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Preservice teachers' understanding of mathematical equivalence

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Abstract

Prior research has shown that many elementary school students hold misconceptions about mathematical equivalence, interpreting the equal sign operationally as an indicator to give an answer or the total. They often fail to correctly solve missing operand problems such as $1 + 5 = \_\_\_ + 2$. The present study extends the research on mathematical equivalence to examine pre-service teachers' performance on equivalence tasks. Results show that some participants failed to correctly solve missing operand problems and chose an operational definition of the equal sign over the correct relational definition. Many participants failed to recognize statements that violate equality and failed to correctly identify equations and operations. These findings suggest that misconceptions of mathematical equivalence can involve confusion about the definition of equation and the meaning of mathematical operation.

Keywords: Mathematics, Mathematical Equivalence, Pre-service teachers

Introduction

In formal mathematics, equivalence is defined as a relation that is reflexive, symmetric, and transitive (Herstein, 1975). In school mathematics (i.e. kindergarten through secondary school), the prevalent form of equivalence is quantitative equivalence which is the concept that two numbers or algebraic expressions represent the same quantity. Symbolically, quantitative equivalence is represented through statements involving the equal sign, =. Quantitative equivalence, or "mathematical equivalence" as it is often termed in prior research, pervades standard school mathematics curriculum in the United States (e.g. National Governors Association Center for Best Practices, Council of Chief State School Officers, 2010; see also Indiana Academic Standards for Mathematics, Indiana Department of Education, 2023; Ohio's Learning Standards for Mathematics, Ohio Department of Education, 2017). For example, elementary students learn the results of arithmetic operations and express these results using the equal sign, such as 2 + 3 = 5. High school students manipulate algebraic expressions involving variables to find equivalent expressions in the process of solving equations.

Considerable research has examined students' understanding of mathematical equivalence and demonstrated pervasive misconceptions (e.g. Kieran, 1981; Knuth, Stephens, McNeil, & Alibali, 2006; McNeil & Alibali, 2005). A common finding is widespread errors on missing operand questions. For example, when given an equation such as $1 + 5 = \_\_\_ + 2$ and asked to write the number that goes in the blank, only approximately 20% of children in the United States aged 7-11 years solve such problems correctly (McNeil, 2014). Common responses to such problems include the sum of the numbers on the left, 6 in the above example, and the sum of all the numbers present, 8 in the above example, (Knuth, McNeil, & Alibali, 2006; McNeil, 2014). These findings have been explained as misconceptions about the equal sign. Students, particularly young elementary school students, tend to interpret the equal sign as a symbol to "do something", such as perform a known procedure or state the result of an arithmetic operation on the numbers present (Kieran, 1981; McNeil & Alibali, 2005). From this perspective, the equal sign is viewed as a symbol for an operation and not correctly as a symbol for a relation. While the percent of students holding a correct relational understanding of the equal sign increases from elementary school to middle school, many older students continue to interpret the equal sign operationally. In a study examining undergraduate students, participants incorrectly responded with sums, as described above, on 6% of missing addend problems (Chesney, McNeil, Brockmole, & Kelley, 2013). When time-limited trials were presented (the problem appeared for only 2 seconds), participants gave operational responses (i.e. responding with sums) on 36% of trials.

The goal of the present research was to examine understanding of mathematical equivalence in pre-service elementary and middle school teachers. Examining pre-service teachers' interpretation of equivalence is important for two reasons. First, pre-service teachers will influence their future students' understanding of equivalence. Prior research has shown that misconceptions about equivalence and the equal sign can hinder students' learning of algebra (Byrd, McNeil, Chesney, & Matthews, 2015). Therefore teachers' understanding of equivalence might be critically important for their students' success in mathematics. Second, much of the focus of college mathematics courses for pre-service teachers is on aspects of numbers, operations, and procedural knowledge, with content based on textbooks such as *Mathematics for Elementary Teachers* by Beckmann[1] (Beckmann, 2018). Pre-service teachers learn procedures to teach computation of arithmetic results using visual and physical models, such as representing 6×7 as a two-dimensional array (e.g. Beckmann, 2018). Pre-service

[1] Universities that use or have used the Beckmann textbook include Iowa State University, Ohio State University, Ohio University, University of Maryland, and University of Wisconsin.

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teachers also learn various informal procedures for arithmetic computations and how to recognize correct and incorrect procedures that elementary students may use to compute arithmetic results using previously learned math facts. For example, the product of 6×7 can be found by using a known fact such as $6 \times 5 = 30$ (e.g. Beckmann, 2018). As a result, more emphasis may be placed on procedures and non-symbolic representations than on the standard symbolic representations of equations.

Another goal of the present study was to examine two less researched aspects of understanding mathematical equivalence: (1) the ability to recognize errors in equivalence statements, termed in this paper as "run-on statements", and (2) the ability to differentiate mathematical statements involving the equal sign from mathematical expressions without an equal sign. To examine ability to recognize errors in equivalence statements, questions presented "run-on statements". For example, to determine the result of 6×7 using 6×5, a student might know that $6 \times 5 = 30$, then add 6 to 30, and then add another 6 to 36. An incorrect way of expressing this is $6 \times 5 = 30 + 6 = 36 + 6 = 42$. The end result of 42 is correct, but the statement generated is not a correct equation. Such "run-on statements" are common incorrect representations that many students make (e.g. Kieran, 1981; Vincent, Bardini, Pierce, & Pearn, 2015). It may be that because college mathematics courses for pre-service teachers emphasize multiple procedures including standard and informal strategies, pre-service teachers may focus on intermediate steps in a procedure and not notice violations of statements of equality such as those in "run-on statements".

This study also examined pre-service teachers' conception of an equation versus an expression. Prior research has attributed weak understanding of mathematical equivalence to misconceptions about the equal sign. However, it is unclear whether inaccuracies in responding to missing operand questions are due to misconceptions about the equal sign or misconceptions about mathematical statements including equations. What portion of students correctly differentiate mathematical statements involving the equal sign from mathematical expressions without an equal sign? If students do not correctly differentiate equations from expressions, what do they categorize as an equation?

In Experiment 1, pre-service teachers were tested on the following three aspects of equivalence: solving missing addend questions similar to those used in previous studies, recognizing errors in "run-on statements", and identifying equations.

Experiment 1

Method

Participants Forty-two undergraduate students (31 female, 10 male, one other identity) majoring in elementary education or middle-school education at a large Midwestern university participated in the present study. At the time of the study, these students were enrolled in a mathematics course which focused on numbers and arithmetic operations and was designed for prospective elementary and middle school teachers.

Material and Design Participants completed a thirteen-question, paper-and-pencil test at their own pace. The test consisted of two missing addend questions asking students to fill in the blanks with the correct number, two run-on questions asking students to judge the correctness of two run-on statements, and nine equation identification questions asking students to circle equations (see Figure 1).

The run-on questions presented a statement string that a hypothetical student might write to find the product of two integers. Both run-on questions showed the correct product on the right of the statement string. One of the run-on questions (test question #3 in Figure 1) presented no equal expressions, and one of the run-on questions (test question #4) presented two equal expressions and two nonequal expressions. Of the equation identification questions, four questions presented equations, four questions presented expressions, and one question presented a single integer.

For questions 1 and 2 below, fill in the blanks with the correct number.

1. $95 + 5 =$ _____ $+ 2$
2. $8 + 2 =$ _____ $+ 5$

For questions 3 and 4, an elementary student is trying to do some arithmetic without a calculator and using some multiplication facts that he/she knows well. Is what the student wrote correct?

3. To find 6×7, the student uses the product 6×5 and writes the following.
   $6 \times 5 = 30 + 7 + 7 = 42$
   Circle: Correct or Incorrect
4. To find 125×11, the student uses the product 125×10 and writes the following.
   $125 \times 10 = 1250 + 125 = 1375$
   Circle: Correct or Incorrect

For questions 5 - 13, circle those that are equations.

5. $345 + 87$
6. $5x^2 - 4x + 3$
7. $5 + 8 = 13$
8. 345
9. $(3x - 5)(x^2 + 2x) = 4$
10. $5^2 - 4$
11. $(x^3 - 2x)(x + 7)$
12. $5x = 12 + 15$
13. $5 + 4 - 2 + 1 = 8$

Figure 1: Test used in Experiment 1.

Results

Accuracy on the missing addend questions was relatively high (*M* = 94.0%, *SD* = 19.8%) and similar to what has been found in previous studies involving undergraduate students (Chesney, et al., 2013). The percent of participants who responded correctly on both missing addend problems was 90.5. However, overall accuracy on judging the correctness of run-on statements was quite low (*M* = 48.8%, *SD* = 30.2%). Only 16.7% of participants responded correctly on both run-on statements. There was a difference in performance between the two questions. Participants were more accurate on question #3 (*M* = 76.2%, *SD* = 43.1%), which showed a string of three nonequal expressions than on question #4 (*M* = 21.4%, *SD* = 41.5%), in which two of three expressions were equal (paired-sample t-test, $t(41) = 5.99$, $p < .001$).

Participants' performance on equation identification (*M* = 87.0%, *SD* = 19.6%) was above chance (one-sample t-test, $t(41) = 12.2$, $p < .001$). There were no correlations between accuracy on equation identification questions, missing addend questions, or run-on questions (Pearson Correlations, *p*s > .09). To deeper examine participants' conception of equations, participants were categorized based on their response strategy to the equation identification questions. Participants were put into one of the response categories shown in Table 1 if at least eight of their nine responses fit the category description. For example, if participants chose only statements with variables (e.g. $5x^2 - 4x + 3$ and $5x = 12 + 15$), then they were categorized as "Variable".

Table 2 presents the percentages of students responding accurately on the Run-on questions split across the equation identification strategy response category. On the first run-on question (test question #3), participants who used the correct strategy for identifying equations were more likely than participants using incorrect strategies to answer correctly, Fisher's exact test, $p < .05$. Of the participants who used the correct equation identification strategy, 86% (25 of 29 participants) answered the first run-on question correctly, while only 54% of the participants who used incorrect equation identification strategies did so. There were no differences in the percentages of accurate responses on the second run-on question (test question #4), Fisher's exact test, $p = .42$.

Table 1: Equation identification response strategies in Experiment 1.

Response Strategy Category	Response Category Description
Correct	An equation is a statement of equivalence using the equal sign
Variable	An equation is any statement with a variable
Variable & equation	An equation is an expression with a variable or a statement of equivalence using the equal sign
Anything but single number	An equation is any statement other than a single number
Other	Arbitrary or unclear

Table 2: Percentages of students responding accurately on run-on statement questions split by equation identification strategy

			Response Strategy on Equation Recognition Questions					
			Correct	Variable	Variable & equation	Anything but single number	Other	Total
Run-on (#3)	Correct	Count	25	2	2	2	1	32
		% of total	59.5%	4.8%	4.8%	4.8%	2.4%	76.2%
	Incorrect	Count	4	2	0	1	3	10
		% of total	9.5%	4.8%		2.4%	7.1%	23.8%
	Total	Count	29	4	2	3	4	42
		% of total	69.0%	9.5%	4.8%	7.1%	9.5%	100%
Run-on (#4)	Correct	Count	5	2	0	1	1	9
		% of total	11.9%	4.8%		2.4%	2.4%	21.4%
	Incorrect	Count	24	2	2	2	3	33
		% of total	57.1	4.8%	4.8%	4.8%	7.1%	78.6%
	Total	Count	29	4	2	3	4	42
		% of total	69.0%	9.5%	4.8%	7.1%	9.5%	100%

These results show that while accuracy on missing operand questions was relatively high, accuracy on the other measures of equivalence was not. In particular, only 17% of participants identified both run-on questions as incorrect and only 69% could correctly discriminate equations from expressions. These findings suggest that not only do many of the participants misinterpret the equal sign, they may not understand the components of mathematical statements and expressions.

Mathematical statements and expressions can involve different components, including numbers, variables, operations, and relations. In school mathematics, operations are primarily the arithmetic operations of addition, subtraction, multiplication, division, and exponentiation. Statements can be formed by including relations such as equality. The goal of Experiment 2 was to further examine pre-service teachers understanding of such components.

Experiment 2

Method

Participants Twenty-three undergraduate students (16 female, 7 male) majoring in elementary education or middle-school education at a large Midwestern university participated in the present study. At the time of the study, these students were enrolled in a mathematics course which focused on numbers and arithmetic operations and was designed for prospective elementary and middle school teachers.

Material and Design Participants completed a short five-question, paper-and-pencil test at their own pace. The test consisted of four multiple-choice questions and one open-ended question (see Figure 2). For the first four questions, participants were shown an equation and asked to choose a label for different components of the equation. The label choices included: number, letter, variable, equation, expression, and word. Then participants were asked to list all of the mathematical operations involved in the equation. The fifth question showed participants an equation with a variable *x* appearing alone on the right and asked participants to select the meaning of the equal sign from four possible choices. The choices included a description of a procedure using the correct order of operations to solve for x (choice a), a description of an incorrect procedure violating the order of operations to solve for x (choice b), a statement that the variable x is an unknown quantity that we need to determine (choice c), and the correct relational interpretation of the equal sign (choice d).

Results

For each question, the percentages of participants giving a specific response or type of response what calculated. One hundred percent of participants correctly identified 8 as a number and x as a variable. However, only 74% of participants correctly labeled $x^2 - x$ as an expression; all of the other 26% of participants labeled this as an equation.

When asked to list all of the mathematical operations involved, only 39% correctly listed addition, subtraction, and multiplication (including or excluding exponentiation was also considered correct). To examine the nature of incorrect responses, participants were categorized as follows. Participants were categorized as "Incorrect Operations" if they listed any incorrect operations such as division or did not list all of the present operations. Participants were categorized as "Other", if they listed something other than an operation. Such responses included PEMDAS (i.e. an abbreviation for the correct order of performing operations), variable, and distribution, which is an arithmetic property involving two operations, not an operation. Participants were placed in "Correct Operations and Other" if they listed all of the correct operations but also included non-operations. They were placed in "Incorrect Operations and Other" if they omitted any relevant operations or included incorrect operations and also included non-operations. Only 39% of participants correctly listed all the operations. Percentages of participants falling into the categories are shown in the lower portion of the far right column of Table 3.

For questions 1 – 4, consider the following:

$$(x^2 - x)(7x + 8) = 0$$

Enter one of the following possibilities into each blank below to make a correct statement. Choose the one that is the best fit.

number	letter	variable
equation	expression	word

1. 8 is a/an _____________
2. x is a/an _____________
3. $x^2 - x$ is a/an _____________
4. List all of the mathematical operations involved in the above.

5. Consider the following: $10 + 5 \times 5 = x$
   What does the symbol = mean?
   Circle the best response from those below:
   a.) multiply 5 and 5 then add that result to 10
   b.) add 10 and 5 then multiply that result by 5
   c.) x is an unknown quantity that we need to determine
   d.) the quantity on the left of the symbol is the same as the quantity on the right of the symbol

Figure 2: Test used in Experiment 2.

When asked to select a best response for the meaning of the equal sign, approximately 56% responded with the relational interpretation. The remaining 44% chose more procedural responses; 17.4% chose the correct procedure for solving for x and 26.1% chose the statement "x is an unknown quantity that we need to determine". No one selected the incorrect procedural response. Table 3 presents a cross tabulation of responses to question #3 (labeling $x^2 - x$) and question #4 (list all operations) split across participants' meaning of the equal sign category. These results show that participants who responded with a relational interpretation of the equal sign tended to be accurate in labeling $x^2 - x$ as an expression and also accurate in listing operations, but differences in the distributions were not significant (Fisher's exact test, *p*s > .19).

Conclusion

The findings of this study demonstrate that some pre-service elementary and middle school teachers have a weak understanding of mathematical equivalence. The results contribute to research on mathematical equivalence and provide further evidence that many students, specifically some undergraduate pre-service teachers, have misconceptions about the equal sign and hold an operational interpretation of it. Only 56% of participants in Experiment 2 chose the correct relational definition of the equal sign, while the remaining participants chose operational responses of performing the given arithmetic or solving for the given variable. While the mean accuracy on the missing operand questions of Experiment 1 was high (94.0%), it is concerning that nearly 10% of the participants did not consistently answer those questions correctly. More interestingly, the present findings extend the examination of mathematical equivalence beyond missing operand questions and misconceptions about the equal sign to demonstrate (1) a context of failure to notice violations in equality and (2) misconceptions about components of equations.

The results of Experiment 1 show that many participants had very poor ability to detect errors in "run-on statements". Only 16% of participants recognized that both run-on statements were incorrect statements of equality. These run-on statements represent possible steps in determining an arithmetic result that are connected incorrectly with equal signs. Not detecting that these statements violate equality suggests that participants were focused more on the operations present and not on the equal sign. This evidence further supports the notion that participants had an operational view of the equal sign.

Table 3: Response frequencies on questions #3 and #4 in Experiment 2 split by responses to meaning of the equal sign (question #5).

Question	Response		Responses to Meaning of the Equal Sign: a) Correct Procedural	b) Incorrect Procedural	c) x is an unknown	d) Correct Relational	Total
#3.) $x^2 - x$ is	Expression	Count	2	-	4	11	17
		% of total	8.7%		17.4%	47.8%	73.9%
	Equation	Count	2	-	2	2	6
		% of total	8.7%		8.7%	8.7%	26.1%
	Total	Count	4	-	6	13	23
		% of total	17.4%		26.1%	56.5%	100%
#4.) List all operations	Correct	Count	1	-	3	5	9
		% of total	4.3%		13.0%	21.7%	39.1%
	Correct Operation & Other	Count	2	-	-	2	4
		% of total	8.7%		0.0%	8.7%	17.4%
	Incorrect Operation	Count	1	-	2	1	4
		% of total	4.3%		8.7%	4.3%	17.4%
	Incorrect Operation & Other	Count	-	-	1	2	3
		% of total	0.0%		4.3%	8.7%	13.0%
	Other	Count	-	-	-	3	3
		% of total	0.0%		0.0%	13.0%	13.0%
	Total	Count	4	-	6	13	23
		% of total	17.4%		26.1%	56.5%	100.0%

Experiment 1 also demonstrated that many participants have an incorrect interpretation of the meaning of equation. Only 69% accurately identified equations. Of the remaining participants, 14% labeled as an equation any item containing a variable. Seven percent labeled any item other than a single number to be an equation. This suggests that many participants view the word equation to mean an expression with an operation or a variable for which one may want to find a solution. Both cases suggest an operational interpretation of equation instead of recognizing that an equation is a relational statement of equivalence, regardless of whether there are operations or variables present.

Experiment 2 further investigated participants' notion of equation by examining interpretations of components of equations. While 100% of participants correctly labeled number and variable, only 74% correctly identified an expression. The remaining 26% labeled the expression as an equation. The findings also suggest that in addition to misconceptions about equivalence, many participants have misconceptions about mathematical operations. Only 39% of participants correctly listed all operations involved in the given equation; 43% listed non-operations such as distribution. Taken together these findings suggest that many participants interpret the equal sign similar to an operation, yet they do not have a clear understanding of what an operation is.

While part of the motivation for examining pre-service teachers' understanding of equivalence stemmed from the focus on operations and procedures in courses and textbooks, it should be noted that textbooks, and likely other course content, do not ignore symbolic representations of equivalence. In *Mathematics for Elementary Teachers* by Beckmann (Beckmann, 2018), for example, mathematical explanations and correct use of the equal sign are presented through equations in every chapter. There is even a warning, "When you use an equal sign, make sure that the quantities before and after the equal sign really are equal to each other." However, many activities and questions focus on numerical calculations without equations, often involving entirely non-symbolic representations. While multiple forms of representations and calculation strategies are important for pre-service teachers to learn, one of the most important representations in mathematics is the symbolic representation of equivalence. It has been well argued that misconceptions about the mathematical equivalence and the equal sign likely originate from extended, overly narrow experiences with arithmetic through elementary school in which children extensively practice writing results of arithmetic operations to the right of an equal sign, such as $3 + 4 =$ ___ (McNeil, 2014). Given that these misconceptions form from experiences over many years, it may not be surprising that pre-service teachers have them. However, the results of present study suggest that in mathematics courses for pre-service teachers more attention should be given to equivalence and correct use of the equal sign.

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