdot\nabla_n^{d-3}P$ plus subleading terms with a non-zero coefficient. Thus, $\mathcal{B}_d$ can be written so that it explicitly depends on every element of the set $\{\mathring{\mathrm{I\!I}},\ldots,\overset{\circ}{\overline{\underline{d-1}}},\mathring{\top}\nabla_n^{d-3}P\}$ with a non-zero coefficient. This completes the proof. $\square$

REMARK 5.2.8. Much of the argument in the proof of Corollary 5.2.7 follows when $d\geq 5$ is odd. In that case, for $n\geq\frac{d+3}{2}$, the fundamental forms are only conformally-invariant when (M,γ,σ) is hyperumbilic. Nonetheless, the higher transverse derivative terms $\nabla_n^k\mathring{\mathrm{I\!I}}^{\mathrm{e}}$ still appear in $\mathcal{B}_d$, so a similar statement as the above corollary would hold in terms of what may be called *pre-fundamental forms*—tensors that are not conformally-invariant but become conformally-invariant when the hypersurface embedding is hyperumbilic. ■

With Theorem 5.2.5, we can now prove a generalization of Lemmas 3.4.6.

LEMMA 5.2.9. *Let $(M,\gamma,\sigma)_{\mathcal{Y}}$ represent a conformally embedded hypersurface and let $\ell\in\mathbb{Z}_{\neq d-1}$ be non-negative. If $\mathring{\mathrm{I\!I}}^{\mathrm{e}}=\mathcal{O}(\sigma^\ell)$, then $n^a\mathring{\mathrm{I\!I}}^{\mathrm{e}}_{ab}=\mathcal{O}(\sigma^{\ell+1})$ and $\nabla\cdot\mathring{\mathrm{I\!I}}^{\mathrm{e}}=\mathcal{O}(\sigma^\ell)$.*

PROOF. First observe that for $\ell\leq d-2$, this is simply Lemma 3.4.6. Then, for $\ell\geq d$, from Theorem 5.2.5, we have that $B\overset{\Sigma}{=}0$. In particular, because the obstruction density vanishes, we know that B is a smooth function of the asymptotic unit defining density σ and hence we can write $B=B|_\Sigma+\sigma B'|_\Sigma+\ldots$. But an examination of the proof of Theorem 5.2.5 suggests that for $\mathring{\mathrm{I\!I}}^{\mathrm{e}}=\mathcal{O}(\sigma^\ell)$ with $\ell\geq d$, we have that

$$B\overset{\Sigma}{=}0\,,\nabla_n B\overset{\Sigma}{=}0\,,\ldots\,,\nabla_n^{\ell-d+1}B\overset{\Sigma}{=}0\,.$$

But then the proof of this lemma is just a straightforward generalization of the proof of Lemma 3.4.6. In that case, we were restricted by the fact that the obstruction did not vanish: here, the obstruction vanishes to exactly the order necessary. Indeed, in this case, the righthand side of Equation (3.21) is $\mathcal{O}(\sigma^{\ell+1})$. Following the remainder of that proof implies that $n\cdot\mathring{\mathrm{I\!I}}^{\mathrm{e}}=\mathcal{O}(\sigma^{\ell+1})$ and that $\nabla\cdot\mathring{\mathrm{I\!I}}^{\mathrm{e}}=\mathcal{O}(\sigma^\ell)$. $\square$

As promised in Section 4.4, we can now prove that Definition 4.4.1 is sensible.

PROPOSITION 5.2.10. *Let $(M, \gamma, \sigma)_{\mathcal{Y}}$ specify an asymptotically singular Yamabe conformal hypersurface embedding and let $n \geq d+1$. Then, for all $\mathring{\underline{\bar{\mathrm{n}}}}$ as defined in Definition 4.4.1, $\mathring{\underline{\bar{\mathrm{n}}}}$ is a fundamental form.*

PROOF. By construction, the tensor $\mathring{\underline{\bar{\mathrm{n}}}}$ has weight $3-n$ and is symmetric and trace-free. Thus, we need only check that the transverse order of the nth fundamental form is indeed $n-1$. From Lemma 4.1.8, we see that the transverse order of δ_k is k so long as

$$w \neq \left\{\tfrac{2k-1-d}{2}, \tfrac{2k-2-d}{2}, \cdots, \tfrac{k+1-d}{2}\right\}.$$

In particular, we are interested in the action of this operator on P_{AB} because $q(\mathring{\mathrm{II}}^{\mathrm{e}}) = P_{AB}$, so we need only verify that δ_{n-2} has transverse order $n-2$ when $w=-1$. Examining the above display with $w=-1$, we have that the transverse order of δ_k is strictly less than k so long as $-\frac{d-1}{2} \leq k \leq d-3$. However, we are only interested in $k \geq n-2 \geq d-1$. Thus, we find that indeed δ_{n-2} acting on weight -1 tractors has transverse order $n-2$, and hence we can write

$$\mathring{\underline{\bar{\mathrm{n}}}} = (\bar{q}^* \circ \bar{r} \circ \mathring{\top})(\alpha \nabla_n^{n-2} P_{AB}) + \mathrm{ltots}_{n-3}(P_{AB}),$$

for some nonzero coefficient α.

Next, we need to verify that the remaining operators required to produce $\mathring{\underline{\bar{\mathrm{n}}}}$ do not decrease the transverse order of the tensor. Using identical arguments to those found in the proof of Proposition 4.2.10, as well as the generalization of Lemma 3.4.6 given in Lemma 5.2.9, we have that

$$\mathring{\underline{\bar{\mathrm{n}}}} = \beta \mathring{\top}(\nabla_n^{n-2} \mathring{\mathrm{II}}^{\mathrm{e}}) + \mathrm{ltots}_{n-3}(\mathring{\mathrm{II}}^{\mathrm{e}}),$$

for some nonzero coefficient β. Thus, from Equation (4.4), we have that

$$\mathring{\top}(\nabla_n^{n-2} \mathring{\mathrm{II}}^{\mathrm{e}}) = \tfrac{d-n}{2(d-2)} \mathring{\top}(\partial_s^{n-1} g_{ab}),$$

and hence has transverse order $n-1$ for $n \geq d+1$. □

CHAPTER 6

Extrinsic Conformal Hypersurface Invariants in Five Dimensions

As an application of the conformal hypersurface calculus and the conformal fundamental forms developed above, we provide a case study of the extrinsic conformal geometry of a hypersurface embedded in a conformal 5-manifold.

6.1. The Extrinsic Paneitz Operator

The key object underlying our computations in this chapter is the extrinsically-coupled Paneitz operator—a specific case of an extrinsically-coupled GJMS operator. First discovered independently by Paneitz [**64**] and independently by Fradkin and Tseytlin [**28**,**29**], the *Paneitz operator* is a fourth order conformally-invariant differential operator with a leading derivative term given by the square of the Laplacian. In particular, on a conformal manifold $(M^n, \boldsymbol{c})$ with $n \geq 3$, the Paneitz operator

$$\hat{P}_4 : \Gamma\left(\mathcal{E}M\left[\tfrac{4-n}{2}\right]\right) \to \Gamma((\mathcal{E}M\left[\tfrac{-4-n}{2}\right])$$

is given in a choice of scale $\tau = [g; t]$ by

$$\hat{P}_4 := \Delta^2 + \nabla_a \circ (4P^{ab} - (n-2)Jg^{ab}) \circ \nabla_b + \tfrac{n-4}{2}\mathcal{Q}_n(g)\,, \tag{6.1}$$

where

$$\mathcal{Q}_n(g) := -\Delta J - 2P^2 + \tfrac{n}{2}J^2\,. \tag{6.2}$$

The above scalar curvature, introduced in [**12**] and in four dimensions known as Branson's Q-curvature $Q_4(g) := \mathcal{Q}_4(g)$, is of particular importance because its integral is a global conformal invariant. Explicitly, if $\tau = [g; t] = [\tilde{g}; \tilde{t}]$ are two choices of scale and $\mathrm{dvol}(g)$ is the volume form for (M^4, g), then

$$\int_M Q_4(g)\,\mathrm{dvol}(g) = \int_M Q_4(\tilde{g})\,\mathrm{dvol}(\tilde{g})\,.$$

Furthermore, there is an explicit relationship between the Paneitz operator and Branson's Q-curvature in four dimensions. If $\tilde{g} = e^{2\omega} g$ for some $\omega \in C^\infty M$, then

$$e^{4\omega} Q_4(\tilde{g}) = Q_4(g) + \hat{P}_4 \omega \,.$$

This scalar curvature was originally introduced to better understand functional determinants in four dimensions similar to those of Polyakov [**66**]. Later work continued to make use of this curvature (and its higher dimensional analogs) in studying problems such as geometric scattering [**21**,**51**]. For more information, see the reviews found at [**18**, **19**, **20**].

As discussed throughout this dissertation, it is generally hard to construct conformal invariants from whole cloth. As such, it pays to have a method to produce invariants such as the Paneitz operator. The holographic method of Fefferman and Graham [**25**, **26**] summarized in Section 5.1 suggests that we can find such an invariant operator by first constructing the asymptotically Poincaré–Einstein manifold $(M \backslash \partial M, g^o) \in \mathrm{APE}_{d-2}$ attached to the conformal manifold $(\Sigma, \bar{\boldsymbol{c}})$ and then finding diffeomorphism-invariant tangential operators on the Poincaré–Einstein manifold that extend smoothly to the boundary. Indeed, one can verify that given any smooth extension φ^{e} of $\varphi \in C^\infty \Sigma^\infty$ to $C^\infty M^5$, we have that

$$\hat{P}_4 \varphi = \tfrac{1}{9} \left[(\Delta^o + 3) \circ (\Delta^o + 4) \circ (\Delta^o + 3) \circ \Delta^o \right] (\varphi^{\mathrm{e}})|_\Sigma \,, \tag{6.3}$$

Our next step is to construct a holographic formula for this operator. First observe that because $(M \backslash \partial M, g^o)$ is conformally compact, there is a corresponding conformally embedded hypersurface specified by (M, γ, σ) such that $g^o := \gamma / \sigma^2$ on $M \backslash \partial M$. A consequence of the Poincaré–Einstein condition is that $(M, \gamma, \alpha\sigma) \in \mathrm{ASY}$ for some constant $\alpha > 0$. Without loss of generality, we will consider cases where $\alpha = 1$. On $M^+ := M \backslash \partial M$, we can then compute in the scale $\sigma = [g^o; 1]$ so that

$$I_A \stackrel{g^o}{=} (1, 0, -\tfrac{1}{d} J^{g^o}) \,.$$

Then, in that same scale, we have that

$$I \cdot \hat{D} \stackrel{g^o}{=} \left(-(\Delta^o + J^{g^o} \underline{w}) - \tfrac{1}{d} J^{g^o} (d + 2\underline{w} - 2) \underline{w} \right) \circ \left(\tfrac{1}{d + 2\underline{w} - 2} \right) \,.$$

Because the embedding is asymptotically singular Yamabe, we have that $Sc^{g^o} = -d(d-1) + \mathcal{O}(\sigma^d)$ and hence $J^{g^o} = -d/2 + \mathcal{O}(\sigma^d)$. Recall that this expression is conformally invariant, meaning that for any defining function s for Σ, there exists a smooth B_s such that $J^{g^o} = -d/2 + s^d B_s$. Thus, simplifying the above formula for $I \cdot \hat{D}$, we have that

$$I \cdot \hat{D} \overset{g^o}{=} \left(-\Delta^o + (1 - \tfrac{2}{d} s^d B_s)(d + \underline{w} - 1)\underline{w}\right) \circ \left(\tfrac{1}{d+2\underline{w}-2}\right) .$$

However, note that because $I \cdot D^k$ has transverse order at most k along Σ, we can therefore write

$$I \cdot \hat{D}^k \overset{g^o}{=} (-1)^k \left[(\Delta^o - (d + \underline{w} - 1)\underline{w}) \circ \left(\tfrac{1}{d+2\underline{w}-2}\right)\right]^k \Big|_\Sigma$$

so long as $k \leq d-1$. In particular, when $k = 4$ and $d \geq 5$, acting on tractors of weight $\frac{5-d}{2}$, along Σ we have that

$$I \cdot \hat{D}^4 \overset{g^o}{=} \tfrac{1}{9} \left[\left(\Delta^o + \tfrac{(d-3)(d+1)}{4}\right) \circ \left(\Delta^o + \tfrac{(d-1)^2}{4}\right) \circ \left(\Delta^o + \tfrac{(d-3)(d+1)}{4}\right) \circ \left(\Delta^o + \tfrac{(d-3)(d-5)}{4}\right)\right] \Big|_\Sigma .$$

But when $d = 5$ this precisely reproduces the operator in Equation (6.3).

Furthermore, we can now verify that this operator is indeed tangential on weight-$\frac{5-d}{2}$ densities via the $\mathfrak{sl}(2)$ algebra of Proposition 3.4.3. To do so, first note that if $I \cdot D^4$ is tangential on such tractors, then so is $I \cdot \hat{D}^4$, so we verify this claim for the simpler operator $I \cdot D^4$. Furthermore, because $I^2 = 1 + \mathcal{O}(\sigma^d)$ and $d \geq 5$, we can treat $I^2 = 1$ identically for the purposes of handling the operator $I \cdot D^4$. To show tangentiality, it is sufficient to show that $(I \cdot D^4 \circ \sigma)(T) = (\sigma \circ \mathsf{O})(T)$ where O is some operator on tractors and $T \in \Gamma\left(\mathcal{T}^\Phi M\left[\frac{3-d}{2}\right]\right)$. (Note that $\sigma T \in \Gamma\left(\mathcal{T}^\Phi M\left[\frac{5-d}{2}\right]\right)$ as desired.)

Phrased in the language of the $\mathfrak{sl}(2)$ algebra, we thus wish to show that $[y^k, x] = 0$ as an operator acting on sections of $T \in \Gamma\left(\mathcal{T}^\Phi M\left[\frac{3-d}{2}\right]\right)$. But a standard result of the algebra states that

$$[y^k, x] = y^{k-1} k(\mathsf{h} - k + 1) .$$

Evaluating on a tractor $T \in \Gamma\left(\mathcal{T}^\Phi M\left[\frac{3-d}{2}\right]\right)$ the operator $(\mathsf{h} - k + 1) = 0$ when $k = 4$, which shows that indeed $I \cdot \hat{D}^4$ is tangential on any tractor $T^{\mathrm{e}} \in \Gamma\left(\mathcal{T}^\Phi M\left[\frac{5-d}{2}\right]\right)$. Thus, we have that

$$\hat{P}_4 T = (I \cdot \hat{D}^4 T^{\mathrm{e}})|_\Sigma$$

is independent of the choice of extension T^{e} for T.

Given the above holographic formula for the operator $\hat{P}_4$, this suggests that one can construct an extrinsically-coupled Paneitz operator via the same operator $I \cdot \hat{D}^4$ but simply loosening the restriction that $(M, \gamma, \sigma) \in \mathrm{APE}_{d-2}$. Rather, we can consider triples $(M, \gamma, \sigma) \in \mathrm{ASY}$ so that the jets of $\bar{\boldsymbol{c}}$ are not fixed by the holographic condition and therefore carry extrinsic embedding data. Observe that in the above construction, we did not require the Poincaré–Einstein condition, but rather only the asymptotic singular Yamabe condition. Therefore, acting on any extension T^{e} of a weight-$\frac{5-d}{2}$ tractor T, we define the extrinsically-coupled Paneitz operator by

$$\hat{P}_4^{\Sigma \hookrightarrow M^d} T := (I \cdot \hat{D}^4 T^{\mathrm{e}})|_\Sigma \,.$$

Generalizing the above calculation leads to a holographic formula for all extrinsically-coupled GJMS operators. We record this result below.

Theorem 6.1.1 (Theorem 7.1 of [**45**]). *Let $(M^d, \gamma, \sigma)_{\mathcal{Y}}$ specify a conformal hypersurface embedding and let k be an integer satisfying $1 \leq k \leq d-1$. Then, the kth order extrinsically-coupled GJMS operator*

$$P_k^{\Sigma \hookrightarrow M^d} : \Gamma\left(\mathcal{T}^\Phi M \left[\tfrac{k-d+1}{2}\right]\right)\big|_\Sigma \to \left(\mathcal{T}^\Phi M \left[\tfrac{-k-d+1}{2}\right]\right)\big|_\Sigma$$

defined by its action on any extension T^{e} of a tractor $T \in \Gamma\left(\mathcal{T}^\Phi \left[\frac{k-d+1}{2}\right]\right)\big|_\Sigma$ by

$$P_k^{\Sigma \hookrightarrow M^d} T = (I \cdot D^k T^{\mathrm{e}})|_\Sigma$$

is a tangential operator. When k is even, a kth order normalized extrinsically-coupled GJMS operator can also be defined by its action on T^{e} via

$$\hat{P}_k^{\Sigma \hookrightarrow M^d} T = (I \cdot \hat{D}^k T^{\mathrm{e}})|_\Sigma$$

which has leading term $(\Delta^\top)^{k/2}$ as well as being tangential.

REMARK 6.1.2. Of note is that in the critical dimension d even, the normalized extrinsically-coupled GJMS operator $\hat{P}_{d-1}^{\Sigma\hookrightarrow M^d}$ annihilates constants. Thus, for a conformal hypersurface embedding specified by $(M^{d'}, \gamma, \sigma)$, acting on densities $\varphi \in \Gamma(\mathcal{E}\Sigma[\frac{d-d'}{2}])|_\Sigma$, we have that

$$P_{d-1}^{\Sigma\hookrightarrow M^{d'}} \varphi = (\bar{\mathsf{O}}^a \circ \bar{\nabla}_a)(\varphi) - \tfrac{d-d'}{2} \mathcal{Q}_{d-1}^{\Sigma\hookrightarrow M^{d'}}(g)\varphi \,, \tag{6.4}$$

where $\bar{\mathsf{O}}^a$ is some hypersurface operator and $\mathcal{Q}_{d-1}^{\Sigma\hookrightarrow M^{d'}}(g)$ is a scalar curvature quantity that depends both on the order $d-1$ of the operator $\hat{P}_{d-1}^{\Sigma\hookrightarrow M^{d'}}$ and the dimension d' of the conformal manifold. A similar statement can be made in general about $\hat{P}_{d-1}^{\Sigma\hookrightarrow M^d}$. ∎

The extrinsically-coupled Paneitz operator underlies numerous extrinsic conformal hypersurface invariants. For computational purposes, it is both instructive and useful to reexpress the extrinsically-coupled Paneitz operator in terms of explicitly tangential tractor operators.

6.2. Useful Tractor Identities

In this section we detail the identities required to produce an explicit tangential formula for the extrinsically-coupled Paneitz operator. Our methods rely heavily on the tractor calculus developed in Chapters 2, 3, and 4. We detail identities involving any weight one density σ and its interactions with the Thomas-D operator and the canonical tractor X. These all follow from straightforward applications of the Leibniz failure. We have employed the symbolic algebra system FORM [**83**] to handle more intricate cases. First we need some simplifying notations.

DEFINITION 6.2.1. Let $\sigma \in \Gamma(\mathcal{E}M[1])$. Then, in dimensions d where the right hand sides below are defined, we define the following quantities:

$$\begin{aligned}
I^\sigma_A &:= \hat{D}_A\sigma \in \Gamma(\mathcal{T}M[0])\,, & K^\sigma_{\mathrm{e}} &:= P^\sigma_{AB}P^{\sigma AB} \in \Gamma(\mathcal{E}M[-2])\,,\\
P^\sigma_{AB} &:= \hat{D}_A I^\sigma_B \in \Gamma(\odot^2_\circ \mathcal{T}M[-1])\,, & \dot{K}^\sigma_{\mathrm{e}} &:= I^\sigma\cdot\hat{D}\, P^\sigma_{AB}P^{\sigma AB} \in \Gamma(\mathcal{E}M[-3])\,,\\
\dot{P}^\sigma_{AB} &:= I^\sigma\cdot\hat{D}\, P^\sigma_{AB} \in \Gamma(\odot^2_\circ \mathcal{T}M[-2])\,, & \ddot{K}^\sigma_{\mathrm{e}} &:= (I^\sigma\cdot\hat{D})^2\, P^\sigma_{AB}P^{\sigma AB} \in \Gamma(\mathcal{E}M[-4])\,,\\
\ddot{P}^\sigma_{AB} &:= I^\sigma\cdot\hat{D}^2\, P^\sigma_{AB} \in \Gamma(\odot^2_\circ \mathcal{T}M[-3])\,, & \dddot{K}^\sigma_{\mathrm{e}} &:= (I^\sigma\cdot\hat{D})^3\, P^\sigma_{AB}P^{\sigma AB} \in \Gamma(\mathcal{E}M[-5])\,.
\end{aligned}$$

∎

We are particularly interested in the case that $(M,\gamma,\sigma)\in$ ASY. In that case we shall often drop the superscript σ. Note that the density $K^{\sigma}_{\rm e}$ and the tractor P^{σ}_{AB} defined above agree with the definitions for the same given in Section 3.4.

6.2.1. Commutators. The simplest identities we require are commutators involving the objects in Definition 6.2.1 and the Thomas-D operator.

LEMMA 6.2.2. *Let σ be any weight $w=1$ density. Acting on tractors of weight w such that $\hat{D}$ as it appears below is well-defined, the following operator identities hold:*

$$\begin{aligned}
\big[\hat{D}_A,\sigma\big] \; &= I_A-\tfrac{2}{h}X_A I\cdot\hat{D}\,,\\
\big[\hat{D}_A,X_B\big] &= h_{AB}-\tfrac{2}{h}X_A\hat{D}_B\,,\\
\big[\hat{D}_A,I_B\big] \; &= P_{AB}-\tfrac{2}{h-2}X_A P_{CB}\hat{D}^C\,,
\end{aligned}$$

where $h:=d+2w$.

PROOF. The results follow from a direct application of Proposition 2.3.1 and Definition 6.2.1. $\square$

LEMMA 6.2.3. *Let $(M,\gamma,\sigma)\in$ ASY. Then, acting on tractors with weight $w=:\frac{1}{2}(h-d)$ such that $\hat{D}$ as it appears below is well-defined, the following operator identities hold:*

$$\Big[I\cdot\hat{D},X^A\Big]=I^A-\tfrac{2\sigma}{h}\,\hat{D}^A\,,\tag{6.5}$$

$$\Big[I\cdot\hat{D},\hat{D}_A\Big]=-P_{AB}\hat{D}^B-I^B\Big[\hat{D}_A,\hat{D}_B\Big]+\tfrac{2}{h-4}\,X_A P^{BC}\hat{D}_B\hat{D}_C\,,\tag{6.6}$$

$$(I\cdot\hat{D})^k\circ\sigma=\tfrac{h-2k}{h}\,\sigma(I\cdot\hat{D})^k+\tfrac{k(h-k+1)}{h}\,(I\cdot\hat{D})^{k-1}+\mathcal{O}(\sigma^{d-k+1})\,,\tag{6.7}$$

$$\Big[I\cdot\hat{D},I_A\Big]=\tfrac{1}{d-2}\,X_A K-\tfrac{2\sigma}{h-2}P_{AB}\hat{D}^B+\mathcal{O}(\sigma^{d-2})\,,\tag{6.8}$$

$$\Big[I\cdot\hat{D},P^{AB}\Big]=\dot{P}^{AB}-\tfrac{2\sigma}{h-4}\,\big(\hat{D}^E P^{AB}\big)\hat{D}_E\,.\tag{6.9}$$

PROOF. As in Lemma 6.2.2, this result follows from Proposition 2.3.1 and Definition 6.2.1. The third identity requires a simple induction argument. The third and fourth identities also require that

$I^2 = 1 + \mathcal{O}(\sigma^d)$. Also, Equation (6.8) uses that $\hat{D}_A I_B = \hat{D}_B I_A$. Note that Equations (6.5), (6.6) and (6.9) in fact hold for any weight one density σ. □

The W-tractor also frequently appears in tractor computations, so it is useful to have one more commutator result.

LEMMA 6.2.4. *Let $(M, \gamma, \sigma) \in$ ASY and let $T^{\Phi AB} \in \Gamma(\mathcal{T}^\Phi M \otimes \wedge^2 \mathcal{T}M[w'])$. Then, acting on a tractor of weight w such that $\hat{D}$ as it appears below is well-defined, the following operator identities hold:*

$$\begin{aligned}
\big[\hat{D}_A, T^{\Phi\sharp}\big] &= (\hat{D}_A T^{\Phi\sharp}) - T^{\Phi}{}_A{}^B \hat{D}_B - \tfrac{2}{h+2w'-2} X_A (\hat{D}_C T^{\Phi\sharp}) \circ \hat{D}^C + \tfrac{2}{h+2w'-2} X_A (\hat{D}_C T^{\Phi C}{}_D) \hat{D}^D\,,\\
\big[I \cdot \hat{D}, T^{\Phi\sharp}\big] &= (I \cdot \hat{D} T^{\Phi\sharp}) - \tfrac{2\sigma}{h+2w'-2} (\hat{D}_C T^{\Phi\sharp}) \circ \hat{D}^C + \tfrac{2\sigma}{h+2w'-2} (\hat{D}_C T^{\Phi C}{}_D) \hat{D}^D\,,
\end{aligned}$$

where $h := d + 2w$.

PROOF. This operator identity is an elementary application of Lemma 2.3.1 while accounting for the action of $T^{\Phi\sharp}$. □

6.2.2. Operator identities along Σ. Here we provide a list of useful operator identities valid along Σ. These identities were proved using Definition 6.2.1, Lemmas 6.2.2 and 6.2.3, and the computer algebra system FORM. The next lemma shows how to convert various operators involving Thomas-D operators to combinations of their tangential parts and Laplace–Robin operators.

LEMMA 6.2.5. *Acting on tractors of weight $w =: \frac{1}{2}(h - d)$, and such that $\hat{D}$ and $\hat{D}^T$ as they appear below are well-defined, the following operator identities hold:*

$$P^{AB} \circ \hat{D}_A \circ \hat{D}_B \stackrel{\Sigma}{=} P^{AB} \hat{D}^T_A \hat{D}^T_B + \tfrac{h-4}{d-2} K I \cdot \hat{D}\,, \tag{6.10}$$

$$P^{AB} \circ \hat{D}_A \circ P^C_B \circ \hat{D}_C \stackrel{\Sigma}{=} P^A_C P^{CB} \hat{D}^T_A \hat{D}^T_B + P_{AB} (\hat{D}^A P^{BC}) \hat{D}^T_C\,, \tag{6.11}$$

$$P^A_C P^{CB} \circ \hat{D}_A \circ \hat{D}_B \stackrel{\Sigma}{=} -\tfrac{1}{h-3} K I \cdot \hat{D}^2 + P^3 I \cdot \hat{D} + P^A_C P^{CB} \hat{D}^T_A \hat{D}^T_B\,, \tag{6.12}$$

$$(\hat{D}^A K) \circ \hat{D}_A \stackrel{\Sigma}{=} \tfrac{2}{h-3} K I \cdot \hat{D}^2 + \dot{K} I \cdot \hat{D} + (\hat{D}^A K) \hat{D}^T_A\,, \tag{6.13}$$

$$
\begin{aligned}
I_A(\hat{D}^C P^{AB}) \circ \hat{D}_C \circ \hat{D}_B \overset{\Sigma}{=} &\tfrac{2h-d-6}{(d-2)(h-3)} K I\cdot\hat{D}^2 + \tfrac{(d-6)(h-d-2)}{2(d-2)(d-4)} \dot{K} I\cdot\hat{D} \\
&- \tfrac{h-6}{d-4} P^3 I\cdot\hat{D} - P^A_C P^{CB} \hat{D}^T_A \hat{D}^T_B \\
&+ \tfrac{h-d-2}{d-4} P_{AB}(\hat{D}^A P^{BC})\hat{D}^T_C + \tfrac{(d-6)(h-d-2)}{2(d-2)(d-4)}(\hat{D}^A K)\hat{D}^T_A\,,
\end{aligned} \tag{6.14}
$$

$$
\begin{aligned}
I_A \ddot{P}^{AB} \circ \hat{D}_B \overset{\Sigma}{=} &\tfrac{2}{(d-2)(h-3)} K I\cdot\hat{D}^2 + \tfrac{2(d-5)}{(d-2)(d-4)} \dot{K} I\cdot\hat{D} - \tfrac{2}{d-4} P^3 I\cdot\hat{D} \\
&+ \tfrac{2}{d-4} P_{AB}(\hat{D}^A P^{BC})\hat{D}^T_C - \tfrac{2}{(d-2)(d-4)}(\hat{D}^A K)\hat{D}^T_A \\
&+ \tfrac{h-d}{(d-2)^2} K^2 + \tfrac{h-d}{2(d-2)} \ddot{K}\,,
\end{aligned} \tag{6.15}
$$

$$
\begin{aligned}
(\hat{D}_A \dot{P}^{AB}) \circ \hat{D}_B \overset{\Sigma}{=} &\tfrac{d-2}{(d-6)(h-3)} K I\cdot\hat{D}^2 + \tfrac{d-4}{2(d-6)} \dot{K} I\cdot\hat{D} \\
&- \tfrac{2}{d-6} P^3 I\cdot\hat{D} + (\hat{D}_A \dot{P}^{AB})\hat{D}^T_B\,,
\end{aligned} \tag{6.16}
$$

$$
\begin{aligned}
\hat{D}_A \circ \dot{P}^{AB} \circ \hat{D}_B \overset{\Sigma}{=} &\tfrac{(d+2)h-8d}{(d-2)(h-3)(h-8)} K I\cdot\hat{D}^2 + \tfrac{(d^2-4d-4)h-8d^2+44d-24}{2(d-2)(d-4)(h-8)} \dot{K} I\cdot\hat{D} \\
&- \tfrac{2(h-6)}{(d-4)(h-8)} P^3 I\cdot\hat{D} + \tfrac{h-6}{h-8} \dot{P}^{AB} \hat{D}_A \hat{D}_B \\
&+ \tfrac{(d-6)(h-d-2)}{(d-2)(d-4)(h-8)}(\hat{D}^A K)\hat{D}^T_A + \tfrac{(d-6)(h-6)}{(d-4)(h-8)}(\hat{D}_A \dot{P}^{AB})\hat{D}^T_B\,.
\end{aligned} \tag{6.17}
$$

PROOF. These identities result from a series of symbolic algebra calculations using FORM. The programs performing these calculations are described in Appendix A.4. □

In addition to Definition 6.2.1 and Lemmas 6.2.2 and 6.2.3, to establish the following lemma for operators involving the Laplace–Robin operator, we also need to use Lemma 6.2.5.

LEMMA 6.2.6. *Acting on tractors of weight $w =: \frac{1}{2}(h-d)$, and such that $\hat{D}$ and $\hat{D}^T$ as they appear below are well-defined, the following operator identities hold:*

$$X^A \circ I\cdot\hat{D} \circ X_A \overset{\Sigma}{=} 0\,, \tag{6.18}$$

$$X^A \circ I\cdot\hat{D}^2 \circ X_A \overset{\Sigma}{=} -\tfrac{h-d}{h}\,, \tag{6.19}$$

$$X^A \circ I\cdot\hat{D}^3 \circ X_A \overset{\Sigma}{=} -\tfrac{3(h-d-2)}{h} I\cdot\hat{D}\,, \tag{6.20}$$

$$X^A \circ I\cdot\hat{D}^4 \circ X_A \overset{\Sigma}{=} -\frac{6(h-d-4)}{h} I\cdot\hat{D}^2 - \frac{2h^3-(2d+20)h^2+(16d+56)h-48d}{(d-2)h(h-2)(h-4)}\,K\,, \tag{6.21}$$

$$X^A \circ I\cdot\hat{D} \circ I_A \overset{\Sigma}{=} 0\,, \tag{6.22}$$

$$X^A \circ I\cdot\hat{D}^2 \circ I_A \overset{\Sigma}{=} 0\,, \tag{6.23}$$

$$X^A \circ I\cdot\hat{D}^3 \circ I_A \overset{\Sigma}{=} \frac{h^2-(d+4)h+8d-8}{(d-2)(h-2)(h-4)} K\,, \tag{6.24}$$

$$\begin{aligned} X^A \circ I\cdot\hat{D}^4 \circ I_A \overset{\Sigma}{=}\ & \frac{4(h-5)(h^2-(d+8)h+10d+4)}{(d-2)(h-2)(h-4)(h-6)}\,K\,I\cdot\hat{D} - \frac{12}{(h-2)(h-4)} P^{AB}\hat{D}^T_A\hat{D}^T_B \\ &+ \frac{3h^2-(3d+12)h+30d-36}{(d-2)(h-2)(h-4)}\dot{K}\,, \end{aligned} \tag{6.25}$$

$$I^A \circ I\cdot\hat{D} \circ X_A \overset{\Sigma}{=} 1\,, \tag{6.26}$$

$$I^A \circ I\cdot\hat{D}^2 \circ X_A \overset{\Sigma}{=} \frac{2(h-1)}{h} I\cdot\hat{D}\,, \tag{6.27}$$

$$I^A \circ I\cdot\hat{D}^3 \circ X_A \overset{\Sigma}{=} \frac{3(h-2)}{h}\,I\cdot\hat{D}^2 + \frac{2h^2-(d+8)h+6d}{h(d-2)(h-2)}\,K\,, \tag{6.28}$$

$$\begin{aligned} I^A \circ I\cdot\hat{D}^4 \circ X_A \overset{\Sigma}{=}\ & \frac{4(h-3)}{h} I\cdot\hat{D}^3 - \frac{4(h-3)(-2h^2+(d+16)h-8d-24)}{(d-2)h(h-2)(h-4)} K I\cdot\hat{D} \\ &- \frac{12}{h(h-2)} P^{AB}\hat{D}^T_A\hat{D}^T_B + \frac{3h^3-(d+28)h^2+(14d+68)h-48d}{(d-2)h(h-2)(h-4)}\dot{K}\,, \end{aligned} \tag{6.29}$$

$$X^A \circ I\cdot\hat{D} \circ \hat{D}_A \overset{\Sigma}{=} \frac{h-d-2}{2} I\cdot\hat{D}\,, \tag{6.30}$$

$$X^A \circ I\cdot\hat{D}^2 \circ \hat{D}_A \overset{\Sigma}{=} \frac{h-d-4}{2} I\cdot\hat{D}^2 + \frac{h-d}{2(d-2)} K\,, \tag{6.31}$$

$$\begin{aligned} X^A \circ I\cdot\hat{D}^3 \circ \hat{D}_A \overset{\Sigma}{=}\ & \frac{h-d-6}{2} I\cdot\hat{D}^3 - \frac{2(2h-d-10)}{(h-4)(h-6)} P^{AB}\hat{D}^T_A\hat{D}^T_B \\ &+ \frac{3h^2-(3d+28)h+22d+52}{2(d-2)(h-6)} K I\cdot\hat{D} + \frac{h-d}{d-2}\dot{K} + \frac{2}{h-4} I^A\hat{D}^B[\hat{D}_A,\hat{D}_B]\,, \end{aligned} \tag{6.32}$$

$$
\begin{aligned}
X^A \circ I\cdot\hat{D}^4 \circ \hat{D}_A \overset{\Sigma}{=}\ & \tfrac{h-d-8}{2} I\cdot\hat{D}^4 \\
&+ \tfrac{3h^5-(3d+83)h^4+(71d+908)h^3-(608d+4932)h^2+(6d^2+2244d+13224)h-48d^2-3024d-13536}{(d-2)(h-3)(h-4)(h-6)(h-8)}\, K I\cdot\hat{D}^2 \\
&+ \tfrac{h-d}{2(d-2)^2} K^2 + \tfrac{3(h-d)}{2(d-2)}\ddot{K} \\
&+ \tfrac{4h^3-(4d+61)h^2+(58d+246)h-210d-108}{(d-2)(h-6)(h-8)} \dot{K} I\cdot\hat{D} \\
&+ \tfrac{6}{h-4} I^A \hat{D}^B I\cdot\hat{D}[\hat{D}_A,\hat{D}_B] + \tfrac{2}{h-6} I^A\hat{D}^B[\hat{D}_A,\hat{D}_B] I\cdot\hat{D} \\
&- \tfrac{4(h-7)}{(h-4)(h-6)} I^A[\hat{D}_A,\hat{D}_B] I_C [\hat{D}^B,\hat{D}^C] \\
&+ \tfrac{2(10h^2-(3d+130)h+24d+408)}{(h-4)(h-6)(h-8)} P^{AB} I^C \hat{D}_A [\hat{D}_B,\hat{D}_C] \\
&+ \tfrac{2(5h^2-(3d+74)h+24d+264)}{(h-4)(h-6)(h-8)} \Big[P^3 I\cdot\hat{D} + P^{AB} I^C [\hat{D}_A,\hat{D}_C] \Big(\hat{D}^T_B + I_B I\cdot\hat{D}\Big)\Big] \\
&- \tfrac{4(4h^2-(2d+53)h+14d+172}{(h-4)(h-6)(h-8)} P^{AB}\hat{D}^T_A \hat{D}^T_B I\cdot\hat{D} \\
&+ \tfrac{2(13h^2-(6d+184)h+48d+624)}{(h-4)(h-6)(h-8)} P^A_C P^{CB} \hat{D}^T_A \hat{D}^T_B \\
&+ \tfrac{2((8d-30)h^2-(3d^2+98d-420)h+24d^2+264d-1392)}{(d-4)(h-4)(h-6)(h-8)} P^{AB}\Big(\hat{D}_A P^C_B\Big)\hat{D}^T_C \\
&- \tfrac{3(h-d)}{(d-2)(h-6)} \Big(\hat{D}^A K\Big)\hat{D}^T_A \\
&- \tfrac{6(2h-d-12)}{(h-4)(h-6)} \dot{P}^{AB}\hat{D}_A\hat{D}_B \\
&- \tfrac{2(d-6)}{(d-4)(h-8)} \Big(\hat{D}_A \dot{P}^{AB}\Big)\hat{D}^T_B\,,
\end{aligned}
\tag{6.33}
$$

$$
\hat{D}_A \circ I\cdot\hat{D}^2 \circ X^A \overset{\Sigma}{=} \tfrac{(h+d)(h+2)}{2h} I\cdot\hat{D}^2 + \tfrac{h^3+(d-6)h^2-2dh+8d}{2(d-2)h(h-4)} K\,,
\tag{6.34}
$$

$$
\begin{aligned}
\hat{D}_A \circ I\cdot\hat{D}^2 \circ \hat{D}^A \overset{\Sigma}{=} & -\tfrac{h-6}{2(h-8)}\dot{K} I\cdot\hat{D} \\
& + \tfrac{2h^3-32h^2+(d+160)h-8d-236}{(h-3)(h-6)(h-8)} K I\cdot\hat{D}^2 \\
& + I^A\hat{D}^B\left(I\cdot\hat{D}[\hat{D}_A,\hat{D}_B] + [\hat{D}_A,\hat{D}_B] I\cdot\hat{D}\right) \\
& - \tfrac{d-2}{h-6} P^{AB} I^C \hat{D}_A[\hat{D}_B,\hat{D}_C] \\
& - \tfrac{h^2+(d-14)h-8d+52}{(h-6)(h-8)}\left(P^3 I\cdot\hat{D} + P^{AB} I^C [\hat{D}_A,\hat{D}_C]\hat{D}_B\right) \\
& + \tfrac{2(d-2)(h-7)}{(h-6)(h-8)} P^{AB}\hat{D}^T_A\hat{D}^T_B I\cdot\hat{D} \\
& - \tfrac{h^2+(2d-14)h-16d+56}{(h-6)(h-8)} P^A_C P^{CB}\hat{D}^T_A\hat{D}^T_B \\
& + \tfrac{2h^2-(d^2-4d+24)h+8d^2-36d+88}{(d-4)(h-6)(h-8)} P^{AB}\left(\hat{D}_A P^C_B\right)\hat{D}^T_C \\
& + \tfrac{d-2}{h-6}\dot{P}^{AB}\hat{D}_A\hat{D}_B \\
& - \tfrac{(d-6)(h-6)}{(d-4)(h-8)}\left(\hat{D}_A\dot{P}^{AB}\right)\hat{D}^T_B\,.
\end{aligned}
\tag{6.35}
$$

PROOF. This lemma was proved sequentially using FORM—generally, the more complex identities rely on the less complex ones. □

With these identities in hand, we are now ready to tackle the central result of this section: an explicitly tangential formula for the extrinsically-coupled Paneitz-operator.

6.3. An Explicit Tangential Tractor Formula for $P_4^{\Sigma\hookrightarrow M^d}$

Observe that the Paneitz operator P_4 intrinsic to a conformal n-manifold $(\Sigma, \boldsymbol{c}_\Sigma)$, acting on weight $2-\frac{n}{2}\neq 0$ densities, can be expressed as

$$
P_4 = \tfrac{8}{n-4}\,\hat{D}^A\circ P_2\circ\hat{D}_A\,,
$$

where P_2 is the *Yamabe operator* (also known as the *conformal Laplacian*) on weight $1-\frac{n}{2}$ tractors defined by $D^A T = -X_A P_2 T$ for $T\in\Gamma(\mathcal{T}M[1-\frac{n}{2}])$; see for example [**32, 37**]. Therefore, to write the holographic formula of Theorem 6.1.1 for $\hat{P}_4^{\Sigma\hookrightarrow M^d}$ explicitly in terms of hypersurface data, our strategy is to convert the operator $(I\cdot\hat{D})^4$ to the form

$$
\hat{P}_4^{\Sigma\hookrightarrow M^d} \overset{\Sigma}{=} \tfrac{8}{d-5}\,\hat{D}^{T\,A}\circ P_2^{\Sigma\hookrightarrow M}\circ\hat{D}^T_A + \text{lower derivative terms.}
$$

Here the *extrinsically-coupled Yamabe operator* $\hat{P}_2^{\Sigma\hookrightarrow M^d}$ is defined on tractors of weight $\frac{3-d}{2}$ by the tangential operator $I \cdot \hat{D}^2$ (again see Theorem 6.1.1). In the case that the operator $\hat{P}_4^{\Sigma\hookrightarrow M^d}$ acts on sections of $\mathcal{T}^\Phi\Sigma\big[\frac{5-d}{2}\big]$, Theorem 3.3.4 can be used to convert tangential Thomas-D operators to hypersurface ones. The result of this computation is given below.

THEOREM 6.3.1. *Let σ be a unit conformal defining density for $\Sigma \hookrightarrow (M, \boldsymbol{c})$,where M has dimension $d \geq 8$. Then, acting on sections of $\Gamma(\mathcal{T}^\Phi M[\frac{5-d}{2}])|_\Sigma$,*

$$\begin{aligned}
\hat{P}_4^{\Sigma\hookrightarrow M^d} \stackrel{\Sigma}{=} & \tfrac{8}{d-5}\hat{D}^{TA} \circ \hat{P}_2^{\Sigma\hookrightarrow M^d} \circ \hat{D}_A^T \\
& + \Big(\tfrac{4(2d-11)}{d-5} L^{AC} L_C^B + \tfrac{4(d-3)(d-6)}{d-5} F^{AB} - \tfrac{8}{d-5} W^{AB\,\sharp}\Big) \circ \hat{D}_A^T \circ \hat{D}_B^T \\
& + \Big(\tfrac{4(2d-11)}{d-5} L_{BC}(\hat{\bar{D}}^B L^{CA}) - \tfrac{2(d^2-8d+17)}{(d-1)(d-2)(d-5)}(\hat{\bar{D}}^A K) + \tfrac{4(3d-16)}{d-5} N_B L_{CD} W^{BCDA} \\
& \qquad - \tfrac{4(3d-16)}{d-5} N_B L_C^A W^{BC\sharp} + \tfrac{4(d-4)}{d-5} N_B N^C (\hat{D}_C W^{BA}{}_{\cdot\cdot})^\sharp\Big) \circ \hat{D}_A^T \\
& + \tfrac{2d^4-21d^3+95d^2-200d+152}{(d-1)(d-2)^2(d-4)^2} K^2 + \tfrac{d-2}{(d-4)^2} L \cdot J - \tfrac{2(d-2)}{(d-4)^2} L^4 + \tfrac{2(2d^3-27d^2+118d-172)}{(d-4)^2} L \cdot F \cdot L \\
& + \tfrac{(3d-16)(d-3)^2}{d-4} F^2 - \tfrac{2(d-5)(d-6)}{(d-4)^2} L^{AC} L^{BD} \bar{W}_{ABCD} + \tfrac{(d-5)(d-8)}{(d-4)(d-7)} (\hat{\bar{D}}_A L_{BC})(\hat{\bar{D}}^A L^{BC}) \\
& + 4 W_N{}^{B\sharp} \circ W_{NB}{}^\sharp - 4 N_B L_{AC} \Big(\hat{D}^A W^{BC}{}_{\cdot\cdot}\Big)^\sharp .
\end{aligned}$$

The above result can in fact be profitably used in dimensions $d = 5, 6, 7$ by a dimensional continuation argument. Because $(I \cdot \hat{D})^4$ is well-defined in all these dimensions, so too, by construction, must be the operator appearing in the theorem. The $d = 5$ case is clearly the most complicated of these continuations, and will be performed in the cases when $\hat{P}_4^{\Sigma\hookrightarrow M^d}$ acts on scalars and the normal tractor; see Corollary 6.4.3 and Theorem 6.5.2. Dimensions $d = 6, 7$ are discussed below.

First consider $d = 6$. While there are no explicit poles here, since the tractor

$$\hat{D}_A W \stackrel{g}{=} -2Y_A W + Z_A^a \nabla_a W - \tfrac{1}{d-6} X_A(\Delta W - 2J^g W)\,,$$

it naively has a pole at $d = 6$. However, note that in the theorem above, the tractor $\hat{D}_A W$ only appears when contracted with N^A or L^{AB}. In both cases, the pole can be eliminated by first working in an arbitrary dimension, there noting that $N^A X_A = 0 = L^{AB} X_A$, and then continuing down to six dimensions.

Now consider the case where $d = 7$, which has one explicit pole with residue $(\hat{\bar{D}}L)^2$, and an implicit one:

$$\hat{\bar{D}}_A K \stackrel{g}{=} -2\bar{Y}_A K + \bar{Z}_A^a \bar{\nabla}_a K - \tfrac{1}{d-7} X_A (\bar{\Delta} K - 2\bar{J}^{\bar{g}} K)\,.$$

However, note that in $d = 7$,

$$(\hat{\bar{D}}L)^2 \stackrel{g}{=} \tfrac{1}{2}\big(\bar{\Delta} - 2\bar{J}^{\bar{g}}\big)K = \tfrac{1}{2}\square_Y K\,,$$

and acting on sections of $\Gamma(\mathcal{E}M[\frac{5-d}{2}])|_\Sigma$,

$$(\hat{\bar{D}}^A K)\hat{D}^T_A \stackrel{g}{=} \tfrac{d-5}{2(d-7)}(\bar{\Delta} - 2\bar{J}^{\bar{g}})K + \text{regular}\,,$$

where "regular" stands for terms that are regular in dimension $d = 7$. Therefore, away from $d = 7$

$$-\tfrac{2(d^2-8d+17)}{(d-1)(d-2)(d-5)}(\hat{\bar{D}}^A K)\hat{D}^T_A + \tfrac{(d-5)(d-8)}{(d-4)(d-7)}(\hat{\bar{D}}L)^2 = \tfrac{d^3-7d^2+8d+8}{2(d-1)(d-2)(d-4)}(\bar{\Delta} - 2\bar{J})K + \text{regular}\,.$$

Thus, the operator $\hat{P}_4^{\Sigma\hookrightarrow M^d}$ can indeed be continued to seven dimensions.

Our next task is to prove the central Theorem 6.3.1. Note that the algorithm presented below can be used to decompose quite general tractor operators into tangential and higher transverse order pieces, the latter captured by powers of the $I\cdot\hat{D}$ operator.

Proof of Theorem 6.3.1. As dictated by Theorem 6.1.1, we begin with the holographic formula $I\cdot\hat{D}^4$, remembering that we will eventually restrict to the hypersurface Σ. Also, we are ultimately interested in this operator acting on tractors of weight $\frac{5-d}{2}$. However, to begin with we will take the weight to be arbitrary, equal to $(h-d)/2$, and such that denominators of the form $h-k$ for certain k are avoided. Later we will employ a weight and dimension continuation argument. Note that all appearances of the operator $\hat{D}$ in what follows are in fact well-defined even when $h = 5$. When problematic poles in the parameter h appear, we will draw the reader's attention to how these are handled.

Our strategy is to perform a series of manipulations converting the operator $I\cdot\hat{D}^4$ to the operator $\hat{D}^T_A \circ I\cdot\hat{D}^2 \circ \hat{D}^{TA}$ plus other terms of transverse order lower than four. The first step is to note that $I\cdot\hat{D} = \hat{D}_A \circ I^A$ (this follows by contracting the last identity in Lemma 6.2.2 with the tractor metric and the fact that $P_A{}^A = 0 = X_A P^{AB}$). Thus

$$I\cdot\hat{D}^4 = \hat{D}_A \circ I^A I\cdot\hat{D}^2 \circ I^B \hat{D}_B\,.$$

We would like to trade the explicit appearances of scale tractors I^A and I^B in the above display for an extension of the tractor first fundamental form $I^{AB}_{\rm ext} := h^{AB} - I^A I^B$. For that we must first bring I^A and I^B together using Equation (6.8), which gives

$$\begin{aligned} I\cdot\hat{D}^4 &= \hat{D}_A \circ I\cdot\hat{D} \circ I^A I^B \circ I\cdot\hat{D} \circ \hat{D}_B + \mathcal{R}_1 \\ &= -\hat{D}_A \circ I\cdot\hat{D} \circ I^{AB}_{\rm ext} \circ I\cdot\hat{D} \circ \hat{D}_B + \mathcal{R}_2\,. \end{aligned} \tag{6.36}$$

Here $\mathcal{R}_1 := \hat{D}_A \circ [I^A, I\cdot\hat{D}] \circ I\cdot\hat{D} \circ I^B \hat{D}_B + \hat{D}_A \circ I\cdot\hat{D} \circ I^A \circ [I\cdot\hat{D}, I^B] \circ \hat{D}_B$ has transverse order no more than three—see Equation (6.8). The second remainder term $\mathcal{R}_2 := \mathcal{R}_1 + \hat{D}_A \circ I\cdot\hat{D}^2 \circ \hat{D}^A$ also has transverse order no more than three which can be verified using the identity $\hat{D}_A\hat{D}^A = 0$ and Proposition 2.6, which shows that the commutator of a pair of Thomas-D operators has transverse order one. Later, to simplify $\mathcal{R}_2$, we will apply Equation (6.35). We will also handle lower transverse order remainder terms later.

We employ the identity

$$I^{AD}_{\rm ext} = I^{AB}_{\rm ext} h_{BC} I^{CD}_{\rm ext} + \mathcal{O}(\sigma^d)\,,$$

to produce a pair of extensions of the tractor first fundamental form. (Also observe from Equation (3.15) that $I_{AB}\hat{D}^B$ is very nearly $\hat{D}^T_A$.) We then use Equation (6.8) and Lemma 6.2.2 to rewrite Equation (6.36) as

$$I\cdot\hat{D}^4 = -I^{AB}_{\rm ext}\hat{D}_B \circ I\cdot\hat{D}^2 \circ I^{\rm ext}_{AC}\hat{D}^C + \mathcal{R}_3\,.$$

Similar arguments to above show that the latest remainder $\mathcal{R}_3$ still has transverse order at most three. Using Proposition 3.3.3, note that $\hat{D}^T_A = I^{\rm ext}_{AB}\hat{D}^B + \frac{1}{d+2w-3} X_A I\cdot\hat{D}^2$. Note that we encounter no poles applying this identity to the above display when $h = 5$. This maneuver produces

$$\begin{aligned} I\cdot\hat{D}^4 = &-\hat{D}^T_A \circ I\cdot\hat{D}^2 \circ \hat{D}^{T\,A} + \tfrac{1}{h-3}\hat{D}^T_A \circ I\cdot\hat{D}^2 \circ X_A I\cdot\hat{D}^2 \\ &+ \tfrac{1}{h-9} X_A I\cdot\hat{D}^4 \circ \hat{D}^{T\,A} - \tfrac{1}{(h-3)(h-9)} X_A I\cdot\hat{D}^4 \circ X^A I\cdot\hat{D}^2 + \mathcal{R}_3\,. \end{aligned} \tag{6.37}$$

Applying identities from Lemma 6.2.6, for the second, third, and fourth terms on the right hand side above, we find

$$\begin{aligned}
\hat{X}^A I\cdot\hat{D}^4 \circ D^T_A &= \tfrac{h(h-7)(h-d-8)}{2(h-3)(h-4)}\, I\cdot\hat{D}^4 + \cdots\,,\\
\hat{D}^T_A \circ I\cdot\hat{D}^2 \circ X^A I\cdot\hat{D}^2 &= \tfrac{(h-5)(h-6)(h+d-10)}{2(h-4)(h-9)}\, I\cdot\hat{D}^4 + \cdots\,,\\
X_A I\cdot\hat{D}^4 \circ X^A I\cdot\hat{D}^2 &= -\tfrac{6(h-d-8)}{h-4} I\cdot\hat{D}^4 + \cdots\,,
\end{aligned}$$

where $\cdots$ represents lower transverse order terms. Then, after collecting all the terms in Equation (6.37) containing $I\cdot\hat{D}^4$ on the left hand side, we have

$$\tfrac{(2h-9)(h+d-10)}{(h-3)(h-4)(h-9)}\, I\cdot\hat{D}^4 = -\hat{D}^T_A \circ I\cdot\hat{D}^2 \circ \hat{D}^{T\,A} + \mathcal{R}_4\,.$$

When $h = 5$, the coefficient on the left hand side is $-\frac{d-5}{8}$. Moreover, at that value of h the operator $I\cdot\hat{D}^2$ in the first term on the right hand side acts on tractors of weight $1-\frac{d-1}{2}$, in which case it is tangential and equals $\hat{P}_2^{\Sigma\hookrightarrow M^d}$. Therefore we have established that

$$\hat{P}_4^{\Sigma\hookrightarrow M^d} \stackrel{\Sigma}{=} \tfrac{8}{d-5}\big(\hat{D}^{T\,A}_{\Sigma} \circ \hat{P}_2^{\Sigma\hookrightarrow M^d} \circ \hat{D}^T_{\Sigma\,A} - \mathcal{R}\big)\,.$$

It remains to compute the operator $\mathcal{R}$ by evaluating $\mathcal{R}_4$ along Σ in the limit $h\to 5$.

We must therefore now discuss the poles in $\mathcal{R}_4$. The issue is that we cannot use Equation (3.3.3) to convert the operator $\hat{D}$ to $\hat{D}^T$ when acting on tractors of weight $1-\frac{d-1}{2}$. However, since we know that the limit $h\to 5$ of the operator $I\cdot\hat{D}^4$ is well-defined, we may employ Equation (3.3.3) at general weights and apply a limiting procedure at the end of our calculations.

Returning to $\mathcal{R}$, we first apply Equation (6.35) to simplify the term $\hat{D}_A I\cdot\hat{D}^2\hat{D}^A$ in $\mathcal{R}_4$. To compute $\mathcal{R}_4$, we employ an algorithm whose starting point $\mathcal{R}_4$ is an operator of transverse order no more than three, that acts on an arbitrary weight tractors, and is evaluated along Σ. Moreover, $\mathcal{R}_4$ is expressed as a sum of "words" (each of which has transverse order no more than three) composed of operator-valued "letters" in the alphabet given by the scale σ, the scale tractor I, the canonical tractor X, the Thomas-D operator, the W tractor, the Thomas-D operator acting on any of the other letters (possibly multiple times), and rational functions of h. Note that the tractor identities derived so far can be used to simplify these words, *e.g.*, $X^A\circ(\hat{D}_A I_B)=0$. Our algorithm manipulates such words and letters. We also introduce the distinguished letter

$$\hat{y} := -I \cdot \hat{D} \,.$$

The aim of the algorithm is to iteratively convert any word of transverse order ℓ into the form

$$\mathsf{Op} \circ \hat{y}^{\ell} \circ f(\mathsf{h}) + \cdots \,,$$

where the terms $\cdots$ have transverse order lower than ℓ and $f(\mathsf{h})$ is some rational function of the operator h. Here Op is some tangential operator that may involve an additional letter $\hat{D}^T$. Let us first sketch the main ideas of the algorithm:

Step 0 of the algorithm takes any word containing rational functions of h and rewrites those words by shifting these operators to the right end of the word. Then, it applies a simplification-type procedure. While in principle, this simplification is not necessary for the algorithm to achieve the goal desired, it significantly reduces computational complexity in the implementation documented in Appendix A.4. This step will be repeated after each of the following steps.

Step 1 takes any word ending in $f_1(\mathsf{h})$ and rewrites it in the order

$$U_1^{\Theta_1} \cdots \circ T_1^{\Phi_1 \sharp} \cdots \circ \hat{D}^T_{A_1} \cdots \circ \hat{D}_{B_1} \cdots \circ \hat{y}^{\ell} \circ f_2(\mathsf{h}) \,,$$

for $\{U_i^{\Theta_i}\}$ some set of multiplicative letters (acting by tensor multiplication by a tractor), $\{T_j^{\Phi_j \sharp}\}$ some set of tractor-valued tractor-endomorphism letters, and $f_2(\mathsf{h})$ some possibly new rational function of the letter h.

Step 2 prepares to combine pairs of letters I^A and $\hat{D}_A$ into $-\hat{y}$ by commuting the corresponding Is to the right of the multiplicative letters, the tractor-valued tractor endomorphisms, and the tangential Thomas-D operators. That is, words containing I^A and $\hat{D}_A$ are manipulated to take the form

$$U_1^{\Theta_1} \cdots \circ T_1^{\Phi_1 \sharp} \cdots \circ \hat{D}^T_{A_1} \cdots \circ I^{B_k} \circ \hat{D}_{B_1} \cdots \hat{D}_{B_k} \cdots \circ \hat{y}^{\ell} \circ f(\mathsf{h}) \,.$$

Step 3 applies the tractor lemmas above to push every I^{B_k} past Thomas-D operators with different indices until it is left-adjacent to its corresponding $\hat{D}_{B_k}$, and then combines these terms to form $-\hat{y}$.

Step 4 reapplies Step 1 (so that this newly formed $\hat{y}$ letter is commuted to the right), leaving us with words of the form

$$U_1^{\Theta_1} \cdots \circ T_1^{\Phi_1 \sharp} \cdots \circ \hat{D}^T_{A_1} \cdots \circ \hat{D}_{B_1} \cdots \circ \hat{y}^\ell \circ f(\mathsf{h}) \,,$$

where no letter $U_i^{\Theta_i}$ is a letter I^{B_i} with corresponding letter $\hat{D}_{B_i}$.

Step 5 rewrites $\hat{D}_{B_1}$ (which by the previous step has no corresponding I^{B_1}) in terms of the tangential Thomas-D operator, $\hat{D}^T_{B_1}$.

Step 6 repeats the previous five steps so long as any letters $\hat{D}$ remain. The output of the algorithm is a linear combination of words of the form

$$U_1^{\Theta_1} \cdots \circ T_1^{\Phi_1 \sharp} \cdots \circ \hat{D}^T_{A_1} \cdots \circ I \cdot \hat{D}^\ell \circ f(\mathsf{h}) \,.$$

We now present the algorithm in full detail.

Step 0:

Step 0a: For every letter pair of an operator $f(\mathsf{h})$ and some tractor-valued operator T : $\Gamma(\mathcal{T}^\Phi M[w]) \to \Gamma(\mathcal{T}^\Theta M[w'])$ with "operator weight" $w' - w$, replace $f(\mathsf{h}) \circ T^\Theta_\Phi$ by $T^\Theta_\Phi \circ f(\mathsf{h} + 2(w' - w))$.

Step 0b: For any word beginning with multiplicative letters, combine those letters to reduce complexity using the definitions found in 6.2.1, the Leibniz failure 2.3.1, and Lemmas 6.2.2 and 6.2.3. For example, one may write $X_A(\hat{D}^B \hat{D}^C I^A) = -P^{BC}$ or $I_A P^{AB} = -\frac{1}{d-2} K X^B$.

Step 1: Repeat the following sub-steps until a full iteration leaves the expression unchanged (*i.e.*, "repeat until termination"):

Step 1a: Rewrite all two-letter pairs $I^A \circ \hat{D}_A$ as $-\hat{y}$.

Step 1b: Rewrite every instance of the letter combination $[\hat{D}_A, \hat{D}_B]$ following Equation (2.6).

Step 1c: Repeat until termination: For every letter of the form $T^{\Phi\sharp}$ and every multiplicative letter $U^\Theta \in \Gamma(\mathcal{T}^\Theta M[w])$, rewrite pairs $T^{\Phi\sharp} \circ U^\Theta$ as

$$(T^{\Phi\sharp} U^\Theta) + U^\Theta \circ T^{\Phi\sharp} \,.$$

Step 1d: Repeat until termination: For some operators Op_1, Op_2, and Op_3, apply Proposition 2.3.1 and Lemmas 6.2.3 and 6.2.4 to write all words of the form $\mathsf{Op}_1 \circ \hat{y} \circ \mathsf{Op}_2 \circ \hat{y}^\ell$ as

$$\mathsf{Op}_1 \circ \mathsf{Op}_2 \circ \hat{y}^{\ell+1} + \mathsf{Op}_3 \circ \hat{y}^\ell \,.$$

By construction Op_3 has fewer $\hat{y}$'s (including pairs $I^A \circ \hat{D}_A$) in it than the operator $\mathsf{Op}_1 \circ \mathsf{Op}_2$. Then apply Step 0.

Step 1e: Repeat until termination: For any operators Op_1, Op_2 (where Op_2 does not contain the letter $\hat{D}$), and Op_3, apply Proposition 2.3.1 and Lemmas 6.2.2 and 6.2.4 to write all words of the form $\mathsf{Op}_1 \circ \hat{D}^{A_1} \circ \mathsf{Op}_2 \circ \hat{D}^{A_2} \cdots \hat{D}^{A_k} \circ \hat{y}^\ell$ as

$$\mathsf{Op}_1 \circ \mathsf{Op}_2 \circ \hat{D}^{A_1} \cdots \hat{D}^{A_k} \circ \hat{y}^\ell + \mathsf{Op}_3 \circ \hat{D}^{A_2} \cdots \hat{D}^{A_k} \circ \hat{y}^\ell \,.$$

By construction Op_3 has no more operators $\hat{D}$ in it than the operator Op_1. Then apply Step 0.

Step 1f: Repeat until termination: Use Proposition 2.3.1 and Lemmas 6.2.2 and 6.2.3 to faithfully replace every combination of letters $\hat{D}^T \circ I$ to the left of every appearance of $\hat{D}$ or $\hat{y}$ with

$$I \circ \hat{D}^T + \mathsf{Op}_1^T \,.$$

By construction the operator Op_1^T is tangential. Similarly, for every combination of letters $\hat{D}^T \circ X$, replace it with

$$X \circ \hat{D}^T + \mathsf{Op}_2^T \,.$$

Again the operator Op_2^T must be tangential. Apply similar identities to letters of the form $\hat{D}^T \circ U^\Phi$ where U^Φ is a multiplicative letter. Then apply Step 0.

Step 2: Repeat until every word containing at least one pair $I^A \circ \mathsf{Op} \circ \hat{D}_A$ is written as $\mathsf{Op}' \circ I^{A_k} \circ \hat{D}_{A_1} \cdots \hat{D}_{A_k}$, where Op and Op' are some combination of letters:

Step 2a: For some operators Op_1 and Op_2, and for every multiplicative letter $U^\Theta \in \Gamma(\mathcal{T}^\Theta M[w])$, rewrite every word of the form $\mathsf{Op}_1 \circ I^A \circ U^\Phi \circ \mathsf{Op}_2$ as

$$\mathsf{Op}_1 \circ U^\Phi \circ I^A \circ \mathsf{Op}_2 \,.$$

Step 2b: For some operators Op_1, Op_2, and Op_3, and for each tractor-valued tractor-endomorphism letter $T^{\Phi\sharp}$, rewrite every word of the form $\mathsf{Op}_1 \circ I^A \circ T^{\Phi\sharp} \circ \mathsf{Op}_2 \circ \hat{D}_A \circ \mathsf{Op}_3$ as

$$\mathsf{Op}_1 \circ T^{\Phi\sharp} \circ I^A \circ \mathsf{Op}_2 \circ \hat{D}_A \circ \mathsf{Op}_3 - \mathsf{Op}_1 \circ T^{\Phi A}{}_B I^B \circ \mathsf{Op}_2 \circ \hat{D}_A \circ \mathsf{Op}_3 \,.$$

Step 2c: For some operators Op_1, Op_2, and Op_3, apply Lemmas 6.2.2 and 6.2.3 as well as the definition of $\hat{D}^T$ in Definition 6.2.1 to rewrite every word of the form $\mathsf{Op}_1 \circ I^A \circ \hat{D}^T \circ \mathsf{Op}_2 \circ \hat{D}_A \circ \mathsf{Op}_3$ as

$$\mathsf{Op}_1 \circ \hat{D}^T \circ I^A \circ \mathsf{Op}_2 \circ \hat{D}_A \circ \mathsf{Op}_3 + \mathsf{Op}'_A \circ \mathsf{Op}_2 \circ \hat{D}^A \circ \mathsf{Op}_3 \,,$$

for some Op'_A that does not contain the letter I^A. This step is designed to only move Is to the right when they can be contracted on to $\hat{D}$'s. Apply Step 0.

Step 3: Repeat until each word has one fewer (or zero) pair(s) of letters I^{A_i} and $\hat{D}_{A_i}$. For some operators Op_1 and Op_2, apply Proposition 2.3.1 and Lemma 6.2.2 to rewrite every word of the form $\mathsf{Op}_1 \circ I_{A_i} \circ \hat{D}^{A_1} \cdots \hat{D}^{A_i} \cdots \hat{D}^{A_k} \circ \hat{y}^\ell$ as

$$-\mathsf{Op}_1 \circ \hat{D}^{A_1} \cdots \hat{y} \cdots \hat{D}^{A_k} \circ \hat{y}^\ell + \mathsf{Op}_2 \circ \hat{y}^\ell \,,$$

where Op_2 is some operator that does not contain I_{A_i}. Apply Step 0.

Step 4: Reapply Step 1.

Step 5: In any given word, by virtue of Proposition 3.3.3, rewrite the left-most letter $\hat{D}_A$ as $\hat{D}^T_A - I_A \circ \hat{y} - X_A \circ \hat{y}^2 \circ \frac{1}{h-3}$.

Step 6: If any word contains the letter $\hat{D}$, repeat Steps 0 through 5.

The remainder of the calculation amounts to rewriting combinations of non-derivative letters

$$U_1^{\Theta_1} \cdots \circ T_1^{\Phi_1\sharp} \cdots$$

in terms of hypersurface tractors via holographic formulæ. This process is generally tedious and relies on dozens of identities arising from the Leibniz failure (both on the hypersurface and in the ambient space), Lemma 6.2.2 (and its direct application to the hypersurface operator $\hat{\bar{D}}$), Lemma 6.2.3, and the Gauss-Thomas formula (3.3.4). Some of these identities can be found in Section 4.5. Finally, we can take the limit $h \to 5$ to resolve $\mathcal{R}$. This entire procedure was implemented using FORM in the file

`FORM-Proofs/General-tensor/Paneitz-tensor-algorithm.frm`.

That the proposed algorithm here terminates requires no proof, since we explicitly verified that six iterations suffices. The result obtained from this computation is below:

$$
\begin{aligned}
-\tfrac{8}{d-5}\mathcal{R} = \; & \Big(\tfrac{4(2d-11)}{d-5} L^{AC} L_C^B + \tfrac{4(d-3)(d-6)}{d-5} F^{AB} - \tfrac{8}{d-5} W^{AB\,\sharp}\Big) \circ \hat{D}_A^T \circ \hat{D}_B^T \\
& + \Big(\tfrac{4(2d-11)}{d-5} L_{BC} (\hat{\bar{D}}^B L^{CA}) - \tfrac{2(d^2-8d+17)}{(d-1)(d-2)(d-5)} (\hat{\bar{D}}^A K) + \tfrac{4(3d-16)}{d-5} N_B L_{CD} W^{BCDA} \\
& \qquad - \tfrac{4(3d-16)}{d-5} N_B L_C^A W^{BC\sharp} + \tfrac{4(d-4)}{d-5} N_B N^C (\hat{D}_C W^{BA}{}_{\cdot\cdot})^\sharp\Big) \circ \hat{D}_A^T \\
& + \tfrac{2d^4-21d^3+95d^2-200d+152}{(d-1)(d-2)^2(d-4)^2} K^2 + \tfrac{d-2}{(d-4)^2} L \cdot J - \tfrac{2(d-2)}{(d-4)^2} L^4 + \tfrac{2(2d^3-27d^2+118d-172)}{(d-4)^2} L \cdot F \cdot L \\
& + \tfrac{(3d-16)(d-3)^2}{d-4} F^2 - \tfrac{2(d-5)(d-6)}{(d-4)^2} L^{AC} L^{BD} \bar{W}_{ABCD} + \tfrac{(d-5)(d-8)}{(d-4)(d-7)} (\hat{\bar{D}}_A L_{BC})(\hat{\bar{D}}^A L^{BC}) \\
& + 4 W_N{}^{B\sharp} \circ W_{NB}{}^\sharp - 4 N_B L_{AC} \Big(\hat{D}^A W^{BC}{}_{\cdot\cdot}\Big)^\sharp .
\end{aligned}
$$

The above expression for $\mathcal{R}$ contains operators of the type $\mathcal{W}^A \circ \hat{D}_A^T \circ \hat{D}_B^T$ which, strictly, are not defined on the weight we are interested in. However, because $\mathcal{W}^A \circ X_A = 0$, as discussed earlier, we have used use the notation $\hat{D}^T$ for the left-most Thomas-D operator. This completes the proof. $\square$

With this formula in hand, we can compute several conformally-invariant Riemannian quantities by translating this formula into a Riemannian formula in different settings. We begin with an extrinsic analog of Branson's Q-curvature.

6.4. The Extrinsic Q-Curvature

As an analog to Branson's Q-curvature described above, there also exists an *extrinsic Q-curvature* $Q_4^{\Sigma \hookrightarrow M^5}$ which also is an integrated invariant and, when restricted to the Poincareé–Einstein setting, reproduces Branson's Q-curvature. This curvature was initially defined by Gover and Waldron [**40**] as

$$Q_4^{\Sigma \hookrightarrow M^5}(g_\tau) := P_4^{\Sigma \hookrightarrow M^5} \log \tfrac{1}{\tau}|_\Sigma \,,$$

where $\tau = [g_\tau; 1]$. Their result [**40**, Theorem 4.7] shows that this extrinsic Q-curvature has the correct analogous conformal variation:

$$Q_4^{\Sigma\hookrightarrow M^5}(e^{2\omega}g) = e^{-4\omega}\left[Q_4^{\Sigma\hookrightarrow M^5}(g) + \hat{P}_4^{\Sigma\hookrightarrow M^5}\bar{\omega}\right]_\Sigma ,$$

where $\omega \in C^\infty M$ and $\bar{\omega} = \omega|_\Sigma$.

Indeed, one can define a *generalized extrinsic Q-curvature* associated to each normalized extrinsically-coupled GJMS operator. A slight modification (for the purposes of coefficient fixing) of that generalization is the following:

$$Q_{d-1}^{\Sigma\hookrightarrow M^d}(g_\tau) := \hat{P}_{d-1}^{\Sigma\hookrightarrow M^d}\log\tfrac{1}{\tau}|_\Sigma .$$

As above, this modification has the desired conformal variation property that

$$Q_{d-1}^{\Sigma\hookrightarrow M^d}(e^{2\omega}g) = e^{-(d-1)\omega}\left[Q_{d-1}^{\Sigma\hookrightarrow M^d}(g) + \hat{P}_{d-1}^{\Sigma\hookrightarrow M^d}\bar{\omega}\right]_\Sigma ,$$

where ω is as above. Furthermore, this result reduces to the results for Q_6 and Q_8 in the Poincaré–Einstein setting computed by [**37**] and matches the description of the generalized Q-curvature described in [**27**]. As in Remark 6.1.2, the above statements can be generalized to the d even dimensional case.

A useful consequence of this definition of the generalized extrinsic Q-curvature is that it agrees with the scalar curvature term that appears in the expression for the extrinsically-coupled Paneitz operator acting on scalars given by $\mathcal{Q}_{d-1}^{\Sigma\hookrightarrow M^{d'}}$. To see this, we need a technical lemma.

LEMMA 6.4.1. *Let τ be a true scale. Then the log density $\log\tau$ obeys*

$$\log\tau = 2\lim_{\varepsilon\to 0}\frac{\tau^{\varepsilon/2}-1}{\varepsilon} \quad \textit{and} \quad \hat{D}\log\tau = 2\lim_{\varepsilon\to 0}\frac{\hat{D}\tau^{\varepsilon/2}}{\varepsilon} .$$

PROOF. The first identity follows from the easily verified limit

$$\log x = 2\lim_{\varepsilon\to 0}\frac{x^{\varepsilon/2}-1}{\varepsilon}$$

valid for any $0 < x \in \mathbb{R}$. The second identity follows a direct computation in a choice of scale $g \in \boldsymbol{c}$. □

A consequence of the above is the desired result:

PROPOSITION 6.4.2. *Let* $(M^d, \gamma, \sigma) \in \mathrm{ASY}$ *specify a conformal hypersurface embedding with* d *odd and let* τ *be a true-scale. Then,*

$$Q_{d-1}^{\Sigma\hookrightarrow M^d}(g_\tau) = \mathcal{Q}_{d-1}^{\Sigma\hookrightarrow M^d}(g_\tau)\,.$$

PROOF. From Lemma 6.4.1 we have that

$$I\cdot\hat{D}\log\tau = 2\lim_{\varepsilon\to 0}\frac{I\cdot\hat{D}\tau^{\varepsilon/2}}{\varepsilon}\,.$$

A similar argument to that employed in the proof of that lemma also establishes that

$$I\cdot\hat{D}^{d-1}\log\tfrac{1}{\tau} = 2\lim_{\varepsilon\to 0}\frac{I\cdot\hat{D}^{d-1}\tau^{-\varepsilon/2}}{\varepsilon}\,.$$

Thus it follows that

$$Q_{d-1}^{\Sigma\hookrightarrow M^d}(g_\tau) \overset{g_\tau}{=} \lim_{\varepsilon\to 0}\frac{2}{\varepsilon}I\cdot\hat{D}^{d-1}\tau^{-\varepsilon/2}\big|_\Sigma\,. \tag{6.38}$$

To extract $\mathcal{Q}_{d-1}^{\Sigma\hookrightarrow M^d}(g_\tau)$ from Equation (6.4), note that the basis of invariants present in the expression for $\hat{P}_{d-1}^{\Sigma\hookrightarrow M^{d'}}\tau^{\frac{d-d'}{2}}$ are stable as the dimension d' varies **[38]** and their coefficients are rational functions of d' and d, so we can treat Equation (6.4) as a universal formula for $\hat{P}_{d-1}^{\Sigma\hookrightarrow M^{d'}}\tau^{\frac{d-d'}{2}}$ with d' an arbitrary parameter. This "dimensional continuation"-type argument is standard, see for example **[37]**. Then, working in a choice of scale $\tau = [g_\tau; 1]$, we have that

$$I\cdot\hat{D}^{d-1}\tau^{\frac{d-d'}{2}} \overset{g_\tau}{=} -\tfrac{d-d'}{2}\mathcal{Q}_{d-1}^{\Sigma\hookrightarrow M^{d'}}(g_\tau)\,,$$

and thus

$$\mathcal{Q}_{d-1}^{\Sigma\hookrightarrow M^d}(g_\tau) \overset{g_\tau}{=} \lim_{d'\to d}\left(\frac{2}{d'-d}I\cdot\hat{D}^{d-1}\tau^{\frac{d-d'}{2}}\right)\,.$$

Substituting $d+\varepsilon$ for d' in the universal formula, we obtain Equation (6.38), as required by the proposition. □

As a consequence, obtaining a Riemannian expression for $Q_4^{\Sigma\hookrightarrow M^5}$ can be done by finding a Riemannian formula for the action of the extrinsically-coupled Paneitz operator on a scalar-valued density $\tau \in \Gamma(\mathcal{E}M[\frac{5-d}{2}])$. However, given the formula for $\hat{P}_4^{\Sigma\hookrightarrow M^d}$ in Theorem 6.3.1, this is explicitly computable.

COROLLARY 6.4.3. *Let $d \geq 5$. Acting on conformal densities of weight $\frac{5-d}{2}$ along Σ, the extrinsically-coupled Paneitz operator is given by*

$$\hat{P}_4^{\Sigma\hookrightarrow M^d} = \bar{\Delta}^2 + \bar{\nabla}^a \circ \big(4\bar{P}_{ab} - (d-3)\bar{J}\bar{g}_{ab} + 8\mathring{\mathrm{II}}^2_{ab} + \tfrac{d^2-4d-1}{2(d-1)(d-2)}K\bar{g}_{ab} + 4(d-2)\mathring{F}_{ab}\big) \circ \bar{\nabla}^b - \tfrac{5-d}{2}\,\mathcal{Q}_4^{\Sigma\hookrightarrow M^d}(g)\,,$$

where

$$\begin{aligned}
\mathcal{Q}_4^{\Sigma\hookrightarrow M^d}(g) :=\; & -\bar{\Delta}\bar{J} - 2\bar{P}^2 + \tfrac{d-1}{2}\bar{J}^2 \\
& + \tfrac{2(d-2)}{d-4}(\bar{\nabla}\cdot\bar{\nabla}\cdot\mathring{F}) + \tfrac{3d^2-9d+4}{2(d-1)(d-2)(d-4)}(\bar{\Delta}K) + \tfrac{2}{d-1}\mathring{\mathrm{II}}\cdot(\bar{\Delta}\mathring{\mathrm{II}}) + \tfrac{4}{d-4}\bar{\nabla}^a(\mathring{\mathrm{II}}_a\cdot\bar{\nabla}\cdot\mathring{\mathrm{II}}) \\
& - \tfrac{2(d-2)}{d-4}\mathring{\mathrm{II}}\cdot C^{\top}_{\hat{n}} - \tfrac{6(d-2)}{(d-1)(d-4)}\mathring{\mathrm{II}}^{ab}\bar{\nabla}^c W^{\top}_{cab\hat{n}} \\
& - \tfrac{2(d-2)(d-5)}{d-4}\mathring{F}\cdot\bar{P} - \tfrac{4(d-6)}{d-4}\mathring{\mathrm{II}}\cdot\bar{P}\cdot\mathring{\mathrm{II}} - \tfrac{d^3+2d^2-27d+44}{2(d-1)(d-2)(d-4)}\bar{J}K \\
& + \tfrac{2(d-2)(d-3)}{d-4}H\mathring{\mathrm{II}}\cdot\mathring{F} - \tfrac{2(d-2)}{d-4}H\mathring{\mathrm{II}}^3 \\
& + \tfrac{2(d+2)}{(d-1)(d-4)}\mathring{\mathrm{II}}^{ad}\mathring{\mathrm{II}}^{bc}\bar{W}_{abcd} + \tfrac{2(d-2)^2}{d-4}\mathring{F}^2 + \tfrac{2(d-2)}{d-4}\mathring{\mathrm{II}}\cdot\mathring{F}\cdot\mathring{\mathrm{II}} + \tfrac{17d^3-86d^2+133d-52}{8(d-1)(d-2)^2(d-4)}K^2\,.
\end{aligned} \tag{6.39}$$

PROOF OF COROLLARY 6.4.3. The proof mainly amounts to an application of Theorem 6.3.1. Because the operator acts on scalar densities, we may use Theorem 3.3.4 to convert operators $\hat{D}^T$ to $\hat{\bar{D}}$ plus lower order terms. The proof then splits into two separate computations. The first expresses $\hat{\bar{D}}^A \circ \hat{P}_2^{\Sigma\hookrightarrow M^d} \circ \hat{\bar{D}}_A$ in terms of Riemannian operators, while the second similarly handles the subleading terms. The entire computation is carried out in FORM (see Appendix A.4): the first computation can be found in the file

`FORM-Proofs/Paneitz-scalar/DbID2Db-scalar.frm`

and the second in

`FORM-Proofs/Paneitz-scalar/Paneitz-scalar-Riemannian.frm`.

The final step uses Equation (4.5) to rewrite $\mathring{\mathrm{IV}}_{ab}$ in terms of $C^{\top}_{\hat{n}(ab)}$ in order that the result can be continued to $d = 5$. Well-definedness of the final result in $d = 5$ can be established by inspection. □

Comparing with Equation (6.1), we see that the first three terms match the first three terms of the operators intrinsic counterpart, and furthermore the first three terms in the multiplicative operator $\mathcal{Q}_4^{\Sigma\hookrightarrow M^d}$ match those of Equation (6.2). One consequence of this formula for $\hat{P}_4^{\Sigma\hookrightarrow M^d}$ on a scalar-valued density is that we obtain a Riemannian formula for the extrinsically-coupled Paneitz operator on a function in five dimensions:

$$\hat{P}_4^{\Sigma\hookrightarrow M^5}\bar{f} = \bar{\Delta}^2\bar{f} + \bar{\nabla}^a \circ \big(4\bar{P}_{ab} - 2\bar{J}\bar{g}_{ab} + 8\mathring{\rm I\!I}^2_{ab} + 12\mathring{F}_{ab} + \tfrac{1}{6}K\bar{g}_{ab}\big) \circ \bar{\nabla}^b\bar{f}\,. \tag{6.40}$$

Second, we obtain a formula for the extrinsic Q-curvature, recorded in the following theorem.

THEOREM 6.4.4. *Let $\Sigma \hookrightarrow (M, \boldsymbol{c})$ be a conformally embedded hypersurface. Then, given $g \in \boldsymbol{c}$ the extrinsic Q-curvature is*

$$Q_4^{\Sigma\hookrightarrow M^5}(g) = Q_4 + Wm + U + Qe\,,$$

where

$$\begin{aligned}
Q_4^{\Sigma} &= -\bar{\Delta}\bar{J} - 2\bar{P}^2 + 2\bar{J}^2\,,\\
Wm :=&\quad \tfrac{1}{2}\mathring{\rm I\!I}\cdot\bar{\Delta}\mathring{\rm I\!I} + \tfrac{4}{3}\bar{\nabla}^a\big(\mathring{\rm I\!I}_a\cdot\bar{\nabla}\cdot\mathring{\rm I\!I}\big) + \tfrac{3}{2}\bar{\Delta}K\\
&\quad - 6\mathring{\rm I\!I}\cdot C_n^{\top} + 4\mathring{\rm I\!I}\cdot\bar{P}\cdot\mathring{\rm I\!I} - \tfrac{7}{2}\bar{J}K - 6H\mathring{\rm I\!I}^3 + 12H\mathring{\rm I\!I}\cdot\mathring{F}\,,\\
U \;:=&\quad 18\mathring{F}\cdot\mathring{F} + 6\mathring{\rm I\!I}\cdot\mathring{F}\cdot\mathring{\rm I\!I} + \tfrac{49}{24}K^2 - \tfrac{9}{2}\mathring{\rm I\!I}^{ab}\bar{\nabla}^c W^{\top}_{cab\hat{n}} + \tfrac{7}{2}\mathring{\rm I\!I}^{ad}\mathring{\rm I\!I}^{bc}\bar{W}_{abcd}\,,\\
Qe \;:=&\quad \tfrac{8}{3}\bar{\nabla}^a\big(\mathring{\rm I\!I}_a\cdot\bar{\nabla}\cdot\mathring{\rm I\!I}\big) + 6\bar{\nabla}\cdot\bar{\nabla}\cdot\mathring{F} - \tfrac{1}{12}\bar{\Delta}K\,,
\end{aligned}$$

and Wm and U are conformally invariant weight -4 conformal densities.

PROOF. Following Lemma 6.4.2, we may set $d = 5$ in the expression for $\mathcal{Q}_4^{\Sigma\hookrightarrow M^d}(g)$ in Equation (6.39) to obtain the quoted result for $Q_4^{\Sigma\hookrightarrow M^5}(g)$.

It only remains to establish that the quantities Wm and U are conformal invariants. To do so, we first consider the weight -4 hypersurface density $L_{AB}\bar{\Box}_Y L^{AB}$ where the hypersurface Yamabe operator is defined acting on weight -1 four-manifold tractors T via

$$-\bar{D}^A T = X^A\bar{\Box}_Y T\,,$$

and L_{AB} is the tractor second fundamental form. It is not difficult to compute (see [**72**]) that

$$\bar{\Box}_Y L^{AB} \stackrel{\bar{g}}{=} \begin{pmatrix} 0 & 0 & \star \\ 0 & \bar{\Delta}\mathring{\text{II}}^{ab} - \frac{4}{3}\bar{\nabla}^{(a}\bar{\nabla}\cdot\mathring{\text{II}}^{b)\circ} - 4\bar{P}^{(a}\cdot\mathring{\text{II}}^{b)\circ} - \bar{J}\,\mathring{\text{II}}^{ab} + \star\,\bar{g}^{ab} & \star \\ \star & \star & \star \end{pmatrix}.$$

Here $\star$ denotes terms that do not contribute to the (manifestly invariant) density $L_{AB}\bar{\Box}_Y L^{AB}$. Applying the Codazzi–Mainardi Equation (3.3), we find

$$L_{AB}\bar{\Box}_Y L^{AB} = \mathring{\text{II}}^{ab}\bar{\nabla}^c W^{\top}_{cab\hat{n}} + \mathring{\text{II}}^{ad}\mathring{\text{II}}^{bc}\bar{W}_{abcd}\,. \tag{6.41}$$

Thus, because $\mathring{\text{II}}^{ad}\mathring{\text{II}}^{bc}\bar{W}_{abcd}$ is conformally invariant, so too is $\mathring{\text{II}}^{ab}\bar{\nabla}^c W^{\top}_{cab\hat{n}}$. Thus we have that U is composed solely of manifestly invariant scalars.

To handle Wm, we first define

$$W_0 := Wm + 24\mathring{F}^2 + 6\,\mathring{\text{II}}\cdot\mathring{F}\,\mathring{\text{II}} + \tfrac{31}{12}K^2 + \tfrac{11}{2}\,\mathring{\text{II}}^{ad}\mathring{\text{II}}^{bc}\bar{W}_{abcd} - \tfrac{13}{2}\,\mathring{\text{II}}^{ab}\bar{\nabla}^c W^{\top}_{cab\hat{n}}\,.$$

Comparing W_0 with $\ddot{K}$ in Equation (4.16) reveals that $W_0 = \frac{3}{2}\ddot{K}$, which is manifestly invariant and thus Wm is also invariant. □

Remark 6.4.5. Equation (6.41) in the proof of Theorem 6.4.4 also establishes that $\bar{\nabla}^c W^{\top}_{c(ab)\hat{n}}$ is invariant in $d = 5$ dimensions; alternatively this is the application to $W^{\top}_{c(ab)\hat{n}}$ of a well-known invariant first order operator on conformal 4-manifolds. ■

One important application of this result is to renormalized volumes and Weyl anomalies. For a conformally compact manifold $(M\backslash\partial M, g^o)$, the volume $\int_{M\backslash\partial M}\mathrm{dvol}(g^o)$ is formally infinite—however, as is typical in quantum theory, often physical observables can be extracted by renormalizing otherwise infinite quantities. The renormalized volume of such a manifold has played an important role in the AdS/CFT correspondence and has attracted significant attention. However, such a renormalized volume is not always definable in a conformally-invariant way: instead, one can only define such a volume up to a conformally-invariant anomaly. Early work by Henningson and Skenderis [**55**] showed that this conformally-invariant anomaly is precisely the Weyl anomaly of the boundary CFT. A result of [**24**] showed that the integral of Branson's Q-curvature is indeed this anomaly when the bulk dimension is five. Furthermore, they showed that the integral of the generalized Q-curvature generates this anomaly in general dimensions.

By generalizing the Q-curvature to its extrinsic counterpart, Gover and Waldron in [**42**] provided a formula for computing the Weyl anomaly for quantum field theories on the boundary of a gravitating theory that is not necessarily Poincaré–Einstein. They showed that, given a conformally embedded hypersurface $\Sigma \hookrightarrow (M, \boldsymbol{c})$ with $M = \Sigma \sqcup M^+$ and a choice of metric $\bar{g} \in \bar{\boldsymbol{c}}$, there is a unique renormalized volume $\mathrm{Vol}_{\mathrm{ren}}(M^+, g^o, \bar{g})$ associated to each representative $\bar{g} \in \bar{\boldsymbol{c}}$. Indeed, this dependence on $\bar{g}$ can be seen explicitly to depend on $\int_\Sigma Q_{d-1}^{\Sigma \hookrightarrow M^d}$ in a result proven in [**43**]. For $\lambda \in \mathbb{R}_+$, the difference in the renormalized volulmes associated to $\bar{g}$ and $\lambda^2 \bar{g}$ obeys

$$\mathrm{Vol}_{\mathrm{ren}}(M^+, g^o, \lambda^2 \bar{g}) - \mathrm{Vol}_{\mathrm{ren}}(M^+, g^o, \bar{g}) \;=\; \frac{\log \lambda}{16} \int_\Sigma \mathrm{dvol}(\bar{g})\, Q_4^{\Sigma \hookrightarrow M^5}(g)\,.$$

We furthermore can make a connection to the Willmore energy and invariants discussed in Section 5.2. As mentioned in that section, the integrated extrinsic Q-curvature $Q_4^{\Sigma \hookrightarrow M^5}$ is indeed a higher Willmore energy

$$E_{\mathrm{Sing}}[\Sigma \hookrightarrow (M, \boldsymbol{c})] = \int_\Sigma \mathrm{dvol}(\bar{g}_\tau) Q_4^{\Sigma \hookrightarrow M^5}(g_\tau)\,. \tag{6.42}$$

Furthermore, the functional variation of this Willmore energy, and hence its associated generalized Willmore invariant, is precisely $\mathcal{B}_5$ as in Definition 5.2.2. Other higher Willmore invariants have been produced in the literature [**45**, **53**, **68**, **85**].

6.5. The Willmore Invariant in $d = 5$

In principle, given the formula for $Q_4^{\Sigma \hookrightarrow M^5}$, we could explicitly vary E_{Sing} in Equation (6.42) and compute the Willmore invariant. However, another key result of Gover and Waldron [**45**] provides a tractor formula for computing this invariant.

Theorem 6.5.1 (Theorem 7.7 of [**45**]). *Let $(M^d, \gamma, \sigma)_{\mathcal{Y}}$ specify a conformally embedded hypersurface. Then, the singular Yamabe obstruction in d dimensions is given by the holographic formula*

$$B \stackrel{\Sigma}{=} \tfrac{2}{d!(d-1)!} \left(\bar{D}_A \circ \top\right) \left(P_{d-1}^{\Sigma \hookrightarrow M^d} N^A + (-1)^{-2} \left[I \cdot D^{d-2} (X^A K_{\mathrm{e}}) \right]\right)\,. \tag{6.43}$$

Because we have a tractor formula for $\hat{P}_4^{\Sigma\hookrightarrow M^d}$ and we know the relationship between $\hat{P}_4^{\Sigma\hookrightarrow M^d}$ and $P_4^{\Sigma\hookrightarrow M^d}$, we can explicitly compute the singular Yamabe obstruction so long as we can compute $\dddot{K}$, which will play an important role in the second term on the right-hand side of Equation (6.43). Indeed, when $d = 5$ we can rewrite this result to obtain an explicit formula:

$$B \stackrel{\Sigma}{=} \tfrac{1}{1440}(\bar{D}_A \circ \top)\big(\hat{P}_4^{\Sigma\hookrightarrow M^5} N^A - 3I \cdot \hat{D}^3(X^A K_{\rm e})\big)\,. \tag{6.44}$$

As mentioned above, in order to compute B in five dimensions, we need both $\hat{P}_4^{\Sigma\hookrightarrow M^5} I$ and $\dddot{K}$. We begin with the former. As above, this result was computed using FORM and is captured in the following result.

Proposition 6.5.2. *Let $d = 5$. Then, given $g \in \boldsymbol{c}$,*

$$\Gamma\big(\mathcal{T}M[-4]\big|_\Sigma\big) \ni \hat{P}_4^{\Sigma\hookrightarrow M^5} N^A \stackrel{g}{=} N^A A + X^A B + \bar{Z}^A_a C^a + \bar{Y}^A D\,,$$

where

$$\begin{aligned} A \ &= -\tfrac{2}{3} Wm - \tfrac{23}{6}K^2 - \tfrac{11}{3}\mathring{\rm I\!I}^{ad}\mathring{\rm I\!I}^{bc}\bar{W}_{abcd} + \tfrac{13}{3}\mathring{\rm I\!I}^{ab}\bar{\nabla}^c W^\top_{cab\hat{n}} - 4\mathring{\rm I\!I}^2 \cdot \mathring{F} - 16\mathring{F}^2\,,\\ C_a &= 8\bar{\nabla}_a(\mathring{\rm I\!I}\cdot\mathring{F}) + \tfrac{10}{3}\mathring{\rm I\!I}_a \cdot \bar{\nabla}K + \tfrac{20}{9}K\bar{\nabla}\cdot\mathring{\rm I\!I}_a\,,\\ D \ &= -24\mathring{\rm I\!I}\cdot\mathring{F}\,, \end{aligned}$$

and the leading hypersurface derivative term of the scalar B is $-\frac{1}{3}\bar{\Delta}\bar{\nabla}\cdot\bar{\nabla}\cdot\mathring{\rm I\!I}$.

Proof. This theorem is also an application of Theorem 6.3.1. Note that the normal tractor has weight zero in all dimensions, while the operator $\hat{P}_4^{\Sigma\hookrightarrow M^d}$ acts on tractors of weight $\frac{5-d}{2}$. Moreover, to use Theorem 6.3.1 in five dimensions we must compute in general d and then continue to $d = 5$. Thus we introduce a scalar density τ of weight $\frac{5-d}{2}$ and instead compute $\hat{P}_4^{\Sigma\hookrightarrow M^d}(N^A\tau)$, continue to $d = 5$, and thereafter set τ to unity. Similarly to the previous proof, handling the term $\hat{D}^{T\,B} \circ \hat{P}_2^{\Sigma\hookrightarrow M^d} \circ \hat{D}^T_B(N^A\tau)$ term is challenging. It is computed in the FORM file

`FORM-Proofs/Paneitz-N/DtID2Dt-N.frm`

The remainder of the computation is performed by the file

`FORM-Proofs/Paneitz-N/Paneitz-N-Riemannian.frm`

See Appendix A.4 for more details on this computation. Finally, we write $\mathring{\rm IV}_{ab}$ in terms of $C^{\top}_{\hat{n}(ab)}$ so that the result is well-defined in $d = 5$. □

REMARK 6.5.3. Observe from the above proposition that the action of the extrinsically-coupled Paneitz operator on the normal tractor is well-defined, despite the apparent singularities in $d = 5$ as per Theorem 6.3.1. ■

Next, we compute $\dddot{K}$.

LEMMA 6.5.4. *Let $d = 5$ and $K_{\rm e}$ be defined as above. Then along Σ, the weight -5 density $\dddot{K}_{\rm e}$ is given by the following sum of conformally invariant terms*

$$\begin{aligned}\dddot{K}_{\rm e}\big|_{\Sigma} = &- 32L^{ab}\mathring{\rm II}^3_{(ab)\circ} - 48L^{ab}(\mathring{\rm II}_{(a}\cdot\mathring{F}_{b)\circ}) + 10L^{ab}(K\mathring{\rm II}_{ab}) - 48W^{\top}_{ABCN}\hat{\bar{D}}^{A}F^{BC}\\ &- 42K\mathring{\rm II}\cdot\mathring{F} - 48\mathring{\rm II}\cdot\mathring{F}^2 - 24\mathring{\rm II}^{bc}(5\mathring{F}^{ad} + W_{\hat{n}}{}^{ad}{}_{\hat{n}})\bar{W}_{abcd} - 48\mathring{\rm II}^{a}_{b}W^{\top}_{acd\hat{n}}W^{bcd\top}{}_{\hat{n}}\\ &+ 8\mathring{\rm II}\cdot\bar{B} - 24(\mathring{F}^{ab} + W_{\hat{n}}{}^{ab}{}_{\hat{n}})\bar{\nabla}^{c}W^{\top}_{cab\hat{n}}\,,\end{aligned}$$

where L^{ab} is the operator defined by Equation (5.5) and

$$W^{\top}_{ABCN}\hat{\bar{D}}^{A}F^{BC} = -\mathring{F}\cdot C^{\top}_{\hat{n}} + W^{\top}_{abc\hat{n}}\bar{\nabla}^{a}\mathring{F}^{bc} - H\mathring{\rm II}^2\cdot\mathring{F} + 2H\mathring{F}^2\,.$$

PROOF. This calculation is a significant exercise in Riemannian hypersurface geometry. We use three facts: First, acting on a weight $w \neq 1 - \frac{d}{2}$ tractor, the operator $I\cdot\hat{D}$ is given by

$$I\cdot\hat{D} \stackrel{g}{=} \nabla_n + w\rho - \tfrac{s}{d+2w-2}(\Delta + wJ)\,.$$

Second, written in terms of the canonical extension of Equation (3.18) given by $\mathring{\rm II}^{\rm e}_{ab} = Z^{A}_{a}Z^{B}_{b}P_{AB}$, we have that $K^{\rm e} = (\mathring{\rm II}^{\rm e})^2$. Using these facts, we can recast $\dddot{K} = I\cdot\hat{D}^3K^{\rm e}$ as a Riemannian operator on tensors, at which point the problem reduces to standard (albeit lengthy) hypersurface calculations, primarily carried out in the FORM program (see Appendix A.4 for more details)

(6.45) `FORM-Proofs/Riemannian-identities/Kddd.frm`

In order to obtain a manifestly invariant result, we note that the operator L^{ab} can act on three independent structures appearing in $\dddot{K}$: namely,

$$\mathring{\rm II}^3_{(ab)\circ},\; 2\,\mathring{\rm II}_{(a}\cdot\mathring{F}_{b)\circ},\; K\,\mathring{\rm II}_{ab}\in\Gamma(\odot^2_\circ T^*\Sigma[4-d])\,.$$

The conversion (in five dimensions) of the result to expressions involving the weight -1 densities $L^{ab}\,\mathring{\rm II}^3_{(ab)\circ}$, $2L^{ab}(\mathring{\rm II}_{(a}\cdot\mathring{F}_{b)\circ})$, and $L^{ab}(K\,\mathring{\rm II}_{ab})$ is also carried out in the above FORM file. The tractor expression $W^\top_{ABCN}\hat{\bar{D}}^A F^{BC}$ can be computed with standard tractor techniques and is also included in the above FORM computation. □

We now have the pieces to assemble the obstruction density in five dimensions. Applying Equation (6.44), Lemma 6.5.4, and Proposition 6.5.2 we obtain the following result.

THEOREM 6.5.5. *Let $d=5$. Then the obstruction density is given by the sum of conformally invariant terms*

$$\begin{aligned}\mathcal{B}_\Sigma = -\tfrac{1}{120}\big[&Ob + 16L^{ab}\,\mathring{\rm II}^3_{(ab)\circ} + 36L^{ab}(\mathring{\rm II}_{(a}\cdot\mathring{F}_{b)\circ}) + \tfrac{1}{6}L^{ab}(K\,\mathring{\rm II}_{ab}) + 54W^\top_{ABCN}\hat{\bar{D}}^A F^{BC}\\
&+3(4W_{\hat n}{}^{ab}{}_{\hat n}+3\mathring{F}^{ab})\bar\nabla^c W^\top_{cab\hat n} - 13\,\mathring{\rm II}\cdot\bar{B} + (12W_{\hat n}{}^{ad}{}_{\hat n}+69\mathring{F}^{ad})\,\mathring{\rm II}^{bc}\bar{W}_{abcd}\\
&+24\,\mathring{\rm II}^a_b W^\top_{acd\hat n}W^{bcd\top}{}_{\hat n} + 60\,\mathring{\rm II}\cdot\mathring{F}^2 + \tfrac{89}{2}K\,\mathring{\rm II}\cdot\mathring{F}\big]\,,\end{aligned}$$

where

$$\begin{aligned}Ob :=\;& \bar\Delta\bar\nabla\cdot\bar\nabla\cdot\mathring{\rm II} + 6\,\bar\nabla\cdot B^\top_{\hat n} + 6\,\mathring{\rm II}\cdot B^\top - 6\,\bar{P}\cdot C^\top_{\hat n}\\
&-6\,\bar{P}^{ab}\bar\nabla^c W^\top_{cab\hat n} + 9\,\bar{P}\cdot\bar\Delta\,\mathring{\rm II} - \bar{J}\,\bar\nabla\cdot\bar\nabla\cdot\mathring{\rm II} + 6\,(\bar\nabla\mathring{\rm II})\cdot(\bar\nabla\bar{P}) + 4\,(\bar\nabla\cdot\mathring{\rm II})\cdot\bar\nabla\bar{J} + 3\,\mathring{\rm II}\cdot\bar\nabla\bar\nabla\bar{J}\\
&+12\,\mathring{\rm II}\cdot\bar\nabla\bar\nabla\cdot\mathring{F} + 15\,\mathring{F}\cdot\bar\Delta\,\mathring{\rm II} + 14\,(\bar\nabla\cdot\mathring{\rm II})\cdot(\bar\nabla\cdot\mathring{F}) + 12\,(\bar\nabla\mathring{\rm II})\cdot(\bar\nabla\mathring{F}) - 18\,W^\top_{abc\hat n}\bar\nabla^a\mathring{F}^{bc}\\
&-\tfrac{5}{6}\,\mathring{\rm II}\cdot\bar\nabla\bar\nabla K - \tfrac{2}{3}(\bar\nabla\cdot\mathring{\rm II})\cdot\mathring{\rm II}\cdot(\bar\nabla\cdot\mathring{\rm II}) - \tfrac{7}{4}(\bar\nabla\cdot\mathring{\rm II})\cdot\bar\nabla K\\
&+24\,\mathring{\rm II}\cdot\bar{P}\cdot\mathring{F} - 9\,\bar{J}\,\mathring{\rm II}\cdot\bar{P} - 15\,\bar{J}\,\mathring{\rm II}\cdot\mathring{F} + \tfrac{5}{3}\,K\,\mathring{\rm II}\cdot\bar{P}\\
&-6(\bar\nabla H)\cdot\big[\tfrac{1}{8}\bar\nabla K + \tfrac{1}{3}\,\mathring{\rm II}\cdot(\bar\nabla\cdot\mathring{\rm II}) - \bar\nabla\cdot\mathring{F}\big]\\
&+6\,H\bar\nabla\cdot\bar\nabla\cdot\mathring{F} - \tfrac{3}{2}H\,\mathring{\rm II}\cdot\bar\Delta\,\mathring{\rm II} - 2H(\bar\nabla\cdot\mathring{\rm II})^2 - \tfrac{3}{4}\,H\bar\Delta K + 12\,H\mathring{F}\cdot\bar{P} + 3\,HK\bar{J}\\
&+\tfrac{3}{2}H\,\mathring{\rm II}^{ab}\bar\nabla^c W^\top_{cab\hat n} + \tfrac{3}{2}H\bar{W}_{abcd}\mathring{\rm II}^{ad}\mathring{\rm II}^{bc} - 12\,H\big[\mathring{\rm II}\cdot C^\top_{\hat n} + \tfrac{1}{2}\,H\,\mathrm{tr}\,\mathring{\rm II}^3 - H\,\mathring{\rm II}\cdot\mathring{F}\big]\in\Gamma(\mathcal{E}\Sigma[-5])\,.\end{aligned}$$

PROOF. In the proof of Proposition 6.5.2 above, we showed how to calculate $\hat{P}_4^{\Sigma\hookrightarrow M^5} N^A$. As noted in Equation (6.44), to calculate the obstruction density $\mathcal{B}_\Sigma$, we should first compute $\hat{P}_4^{\Sigma\hookrightarrow M^5} N^A - 3I\cdot\hat{D}^3(X^A K_{\rm e})$ and in turn compute

$$\tfrac{1}{1440}(\bar{D}_A \circ \top)\big(\hat{P}_4^{\Sigma\hookrightarrow M^5} N^A - 3I\cdot\hat{D}^3(X^A K_{\rm e})\big)\,.$$

To do so, we start with $I\cdot\hat{D}^3(X^A K_{\rm e})$, and employ a combination of the tractor and hypersurface calculi developed above. This is carried out in the FORM program

`FORM-Proofs/Paneitz-N/ID3xK.frm`

Note that this computation involves $\dddot{K}$ and thus requires Lemma 6.5.4. Combining this result with that for $\hat{P}_4^{\Sigma\hookrightarrow M^5} N^A$, we can directly evaluate the obstruction; this is carried out in the FORM program

`FORM-Proofs/Paneitz-N/Obstruction-d5.frm`.

See Appendix A.4 for more details. □

REMARK 6.5.6. Observe from Theorem 6.5.5 that when $\mathring{\rm II} = \mathring{\rm III} = \mathring{\rm IV}$, we have that $\mathcal{B}_5 = 0$, as predicted by Theorem 5.2.5. Furthermore, a fifth pre-fundamental form contains $B^\top_{(ab)}$, which is evidently contained in the expression for $\mathcal{B}_5$, as predicted by Remark 5.2.8. ■

6.6. Existence of an Extrinsically-Coupled Conformally-Invariant Laplacian Powers

A well-known result of Graham [**46**] showed that on a four-dimensional conformal manifold $(M^4, \boldsymbol{c})$, there exists no conformally invariant sixth-order cubed-Laplacian differential operator. A generalization of this result, produced by Gover and Hirachi in [**35**], showed that there exists such nonexistence theorems for all operators of the form Δ^k on $(M^d, \boldsymbol{c})$ for $k > d/2$. Importantly, these are results about operators intrinsic to a conformal manifold and say nothing about the existence (or lack thereof) for such operators when the conformal manifold $(\Sigma^{d-1}, \bar{\boldsymbol{c}})$ is embedded in a larger conformal manifold $(M^d, \boldsymbol{c})$. Indeed, in this section we will show that such operators can indeed exist by providing two examples. We begin by studying the lowest order case: a squared-Laplacian operator on a conformal two-manifold.

6.6.1. Conformally Invariant Laplacian Squares in Two Dimensions. On a manifold $(M^n, \boldsymbol{c})$ with $n \geq 3$, the Paneitz operator of Equation (6.1) expressed in terms of the trace-free Ricci tensor and scalar curvature is given by

$$P_4 = \Delta^2 - \tfrac{(n-1)^2-5}{2n(n-1)} \nabla_a \circ Sc \circ \nabla^a - \tfrac{(n+2)(n-2)(n-4)}{16n(n-1)^2} Sc^2 - \tfrac{n-4}{4(n-1)} \big((\Delta Sc) + \tfrac{(n-2)(n+2)}{4n(n-1)} Sc^2 \big)$$
$$+ \tfrac{1}{n-2} \Big(4 \nabla_a \circ \mathring{Ric}^{ab} \circ \nabla_b - \tfrac{n-4}{n-2} \mathring{Ric}^2 \Big) .$$

Trouble in dimension two is signalled by the pole at $n = 2$ in this formula, although the residue vanishes because the trace-free Ricci tensor is identically zero in this dimension. In fact, given only the intrinsic data $(M^2, \boldsymbol{c})$, there is no natural conformally invariant linear differential operator with leading terms Δ^2: It is not difficult to check that there is no choice of parameters α, β, γ and δ such that $\Delta^2 + \alpha Sc \Delta + \beta (\nabla_a Sc) \nabla^a + \gamma (\Delta Sc) + \delta Sc^2$ is an invariant operator acting on weight one densities.

One way to view the failure of the Paneitz operator in dimension two to exist is to observe that the numerator of the $n = 2$ pole becomes conformally invariant (by virtue of vanishing identically) in $n = 2$ dimensions. Notice, however, that away from $n = 2$ dimensions, $\mathring{Ric}/(n-2)$ transforms under conformal changes of metric $\hat{g} = e^{2\omega} g$ by a shift of $\nabla_{(a} \nabla_{b)\circ} \omega + (\nabla_{(a} \omega) \nabla_{b)\circ} \omega$. Indeed, away from two dimensions, this conformal transformation cancels with the conformal transformation of other terms. However, in two dimensions, these conformal transformations go uncancelled.

To produce an invariant P_4 operator in two dimensions, additional data in the form of a suitable tensor with the transformation property $\nabla_{(a} \nabla_{b)\circ} \omega + (\nabla_{(a} \omega) \nabla_{b)\circ} \omega$ is necessary. The data of a tensor transforming this way is called a Möbius structure and is also required to write down a tractor connection for two-dimensional conformal manifolds [**14**]. A natural way to generate a Möbius structure is by embedding the 2-manifold Σ in some ambient conformal manifold. The following result relies on this mechanism.

LEMMA 6.6.1. *Let $d = 3$. Then the mapping*

$$f \mapsto \bar{\Delta}^2 f + 4 \bar{\nabla}^a \circ \mathcal{P}_{ab} \circ \bar{\nabla}^b f$$
$$+ \Big[2 \bar{\nabla} \cdot \bar{\nabla} \cdot \mathcal{P} - \bar{\Delta} \mathcal{J} + 2 \mathcal{P}^2 - \mathcal{J}^2 + 2 \bar{\nabla}_a \left(\mathring{\mathrm{I\!I}}^{ab} \bar{\nabla} \cdot \mathring{\mathrm{I\!I}}_b \right) + 2 (\bar{\nabla} \cdot \mathring{\mathrm{I\!I}})^2 + \tfrac{1}{4} K^2 \Big] f$$

with $\mathcal{J} := \mathcal{P}_a^a$ *where*

$$\mathcal{P}_{ab} := P_{ab}^\top + H\mathring{\mathrm{I\!I}}_{ab} + \tfrac{1}{2}H^2\bar{g}_{ab} - \tfrac{1}{2}K\bar{g}_{ab}\,,$$

defines a conformal squared Laplacian operator

$$\Gamma(\mathcal{E}\Sigma[1]) \to \Gamma(\mathcal{E}\Sigma[-3])\,.$$

Proof. The claimed conformal variation can be verified by direct computation. Alternatively, there is a simple dimensional continuation argument. Consider, a $d-1$ dimensional hypersurface Σ embedded in a conformal manifold $(M, \boldsymbol{c})$ with $d \geq 4$. The operator

$$\hat{\bar{D}}^A \circ \hat{P}_2^{\Sigma\hookrightarrow M^d} \circ \hat{\bar{D}}_A \tag{6.46}$$

maps $\Gamma(\mathcal{E}\Sigma[\frac{5-d}{2}]) \to \Gamma(\mathcal{E}\Sigma[-\frac{3+d}{2}])$, where $\hat{P}_2^{\Sigma\hookrightarrow M^d}$ is defined in Theorem 6.1.1 and is given by

$$\hat{P}_2^{\Sigma\hookrightarrow M^d} = \Delta^\top + \tfrac{3-d}{2}\big(\bar{J} - \tfrac{1}{2(d-2)}K\big)\,.$$

The operator (6.46) can be expressed in terms of the Levi-Civita connection in a choice of scale as follows

$$\begin{aligned}
\hat{\bar{D}}^A P_2^{\Sigma\hookrightarrow M}\hat{\bar{D}}_A = \tfrac{5-d}{2}\bar{\Delta}^2 \\
&+ \bar{\nabla}^a \circ \Big[-2(d-5)\bar{P}_{ab} - 4(d-4)\mathring{F}_{ab} - 2\mathring{\mathrm{I\!I}}^2_{ab} - \tfrac{(d-3)^2(d-7)}{4(d-1)(d-2)}K\bar{g}_{ab} \\
&\qquad + \tfrac{1}{2}(d-3)(d-5)\bar{J}\bar{g}_{ab}\Big] \circ \bar{\nabla}^b \\
&- \tfrac{d-5}{2}\Big(-\tfrac{d-5}{2}\bar{\Delta}\bar{J} - (d-5)\bar{P}^2 + \tfrac{1}{4}(d-1)(d-5)\bar{J}^2 + \tfrac{(d-3)^2}{4(d-1)(d-2)}\bar{\Delta}K \\
&- 2(d-5)\mathring{F}\cdot\bar{P} - (d-5)\mathring{F}^2 - \tfrac{(d-3)^2(d-5)}{4(d-1)(d-2)}K\bar{J} - \tfrac{d-5}{4(d-1)(d-2)^2}K^2 + 2\bar{\nabla}\cdot\bar{\nabla}\mathring{F} \\
&+ \tfrac{2}{d-2}\bar{\nabla}^a(\mathring{\mathrm{I\!I}}_{ab}\bar{\nabla}\cdot\mathring{\mathrm{I\!I}}^b) - \tfrac{d-5}{(d-2)^2}(\bar{\nabla}\cdot\mathring{\mathrm{I\!I}})^2\Big)\,.
\end{aligned}$$

As written, this identity cannot be dimensionally continued because neither $\bar{P}$ nor $\mathring{F}$ are defined in hypersurface dimension $d-1=2$. Instead, observe that for all $d \geq 4$,

$$\mathring{F} = P^\top - \mathring{\bar{P}} + H\mathring{\mathrm{I\!I}} - \tfrac{1}{d-1}\,\bar{g}\big(\bar{J} - \tfrac{d-1}{2}H^2 + \tfrac{1}{2(d-2)}K\big) \quad \text{and} \quad \mathring{\bar{P}} = \overline{\mathring{Ric}}/(d-3)\,.$$

Note that $\overline{\mathring{Ric}}$ is well defined in all dimensions so can be dimensionally continued to hypersurface dimension $d-1=2$ (where it vanishes). Also $\bar{J}$ is defined in hypersurface dimension $d-1=2$ by the identity $\bar{J}=\overline{Sc}/(2(d-2))$. Thus, in these terms, we have

$$\begin{aligned}
\hat{\bar{D}}^A P_2^{\Sigma\hookrightarrow M}\hat{\bar{D}}_A = {}& \tfrac{5-d}{2}\bar{\Delta}^2 \\
& + \bar{\nabla}^a \circ \Big[4(4-d)P^\top_{ab} + 2\overline{\mathring{Ric}}_{ab} - 2\,\mathring{\mathrm{I\!I}}^2_{ab} - \tfrac{d^2-12d+31}{4(d-2)}K\bar{g}_{ab} + \tfrac{(d-3)^2}{2(d-1)}\bar{J}\bar{g}_{ab} \\
& \qquad + 4(4-d)H\,\mathring{\mathrm{I\!I}}_{ab} + 2(4-d)H^2\bar{g}_{ab}\Big] \circ \bar{\nabla}^b \\
& - \tfrac{d-5}{2}\Big((5-d)(P^\top)^2 + \tfrac{(d-1)(d-5)}{4}\bar{J}^2 - \tfrac{d-1}{2}\bar{\Delta}J + 2\bar{\nabla}\cdot\bar{\nabla}\cdot P^\top \\
& + \tfrac{(d-1)(d-5)}{4}H^4 - \tfrac{(2d-3)(d-5)}{2(d-2)}H^2K - (d-5)H^2\bar{J} - 2(d-5)H\,\mathring{\mathrm{I\!I}}\cdot P^\top + 2H\bar{\nabla}\cdot\bar{\nabla}\cdot\mathring{\mathrm{I\!I}} \\
& + 2H\bar{\Delta}H - \tfrac{(d-5)^2}{4(d-2)}K\bar{J} + 2\,\mathring{\mathrm{I\!I}}\cdot\bar{\nabla}\bar{\nabla}H + 4(\bar{\nabla}\cdot\mathring{\mathrm{I\!I}})\cdot\bar{\nabla}H - \tfrac{(d-5)}{(d-2)^2}(\bar{\nabla}\cdot\mathring{\mathrm{I\!I}})^2 \\
& + \tfrac{(d-5)}{4(d-2)}\bar{\Delta}K + 2(\bar{\nabla}H)^2 + \tfrac{2}{d-2}\bar{\nabla}^a(\mathring{\mathrm{I\!I}}_{ab}\bar{\nabla}\cdot\mathring{\mathrm{I\!I}}^b)\Big)\,.
\end{aligned}$$

Taking the limit where $d\to 3$ so that $\overline{\mathring{Ric}}=0$ and defining $\mathcal{P} := P^\top + H\,\mathring{\mathrm{I\!I}} + \frac{1}{2}H^2\bar{g} - \frac{1}{2}K\bar{g}$ completes the proof. $\square$

That such an operator exists should not be surprising: indeed, the Laplacian power operators fail to exist because in certain dimensions, there do not exist tensors that transform in the required way. In this case, the required tensor is the Schouten tensor, but that tensor does not exist in two dimensions. However, by embedding the conformal two-manifold in a conformal three-manifold, one can instead use the bulk Schouten tensor and project it to the hypersurface to produce such an invariant operator.

6.6.2. Conformally Invariant Laplacian Cubes in Four Dimensions. As proved in [**46**], there is no such conformally-invariant operator of the form Δ^3+more on a conformal-four manifold. Nonetheless, as demonstrated by the previous section, it may be possible to construct such an operator using extrinsic curvatures. Indeed, the operator

$$\hat{D}^{TA}\circ\hat{P}_4^{\Sigma\hookrightarrow M^d}\circ\hat{D}^T_A \tag{6.47}$$

holographically defines a mapping

$$\Gamma\big(\mathcal{E}\Sigma\big[\tfrac{7-d}{2}\big]\big)\longrightarrow\Gamma\big(\mathcal{E}\Sigma\big[\tfrac{-5-d}{2}\big]\big)\,,$$

with leading derivative term proportional to $\bar{\Delta}^3$; see [**44**]. When $d=5$ (so Σ is a four manifold) this defines a sixth order Laplacian power $P_6^{\Sigma\hookrightarrow M}$. An explicit Riemannian formula for $P_6^{\Sigma\hookrightarrow M}$ follows as a direct corollary of Theorem 6.3.1, however this necessarily involves many terms, see for example already the result of [**37**] for the intrinsic sixth order GJMS operator on conformal manifolds of dimension five and higher. However, when the conformal hypersurface embedding is 4-umbilic, it is possible to write down a relatively compact formula for this operator. This does not contradict Graham's nonexistence result because that result relies on the fact that the Bach tensor intrinsic to a four manifold is a conformal invariant; indeed, the intrinsic Bach tensor explicitly appears as the residue of a $1/(d-5)$ pole in the intrinsic result of [**37**]. However, the operator $P_6^{\Sigma\hookrightarrow M}$ above contains data not fixed by the intrinsic conformal geometry of Σ (because the ambient Bach tensor is not fixed by 4-umbilic embeddings) which allows for the replacement of the intrinsic Bach tensor by the ambient Bach tensor with the same transformation property. We thus have the following result.

THEOREM 6.6.2. *Let $\Sigma\hookrightarrow(M^5,\boldsymbol{c})$ be 4-umbilic. Then the mapping*

$$\begin{aligned}
f\mapsto{}&\bar{\Delta}^3 f-3\bar{J}\bar{\Delta}^2 f+16\bar{P}\cdot\bar{\nabla}\bar{\nabla}\bar{\Delta}f+10(\bar{\nabla}\bar{J})\cdot\bar{\nabla}\bar{\Delta}f+16(\bar{\nabla}\bar{P})\cdot(\bar{\nabla}\bar{\nabla}\bar{\nabla}f)\\
&+\Big((\bar{\Delta}\bar{J})\bar{g}+20(\bar{\nabla}\bar{\nabla}\bar{J})-\bar{J}^2\bar{g}-16\bar{J}\bar{P}-24\bar{P}^2\bar{g}+32\bar{W}(\cdot,\bar{P},\cdot)+144\bar{P}^2+16B^\top\Big)\cdot\bar{\nabla}\bar{\nabla}f\\
&+\Big(8(\bar{\nabla}\bar{\Delta}\bar{J})-14\bar{J}(\bar{\nabla}\bar{J})+72(\bar{\nabla}\bar{J})\cdot\bar{P}+32(\bar{\nabla}\bar{P}^2)-80\bar{C}(\cdot,\bar{P})+16(\bar{\nabla}\cdot B^\top)\Big)\cdot\bar{\nabla}f\\
&+\Big((\bar{\Delta}^2\bar{J})+3\bar{J}^3-24\bar{J}\bar{P}^2-5\bar{J}(\bar{\Delta}\bar{J})+8\bar{P}\cdot\bar{W}(\cdot,\bar{P},\cdot)+48\bar{P}^3+16\bar{P}\cdot(\bar{\nabla}\bar{\nabla}\bar{J})\\
&\quad+8(\bar{\nabla}\bar{P})^2-8(\bar{\nabla}\bar{P})\cdot\bar{C}+2(\bar{\nabla}\bar{J})^2-16\bar{P}\cdot\bar{B}+16\bar{P}\cdot B^\top+8(\bar{\nabla}\cdot\bar{\nabla}\cdot B^\top)\Big)f
\end{aligned}$$

defines a conformal cubed-Laplacian operator

$$\Gamma(\mathcal{E}\Sigma[1])\to\Gamma(\mathcal{E}\Sigma[-5])\,.$$

PROOF. Because $\Sigma \hookrightarrow (M^5, \boldsymbol{c})$ is a 4-umbilic embedding, from Definition 4.2.12, we have that $\mathring{\rm II}_{ab} = \mathring{F}_{ab} = C^\top_{\hat{n}(ab)} = 0$. Consequently, from Theorem 3.3.4, we have that $\hat{D}^T \overset{\Sigma}{=} \hat{\bar{D}}$. Note that another straightforward consequence is that $C_{a\hat{n}\hat{n}} \overset{\Sigma}{=} B_{\hat{n}a} \overset{\Sigma}{=} 0$ (see [**34**, Proposition 4.3]), which implies that $W_{NABC} \overset{\Sigma}{=} 0$. Thus, to prove the theorem, from Equation (6.47), it suffices to compute $\hat{\bar{D}}^A \circ \hat{P}_4^{\Sigma \hookrightarrow M^d} \circ \hat{\bar{D}}_A$. According to Theorem 6.3.1, in this case we have that

$$\hat{P}_4^{\Sigma \hookrightarrow M^d} \overset{\Sigma}{=} \tfrac{8}{d-5}\left(\hat{\bar{D}}^A \circ P_2^{\Sigma \hookrightarrow M} \circ \hat{\bar{D}}_A - W^{AB\sharp} \circ \hat{\bar{D}}_A \circ \hat{\bar{D}}_B + \tfrac{d-4}{2} N_B N^C (\hat{D}_C W^{BA}{}_{\cdot\cdot})^\sharp \circ \hat{\bar{D}}_A\right) .$$

We explicitly compute the operator $\hat{\bar{D}}^A \circ \hat{P}_4^{\Sigma \hookrightarrow M^d} \circ \hat{\bar{D}}_A$ using FORM to compute in dimension d, and then continue to $d = 5$. First, we compute

$$(\hat{\bar{D}}^B \circ \hat{P}_2^{\Sigma \hookrightarrow M^d} \circ \hat{\bar{D}}_B \circ \hat{\bar{D}}_A) f$$

for $f \in \Gamma(\mathcal{E}\Sigma[\frac{7-d}{2}])$ using the FORM file

`FORM-Proofs/P6/DbP2Db-Dbf.frm`

To reduce computational complexity, we take the resulting expression and feed it into the following file which completes the calculation:

`FORM-Proofs/P6/Paneitz-Dbf-Riemannian-PE.frm`.

See Appendix A.4 for more details. This outputs the result displayed in the theorem. □

Following the discussion above, the appearance of the projected bulk Bach tensor $B^\top$ in Theorem 6.6.2 parallels that of $P^\top$ in Lemma 6.6.1. This is part of a more general picture linked to the conformal fundamental forms. Just as a third fundamental form, in the guise of the Fialkow tensor, was used to construct a tensor of the form $P^\top + \cdots$ with the same transformation properties $\bar{P}$ in dimensions $d \geq 4$, the higher fundamental forms can be used for the same purpose. In particular, in dimension $d = 6$, a conditional fundamental form is given by

$$\mathring{B}^\top - \tfrac{d-4}{d-5}\bar{B} + \cdots$$

and is invariant, and hence can be used to construct a tensor with the same transformation properties as $\bar{B}$ (when $d \geq 6$). A dimensional continuation argument can then be used to extract the tensor in $d = 5$ dimensions required in the above theorem.

CHAPTER 7

Conclusion

This dissertation has provided a codification of known conformal hypersurface results as well as new results that complete the overall picture. Furthermore, we generalized higher order transverse derivative operators to tensor-valued densities (such as trace-free symmetric rank-2 tensors). These operators, together with geometric holography, allowed for the construction of a new family of conformally-invariant tensors generalizing the trace-free second fundamental form, which were used to characterize extrinsic data encoded by a conformal hypersurface embedding. We used these newly-developed tools to analyze the family of asymptotically Poincarè–Einstein manifolds and characterize the Willmore invariant in terms of the conformal fundamental forms (or their pre-invariant analogs). Then we applied these notions to compute useful and interesting conformal invariants for hypersurfaces embedded in conformal five-manifolds.

While the picture developed in this dissertation is nearly complete, there are a few missing pieces that have yet to be resolved. First and perhaps most urgently, it would be particularly useful to prove the existence (or non-existence) of a dth conformal fundamental form for a conformally-embedded hypersurface $(M^d, \boldsymbol{\gamma}, \sigma)$ for $d \geq 6$ even. This would complete the picture of the conformal fundamental forms in even dimensions. One particular application for a completed picture of conformal fundamental forms would be a description of even-dimensional conformal manifolds that are Fefferman-Graham flat, *i.e.* those with vanishing Fefferman-Graham tensors. As an example, observe that the Bach tensor (the Fefferman-Graham tensor in $d = 4$) can be decomposed along a hypersurface into a term containing the fifth fundamental form and terms containing the hypersurface divergences of third and fourth fundamental forms. In this case, the fourth fundamental form has been shown to exist in $d = 4$ (as a special case), so we can characterize the Bach-flat condition in terms of differential equations on conformal fundamental forms on the set of tensors $\{\mathring{\mathrm{II}}, \mathring{\mathrm{III}}, \mathring{\mathrm{IV}}, \mathring{\mathrm{V}}\}$. When we consider the Fefferman-Graham tensor in $d \geq 6$, a similar construction could be made, so long as a dth fundamental form can be shown to exist.

An additional application of the conformal fundamental forms would be to provide further characterizations of the generalized extrinsic Q-curvatures and the generalized Willmore invariants. While we have proven that the generalized Willmore invariants can be characterized by a set of conformal fundamental forms (and one non-invariant tensor), it would be useful to also characterize the types of hypersurface operators that can appear acting on conformal fundamental forms in manifestly-invariant expressions for the generalized Willmore invariants. Low-lying results suggest that the Q-curvatures can be characterized in terms of the conformal fundamental forms as well as the integrand of an integrated topological invariant of the hypersurface.

Also of interest are global phenomena on conformal manifolds. All of the hypersurface invariants we described in Chapter 6 were local invariants, but using conformal geometry to study global properties of the bulk manifold is also particularly interesting. As an example, the renormalized volume of a hypersurface embedded in a conformally-compact manifold is a nonlocal quantity that captures important information about the hypersurface, and yet it is typically not conformally-invariant. Nonetheless, when the Weyl anomaly vanishes, the renormalized volume is indeed a conformal invariant but still cannot be written in terms of local hypersurface invariants. However, because infinite order solutions to the singular Yamabe problem necessarily contain nonlocal information about the bulk manifold, higher fundamental forms beyond transverse order $d-1$ must also contain such nonlocal information. Hence, we may be able to describe nonlocal—or in particular global—invariants with these higher-order fundamental forms.

Finally, and perhaps most speculatively, there has been recent interest in higher-codimension conformal submanifold embeddings [**3**, **63**, **68**]. All of the machinery developed in Chapters 3 and 4 is specific to the hypersurface (codimension-1) context, but likely could be generalized to higher-codimension considerations should the need arise. Indeed, the original motivation for much of the development of the hypersurface tractor calculus and the conformal fundamental forms stemmed from observing certain patterns in formulæ for certain conformal hypersurface invariants. Consequently, one might expect that similar observations could be made for conformal submanifold invariants with higher codimension which would lead to a similarly generalized conformal hypersurface calculus. The machinery already developed should generalize relatively easily, should such tools be needed.

APPENDIX A

FORM Documentation

A.1. Introduction to FORM

FORM is an algebraic manipulation software developed by Jos Vermaseren [**83**] for the computation of the higher loop Feynman diagrams needed for experiments at CERN, and is funded by FOM, the Dutch granting agency for physics research. In Section 4.5 and Chapter 6, it was used to symbolically manipulate large expressions involving tractor tensors, Riemannian tensors, and scalar curvatures. Thanks to a combination of speed, a what-you-see-is-what-you-get programming philosophy, and a natural implementation of Einstein summations, FORM lends itself to large tensor and differential operator computations.

The general structure of our FORM programs are as follows: The FORM file `xxx.frm` begins with the line `#-` (which suppresses outputting the source code to the terminal) and ends with the line `.end`. Thereafter follows a series of declarations of symbol and object names. Next are declarations of one or more *expressions*. Mathematically, FORM expressions are elements of some associative, unital algebra and are formed from the symbols and object names specified. After specifying an expression, a series of manipulations can be performed, most often by application of the `id` operation: this replaces any instance of a specific product of symbols and objects with some other combination (not necessarily a product) of symbols and objects. Once all desired manipulations are completed, the resulting expression can be output to the command line or saved to an external file. The FORM file can then be run by entering `form xxx.frm` into a command line of choice (after installing the FORM binary—see the user manual [**84**]). Below, we provide a basic example of the type of calculation used in the paper.

```
1  #-
2  symbol d,x;
3  dimension d;
4  index A,B;
5
6  function start,end,h;
7
8  ntensor X,I,Dhat,P;
9
10 local [X.P] = start*X(A)*P(A,B)*end;
11 sum A;
12
13 id P(A?,B?)*end = Dhat(A)*I(B)*end;
14 id X(A?)*Dhat(A?) = (1/2)*(h(0)-d);
15 id h(x?)*I(A?) = I(A)*h(x);
16 id h(x?)*end = end*(d+x);
17
18 print;
19 .end
```

The FORM output of this calculation is

```
FORM 4.2.1 (Aug 10 2020, v4.2.1-29-g557be9d) 64-bits  Run: Tue May 25
    10:44:03 2021
    #-

Time =       0.00 sec    Generated terms =          2
           [X.P]         Terms in output =          0
                         Bytes used      =          4

   [X.P] = 0;

  0.00 sec out of 0.00 sec
```

The penultimate line that reads `[X.P] = 0;` is the output of the calculation. In fact, the above is a proof of the tractor identity $X^A P_{AB} = 0$. A detailed explanation follows.

- Line 2 declares *symbols* `d` and `x`. Symbols are commuting objects that take no inputs and are used either as free parameters or to specify generic arguments of functions. Here the the symbol `d` represents the dimension of the ambient manifold while the symbol `x` is a dummy variable. Line 3 indicates that if the built-in Kronecker delta `d_(a,b)` is ever traced, the output is `d`. While the Kronecker delta never appears in this program, it is good practice to specify the dimension.

- Line 4 specifies a type of object called an *index*. An index is a special type of symbol that can be summed according to the Einstein convention and can only appear as the argument of either a vector object (never utilized in this document) or a tensor object. Hence, the objects `A` and `B` are special symbols that can only appear as arguments of tensor objects.

- Line 6 defines the set of *functions* utilized in this computation. Functions can have any number of arguments, including none. To FORM, a "function" is a generic object that can take any symbol as an argument—in this document, functions are used to represent multiplicative or differential operators. By default, specified functions do not commute with any other functions or non-commuting tensors. For bookkeeping purposes, we have introduced the functions `start` and `end` without arguments to demarcate the left- and right-most ends of a given expression. In our computations, the function `h` is given one argument and represents the operator $h := d + 2\underline{w}$ [, Section 2.3]. This function evaluated on some value x represents $h(x) := d + 2\underline{w} + x$.

- Line 8 declares *non-commuting tensor* objects. FORM is case-sensitive, so in this example, `X` represents a tensor whereas `x` is a symbol. Here we will give the tensor `X` only one argument because it represents the canonical tractor X^A which is a rank 1 tensor. FORM does not natively distinguish between upper and lower indices since there is no need to do so if one keeps track of what type of tensor any given object represents. Contraction may still be implied by repeated indexing. Here the tensor `I` will be used with only one index to represent the scale tractor I^A. The tensor `Dhat` will also have one index and represents the hatted Thomas-D *operator* $\hat{D}^A$. The final tensor `P` will be used with two indices to represent the tractor $P_{AB} := (\hat{D}_A I_B)$. Because these are non-commuting objects, they all can be used to represent operators, (differential or multiplicative).

- Line 10 defines the quantity we wish to compute/manipulate: here we locally define the expression `[X.P]` to denote the product/composition of objects `start*X(A)*P(A,B)*end`. In FORM, any string of characters in square brackets `[...]` may be used to label any type of object. Note that this code uses the marker `end` to play the role of 1 in identities such as $(\hat{D}_A I_B) = (\hat{D}_A \circ I_B)(1)$. Here `start` was not really needed, but will be useful for more intricate computations. The sequence of objects `X(A)*P(A,B)` in this line represents the tractor expression $X^A P_{AB}$.

In any FORM program—just as for most index computations—we must avoid reusing repeated dummy indices. This is conveniently handled in line 10 by the statement `sum A;` which indicates that the index `A` is to be internally summed via the Einstein convention and replaced with an *internal* index whose name is the next in a sequence of internal indices. This ensures that `A` is free to be used again without reusing internal indices. In this case, the resulting expression after summation looks like `start*X(N1_?)*P(N1_?,B)*end` - here, `N1_?` is the first index in the mentioned internal sequence. The next index in the sequence is `N2_?`, *etc.*

Additionally, note that more than one expression can be *active* at any one time—this is handled by multiple expression declarations. The manipulations that follow are performed on every active expression simultaneously.

- Lines 13 through 16 perform the actual computation via `id` statements. These statements are typically written as `id monomial = replaced_subexpression;`. When FORM encounters this statement, it searches all active expressions in memory for any instance of `monomial` and replaces it with `replaced_subexpression`, being careful to rename internal indices if they would overlap with other already-present indices:

  13 This line replaces all instances of the tensor object `P(A?,B?)*end` within active expressions with the monomial `Dhat(A)*I(B)*end`—that is, it applies the tractor identity $P^{AB} = \hat{D}^A I^B$. Note that on the left-hand side of `=`, the indices `A` and `B` are both followed by the symbol `?`, which indicates that the preceding object is a wildcard of a specified type. In this case, because the preceding objects for both `?`s are indices, the operation `id` searches all active expressions for all instances of `P(*,**)` where `*` and `**` are any declared or internal indices. Once it finds one of these monomials, it replaces it with the provided replacement subexpression *using the objects matched by the wildcard*. Here, because the expression does *not* contain `P(A,B)*end` but *does* contain `P(N1_?,B)*end`, this monomial would be replaced by `Dhat(N1_?)*I(B)*end`. Note that the wildcard symbol can also be applied to functions and tensors—for example, `P?(A?,B?)` would match any two-index tensor object.

  14 This line applies the tractor identity $X_A \hat{D}^A = \underline{w} = \frac{\mathsf{h}-d}{2}$. To do so, FORM finds all instances of `X(*)*Dhat(*)` (where both `*`s represent the *same* index) in the active expression and replaces them with `(1/2)*(h(0)-d)`.

  15 This line replaces any occurence of `h(*)*I(**)` with `I(**)*h(*)`. Here, because `x?` is a wildcarded symbol, `*` could be any polynomial in declared symbols. Also, since `A?` is a wildcarded index, FORM will match `**` when this is any declared or internal index.

This identification is the blueprint for an oft-needed identity because the operator $\mathsf{h} := d + 2\underline{w}$ composed with a weight w object T obeys $h(x) \circ T = T \circ h(x + 2w)$. In this case, the scale tractor has weight 0, so $\mathsf{h}(x)$ passes through I unchanged—however, in the future, we will specify how $\mathsf{h}(x)$ interacts with other objects of non-zero weights.

16 The identification `h(x?)*end = end*(d+x);` detects any instance of the function `h(*)` at the right-most end of a given expression and replaces it with `d+*` for any polynomial combinations of symbols `*`, being careful not to lose the placeholder `end` in case it is needed later. This amounts to the identity $\mathsf{h}(x)(1) = (d + 2\underline{w}(1) + x) = d + x$.

- Line 18 tells FORM to output the current expression to the command line, and the final statement `.end` terminates the program.

A.2. Additional FORM Tips and Tricks

Here we provide a list of additional FORM functionalities that are useful for our computations; see [**84**] for a more complete description.

Symmetric, Antisymmetric. A function or tensor can be specified to be symmetric or antisymmetric when the object name declaration is followed by `(symmetric)` or `(antisymmetric)`. This indicates that the arguments of the declared object are either *all* symmetric or *all* antisymmetric. For example, declaring `tensor P(symmetric)` ensures that, in the context of FORM's pattern-matching functionality, `P(A,B) = P(B,A)`. Similarly, declaring `tensor P(antisymmetric)` enforces that `P(A,B)=-P(B,A)`.

Symmetrize, Antisymmetrize. Even though many of the tensors we define in our FORM code have various symmetries, FORM is not particularly adept at handling specific symmetries of tensors, and thus occasionally needs to be prodded to do some simplification via explicit instructions. To that end, the `Symmetrize` and `Antisymmetrize` commands can be useful. While these commands can be implemented in many ways, we used two particular sets of arguments in our computations. Given a declared tensor (with no declared symmetries) `T`, the command `Symmetrize T 1,3;` explicitly symmetrizes the tensor `T` in its first and third indices everywhere that the tensor has at least three indices. Alternatively, the command `Symmetrize T:4 1,2;` will only symmetrize the tensor `T` in the first and second indices when the tensor has exactly four indices—otherwise it does nothing. The `Antisymmetrize` command has the exact same argument types.

The Argument Field. For identifying complicated function or tensor monomials, sometimes it is useful to be able to pattern match an arbitrary or unknown number of arguments; argument field wildcards enable this. This is implemented by *preceeding* an argument of the appropriate type by `?`. When using an argument field wildcard, the symbol `?` is included on both the left and right side of `=`. For example, one might write `id X(?A)= I(?A);`, which would match `X(A)`, `X(A,B,C)`, *and* `X`, replacing them with `I(A)`, `I(A,B,C)`, and `I`, respectively (note that the argument field wildcard can match the empty set). Argument field wildcards and standard wildcards can both be applied but not simultaneously: `X(?A,B?)` is allowed but `X(?A?)` is not. Note that the pattern-matching functionality of `id` is not compatible with argument field wildcards in (anti)symmetric objects.

Another useful feature associated with argument fields is the `nargs_` association, which can count the number of arguments that a wildcard field matches. Consider the line `id f(?A)= nargs_(?A)`. The keyword `nargs_` takes as input the argument field and returns an integer equaling the number of arguments contained in that argument field. For example, the above line of code would match `f(a,b,c)` to `3`.

PolyRatFun. Given a single commuting function `f`, upon declaring `PolyRatFun f;`, FORM treats `f(x,y)` as if it were the rational function $f(x, y) = x/y$, where x and y are polynomials in declared symbols. Moreover the coefficient of every monomial is expressed this way, so that `3*X(A)` would become `f(3,1)*X(A)`. After this declaration, the function `f` is restricted to having only two arguments. This allows FORM to simplify rational functions, for example `f(x,y)*f(a,b)` automatically becomes `f(a*x,b*y)`.

When a PolyRatFun depends only on one declared symbol, say `x`, FORM performs a series expansion to order n about $x = 0$ when the statement `PolyRatFun drat(expand,x,n);` is included. For example, declaring `PolyRatFun drat(expand,x,3);` converts `f(1,1-x)` to `f(1+x+x^2+x^3)`.

Renumber. The `renumber` command attempts to relabel dummy indices in order to reduce the number of terms. Our implementation `renumber 1;` indicates to FORM that it should try every permutation of dummy indices to try to reduce the number of terms in the output—this typically results in a dramatic simplification of the result.

Sets. A declared set is an ordered list of objects with the same type. Given a list of declared tensors `ntensor T1,T2,T3,T4`, one might write `Set xyz:T1,T2,T3`. Doing so associates to the ordered subset of tensors `T1,T2,T3` the name `xyz`. We can then restrict wildcard matches to elements of a set by following `?` with the set name; for example, `T1?xyz` only matches tensors `T1`, `T2`, or `T3` and *not* `T4`. This functionality can be further extended by associating two sets to one another. Given declared tensors `ntensor T1,T2,T3,T4,U1,U2,U3,U4` and the set `Set abc:U1,U2,U3`, by using two wildcards sequentially, one can specify that a tensor should be matched with elements of the first set and replaced by corresponding elements of the second set. So, given an expression `T1 + T2 + T3 + T4`, the pattern-matching statement `id T1?xyz?abc = T1;` would result in the expression `U1 + U2 + U3 + T4`.

Repeat. The keyword `repeat` can appear paired with `endrepeat` surrounding a block of code, or directly preceeding an executable statement (like `repeat id xyz = abc;`). The basic functionality is that of a standard while-loop that executes until no active expressions change. If a repeat statement is in danger of entering an infinite loop, FORM will terminate the program.

If. The executable statement `if (condition)` (paired with `endif`) encloses a block of code that is executed conditionally. A useful condition is the `occurs` argument, which takes as input a list of objects and returns `1` when at least one of its arguments is present in the expression and returns `0` otherwise. The resulting value can be compared with a numerical value to determine whether or not the condition is met. For example `if (occurs(x)=1) id y = 1;` would replace the expression `y*x^2` by `x^2`. Another useful implementation of `if` is when it is paired with the `match` argument. In that case, the `if` command returns a boolean `1` or `0` if FORM's pattern-matching machinery would find a match in a term in a given expression. For example, if FORM's active expression was `f(a,b,b)+ f(a,b,c)`, then the line `if(match(f(a?,b?,b?))== 1)id f(a?,b?,c?)= g(a,b,c);` would turn our original expression into `g(a,b,b)+ f(a,b,c)`.

Once. The `id` statement acts simultaneously on *all* matched monomials. By including the keyword `once` as in `id once xyz = abc`, FORM replaces only the first instance of the matched pattern in a given monomial with the replacement subexpression. This functionality is useful to avoid duplicate indices when new indices are introduced in a tensor calculation. For example, given an expression `a*a*a`, the line `id a = b;` results in the expression `b*b*b`, while the line `id once a = b;` would result in the expression `b*a*a`.

Sort. The `.sort` command is one of a few types of module-ending statements. A *module* is a block of statements. A FORM program is composed of modules, and a given module must contain statements in a specific order: declarations, specifications, definitions, executable statements, and finally output specifications. In general, a module can be ended by several module-ending statements, however `.sort` is the most common—it executes all of the lines in the module and prepares the output for the next module. Another module-ending statement is `.end`; it performs the function of `.sort` but also terminates the program.

Delete. The command `delete` can be particularly useful for deleting stored global expressions via the command `delete storage;`. When called this command removes from memory all globally-stored expressions. By deleting the stored global expressions when we no longer need them in a particular computation, those global expressions can be imported into many different linked computations.

Hide. The `hide` specification statement takes all active expressions, stores them in a hidden auxiliary file, and removes them from the active expressions list. These expressions can be unhidden later in the same program via the `unhide` statement. This is useful when one wishes to either not display a certain expression as final output or leave it unaffected by later computations. The command `hide` typically appears between a pair of `.sort` statements. The `hide` statement hides all active expressions by default, but by providing a comma-separated list after `hide`, one can selectively hide expressions—and similarly, one can selectively unhide expressions. An example application of the hide command appears in the following FORM code snippet:

```
local expr1 = X;
id X = Y;
.sort;
hide;
.sort;

local expr2 = Y + expr1;
id Y = Z;
.sort;
unhide;
.sort;
print;
```

Assuming everything was properly declared, the output of this FORM code is displayed below:

```
expr1 = Y
expr2 = 2*Z
```

Because `expr1` was hidden when `id Y = Z;` was executed, that expression remains unchanged, whereas any `Y` present in `expr2` is replaced with `Z`.

Bracket. The `bracket` statement is an output control specification that groups specified terms when printing output. The statement takes arguments by a comma-separated list of symbols, functions, tensors, and/or sets and factors these objects out of the current expressions. This statement can be useful for bug-detection as well as result presentation.

Preprocessor Variables. Before any of the statements are executed by FORM's compiler, the program is read by the preprocessor and certain preprocessor instructions are executed (such as editing the input stream to the FORM compiler). Each preprocessor instruction begins with the character `#`. One such preprocessor instruction is the declaration and assignment of a *preprocessor variable* by the instruction `#define xyz "abc"`. This instruction indicates to the preprocessor that, every time the preprocessor encounters the character string `'xyz'` after the initial declaration, it should replace that string with the string `abc`. The replacement string can contain FORM symbols, rational numbers, or any combination of these. More intricate preprocessor routines where FORM self-generates a substantial piece of code are also possible.

Procedures. Using the `procedure` preprocessor instruction, one can define what would be called a subroutine in other programming languages. In general, it does not matter where procedures are defined although it is often useful to define a procedure directly after the initial declarations are made. To define a procedure, one begins a block of code with the instruction `#procedure xyz(args):` and ends the procedure with the instruction `#endprocedure`. Note that the procedure created by these instructions is named `xyz` and takes arguments `args`. For the most part, the procedures used here will not take arguments and so will be initialized as `procedure xyz():`. The instruction `#call xyz()` calls a previously defined procedure, inserting the executable statement block where the procedure was called. Procedures can be called inside of other procedures. We provide an example below:

```
#-
function A,B,C;

#procedure square()
id A = A*A;
id B = B*B;
id C = C*C;
#end procedure

local expr = A + B + C;
#call square()
print;
.end
```

This code will output `expr = A*A + B*B + C*C`.

Headers and Procedure Files. Such procedures described above can be included in an external directory in the form of *header* and *procedure* files. These are files that do not perform any specific calculations, but instead contain one or more procedures that can be used in essentially any computation. This reduces redundancy and centralizes the more basic identities that are used in many of these FORM computations. In order to use these files in a given computation, two steps are needed. First, one must include that directory when calling FORM to run; to do so, feed FORM the `-p DIRECTORY-NAME` option at run-time. The second step is to include in the FORM code itself lines that indicate which header (or procedure) files are to be included in that specific program. This is done with a line such as `#include - DIRECTORY/FILENAME.h;`.

A.3. $\ddot{K}$ FORM Computation

The following FORM program is used to compute $\ddot{K}$ as given in Equation (4.16) in Proposition 4.5.8. To calculate this curvature, we implement an identity for $I \cdot \hat{D}$ found in [**44**] and then perform a series of Riemannian manipulations to reexpress the curvature in terms of curvatures intrinsic to the hypersurface and higher fundamental forms. We begin with a list of declarations.

A.3.1. Declarations.

The following are the variable and object declarations for the program used to compute $\ddot{K}$.

```
#-
symbol d,w;
dimension d;
index a,b,c,e,f,ap,bp,cp,fp,A,B,C,E;

function sigma, [1/h], K, Kext, Jb, RhoNN, J, H, dnJ, r, dnr, [I.Dh],
    [I.D], [ID2], dn, LapT, LapB, start,end;
cfunction s, [1/d], k, jb, rhoNN, j, Hc, rC, dnrC, drat;

PolyRatFun drat;

ntensor n, del, delt, delb, Rho(symmetric), RhoT(symmetric),
    RhoB(symmetric), RhoN, RhoNT, dbRhoNT, [Gb_](symmetric), dnRho,
    Riemann, dnRn, Ric(symmetric), Weyl, Weyln, Weylnt, Wn(symmetric),
    Weylb, Weylt, IInc(symmetric), IIO(symmetric), IIOe(symmetric),
    dnIIOe(symmetric), dn2IIOe(symmetric), dbIIO, FNo(symmetric),
    FO(symmetric), [d^n^], dbH, dr, Cotton, Ten;

ctensor nC, rho(symmetric), rhoT(symmetric), rhoB(symmetric), rhoN,
    rhoNT, Fno(symmetric), Fo(symmetric), [gb_](symmetric), riemann,
    riemannB, ric(symmetric), weyl, weyln, weylnt, wnc(symmetric),
    weylt, weylb, II(symmetric), IIo(symmetric), dbII, dbIIo, [d^n^C],
    dbHc, IIIo(symmetric), drC, cotton, dbrhoNT, TenC;

Set noncommF: sigma,r, K,Kext,H, dnr, RhoNN,J,Jb;
Set commF:    s,    rC,k,k,   Hc,dnrC,rhoNN,j,jb;

Set noncommT: n,  Rho, RhoT, RhoB, RhoN, RhoNT, dbRhoNT, [Gb_], Riemann,
    Ric, Weyl, Weyln, Weylnt, Weylb, Weylt, Wn,  IInc, IIO, IIOe, dbIIO,
    FNo, FO, [d^n^],  dbH,  dr,  Cotton;
Set commT:    nC, rho, rhoT, rhoB, rhoN, rhoNT, dbrhoNT, [gb_], riemann,
    ric, weyl, weyln, weylnt, weylb, weylt, wnc, II,   IIo, IIo,  dbIIo,
    Fno, Fo, [d^n^C], dbHc, drC, cotton;

Set hypT: rhoB, rhoNT, rhoT, Fno, Fo, [gb_], riemannB, weylnt, wnc,
    weylt, weylb, II, IIo, dbII, dbIIo, dbHc, IIIo, dbrhoNT;
Set trfr: IIo, IIIo, Fo, wnc;

#define dpp "5"
```

Provided below in Table A.1 are the corresponding mathematical objects for FORM's variables. Besides the differences in the lists of declared functions and tensors, the structure above is identical to the structure in the declarations for the previous program. The only difference to note is that, because the calculation that follows is a strictly Riemannian calculation, we need not define a separate symbol for dimension of the tractor metric, so we can use the Kronecker delta as the metric of the ambient manifold. In that case, the dimension is simply declared as `d`.

Symbol	Object
d	d
Function	**Mathematical Object**
sigma	σ
[1/h](x)	$\frac{1}{\mathtt{h}+x}$
K	K
Kext	K^{e}
Jb	$\bar{J}$
RhoNN	P_{nn}
J	J
H	H
dnJ	$\nabla_n J$
r	$\rho := -\frac{1}{d}(\Delta\sigma + J\sigma)$
dnr	$\nabla_n \rho$
[I.Dh]	$I\cdot\hat{D}$
[I.D]	$I\cdot D$
[ID2]	$I\cdot D^2$
dn	∇_n
LapT	$\Delta^\top$
LapB	$\bar{\Delta}$
[1/d](x)	$\frac{1}{d+x}$

Tensor	Object
n(a)	n_a
[Gb_](a,b)	$g_{ab} - n_a n_b$
del(a)	∇_a
delt(a)	$\nabla_a^\top$
delb(a)	$\bar{\nabla}_a$
Rho(a,b)	P_{ab}
RhoT(a,b)	$P_{ab}^\top$
RhoB(a,b)	$\bar{P}_{ab}$
RhoN(a)	P_{na}
RhoNT(a)	$P_{na}^\top$
dbRhoNT(a,...,b,c)	$\bar{\nabla}_a \dots \bar{\nabla}_b P_{\hat{n}c}^\top$
dnRho(a,b)	$\nabla_n P_{ab}$
Riemann(a,b,c,d)	R_{abcd}
dnRn(a,b)	$\nabla_n n^c n^d R_{cabd}$
Ric(a,b)	Ric_{ab}
Weyl(a,b,c,d)	W_{abcd}
Weyln(a,b,c)	W_{abcn}
Weylnt(a,b,c)	$W_{abcn}^\top$
Wn(a,b)	W_{nabn}
Weylb(a,b,c,d)	$\bar{W}_{abcd}$
Weylt(a,b,c,d)	$W_{abcd}^\top$
IInc(a,b)	$\mathrm{I\!I}_{ab}$
II0(a,b)	$\mathring{\mathrm{I\!I}}_{ab}$
II02(a,b)	$\mathring{\mathrm{I\!I}}_a^c \mathring{\mathrm{I\!I}}_{cb}$
II0e(a,b)	$\mathring{\mathrm{I\!I}}_{ab}^{\mathrm{e}}$
dnII0e(a,b)	$\nabla_n \mathring{\mathrm{I\!I}}_{ab}^{\mathrm{e}}$
dn2II0e(a,b)	$\nabla_n^2 \mathring{\mathrm{I\!I}}_{ab}^{\mathrm{e}}$
dbII0(a,...,b,cd)	$\bar{\nabla}_a \dots \bar{\nabla}_b \mathring{\mathrm{I\!I}}_{cd}$
FNo(a,b)	F_{ab}
F0(a,b)	$\mathring{F}_{ab}$
[d^n^](a,...,b,c)	$\nabla_a \dots \nabla_b n_c$
dbH(a,...,b)	$\bar{\nabla}_a \dots \bar{\nabla}_b H$
dr(a,...,b)	$\nabla_a \dots \nabla_b \rho$
Cotton(a,b,c)	C_{abc}

TABLE A.1. FORM symbols and their corresponding mathematical objects.

A.3.2. Procedures. As in the previous program, we also provide a set of procedures that will be used in this program. Three of the procedures are of the same type as in the previous program, and indeed one of them is identical. We list these procedures here and explain any new operations.

The first procedure is the `makeCommute()` procedure, which performs the same function as the like-named procedure above.

```
#procedure makeCommute()
repeat;
id start*sigma?noncommF?commF = sigma*start;
id start*n?noncommT?commT(?A) = n(?A)*start;
id start*II0?noncommT?commT(A?,B?) = II0(A,B)*start;
endrepeat;
#endprocedure
```

The next procedure is called `sigmaIdentities()` and, as before, it performs the role of simplifying and rewriting the expression in terms of hypersurface quantities along Σ using commuting variables.

```
#procedure sigmaIdentities()
repeat;
id s = 0;
id nC(A?)*nC(A?) = 1;
id nC(a?)*dbHc?hypT(?A,a?,?B) = 0;
id nC(a?)*IIo?hypT(a?,b?) = 0;
id nC(a?)*start*delt(a?) = 0;

id IIo?trfr(a?,a?) = 0;

id [gb_](a?,b?) = d_(a,b) - nC(a)*nC(b);
```

We use g instead of $\bar{g}$ in this program.

```
id dbHc?hypT(?A,a?,?B)*start*del(a?) = dbHc(?A,a,?B)*start*delt(a);
id IIo?hypT(a?,b?)*start*del(a?) = IIo(a,b)*start*delt(a);
```

Contraction with hypersurface tensors projects $\nabla \mapsto \nabla^{\top}$.

```

id rho(a?,a?) = j;
id rhoB(a?,a?) = jb;
id Fno(a?,a?) = (1/2)*[1/d](-2)*k;
id weyl(?A,a?,?B,a?,?C) = 0;
id weylb(?A,a?,?B,a?,?C) = 0;

id nC(a?)*nC(b?)*riemann(a?,b?,c?,e?) = 0;
id nC(a?)*nC(b?)*riemann(c?,e?,a?,b?) = 0;
```

Symmetries of the Riemann tensor.

```

id [d^n^C](a?,b?) = II(a,b) + Hc*nC(a)*nC(b);
id II(a?,b?) = IIo(a,b) + Hc*[gb_](a,b);
id rC = -Hc;
id drC(a?) = -dbHc(a) + nC(a)*dnrC;
```

$\nabla\rho = \nabla^{\top}\rho + n\nabla_{n}\rho \stackrel{\Sigma}{=} -\bar{\nabla}H + \hat{n}\nabla_{\hat{n}}\rho$

```

id dnrC = [1/d](-2)*k + rhoNN;
```

$\nabla_{n}\rho \stackrel{\Sigma}{=} \frac{K}{d-2} + P_{\hat{n}\hat{n}}$, see [44, Equation (3.11)].

```

id IIo(a?,b?)*drC(a?,b?) = -IIo(a,b)*dbHc(a,b) + IIo(a,b)*II(a,b)*dnrC;

id nC(a?)*nC(b?)*[d^n^C](a?,b?,c?) = -drC(c) - nC(c)*dnrC;
```

Apply $\nabla_{n} n_{a} = -\sigma\nabla_{a}\rho - \rho n_{a}$, `repeat` loop then implements above formulæ for $\nabla_{a}\rho$.

```
id IIo(a?,b?)*IIo(a?,b?) = k;
id once wnc(a?,b?) = IIo(a,c)*IIo(c,b) - [1/d](-1)*k*[gb_](a,b) - (d-3)*Fo(a,b);
sum c;
```

Trace-free Fialkow equation.

```
id Fno(a?,b?) = Fo(a,b) + (1/2)*[1/d](-1)*[1/d](-2)*k*[gb_](a,b);
id Fo(a?,b?) = -[1/d](-3)*IIIo(a,b);
id dbIIo(?A,a?,a?) = 0;

id riemann(a?,b?,c?,e?) = weyl(a,b,c,e) + d_(a,c)*rho(b,e) - d_(b,c)*rho(a,e) -
     d_(a,e)*rho(b,c) + d_(b,e)*rho(a,c);

id rho(a?,b?) = rhoT(a,b) + nC(a)*rhoNT(b) + nC(b)*rhoNT(a) + nC(a)*nC(b)*rhoNN;
id rhoT(a?,b?) = Fno(a,b) + rhoB(a,b) - Hc*IIo(a,b) - (1/2)*[gb_](a,b)*Hc^2;
id once rhoNT(b?) = [1/d](-2)*dbIIo(a,a,b) - dbHc(b);
sum a;
id j = rhoNN - ( - jb + drat('dpp'-1,1)*(1/2)*Hc^2 - (1/2)*drat(1,'dpp'-2)*k);

id once weyl(a?,b?,c?,e?) = ([gb_](a,ap)+nC(a)*nC(ap))*
     ([gb_](b,bp)+nC(b)*nC(bp))* ([gb_](c,cp)+nC(c)*nC(cp))*
     ([gb_](e,fp)+nC(e)*nC(fp))* weyl(ap,bp,cp,fp);
sum ap,bp,cp,fp;
id [gb_](a?,ap?)* [gb_](b?,bp?)* [gb_](c?,cp?)* [gb_](e?,fp?)*
     weyl(ap?,bp?,cp?,fp?) = weylt(a,b,c,e);
```

Project ambient Weyl tensor to Σ.

```
id weylt(a?,b?,c?,e?) = weylb(a,b,c,e) - IIo(a,c)*IIo(e,b) + IIo(a,e)*IIo(c,b) -
     [gb_](a,c)*Fno(e,b) + [gb_](a,e)*Fno(c,b) + [gb_](b,c)*Fno(e,a) -
     [gb_](b,e)*Fno(c,a);
```

Trace-free Gauß equation.

```

id nC(a?)*weyl(a?,b?,c?,e?) = weyln(e,c,b);
id nC(b?)*weyl(a?,b?,c?,e?) = weyln(c,e,a);
id nC(c?)*weyl(a?,b?,c?,e?) = -weyln(a,b,e);
id nC(e?)*weyl(a?,b?,c?,e?) = weyln(a,b,c);
id nC(a?)*weyln(a?,b?,c?) = wnc(b,c);
id nC(a?)*weyln(b?,a?,c?) = -wnc(b,c);
id nC(c?)*weyln(a?,b?,c?) = 0;
endrepeat;
#endprocedure
```

Uniformize contractions of $\hat{n}$ into Weyl tensor .

The next procedure `commuteThingsLeft()` uses the Gauß formula and the Leibniz property of the tangential Levi-Civita connection to commute hypertensor objects left of $\nabla^{\top}$. While the bulk of the procedure is nearly identical to code documented earlier. Here we use the `if` executable statement, to ensure that the procedure only applies hypersurface identities to the expression when there are no ambient connections (in the form of `dn` and `del(a)`) present.

```
#procedure commuteThingsLeft()
if (occurs(dn,del)=0);    } Returns true when neither dn nor del(a) are present.
repeat;

id IInc(a?,b?) = II0(a,b) + H*(d_(a,b) - n(a)*n(b));
id Rho(a?,b?) = RhoT(a,b) + n(a)*RhoNT(b) + n(b)*RhoNT(a) +
     n(a)*n(b)*RhoNN;
repeat;
id once RhoNT(b?) = [1/d](-2)*dbII0(a,a,b) - dbH(b);
sum a;
endrepeat;
id RhoNN = J - Jb + drat('dpp'-1,1)*(1/2)*H^2 - (1/2)*drat(1,'dpp'-2)*K;


id delt(a?)*H = dbH(a) + H*delt(a);
id delt(a?)*n(b?) = IInc(a,b) + n(b)*delt(a);
id once delt(a?)*dbH(b?) = dbH(a,b) - n(b)*IInc(a,e)*dbH(e) +
     dbH(b)*delt(a);                    This loop is only executed
                                        when the above condition
                                        is satisfied.
sum e;
id delt(a?)*[Gb_](b?,c?) = [Gb_](b,c)*delt(a) - n(b)*IInc(a,c) -
     n(c)*IInc(a,b);
id once delt(a?)*II0(b?,c?) = dbII0(a,b,c) - n(b)*IInc(a,e)*II0(e,c) -
     n(c)*IInc(a,e)*II0(b,e) + II0(b,c)*delt(a);
sum e;
id once delt(a?)*dbII0(b?,c?,e?) = dbII0(a,b,c,e) -
     n(b)*IInc(a,ap)*dbII0(ap,c,e) - n(c)*IInc(a,ap)*dbII0(b,ap,e) -
     n(e)*IInc(a,ap)*dbII0(b,c,ap) + dbII0(b,c,e)*delt(a);
sum ap;

#call makeCommute()
#call sigmaIdentities()
endrepeat;
endif;    } Marks then end of the conditional execution block.
#endprocedure
```

The final procedure `commuteDnRight()` effectively commutes ∇_n to the right of ambient tensors using the Leibniz property. Note that unlike the previous procedures, this procedure is applicable even when objects are not evaluated along the hypersurface Σ; like all of the other manipulating procedures, this procedure is repeated until no changes are made in the active expressions.

```
#procedure commuteDnRight()
repeat;
id dn*r = dnr + r*dn;
id dn*sigma = (1-2*r*sigma) + sigma*dn;
id dn*n(a?) = -sigma*dr(a) - r*n(a) + n(a)*dn;
id dn*II0e(a?,b?) = dnII0e(a,b) + II0e(a,b)*dn;
id dn*Rho(a?,b?) = dnRho(a,b) + Rho(a,b)*dn;
endrepeat;
#endprocedure
```

Define dnr by $\nabla_n \rho$.

Apply $\nabla_n \sigma = n^2 = 1 - 2\sigma\rho$.

$\nabla_n n = \frac{1}{2}\nabla_a n^2 = -\sigma\nabla\rho - \rho n$.

Definitions of dnII0e and dnRho.

A.3.3. Riemannian Computation of $\ddot{K}$. We begin by converting the tractor expression $\ddot{K} := I \cdot \hat{D}^2 K^{\mathrm{e}}$ into a Riemannian expression.

```
local Kdd = start*[I.Dh]^2*Kext*end;

id [I.Dh] = [I.D]*[1/h](-2);
repeat;
id [1/h](x?)*[I.D] = [I.D]*[1/h](x-2);
id [1/h](x?)*Kext = Kext*[1/h](x-4);
id [1/h](x?)*end = [1/d](x)*end;
endrepeat;
id [I.D]^2 = [ID2];

id [ID2] = -(d-8)*(LapT -2*(Jb - (1/2)*K*[1/d](-2)) - (d-7)*(dn^2
     +2*(2*H*dn - RhoNN - [1/d](-2)*K +5*H^2*(1/2))));
sum a,b;
#call makeCommute()
```

Apply definition of $\ddot{K}$.

Reexpress $I \cdot \hat{D}$ as $I \cdot D \circ \frac{1}{\mathrm{h}-2}$
and evaluate weight-dependent operators.

Apply Equation 4.7 of [44]
to weight -2 scalars.

Next, we perform Riemannian computations reexpressing terms containing normal derivatives and gradients in terms of fundamental forms and other hypersurface tensors.

```
id once Kext = II0e(a,b)*II0e(a,b);
sum a,b;
```

Reexpressing K^{e} in terms of its canonical extension $(\mathring{\mathrm{I\!I}}^{\mathrm{e}})^2$.

```

#call commuteDnRight()
id dnII0e(a?,b?)*II0e(a?,b?) = II0e(a,b)*dnII0e(a,b);
id dn2II0e(a?,b?)*II0e(a?,b?) = II0e(a,b)*dn2II0e(a,b);
id dn2II0e(a?,b?)*end = dn*dnII0e(a,b)*end;
```

Commute ∇_n right then reorder.

```

repeat;
id once dnII0e(a?,b?) = n(c)*n(e)*Riemann(c,a,b,e) - II0e(a,c)*II0e(c,b)
     + r*II0e(a,b) - (n(a)*dr(b) + n(b)*dr(a)) + d_(a,b)*dnr + Rho(a,b) +
     sigma*(dnRho(a,b) - 3*r*Rho(a,b) + II0e(c,a)*Rho(b,c) +
     II0e(c,b)*Rho(c,a) - dr(a,b));
sum c,e;
endrepeat;
```

Use [**31**, Equation (6.4)] for $\nabla_n \mathring{\mathrm{I\!I}}^{\mathrm{e}}$. Because at most two normal derivatives of $\mathring{\mathrm{I\!I}}^{\mathrm{e}}$ appear, only need $\nabla_n \mathring{\mathrm{I\!I}}^{\mathrm{e}}$ accurate to order σ.

```

#call makeCommute()
#call sigmaIdentities()

id dn*n(a?)*n(e?)*Riemann(a?,b?,c?,e?)*end = dnRn(b,c)*end;
```

Apply definition.

```

#call commuteDnRight()
#call makeCommute()
#call sigmaIdentities()
```

Push ∇_n right again because $\nabla_n \circ (\nabla_n \mathring{\mathrm{I\!I}}^{\mathrm{e}})$ contains simplifiable monomials.

```

id once dnII0e(a?,b?) = n(c)*n(e)*Riemann(c,a,b,e) - II0e(a,c)*II0e(c,b)
     + r*II0e(a,b) - (n(a)*dr(b) + n(b)*dr(a)) + d_(a,b)*dnr + Rho(a,b) +
     sigma*(dnRho(a,b) - 3*r*Rho(a,b) + II0e(c,a)*Rho(b,c) +
     II0e(c,b)*Rho(c,a) - dr(a,b));
sum c,e;
#call makeCommute()
#call sigmaIdentities()
```

Pushing ∇_n right in $\nabla_n \circ \mathring{\mathrm{I\!I}}^{\mathrm{e}}$, requires reapplication of the identity above.

```

id once start*dnRn(a?,b?)*end = start*(H*n(c)*n(e)*Riemann(c,a,b,e) +
     (II0e(c,e) + H*d_(c,e))*Riemann(e,a,b,c) + n(c)*del(b)*Ric(a,c) -
     (d-2)*dnRho(a,b) - d_(a,b)*dnJ + delt(e)*n(f)*Riemann(f,b,e,a))*end;
sum c,e,f;
#call makeCommute()
#call sigmaIdentities()
```

Apply general dimension d analog of the identity for $\nabla_n(n^c n^d R_{cabd})$ given in [**31**].

```

id Riemann(a?,b?,c?,e?) = Weyl(a,b,c,e) + d_(a,c)*Rho(b,e) -
     d_(b,c)*Rho(a,e) - d_(a,e)*Rho(b,c) + d_(b,e)*Rho(a,c);
id Ric(a?,b?) = (d-2)*Rho(a,b) + J*d_(a,b);
```

Convert Riemann to Weyl and Schouten tensors.

```

id once dnRho(a?,b?)*end = n(c)*del(c)*Rho(a,b)*end;
sum c;
id del(a?)*Rho(b?,c?) = Cotton(a,b,c) + del(b)*Rho(a,c);
#call makeCommute()
#call sigmaIdentities()
```

Convert $\nabla_a P_{bc}$ to Cotton tensor $C_{abc} := \nabla_a P_{bc} - \nabla_b P_{ac}$.

At this point in the program, all ambient gradients and normal derivatives have been evaluated in terms of objects along the hypersurface or higher fundamental forms. The final segment of the program uses procedures `commuteThingsLeft()` and hypersurface identities to simplify the result.

```
id LapT*IIOe(a?,b?)*IIOe(a?,b?)*end = LapB*IIOe(a,b)*IIOe(a,b)*end;
id start*LapB*IIOe(a?,b?)*IIOe(a?,b?) = start*LapB*K;
```

$(\mathring{\mathrm{I\!I}}^{e})^2|_\Sigma = K$ and $\Delta^\top K = \bar{\Delta} K$.

```
#call commuteThingsLeft()
```

Push $\nabla^\top$ as far right as possible.

```

id nC(a?)*start*delt(b?)*RhoT(a?,c?) = -II(a,b)*start*RhoT(a,c);
id nC(a?)*start*delt(b?)*Weyl(a?,ap?,bp?,cp?) =
     start*delt(b)*Weyln(cp,bp,ap) - II(a,b)*start*Weyl(a,ap,bp,cp);
```

Commute n right whenever it can be contracted with hypersurface tensors (which yields zero).

```

if (occurs(dn,del)=0);
id Weyln(a?,b?,c?) = Weylnt(a,b,c) + n(a)*Wn(b,c) - n(b)*Wn(a,c);
id once Weylnt(a?,b?,c?) = dbIIO(a,b,c) - dbIIO(b,a,c) +
     [1/d](-2)*(dbIIO(e,e,a)*[Gb_](b,c) - dbIIO(e,e,b)*[Gb_](a,c));
sum e;
endif;
```

`if` command ensure absence of ambient gradients or normal derivatives; in this case rewrite $W_{abc\hat{n}}$ as $W^\top_{abc\hat{n}}$ + more and apply the Codazzi-Mainardi equation.

```

#call commuteThingsLeft()
```

Push $\nabla^\top$ as far right as possible.

```

id once dbIIo(c?,a?,c?,b?) = riemannB(c,a,c,e)*IIo(e,b) +
     riemannB(c,a,b,e)*IIo(c,e) + dbIIo(a,c,c,b);
sum e;
id riemannB(a?,b?,c?,e?) = weylb(a,b,c,e) + [gb_](a,c)*rhoB(b,e) -
     [gb_](b,c)*rhoB(a,e) - [gb_](a,e)*rhoB(b,c) + [gb_](b,e)*rhoB(a,c);
#call sigmaIdentities()
```

Reorder hypersurface gradients modulo the hypersurface Riemann curvature, and convert to hypersurface Weyl and Schouten tensors.

```

id once IIo(a?,b?) = TenC(a,b);
id TenC(a?,b?)*dbIIo(b?,c?,c?,a?) = TenC(a,b)*dbIIo(a,c,c,b);
id TenC(a?,b?)*IIIo(b?,c?) = TenC(b,a)*IIIo(b,c);
.sort;
id TenC(a?,b?) = IIo(a,b);
```

Help FORM simplify contractions with symmetric tensors.

```

id del(a?)*end = 0;
id delt(a?)*end = 0;
id delb(a?)*end = 0;
id delt(a?)*start*end = 0;
id dn*end = 0;
```

Eliminate all derivatives on far right of the expression, *i.e.* $\nabla(1) = 0$.

```

id start*end = 1;
```

Remove unneeded `start` and `end` markers.

```

id [1/d](x?) = drat(1,'dpp'+x);
id d = drat('dpp',1);
```

Convert coefficients `d` and `[1/d](x)` to the PolyRatFun `drat`.

```
Format 180;
print +s;
.end
```

The above code has been assembled into the FORM file `Kdd.frm` attached to this document. The FORM file, when run, outputs the following result:

```
 Kdd =
     + k^2*drat(31,18)
     + k*jb*drat(2,1)
     + Hc*IIo(N1_?,N2_?)*IIo(N1_?,N3_?)*IIo(N2_?,N3_?)*drat(-4,1)
     + Hc*IIo(N1_?,N2_?)*IIIo(N1_?,N2_?)*drat(-4,1)
     + nC(N1_?)*IIo(N2_?,N3_?)*cotton(N1_?,N2_?,N3_?)*drat(-4,1)
     + rhoB(N1_?,N2_?)*IIo(N1_?,N3_?)*IIo(N2_?,N3_?)*drat(20,1)
     + weylb(N1_?,N2_?,N3_?,N4_?)*IIo(N1_?,N4_?)*IIo(N2_?,N3_?)*drat(8,1)
     + IIo(N1_?,N2_?)*IIo(N1_?,N3_?)*IIIo(N2_?,N3_?)*drat(-2,1)
     + IIo(N1_?,N2_?)*dbIIo(N1_?,N3_?,N3_?,N2_?)*drat(20,3)
     + IIo(N1_?,N2_?)*dbIIo(N3_?,N3_?,N1_?,N2_?)*drat(-4,1)
     + dbIIo(N1_?,N1_?,N2_?)*dbIIo(N3_?,N3_?,N2_?)*drat(8,9)
     + IIIo(N1_?,N2_?)*IIIo(N1_?,N2_?)*drat(4,1)
     + start*LapB*K*end*drat(1,1)
    ;
```

This result matches the quoted expression in Equation (4.16) for $\ddot{K}$ when $d = 5$, so the proof is complete.

A.4. FORM Code for Proofs in Chapter 6

To prove the results in Chapter 6, we also used FORM computations. These can be found in the ancillary files in the arXiv entry for [**10**]. In particular, we prove Theorem 6.3.1, Corollary 6.4.3, Proposition 6.5.2, Lemma 6.5.4, Theorem 6.5.5, and Theorem 6.6.2. Most of the computations performed in the files mentioned use the methods described above—but here, we use header and procedure files for centralization.

A.4.1. Declarations. In the files `Headers/symbol-index-declarations.h`, `Headers/function-declarations.h`, `Headers/tractor-declarations.h`, and `Headers/Riem-declarations.h`, we declared a number of objects that are associated to particular variables, functions, tractors, and tensors, respectively. As above, we provide tables of these declarations and their mathematical counterparts. Much of these are duplicated, but we include them anyway.

First, note that our convention is that (with few exceptions) indices that are enclosed in square brackets `[*]` are to be viewed as "free" or floating indices that are not contracted onto other indices, and the same set of indices without the square brackets are to be viewed as dummy indices. We use lower-case letters at the beginning of the alphabet (except "d") to represent Riemannian indices and upper-case letters at the beginning of the alphabet (except "D") to represent tractor indices. One exception is that we typically use upper-case letters for argument fields, regardless of the type of index that those argument fields represent.

The symbols `d` and `db` represent the dimension of the bulk manifold and the hypersurface respectively. The symbols `x,ep,alpha,beta,invep,invalpha` are dummy variables that typically represent real numbers. In particular, `ep` is often used for dimensional continuation purposes as ε, `invep` is its inverse ε^{-1}, and similarly for `alpha` and `invalpha`.

Some of the scalars and functions not listed in the below table are documented in the FORM file itself. Table A.2 contains all of the functions and scalars declared in `Headers/function-declarations.h`.

Function	Mathematical Object
`H`	H
`Jb`	$\bar{J}$
`r`	ρ
`dnr`	$\nabla_n \rho$
`dn2r`	$\nabla_n^2 \rho$
`dn3r`	$\nabla_n^3 \rho$
`J`	J
`dnJ`	$\nabla_n J$
`dn2J`	$\nabla_n^2 J$
`RhoNN`	P_{nn}
`BachNN`	B_{nn}
`sigma`	σ
`K`	K
`Kd`	$\dot{K}$
`Kdd`	$\ddot{K}$
`Kddd`	$\dddot{K}$
`DbDbPdT`	$\hat{\bar{D}}^A \hat{\bar{D}}^B \dot{P}^t_{AB}$
`f`	f
`[tau^(5/2-d/2)]`	$\tau^{\frac{5-d}{2}}$

Function	Mathematical Object
`h(x)`	$\mathsf{h} + x$
`[1/h](x)`	$\frac{1}{\mathsf{h}+x}$
`[I.D]`	$I \cdot D$
`[I.Dh]`	$I \cdot \hat{D}$
`[Dth.IDh2.Dth]`	$\hat{D}^{T\,A} I \cdot \hat{D}^2 \hat{D}^T_A$
`dn`	∇_n
`Lap`	Δ
`LapT`	$\Delta^\top$
`LapB`	$\bar{\Delta}$
`R1`	R_1
`R2`	R_2
`R3`	R_3
`R4`	R_4
`R`	$\mathcal{R}$

TABLE A.2. FORM functions and their corresponding mathematical objects.

Table A.3 contains all of the tractors declared in `Headers/tractor-declarations.h`. Tables A.4 and A.5 contains all of the Riemannian tensors declared in `Headers/Riem-declarations.h`.

A.4.2. Execution. To run the computations, first ensure that FORM is in the path. The version of FORM required to run these computations requires a unix-based operating system, so if one wishes to run these computations on a Windows machine, they must use Cygwin and install FORM there. Once FORM is installed in the path, executing (as a bash script) the computations is as simple as calling `bash FileName.sh`. To clear the saved files, one should execute the script `DeleteData.sh`. Each script is named according to the result it proves, with the exception of `PreliminaryComputations.sh`, which provides lower-level identities to be used elsewhere.

Tensor	Object
`D(A)`	D_A
`I(A)`	I_A
`[I](A,B)`	$h_{AB} - I_A I_B$
`Dhat(A)`	$\hat{D}_A$
`DhTemp(A)`	$\hat{D}_A$
`Dth(A)`	$\hat{D}_A^T$
`DthTemp(A)`	$\hat{D}_A^T$
`Dthext(A)`	$\hat{D}_A^{T\mathrm{e}}$
`Db(A)`	$\bar{D}_A$
`Dbh(A)`	$\hat{\bar{D}}_A$
`P(A,B)`	P_{AB}
`DK(A,...,C)`	$\hat{D}_C \cdots \hat{D}_A K$
`DtK(A,...,C)`	$\hat{D}_C^T \cdots \hat{D}_A^T K$
`DKd(A,...,C)`	$\hat{D}_C \cdots \hat{D}_A \dot{K}$
`DtKd(A,...,C)`	$\hat{D}_C^T \cdots \hat{D}_A^T \dot{K}$
`DbKd(A,...,C)`	$\hat{\bar{D}}_C \cdots \hat{\bar{D}}_A \dot{K}$
`DKdd(A,...,C)`	$\hat{D}_C \cdots \hat{D}_A \ddot{K}$
`DtP(B,C,A)`	$\hat{D}_A^T P_{BC}$
`DP(C,D,A,...,B)`	$\hat{D}_B \cdots \hat{D}_A P_{CD}$
`Pd(A,B)`	$\dot{P}_{AB}$
`Pdd(A,B)`	$\ddot{P}_{AB}$
`Pddd(A,B)`	$\dddot{P}_{AB}$
`DPd(C,D,A,...,B)`	$\hat{D}_B \cdots \hat{D}_A \dot{P}_{CD}$
`DPdd(C,D,A,...,B)`	$\hat{D}_B \cdots \hat{D}_A \ddot{P}_{CD}$
`Pdt(A,B)`	$\mathring{\top}(\dot{P}_{AB})$
`PdT(A,B)`	$\dot{P}_{AB}^t := (\bar{r} \circ \mathring{\top})(\dot{P}_{AB})$
`DbPdT(A)`	$\hat{\bar{D}}^B \dot{P}_{AB}^t$
`[Dh, Dh](A,B)`	$[\hat{D}_A, \hat{D}_B]$
`[Dbh,Dbh](A,B)`	$[\hat{\bar{D}}_A, \hat{\bar{D}}_B]$

Tensor	Object
`W(A,B,C,D)`	W_{ABCD}
`Wt(A,B,C,D)`	$W_{ABCD}^{\top}$
`Wb(A,B,C,D)`	$\bar{W}_{ABCD}$
`Ln(A,B)`	L_{AB}
`DtLn(C,D,A,...,B)`	$\hat{D}_B^T \cdots \hat{D}_A^T L_{CD}$
`DbLn(C,D,A,...,B)`	$\hat{\bar{D}}_B \cdots \hat{\bar{D}}_A L_{CD}$
`Fn(A,B)`	F_{AB}
`DbFn(C,D,A,...,B)`	$\hat{\bar{D}}_B \cdots \hat{\bar{D}}_A F_{CD}$
`Jn(A,B)`	J_{AB}
`DbK(A,...,B)`	$\hat{\bar{D}}_B \cdots \hat{\bar{D}}_A K$
`Gamma(A,B,C)`	Γ_{ABC}
`Wn(A,B,C)`	W_{IABC}
`Wd(A,B,C,D)`	$\dot{W}_{ABCD}$
`Wdd(A,B,C,D)`	$\ddot{W}_{ABCD}$
`Wddd(A,B,C,D)`	$\dddot{W}_{ABCD}$
`DW(A,B,C,D,E,...,F)`	$\hat{D}_F \cdots \hat{D}_E W_{ABCD}$
`DWd(A,B,C,D,E,...,F)`	$\hat{D}_F \cdots \hat{D}_E \dot{W}_{ABCD}$
`DWdd(A,B,C,D,E,...,F)`	$\hat{D}_F \cdots \hat{D}_E \ddot{W}_{ABCD}$
`Wnn(A,B)`	W_{IABI}
`[I,I.Dh](A)`	$[I_A, I\cdot\hat{D}]$
`[I.Dh,I](A)`	$[I\cdot\hat{D}, I_A]$
`[hashTr](A,B)`	$(h_A \boldsymbol{.} h_B \boldsymbol{.})^\sharp$
`[hashTrB](A,B)`	$(\bar{h}_A \boldsymbol{.} \bar{h}_B \boldsymbol{.})^\sharp$
`X(A)`	X_A
`Y(A)`	Y_A
`Z(A,a)`	Z_A^a
`Yb(A)`	$\bar{Y}_A$
`Zb(A,a)`	$\bar{Z}_A^a$

TABLE A.3. FORM tensors representing tractors and the corresponding tractors (or tractor-valued operators). Note that most of these have corresponding "FORM-commuting" counterparts. The names of these counterparts can be found in the set `commTracs`.

Tensor	Object
`[G_](a,b)`	g_{ab}
`[Gb_](a,b)`	$\bar{g}_{ab}$
`n(a)`	n_a
`[d^n^](a,...,b,c)`	$\nabla_a \cdots \nabla_b n_c$
`IInc(a,b)`	II_{ab}
`II0(a,b)`	$\mathring{\mathrm{II}}_{ab}$
`II02(a,b)`	$\mathring{\mathrm{II}}_a^c \mathring{\mathrm{II}}_{cb}$
`II03(a,b)`	$\mathring{\mathrm{II}}_a^c \mathring{\mathrm{II}}_c^d \mathring{\mathrm{II}}_{db}$
`dtII0(a,...,b,c,d)`	$\nabla_a^\top \cdots \nabla_b^\top \mathring{\mathrm{II}}_{cd}$
`dbII0(a,...,b,c,d)`	$\bar{\nabla}_a \cdots \bar{\nabla}_b \mathring{\mathrm{II}}_{cd}$
`dbivII0(a)`	$\bar{\nabla} \cdot \mathring{\mathrm{II}}_a$
`FNC(a,b)`	F_{ab}
`F0(a,b)`	$\mathring{F}_{ab}$
`dtF0(a,...,b,c,d)`	$\nabla_a^\top \cdots \nabla_b^\top \mathring{F}_{cd}$
`dbF0(a,...,b,c,d)`	$\bar{\nabla}_a \cdots \bar{\nabla}_b \mathring{F}_{cd}$
`dbivF0(a)`	$\bar{\nabla} \cdot \mathring{F}$
`IV0(a,b)`	$\mathring{\mathrm{IV}}_{ab}$
`dbIV0(a,...,b,c,d)`	$\bar{\nabla}_a \cdots \bar{\nabla}_b \mathring{\mathrm{IV}}_{cd}$
`Riemann(a,b,c,d)`	R_{abcd}
`RiemannB(a,b,c,d)`	$\bar{R}_{abcd}$
`Weyl(a,b,c,d)`	W_{abcd}
`Weyln(a,b,c)`	W_{abcn}
`Weylnt(a,b,c)`	$W_{abcn}^\top$
`Weylnn(a,b)`	W_{nabn}
`Weylt(a,b,c,d)`	$W_{abcd}^\top$
`dbivWeylnt(a,b)`	$\bar{\nabla}^c W_{a(bc)\hat{n}}^\top$
`Weylb(a,b,c,d)`	$\bar{W}_{abcd}$
`Ric(a,b)`	Ric_{ab}
`Rho(a,b)`	P_{ab}
`RhoN(a)`	P_{an}
`RhoNT(a)`	$P_{an}^\top$
`RhoT(a,b)`	$P_{ab}^\top$

Tensor	Object
`RhoB(a,b)`	$\bar{P}_{ab}$
`dbRhoB(a,...,b,c,d)`	$\bar{\nabla}_a \cdots \bar{\nabla}_b \bar{P}_{cd}$
`Cotton(a,b,c)`	C_{abc}
`CottonT(a,b,c)`	$C_{abc}^\top$
`CottonB(a,b,c)`	$\bar{C}_{abc}$
`CottonN(a,b)`	C_{nab}
`CottonNT(a,b)`	$C_{nab}^\top$
`CottonNTS(a,b)`	$C_{n(ab)}^\top$
`CottonNL(a,b)`	C_{abn}
`CottonNLT(a,b)`	$C_{abn}^\top$
`CottonNN(a)`	C_{ann}
`delK(a,...,b)`	$\nabla_a \cdots \nabla_b K$
`deltK(a,...,b)`	$\nabla_a^\top \cdots \nabla_b^\top K$
`delbK(a,...,b)`	$\bar{\nabla}_a \cdots \bar{\nabla}_b K$
`deltRhoNN(a,...,b)`	$\nabla_a^\top \cdots \nabla_b^\top P_{nn}$
`delbRhoNN(a,...,b)`	$\bar{\nabla}_a \cdots \bar{\nabla}_b P_{nn}$
`dRho(a,...,b,c,d)`	$\nabla_a \cdots \nabla_b P_{cd}$
`dnRho(a,b)`	$\nabla_n P_{ab}$
`dn2Rho(a,b)`	$\nabla_n^2 P_{ab}$
`dn3Rho(a,b)`	$\nabla_n^3 P_{ab}$
`dnWeylnn(a,b)`	$\nabla_n W_{nabn}$
`dn2Weylnn(a,b)`	$\nabla_n^2 W_{nabn}$
`dbH(a,...,b)`	$\bar{\nabla}_a \cdots \bar{\nabla}_b H$
`Bach(a,b)`	B_{ab}
`BachT(a,b)`	$B_{ab}^\top$
`BachB(a,b)`	$\bar{B}_{ab}$
`BachN(a)`	B_{an}
`BachNT(a)`	$B_{an}^\top$
`dJ(a,...,b)`	$\nabla_a \cdots \nabla_b J$
`dtJ(a,...,b)`	$\nabla_a^\top \cdots \nabla_b^\top J$
`dbJ(a,...,b)`	$\bar{\nabla}_a \cdots \bar{\nabla}_b \bar{J}$
`dr(a,...,b)`	$\nabla_a \cdots \nabla_b \rho$

TABLE A.4. FORM tensors representing Riemmanian tensors and the corresponding tensors (or tensor-valued operators). Note that most of these have corresponding FORM-commuting counterparts. The names of these counterparts can be found in the set `commTens`.

Tensor	Object
`dCotton(a,b,c,d)`	$\nabla_a C_{bcd}$
`dWeyl(a,b,c,d,e)`	$\nabla_a W_{bcde}$
`dBach(a,b,c)`	$\nabla_a B_{bc}$
`[hashR](a,b)`	$(g_a \,\bullet\, g_b \,\bullet)^\sharp$
`[hashRB](a,b)`	$(\bar{g}_a \,\bullet\, \bar{g}_b \,\bullet)^\sharp$
`IIOe(a,b)`	$\mathring{\mathrm{II}}^{\mathrm{e}}_{ab}$

Tensor	Object
`dnIIOe(a,b)`	$\nabla_n \mathring{\mathrm{II}}^{\mathrm{e}}_{ab}$
`dn2IIOe(a,b)`	$\nabla_n^2 \mathring{\mathrm{II}}^{\mathrm{e}}_{ab}$
`dn3IIOe(a,b)`	$\nabla_n^3 \mathring{\mathrm{II}}^{\mathrm{e}}_{ab}$
`dbf(a,...,b)`	$\bar{\nabla}_a \cdots \bar{\nabla}_b f$
`grdivIIoS(a,b)`	$\bar{\nabla}_{(a} \bar{\nabla} \cdot \mathring{\mathrm{II}}_{b)}$
`grdivIIoA(a,b)`	$\bar{\nabla}_{[a} \bar{\nabla} \cdot \mathring{\mathrm{II}}_{b]}$

TABLE A.5. FORM tensors representing Riemmanian tensors and the corresponding tensors (or tensor-valued operators). Note that most of these have corresponding FORM-commuting counterparts. The names of these counterparts can be found in the set `commTens`.

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