THE p-ADIC GELFAND-KAPRANOV-ZELEVINSKY HYPERGEOMETRIC COMPLEX

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ABSTRACT. To a torus action on a complex vector space, Gelfand, Kapranov and Zelevinsky introduce a system of differential equations, called the GKZ hypergeometric system. Its solutions are GKZ hypergeometric functions. We study the p-adic counterpart of the GKZ hypergeometric system. In the language of dagger spaces introduced by Grosse-Klönne, the p-adic GKZ hypergeometric complex is a twisted relative de Rham complex of meromorphic differential forms with logarithmic poles for an affinoid toric dagger space over the dagger unit polydisc. It is a complex of of $\mathcal{O}^\dagger$-modules with integrable connections and with Frobenius structures defined on the dagger unit polydisc such that traces of Frobenius on fibers at Techmüller points define the hypergeometric function over the finite field introduced by Gelfand and Graev.

Key words: GKZ hypergeometric system, $\mathcal{D}^\dagger$-modules, twisted de Rham complex, Dwork trace formula.

Mathematics Subject Classification: Primary 14F30; Secondary 11T23, 14G15, 33C70.

INTRODUCTION

0.1. **The GKZ hypergeometric system.** Let

$$A = \begin{pmatrix} w_{11} & \cdots & w_{1N} \\ \vdots & & \vdots \\ w_{n1} & \cdots & w_{nN} \end{pmatrix}$$

be an $(n\times N)$-matrix of rank n with integer entries. Denote the column vectors of A by $\mathbf{w}_1, \ldots, \mathbf{w}_N \in \mathbb{Z}^n$. It defines an action of the n-dimensional torus $\mathbb{T}^n_{\mathbb{Z}} = \operatorname{Spec} \mathbb{Z}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ on the N-dimensional affine space $\mathbb{A}^N_{\mathbb{Z}} = \operatorname{Spec} \mathbb{Z}[x_1, \ldots, x_N]$:

$$\mathbb{T}^n_{\mathbb{Z}} \times \mathbb{A}^N_{\mathbb{Z}} \to \mathbb{A}^N_{\mathbb{Z}}, \quad \Big((t_1, \ldots, t_n), (x_1, \ldots, x_N)\Big) \mapsto (t_1^{w_{11}} \cdots t_n^{w_{n1}} x_1, \ldots, t_1^{w_{1N}} \cdots t_n^{w_{nN}} x_N).$$

Let $\gamma_1, \ldots, \gamma_n \in \mathbb{C}$. In [10], Gelfand, Kapranov and Zelevinsky define the *A-hypergeometric system* to be the system of differential equations

$$\begin{aligned} &\textstyle\sum_{j=1}^N w_{ij} x_j \frac{\partial f}{\partial x_j} + \gamma_i f = 0 \quad (i = 1, \ldots, n), \\ &\textstyle\prod_{\lambda_j > 0} \left(\frac{\partial}{\partial x_j}\right)^{a_j} f = \prod_{\lambda_j < 0} \left(\frac{\partial}{\partial x_j}\right)^{-a_j} f, \end{aligned} \tag{0.1.1}$$

where for the second system of equations, $(\lambda_1, \ldots, \lambda_N) \in \mathbb{Z}^N$ goes over the family of integral linear relations

$$\sum_{j=1}^N \lambda_j \mathbf{w}_j = 0$$

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among $\mathbf{w}_1, \ldots, \mathbf{w}_N$. We call the A-hypergeometric system as the *GKZ hypergeometric system.* An integral representation of a solution of the GKZ hypergeometric system is given by

$$f(x_1, \ldots, x_N) = \int_\Sigma t_1^{\gamma_1} \cdots t_n^{\gamma_n} e^{\sum_{j=1}^N x_j t_1^{w_{1j}} \cdots t_n^{w_{nj}}} \frac{dt_1}{t_1} \cdots \frac{dt_n}{t_n} \tag{0.1.2}$$

where Σ is a real n-dimensional cycle in $\mathbb{T}^n$. Confer [1, equation (2.6)], [4, section 3] and [8, Corollary 2 in §4.2].

0.2. **The GKZ hypergeometric function over finite fields.** Let p be a prime number, q a power of p, $\mathbb{F}_q$ the finite field with q elements, $\psi : \mathbb{F}_q \to \overline{\mathbb{Q}}^*$ a nontrivial additive character, and $\chi_1, \ldots, \chi_n : \mathbb{F}_q^* \to \overline{\mathbb{Q}}^*$ multiplicative characters. In [7] and [9], Gelfand and Graev define the *hypergeometric function over the finite field* to be the function defined by the family of twisted exponential sums

$$\mathrm{Hyp}(x_1, \ldots, x_N) = \sum_{t_1, \ldots, t_n \in \mathbb{F}_q^*} \chi_1(t_1) \cdots \chi_n(t_n) \psi\Big(\sum_{j=1}^N x_j t_1^{w_{1j}} \cdots t_n^{w_{nj}}\Big), \tag{0.2.1}$$

where $(x_1, \ldots, x_N)$ varies in $\mathbb{A}^N(\mathbb{F}_q)$. It is an arithmetic analogue of the expression (0.1.2).

In [5], we introduce the ℓ-adic GKZ hypergeometric sheaf Hyp which is a perverse sheaf on $\mathbb{A}^N_{\mathbb{F}_q}$ such that for any rational point $x = (x_1, \ldots, x_N) \in \mathbb{A}^N(\mathbb{F}_q)$, we have

$$\mathrm{Hyp}(x_1, \ldots, x_N) = (-1)^{n+N} \mathrm{Tr}(\mathrm{Frob}_x, \mathrm{Hyp}_{\bar{x}}),$$

where Frob_x is the geometric Frobenius at x. In this paper, we study the p-adic counterpart of the GKZ hypergeometric system. It is a complex of $\mathcal{O}^\dagger$-modules with integrable connections and with Frobenius structures defined on the dagger space ([11]) corresponding to the unit polydisc so that traces of Frobenius on fibers at Techmüller points are given by $\mathrm{Hyp}(x_1, \ldots, x_N)$.

0.3. **The p-adic GKZ hypergeometric complex.** For any $\mathbf{v} = (v_1, \ldots, v_N) \in \mathbb{Z}^N_{\geq 0}$ and $\mathbf{w} = (w_1, \ldots, w_n) \in \mathbb{Z}^n$, write

$$\mathbf{x}^\mathbf{v} = x_1^{v_1} \cdots x_N^{v_N}, \quad \mathbf{t}^\mathbf{w} = t_1^{w_1} \cdots t_n^{w_n}, \quad |\mathbf{v}| = v_1 + \cdots + v_N.$$

Let K be a finite extension of $\mathbb{Q}_p$ containing an element π satisfying

$$\pi^{p-1} + p = 0.$$

Denote by $|\cdot|$ the p-adic norm on K defined by $|a| = p^{-\mathrm{ord}_p(a)}$. For each real number $r > 0$, consider the algebras

$$\begin{aligned} K\{r^{-1}\mathbf{x}\} &= \{\sum_{\mathbf{v} \in \mathbb{Z}^N_{\geq 0}} a_\mathbf{v} \mathbf{x}^\mathbf{v} : \ a_\mathbf{v} \in K, \ |a_\mathbf{v}| r^{|\mathbf{v}|} \text{ are bounded}\}, \\ K\langle r^{-1}\mathbf{x}\rangle &= \{\sum_{\mathbf{v} \in \mathbb{Z}^N_{\geq 0}} a_\mathbf{v} \mathbf{x}^\mathbf{v} : \ a_\mathbf{v} \in K, \ \lim_{|\mathbf{v}| \to \infty} |a_\mathbf{v}| r^{|\mathbf{v}|} = 0\}. \end{aligned}$$

They are Banach K-algebras with respect to the norm

$$\|\sum_{\mathbf{v} \in \mathbb{Z}^N_{\geq 0}} a_\mathbf{v} \mathbf{x}^\mathbf{v}\|_r = \sup |a_\mathbf{v}| r^{|\mathbf{v}|}.$$

We have $K\langle r^{-1}\mathbf{x}\rangle \subset K\{r^{-1}\mathbf{x}\}$. Elements in $K\langle r^{-1}\mathbf{x}\rangle$ are exactly those power series converging in the closed polydisc $\{(x_1, \ldots, x_N) : \ x_i \in \overline{\mathbb{Q}}_p, \ |x_i| \leq r\}$. Moreover, for any $r < r'$, we have

$$K\{r'^{-1}\mathbf{x}\} \subset K\langle r^{-1}\mathbf{x}\rangle \subset K\{r^{-1}\mathbf{x}\}.$$

Let

$$K\langle \mathbf{x}\rangle^\dagger = \bigcup_{r>1} K\{r^{-1}\mathbf{x}\} = \bigcup_{r>1} K\langle r^{-1}\mathbf{x}\rangle.$$

$K\{\mathbf{x}\}^\dagger$ is the ring of *over-convergent* power series, that is, series converging in closed polydiscs of radii >1.

Let Δ be the convex hull of $\{0, \mathbf{w}_1, \ldots, \mathbf{w}_N\}$ in $\mathbb{R}^n$, and let δ be the convex polyhedral cone generated by $\{\mathbf{w}_1, \ldots, \mathbf{w}_N\}$. For any $\mathbf{w} \in \delta$, define

$$d(\mathbf{w}) = \inf\{a > 0 :\ \mathbf{w} \in a\Delta\}.$$

We have

$$d(a\mathbf{w}) = ad(\mathbf{w}), \quad d(\mathbf{w} + \mathbf{w}') \leq d(\mathbf{w}) + d(\mathbf{w}')$$

whenever $a \geq 0$ and $\mathbf{w}, \mathbf{w}' \in \delta$. There exists an integer $d > 0$ such that we have $d(\mathbf{w}) \in \frac{1}{d}\mathbb{Z}$ for all $\mathbf{w} \in \mathbb{Z}^n$. For any real numbers $r > 0$ and $s \geq 1$, define

$$\begin{aligned}
L(r,s) &= \{\sum_{\mathbf{w}\in\mathbb{Z}^n\cap\delta} a_{\mathbf{w}}(\mathbf{x})\mathbf{t}^{\mathbf{w}} :\ a_{\mathbf{w}}(\mathbf{x}) \in K\{r^{-1}\mathbf{x}\},\ \|a_{\mathbf{w}}(\mathbf{x})\|_r s^{d(\mathbf{w})} \text{ are bounded}\} \\
&= \{\sum_{\mathbf{v}\in\mathbb{Z}_{\geq 0}^N,\ \mathbf{w}\in\mathbb{Z}^n\cap\delta} a_{\mathbf{v}\mathbf{w}}\mathbf{x}^{\mathbf{v}}\mathbf{t}^{\mathbf{w}} :\ a_{\mathbf{v}\mathbf{w}} \in K,\ |a_{\mathbf{v}\mathbf{w}}| r^{|\mathbf{v}|} s^{d(\mathbf{w})} \text{ are bounded}\}, \\
L^\dagger &= \bigcup_{r>1,\ s>1} L(r,s).
\end{aligned}$$

Note that $L(r,s)$ and $L^\dagger$ are rings. Let

$$F(\mathbf{x},\mathbf{t}) = \sum_{j=1}^N x_j t_1^{w_{1j}} \cdots t_n^{w_{nj}},$$

Consider the *twisted de Rham complex* $C^\cdot(L^\dagger)$ defined as follows: We set

$$C^k(L^\dagger) = \{\sum_{1\leq i_1<\cdots<i_k\leq n} f_{i_1\ldots i_k} \frac{\mathrm{d}t_{i_1}}{t_{i_1}} \wedge \cdots \wedge \frac{\mathrm{d}t_{i_k}}{t_{i_k}} :\ f_{i_1\ldots i_k} \in L^\dagger\} \cong L^{\dagger\binom{n}{k}}$$

with differential $d : C^k(L^\dagger) \to C^{k+1}(L^\dagger)$ given by

$$\begin{aligned}
d(\omega) &= \Big(t_1^{\gamma_1}\cdots t_n^{\gamma_n}\exp(\pi F(\mathbf{x},\mathbf{t}))\Big)^{-1} \circ \mathrm{d}_{\mathbf{t}} \circ \Big(t_1^{\gamma_1}\cdots t_n^{\gamma_n}\exp(\pi F(\mathbf{x},\mathbf{t}))\Big)(\omega) \\
&= \mathrm{d}_{\mathbf{t}}\omega + \sum_{i=1}^n \Big(\gamma_i + \pi\sum_{j=1}^N w_{ij}x_j\mathbf{t}^{\mathbf{w}_j}\Big)\frac{\mathrm{d}t_i}{t_i} \wedge \omega
\end{aligned}$$

for any $\omega \in C^k(L^\dagger)$, where $\mathrm{d}_{\mathbf{t}}$ is the exterior derivative with respect to the $\mathbf{t}$ variable. For each $j \in \{1, \ldots, N\}$, define $\nabla_{\frac{\partial}{\partial x_j}} : C^\cdot(L^\dagger) \to C^\cdot(L^\dagger)$ by

$$\begin{aligned}
\nabla_{\frac{\partial}{\partial x_j}}(\omega) &= \Big(t_1^{\gamma_1}\cdots t_n^{\gamma_n}\exp(\pi F(\mathbf{x},\mathbf{t}))\Big)^{-1} \circ \frac{\partial}{\partial x_j} \circ \Big(t_1^{\gamma_1}\cdots t_n^{\gamma_n}\exp(\pi F(\mathbf{x},\mathbf{t}))\Big) \\
&= \frac{\partial\omega}{\partial x_j} + \pi\mathbf{t}^{\mathbf{w}_j}\omega.
\end{aligned}$$

Since $\frac{\partial}{\partial x_j}$ commutes with $\mathrm{d}_{\mathbf{t}}$, $\nabla_{\frac{\partial}{\partial x_j}}$ commutes with $d : C^k(L^\dagger) \to C^{k+1}(L^\dagger)$. We have integrable connections

$$\nabla : C^\cdot(L^\dagger) \to C^\cdot(L^\dagger) \otimes_{K\langle\mathbf{x}\rangle^\dagger} \Omega^1_{K\langle\mathbf{x}\rangle^\dagger}$$

defined by

$$\nabla(\omega)=\sum_{j=1}^N \nabla_{\frac{\partial}{\partial x_j}}(\omega)\otimes \mathrm{d}x_j,$$

where $\Omega^1_{K\langle \mathbf{x}\rangle^\dagger}$ is the free $K\{\mathbf{x}\}^\dagger$-module with basis $\mathrm{d}x_1,\ldots,\mathrm{d}x_N$.

Consider the lifting of the Frobenius correspondence in the variable $\mathbf{t}$ defined by

$$\Phi(f(\mathbf{x},\mathbf{t}))=f(\mathbf{x},\mathbf{t}^q).$$

One verifies directly that $\Phi(L(r,s))\subset L(r,\sqrt[q]{s})$ and hence $\Phi(L^\dagger)\subset L^\dagger$. It induces maps $\Phi: C^k(L^\dagger)\to C^k(L^\dagger)$ on differential forms commuting with $\mathrm{d}_{\mathbf{t}}$:

$$\Phi\Big(\sum_{1\leq i_1<\cdots<i_k\leq n} f_{i_1\ldots i_k}(\mathbf{x},\mathbf{t})\frac{\mathrm{d}t_{i_1}}{t_{i_1}}\wedge\cdots\wedge\frac{\mathrm{d}t_{i_k}}{t_{i_k}}\Big)=\sum_{1\leq i_1<\cdots<i_k\leq n} q^k f_{i_1\ldots i_k}(\mathbf{x},\mathbf{t}^q)\frac{\mathrm{d}t_{i_1}}{t_{i_1}}\wedge\cdots\wedge\frac{\mathrm{d}t_{i_k}}{t_{i_k}}$$

Suppose furthermore that $\gamma_1,\ldots,\gamma_n\in\frac{1}{1-q}\mathbb{Z}$ and $(\gamma_1,\ldots,\gamma_n)\in\delta$. Consider the maps $F: C^k(L^\dagger)\to C^k(L^\dagger)$ defined by

$$F = \Big(t_1^{\gamma_1}\cdots t_n^{\gamma_n}\exp(\pi F(\mathbf{x},\mathbf{t}))\Big)^{-1}\circ\Phi\circ\Big(t_1^{\gamma_1}\cdots t_n^{\gamma_n}\exp(\pi F(\mathbf{x}^q,\mathbf{t}))\Big) \tag{0.3.1}$$

$$= \Big(t_1^{\gamma_1(q-1)}\cdots t_n^{\gamma_n(q-1)}\exp\big(\pi F(\mathbf{x}^q,\mathbf{t}^q)-\pi F(\mathbf{x},\mathbf{t})\big)\Big)\circ\Phi. \tag{0.3.2}$$

Even though $t_1^{\gamma_1}\cdots t_n^{\gamma_n}\exp(\pi F(\mathbf{x},\mathbf{t}))$ does not lie in $L^\dagger$ and multiplication by it does not define an endomorphism on $C^\cdot(L^\dagger)$, the next Lemma 0.4 (i) shows that $t_1^{\gamma_1(q-1)}\cdots t_n^{\gamma_n(q-1)}\exp\Big(\pi F(\mathbf{x}^q,\mathbf{t}^q)-\pi F(\mathbf{x},\mathbf{t})\Big)$ lie in $L^\dagger$, and hence the expression (0.3.2) shows that F defines endomorphism on each $C^k(L^\dagger)$.

Lemma 0.4.
(i) $t_1^{\gamma_1(q-1)}\cdots t_n^{\gamma_n(q-1)}\exp\big(\pi F(\mathbf{x}^q,\mathbf{t}^q)-\pi F(\mathbf{x},\mathbf{t})\big)$ *and* $t_1^{\gamma_1(1-q)}\cdots t_n^{\gamma_n(1-q)}\exp\big(\pi F(\mathbf{x},\mathbf{t})-\pi F(\mathbf{x}^q,\mathbf{t}^q)\big)$ *lie in* $L(r,r^{-1}p^{\frac{p-1}{pq}})$ *for any* $0<r\leq p^{\frac{p-1}{pq}}$.

(ii) Let $C^{(1)\cdot}(L^\dagger)$ *be the twisted de Rham complex so that* $C^{(1),j}(L^\dagger)=C^j(L^\dagger)$ *for each* k, *and* $d^{(1)}: C^{(1),k}\to C^{(1),k+1}$ *is given by*

$$\begin{aligned} d^{(1)} &= \Big(t_1^{\gamma_1}\cdots t_n^{\gamma_n}\exp(\pi F(\mathbf{x}^q,\mathbf{t}))\Big)^{-1}\circ\mathrm{d}_{\mathbf{t}}\circ\Big(t_1^{\gamma_1}\cdots t_n^{\gamma_n}\exp(\pi F(\mathbf{x}^q,\mathbf{t}))\Big)\\ &= \mathrm{d}_{\mathbf{t}}+\sum_{i=1}^n\Big(\gamma_i+\pi\sum_{j=1}^N w_{ij}x_j^q\mathbf{t}^{\mathbf{w}_j}\Big)\frac{\mathrm{d}t_i}{t_i}.\end{aligned}$$

Let $\nabla^{(1)}$ *be the connection on* $C^{(1)\cdot}(L^\dagger)$ *defined by*

$$\begin{aligned}\nabla^{(1)}_{\frac{\partial}{\partial x_j}} &= \Big(t_1^{\gamma_1}\cdots t_n^{\gamma_n}\exp(\pi F(\mathbf{x}^q,\mathbf{t}))\Big)^{-1}\circ\frac{\partial}{\partial x_j}\circ\Big(t_1^{\gamma_1}\cdots t_n^{\gamma_n}\exp(\pi F(\mathbf{x}^q,\mathbf{t}))\Big)\\ &= \frac{\partial}{\partial x_j}+q\pi x_j^{q-1}\mathbf{t}^{\mathbf{w}_j}.\end{aligned}$$

Then F *defines a horizontal morphism of complexes of* $K\langle\mathbf{x}\rangle^\dagger$*-modules with connections*

$$F:(C^{(1)\cdot}(L^\dagger),\nabla^{(1)})\to(C^\cdot(L^\dagger),\nabla).$$

(iii) Let $E(0,1)^N$ be the closed unit polydisc with the dagger structure sheaf ([11]) associated to the algebra $K\langle\mathbf{x}\rangle^\dagger$, and let Fr *be the lifting*

$$\mathrm{Fr}: E(0,1)^N \to E(0,1)^N, \quad (x_1,\ldots,x_N) \to (x_1^q,\ldots,x_N^q)$$

of the geometric Frobenius correspondence. We have an isomorphism

$$\mathrm{Fr}^*(C^\cdot(L^\dagger),\nabla) \cong (C^{(1)\cdot}(L^\dagger),\nabla^{(1)}).$$

Proof. (i) Write $\exp(\pi z - \pi z^q) = 1 + \sum_{i=1}^\infty c_i z^i$. We have $|c_i| \leq p^{-\frac{p-1}{pq}i}$ by [15, Theorem 4.1]. Write

$$\begin{aligned}\exp(\pi z^q - \pi z) &= 1 - (\sum_{i=1}^\infty c_i z^i) + (\sum_{i=1}^\infty c_i z^i)^2 - \cdots \\ &= 1 + \sum_{i=1}^\infty c_i' z^i.\end{aligned}$$

Then we also have the estimate $|c_i'| \leq p^{-\frac{p-1}{pq}i}$. For the monomial $x_j \mathbf{t}^{\mathbf{w}_j}$, we have

$$\exp\left(\pi(x_j\mathbf{t}^{\mathbf{w}_j})^q - \pi x_j \mathbf{t}^{\mathbf{w}_j}\right) = \sum_{i=0}^\infty c_i' x_j^i \mathbf{t}^{i\mathbf{w}_j},$$

$$\|c_i' x_j^i\|_r \leq p^{-\frac{p-1}{pq}i} r^i = \left(r^{-1}p^{\frac{p-1}{pq}}\right)^{-i} \leq \left(r^{-1}p^{\frac{p-1}{pq}}\right)^{-d(i\mathbf{w}_j)}$$

Here for the last inequality, we use the fact that $d(i\mathbf{w}_j) \leq i$ and the assumption that $r \leq p^{\frac{p-1}{pq}}$. So we have $\exp\left(\pi(x_j\mathbf{t}^{\mathbf{w}_j})^q - \pi x_j\mathbf{t}^{\mathbf{w}_j}\right) \in L(r, r^{-1}p^{\frac{p-1}{pq}})$. Since $r^{-1}p^{\frac{p-1}{pq}} \geq 1$, the space $L(r, r^{-1}p^{\frac{p-1}{pq}})$ is a ring. So $t_1^{\gamma_1(q-1)}\cdots t_n^{\gamma_n(q-1)} \exp\left(\pi F(\mathbf{x}^q,\mathbf{t}^q) - \pi F(\mathbf{x},\mathbf{t})\right)$ lies in $L(r, r^{-1}p^{\frac{p-1}{pq}})$. Similarly $t_1^{\gamma_1(1-q)}\cdots t_n^{\gamma_n(1-q)} \exp\left(\pi F(\mathbf{x},\mathbf{t}) - \pi F(\mathbf{x}^q,\mathbf{t}^q)\right)$ lies in $L(r, r^{-1}p^{\frac{p-1}{pq}})$.

(ii) Using the fact that $\Phi \circ \mathrm{d}_{\mathbf{t}} = \mathrm{d}_{\mathbf{t}} \circ \Phi$ and $\Phi \circ \frac{\partial}{\partial x_j} = \frac{\partial}{\partial x_j} \circ \Phi$, one checks that $F \circ d^{(1)} = d \circ F$ and $F \circ \nabla^{(1)} = \nabla \circ F$.

(iii) Consider the K-algebra homomorphism

$$K\langle y_1,\ldots,y_N\rangle^\dagger \to K\langle x_1,\ldots,x_N\rangle^\dagger, \quad y_j \mapsto x_j^q.$$

This makes $K\langle\mathbf{x}\rangle^\dagger$ a finite $K\langle\mathbf{y}\rangle^\dagger$-algebra. We have a canonical isomorphism

$$\tilde{L}^\dagger \otimes_{K\langle\mathbf{y}\rangle^\dagger} K\langle\mathbf{x}\rangle^\dagger \stackrel{\cong}{\to} L^\dagger,$$

where $\tilde{L}^\dagger$ is defined in the same way as $L^\dagger$ except that we change the variables from x_j to y_j. The connection ∇ on $\tilde{L}^\dagger$ defines a connection on $\tilde{L}^\dagger \otimes_{K\langle\mathbf{y}\rangle^\dagger} K\langle\mathbf{x}\rangle^\dagger$ via the Leibniz rule. Via the above isomorphism, it defines the connection $\mathrm{Fr}^*\nabla$ on $L^\dagger$. Let's verify that it coincides with the connection $\nabla^{(1)}$ on $L^\dagger$. Any element in $L^\dagger$ can be written as a finite sum of elements of the form

$f(\mathbf{x})g(\mathbf{y},\mathbf{t})$ with $f(\mathbf{x})\in K[\mathbf{x}]$ and $g(\mathbf{y},\mathbf{t})\in\tilde{L}^\dagger$. By the Leibniz rule, we have

$$\begin{aligned}
(\mathrm{Fr}^*\nabla)_{\frac{\partial}{\partial x_j}}(f(\mathbf{x})g(\mathbf{y},\mathbf{t})) &= \frac{\partial}{\partial x_j}(f(\mathbf{x}))g(\mathbf{y},\mathbf{t})+f(\mathbf{x})\Big(\nabla(g(\mathbf{y},\mathbf{t})),\frac{\partial}{\partial x_j}\Big)\\
&= \frac{\partial}{\partial x_j}(f(\mathbf{x}))g(\mathbf{y},\mathbf{t})+f(\mathbf{x})\Big(\sum_m \nabla_{\frac{\partial}{\partial y_m}}(g(\mathbf{y},\mathbf{t}))\mathrm{d}y_m,\frac{\partial}{\partial x_j}\Big)\\
&= \frac{\partial}{\partial x_j}(f(\mathbf{x}))g(\mathbf{y},\mathbf{t})+f(\mathbf{x})\nabla_{\frac{\partial}{\partial y_j}}(g(\mathbf{y},\mathbf{t}))qx_j^{q-1}\\
&= \frac{\partial}{\partial x_j}(f(\mathbf{x}))g(\mathbf{y},\mathbf{t})+f(\mathbf{x})\Big(\frac{\partial}{\partial y_j}(g(\mathbf{y},\mathbf{t}))+\pi\mathbf{t}^{\mathbf{w}_j}g(\mathbf{y},\mathbf{t})\Big)qx_j^{q-1}\\
&= \frac{\partial}{\partial x_j}(f(\mathbf{x}))g(\mathbf{y},\mathbf{t})+f(\mathbf{x})\frac{\partial}{\partial x_j}(g(\mathbf{y},\mathbf{t}))+q\pi x_j^{q-1}\mathbf{t}^{\mathbf{w}_j}f(\mathbf{x})g(\mathbf{y},\mathbf{t})\\
&= \frac{\partial}{\partial x_j}\Big(f(\mathbf{x})g(\mathbf{y},\mathbf{t})\Big)+q\pi x_j^{q-1}\mathbf{t}^{\mathbf{w}_j}f(\mathbf{x})g(\mathbf{y},\mathbf{t})\\
&= \nabla^{(1)}(f(\mathbf{x})g(\mathbf{y},\mathbf{t})).
\end{aligned}$$

This proves our assertion. Similarly, one verifies that the connection $\mathrm{Fr}^*\nabla$ on $\mathrm{Fr}^*C^j(\tilde{L}^\dagger)$ can be identified with the connection $\nabla^{(1)}$ on $C^\cdot(L^\dagger)$. □

Definition 0.5. Suppose $\gamma_1,\ldots,\gamma_n\in\frac{1}{1-q}\mathbb{Z}$ and $(\gamma_1,\ldots,\gamma_n)\in\delta$. The *p-adic GKZ hypergeometric complex* is defined to be the tuple $(C^\cdot(L^\dagger),\nabla,F)$ consisting of the complex $C^\cdot(L^\dagger)$ of $K\langle\mathbf{x}\rangle^\dagger$-module modules with the connection ∇ and the horizontal morphism $F:\mathrm{Fr}^*(C^\cdot(L^\dagger),\nabla)\to(C^\cdot(L^\dagger),\nabla)$.

0.6. **The GKZ hypergeometric $\mathcal{D}^\dagger$-module.** Let

$$\mathcal{D}^\dagger=\bigcup_{r>1,\,s>1}\{\sum_{\mathbf{v}\in\mathbb{Z}_{\geq 0}^N} f_{\mathbf{v}}(\mathbf{x})\frac{\partial^{\mathbf{v}}}{\pi^{|\mathbf{v}|}}:\ f_{\mathbf{v}}(\mathbf{x})\in K\{r^{-1}\mathbf{x}\},\ \|f_{\mathbf{v}}(\mathbf{x})\|_r s^{|\mathbf{v}|}\text{ are bounded}\},$$

where for any $\mathbf{v}=(v_1,\ldots,v_N)\in\mathbb{Z}_{\geq 0}^N$, we set $\partial^{\mathbf{v}}=\frac{\partial^{v_1+\cdots+v_N}}{\partial x_1^{v_1}\cdots\partial x_N^{v_N}}$. $\mathcal{D}^\dagger$ is a ring of differential operators possibly of infinite orders. This $\mathcal{D}^\dagger$ is also used in [14]. Let $\mathcal{D}^\dagger_{\mathbb{P}^N,\mathbb{Q}}(\infty)$ be the sheaf of differential operators of finite level and of infinite order on the formal projective space $\mathbb{P}^N$ over the integer ring of K with over-convergent poles along the ∞ divisor. For the definition of this sheaf, see [3]. By [13], we have

$$\Gamma(\mathbb{P}^N,\mathcal{D}^\dagger_{\mathbb{P}^N,\mathbb{Q}}(\infty))=\bigcup_{r>1,\,s>1}\{\sum_{\mathbf{v}\in\mathbb{Z}_{\geq 0}^N} f_{\mathbf{v}}(\mathbf{x})\frac{\partial^{\mathbf{v}}}{\mathbf{v}!}:\ f_{\mathbf{v}}(\mathbf{x})\in K\{r^{-1}\mathbf{x}\},\ \|f_{\mathbf{v}}(\mathbf{x})\|_r s^{|\mathbf{v}|}\text{ are bounded}\},$$

where $\mathbf{v}!=v_1!\cdots v_N!$. In section 1, we prove the following proposition.

Proposition 0.7. *We have $\mathcal{D}^\dagger=\Gamma(\mathbb{P}^N,\mathcal{D}^\dagger_{\mathbb{P}^N,\mathbb{Q}}(\infty))$.*

In particular, by the result in [13], $\mathcal{D}^\dagger$ is a coherent ring. Let $\frac{\partial}{\partial x_j}\in\mathcal{D}^\dagger$ act via $\nabla_{\frac{\partial}{\partial x_j}}$. Then $L^\dagger$ is a left $\mathcal{D}^\dagger$-module, and the twisted de Rham complex $C^\cdot(L^\dagger)$ is a complex of $\mathcal{D}^\dagger$-modules. The cohomology groups $H^k(C^\cdot(L^\dagger))$ are also left $\mathcal{D}^\dagger$-modules.

Let

$$\begin{aligned}
C(A) &= \{k_1\mathbf{w}_1+\cdots+k_N\mathbf{w}_N:\ k_i\in\mathbb{Z}_{\geq 0}\},\\
L^{\dagger\prime} &= \bigcup_{r>1,\,s>1}\{\sum_{\mathbf{w}\in C(A)} a_{\mathbf{w}}(\mathbf{x})\mathbf{t}^{\mathbf{w}}:\ a_{\mathbf{w}}(\mathbf{x})\in K\{r^{-1}\mathbf{x}\},\ \|a_{\mathbf{w}}(\mathbf{x})\|_r s^{d(\mathbf{w})}\text{ are bounded}\}.
\end{aligned}$$

$C(A)$ is a submonoid of $\mathbb{Z}^n \cap \delta$, and $L^{\dagger\prime}$ is both a subring and a $\mathcal{D}$-submodule of $L^\dagger$. Let

$$C^k(L^{\dagger\prime}) = \{\sum_{1\leq i_1<\cdots<i_k\leq n} f_{i_1\ldots i_k}\frac{\mathrm{d}t_{i_1}}{t_{i_1}}\wedge\cdots\wedge\frac{\mathrm{d}t_{i_k}}{t_{i_k}}:\ f_{i_1\ldots i_k}\in L^{\dagger\prime}\}\cong L^{\dagger\prime\binom{n}{j}}.$$

Note that $d: C^k(L^\dagger)\to C^{k+1}(L^\dagger)$ (resp. $\nabla_{\frac{\partial}{\partial x_j}}$) maps $C^k(L^{\dagger\prime})$ to $C^{k+1}(L^{\dagger\prime})$ (resp. $C^k(L^{\dagger\prime})$). So $C^{\cdot}(L^{\dagger\prime})$ is a subcomplex of $\mathcal{D}^\dagger$-modules of $C^{\cdot}(L^\dagger)$. Let

$$F_{i,\gamma} = t_i\frac{\partial}{\partial t_i}+\gamma_i+\pi\sum_{j=1}^N w_{ij}x_j\mathbf{t}^{\mathbf{w}_j}.$$

It follows from the definition of the twisted de Rham complex that the homomorphism

$$L^{\dagger\prime}\to C^n(L^{\dagger\prime}),\quad f\mapsto f\frac{\mathrm{d}t_1}{t_1}\wedge\cdots\wedge\frac{\mathrm{d}t_n}{t_n}$$

induces an isomorphism

$$L^{\dagger\prime}/\sum_{i=1}^n F_{i,\gamma}L^{\dagger\prime}\cong H^n(C^{\cdot}(L^{\dagger\prime})).$$

Let's give an explicit presentation of the $\mathcal{D}^\dagger$-module $H^n(C^{\cdot}(L^{\dagger\prime}))$. Let

$$\begin{aligned}
\Lambda &= \{\lambda=(\lambda_1,\ldots,\lambda_N)\in\mathbb{Z}^N:\ \sum_{j=1}^N\lambda_j\mathbf{w}_j=0\},\\
\Box_\lambda &= \prod_{\lambda_j>0}\left(\frac{1}{\pi}\frac{\partial}{\partial x_j}\right)^{\lambda_j}-\prod_{\lambda_j<0}\left(\frac{1}{\pi}\frac{\partial}{\partial x_j}\right)^{-\lambda_j}\quad(\lambda\in\Lambda)\\
E_{i,\gamma} &= \sum_{j=1}^N w_{ij}x_j\frac{\partial}{\partial x_j}+\gamma_i\ (i=1,\ldots,n),
\end{aligned}$$

Consider the map

$$\varphi:\mathcal{D}^\dagger\to L^{\dagger\prime},\quad \sum_{\mathbf{v}\in\mathbb{Z}_{\geq0}^N}f_{\mathbf{v}}(\mathbf{x})\frac{\partial^{\mathbf{v}}}{\pi^{|\mathbf{v}|}}\mapsto(\sum_{\mathbf{v}\in\mathbb{Z}_{\geq0}^N}f_{\mathbf{v}}(\mathbf{x})\frac{\partial^{\mathbf{v}}}{\pi^{|\mathbf{v}|}})\cdot1=\sum_{\mathbf{v}\in\mathbb{Z}_{\geq0}^N}f_{\mathbf{v}}(\mathbf{x})\mathbf{t}^{v_1\mathbf{w}_N+\cdots+v_N\mathbf{w}_N}.$$

It is a homomorphism of $\mathcal{D}^\dagger$-modules. In §1, we prove the following theorems.

Theorem 0.8. *φ induces isomorphisms*

$$\begin{aligned}
\mathcal{D}^\dagger/\sum_{\lambda\in\Lambda}\mathcal{D}^\dagger\Box_\lambda &\overset{\cong}{\to} L^{\dagger\prime},\\
\mathcal{D}^\dagger/(\sum_{i=1}^n\mathcal{D}^\dagger E_{i,\gamma}+\sum_{\lambda\in\Lambda}\mathcal{D}^\dagger\Box_\lambda) &\overset{\cong}{\to} L^{\dagger\prime}/\sum_{i=1}^n F_{i,\gamma}L^{\dagger\prime}\cong H^n(C^{\cdot}(L^{\dagger\prime})).
\end{aligned}$$

Moreover, there exist finitely many $\mu^{(1)},\ldots,\mu^{(m)}\in\Lambda$ such that

$$\sum_{i=1}^m\mathcal{D}^\dagger\Box_{\mu^{(i)}}=\sum_{\lambda\in\Lambda}\mathcal{D}^\dagger\Box_\lambda.$$

Theorem 0.9. *$C^{\cdot}(L^\dagger)$ and $C^{\cdot}(L^{\dagger\prime})$ are complexes of coherent $\mathcal{D}^\dagger$-modules.*

Definition 0.10. The *GKZ hypergeometric $\mathcal{D}^\dagger$-module* is defined to be the left $\mathcal{D}^\dagger$-module

$$\mathcal{D}^\dagger/(\sum_{i=1}^n \mathcal{D}^\dagger E_{i,\gamma} + \sum_{\lambda\in\Lambda} \mathcal{D}^\dagger \Box_\lambda) \cong H^n(C^\cdot(L^{\dagger\prime})).$$

The GKZ hypergeometric $\mathcal{D}^\dagger$-module is the p-adic analogue of the (complex) hypergeometric D-module ([1]) associated to the GKZ hypergeometric system of differential equations (0.1.1).

0.11. **Fibers of the GKZ hypergeometric complex.** Let $\mathbf{a} = (a_1, \ldots, a_N)$ be a point in the closed unit polydisc $E(0,1)$, where $a_i \in K'$ for some finite extension K' of K. Let's specialize at $\mathbf{x} = \mathbf{a}$, that is, apply the functor - $\otimes_{K\langle \mathbf{x}\rangle^\dagger} K'$, where K' is regarded as a $K\langle \mathbf{x}\rangle^\dagger$-algebra via the homomorphism

$$K\langle \mathbf{x}\rangle^\dagger \to K', \quad x_i \mapsto a_i.$$

Let

$$L_0^\dagger \quad = \quad \bigcup_{s>1} \{ \sum_{\mathbf{w}\in\mathbb{Z}^n\cap\delta} a_{\mathbf{w}} t^{\mathbf{w}} :\ a_{\mathbf{w}} \in K',\ |a_{\mathbf{w}}| s^{d(\mathbf{w})} \text{ are bounded}\}.$$

In section 1, we prove the following.

Lemma 0.12. *$L^\dagger$ is flat over $K\langle \mathbf{x}\rangle^\dagger$ and*

$$L^\dagger \otimes_{K\langle \mathbf{x}\rangle^\dagger} K' \cong L_0^\dagger.$$

Consider the twisted de Rham complex $C^\cdot(L_0^\dagger)$ defined as follows: We set

$$C^k(L_0^\dagger) = \{ \sum_{1\leq i_1<\cdots<i_k\leq n} f_{i_1\ldots i_k} \frac{\mathrm{d}t_{i_1}}{t_{i_1}} \wedge \cdots \wedge \frac{\mathrm{d}t_{i_k}}{t_{i_k}} :\ f_{i_1\ldots i_k} \in L_0^\dagger \} \cong L_0^{\dagger \binom{n}{k}}$$

with differential $d : C^k(L_0^\dagger) \to C^{k+1}(L_0^\dagger)$ given by

$$\begin{aligned} d(\omega) \quad &= \quad \Big(t_1^{\gamma_1}\cdots t_n^{\gamma_n} \exp(\pi F(\mathbf{a},\mathbf{t}))\Big)^{-1} \circ \mathrm{d}_{\mathbf{t}} \circ \Big(t_1^{\gamma_1}\cdots t_n^{\gamma_n} \exp(\pi F(\mathbf{a},\mathbf{t}))\Big)(\omega) \\ &= \quad \mathrm{d}_{\mathbf{t}}\omega + \sum_{i=1}^n \Big(\gamma_i + \pi \sum_{j=1}^N w_{ij} a_j \mathbf{t}^{\mathbf{w}_j}\Big) \frac{\mathrm{d}t_i}{t_i} \wedge \omega \end{aligned}$$

for any $\omega \in C^k(L_0^\dagger)$. By Lemma 0.12, we have the following corollary.

Corollary 0.13. *In the derived category of complexes of $K\langle \mathbf{x}\rangle^\dagger$-modules, we have*

$$C^\cdot(L^\dagger) \otimes^L_{K\langle \mathbf{x}\rangle^\dagger} K' \cong C^\cdot(L_0^\dagger).$$

The specialization of Φ at $\mathbf{a}$ is the lifting of the Frobenius correspondence defined by

$$\Phi_{\mathbf{a}} : L_0^\dagger \to L_0^\dagger, \quad f(\mathbf{t}) \mapsto f(\mathbf{t}^q).$$

It induces the maps $\Phi_{\mathbf{a}} : C^k(L^\dagger) \to C^k(L^\dagger)$ on differential forms commuting with $\mathrm{d}_{\mathbf{t}}$:

$$\Phi_{\mathbf{a}}\Big(\sum_{1\leq i_1<\cdots<i_k\leq n} f_{i_1\ldots i_k}(\mathbf{t}) \frac{\mathrm{d}t_{i_1}}{t_{i_1}} \wedge \cdots \wedge \frac{\mathrm{d}t_{i_k}}{t_{i_k}} \Big) = \sum_{1\leq i_1<\cdots<i_k\leq n} q^k f_{i_1\ldots i_j}(\mathbf{t}^q) \frac{\mathrm{d}t_{i_1}}{t_{i_1}} \wedge \cdots \wedge \frac{\mathrm{d}t_{i_k}}{t_{i_k}}$$

The specialization of $F : C^\cdot(L^\dagger) \to C^\cdot(L^\dagger)$ at $\mathbf{a}$ is given by

$$\begin{aligned} F_a \quad &= \quad \Big(t_1^{\gamma_1}\cdots t_n^{\gamma_n} \exp(\pi F(\mathbf{a},\mathbf{t}))\Big)^{-1} \circ \Phi_{\mathbf{a}} \circ \Big(t_1^{\gamma_1}\cdots t_n^{\gamma_n} \exp(\pi F(\mathbf{a}^q,\mathbf{t}))\Big) \\ &= \quad \Big(t_1^{\gamma_1(q-1)}\cdots t_n^{\gamma_n(q-1)} \exp\big(\pi F(\mathbf{a}^q,\mathbf{t}^q) - \pi F(\mathbf{a},\mathbf{t})\big)\Big) \circ \Phi_{\mathbf{a}}. \end{aligned}$$

By Lemma 0.4 (i), $t_1^{\gamma_1(q-1)}\cdots t_n^{\gamma_n(q-1)}\exp\big(\pi F(\mathbf{a}^q,\mathbf{t}^q)-\pi F(\mathbf{a},\mathbf{t})\big)$ lie in $L_0^\dagger$, and hence F_a defines an endomorphism on each $C^k(L_0^\dagger)$.

From now on, we assume that $\mathbf{a}$ is a Techmüller point, that is, $a_j^q=a_j$ $(j=1,\ldots,N)$. Then $\mathbf{a}$ is a fixed point of Fr. In this case $F_{\mathbf{a}}: C^\cdot(L_0^\dagger)\to C^\cdot(L_0^\dagger)$ commutes with $d: C^j(L_0^\dagger)\to C^{j+1}(L_0^\dagger)$ and hence is a chain map.

Consider the operator $\Psi_{\mathbf{a}}: L_0^\dagger\to L_0^\dagger$ defined by

$$\Psi_{\mathbf{a}}(\sum_{\mathbf{w}} c_{\mathbf{w}}\mathbf{t}^{\mathbf{w}})=\sum_{\mathbf{w}} c_{q\mathbf{w}}\mathbf{t}^{\mathbf{w}}.$$

We extend it to differential forms by

$$\Psi_{\mathbf{a}}\Big(\sum_{1\leq i_1<\cdots<i_k\leq n} f_{i_1\ldots i_k}(\mathbf{t})\frac{\mathrm{d}t_{i_1}}{t_{i_1}}\wedge\cdots\wedge\frac{\mathrm{d}t_{i_k}}{t_{i_k}}\Big)=\sum_{1\leq i_1<\cdots<i_k\leq n} q^{-k}\Psi_{\mathbf{a}}(f_{i_1\ldots i_j}(\mathbf{t}))\frac{\mathrm{d}t_{i_1}}{t_{i_1}}\wedge\cdots\wedge\frac{\mathrm{d}t_{i_k}}{t_{i_k}}.$$

It commutes with $\mathrm{d}_{\mathbf{t}}$. Let $G_{\mathbf{a}}: C^\cdot(L_0^\dagger)\to C^\cdot(L_0^\dagger)$ be the map defined by

$$\begin{aligned} G_{\mathbf{a}} &= \Big(t_1^{\gamma_1}\cdots t_n^{\gamma_n}\exp(\pi F(\mathbf{a},\mathbf{t}))\Big)^{-1}\circ\Psi_{\mathbf{a}}\circ\Big(t_1^{\gamma_1}\cdots t_n^{\gamma_n}\exp(\pi F(\mathbf{a},\mathbf{t}))\Big)\\ &= \Psi_{\mathbf{a}}\circ\Big(t_1^{\gamma_1(1-q)}\cdots t_n^{\gamma_n(1-q)}\exp\big(\pi F(\mathbf{a},\mathbf{t})-\pi F(\mathbf{a}^{\mathbf{a}},\mathbf{t}^q)\big)\Big).\end{aligned}$$

Here by Lemma 0.4 (i), $t_1^{\gamma_1(1-q)}\cdots t_n^{\gamma_n(1-q)}\exp\big(\pi F(\mathbf{a},\mathbf{t})-\pi F(\mathbf{a},\mathbf{t}^q)\big)$ lies in $L_0^\dagger$ and hence $G_{\mathbf{a}}$ defines an operator on $C^\cdot(L_0^\dagger)$. Then $G_{\mathbf{a}}$ commutes with $d: C^k(L_0^\dagger)\to C^{k+1}(L_0^\dagger)$. We thus get a chain map $G_{\mathbf{a}}: C^\cdot(L_0^\dagger)\to C^\cdot(L_0^\dagger)$.

Lemma 0.14. *We have $G_{\mathbf{a}}\circ F_{\mathbf{a}}=\mathrm{id}$ and $F_{\mathbf{a}}\circ G_{\mathbf{a}}$ is homotopic to* id. *In particular, $F_{\mathbf{a}}$ and $G_{\mathbf{a}}$ induce isomorphisms on $H^\cdot(C^\cdot(L_0^\dagger))$.*

In section 3, we show that each $G_{\mathbf{a}}: C^k(L_0^\dagger)\to C^k(L_0^\dagger)$ is a nuclear operator and hence the homomorphism on each $H^k(C^\cdot(L_0^\dagger))$ induced by $G_{\mathbf{a}}$ is also nuclear. We can talk about their traces and characteristic power series. But $F_{\mathbf{a}}$ does not have this property. Let

$$\begin{aligned}\mathrm{Tr}\big(G_{\mathbf{a}},C^\cdot(L_0^\dagger)\big) &= \sum_{k=0}^n(-1)^k\mathrm{Tr}\big(G_{\mathbf{a}},C^k(L_0^\dagger)\big)\\ &= \sum_{k=0}^n(-1)^k\mathrm{Tr}\big(G_{\mathbf{a}},H^k(C^\cdot(L_0^\dagger))\big)\\ &= \sum_{k=0}^n(-1)^k\mathrm{Tr}\big(F_{\mathbf{a}}^{-1},H^k(C^\cdot(L_0^\dagger))\big),\\ \det\big(I-TG_{\mathbf{a}},C^\cdot(L_0^\dagger)\big) &= \prod_{k=0}^n\det\big(I-TG_{\mathbf{a}},C^k(L_0^\dagger)\big)^{(-1)^k}\\ &= \prod_{k=0}^n\det\big(I-TG_{\mathbf{a}},H^k(C^\cdot(L_0^\dagger))\big)^{(-1)^k}\\ &= \prod_{k=0}^n\det\big(I-TF_{\mathbf{a}}^{-1},H^k(C^\cdot(L_0^\dagger))\big)^{(-1)^k}.\end{aligned}$$

Let $\chi:\mathbb{F}_q^*\to\overline{\mathbb{Q}}_p$ be the Techmüller character which maps each u in $\mathbb{F}_q^*$ to its Techmüller lifting. By [15, Theorems 4.1 and 4.3], the formal power series $\theta(z)=\exp(\pi z-\pi z^p)$ converges in a disc of radius > 1, and its value $\theta(1)$ at $z = 1$ is a primitive p-th root of unity in K. Let $\psi : \mathbb{F}_q \to K^*$ be the additive character defined by

$$\psi(\bar{a}) = \theta(1)^{\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\bar{a})}$$

for any $\bar{a} \in \mathbb{F}_q$. Let $\bar{a}_j \in \mathbb{F}_q$ be the residue class $a_j \mod p$, let

$$\begin{aligned} & S_m(F(\bar{\mathbf{a}}, \mathbf{t})) \\ = & \sum_{\bar{u}_1,\ldots,\bar{u}_n\in\mathbb{F}_{q^m}^*} \chi_1(\mathrm{Norm}_{\mathbb{F}_q^m/\mathbb{F}_q}(\bar{u}_1))\cdots\chi_n(\mathrm{Norm}_{\mathbb{F}_q^m/\mathbb{F}_q}(\bar{u}_n))\psi\Big(\mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}\Big(\sum_{j=1}^N \bar{a}_j\bar{u}_1^{w_{1j}}\cdots\bar{u}_n^{w_{nj}}\Big)\Big) \end{aligned}$$

be the twisted exponential sums for the multiplicative characters $\chi_i = \chi^{(1-q)\gamma_i}$, the nontrivial additive character $\psi : \mathbb{F}_q \to \mathbb{C}_p^*$, and the polynomial $F(\bar{\mathbf{a}}, \mathbf{t})$, and let

$$L(F(\bar{\mathbf{a}}, \mathbf{t}), T) = \exp\Big(\sum_{m=1}^\infty S_m(F(\bar{\mathbf{a}}, \mathbf{t}))\frac{T^m}{m}\Big)$$

be the L-function for the twisted exponential sums. The following theorem is well-known in Dwork's theory. Its proof is given in section 2 for completeness.

Theorem 0.15. *Suppose $\gamma_1, \ldots, \gamma_n \in \frac{1}{1-q}\mathbb{Z}$, $\gamma = (\gamma_1, \ldots, \gamma_n) \in \delta$, and suppose K' contains all $(q-1)$-th root of unity. Let $\mathbf{a} = (a_1, \ldots, a_n)$ be a Techmüller point, that is, $a_j^q = a_j$. Then each $G_{\mathbf{a}} : C^k(L_0^\dagger) \to C^k(L_0^\dagger)$ is nuclear. Moreover, we have*

$$\begin{aligned} S_m(F(\bar{\mathbf{a}}, \mathbf{t})) &= \mathrm{Tr}\big((q^nG_{\mathbf{a}})^m, C^{\cdot}(L_0^\dagger)\big) \\ &= \sum_{k=0}^n (-1)^k\mathrm{Tr}\big((q^nF_{\mathbf{a}}^{-1})^m, H^k(C^{\cdot}(L_0^\dagger))\big), \\ L(F(\bar{\mathbf{a}}, \mathbf{t}), T) &= \det\big(I - q^nTG_{\mathbf{a}}, C^{\cdot}(L_0^\dagger)\big)^{-1} \\ &= \prod_{k=0}^n \det\big(I - q^nTF_{\mathbf{a}}^{-1}, H^k(C^{\cdot}(L_0^\dagger))\big)^{(-1)^{k+1}}, \end{aligned}$$

In [2], Adolphson shows that $L(F(\bar{\mathbf{a}}, \mathbf{t}), T)$ depends analytically on the parameters $\mathbf{a}$ and γ.

0.16. **The GKZ hypergeometric F-crystal.** It follows from the definition of the twisted de Rham complex that the homomorphism

$$L^\dagger \to C^n(L^\dagger), \quad f \mapsto f\frac{\mathrm{d}t_1}{t_1}\wedge\cdots\wedge\frac{\mathrm{d}t_n}{t_n}$$

induces an isomorphism

$$L^\dagger/\sum_{i=1}^n F_{i,\gamma}L^\dagger \cong H^n(C^{\cdot}(L^\dagger)).$$

∇ defines a connection on $H^n(C^{\cdot}(L^\dagger))$, and F defines a horizontal morphism

$$F : \mathrm{Fr}^*(H^n(C^{\cdot}(L^\dagger)), \nabla) \to (H^n(C^{\cdot}(L^\dagger)), \nabla).$$

Let U be the affinoid subdomain of the closed unit polydisc $E(0,1)^N$ parametrizing those points $\mathbf{a} = (a_1, \ldots, a_N)$ so that $F(\bar{\mathbf{a}}, \mathbf{t}) = \sum_{j=1}^N \bar{a}_j\mathbf{t}^{\mathbf{w}_j}$ is *non-degenerate* in the sense that for any face τ of Δ not containing the origin, the system of equations

$$\frac{\partial}{\partial t_1}F_\tau(\bar{\mathbf{a}}, \mathbf{t}) = \cdots = \frac{\partial}{\partial t_n}F_\tau(\bar{\mathbf{a}}, \mathbf{t}) = 0$$

has no solution in $(\overline{\mathbb{F}}_p^*)^n$, where $F_\tau(\bar{\mathbf{a}}, \mathbf{t}) = \sum_{\mathbf{w}_j \in \tau} \bar{a}_j \mathbf{t}^{\mathbf{w}_j}$. When restricted to U, we have

$$H^k(C^{\cdot}(L^\dagger)) = 0$$

for $k \neq n$, and $H^n(C^{\cdot}(L^\dagger))$ defines a vector bundle on U of rank $n!\mathrm{vol}(\Delta)$. Denote this vector bundle by Hyp.

Definition 0.17. We define the *GKZ hypergeometric crystal* to be $(\mathrm{Hyp}, \nabla, F)$.

Let $\mathbf{a} = (a_1, \ldots, a_N)$ be a point in U with coordinates in K', and let $\mathrm{Hyp}(\mathbf{a})$ be the fiber of Hyp at $\mathbf{a}$. By Corollary 0.13, the fact $C^k(L^\dagger) = 0$ for $k > n$, and the fact that $- \otimes_{K\langle \mathbf{x} \rangle^\dagger} K'$ is right exact, we have

$$H^n(C^{\cdot}(L^\dagger)) \otimes_{K\langle \mathbf{x} \rangle^\dagger} K' \cong H^n(C^{\cdot}(L_0^\dagger)).$$

So we have

$$\mathrm{Hyp}(\mathbf{a}) \cong L_0^\dagger / \sum_{i=1}^n F_{i,\gamma,\mathbf{a}} L_0^\dagger,$$

where $F_{i,\gamma,\mathbf{a}} = t_i \frac{\partial}{\partial t_i} + \gamma_i + \pi \sum_{j=1}^N w_{ij} a_j \mathbf{t}^{\mathbf{w}_j}$. If $\mathbf{a}$ is a Techmüller point, then we have

$$\begin{aligned} S_m(F(\bar{\mathbf{a}}, \mathbf{t})) &= (-1)^n \mathrm{Tr}\big((q^n F_{\mathbf{a}}^{-1})^m, \mathrm{Hyp}(\mathbf{a})\big), \\ L(F(\bar{\mathbf{a}}, \mathbf{t}), T) &= \det\big(I - q^n T F_{\mathbf{a}}^{-1}, \mathrm{Hyp}(\mathbf{a})\big)^{(-1)^{n-1}}. \end{aligned}$$

Let $\mathbf{a} = (a_1, \ldots, a_N)$ and $\mathbf{b} = (b_1, \ldots, b_N)$ be points in U with coordinates in K', and let

$$T_{\mathbf{a},\mathbf{b}} : \mathrm{Hyp}(\mathbf{a}) \stackrel{\cong}{\to} \mathrm{Hyp}(\mathbf{b})$$

be the parallel transport for Hyp. It is well-defined if $|b_i - a_i| < 1$ for all i. It can be described as follows: For any formal power series $f(t) \in \overline{\mathbb{Q}}_p[[\mathbb{Z}^n \cap \delta]]$, we have

$$\begin{aligned} \nabla_{\frac{\partial}{\partial x_j}} \Big(\exp(-\pi F(\mathbf{x}, \mathbf{t})) f(t) \Big) &= \exp(-\pi F(\mathbf{x}, \mathbf{t})) \circ \frac{\partial}{\partial x_j} \circ \exp(\pi F(\mathbf{x}, \mathbf{t})) \Big(\exp(-\pi F(\mathbf{x}, \mathbf{t})) f(t) \Big) \\ &= 0. \end{aligned}$$

So $\exp(-\pi F(\mathbf{x}, \mathbf{t})) f(t)$ is horizontal with respect to ∇. But it is only a formal horizontal section since it may not lie in $L^\dagger$. Formally, $T_{\mathbf{a},\mathbf{b}}$ maps $\exp(-\pi F(\mathbf{a}, \mathbf{t})) f(t)$ to $\exp(-\pi F(\mathbf{b}, \mathbf{t})) f(t)$. So $T_{\mathbf{a},\mathbf{b}} : \mathrm{Hyp}(\mathbf{a}) \stackrel{\cong}{\to} \mathrm{Hyp}(\mathbf{b})$ can be identified with the isomorphism

$$T_{\mathbf{a},\mathbf{b}} : L_0^\dagger / \sum_{i=1}^n F_{i,\gamma,\mathbf{a}} L_0^\dagger \to L_0^\dagger / \sum_{i=1}^n F_{i,\gamma,\mathbf{b}} L_0^\dagger, \quad g(t) \mapsto \exp\big(\pi F(\mathbf{a}, \mathbf{t}) - \pi F(\mathbf{b}, \mathbf{t})\big) g(t).$$

This is well-defined if $|b_i - a_i| < 1$ for all i since we then have $\exp\big(\pi F(\mathbf{a}, \mathbf{t}) - \pi F(\mathbf{b}, \mathbf{t})\big) \in L_0^\dagger$.

Since $F : \mathrm{Fr}^*(\mathrm{Hyp}, \nabla) \to (\mathrm{Hyp}, \nabla)$ is a horizontal morphism, we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Hyp}(\mathbf{a}^q) & \stackrel{T_{\mathbf{a}^q,\mathbf{x}^q}}{\to} & \mathrm{Hyp}(\mathbf{x}^q) \\ {\scriptstyle F_{\mathbf{a}}} \downarrow & & \downarrow {\scriptstyle F_{\mathbf{x}}} \\ \mathrm{Hyp}(\mathbf{a}) & \stackrel{T_{\mathbf{a},\mathbf{x}}}{\to} & \mathrm{Hyp}(\mathbf{x}). \end{array}$$

Let $\{e_1(\mathbf{x}), \ldots, e_M(\mathbf{x})\}$ be a local basis for Hyp over U. Write

$$\begin{aligned} (q^n F_{\mathbf{x}}^{-1}) \Big(e_1(\mathbf{x}), \ldots, e_M(\mathbf{x}) \Big) &= (e_1(\mathbf{x}^q), \ldots, e_M(\mathbf{x}^q)) Q(\mathbf{x}), \\ T_{\mathbf{a},\mathbf{x}}(e_1(\mathbf{a}), \ldots, e_M(\mathbf{a})) &= (e_1(\mathbf{x}), \ldots, e_M(\mathbf{x})) P(\mathbf{x}) \end{aligned}$$

where $P(\mathbf{x})$ and $Q(\mathbf{x})$ are matrices of power series. Then we have

$$Q(\mathbf{x}) = P(\mathbf{x}^q)Q(\mathbf{a})P(\mathbf{x})^{-1}$$

and hence

$$(-1)^n S_m(F(\bar{\mathbf{x}}, \mathbf{t})) = \mathrm{Tr}\big((P(\mathbf{x}^q)Q(\mathbf{a})P(\mathbf{x})^{-1})^m\big), \tag{0.17.1}$$

$$L(F(\bar{\mathbf{x}}, \mathbf{t}), T)^{(-1)^{n+1}} = \det\big(I - TP(\mathbf{x}^q)Q(\mathbf{a})P(\mathbf{x})^{-1}\big) \tag{0.17.2}$$

whenever $x_j^{q-1} = 1$ and $a_j^{q-1} = 1$. Write

$$\nabla_{\frac{\partial}{\partial x_j}}\Big(e_1(\mathbf{x}), \ldots, e_M(\mathbf{x})\Big) = (e_1(\mathbf{x}), \ldots, e_M(\mathbf{x}))A_j(\mathbf{x}).$$

As $\nabla_{\frac{\partial}{\partial x_j}}(T_{\mathbf{a},\mathbf{x}}(e_k(\mathbf{a}))) = 0$ for all k, $P(\mathbf{x})$ satisfies the system of differential equations

$$\frac{\partial}{\partial x_j}(P(\mathbf{x})) + A_j(\mathbf{x})P(\mathbf{x}) = 0. \tag{0.17.3}$$

Equations (0.17.1)-(0.17.3) give formulas for calculating the exponential sums and the L-function using a solution of a system of differential equations.

1. $\mathcal{D}^\dagger$-modules

Lemma 1.1. *Let m be a positive integer and let*

$$m = a_0 + a_1 p + a_2 p^2 + \cdots$$

be its p-expansion, where $0 \le a_i \le p-1$ for all i. Define

$$\sigma(m) = a_0 + a_1 + a_2 + \cdots .$$

(i) We have

$$\mathrm{ord}_p\Big(\frac{\pi^m}{m!}\Big) = \frac{\sigma(m)}{p-1}.$$

(ii) For any real number $\epsilon > 0$, there exists $\delta > 0$ such that

$$\sigma(m) \le \epsilon m + \delta.$$

Proof. (i) We have

$$\begin{aligned}
\mathrm{ord}_p(m!) &= \Big[\frac{m}{p}\Big] + \Big[\frac{m}{p^2}\Big] + \cdots \\
&= (a_1 + a_2 p + \cdots) + (a_2 + a_3 p + \cdots) + \cdots \\
&= a_1 + a_2(1+p) + a_3(1+p+p^2) + \cdots \\
&= \frac{a_1(p-1)}{p-1} + \frac{a_2(p^2-1)}{p-1} + \frac{a_3(p^3-1)}{p-1} + \cdots \\
&= \frac{m - \sigma(m)}{p-1}.
\end{aligned}$$

So we have

$$\mathrm{ord}_p\Big(\frac{\pi^m}{m!}\Big) = \frac{m}{p-1} - \frac{m-\sigma(m)}{p-1} = \frac{\sigma(m)}{p-1}.$$

(ii) Choose M sufficiently large so that for any $x \ge M$, we have

$$(p-1)(x+1) \le \epsilon p^x.$$

Let

$$m = a_0 + a_1 p + \cdots + a_l p^l$$

be the expansion of m, where $0 \leq a_i \leq p-1$ and $a_l \neq 0$. If $m \geq p^M$, then we have $l \geq M$ and hence $(p-1)(l+1) \leq \epsilon p^l$. So we have

$$\sigma(m) = a_0 + a_1 + \cdots + a_l \leq (p-1)(l+1) \leq \epsilon p^l \leq \epsilon m$$

for any $m \geq p^M$. Take $\delta = \max(\sigma(1), \ldots, \sigma(p^M))$. Then we have $\sigma(m) \leq \epsilon m + \delta$ for all m. ☐

1.2. **Proof of Proposition 0.7.** Set

$$\mathcal{B}^\dagger = \bigcup_{r>1,\, s>1} \{\sum_{\mathbf{v}\in\mathbb{Z}_{\geq 0}^N} f_{\mathbf{v}}(\mathbf{x})\frac{\partial^{\mathbf{v}}}{\mathbf{v}!} :\ f_{\mathbf{v}}(\mathbf{x}) \in K\{r^{-1}\mathbf{x}\},\ \|f_{\mathbf{v}}(\mathbf{x})\|_r s^{|\mathbf{v}|} \text{ are bounded}\}.$$

Let's prove $\mathcal{B}^\dagger = \mathcal{D}^\dagger$. Given $\sum_{\mathbf{v}\in\mathbb{Z}_{\geq 0}^N} f_{\mathbf{v}}(\mathbf{x})\frac{\partial^{\mathbf{v}}}{\mathbf{v}!}$ in $\mathcal{B}^\dagger$, choose real numbers $r > 1, s > 1$ and $C > 0$ such that

$$\|f_{\mathbf{v}}(\mathbf{x})\|_r s^{|\mathbf{v}|} \leq C.$$

We have

$$\sum_{\mathbf{v}\in\mathbb{Z}_{\geq 0}^N} f_{\mathbf{v}}(\mathbf{x})\frac{\partial^{\mathbf{v}}}{\mathbf{v}!} = \sum_{\mathbf{v}\in\mathbb{Z}_{\geq 0}^N} \Big(f_{\mathbf{v}}(\mathbf{x})\frac{\pi^{|\mathbf{v}|}}{\mathbf{v}!}\Big)\frac{\partial^{\mathbf{v}}}{\pi^{|\mathbf{v}|}}.$$

By Lemma 1.1 (i), we have $\mathrm{ord}_p\Big(\frac{\pi^{|\mathbf{v}|}}{\mathbf{v}!}\Big) \geq 0$. Hence

$$\Big\|f_{\mathbf{v}}(\mathbf{x})\frac{\pi^{|\mathbf{v}|}}{\mathbf{v}!}\Big\|_r s^{|\mathbf{v}|} \leq \|f_{\mathbf{v}}(\mathbf{x})\|_r s^{|\mathbf{v}|} \leq C.$$

So $\sum_{\mathbf{v}\in\mathbb{Z}_{\geq 0}^N} f_{\mathbf{v}}(\mathbf{x})\frac{\partial^{\mathbf{v}}}{\mathbf{v}!}$ lies in $\mathcal{D}^\dagger$.

Conversely, given $\sum_{\mathbf{v}\in\mathbb{Z}_{\geq 0}^N} f_{\mathbf{v}}(\mathbf{x})\frac{\partial^{\mathbf{v}}}{\pi^{|\mathbf{v}|}}$ in $\mathcal{D}^\dagger$, choose real numbers $r > 1, s > 1$ and $C > 0$ such that

$$\|f_{\mathbf{v}}(\mathbf{x})\|_r s^{|\mathbf{v}|} \leq C.$$

We have

$$\sum_{\mathbf{v}\in\mathbb{Z}_{\geq 0}^N} f_{\mathbf{v}}(\mathbf{x})\frac{\partial^{\mathbf{v}}}{\pi^{|\mathbf{v}|}} = \sum_{\mathbf{v}\in\mathbb{Z}_{\geq 0}^N} \Big(f_{\mathbf{v}}(\mathbf{x})\frac{\mathbf{v}!}{\pi^{|\mathbf{v}|}}\Big)\frac{\partial^{\mathbf{v}}}{\mathbf{v}!}.$$

Choose $\epsilon > 0$ so that

$$s > p^{\frac{\epsilon}{p-1}},$$

and choose δ as in Lemma 1.1 (ii). We have

$$\mathrm{ord}_p\Big(\frac{\pi^{|\mathbf{v}|}}{\mathbf{v}!}\Big) \leq \frac{\epsilon|\mathbf{v}| + \delta n}{p-1}.$$

Let $s' = sp^{-\frac{\epsilon}{p-1}} > 1$ and let $C' = Cp^{\frac{\delta n}{p-1}}$. We have

$$\begin{aligned}
\Big\|f_{\mathbf{v}}(\mathbf{x})\frac{\mathbf{v}!}{\pi^{|\mathbf{v}|}}\Big\|_r s'^{|\mathbf{v}|} &\leq \|f_{\mathbf{v}}(\mathbf{x})\|_r p^{\frac{\epsilon|\mathbf{v}|+\delta n}{p-1}} s'^{|\mathbf{v}|} \\
&= \|f_{\mathbf{v}}(\mathbf{x})\|_r s^{|\mathbf{v}|} p^{\frac{\delta n}{p-1}} \\
&\leq C'.
\end{aligned}$$

So $\sum_{\mathbf{v}\in\mathbb{Z}_{\geq 0}^N} f_{\mathbf{v}}(\mathbf{x})\frac{\partial^{\mathbf{v}}}{\mathbf{v}!}$ lies in $\mathcal{B}^\dagger$.

Lemma 1.3. *Let S be any subset of $\mathbb{Z}^n_{\geq 0}$. There exists a finite subset S_0 of S such that $S \subset \bigcup_{\mathbf{v}\in S_0}(\mathbf{v}+\mathbb{Z}^n_{\geq 0})$.*

Proof. We use induction on n. When $n=1$, we have $S\subset v+\mathbb{Z}_{\geq 0}$, where v is the minimal element in $S\subset \mathbb{Z}_{\geq 0}$. Suppose the assertion holds for any subset of $\mathbb{Z}^n_{\geq 0}$, and let S be a subset of $\mathbb{Z}^{n+1}_{\geq 0}$. If S is empty, our assertion holds trivially. Otherwise, we fix an element $\mathbf{a}=(a_1,\ldots,a_{n+1})$ in S. For any $i\in\{1,\ldots,n+1\}$ and any $0\leq b_i\leq a_i$, let

$$S_{i,b_i}=\{(c_1,\ldots,c_{n+1})\in S:\ c_i=b_i\}.$$

By the induction hypothesis, there exists a finite subset $T_{i,b_i}\subset S_{i,b_i}$ such that

$$S_{i,b_i}\subset \bigcup_{\mathbf{v}\in T_{i,b_i}}(\mathbf{v}+\mathbb{Z}^{n+1}_{\geq 0}).$$

We have

$$\begin{aligned} S &\subset \Big(\bigcup_{1\leq i\leq n+1}\ \bigcup_{0\leq b_i\leq a_i} S_{i,b_i}\Big)\bigcup(\mathbf{a}+\mathbb{Z}^{n+1}_{\geq 0})\\ &\subset \Big(\bigcup_{1\leq i\leq n+1}\ \bigcup_{0\leq b_i\leq a_i}\ \bigcup_{\mathbf{v}\in T_{i,b_i}}(\mathbf{v}+\mathbb{Z}^{n+1}_{\geq 0})\Big)\bigcup(\mathbf{a}+\mathbb{Z}^{n+1}_{\geq 0}). \end{aligned}$$

We can take $S_0=\bigcup_{1\leq i\leq n+1}\bigcup_{0\leq b_i\leq a_i}T_{i,b_i}\bigcup\{\mathbf{a}\}$. $\square$

Lemma 1.4.

(i) The ring homomorphism

$$\phi: K\langle \mathbf{x},\mathbf{y}\rangle^\dagger\to L^{\dagger\prime},\quad \sum_{\mathbf{v}\in\mathbb{Z}^N_{\geq 0}} f_{\mathbf{v}}(\mathbf{x})\mathbf{y}^{\mathbf{v}}\mapsto \sum_{\mathbf{v}\in\mathbb{Z}^N_{\geq 0}} f_{\mathbf{v}}(\mathbf{x})\mathbf{t}^{v_1\mathbf{w}_N+\cdots+v_N\mathbf{w}_N}$$

is surjective, where $\mathbf{y}=(y_1,\ldots,y_N)$ and

$$K\langle\mathbf{x},\mathbf{y}\rangle^\dagger=\bigcup_{r>1,\ s>1}\{\sum_{\mathbf{v}\in\mathbb{Z}^N_{\geq 0}} f_{\mathbf{v}}(\mathbf{x})\mathbf{y}^{\mathbf{v}}:\ f_{\mathbf{v}}(\mathbf{x})\in K\{r^{-1}x\}\ \textit{and}\ \|f_{\mathbf{v}}(x)\|_r s^{|\mathbf{v}|}\ \textit{is bounded}\}.$$

(ii) The homomorphism of $\mathcal{D}^\dagger$-modules

$$\varphi:\mathcal{D}^\dagger\to L^{\dagger\prime},\quad \sum_{\mathbf{v}\in\mathbb{Z}^N_{\geq 0}} f_{\mathbf{v}}(\mathbf{x})\frac{\partial^{\mathbf{v}}}{\pi^{|\mathbf{v}|}}\mapsto (\sum_{\mathbf{v}\in\mathbb{Z}^N_{\geq 0}} f_{\mathbf{v}}(\mathbf{x})\frac{\partial^{\mathbf{v}}}{\pi^{|\mathbf{v}|}})\cdot 1=\sum_{\mathbf{v}\in\mathbb{Z}^N_{\geq 0}} f_{\mathbf{v}}(\mathbf{x})\mathbf{t}^{v_1\mathbf{w}_N+\cdots+v_N\mathbf{w}_N}$$

is surjective.

Proof. Decompose Δ into a finite union $\bigcup_\tau \tau$ so that each τ is a simplicial complex of dimension n with vertices $\{0,\mathbf{w}_{i_1},\ldots,\mathbf{w}_{i_n}\}$ for some subset $\{i_1,\ldots,i_n\}\subset\{1,\ldots,N\}$. For each τ, let $\delta(\tau)$ be the cone generated by τ, and let

$$\begin{aligned} B(\tau) &= \mathbb{Z}^n\cap\{c_1\mathbf{w}_{i_1}+\cdots+c_n\mathbf{w}_{i_n}:\ 0\leq c_i\leq 1\},\\ C(\tau) &= \{k_1\mathbf{w}_{i_1}+\cdots+k_n\mathbf{w}_{i_n}:\ k_i\in\mathbb{Z}_{\geq 0}\}. \end{aligned}$$

Being a discrete bounded set, $B(\tau)$ is finite. Every element $\mathbf{w}\in\mathbb{Z}^n\cap\delta(\tau)$ can be written uniquely as

$$\mathbf{w}=b(\mathbf{w})+c(\mathbf{w})$$

with $b(\mathbf{w})\in B(\tau)$ and $c(\mathbf{w})\in C(\tau)$. So we have $\mathbb{Z}^n\cap\delta(\tau)=\bigcup_{\mathbf{w}\in B(\tau)}(\mathbf{w}+C(\tau))$, and hence

$$C(A)=\bigcup_\tau(C(A)\cap\delta(\tau))=\bigcup_\tau\ \bigcup_{\mathbf{w}\in B(\tau)}\big(C(A)\cap(\mathbf{w}+C(\tau))\big).$$

For each $C(A)\cap(\mathbf{w}+C(\tau))$, the map

$$\mathbb{Z}_{\geq 0}^n\to\mathbf{w}+C(\tau),\quad (k_1,\ldots,k_n)\mapsto \mathbf{w}+k_1\mathbf{w}_{i_1}+\cdots+k_n\mathbf{w}_{i_n}$$

is a bijection. Applying Lemma 1.3 to the inverse image of $C(A)\cap(\mathbf{w}+C(\tau))$, we can find finitely many $\mathbf{u}_1,\ldots,\mathbf{u}_m\in C(A)\cap(\mathbf{w}+C(\tau))$ such that

$$C(A)\cap(\mathbf{w}+C(\tau))=\bigcup_{i=1}^m(\mathbf{u}_i+C(\tau)).$$

We thus decompose $C(A)$ into a finite union of subsets of the form $\mathbf{u}+C(\tau)$ such that τ is a simplicial complex of dimension n with vertices $\{0,\mathbf{w}_{i_1},\ldots,\mathbf{w}_{i_n}\}$ for some subset $\{i_1,\ldots,i_n\}\subset\{1,\ldots,N\}$, and $\mathbf{u}\in C(A)\cap(\mathbf{w}+C(\tau))$ for some $\mathbf{w}\in B(\tau)$. Elements in $L^{\dagger\prime}$ is a sum of elements of the form $\sum_{\mathbf{w}\in\mathbf{u}+C(\tau)}a_{\mathbf{w}}(\mathbf{x})\mathbf{t}^{\mathbf{w}}$, where $a_{\mathbf{w}}(\mathbf{x})\in K\{r^{-1}\mathbf{x}\}$ and $\|a_{\mathbf{w}}(\mathbf{x})\|_r s^{d(\mathbf{w})}$ are bounded for some $r,s>1$. To prove $\phi:K\langle\mathbf{x},\mathbf{y}\rangle^\dagger\to L^{\dagger\prime}$ is surjective, it suffices to show $\sum_{\mathbf{w}\in\mathbf{u}+C(\tau)}a_{\mathbf{w}}(\mathbf{x})\mathbf{t}^{\mathbf{w}}$ lies in the image of ϕ. Write $\mathbf{u}=c_1\mathbf{w}_1+\cdots+c_N\mathbf{w}_N$, where $c_i\in\mathbb{Z}_{\geq0}$. A preimage for $\sum_{\mathbf{w}\in\mathbf{u}+C(\tau)}a_{\mathbf{w}}(\mathbf{x})\mathbf{t}^{\mathbf{w}}$ is

$$\sum_{v_1,\ldots,v_n\geq0}a_{\mathbf{u}+v_1\mathbf{w}_{i_1}+\cdots+v_n\mathbf{w}_{i_n}}(\mathbf{x})y_{i_1}^{c_{i_1}+v_1}\cdots y_{i_n}^{c_{i_n}+v_n}\prod_{j\in\{1,\ldots,N\}-\{i_1,\ldots,i_n\}}y_j^{c_j}.$$

Here to verify this element lies in $K\langle\mathbf{x},\mathbf{y}\rangle^\dagger$, we use the fact that

$$d(\mathbf{u}+v_1\mathbf{w}_{i_1}+\cdots+v_n\mathbf{w}_{i_n})=d(\mathbf{u})+v_1+\cdots+v_n$$

since $\mathbf{u},\mathbf{w}_{i_1},\ldots,\mathbf{w}_{i_n}$ all lie in the simplicial cone $\delta(\tau)$. This prove $\phi:K\langle\mathbf{x},\mathbf{y}\rangle^\dagger\to L^{\dagger\prime}$ is surjective. It implies that $\varphi:\mathcal{D}^\dagger\to L^{\dagger\prime}$ is also surjective. □

1.5. **Proof of Theorem 0.8.** We have shown that φ is surjective in the proof of Lemma 1.4.

The ring $D=K\left[\frac{\partial}{\partial x_1},\ldots,\frac{\partial}{\partial x_N}\right]$ of algebraic differential operators with constant coefficients is isomorphic to the polynomial ring and is noetherian. So we can finitely many $\mu^{(1)},\ldots,\mu^{(m)}\in\Lambda$ such that $\Box_{\mu^{(1)}},\ldots,\Box_{\mu^{(m)}}$ generate the ideal $\sum_{\lambda\in\Lambda}D\Box_\lambda$ of D. Then they also generate the left ideal $\sum_{\lambda\in\Lambda}\mathcal{D}^\dagger\Box_\lambda$ of $\mathcal{D}^\dagger$. Suppose $\sum_{\mathbf{v}}f_{\mathbf{v}}(\mathbf{x})\frac{\partial^{\mathbf{v}}}{\pi^{|\mathbf{v}|}}$ lies in the kernel of φ, that is,

$$\sum_{\mathbf{v}}f_{\mathbf{v}}(\mathbf{x})\mathbf{t}^{v_1\mathbf{w}_1+\cdots+v_N\mathbf{w}_N}=0,$$

where $f_{\mathbf{v}}(\mathbf{x})\in K\{r^{-1}\mathbf{x}\}$ and $\|f_{\mathbf{v}}(\mathbf{x})\|_r s^{|\mathbf{v}|}$ are bounded for some $r,s>1$. For each $\mathbf{w}\in C(A)$, let

$$S_{\mathbf{w}}=\{\mathbf{v}\in\mathbb{Z}_{\geq0}^n:\ \mathbf{w}=v_1\mathbf{w}_1+\cdots+v_N\mathbf{w}_N\}.$$

Then we have

$$\sum_{\mathbf{v}\in S_{\mathbf{w}}}f_{\mathbf{v}}(\mathbf{x})=0.$$

For each nonempty $S_{\mathbf{w}}$, fix an element $\mathbf{v}^{(0)}=(v_1^{(0)},\ldots,v_N^{(0)})\in S_{\mathbf{w}}$. For any $\mathbf{v}\in S_{\mathbf{w}}$, let $\lambda_{\mathbf{v}}=\mathbf{v}-\mathbf{v}^{(0)}$. We have $\lambda_{\mathbf{v}}\in\Lambda$. Write

$$\Box_{\lambda_{\mathbf{v}}}=P_{\mathbf{v},1}\Box_{\mu^{(1)}}+\cdots+P_{\mathbf{v},m}\Box_{\mu^{(m)}}$$

for some differential operators $P_{\mathbf{v},1},\ldots,P_{\mathbf{v},m}\in D$. We have

$$\begin{aligned}
\frac{\partial^{\mathbf{v}}}{\pi^{|\mathbf{v}|}}-\frac{\partial^{\mathbf{v}^{(0)}}}{\pi^{|\mathbf{v}^{(0)}|}} &= \frac{\partial^{\min(\mathbf{v},\mathbf{v}^{(0)})}}{\pi^{\sum_j \min(v_j,v_j^{(0)})}}\Big(\prod_{v_j>v_j^{(0)}}\Big(\frac{1}{\pi}\frac{\partial}{\partial x_j}\Big)^{v_j-v_j^{(0)}}-\prod_{v_j<v_j^{(0)}}\Big(\frac{1}{\pi}\frac{\partial}{\partial x_j}\Big)^{v_j^{(0)}-v_j}\Big)\\
&= \frac{\partial^{\min(\mathbf{v},\mathbf{v}^{(0)})}}{\pi^{\sum_j \min(v_j,v_j^{(0)})}}\Box_{\lambda_{\mathbf{v}}}\\
&= \frac{\partial^{\min(\mathbf{v},\mathbf{v}^{(0)})}}{\pi^{\sum_j \min(v_j,v_j^{(0)})}}(P_{\mathbf{v},1}\Box_{\mu^{(1)}}+\cdots+P_{\mathbf{v},m}\Box_{\mu^{(m)}}),\\
\sum_{\mathbf{v}} f_{\mathbf{v}}(\mathbf{x})\frac{\partial^{\mathbf{v}}}{\pi^{|\mathbf{v}|}} &= \sum_{\mathbf{w}\in C(A)}\sum_{\mathbf{v}\in S_{\mathbf{w}}} f_{\mathbf{v}}(\mathbf{x})\frac{\partial^{\mathbf{v}}}{\pi^{|\mathbf{v}|}}\\
&= \sum_{\mathbf{w}\in C(A)}\sum_{\mathbf{v}\in S_{\mathbf{w}}} f_{\mathbf{v}}(\mathbf{x})\Big(\frac{\partial^{\mathbf{v}}}{\pi^{|\mathbf{v}|}}-\frac{\partial^{\mathbf{v}^{(0)}}}{\pi^{|\mathbf{v}^{(0)}|}}\Big)\\
&= \sum_{\mathbf{w}\in C(A)}\sum_{\mathbf{v}\in S_{\mathbf{w}}} f_{\mathbf{v}}(\mathbf{x})\frac{\partial^{\min(\mathbf{v},\mathbf{v}^{(0)})}}{\pi^{\sum_j \min(v_j,v_j^{(0)})}}(P_{\mathbf{v},1}\Box_{\mu^{(1)}}+\cdots+P_{\mathbf{v},m}\Box_{\mu^{(m)}})\\
&= \sum_{k=1}^{m}\Big(\sum_{\mathbf{w}\in C(A)}\sum_{\mathbf{v}\in S_{\mathbf{w}}} f_{\mathbf{v}}(\mathbf{x})\frac{\partial^{\min(\mathbf{v},\mathbf{v}^{(0)})}}{\pi^{\sum_j \min(v_j,v_j^{(0)})}}P_{\mathbf{v},k}\Big)\Box_{\mu^{(k)}}
\end{aligned}$$

One can verify that $\varphi(\Box_\lambda)=0$ for all $\lambda\in\Lambda$. So we have

$$\ker\varphi=\sum_{k=1}^{m}\mathcal{D}^\dagger\Box_{\mu^{(k)}}=\sum_{\lambda\in\Lambda}\mathcal{D}^\dagger\Box_\lambda.$$

For any $g_i\in L^{\dagger\prime}$ $(i=1,\ldots,n)$, choose $P_i\in\mathcal{D}^\dagger$ such that $\varphi(P_i)=g_i$. One can check directly that $E_{i,\gamma}(1)=F_{i,\gamma}(1)$. Moreover, $F_{i,\gamma}$ commutes with each $\nabla_{\frac{\partial}{\partial x_j}}$ and hence with P_i. So we have

$$\varphi(\sum_i P_iE_{i,\gamma})=\sum_i P_iE_{i,\gamma}(1)=\sum_i P_iF_{i,\gamma}(1)=\sum_i F_{i,\gamma}P_i(1)=\sum_i F_{i,\gamma}\varphi(P_i)=\sum_i F_{i,\gamma}g_i.$$

So we have

$$\varphi(\sum_i\mathcal{D}^\dagger E_{i,\gamma})=\sum_i F_{i,\gamma}L^{\dagger\prime}.$$

Together with the fact that φ is surjective and $\ker\varphi=\sum_{\lambda\in\Lambda}\mathcal{D}^\dagger\Box_\lambda$, we get

$$\mathcal{D}^\dagger/\sum_{\lambda\in\Lambda}\mathcal{D}^\dagger\Box_\lambda\cong L^{\dagger\prime},\quad \mathcal{D}^\dagger/(\sum_{i=1}^{n}\mathcal{D}^\dagger E_{i,\gamma}+\sum_{\lambda\in\Lambda}\mathcal{D}^\dagger\Box_\lambda)\cong L^{\dagger\prime}/\sum_{i=1}^{n}F_{i,\gamma}L^{\dagger\prime}.$$

1.6. **Proof of Theorem 0.9.** It is known that $\mathcal{D}^\dagger$ is coherent ([13]). So by Theorem 0.8, $\mathcal{L}^{\dagger\prime}$ is a coherent $\mathcal{D}^\dagger$-module.

Keep the notation in the proof of Lemma 1.4. Decompose Δ into a finite union $\bigcup_\tau\tau$ so that each τ is a simplicial complex of dimension n with vertices $\{0,\mathbf{w}_{i_1},\ldots,\mathbf{w}_{i_n}\}$ for some subset $\{i_1,\ldots,i_n\}\subset\{1,\ldots,N\}$. Let $B=\bigcup_\tau B(\tau)$ which is a finite set. Consider the map

$$\psi:\bigoplus_{\beta\in B}L^{\dagger\prime}\to L^\dagger,\quad (f_\beta)\mapsto\sum_{\beta\in B}f_\beta\mathbf{t}^\beta.$$

Note that this a homomorphism of $\mathcal{D}^\dagger$-modules. We will prove ψ is surjective and $\ker\psi$ is a finitely generated $\mathcal{D}^\dagger$-module. Combined with the fact that $\mathcal{L}^{\dagger\prime}$ is a coherent $\mathcal{D}^\dagger$-module, this implies that $\mathcal{L}^\dagger$ is a coherent $\mathcal{D}^\dagger$-module.

We have $\mathbb{Z}^n\cap\delta=\bigcup_\tau(\mathbb{Z}^n\cap\delta(\tau))$. To prove ψ is surjective, it suffices to show every element in $L^\dagger$ of the form $\sum_{\mathbf{w}\in\mathbb{Z}^n\cap\delta(\tau)}a_{\mathbf{w}}(x)t^{\mathbf{w}}$ lies in the image of ψ, where $a_{\mathbf{w}}(\mathbf{x})\in K\{r^{-1}\mathbf{x}\}$ and $\|a_{\mathbf{w}}(\mathbf{x})\|_r s^{d(\mathbf{w})}$ are bounded for some $r,s>1$. Every element $\mathbf{w}\in\mathbb{Z}^n\cap\delta(\tau)$ can be written uniquely as

$$\mathbf{w}=b(\mathbf{w})+c(\mathbf{w})$$

with $b(\mathbf{w})\in B(\tau)$ and $c(\mathbf{w})\in C(\tau)$. We have

$$\sum_{\mathbf{w}\in\mathbb{Z}^n\cap\delta(\tau)}a_{\mathbf{w}}(x)t^{\mathbf{w}}=\sum_{\beta\in B(\tau)}\Big(\sum_{\mathbf{w}\in\mathbb{Z}^n\cap\delta(\tau),\ b(\mathbf{w})=\beta}a_{\mathbf{w}}(x)t^{c(\mathbf{w})}\Big)\mathbf{t}^\beta.$$

Note that $\sum\limits_{\mathbf{w}\in\mathbb{Z}^n\cap\delta(\tau),\ b(\mathbf{w})=\beta}a_{\mathbf{w}}(x)t^{c(\mathbf{w})}$ lie in $L^{\dagger\prime}$. To see this, we use the fact that

$$d(\mathbf{w})=d(b(\mathbf{w}))+d(c(\mathbf{w}))$$

since $b(\mathbf{w})$ and $c(\mathbf{w})$ all lie in the simplicial cone $\delta(\tau)$. Thus ψ is surjective.

Given $\beta',\beta''\in B$, set

$$\begin{aligned}L_{\beta',\beta''}&=\{f\in L^{\dagger\prime}:\ f\mathbf{t}^{\beta'-\beta''}\in L^{\dagger\prime}\},\\ S_{\beta',\beta''}&=\{\mathbf{w}\in C(A):\mathbf{w}+\beta'-\beta''\in C(A)\}.\end{aligned}$$

Note that elements in $L_{\beta',\beta''}$ are of the form $\sum_{\mathbf{w}\in S_{\beta',\beta''}}a_{\mathbf{w}}(\mathbf{x})\mathbf{t}^{\mathbf{w}}$ with $a_{\mathbf{w}}(\mathbf{x})\in K\{r^{-1}\mathbf{x}\}$ and $\|a_{\mathbf{w}}(\mathbf{x})\|_r s^{d(\mathbf{w})}$ bounded for some $r,s>1$. We have $S_{\beta',\beta''}+\mathbf{w}_j\subset S_{\beta',\beta''}$ for all j, and $L_{\beta',\beta''}$ is a $\mathcal{D}^\dagger$-submodule of $L^{\dagger\prime}$. For any $f\in L_{\beta',\beta''}$ and $\beta\in B$, let

$$\iota_{\beta',\beta''}(f)_\beta=\begin{cases}f & \text{if } \beta=\beta',\\ -ft^{\beta'-\beta''} & \text{if } \beta=\beta'',\\ 0 & \text{if } \beta\in B\backslash\{\beta',\beta''\}.\end{cases}$$

Then the map

$$\iota_{\beta',\beta''}:L_{\beta',\beta''}\to\bigoplus_{\beta\in B}L^{\dagger\prime},\quad f\mapsto(\iota_{\beta',\beta''}(f)_\beta)_{\beta\in B}$$

is a homomorphism of $\mathcal{D}^\dagger$-modules and its image is contained in $\ker\psi$. We will prove each $L_{\beta',\beta''}$ is a finitely generated $\mathcal{D}^\dagger$-module, and

$$\ker\psi=\sum_{\beta',\beta''}\iota_{\beta',\beta''}(L_{\beta',\beta''}).$$

It follows that $\ker\psi$ is a finitely generated $\mathcal{D}^\dagger$-module.

We have

$$S_{\beta',\beta''}=\bigcup_\tau(S_{\beta',\beta''}\cap\delta(\tau))=\bigcup_\tau\bigcup_{\mathbf{w}\in B(\tau)}(S_{\beta',\beta''}\cap(\mathbf{w}+C(\tau))).$$

Again by Lemma 1.3, for each $S_{\beta',\beta''}\cap(\mathbf{w}+C(\tau))$, we can find finitely many $\mathbf{u}_1,\ldots,\mathbf{u}_m\in S_{\beta',\beta''}\cap(\mathbf{w}+C(\tau))$ such that

$$S_{\beta',\beta''}\cap(\mathbf{w}+C(\tau))=\bigcup_{i=1}^m(\mathbf{u}_i+C(\tau)).$$

We thus decompose $S_{\beta',\beta''}$ into a finite union of subsets of the form $\mathbf{u}+C(\tau)$ such that τ is a simplicial complex of dimension n with vertices $\{0,\mathbf{w}_{i_1},\ldots,\mathbf{w}_{i_n}\}$ for some subset $\{i_1,\ldots,i_n\}\subset$

$\{1,\ldots,N\}$, and $\mathbf{u}\in S_{\beta',\beta''}\cap(\mathbf{w}+C(\tau))$ for some $\mathbf{w}\in B(\tau)$. We claim that $L_{\beta',\beta''}$ is generated by these $\mathbf{t}^{\mathbf{u}}$ as a $\mathcal{D}^\dagger$-module. Indeed, elements in $L_{\beta',\beta''}$ is a sum of elements of the form $\sum_{\mathbf{w}\in\mathbf{u}+C(\tau)}a_{\mathbf{w}}(\mathbf{x})\mathbf{t}^{\mathbf{w}}$. We have

$$\sum_{\mathbf{w}\in\mathbf{u}+C(\tau)}a_{\mathbf{w}}(\mathbf{x})\mathbf{t}^{\mathbf{w}}=\sum_{v_1,\ldots,v_n\geq 0}a_{\mathbf{u}+v_1\mathbf{w}_{i_1}+\cdots+v_n\mathbf{w}_{i_n}}(\mathbf{x})\Big(\frac{1}{\pi}\frac{\partial}{\partial x_{i_1}}\Big)^{v_1}\cdots\Big(\frac{1}{\pi}\frac{\partial}{\partial x_{i_m}}\Big)^{v_n}\cdot\mathbf{t}^{\mathbf{u}}.$$

Suppose $(f_\beta^{(0)})\in\bigoplus_{\beta\in B}L^{\dagger\prime}$ is an element in $\ker\psi$. We then have

$$\sum_{\beta\in B}f_\beta^{(0)}\mathbf{t}^\beta=0.$$

Write $B=\{\beta_1,\ldots,\beta_k\}$, and write

$$f_\beta^{(0)}=\sum_{\mathbf{w}\in C(A)}a_{\beta\mathbf{w}}(\mathbf{x})\mathbf{t}^{\mathbf{w}}.$$

Define

$$\begin{aligned}f_\beta^{(1)}&=\sum_{\mathbf{w}\in C(A),\ \mathbf{w}+(\beta-\beta_1)\notin C(A)}a_{\beta\mathbf{w}}(\mathbf{x})t^{\mathbf{w}},\\ g_\beta^{(1)}&=\sum_{\mathbf{w}\in C(A),\ \mathbf{w}+(\beta-\beta_1)\in C(A)}a_{\beta\mathbf{w}}(\mathbf{x})t^{\mathbf{w}}.\end{aligned}$$

In particular, $f_{\beta_1}^{(1)}$ is 0 since it is a sum over the empty set. We have $g_\beta^{(1)}\in L_{\beta,\beta_1}$ and

$$(f_\beta^{(0)})-\sum_{\beta\in B\backslash\{\beta_1\}}\iota_{\beta,\beta_1}(g_\beta^{(1)})=(f_\beta^{(1)}).\tag{1.6.1}$$

To verify this equation, we show it holds componentwisely. The equation clearly holds for those component $\beta\neq\beta_1$. Note that $L^{\dagger\prime}$ is a direct factor of $L^\dagger$ in a canonical way as an abelian group. Applying the projection $L^\dagger\to L^{\dagger\prime}$ to the equation

$$\sum_{\beta\in B}f_\beta^{(0)}\mathbf{t}^{\beta-\beta_1}=0,$$

we get

$$f_{\beta_1}^{(0)}+\sum_{\beta\in B\backslash\{\beta_1\}}g_\beta^{(1)}\mathbf{t}^{\beta-\beta_1}=0.$$

This is exactly the β_1 component of the equation 1.6.1.

In general, for $i=1,\ldots,k$, we define

$$\begin{aligned}f_\beta^{(i)}&=\sum_{\mathbf{w}\in C(A),\ \mathbf{w}+(\beta-\beta_1)\notin C(A),\ldots,\ \mathbf{w}+(\beta-\beta_i)\notin C(A)}a_{\beta\mathbf{w}}(\mathbf{x})t^{\mathbf{w}},\\ g_\beta^{(i)}&=\sum_{\mathbf{w}\in C(A),\ \mathbf{w}+(\beta-\beta_1)\notin C(A),\ldots,\ \mathbf{w}+(\beta-\beta_{i-1})\notin C(A),\mathbf{w}+(\beta-\beta_i)\in C(A)}a_{\beta\mathbf{w}}(\mathbf{x})t^{\mathbf{w}}.\end{aligned}$$

We have $g_\beta^{(i)}\in L_{\beta,\beta_i}$ and

$$(f_\beta^{(i-1)})-\sum_{\beta\in B}\iota_{\beta,\beta_i}(g_\beta^{(i)})=(f_\beta^{(i)}).$$

We have $f_\beta^{(k)}=0$ for all $\beta\in B=\{\beta_1,\ldots,\beta_n\}$. So we have

$$(f_\beta^{(0)})=\sum_{i=1}^k\sum_{\beta\in B}\iota_{\beta,\beta_i}(g_\beta^{(i)}).$$

Hence $\ker\psi=\sum_{\beta',\beta''}\iota_{\beta',\beta''}(L_{\beta',\beta''})$.

1.7. **Proof of Lemma 0.12.** Let R be the integer ring of K, and let

$$\begin{aligned}
R\langle\mathbf{x}\rangle^\dagger &= \bigcup_{r>1}\{\sum_{\mathbf{v}\in\mathbb{Z}_{\geq 0}^N}a_{\mathbf{v}}\mathbf{x}^{\mathbf{v}}:\ a_{\mathbf{v}}\in R,\ |a_{\mathbf{v}}|r^{|\mathbf{v}|}\text{ are bounded }\},\\
R\langle\mathbf{x},\mathbf{y}\rangle^\dagger &= \bigcup_{r>1,\ s>1}\{\sum_{\mathbf{u},\mathbf{v}\in\mathbb{Z}_{\geq 0}^N}a_{\mathbf{u}\mathbf{v}}\mathbf{x}^{\mathbf{u}}\mathbf{y}^{\mathbf{v}}:\ a_{\mathbf{u}\mathbf{v}}\in R,\ |a_{\mathbf{u}\mathbf{v}}|r^{|\mathbf{u}|}s^{|\mathbf{v}|}\text{ are bounded }\},\\
L_R^\dagger &= \bigcup_{r>1,s>1}\{\sum_{\mathbf{v}\in\mathbb{Z}_{\geq 0}^N,\ \mathbf{w}\in\mathbb{Z}^n\cap\delta}a_{\mathbf{v}\mathbf{w}}\mathbf{x}^{\mathbf{v}}\mathbf{t}^{\mathbf{w}}:\ a_{\mathbf{v}\mathbf{w}}\in R,\ |a_{\mathbf{v}\mathbf{w}}|r^{|v|}s^{d(\mathbf{w})}\text{ are bounded}\},\\
L_R^{\dagger\prime} &= \bigcup_{r>1,s>1}\{\sum_{\mathbf{v}\in\mathbb{Z}_{\geq 0}^N,\ \mathbf{w}\in C(A)}a_{\mathbf{v}\mathbf{w}}\mathbf{x}^{\mathbf{v}}\mathbf{t}^{\mathbf{w}}:\ a_{\mathbf{v}\mathbf{w}}\in R,\ |a_{\mathbf{v}\mathbf{w}}|r^{|v|}s^{d(\mathbf{w})}\text{ are bounded}\}.
\end{aligned}$$

We have

$$K\langle\mathbf{x}\rangle^\dagger\cong R\langle\mathbf{x}\rangle^\dagger\otimes_R K,\quad L^\dagger\cong L_R^\dagger\otimes_R K.$$

To prove $L^\dagger$ is flat over $K\langle\mathbf{x}\rangle^\dagger$, it suffices to show $L_R^\dagger$ is flat over $R\langle\mathbf{x}\rangle^\dagger$.

Keep the notation in the proof of Lemma 1.4 and 1.6. The same proof shows that the following homomorphisms

$$\begin{aligned}
&\bigoplus_{\beta\in B}L_R^{\dagger\prime}\to L_R^\dagger, &&(f_\beta)\mapsto\sum_{\beta\in B}f_\beta\mathbf{t}^\beta,\\
&R\langle\mathbf{x},\mathbf{y}\rangle^\dagger\to L_R^{\dagger\prime}, &&\sum_{\mathbf{v}\in\mathbb{Z}_{\geq 0}^N}f_{\mathbf{v}}(\mathbf{x})\mathbf{y}^{\mathbf{v}}\mapsto\sum_{\mathbf{v}\in\mathbb{Z}_{\geq 0}^N}f_{\mathbf{v}}(\mathbf{x})t^{v_1\mathbf{w}_N+\cdots+v_N\mathbf{w}_N}
\end{aligned}$$

are surjective. It is known that $R\langle\mathbf{x},\mathbf{y}\rangle^\dagger$ is a noetherian ring by [6]. It follows that $L_R^\dagger$ is also noetherian. We have

$$L_R^\dagger/\pi^kL_R^\dagger\cong(R/\pi^k)[\mathbf{x}][\mathbb{Z}^n\cap\delta],\quad R\langle\mathbf{x}\rangle^\dagger/\pi^kR\langle x\rangle^\dagger\cong(R/\pi^k)[\mathbf{x}].$$

So $L_R^\dagger/\pi^kL_R^\dagger$ is flat over $R\langle\mathbf{x}\rangle^\dagger/\pi^kR\langle\mathbf{x}\rangle^\dagger$ for all k. By [12, IV Théorème 5.6], $L_R^\dagger$ is flat over $R\langle\mathbf{x}\rangle^\dagger$.

Finally let's prove $L^\dagger\otimes_{K\langle\mathbf{x}\rangle^\dagger}K'\cong L_0^\dagger$. One can verify directly that in the case where $K'=K$, the homomorphism

$$L^\dagger\to L_0^\dagger,\quad\sum_{\mathbf{w}\in\mathbb{Z}^n\cap\delta}a_{\mathbf{x}}(\mathbf{x})\mathbf{t}^{\mathbf{w}}\mapsto\sum_{\mathbf{w}\in\mathbb{Z}^n\cap\delta}a_{\mathbf{x}}(0)\mathbf{t}^{\mathbf{w}}$$

is surjective with kernel $(x_1,\ldots,x_N)L^\dagger$. This proves our assertion in the case where $K=K'$ and $\mathbf{a}=(0,\ldots,0)$. In general, we have an isomorphism $L^\dagger\otimes_K K'\cong L_{K'}^\dagger$, where

$$L_{K'}^\dagger=\bigcup_{r>1,s>1}\{\sum_{\mathbf{v}\in\mathbb{Z}_{\geq 0}^N,\ \mathbf{w}\in\mathbb{Z}^n\cap\delta}a_{\mathbf{v}\mathbf{w}}\mathbf{x}^{\mathbf{v}}\mathbf{t}^{\mathbf{w}}:\ a_{\mathbf{v}\mathbf{w}}\in K',\ |a_{\mathbf{v}\mathbf{w}}|r^{|v|}s^{d(\mathbf{w})}\text{ are bounded}\}.$$

By base change from K to K' and using this isomorphism, we can reduce to the case where $K'=K$. Then using the automorphism

$$K'\langle\mathbf{x}\rangle\to K'\langle\mathbf{x}\rangle,\quad x_i\mapsto x_i-a_i,$$

we can reduce to the case where $\mathbf{a} = (0, \dots, 0)$.

1.8. **Proof of Lemma 0.14.** We first work with de Rham complexes and later with twisted de Rham complexes. We have

$$\Psi_{\mathbf{a}} \circ \Phi_{\mathbf{a}} = \mathrm{id}$$

on $C^{\cdot}(L_0^\dagger)$. Since K contains the primitive root of unity $\theta(1)$, it contains all q-th roots of unity. Let μ_q be the group of q-th roots of unity in K. For any $\zeta = (\zeta_1, \dots, \zeta_n) \in \mu_q^n$, write

$$\zeta\mathbf{t} = (\zeta_1 t_1, \dots, \zeta_n t_n).$$

We have

$$\sum_{\zeta \in \mu_q^n} \zeta^{\mathbf{w}} = \begin{cases} q^n & \text{if } q|\mathbf{w}, \\ 0 & \text{otherwise.} \end{cases}$$

So we have

$$\Phi_{\mathbf{a}} \circ \Psi_{\mathbf{a}}(\sum_{\mathbf{w}} c_{\mathbf{w}} \mathbf{t}^{\mathbf{w}}) = \sum_{\mathbf{w}} c_{q\mathbf{w}} \mathbf{t}^{q\mathbf{w}} = \frac{1}{q^n} \sum_{\zeta \in \mu_q^n} \sum_{\mathbf{w}} c_{\mathbf{w}} (\zeta\mathbf{t})^{\mathbf{w}}.$$

Let Θ_ζ be the endomorphism on differential forms defined by

$$\Theta_\zeta\Big(\sum_{1 \le i_1 < \cdots < i_k \le n} f_{i_1 \dots i_k}(\mathbf{t}) \frac{\mathrm{d}t_{i_1}}{t_{i_1}} \wedge \cdots \wedge \frac{\mathrm{d}t_{i_k}}{t_{i_k}}\Big) = \sum_{1 \le i_1 < \cdots < i_k \le n} f_{i_1 \dots i_k}(\zeta\mathbf{t}) \frac{\mathrm{d}t_{i_1}}{t_{i_1}} \wedge \cdots \wedge \frac{\mathrm{d}t_{i_k}}{t_{i_k}}.$$

It commutes with $\mathrm{d}_{\mathbf{t}}$. We have

$$\Phi_{\mathbf{a}} \circ \Psi_{\mathbf{a}} = \frac{1}{q^n} \sum_{\zeta \in \mu_q^n} \Theta_\zeta.$$

Let's show $\Phi_{\mathbf{a}} \circ \Psi_{\mathbf{a}}$ is homotopic to id. It suffices to that Θ_ζ is homotopic to id for each $\zeta \in \mu_q^n$. Let

$$L_T^\dagger = \bigcup_{r>1,\, s>1} \{ \sum_{\mathbf{w} \in \mathbb{Z}^n \cap \delta} a_{\mathbf{w}}(T) \mathbf{t}^{\mathbf{w}} : \ a_{\mathbf{w}}(T) \in K\{r^{-1}T\}, \ \|a_{\mathbf{w}}(T)\|_r s^{d(\mathbf{w})} \text{ are bounded} \}.$$

Consider the de Rham complex $(C^{\cdot}(L_T^\dagger), \mathrm{d})$ so that $C^k(L_T^\dagger)$ is the space of k-forms which can be written as a sum of products of $\mathrm{d}T$, $\frac{\mathrm{d}t_1}{t_1}, \dots, \frac{\mathrm{d}t_n}{t_n}$ and functions in $L_T^\dagger$, and $\mathrm{d} : C^k(L_T^\dagger) \to C^{k+1}(L_T^\dagger)$ is the usual exterior derivative of differential forms. The substitution

$$t_i \to (1 + (\zeta_i - 1)T) t_i \quad (i = 1, \dots, n)$$

induces a chain map

$$\iota : (C^{\cdot}(L_0^\dagger), \mathrm{d}_t) \to (C^{\cdot}(L_T^\dagger), \mathrm{d}).$$

Here we use the fact that $\zeta_i \equiv 1 \mod p$ so that each $1 + (\zeta_i - 1)T$ is a unit in $L_T^\dagger$. In particular, $\frac{\mathrm{d}\big((1+(\zeta_i-1)T)t_i\big)}{(1+(\zeta_i-1)T)t_i}$ lies in $C^{\cdot}(L_T^\dagger)$. The evaluation at $T = 0$ (resp. $T = 1$) induces a chain map

$$\mathrm{ev}_0 : (C^{\cdot}(L_T^\dagger), \mathrm{d}) \to (C^{\cdot}(L_0^\dagger), \mathrm{d}_t) \quad (\text{resp. } \mathrm{ev}_1 : (C^{\cdot}(L_T^\dagger), \mathrm{d}) \to (C^{\cdot}(L_0^\dagger), \mathrm{d}_t)).$$

We have

$$\mathrm{ev}_1 \circ \iota = \Theta_\zeta, \quad \mathrm{ev}_0 \circ \iota = \mathrm{id}.$$

To prove Θ_ζ is homotopic to identity, it suffices to show ev_1 is homotopic to ev_0. Note that $\int_0^T g(T,\mathbf{t})\mathrm{d}T$ lies in $L_0^\dagger$ for any $g(T,\mathbf{t})\in L_T^\dagger$. Define $\Xi: C^k(L_T^\dagger)\to C^{k-1}(L_0^\dagger)$ by

$$\Xi\Big(f(T,\mathbf{t})\frac{\mathrm{d}t_{i_1}}{t_{i_1}}\wedge\cdots\wedge\frac{\mathrm{d}t_{i_k}}{t_{i_k}}\Big)=0,$$
$$\Xi\Big(g(T,\mathbf{t})\mathrm{d}T\wedge\frac{\mathrm{d}t_{j_1}}{t_{j_1}}\wedge\cdots\wedge\frac{\mathrm{d}t_{j_{k-1}}}{t_{j_{k-1}}}\Big)=\Big(\int_0^1 g(T,\mathbf{t})\mathrm{d}T\Big)\frac{\mathrm{d}t_{j_1}}{t_{j_1}}\wedge\cdots\wedge\frac{\mathrm{d}t_{j_{k-1}}}{t_{j_{k-1}}}.$$

Then we have

$$\mathrm{d}_{\mathbf{t}}\Xi+\Xi\mathrm{d}=\mathrm{ev}_1-\mathrm{ev}_0.$$

We now consider the twisted de Rham complexes. Let

$$F_{\mathbf{a}},G_{\mathbf{a}},T_\zeta,L,E_0,E_1,H$$

be the conjugates of

$$\Phi_{\mathbf{a}},\Psi_{\mathbf{a}},\Theta_\zeta,\iota,\mathrm{ev}_0,\mathrm{ev}_1,\Xi$$

by $t_1^{\gamma_1}\cdots t_n^{\gamma_n}\exp(\pi F(\mathbf{a},\mathbf{t}))$ respectively. One verifies that they are defined on $C^\cdot(L_0^\dagger)$ or $C^\cdot(L_T^\dagger)$. By the discussion above for the untwisted de Rham complexes, we have

$$G_{\mathbf{a}}F_{\mathbf{a}}=\mathrm{id},\quad F_{\mathbf{a}}G_{\mathbf{a}}=\frac{1}{q^n}\sum_{\zeta\in\mu_q^n}T_\zeta,$$
$$E_1\circ L=T_\zeta,\quad E_0\circ L=\mathrm{id},\quad dH+Hd=E_1-E_1.$$

It follows that each T_ζ is homotopic to identity and hence $F_{\mathbf{a}}G_{\mathbf{a}}$ is also homotopic to identity.

2. Dwork's theory

2.1. Let

$$\theta(z)=\exp(\pi z-\pi z^p),\quad \theta_m(z)=\exp(\pi z-\pi z^{p^m})=\prod_{i=0}^{m-1}\theta(z^{p^i}).$$

Then $\theta_m(z)$ converges in a disc of radius >1, and the value $\theta(1)=\theta(z)|_{z=1}$ of the power series $\theta(z)$ at $z=1$ is a primitive p-th root of unity in K ([15, Theorems 4.1 and 4.3]). Let $\bar u\in\mathbb{F}_{p^m}$ and let $u\in\overline{\mathbb{Q}}_p$ be its Techmüller lifting, that is, $u^{p^m}=u$ and $u\equiv\bar u\mod p$. Then we have ([15, Theorem 4.4])

$$\theta_m(u)=\theta(1)^{\mathrm{Tr}_{\mathbb{F}_{p^m}/\mathbb{F}_p}(\bar u)}.$$

From now on, we denote elements in finite fields by letters with bars such as $\bar u,\bar a_j,\bar u_i$ etc and denote their Techmüller liftings by the same letters without bars such as u,a_j,u_i etc. Let $\psi_m:\mathbb{F}_{q^m}\to K^*$ be the additive character defined by

$$\psi_m(\bar u)=\theta(1)^{\mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_p}(\bar u)}.$$

Then we have

$$\psi_m(\bar u)=\exp(\pi z-\pi z^{q^m})|_{z=u}.$$

Denote ψ_1 by ψ. We have $\psi_m=\psi\circ\mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}$. Let $\bar a_1,\ldots,\bar a_N\in\mathbb{F}_q$. For any $\bar u_1,\ldots,\bar u_n\in\mathbb{F}_{q^m}^*$, we have

$$(2.1.1)\quad \begin{aligned}\psi\Big(\mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\textstyle\sum_{j=1}^N\bar a_j\bar u_1^{w_{1j}}\cdots\bar u_n^{w_{nj}})\Big)&=\textstyle\prod_{j=1}^N\psi_m(\bar a_j\bar u_1^{w_{1j}}\cdots\bar u_n^{w_{nj}})\\&=\textstyle\prod_{j=1}^N\exp(\pi z-\pi z^{q^m})\big|_{z=a_ju_1^{w_{1j}}\cdots u_n^{w_{nj}}}.\end{aligned}$$

Let $\chi : \mathbb{F}_q^* \to \overline{\mathbb{Q}}_p^*$ be the Techmüller character, that is, $\chi(\bar{u}) = u$ is the Techmüller lifting of $\bar{u} \in \mathbb{F}_q$. It is a generator for the group of multiplicative characters on $\mathbb{F}_q$. Any multiplicative character $\mathbb{F}_q^* \to \overline{\mathbb{Q}}_p^*$ is of the form $\chi_\gamma = \chi^{\gamma(1-q)}$ for some rational number $\gamma \in \frac{1}{1-q}\mathbb{Z}$. Moreover, for any $\bar{u} \in \mathbb{F}_{q^m}$, we have

$$\chi_\gamma(\mathrm{Norm}_{\mathbb{F}_q^m/\mathbb{F}_q}(\bar{u})) = (u^{1+q+\cdots+q^{m-1}})^{\gamma(1-q)} = u^{\gamma(1-q^m)}, \tag{2.1.2}$$

Let $\gamma_1, \ldots, \gamma_n \in \frac{1}{1-q}\mathbb{Z}$. Set $\chi_i = \chi^{\gamma_i(1-q)}$ $(i = 1, \ldots, n)$.

Consider the twisted exponential sum

$$S_m(F(\bar{\mathbf{a}}, \mathbf{t})) = \sum_{\bar{u}_1,\ldots,\bar{u}_n \in \mathbb{F}_{q^m}^*} \chi_1(\mathrm{Norm}_{\mathbb{F}_q^m/\mathbb{F}_q}(\bar{u}_1)) \cdots \chi_n(\mathrm{Norm}_{\mathbb{F}_q^m/\mathbb{F}_q}(\bar{u}_n)) \psi\Big(\mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q}\Big(\sum_{j=1}^N \bar{a}_j \bar{u}_1^{w_{1j}} \cdots \bar{u}_n^{w_{nj}}\Big)\Big).$$

Write $\exp(\pi z - \pi z^{q^m}) = \sum_{i=1}^\infty c_i z^i$. By the equations (2.1.1) and (2.1.2), we have

$$\begin{aligned}
& S_m(F(\bar{\mathbf{a}}, \mathbf{t})) \\
= & \sum_{u_i^{q^m-1}=1} u_1^{\gamma_1(1-q^m)} \cdots u_n^{\gamma_n(1-q^m)} \prod_{j=1}^N \exp(\pi z - \pi z^{q^m})|_{z=a_j u_1^{w_{1j}} \cdots u_n^{w_{nj}}} \\
= & \sum_{u_i^{q^m-1}=1} u_1^{\gamma_1(1-q^m)} \cdots u_n^{\gamma_n(1-q^m)} \prod_{j=1}^N \Big(\sum_{i=1}^\infty c_i (a_j u_1^{w_{1j}} \cdots u_n^{w_{nj}})^i\Big) \\
= & \sum_{u_i^{q^m-1}=1} \left(t_1^{\gamma_1(1-q^m)} \cdots t_n^{\gamma_n(1-q^m)} \prod_{j=1}^N \Big(\sum_{i=1}^\infty c_i (a_j t_1^{w_{1j}} \cdots t_n^{w_{nj}})^i\Big)\right) |_{t_i=u_i} \\
= & \sum_{u_i^{q^m-1}=1} \left(t_1^{\gamma_1(1-q^m)} \cdots t_n^{\gamma_n(1-q^m)} \prod_{j=1}^N \exp\left(\pi a_j t_1^{w_{1j}} \cdots t_n^{w_{nj}} - \pi a_j t_1^{q^m w_{1j}} \cdots t_n^{q^m w_{nj}}\right)\right) |_{t_i=u_i} \\
= & \sum_{u_i^{q^m-1}=1} \left(t_1^{\gamma_1(1-q^m)} \cdots t_n^{\gamma_n(1-q^m)} \exp\left(\pi F(\mathbf{a}, \mathbf{t}) - \pi F(\mathbf{a}, \mathbf{t}^{q^m})\right)\right) |_{t_i=u_i}.
\end{aligned}$$

We thus have

$$S_m(F(\bar{\mathbf{a}}, \mathbf{t})) = \sum_{u_i^{q^m-1}=1} \left(t_1^{\gamma_1(1-q^m)} \cdots t_n^{\gamma_n(1-q^m)} \exp\left(\pi F(\mathbf{a}, \mathbf{t}) - \pi F(\mathbf{a}, \mathbf{t}^{q^m})\right)\right) |_{t_i=u_i}. \tag{2.1.3}$$

2.2. Let K' be a finite extension of K containing all q-th roots of unity. Set

$$L(s)_0 = \{\sum_{\mathbf{w} \in \mathbb{Z}^n \cap \delta} a_{\mathbf{w}} t^{\mathbf{w}} : a_{\mathbf{w}} \in K',\ |a_{\mathbf{w}}| s^{d(\mathbf{w})} \text{ are bounded}\}.$$

We have $L_0^\dagger = \bigcup_{s>1} L(s)_0$. Note that $L(s)_0$ $(s \geq 1)$ and $L_0^\dagger$ are rings. Each $L(s)_0$ is a Banach space with respect to the norm

$$\|\sum_{\mathbf{w} \in \mathbb{Z}^n \cap \delta} a_{\mathbf{w}} t^{\mathbf{w}}\| = \sup_{\mathbf{w} \in \mathbb{Z}^n \cap \delta} |a_{\mathbf{w}}| s^{d(\mathbf{w})}.$$

Theorem 2.3 (Dwork trace formula)**.** *The operator $G_{\mathbf{a}} :\ L_0^\dagger \to L_0^\dagger$ is nuclear, and we have*

$$(q^m - 1)^n \mathrm{Tr}(G_{\mathbf{a}}^m, L_0^\dagger) = \sum_{u_i^{q^m-1}=1} \left(t_1^{\gamma_1(1-q^m)} \cdots t_n^{\gamma_n(1-q^m)} \exp\left(\pi F(\mathbf{a}, \mathbf{t}) - \pi F(\mathbf{a}, \mathbf{t}^{q^m})\right)\right) |_{t_i=u_i}.$$

Proof. For any real number $s \geq 1$, define

$$\tilde{L}(s)_0 = \{ \sum_{\mathbf{w} \in \mathbb{Z}^n \cap \delta} a_{\mathbf{w}} t^{\mathbf{w}} : \ a_{\mathbf{w}} \in K', \ \lim_{d(\mathbf{w}) \to \infty} |a_{\mathbf{w}}| s^{d(\mathbf{w})} = 0 \}.$$

For any $s < s'$, we have

$$L(s')_0 \subset \tilde{L}(s)_0 \subset L(s)_0,$$

and $L_0^\dagger = \bigcup_{s>1} \tilde{L}(s)_0$. Endow $\tilde{L}(s)_0$ with the norm

$$\| \sum_{\mathbf{w} \in \mathbb{Z}^n \cap \delta} a_{\mathbf{w}} t^{\mathbf{w}} \| = \sup_{\mathbf{w} \in \mathbb{Z}^n \cap \delta} |a_{\mathbf{w}}| s^{d(\mathbf{w})}.$$

Then $\tilde{L}(s)_0$ is a Banach space with the orthogonal basis $\{t^{\mathbf{w}}\}_{\mathbf{w} \in \mathbb{Z}^n \cap \delta}$. The inclusion $L(s')_0 \hookrightarrow \tilde{L}(s)_0$ is completely continuous. Indeed, choose $s < s'' < s'$. We can factorize this inclusion as the composite

$$L(s')_0 \hookrightarrow \tilde{L}(s'')_0 \hookrightarrow \tilde{L}(s)_0.$$

It suffices to verify the inclusion $i : \tilde{L}(s'')_0 \hookrightarrow \tilde{L}(s)_0$ is completely continuous. Indeed, let L_S be the finite dimensional K'-vector space spanned by a finite subset S of $\{t^{\mathbf{w}}\}_{\mathbf{w} \in \mathbb{Z}^n \cap \delta}$, and let

$$i_S : \tilde{L}(s'')_0 \to \tilde{L}(s)_0$$

be the composite of the projection $\tilde{L}(s'')_0 \to L_S$ and the inclusion $L_S \hookrightarrow \tilde{L}(s)_0$. One can verify that

$$\|i_S - i\| \leq \sup_{\mathbf{w} \notin S} \left(\frac{s}{s''}\right)^{d(\mathbf{w})}.$$

So i_S converges to i as S goes over all finite subsets of $\{t^{\mathbf{w}}\}_{\mathbf{w} \in \mathbb{Z}^n \cap \delta}$. Moreover i_S has finite ranks. So i is completely continuous.

Let $H(\mathbf{t}) = t_1^{\gamma_1(1-q)} \cdots t_n^{\gamma_n(1-q)} \exp\big(\pi F(\mathbf{a}, \mathbf{t}) - \pi F(\mathbf{a}, \mathbf{t}^q)\big)$. By Lemma 0.4, we have have $H_q(\mathbf{t}) \in L(p^{\frac{p-1}{pq}})_0$. For any $s \geq 1$, we have $\Psi_{\mathbf{a}}(L(s)_0) \subset L(s^q)_0$. Consider the operator

$$\begin{aligned} G_{\mathbf{a}} &= \Big(t_1^{\gamma_1} \cdots t_n^{\gamma_n} \exp(\pi F(\mathbf{a}, \mathbf{t}))\Big)^{-1} \circ \Psi_{\mathbf{a}} \circ \Big(t_1^{\gamma_1} \cdots t_n^{\gamma_n} \exp(\pi F(\mathbf{a}, \mathbf{t}))\Big) \\ &= \Psi_{\mathbf{a}} \circ \Big(t_1^{\gamma_1(1-q)} \cdots t_n^{\gamma_n(1-q)} \exp\big(\pi F(\mathbf{a}, \mathbf{t}) - \pi F(\mathbf{a}, \mathbf{t}^q)\big)\Big). \end{aligned}$$

If $1 < s < p^{\frac{p-1}{p}}$, then $G_{\mathbf{a}}$ induces a map $G_{\mathbf{a}} : \ \tilde{L}(s)_0 \to \tilde{L}(s)_0$. It is the composite

$$\tilde{L}(s)_0 \hookrightarrow L(s)_0 \stackrel{H(\mathbf{t})}{\to} L\Big(\min\Big(s, p^{\frac{p-1}{pq}}\Big)\Big)_0 \stackrel{\Psi_{\mathbf{a}}}{\to} L\Big(\min\Big(s^q, p^{\frac{p-1}{p}}\Big)\Big)_0 \hookrightarrow \tilde{L}(s)_0.$$

$G_{\mathbf{a}} : \ \tilde{L}(s)_0 \to \tilde{L}(s)_0$ is completely continuous since the last inclusion in the above composite is completely continuous. In particular, it is nuclear ([15, Theorem 6.9]). Write

$$t_1^{\gamma_1(1-q)} \cdots t_n^{\gamma_n(1-q)} \exp\big(\pi F(\mathbf{a}, \mathbf{t}) - \pi F(\mathbf{a}, \mathbf{t}^q)\big) = \sum_{\mathbf{w}} c_{\mathbf{w}} t^{\mathbf{w}}.$$

We have

$$\begin{aligned} G_{\mathbf{a}}(t^{\mathbf{u}}) &= \Psi_{\mathbf{a}}(\sum_{\mathbf{w}} c_{\mathbf{w}} t^{\mathbf{w}+\mathbf{u}}) \\ &= \Psi_{\mathbf{a}}(\sum_{\mathbf{w}} c_{\mathbf{w}-\mathbf{u}} t^{\mathbf{w}}) \\ &= \sum_{\mathbf{w}} c_{q\mathbf{w}-\mathbf{u}} t^{\mathbf{w}}, \end{aligned}$$

where $c_{q\mathbf{w}-\mathbf{u}}$ is nonzero only if $\mathbf{u}, \mathbf{w}, \mathbf{qw}-\mathbf{u} \in \delta$. The matrix of $G_{\mathbf{a}}$ on $\tilde{L}(s)_0$ with respect to the orthogonal basis $\{t^{\mathbf{w}}\}$ is $(c_{q\mathbf{w}-\mathbf{u}})$. By [15, Theorem 6.10] we have

$$\mathrm{Tr}(G_{\mathbf{a}}, \tilde{L}(s)_0) = \sum_{\mathbf{u}} c_{q\mathbf{u}-\mathbf{u}}.$$

In particular, $\mathrm{Tr}(G_{\mathbf{a}}, \tilde{L}(s)_0)$ is independent of s. Similarly, $\mathrm{Tr}(G_{\mathbf{a}}^m, \tilde{L}(s)_0)$ and

$$\det(I - TG_{\mathbf{a}}, \tilde{L}(s)_0) = \exp\Big(-\sum_{m=1}^{\infty} \frac{\mathrm{Tr}(G_{\mathbf{a}}^m, \tilde{L}(s)_0)}{m} T^m\Big)$$

are independent of s. For any monic irreducible polynomial $f(T) \in K'[T]$ with nonzero constant term, write ([15, Theorem 6.9])

$$\tilde{L}(s)_0 = N(s)_f \bigoplus W(s)_f,$$

where $N(s)_f$ and $W(s)_f$ are $G_{\mathbf{a}}$-invariant spaces, $N(s)_f$ is finite dimensional over K', $f(G_{\mathbf{a}})$ is nilpotent on $N(s)_f$ and bijective on $W(s)_f$. We have

$$N(s)_f = \bigcup_{m=1}^{\infty} \ker\,(f(G_{\mathbf{a}}))^m, \quad W(s)_f = \bigcap_{m=1}^{\infty} \mathrm{im}\,(f(G_{\mathbf{a}}))^m.$$

For any pair $s < s'$, we have

$$\tilde{L}(s')_0 \subset \tilde{L}(s)_0, \quad N(s')_f \subset N(s)_f, \quad W(s')_f \subset W(s)_f.$$

Let $N_f = \bigcup_{1<s<p^{\frac{p-1}{p}}} N(s)_f$ and $W_f = \bigcup_{1<s<p^{\frac{p-1}{p}}} W(s)_f$. Then

$$L_0^{\dagger} = N_f \bigoplus W_f,$$

N_f and W_f are $G_{\mathbf{a}}$-invariant, $f(G_{\mathbf{a}})$ is nilpotent on N_f and bijective on W_f. Since $\det(I - TG_{\mathbf{a}}, \tilde{L}(s)_0)$ is independent of s, all $N(s)_f$ have the same dimension, and hence we have $N_f = N(s)_f$ for all $1 < s < p^{\frac{p-1}{p}}$. This shows that $G_{\mathbf{a}} : L_0^{\dagger} \to L_0^{\dagger}$ is nuclear and

$$\mathrm{Tr}(G_{\mathbf{a}}, L_0^{\dagger}) = \sum_{\mathbf{u}} c_{q\mathbf{u}-\mathbf{u}}.$$

On the other hand, we have

$$\sum_{u^{q-1}=1} u^w = \begin{cases} q-1 & \text{if } q-1 | w, \\ 0 & \text{otherwise.} \end{cases}$$

So we have

$$\begin{aligned} & \sum_{u_i^{q-1}=1} \Big(t_1^{\gamma_1(1-q)} \cdots t_n^{\gamma_n(1-q)} \exp\big(\pi F(\mathbf{a}, \mathbf{t}) - \pi F(\mathbf{a}, \mathbf{t}^q)\big)\Big)|_{t_i=u_i} \\ = & \sum_{\mathbf{w}} \sum_{u_i^{q-1}=1} c_{\mathbf{w}} u_1^{w_1} \cdots u_n^{w_n} \\ = & (q-1)^n \sum_{\mathbf{u}} c_{(q-1)\mathbf{u}}. \end{aligned}$$

We thus get

$$(q-1)^n \mathrm{Tr}(G_{\mathbf{a}}, L_0^{\dagger}) = \sum_{u_i^{q-1}=1} \Big(t_1^{\gamma_1(1-q)} \cdots t_n^{\gamma_n(1-q)} \exp\big(\pi F(\mathbf{a}, \mathbf{t}) - \pi F(\mathbf{a}, \mathbf{t}^q)\big)\Big)|_{t_i=u_i}.$$

This proves the theorem for $m=1$. We have

$$\begin{aligned} G_{\mathbf{a}}^m &= \Big(t_1^{\gamma_1}\cdots t_n^{\gamma_n}\exp(\pi F(\mathbf{a},\mathbf{t}))\Big)^{-1}\circ\Psi_{\mathbf{a}}^m\circ\Big(t_1^{\gamma_1}\cdots t_n^{\gamma_n}\exp(\pi F(\mathbf{a},\mathbf{t}))\Big)\\ &= \Psi_{\mathbf{a}}^m\circ\Big(t_1^{\gamma_1(1-q^m)}\cdots t_n^{\gamma_n(1-q^m)}\exp\big(\pi F(\mathbf{a},\mathbf{t})-\pi F(\mathbf{a},\mathbf{t}^{q^m})\big)\Big).\end{aligned}$$

So the assertion for general m follows from the case $m=1$. □

2.4. **Proof of Theorem 0.15.** By the equation (2.1.3) and the Dwork trace formula 2.3, we have

$$\begin{aligned} S_m(F(\bar{\mathbf{a}},\mathbf{t})) &= (q^m-1)^n\mathrm{Tr}(G_{\mathbf{a}}^m, L_0^\dagger)\\ &= \sum_{k=0}^n\binom{n}{k}(-1)^k(q^m)^{n-k}\mathrm{Tr}(G_{\mathbf{a}}^m, L_0^\dagger)\\ &= \sum_{k=0}^n(-1)^k\mathrm{Tr}\Big((q^{n-k}G_{\mathbf{a}})^m, L_0^{\dagger\binom{n}{k}}\Big).\end{aligned}$$

For the L-function, we have

$$\begin{aligned} L(F(\bar{\mathbf{a}},\mathbf{t}),T) &= \exp\Big(\sum_{m=1}^\infty S_m(F(\bar{\mathbf{a}},\mathbf{t}))\frac{T^m}{m}\Big)\\ &= \exp\Big(\sum_{m=1}^\infty\sum_{k=0}^n(-1)^k\mathrm{Tr}\Big((q^{n-k}G_{\mathbf{a}})^m, L_0^{\dagger\binom{n}{k}}\Big)\frac{T^m}{m}\Big)\\ &= \prod_{k=0}^n\exp\Big((-1)^k\sum_{m=1}^\infty\mathrm{Tr}\Big((q^{n-k}G_{\mathbf{a}})^m, L_0^{\dagger\binom{n}{k}}\Big)\frac{T^m}{m}\Big)\\ &= \prod_{k=0}^n\det\Big(I-Tq^{n-k}G_{\mathbf{a}}, L_0^{\dagger\binom{n}{k}}\Big)^{(-1)^{k+1}}\end{aligned}$$

This prove Theorem 0.15.

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