Title
E.E. REVIEW COURSE - LECTURE V.

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Publication Date
1952-03-31

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UCRL 1769

ELECTRICAL ENGINEERING REVIEW COURSE

Lecture V
March 31, 1952
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(Notes by: A. Du Bois, W. Eaton)

ELECTROSTATICS

Dielectric material in a parallel plate condenser.

When a dielectric (or insulator) is placed between the plates of a parallel plate condenser, the electric field existing between those plates causes dipoles to form. The dielectric is said to be polarized. This polarization $\vec{P}$ consists of many dipoles, each consisting of equal and opposite charges, q, separated by a displacement $\vec{d}$.

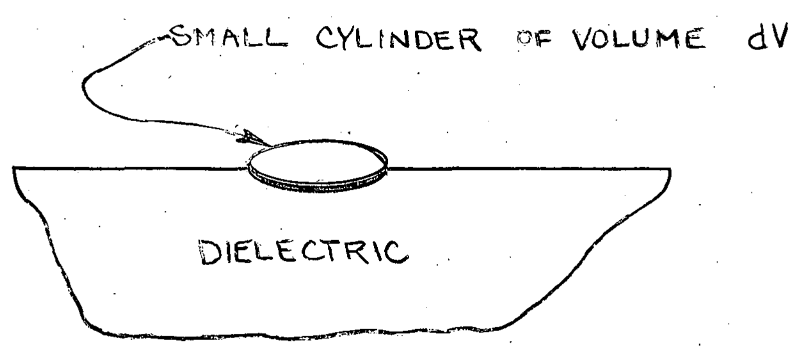
The polarization of a small volume within the dielectric may be defined as 1.) the electrical moment per unit volume; or 2.) the net quantity of electricity flowing through unit area normal to the surface of this volume.

The net charge density ρ' at any point in the dielectric is given by:

$$\text{div } \vec{P} = -\rho'$$

If $\vec{P}$ is uniform, div $\vec{P} = 0$ and the net charge in any small volume remains zero.

At the surface of the dielectric $\vec{P}$ is discontinuous and div $\vec{P} = |\vec{P}|$; therefore we will find a charge q accumulated there. Take a very short cylinder at the surface of the dielectric, as shown in the sketch below. The surface area of the cylinder will be essentially the area, 2dS, of its ends and we may assume that all the lines of flux P, flowing into or out of the cylinder pass through these ends.



Since: $\operatorname{div} \vec{P} = |\vec{P}|$

$$|\vec{P}| \cdot dS = \rho' dv = dq$$

Therefore:

$$\frac{dq}{ds} = |\vec{P}| = \sigma'$$ where σ' = surface charge due to polarization.

DISPLACEMENT VECTOR

We derived, in a previous lecture, the expression $k_o \operatorname{div} \vec{E} = \rho$ for a condenser with free space between the plates. The insertion of a dielectric does not change the equation except that we must consider both the density of the free charges, ρ, and the density of those charges, ρ', which appear automatically when the dielectric is polarized. Therefore we will now write the equation as:

$$k_o \operatorname{div} \vec{E} = \rho + \rho'$$

But: $\operatorname{div} \vec{P} = -\rho'$

Therefore: $k_o \operatorname{div} \vec{E} = \rho - \operatorname{div} \vec{P}$

or: $\operatorname{div} (k_o\vec{E} + \vec{P}) = \rho$

Now we will define a new vector $\vec{D}$ which has been called the "electric displacement vector".

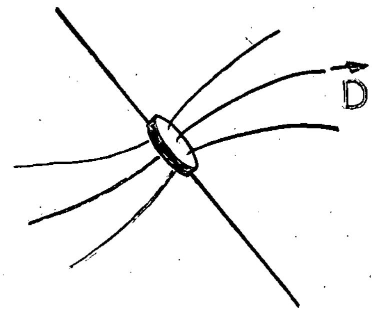
$$\vec{D} = k_o \vec{E} + \vec{P}$$

We see that $\operatorname{div} \vec{D} = \rho$, i.e., the electric displacement vector $\vec{D}$ depends upon the free charges and is unaffected by the charges due to polarization.

Most materials are non-isotropic, and it is easier to distort their crystal lattices in certain directions than in other directions. As a result of this non-isotropy the polarization $\vec{P}$ may not be in the direction of the electric field $\vec{E}$ and consequently D will be parallel to neither. However, in an isotropic material $\vec{P}$ will be collinear with $\vec{E}$, and $\vec{D}$ will be collinear with both.

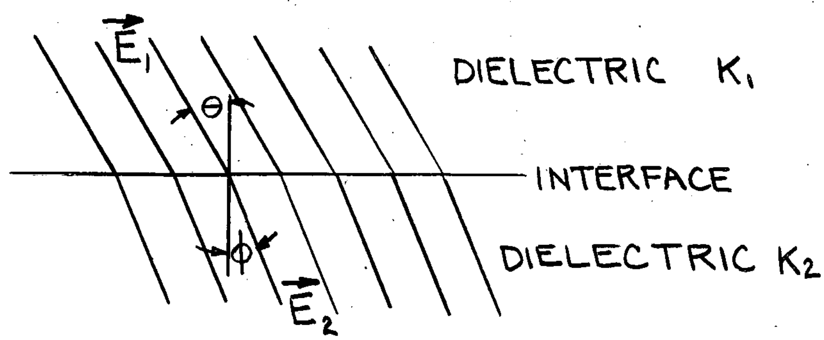
By using the short cylinder devise we can show that the normal component of $\vec{D}$ must be continuous across a boundary where there are no free charges, q, on that boundary. Take a cylinder of volume dV which is very short so that the area of its surface is essentially equal to the area of its ends, and let these ends be parallel to the boundary. From the expression $\operatorname{div} \vec{D} = \rho$ we see that if there are no free charges in the volume dV the normal component of D entering one end must be the same as the normal component of $\vec{D}$ leaving the other end.

(See sketch on following page)



Similarly, from the expression curl $\vec{E} = 0$, we see that the tangential component of $\vec{E}$ must be continuous across an uncharged boundary.

The law of refraction for lines of force can now be shown.



$\vec{D}_1 = K\vec{E}_1$ and $\vec{D}_2 = K\vec{E}_2$

Normal component of $\vec{D}$ is continuous: $K_1\vec{E}_1 \cos\theta = K_2\vec{E}_2 \cos\phi$

Tangential component of $\vec{E}$ is continuous: $\vec{E}_1 \sin\theta = \vec{E}_2 \sin\phi$

$$\therefore \quad \frac{\tan\theta}{\tan\phi} = \frac{K_1}{K_2}$$

ENERGY IN ELECTRIC FIELD

The energy associated with a system of point charges is equal to the total work required to bring them from infinity.

$$W = \sum W_1 + W_2 + W_3 + \ldots\ldots\ldots W_n$$

W = energy of the system

W_n = energy required to bring up the Nth charge, e_n.

$W_1 = 0$

$W_2 = \frac{e_1 e_2}{r_{1-2}}$

$W_3 = \frac{e_1 e_3}{r_{1-3}} + \frac{e_2 e_3}{r_{2-3}}$

$W_4 = \frac{e_1 e_4}{r_{1-4}} + \frac{e_2 e_4}{r_{2-4}} + \frac{e_3 e_4}{r_{3-4}}$

$\therefore W = 1/2 \sum e_i V_i$ (1.)

$V_i =$ Voltage due to aggregate

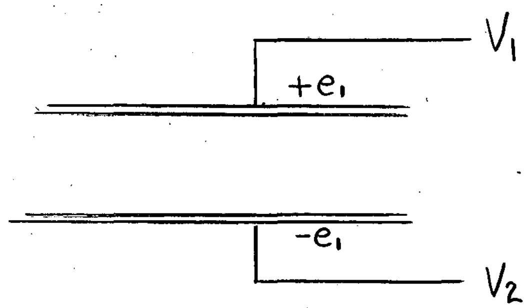
However, the energy is associated with the electric space field rather than with the individual charges.

The energy density in free space is $U = \frac{k_o}{2} E^2$ and the total energy of the system is $W = \int_v U \, dv$ where v = volume (2.)

(This expression is valid even if $\vec{E}$ is a function of time.)

The energy of formation of the point charges is infinite and the value of the integral would be infinite if the total energy density were considered. However, the above expression for U does not include this energy of formation of the point charges and equation (1.) is equivalent to equation (2.)

Example I: Parallel plate condenser with voltage difference $V_1 - V_2$, plates separated by vacuum.



By equation (1.)

$$W = 1/2 \sum e_i V_i$$

$$W = 1/2 \left[e_1 (V_1 - V_2) \right] \qquad \text{Let } \overline{V} = V_1 - V_2$$

$$W = \frac{e_1}{2} \overline{V}$$

But capacity $C = \frac{\overline{V}}{e}$

$$W = 1/2\, C\, \overline{V}^2$$

By equation (2.)

$$W = \int_v U\, dv$$

$$U = \frac{k_o E^2}{2}$$

But: $\vec{E} = \frac{\overline{V}}{d}$

$$U = \frac{k_o \overline{V}^2}{2d^2}$$

Then $W = \int_v \frac{k_o \overline{V}^2}{2d^2} dv$

$v = Ad$

d = Distance between plates.

A = Area of condenser plates.

$$W = \frac{k_o \overline{V}^2 (Ad)}{2d^2}$$

$$W = 1/2 \frac{k_o A}{d} \overline{V}^2$$

But $C = \frac{k_o A}{d}$

$$W = 1/2\, C\, \overline{V}^2$$

Example II: Same condenser with dielectric inserted between plates, dielectric constant = K

$W = 1/2\ C\ \overline{V}^2$ is still valid but value of C is changed.

$$C = \frac{K\ k_o\ A}{d}$$

$$W = 1/2\ \frac{K\ k_o\ A}{d}\ \overline{V}^2$$

$$\overline{V} = E\ d$$

$$W = 1/2\ \ K\ k_o\ E^2\ (Ad)$$

$$U = \frac{W}{\text{Volume}} = \frac{1/2\ \ K\ k_o\ E\ (Ad)}{Ad} = 1/2\ \ K\ k_o\ E^2$$

$\vec{D} = K\ k_o\ \vec{E}$ (if dielectric is isotropic)

$$U = 1/2\ \ \vec{D} \cdot \vec{E}$$

Therefore to obtain large energy storage (large capacity) use low spacing between plates, high voltage, and high dielectric constant.

FORCES ON DIELECTRIC

From the potential energy function, $W = \int_V U\ dv$, forces on the dielectric can be determined. Since the stable state for any system is obtained when the potential energy = minimum, the charges will try to move to that configuration which make W = min. The force acting on a charge is proportional to and in the direction of the gradient of W.

Let us see what happens if dielectric is placed between the plates of a condenser when the charge e_1 on that condenser is a constant.

WITHOUT DIELECTRIC

$$W = \frac{\vec{D}\cdot\vec{E}}{2} Ad$$

$$\sigma = \frac{e_1}{A}$$

$$\vec{E} = \frac{e_1}{k_o A}$$

$$\vec{D} = k_o \vec{E} = \frac{e_1}{A}$$

$$\text{so } W = \frac{e_1^2}{2 k_o} \frac{d}{A}$$

WITH DIELECTRIC

$$W = \frac{\vec{D}\circ\vec{E}}{2} A \circ d$$

$$\vec{E} = \frac{e_1}{K k_o A}$$

$$\vec{D} = \frac{e_1}{A}$$

$$W = \frac{e_1^2}{2 K k_o} \frac{d}{A}$$

It is obvious that W is decreased by the ratio 1/K when the dielectric is inserted.

As a result the force on the dielectric tends to pull it into the condenser. The energy expended on the dielectric is obtained from the electric field $\vec{E}$ which has decreased by the factor 1/K.

The force pulling the condenser plates together $F_p = e_1 \vec{E}$ is also reduced by the factor 1/K but since these plates do not move no energy is expended on them.

If the condenser had been connected across a battery the voltage, rather than the charge, would be constant.

$$W = 1/2\ k_o\ V^2 \frac{d}{A} \quad \text{without dielectric}$$

$$W = 1/2\ K\ k_o\ V^2 \frac{d}{A} \quad \text{with dielectric}$$

The potential energy of the condenser has increased, however, this energy was derived from the battery rather than from an external force on the dielectric.

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