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Posets, Polytopes, and Positroids

by

Anastasia Maria Chavez

A dissertation submitted in partial satisfaction of the

requirements for the degree of

Doctor of Philosophy

in

Mathematics

in the

Graduate Division

of the

University of California, Berkeley

Committee in charge:

Professor Lauren K. Williams, Chair
Professor Federico Ardila
Professor Daniel I. Tataru
Professor Rodolfo Mendoza-Denton

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Abstract

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This dissertation explores questions about posets and polytopes through the lenses of positroids and geometry. The introduction and study of positroids, a special class of matroids, was pioneered by Postnikov in his study of the totally nonnegative Grassmannian and has subsequently been applied to various fields such as cluster algebras, physics, and free probability. Postnikov showed that positroids, the matroids realized by full rank $k \times n$ real matrices whose maximal minors are nonnegative, are in bijection with several combinatorial objects: Grassmann necklaces, decorated permutations, Le-diagrams and plabic graphs.

In the first chapter, following work of Skandera and Reed, we define the unit interval positroid arising from a unit interval order poset via its associated antiadjacency matrix. We give a simple description of the decorated permutation representation of a unit interval positroid, and show it can be recovered from the Dyck path drawn on the associated antiadjacency matrix. We also describe the unit interval positroid cells in the totally nonnegative Grassmannian and their adjacencies. Finally, we provide a new description of the f-vector of posets.

The second chapter concerns the f-vector of a d-dimensional polytope P, which stores the number of faces of each dimension. When P is a simplicial polytope the Dehn–Sommerville relations condense the f-vector into the g-vector, which has length $\lceil \frac{d+1}{2} \rceil$. Thus, to determine the f-vector of P, we only need to know approximately half of its entries. This raises the question: Which $\left(\lceil \frac{d+1}{2} \rceil\right)$-subsets of the f-vector of a general simplicial polytope are sufficient to determine the whole f-vector? We prove that the answer is given by the bases of the Catalan matroid.

In the final chapter, we explore the combinatorial structure of Knuth equivalence graphs G_λ. The vertices of G_λ are the permutations whose insertion tableau is a fixed tableau of shape λ, and the edges are given by local Knuth moves on the permutations. The graph G_λ is the 1-skeleton of a cubical complex C_λ, and one can ask whether it is CAT(0); this is a desirable metric property that allows us to describe the combinatorial structure of G_λ very explicitly. We prove that C_λ is CAT(0) if and only if λ is a hook.

To my parents for their unconditional love and support.

To my husband for his unwavering faith in me.

To my daughters who earned this degree as much as I did.

Contents

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I'll take my Oscar now.

Chapter 1

Introduction

Welcome to my dissertation! Please, find a seat and get comfortable. Perhaps some paper, a pencil, snacks, and a drink are in order if you plan to blow through this in one go. If not, then may I suggest just a drink and snacks. Got some? Great, then here we go.

As advertised, this dissertation is an exploration of the following combinatorial topics: posets, polytopes, and positroids. In particular, we follow a theme of viewing questions about posets and polytopes through a combinatorial lens colored by positroids and geometry.

In Chapters 2 and 3, we address questions regarding posets and polytopes using the theory of positroids. The introduction and study of positroids was established by Postnikov [39] through the study of the totally nonnegative Grassmannian and has subsequently been applied to various fields such as cluster algebras [46], Schubert calculus [29], electrical networks [33, 39], physics [7, 30], free probability [6], and total positivity [39]. A positroid can be represented classically as the matroid of a full rank $k \times n$ real matrix whose maximal minors are nonnegative. Postnikov proved positroids are in bijection with several other combinatorial objects: Grassmann necklaces, decorated permutations, Le-diagrams and plabic graphs [39].

In Chapter 2, we use the various positroid representations and accompanying theory to describe the positroid arising from unit interval order posets. Unit interval orders, earning their name because they can be represented by relations on unit intervals, are characterized by being simultaneously $(3+1)$-free and $(2+2)$-free. It was shown by Skandera and Reed [47] that a unit interval order can be labeled so that the associated antiadjacency matrix has nonnegative minors. By applying a lemma of Postnikov [39], we define the unit interval positroid. We then provide a simple description of the decorated permutation representation and show it can be recovered from a Dyck path drawn on the associated antiadjacency matrix. We also describe the Le-diagram representation which determines the dimension of the unit interval positroid cells in the totally nonnegative Grassmannian. Moreover, we show adjacent unit interval positroid cells are determined easily from the associated decorated permutations. Finally, we provide an interpretation of the f-vector of posets. This is joint work with Felix Gotti [17].

The well-known g-theorem, conjectured by McMullen [35] and subsequently proven by

Billera–Lee, and Stanley [10, 9, 51], provides a complete characterization of the f-vectors of simplicial polytopes. By way of the Dehn–Sommerville relations, the f-vector of a d-simplicial polytope can be reduced to the g-vector with length $\lceil \frac{d+1}{2} \rceil$. This implies that if one wishes to know all of the f-vector, one needs only to know about half of its entries. This begs the following question: Which $\left(\lceil \frac{d+1}{2} \rceil\right)$-subsets of the f-vector of a general simplical polytope are sufficient to determine the whole f-vector? In Chapter 3 we address this question by defining the Dehn–Sommerville matroid, whose bases are precisely the subsets in question. We show the Dehn–Sommerville matroid is essentially isomorphic to the Catalan matroid, proving the sufficient subsets of the f-vector desired are determined by the upstep sets of Dyck paths. This is joint work with Nicole Yamzon [19].

Informally, a reconfigurable system is a collection of states with a set of reversible moves that are used to navigate from one state to another. This system gives rise to a cubical complex, and one might ask whether it exhibits a desirable geometric property called CAT(0). Such CAT(0) cubical complexes were shown by Ardila–Owen–Sullivant [5] to be characterized by partially ordered sets with inconsistent pairs. In Chapter 4 we explore the combinatorial structure of reconfigurable systems generated by Knuth relations acting on permutations of S_n. The 1-skeleton of the resulting cubical complex C_λ is known as a Knuth equivalence graph G_λ, which is indexed by partitions λ of n. Our main result is that C_λ is a CAT(0) cubical complex if and only if λ is a hook. This is joint work with John Guo [18].

Chapter 2

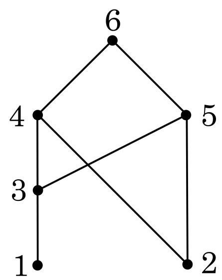
Dyck Paths and Positroids from Unit Interval Orders

2.1 Introduction

A *unit interval order* is a partially ordered set that captures the order relations among a collection of unit intervals on the real line. Unit interval orders were introduced by Luce [32] to axiomatize a class of utilities in the theory of preferences in economics. Since then they have been systematically studied (see [20, 21, 22, 23, 47] and references therein). These posets exhibit many interesting properties. For example, they can be characterized as the posets that are simultaneously $(\mathbf{3}+\mathbf{1})$-free and $(\mathbf{2}+\mathbf{2})$-free. Moreover, it is well known that the number of non-isomorphic unit interval orders on $[n]$ equals $\frac{1}{n+1}\binom{2n}{n}$, the n-th Catalan number (see [20, Section 4] or [48, Exercise 2.180]).

Motivated by the desire to understand the f-vectors of various classes of posets, Skandera and Reed [47] showed that one can label the elements of a unit interval order from 1 to n so that its $n \times n$ antiadjacency matrix is totally nonnegative (i.e., has all its minors nonnegative) and its zero entries form a right-justified Young diagram located strictly above the main diagonal and anchored in the upper-right corner. The zero entries of such a matrix are separated from the one entries by a Dyck path joining the upper-left corner to the lower-right corner. Motivated by this observation, we call such matrices *Dyck matrices*. The Hasse diagram and the antiadjacency (Dyck) matrix of a canonically labeled unit interval order are shown in Figure 2.1.

On the other hand, it then follows from work of Postnikov [39] that $n \times n$ Dyck matrices can be regarded as representing rank n *positroids* on the ground set $[2n]$. Positroids, which are special matroids, were introduced and classified by Postnikov in his study of the totally nonnegative part of the Grassmannian [39]. He showed that positroids are in bijection with various interesting families of combinatorial objects, including decorated permutations and Grassmann necklaces. Positroids and the nonnegative Grassmannian have been the subject of a great deal of recent work, with connections and applications to cluster algebras [46],



$$\longmapsto \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Figure 2.1: A unit interval order with a labeling that respects altitude on [6] and its antiadjacency matrix, which exhibits its *Dyck path*, i.e., the Dyck path separating its one entries from its zero entries.

scattering amplitudes [7], soliton solutions to the KP equation [30], and free probability [6].

In this chapter we characterize the positroids that arise from unit interval orders, which we call *unit interval positroids*. We show that the decorated permutations associated to rank n unit interval positroids are $2n$-cycles in bijection with Dyck paths of length $2n$. The following theorem is a formal statement of our main result.

Theorem 2.4.4. *A decorated permutation π represents a unit interval positroid on $[2n]$ if and only if π is a $2n$-cycle $(1\ j_1\ \ldots\ j_{2n-1})$ satisfying the following two conditions:*

1. *in the sequence $(1, j_1, \ldots, j_{2n-1})$ the elements $1, \ldots, n$ appear in increasing order while the elements $n+1, \ldots, 2n$ appear in decreasing order;*

2. *for every $1 \le k \le 2n-1$, the set $\{1, j_1, \ldots, j_k\}$ contains at least as many elements of the set $\{1, \ldots, n\}$ as elements of the set $\{n+1, \ldots, 2n\}$.*

In particular, there are $\frac{1}{n+1}\binom{2n}{n}$ unit interval positroids on $[2n]$.

The decorated permutation associated to a unit interval positroid on $[2n]$ naturally encodes a Dyck path of length $2n$. Here we provide a recipe to read this decorated permutation directly from the antiadjacency matrix of the unit interval order.

Theorem 2.1.1. *Let P be a unit interval order on $[n]$ whose labeling respects altitude and A the antiadjacency matrix of P. If we number the n vertical steps of the Dyck path of A from bottom to top in increasing order with $\{1, \ldots, n\}$ and the n horizontal steps from left to right in increasing order with $\{n+1, \ldots, 2n\}$, then we obtain the decorated permutation associated to the unit interval positroid induced by P by reading the Dyck path in northwest direction.*

Example 2.1.2. The vertical assignment on the left of Figure 2.2 shows a set I of unit intervals along with unit interval order P on [5] whose labeling respects altitude describing the order relations among the intervals in I (see Theorem 2.2.2). The vertical assignment on the right illustrates the recipe given in Theorem 2.1.1 to read the decorated permutation

$\pi = (1, 2, 10, 3, 9, 4, 8, 7, 5, 6)$ associated to the unit interval positroid induced by P directly from the antiadjacency matrix. Note that the decorated permutation π is a 10-cycle satisfying conditions (1) and (2) of our main theorem. The solid and dashed arrows represent functions that we will introduce later.

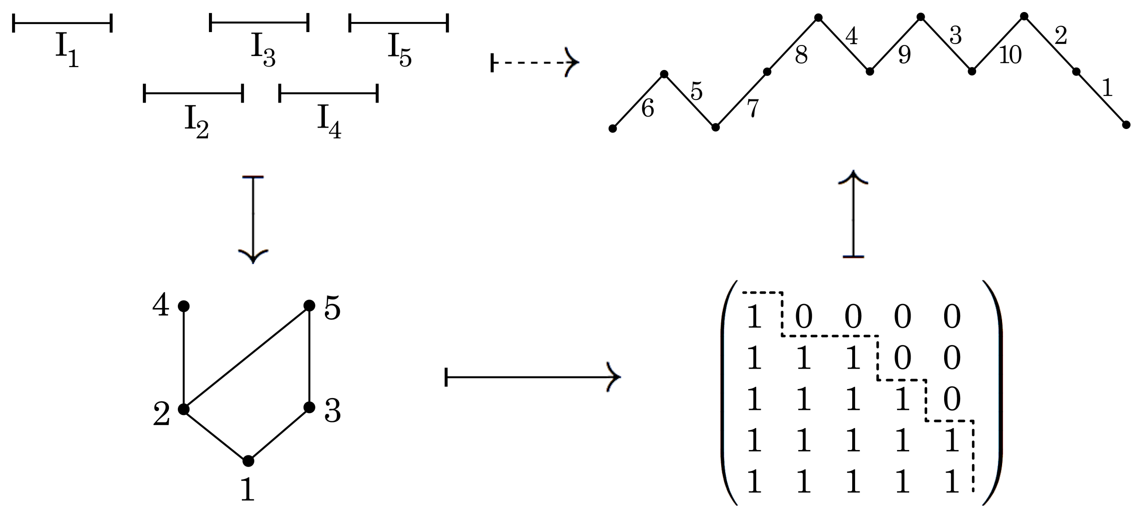


Figure 2.2: Following the solid mappings: unit interval representation I, its unit interval order P, the antiadjacency matrix $\varphi(P)$, and the Dyck path of $\varphi(P)$ showing the decorated permutation $\pi = (1, 2, 10, 3, 9, 4, 8, 7, 5, 6)$.

This chapter is organized as follows. In Section 2.2 we establish the notation and present the fundamental concepts and objects used. In Section 2.3, we introduce altitude respecting labelings and altitude respecting interval representations of unit interval orders. Also, we use altitude respecting labelings to exhibit an explicit bijection from the set of non-isomorphic unit interval orders on $[n]$ to the set of $n \times n$ Dyck matrices. Section 2.4 is dedicated to the description of the unit interval positroids via their decorated permutations, which yields the direct implication of the main theorem. In Section 2.5, we show how to read the decorated permutation associated to a unit interval positroid from either an antiadjacency matrix or an altitude respecting interval representation of the corresponding unit interval order, which allows us to complete the proof of Theorem 2.4.4. In section 2.6, we describe the positroid cells indexed by Le-diagrams associated to the unit interval positroids in the totally nonnegative Grassmannian and prove they have dimension $2n-1$. In section 2.7 we describe when two unit interval positroid cells are adjacent. Finally, in section 2.8, we interpret the f-vectors of naturally labeled posets in terms of special Dyck paths.

2.2 Background and Notation

For ease of notation, when $(P, <_P)$ is a partially ordered set (*poset* for short), we just write P, tacitly assuming that the order relation on P is to be denoted by the symbol $<_P$. We assume all posets used henceforth to be finite.

Definition 2.2.1. A poset P is a *unit interval order* if there exists a bijective map $i \mapsto [q_i, q_i + 1]$ from P to a set $S = \{[q_i, q_i + 1] \mid 1 \le i \le n,\ q_i \in \mathbb{R}\}$ of closed unit intervals of the real line such that for distinct $i, j \in P$, $i <_P j$ if and only if $q_i + 1 < q_j$; that is, $[q_i, q_i + 1]$ is completely to the left of $[q_j, q_j + 1]$. We then say that S is an *interval representation* of P.

For each $n \in \mathbb{N}$, we denote by $\mathcal{U}_n$ the set of all non-isomorphic unit interval orders of cardinality n. For nonnegative integers n and m, let $\mathbf{n} + \mathbf{m}$ denote the poset which is the disjoint sum of an n-element chain and an m-element chain. Let P and Q be two posets. We say that Q is an *induced* subposet of P if there exists an injective map $f : Q \to P$ such that for all $r, s \in Q$ one has $r <_Q s$ if and only if $f(r) <_P f(s)$. By contrast, P is a *Q-free* poset if P does not contain any induced subposet isomorphic to Q. The following theorem provides a useful characterization of the elements of $\mathcal{U}_n$.

Theorem 2.2.2. *[44, Theorem 2.1] A poset is a unit interval order if and only if it is simultaneously* $(\mathbf{3} + \mathbf{1})$*-free and* $(\mathbf{2} + \mathbf{2})$*-free.*

If the poset P has cardinality n, then a bijective function $\ell : P \to [n]$ is called an *n-labeling* of P. We call P an *n-labeled* poset once it has been identified with $[n]$ via ℓ. The n-labeled poset P is *naturally labeled* if $i <_P j$ implies that $i < j$ as integers for all $i, j \in P$.

Example 2.2.3. The figure below depicts the 6-labeled unit interval order introduced in Figure 2.1 with a corresponding interval representation.

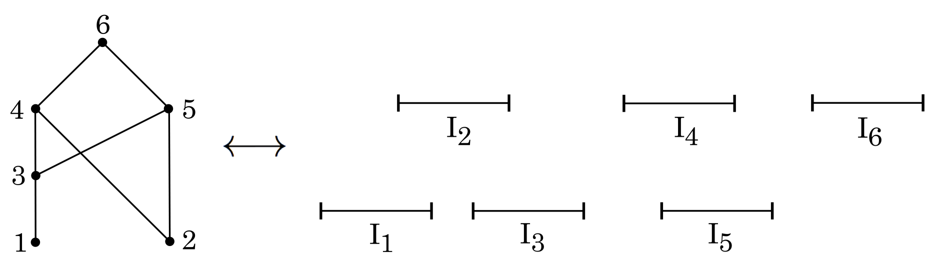


Figure 2.3: A 6-labeled unit interval order and one of its interval representations.

Another useful way of representing an n-labeled unit interval order is through its *antiadjacency matrix.*

Definition 2.2.4. If P is an n-labeled poset, then the *antiadjacency matrix* of P is the $n \times n$ binary matrix $A = (a_{i,j})$ with $a_{i,j} = 0$ if $i <_P j$ and $a_{i,j} = 0$ otherwise.

Recall that a binary square matrix is said to be a *Dyck matrix* if its zero entries form a right-justified Young diagram strictly above the main diagonal and anchored in the upper-right corner. All minors of a Dyck matrix are nonnegative (see, for instance, [2]). We denote the set of all $n \times n$ Dyck matrices as $\mathcal{D}_n$. As presented in [47], every unit interval order P can be labeled to respect the altitude of P so that its antiadjacency matrix is a Dyck matrix (details provided in Section 2.3). This yields a natural map $\varphi\colon \mathcal{U}_n \to \mathcal{D}_n$ that is a bijection (see Theorem 2.3.4). In particular, $|\mathcal{D}_n|$ is the n-th Catalan number, which can also be deduced from the one-to-one correspondence between Dyck matrices and their Dyck paths.

Let $\mathrm{Mat}_{d,n}^{\geq 0}$ denote the set of all full rank $d \times n$ real matrices with nonnegative maximal minors. Given a totally nonnegative real $n \times n$ matrix A, there is a natural assignment $A \mapsto \phi(A)$, where $\phi(A) \in \mathrm{Mat}_{n,2n}^{\geq 0}$.

Lemma 2.2.5. *[39, Lemma 3.9][1] For an $n \times n$ real matrix $A = (a_{i,j})$, consider the $n \times 2n$ matrix $B = \phi(A)$, where*

$$\begin{pmatrix} a_{1,1} & \dots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{n-1,1} & \dots & a_{n-1,n} \\ a_{n,1} & \dots & a_{n,n} \end{pmatrix} \xmapsto{\phi} \begin{pmatrix} 1 & \dots & 0 & 0 & (-1)^{n-1}a_{n,1} & \dots & (-1)^{n-1}a_{n,n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 1 & 0 & -a_{2,1} & \dots & -a_{2,n} \\ 0 & \dots & 0 & 1 & a_{1,1} & \dots & a_{1,n} \end{pmatrix}.$$

Under this correspondence, $\Delta_{I,J}(A) = \Delta_{(n+1-[n]\setminus I)\cup(n+J)}(B)$ for all $I, J \subseteq [n]$ satisfying $|I| = |J|$ (here $\Delta_{I,J}(A)$ is the minor of A determined by the rows I and columns J, and $\Delta_K(B)$ is the maximal minor of B determined by columns K).

Using Lemma 2.2.5 and the aforementioned map $\varphi\colon \mathcal{U}_n \to \mathcal{D}_n$, we can assign via $\phi \circ \varphi$ a matrix of $\mathrm{Mat}_{n,2n}^{\geq 0}$ to each unit interval order of cardinality n. In turn, every real matrix of $\mathrm{Mat}_{n,2n}^{\geq 0}$ gives rise to a positroid, a special representable matroid which has a very rich combinatorial structure.

Matroids are combinatorial objects that abstract the notion of independence. We provide the definition in terms of bases; there are several other equivalent axiomatic definitions available. For a more complete study of matroids see Oxley [37].

Definition 2.2.6. A *matroid* $\mathcal{M}$ is a pair $(E, \mathcal{B})$ consisting of a finite set E and a nonempty collection of subsets $\mathcal{B} = \mathcal{B}(\mathcal{M})$ of E, called the *bases* of $\mathcal{M}$, that satisfy the following properties:

(B1) $\mathcal{B} \neq \emptyset$

(B2) (Basis exchange axiom) If $B_1, B_2 \in \mathcal{B}$ and $b_1 \in B_1 - B_2$, then there exists an element $b_2 \in B_2 - B_1$ such that $B_1 - \{b_1\} \cup \{b_2\} \in \mathcal{B}$.

Any two bases of M have the same size [37], which we denote by $r(M)$ and call the *rank* of M.

[1] There is a typo in the entries of the matrix B in [39, Lemma 3.9].

Definition 2.2.7. For $d, n \in \mathbb{N}$ such that $d \leq n$, let $A \in \mathrm{Mat}^{\geq 0}_{d,n}$ whose columns are denoted by $A_1, \ldots, A_n$. The subsets B of $[n]$ such that $\{A_b \mid b \in B\}$ is a basis for $\mathbb{R}^d$ are the bases of a matroid $M(A)$. Such a matroid is called a *positroid*.

Each unit interval order P (labeled so that its antiadjacency matrix is a Dyck matrix) induces a positroid via Lemma 2.2.5, namely, the positroid represented by the matrix $\phi(\varphi(P))$.

Definition 2.2.8. A positroid on $[2n]$ induced by a unit interval order is called *unit interval positroid*.

We denote by $\mathcal{P}_n$ the set of all unit interval positroids on the ground set $[2n]$. The function $\rho \circ \phi \circ \varphi \colon \mathcal{U}_n \to \mathcal{P}_n$ that sends a unit interval order to its unit interval positroid, where $\rho(B)$ is the positroid represented by $B \in \mathrm{Mat}^{\geq 0}_{n,2n}$, plays a fundamental role in this chapter. Indeed, we will end up proving that it is a bijection (see Theorem 2.5.4).

Several families of combinatorial objects, in bijection with positroids, were introduced in [39] to study the totally nonnegative Grassmannian: decorated permutations, Grassmann necklaces, Le-diagrams, and plabic graphs. We use *decorated permutations*, obtained from *Grassmann necklaces*, to provide a compact and elegant description of unit interval positroids.

In the next definition subindices are considered modulo n.

Definition 2.2.9. Let $d, n \in \mathbb{N}$ such that $d \leq n$. An n-tuple $(I_1, \ldots, I_n)$ of d-subsets of $[n]$ is called a *Grassmann necklace* of type (d, n) if for every $i \in [n]$ the following conditions hold:

- $i \in I_i$ implies $I_{i+1} = (I_i \setminus \{i\}) \cup \{j\}$ for some $j \in [n]$;
- $i \notin I_i$ implies $I_{i+1} = I_i$.

For $i \in [n]$, the total order $<_i$ on $[n]$ defined by $i <_i \cdots <_i n <_i 1 <_i \cdots <_i i-1$ is called the *shifted linear i-order*. For a matroid $M = ([n], \mathcal{B})$ of rank d, one can define the sequence $\mathcal{I}(M) = (I_1, \ldots, I_n)$, where I_i is the lexicographically minimal ordered basis of M with respect to the shifted linear i-order. It was proved in [39, Section 16] that the sequence $\mathcal{I}(M)$ is a Grassmann necklace of type (d, n). We call $\mathcal{I}(M)$ the Grassmann necklace *associated* to M. When M is a positroid we can recover M from its Grassmann necklace (see, e.g., [36] and [39]).

For $i \in [n]$, the *Gale i-order* on $\binom{[n]}{d}$ with respect to $<_i$ is the partial order $\prec_i$ defined in the following way. If $S = \{s_1 <_i \cdots <_i s_d\} \subseteq [n]$ and $T = \{t_1 <_i \cdots <_i t_d\} \subseteq [n]$, then $S \prec_i T$ if and only if $s_j <_i t_j$ for each $j \in [d]$.

Theorem 2.2.10. *[36, Theorem 6] For $d, n \in \mathbb{N}$ such that $d \leq n$, let $\mathcal{I} = (I_1, \ldots, I_n)$ be a Grassmann necklace of type (d, n). Then*

$$\mathcal{B}(\mathcal{I}) = \left\{ B \in \binom{[n]}{d} \;\middle|\; I_j \prec_j B \text{ for every } j \in [n] \right\}$$

is the collection of bases of a positroid $M(\mathcal{I}) = ([n], \mathcal{B}(\mathcal{I}))$, where $\prec_i$ is the Gale i-order on $\binom{[n]}{d}$. Moreover, $M(\mathcal{I}(M)) = M$ for all positroids M.

Therefore a natural bijection exists between positroids on $[n]$ of rank d and Grassmann necklaces of type (d, n). However, *decorated permutations*, also in one-to-one correspondence with positroids, will provide a more succinct representation.

Definition 2.2.11. A *decorated permutation* of $[n]$ is an element $\pi \in S_n$ whose fixed points j are marked either "clockwise"(denoted by $\pi(j) = \underline{j}$) or "counterclockwise" (denoted by $\pi(j) = \overline{j}$).

A *weak i-excedance* of a decorated permutation $\pi \in S_n$ is an index $j \in [n]$ satisfying $j <_i \pi(j)$ or $\pi(j) = \overline{j}$. It is easy to see that the number of weak i-excedances does not depend on i, so we just call it the number of *weak excedances*.

To every Grassmann necklace $\mathcal{I} = (I_1, \ldots, I_n)$ one can associate a decorated permutation $\pi_{\mathcal{I}}$ as follows:

- if $I_{i+1} = (I_i \setminus \{i\}) \cup \{j\}$, then $\pi_{\mathcal{I}}(j) = i$;
- if $I_{i+1} = I_i$ and $i \notin I_i$, then $\pi_{\mathcal{I}}(i) = \underline{i}$;
- if $I_{i+1} = I_i$ and $i \in I_i$, then $\pi_{\mathcal{I}}(i) = \overline{i}$.

The assignment $\mathcal{I} \mapsto \pi_{\mathcal{I}}$ defines a one-to-one correspondence between the set of Grassmann necklaces of type (d, n) and the set of decorated permutations of $[n]$ having exactly d weak excedances.

Proposition 2.2.12. *[6, Proposition 4.6] The map $\mathcal{I} \mapsto \pi_{\mathcal{I}}$ is a bijection between the set of Grassmann necklaces of type (d, n) and the set of decorated permutations of $[n]$ having exactly d weak excedances.*

Definition 2.2.13. If P is a positroid and $\mathcal{I}$ is the Grassmann necklace associated to P, then we call $\pi_{\mathcal{I}}$ the decorated permutation *associated* to P.

2.3 Labelings on Unit Interval Orders that Respect Altitude

In this section we introduce the concept of a poset whose labeling *respects altitude*, and we use it to exhibit an explicit bijection from the set $\mathcal{U}_n$ of non-isomorphic unit interval orders of cardinality n to the set $\mathcal{D}_n$ of $n \times n$ Dyck matrices.

Given a poset P and $i \in P$, we denote the *order ideal* and the *dual order ideal* of i by Λ_i and V_i, respectively. The *altitude* of P is the map $\alpha\colon P \to \mathbb{Z}$ defined by $i \mapsto |\Lambda_i| - |\mathrm{V}_i|$. An n-labeled poset P *respects altitude*, or is altitude respecting, if for all $i, j \in P$, then

$\alpha(i) < \alpha(j)$ implies $i < j$ (as integers). Notice that every poset can be labeled by the set $[n]$ such that, as an n-labeled poset, it respects altitude.

Each altitude respecting n-labeled poset is, in particular, naturally labeled. The next proposition characterizes altitude respecting n-labeled unit interval orders in terms of their antiadjacency matrices.

Proposition 2.3.1. *[47, Proposition 5] An n-labeled unit interval order is altitude respecting if and only if its antiadjacency matrix is a Dyck matrix.*

The above proposition indicates that the antiadjacency matrices of altitude respecting $[n]$-labeled unit interval orders are quite special. In addition, altitude respecting $[n]$-labeled unit interval orders have very convenient interval representations.

Proposition 2.3.2. *Let P be an n-labeled unit interval order. Then the labeling of P respects altitude if and only if there exists an interval representation $\{[q_i, q_i + 1] \mid 1 \le i \le n\}$ of P such that $q_1 < \cdots < q_n$.*

Proof. Let $\alpha \colon P \to \mathbb{Z}$ be the altitude map of P. For the forward implication, suppose that the n-labeling of P is canonical. The existence of an interval representation of P is guaranteed by Theorem 2.2.2. Among all interval representations of P, assume that $\{[q_i, q_i+1] \mid 1 \le i \le n\}$ gives the maximum $m \in [n]$ such that $q_1 < \cdots < q_m$. Suppose, by way of contradiction, that $m < n$. The maximality of m implies that $q_m > q_{m+1}$. This, along with the fact that $\alpha(m) \le \alpha(m+1)$, ensures that $q_m \in (q_{m+1}, q_{m+1} + 1)$. Similarly, $q_i + 1 \notin (q_{m+1}, q_m)$ for any $i \in [n]$; otherwise

$$\alpha(m+1) = |\Lambda_{m+1}| - |\mathrm{V}_{m+1}| < |\Lambda_m| - |\mathrm{V}_{m+1}| \le |\Lambda_m| - |\mathrm{V}_m| = \alpha(m)$$

would contradict that the n-labeling of P respects altitude. An analogous argument guarantees that $q_i \notin (q_{m+1} + 1, q_m + 1)$ for any $i \in [n]$.

Now take k to be the smallest integer in $[m]$ such that $q_j > q_{m+1}$ for all j with $k \le j \le m$, and take $\sigma = (k, k+1, \ldots, m, m+1) \in S_n$. We will show that $S = \{[p_i, p_i + 1] \mid 1 \le i \le n\}$, where $p_i = q_{\sigma(i)}$, is an interval representation of P. Take $i, j \in P$ such that $i <_P j$. Since i and j are comparable in P, at least one of them must be fixed by σ; say $\sigma(i) = i$. If $\sigma(j) = j$, then $p_i + 1 = q_i + 1 < q_j = p_j$. Also, if $\sigma(j) \ne j$, then $q_i + 1 < q_j \in (q_{m+1}, q_m)$. It follows from $q_i + 1 < q_m$ that $p_i + 1 = q_i + 1 < q_{m+1} < q_{\sigma(j)} = p_j$. The case of $\sigma(j) = j$ can be argued similarly. Thus, S is an interval representation of P. As $q_1 < \cdots < q_m$, the definition of k implies that $p_1 < \cdots < p_{m+1}$, which contradicts the maximality of m. Hence $m = n$, and the direct implication follows.

Conversely, note that if $\{[q_i, q_i + 1] \mid 1 \le i \le n\}$ is an interval representation of P satisfying that $q_1 < \cdots < q_n$, then for every $m \in [n-1]$, we have $|\Lambda_m| \le |\Lambda_{m+1}|$ and $|\mathrm{V}_m| \ge |\mathrm{V}_{m+1}|$, so

$$\alpha(m) = |\Lambda_m| - |\mathrm{V}_m| \le |\Lambda_{m+1}| - |\mathrm{V}_{m+1}| = \alpha(m+1),$$

which means that the labeling of P respects altitude. $\square$

If P is an altitude respecting n-labeled unit interval order, and $I = \{[q_i, q_i{+}1] \mid 1 \leq i \leq n\}$ is an interval representation of P satisfying $q_1 < \cdots < q_n$, then we say that I is a *altitude respecting* interval representation of P.

Note that the image (as a multiset) of the altitude map does not depend on the labels but only on the isomorphism class of a poset. On the other hand, the altitude map α_P of an altitude respecting n-labeled unit interval order P satisfies $\alpha_P(1) \leq \cdots \leq \alpha_P(n)$. Thus, if Q is an altitude respecting n-labeled unit interval order isomorphic to P, then

$$(\alpha_P(1), \ldots, \alpha_P(n)) = (\alpha_Q(1), \ldots, \alpha_Q(n)), \tag{2.1}$$

where α_Q is the altitude map of Q. Let A_P and A_Q be the antiadjacency matrices of P and Q, respectively. As $\alpha_P(1) = \alpha_Q(1)$, the first rows of A_P and A_Q are equal. Since the number of zeros in the i-th column (resp., i-th row) of A_P is precisely $|\mathrm{V}_i(P) - 1|$ (resp., $|\Lambda_i(P)| - 1$), and similar statement holds for Q, the next lemma follows immediately by using (2.1) and induction on the row index of A_P and A_Q.

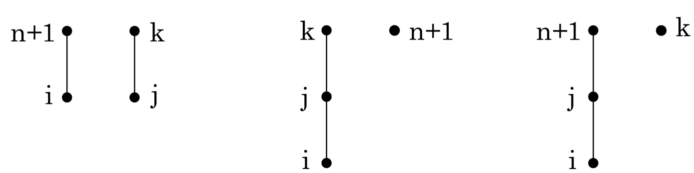
Lemma 2.3.3. *If two altitude respecting labeled unit interval orders are isomorphic, then they have the same antiadjacency matrix.*

Now we can define a map $\varphi\colon \mathcal{U}_n \to \mathcal{D}_n$, by assigning to each unit interval order its antiadjacency matrix with respect to any of its altitude respecting labelings. By Lemma 2.3.3, this map is well defined.

Theorem 2.3.4. *[47] For each natural n, the map $\varphi\colon \mathcal{U}_n \to \mathcal{D}_n$ is a bijection.*

Proof. Since $|\mathcal{U}_n| = |\mathcal{D}_n| = \frac{1}{n+1}\binom{2n}{n}$, it suffices to argue that φ is surjective. We proceed by induction on n. The case $n = 1$ is immediate as $|\mathcal{U}_1| = |\mathcal{D}_1| = 1$. Suppose that surjectivity holds for every $k \leq n$ and, to check that $\varphi\colon \mathcal{U}_{n+1} \to \mathcal{D}_{n+1}$ is surjective, take $D = (d_{i,j}) \in \mathcal{D}_{n+1}$. Let D' be the submatrix of D consisting of the first n columns and the first n rows. As D' is an $n \times n$ Dyck matrix, there is an altitude respecting n-labeled unit interval order P' whose antiadjacency matrix is D'. Define P to be the $(n+1)$-labeled poset obtained by adding an element labeled by $n+1$ to P' with exactly the following order relations: $i <_P n+1$ if and only if either $i = n+1$ or $d_{i,n+1} = 0$. Note that $n+1$ is a maximal element in P and that the antiadjacency matrix of P is precisely D.

We are done once we check that P is unit interval order with an altitude respecting labeling. Since $\alpha_P(1) \leq \cdots \leq \alpha_P(n+1)$, the labeling of P respects altitude. We now show P is a unit interval order. Because P' happens to be a unit interval order, it suffices to check that for any $i, j, k \in [n]$ none of the posets



is an induced subposet of P. The first and the second subposets in the above figure cannot be induced because $j <_P n+1$ for every non-maximal element j of P'. Let Q denote the third subposet shown above. If $k <_P n+1$, then Q cannot be induced. Suppose then that k is not comparable with $n+1$ in P. In this case, k is maximal in P. As j is not maximal in P and the labeling of P is canonical, $i < j < k$ as integers. Since $i <_P j$, one has that $d_{i,j} = 0$ and so $d_{i,k} = 0$. Thus, $i <_P k$, which implies that Q is not an induced subposet of P. Hence P is an altitude respecting $(n+1)$-labeled unit interval order, which concludes the proof. □

2.4 Description of Unit Interval Positroids

We will now describe the decorated permutation associated to a unit interval positroid. Throughout this section A is an $n \times n$ Dyck matrix and $B = (b_{i,j}) = \phi(A)$ is as in Lemma 2.2.5. We will consider the indices of the columns of B modulo $2n$. Furthermore, let P be the unit interval positroid represented by B, and let $\mathcal{I}_P$ and π^{-1} be the Grassmann necklace and the decorated permutation, respectively, associated to P.

Lemma 2.4.1. *For $1 < i \leq 2n$, the i-th entry of $\mathcal{I}_P$ does not contain $i-1$.*

Proof. It is not hard to verify that every matrix resulting from removing one column from B still has rank n. As the matrix obtained by removing the $(i-1)$-th column from B has rank n, it contains n linearly independent columns. Therefore the lemma follows directly from the $<_i$-minimality of $I_i \in \mathcal{I}_P$. □

For the remainder of this section let B_j denote the j-th column of B. As a direct consequence of Lemma 2.4.1, we have that π (and therefore π^{-1}) does not fix any point. As a result, the next lemma immediately follows from the way π^{-1} is produced from the Grassmann necklace $\mathcal{I}_P$ (see the end of Section 2.2).

Lemma 2.4.2. *For $i \in \{1, \dots, 2n\}$, $\pi(i)$ equals the minimum $j \in [2n]$ with respect to the i-order such that $B_i \in \operatorname{span}(B_{i+1}, \dots, B_j)$.*

The *set of principal indices* of B is the subset of $\{n+1, \dots, 2n\}$ defined by

$$J = \{j \in \{n+1, \dots, 2n\} \mid B_j \neq B_{j-1}\}.$$

We associate to B the *weight* map $\omega\colon \{n+1, \dots, 2n\} \to [n]$ defined by $\omega(j) = \max\{i \mid b_{i,j} \neq 0\}$; more explicitly, we obtain that $\omega(j) = \#$ of nonzero entries in column j of B. Since the last row of the antiadjacency matrix A has all its entries equal to 1, the map ω is well defined. We now give an explicit expression of π and then provide a description of the decorated permutation of a unit interval positriod.

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \qquad \phi(A) = \left(\begin{array}{ccccc|ccccc} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$J = \{6, 7, 9, 10\} \qquad \begin{aligned} \omega(6) &= 5 \\ \omega(7) &= \omega(3) = 4 \\ \omega(9) &= 3 \\ \omega(10) &= 2. \end{aligned}$$

i	1	2	3	4	5	6	7	8	9	10
$\pi(i)$	6	1	10	9	7	5	8	4	3	2
Case	1	2	3	3	3	5	4	5	5	5

Figure 2.4: A Dyck matrix A with its associated unit interval positroid $\phi(A)$. The set J and values $\omega(i)$ are used to determine $\pi(i)$, by the indicated case, for all i.

Proposition 2.4.3. *For $i \in \{1, \ldots, 2n\}$,*

$$\pi(i) = \begin{cases} n+1 & \text{if } i = 1 \\ i-1 & \text{if } 1 < i \leq n \text{ and } \omega(j) \neq i-1 \text{ for all } j \in J \\ j & \text{if } 1 < i \leq n \text{ and } \omega(j) = i-1 \text{ for } j \in J \\ i+1 & \text{if } n < i < 2n \text{ and } i+1 \notin J \\ \omega(i) & \text{if } n < i \text{ and either } i = 2n \text{ or } i+1 \in J \end{cases}$$

is the inverse of the decorated permutation π^{-1} of a unit interval positroid.

Proof. We prove each case in the order presented above. Refer to Figure 2.4 for an example of how to calculate J, $\omega(i)$, and what case is used to determine $\pi(i)$ for every i.

Case 1: This follows immediately from Lemma 2.4.2.

Case 2: Assume $1 < i \leq n$. Suppose that $\omega(j) \neq i-1$ for every $j \in J$. Since $I_i \in \mathcal{I}_P$ is Gale-least, then $i, \ldots, n \in I_i$. Note $i-1 \notin I_i$ by Lemma 2.4.1. Since no $j \in J$ has weight $i-1$, the $(i-1)$-th and i-th rows of the maximal submatrix of B determined by the column index set $\{n+1, \ldots, 2n\}$ are equal. Consequently, $\pi(i) = i-1$; otherwise the associated maximal submatrix of B determined by I_i would have the i-th and $(i+1)$-th rows identical.

Case 3: Suppose that $1 < i \leq n$ such that $\omega(j) = i-1$ for some $j \in J$ (note that j is unique). As before, $i, \ldots, n+1 \in I_i$ (because $i > 1$). Each column B_k, for $n < k \leq 2n$ such that $\omega(k) = i-1$, is a linear combination of the columns $B_i, \ldots, B_{n+1}$. Therefore such indices k are not in I_i. By Lemma 2.4.1, it follows that $i-1 \notin I_i$. Thus, $\pi(i) = j$, where $j \in [2n]$ satisfies that $\omega(j) = i-1$; otherwise the $(i-1)$-th row of the submatrix of B determined by $I_{(i+1)}$ would be full of zeros. Since $I_{(i+1)}$ is Gale-least, we have that $j \in J$.

Case 4: Let $i \in \{n+1, \ldots, 2n-1\}$. If $i+1 \notin J$, i.e., $B_i = B_{i+1}$, then $\{B_i, B_{i+1}\}$ is linearly dependent. It follows from Lemma 2.4.2 that $\pi(i) = i+1$.

Case 5: For $n < i$, assume either $i = 2n$ or $i+1 \in J$. Suppose first that $i < 2n$. Then B_{i+1} results from replacing m ($m > 0$) of the last nonzero entries of B_i by zeros. Since $i+1 \in J$, then $i, i+1 \in I_i$. Notice the columns $B_i, B_{i+1}, B_{\omega(i+1)+1}, \ldots, B_{\omega(i)}$ are linearly dependent, implying not all indices $\omega(i+1)+1, \ldots, \omega(i)$ can be in I_i. On the other hand, at most one index in $\omega(i+1)+1, \ldots, \omega(i)$ is missing from I_i; this is because the submatrix of B determined by the row-index set $\{\omega(i+1)+1, \ldots, \omega(i)\}$ and the column-index set $\{n+1, \ldots, 2n\}$ has rank 1. By the i-order of I_i, $\omega(i) \notin I_i$. Thus, $\pi(i) = \omega(i)$; this is the only way to avoid having all zero entries in the $\omega(i)$-th row of the maximal submatrix whose columns are indicated by I_i. A similar argument holds for $i = 2n$, provided that we extend the domain of ω to $[2n+1]$ and set $\omega(2n+1) = 0$. $\square$

In the following section we prove Theorem 2.5.1 which describes the decorated permutation of a unit interval positroid by showing it can be read directly from the associated antiadjacency matrix. Once this is established, the following characterization of decorated permutations of unit interval positroids follows immediately.

Theorem 2.4.4. *The decorated permutation π^{-1} of a unit interval positroid is a $2n$-cycle $(1\ j_1\ \ldots\ j_{2n-1})$ satisfying the conditions:*

1. *the elements $1, \ldots, n$ appear in increasing order while the elements $n+1, \ldots, 2n$ appear in decreasing order;*

2. *for every $1 \leq k \leq 2n-1$, the set $\{1, j_1, \ldots, j_k\}$ contains at least as many elements of the set $\{1, \ldots, n\}$ as elements of the set $\{n+1, \ldots, 2n\}$.*

2.5 A Direct Way to Read The Unit Interval Positroid

Throughout this section, let P be an altitude preserving n-labeled unit interval order with antiadjacency matrix A. Also, let $I = \{[q_i, q_i+1] \mid 1 \leq i \leq n\}$ be an altitude preserving interval representation of P (i.e., $q_1 < \cdots < q_n$); Proposition 2.3.2 ensures the existence of such an interval representation. In this section we describe a way to obtain the decorated permutation associated to the unit interval positroid induced by P directly from either A

or I. Such a description will reveal that the function $\rho \circ \phi \circ \varphi \colon \mathcal{U}_n \to \mathcal{P}_n$ introduced in Section 2.2 is a bijection (Theorem 2.5.4).

The north and east borders of the Young diagram formed by the nonzero entries of A give a path of length $2n$ that we call the *Dyck path* of A. Let $B = (I_n|A') = \phi(A)$, where ϕ is the map introduced in Lemma 2.2.5. Let us call the *inverted path* of A the path consisting of the south and east borders of the Young diagram formed by the nonzero entries of A'. Example 2.5.2 sheds light upon the statement of the next theorem, which describes a way to find the decorated permutation associated to the unit interval positroid induced by P directly from A.

Theorem 2.5.1. *Label the n vertical steps of the Dyck path of A from bottom to top in increasing order with $\{1, \dots, n\}$ and the n horizontal steps from left to right in increasing order with $\{n+1, \dots, 2n\}$. Then the decorated permutation associated to the unit interval positroid induced by P is obtained by reading the Dyck path in the northwest direction.*

Proof. Let π^{-1} be the decorated permutation associated to the unit interval positroid induced by P. Label the n vertical steps of the inverted path of P from top to bottom in increasing order using the label set $[n]$, and we label the n horizontal steps from left to right in increasing order using the label set $\{n+1, \dots, 2n\}$ (see Example 2.5.2). Proving the theorem amounts to showing that we can obtain π (the inverse of the decorated permutation) by reading the inverted path in the northeast direction. Let $(s_1, s_2, \dots, s_{2n})$ be the finite sequence obtained by reading the inverted path in northeast direction. Since the first step of the inverted path is horizontal and the last step of the inverted path is vertical, $s_1 = n+1$ and $s_{2n} = 1$. Thus, it suffices to check that $\pi(s_k) = s_{k+1}$ for $k = 1, \dots, 2n-1$.

Suppose that the k-th step of the inverted path is horizontal, and so located right below the last nonzero entry of the s_k-th column of B. If the $(k+1)$-th step is also horizontal, then $s_k + 1 \notin J$, so Case 4 of Proposition 2.4.3 gives $\pi(s_k) = s_k + 1 = s_{k+1}$. On the other hand, if the $(k+1)$-th step is vertical, then $s_k = 2n$ or $s_k + 1$ is in the set of principal indices J of B, so Case 5 gives $\pi(s_k) = \omega(s_k)$, the number of vertical steps from the top to s_k, namely, s_{k+1}. Hence $\pi(s_k) = s_{k+1}$.

Assume that the k-th step of the inverted path is vertical. This implies that $1 \le s_k \le n$. If the $(k+1)$-th step is also vertical, then $s_{k+1} = s_k - 1$. Because steps k and $k+1$ are both vertical, A' does not contain any column with weight $s_k - 1$. Thus, we are in Case 2 and $\pi(s_k) = s_k - 1 = s_{k+1}$. Finally, if the $(k+1)$-th step is horizontal, $s_k - 1 = \omega(s_{k+1})$ where $S_{k+1} \in J$, so case 3 gives $\pi(s_k) = s_{k+1}$. □

Example 2.5.2. Figure 2.5 displays the antiadjacency matrix A of the altitude respecting 5-labeled unit interval order P introduced in Example 2.1.2 and the matrix $\phi(A)$. Both show their respective Dyck and inverted path, which encodes the decorated permutation $\pi = (1, 2, 10, 3, 9, 4, 8, 7, 5, 6)$ associated to the positroid induced by P.

As a consequence of Theorem 2.5.1, we can deduce that the map $\rho \circ \phi \circ \varphi \colon \mathcal{U}_n \to \mathcal{P}_n$, where ρ, ϕ, and φ are as defined in Section 2.2 and Section 2.3, is indeed a bijection.

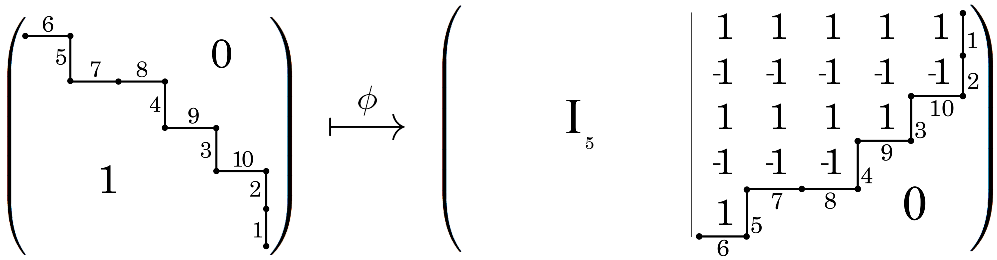


Figure 2.5: Dyck matrix A and its image $\phi(A)$ exhibiting the decorated permutation π along their Dyck path and inverted path, respectively.

Lemma 2.5.3. *The set of* $2n$*-cycles* $(1\ j_1\ \ldots\ j_{2n-1})$ *satisfying conditions (1) and (2) of Theorem 2.4.4 is in bijection with the set of Dyck paths of length* $2n$*.*

Proof. We can assign a Dyck path D of length $2n$ to the $2n$-cycle $(1 = j_0\ j_1\ \ldots\ j_{2n-1})$ by thinking of the entries $j_i \in \{1, \ldots, n\}$ as ascending steps of D and the entries $j_i \in \{n+1, \ldots, 2n\}$ as descending steps of D. The fact that such an assignment yields the desired bijection is straightforward. □

Corollary 2.5.4. *The map* $\rho \circ \phi \circ \varphi \colon \mathcal{U}_n \to \mathcal{P}_n$ *is a bijection.*

Corollary 2.5.5. *The number of unit interval positroids on the ground set* $[2n]$ *equals the* n*-th Catalan number.*

We conclude this section by describing how to decode the decorated permutation associated to the unit interval positroid induced by P directly from its altitude respecting representation I. Labeling the left and right endpoints of the intervals $[q_i, q_i + 1] \in I$ by $-$ and $+$, respectively, we obtain a $2n$-tuple consisting of pluses and minuses by reading from the real line the labels of the endpoints of all such intervals. On the other hand, we can have another *plus-minus* $2n$-tuple if we replace the horizontal and vertical steps of the Dyck path of A by $-$ and $+$, respectively, and then read it in southeast direction as indicated in the following example.

Example 2.5.6. The figure below shows the antiadjacency matrix of the altitude respecting 5-labeled unit interval order P showed in Example 2.1.2 and an altitude respecting interval representation of P, both encoding the plus-minus 10-tuple $(-, +, -, -, +, -, +, -, +, +)$, as described in the previous paragraph.

Lemma 2.5.7. *Let* $\mathbf{a}_n = (a_1, \ldots, a_{2n})$ *and* $\mathbf{b}_n = (b_1, \ldots, b_{2n})$ *be the* $2n$*-tuples with entries in* $\{+, -\}$ *obtained by labeling the steps of the Dyck path of* A *and the endpoints of all intervals in* I*, respectively, in the way described above. Then* $\mathbf{a}_n = \mathbf{b}_n$*.*

Proof. Let us proceed by induction on the cardinality n of P. When $n = 1$, both $\mathbf{a}_1$ and $\mathbf{b}_1$ are equal to $(-, +)$ and so $\mathbf{a}_1 = \mathbf{b}_1$. Suppose now that the statement of the lemma is

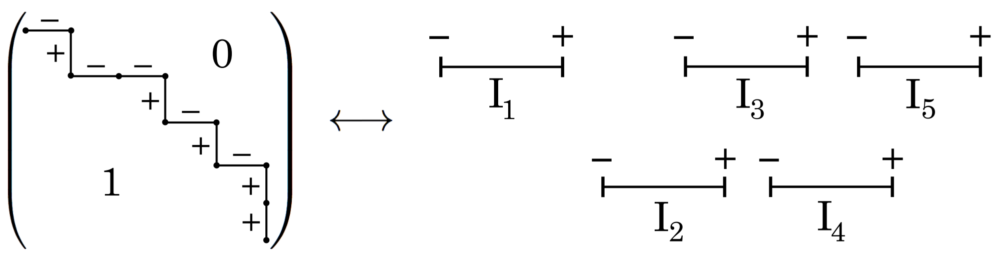


Figure 2.6: Dyck matrix and altitude respecting interval representation of P encoding the 10-tuple $(-,+,-,-,+,-,+,-,+,+)$.

true for every altitude respecting n-labeled unit interval order, and assume that P is a unit interval order with an altitude respecting labeling by $[n+1]$ with antiadjacency matrix A and altitude respecting interval representation I. Set $m = |\Lambda_{n+1}| - 1$. By Proposition 2.3.1, the poset $P \backslash \{n+1\}$ is a unit interval order with an altitude respecting labeling by $[n]$; therefore its associated plus-minus $2n$-tuples $\mathbf{a}'_n$ and $\mathbf{b}'_n$ are equal. Observe, in addition, that $\mathbf{b}_{n+1}$ can be recovered from $\mathbf{b}'_n$ by inserting the $-$ corresponding to the left endpoint of q_{n+1} (labeled by $2n+2$) in the position $m+n+1$ (there are n left interval endpoints and m right interval endpoints to the left of q_{m+1} in I) and adding the $+$ corresponding to the right endpoint of q_{n+1} (labeled by 1) at the end. On the other hand, $\mathbf{a}_{n+1}$ can be recovered from $\mathbf{a}'_n$ by inserting the $-$ corresponding to the rightmost horizontal step of the Dyck path of A in the position $m+n+1$ (there are n horizontal steps and m vertical steps before the last horizontal step of the Dyck path) and placing the $+$ corresponding to the vertical step labeled by 1 in the last position. Hence $\mathbf{a}_{n+1} = \mathbf{b}_{n+1}$, and the lemma follows by induction. □

As a consequence of Theorem 2.5.1 and Lemma 2.5.7, one may read the decorated permutation associated to the unit interval positroid induced by P directly from I.

Corollary 2.5.8. *Labeling the left and right endpoints of the intervals $[q_i, q_i + 1]$ by $n+i$ and $n+1-i$, respectively, we obtain the decorated permutation associated to the positroid induced by P by reading the label set $\{1, \dots, 2n\}$ from the real line from right to left.*

Proof. By Lemma 2.5.7, the $2n$-tuple resulting from reading the set $\{1, \dots, 2n\}$ as indicated in Corollary 2.5.8 equals the $2n$-tuple resulting from reading the same set from the Dyck path of A in northwest direction, as described in Theorem 2.5.1. Hence the corollary follows immediately from Theorem 2.5.1. □

Example 2.5.9. The diagram in Figure 2.7 illustrates how to label the endpoints of an altitude respecting interval representation of the 6-labeled unit interval order P shown in Figure 2.1. By reading the labels from the real line (from right to left) one can obtain the decorated permutation $\pi = (1, 12, 2, 3, 11, 10, 4, 5, 9, 6, 8, 7)$ associated to the positroid induced by P.

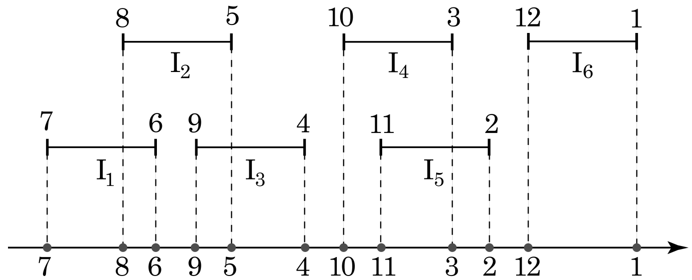


Figure 2.7: Decorated permutation π encoded in a canonical interval representation of P.

2.6 The Totally Nonnegative Grassmannian and Le-diagrams

The space of all k-dimensional subspaces of $\mathbb{R}^n$ is called the *real Grassmannian* $Gr_{k,n}$. An element of $Gr_{k,n}$ can classically be viewed as a full rank $k \times n$ matrix modulo left multiplication by nonsingular $k \times k$ matrices. The *totally nonnegative Grassmannian* $Gr^+_{k,n}$ is the restriction of $Gr_{k,n}$ to the space of full-rank $k \times n$ matrices with nonnegative maximal minors. Postnikov constructed a stratification of $Gr^+_{k,n}$ into *positroid cells, or strata* [39]. The cells are in bijection with several families of combinatorial objects, including Grassmann necklaces, decorated permutations and Le-diagrams [39]. Many properties, such as cell dimension and cell adjacency, can be described in terms of these objects. In this section we prove the dimension of each unit interval positroid cell of a unit interval positroid of rank n is equal to $2n-1$ via Le-diagrams. We address cell adjacency in Section 2.7.

The Totally Nonnegative Grassmannian and Positroid Cells

By definition an element of $Gr_{k,n}$ is not uniquely represented. In particular, two $k \times n$ matrices can represent an element of $Gr_{k,n}$ if and only if one can be obtained from the other by row operations. In order to describe Postnikov's positroid stratification, we introduce the Plücker embedding. Let $\binom{[n]}{k}$ be the set of all k-element subsets of $[n]$. For $I \in \binom{[n]}{k}$, define the *Plücker coordinate* $\Delta_I(A)$ to be the maximal minor of the column set I in the $k \times n$ matrix A. Then the *Plücker embedding* $Gr_{k,n} \hookrightarrow \mathbb{RP}^{\binom{n}{k}-1}$, induced by the map $A \mapsto (\Delta_I(A))$ for I ranging over $\binom{[n]}{k}$, realizes the Grassmannian as a subset of the projective space $\mathbb{RP}^{\binom{n}{k}-1}$.

A stratification of $Gr_{k,n}$ can be given in terms of matroids, called the *matroid stratification*, or *Gelfand-Serganova stratification*. This naturally leads to a definition of the positroid stratification. Given an element $A \in Gr_{k,n}$, there is an associated matroid $\mathcal{M}_A = ([n], \mathcal{B})$ whose bases are the k-subsets $\mathcal{B} \subset \binom{[n]}{k}$ such that $\Delta_I(A) \neq 0$ for $I \in \mathcal{B}$.

Definition 2.6.1. For $\mathcal{B} \subset \binom{[n]}{k}$, let $\mathcal{M} = ([n], \mathcal{B})$ be a matroid. Define the *matroid stratum* $S_{\mathcal{M}}$ to be

$$S_{\mathcal{M}} = \{A \in Gr_{k,n} \mid \Delta_I(A) \neq 0 \text{ if and only if } I \in \mathcal{B}\}.$$

Remark 2.6.2. The matroids $\mathcal{M}$ with nonempty strata $S_{\mathcal{M}}$ are called *realizable* over $\mathbb{R}$.

Recall $\mathrm{Mat}_{k,n}^{\geq 0}$ is the space of real $k \times n$ matrices of rank k with nonnegative maximal minors, and consider the group GL_k^+ of $k \times k$ real matrices with positive determinant. Then the *totally nonnegative Grassmannian* of k-dimensional subspaces in $\mathbb{R}^n$ is defined to be the quotient $Gr_{k,n}^+ = GL_k^+ \backslash \mathrm{Mat}_{k,n}^{\geq 0}$.

A cellular decomposition of $Gr_{k,n}^+$ is achieved by specifying which maximal minors are strictly positive and which are equal to zero [39]. This is Postnikov's positroid stratification, defined as follows. Recall $\mathcal{I}_{\mathcal{M}}$ denotes the Grassmann necklace of type (k, n) of a given positroid $\mathcal{M}$.

Definition 2.6.3. Let $\mathcal{I} = \{I_1, \ldots, I_n\}$ be a Grassmann necklace of type (k, n). Define the *positroid stratum* $S_{\mathcal{I}}$ to be

$$S_{\mathcal{I}} = \{A \in Gr_{k,n}^+ \mid \mathcal{I}(A) = \mathcal{I}\}.$$

Le-diagrams

We consider a description of the cellular decomposition of $Gr_{k,n}^+$ in terms of the associated Le-diagrams [6].

Definition 2.6.4. Fix d and n. For any partition λ, let Y_λ denote the Young diagram associated to λ. A ⅃*-diagram* (or *Le-diagram*) L of shape λ and type (d, n) is a Young diagram Y_λ contained in a $d \times (n - d)$ rectangle, whose boxes are filled with 0s and +s in such a way that the ⅃*-property* is satisfied: there is no 0 which has a + above it in the same column and a + to its left in the same row. See Figure 2.8 for an example of a ⅃-diagram.

+	0	0	0	+
0	0	0	0	+
0	+	0	0	+
0	0	+	0	+
0	0	0	+	+

Figure 2.8: A Le-diagram of shape (5^5) and type $(5, 10)$.

Lemma 2.6.5. *[6] The following algorithm is a bijection between ⅃-diagrams of type (d, n) and decorated permutations on n letters with d weak excedences.*

(1) Replace each + in the ⅃-diagram L with an elbow joint, and each 0 in L with a cross.

(2) Note that the south and east border of Y_λ gives rise to a length-n path from the northeast corner to the southwest corner of the $d \times (n-d)$ rectangle. Label the edges of this path with the numbers 1 through n.

(3) Now label the edges of the north and west border of Y_λ so that opposite horizontal edges and opposite vertical edges have the same label.

(4) View the resulting "pipe dream" as a permutation $\pi \in S_n$, by following the "pipes": from the northwest border to the southeast border of the Young diagram. If the pipe originating at label i ends at label j, we define $\pi(i) = j$.

(5) If $\pi(j) = j$ and j labels two horizontal (respectively, vertical) edges of Y_λ, then $\pi(j) := \underline{j}$ (respectively, $\pi(j) := \overline{j}$).

Figure 2.9 shows how to recover the decorated permutation $(1, 10, 2, 3, 9, 4, 8, 5, 7, 6)$ from the ⅃-diagram of Figure 2.8.

Since each unit interval positroid can be represented by a ⅃-diagram L, it follows immediately that each unit interval positroid cell in $Gr^+_{n,2n}$ can be indexed by L. Let $|L|$ equal the number of +s of the ⅃-diagram. Postnikov proved that the positroid cell indexed by L has dimension $|L|$ [39, Theorem 4.6]. We show that the unit interval positroid cells in $Gr^+_{n,2n}$ have dimension $2n-1$. To this end, we now describe the ⅃-diagram associated with a unit interval positroid.

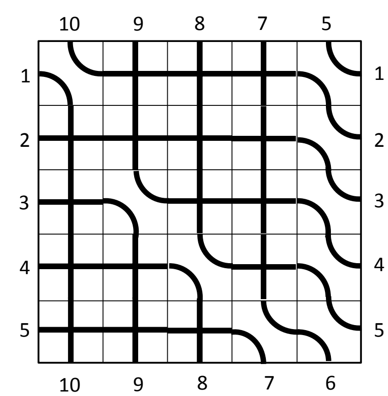


Figure 2.9: A pipe dream on the Le-diagram in Figure 2.8 to recover the decorated permutation $(1, 10, 2, 3, 9, 4, 8, 5, 7, 6)$.

Theorem 2.6.6. *The $n \times n$ ⅃-diagram representation of a unit interval positroid P of rank n with its associated Dyck path D is as follows:*

- *On a Young diagram of shape (n^n), draw the Dyck path D of length $2n$ from the SE-corner to the NW-corner that stays below the diagonal.*
- *Place $+$s in the right most column and in the boxes immediately above every horizontal step of the Dyck path.*
- *Fill the remaining boxes with 0s.*

Recall Proposition 2.4.3 which explicitly describes the decorated permutation of a unit interval positroid. It is enough to establish that the ⅃-diagram constructed in Theorem 2.6.6 produces the decorated permutation (equivalently its inverse) of a unit interval positroid. With this established, Postnikov's bijection implies the ⅃-diagram is unique.

Proof. Given a diagram L as constructed above, we first show that it satisfies the ⅃-property, then we show it produces the decorated permutation of a unit intrval positroid.

To prove L is a ⅃-diagram, assume on the contrary that the ⅃-property is not satisfied. Then there is a 0 with a + above it in the same column and to its left in the same row. This implies there must be a south step in the path drawn on the diagram contradicting it is a Dyck path (a Dyck path from the SE corner to the NW corner staying below the diagonal consists only of west and north steps). Thus L is a ⅃-diagram.

To prove L produces decorated permutation π^{-1} associated to a unit interval positroid, we show that the decorated permutation given by the pipe dream satisfies Proposition 2.4.3.

Apply Lemma 2.6.5 to form the associated pipe dream. To utilize Proposition 2.4.3 we must determine π, the inverse of the decorated permutation. Thus, we will read the pipe dream beginning from the east and south border.

Note that the position of a box in the ⅃-diagram is given by the edge labels. Thus (i, j) in the diagram corresponds to row with edge label i and column with edge label j.

Case 1: $i = 1$. By construction, 1 is adjacent to an elbow joint that leads directly to the edge labeled $n + 1$. Thus $\pi(1) = n + 1$.

Case 2: $1 < i \leq n$ and $\omega(j) \neq i - 1$ for all $j \in J$. A path leaving i first travels through an elbow joint to row $i - 1$ such that $\omega(j) = i - 1$. Without a column j, this implies row i contains all crosses to the left of the first elbow joint. Thus the path ends at edge label $i - 1$.

Case 3: $i \leq n$ where $\omega(j) = i - 1$ for $j \in J$. Since $i < n$, the path beginning at i passes through two vertically adjacent elbow joints to position $(i - 1, i + 2)$. Since $\omega(j) = i - 1$ there must be an elbow joint in position $(\omega(j), j)$. Also, $j \in J$ means this will be the first elbow joint the path meets after its initial steps. Since column j has only crosses in all other positions, the path must travel north to edge label j.

Case 4: $n < i < 2n$ and $i+1 \notin J$. This implies that i and $i+1$ are consecutive horizontal steps in the Dyck path, so the elbow joints in each column are adjacent. Thus the path from i moves up through any crosses, follows the elbow joint in column i to the elbow joint in column $i + 1$ then continues up through the crosses to end at edge label $i + 1$.

Case 5: $i > n$ and either $i = 2n$ or $i + 1 \in J$. If $i = 2n$, then $\omega(i)$ determines where the second to last step in the Dyck path occurs. Thus, an elbow joint exists in position $(\omega(i), 2n)$

and crosses in all other entries of the column. Thus the path leading from $2n$ must travel up through the elbow joint to end at edge label $\omega(i)$.

If $i+1 \in J$, then there is an elbow joint in the column of i at position $(\omega(i), i)$ with a cross in all entries to its left. Thus, the path leaving i will head up through crosses until it meets the elbow then heads left through the remaining crosses and ends at position $\omega(i)$.

This establishes that the set of Ⅎ-diagrams is in bijection with the set of decorated permutation representations of unit interval positroids, as desired. □

Corollary 2.6.7. *Every unit interval positroid cell in $Gr^+_{n,2n}$ has dimension $2n-1$.*

Proof. This follows immediately from Theorem 2.6.6 since the Ⅎ-diagram L indexing each unit interval positroid cell has $|L| = 2n-1$. □

2.7 Adjacent Unit Interval Positroid Cells

In this section we describe when two unit interval order positroid cells are adjacent in terms of the associated decorated permutations.

One may view the set of positroid cells as a graded poset with a simple cover relation. One cell covers another if the closure of the first contains the second. Moreover, the rank of a cell is just its dimension. We may also use decorated permutations to describe the cover relations, which we define next.

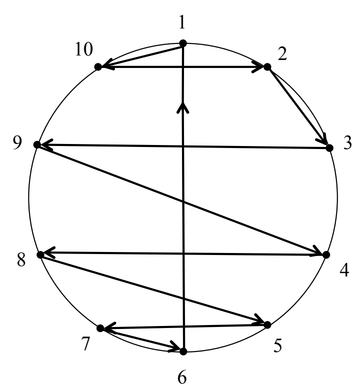


Figure 2.10: A chord diagram of the decorated permutation $(1, 10, 2, 3, 9, 4, 8, 5, 7, 6)$.

A decorated permutation can be represented as a *chord diagram* in the following way. Place n equally spaced points around a circle and label them from 1 to n in clockwise order. If $\pi(i) = j$ then this is represented by a directed arrow, or chord, from i to j. If $\pi(i) = i$ then we draw a chord from i to i (i.e. a loop), and orient it either clockwise or counterclockwise, according to its decoration.

We call a pair of chords in a chord diagram that intersect inside the circle or on its boundary a *crossing*. Referring to π_1 in Figure 2.11, when there are no other chords from

$\mathrm{Arc}(C, A)$ to $\mathrm{Arc}(B, D)$ then the crossing is called a *simple crossing.* A *degenerate simple crossing* occurs on the boundary. A *simple alignment* is a pair of chords that do not cross and there are no other chords from $\mathrm{Arc}(C, A)$ to $\mathrm{Arc}(B, D)$. Denote a simple alignment as $(A, B)||(C, D)$ when chord A, B is aligned with chord C, D, as illustrated by π_2 in Figure 2.11.

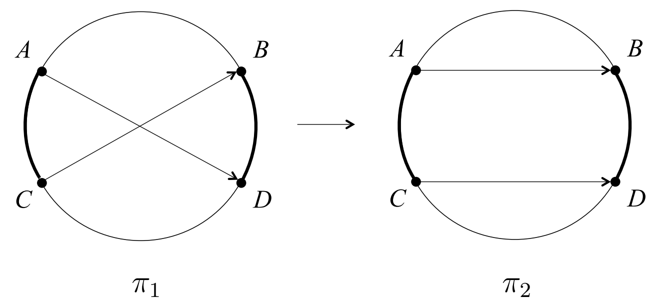


Figure 2.11: Cover relation for chord diagram representations of decorated permutations π_1 and π_2, illustrating a simple crossing and a simple alignment respectively.

We can now define a partial order on the set of decorated permutations. For two decorated permutations π_1 and π_2 of the same size n, we say that π_1 *covers* π_2 if the chord diagram of π_1 contains a pair of chords that forms a simple crossing and the chord diagram of π_2 is obtained by changing them to the pair of chords that forms a simple alignment.

We say two positroids (or positroid cells) are *adjacent* if, as elements in the graded poset, they cover the same element. That is, π_1 and π_2 are adjacent if there exists π_3 such that π_1 and π_2 both cover π_3. See Figures 2.11 and 2.12 for illustrations of the different cover relations for chord diagram representations of positroid cells.

We now give a condition on the decorated permutations for when two unit interval positroids are adjacent.

Theorem 2.7.1. *Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be rank n unit interval positroids and π_1 and π_2 their respective decorated permutations. Then $\mathcal{P}_1$ and $\mathcal{P}_2$ are adjacent if there exists $i \in [2n]\backslash\{1, n+1\}$ such that when i is removed from the cycle notation of π_1 and π_2 the resulting cycles are equal.*

Proof. To prove the forward direction, assume $\mathcal{P}_1$ and $\mathcal{P}_2$ are adjacent unit interval positroids. By Lemma 2.5.3 the associated decorated permutations π_1 and π_2 are cycles of length $2n$ and have no fixed points. It follows that their respective chord diagrams, C_{π_1} and C_{π_2}, always have a directed edge from $n+1$ to 1 and the only nondegenerate simple crossings occur along this edge.

We will now argue that any two unit interval positroid cells are adjacent only via a degenerate crossing. By way of contradiction, assume that π_1 and π_2 both cover π_3 via a simple crossing. Say C_{π_1} contains the simple crossing of chords j_1, i_1 and $n, 1$, and C_{π_2}

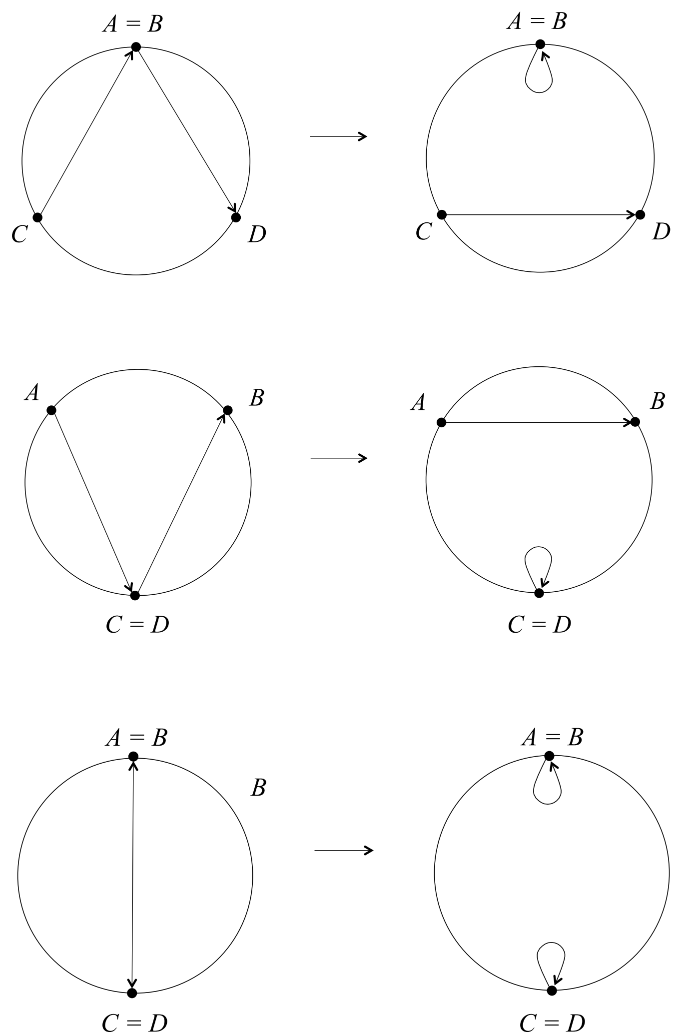


Figure 2.12: Degenerate covering relations

contains the simple crossing of chords j_2, i_2 and $n, 1$, where $1 < i_2 < j_1 < n < j_2 < i_1$. See Figure 2.13 for an illustration of this construction.

Applying the cover relation to the respective crossings, we see that C_{π_3} must contain two pairs of alignments: $(i_1, b)||(a, j_1)$ and $(i_2, b)||(a, j_2)$. But, as seen in Figure 2.13, neither alignment is simple. This implies the crossings in π_1 and π_2 were not simple, a contradiction.

By the above argument we have that the adjacent unit interval positroid cells π_1 and π_2 both cover a decorated permutation π_3 via a degenerate cover relation. This implies π_3 is the product of a fixed point (k) and a $2n-1$ cycle with the following description. The fixed point k has decoration $\pi_3(k) = \bar{k}$ for $1 < k < n+1$ and $\pi_3(k) = \underline{k}$ for $n+1 < k \leq 2k$. The $2n-1$ cycle is the cycle produced by removing k from the cycle notation of π_1 (and equivalently by removing k from the cycle notation of π_2). Note the decoration follows immediately from

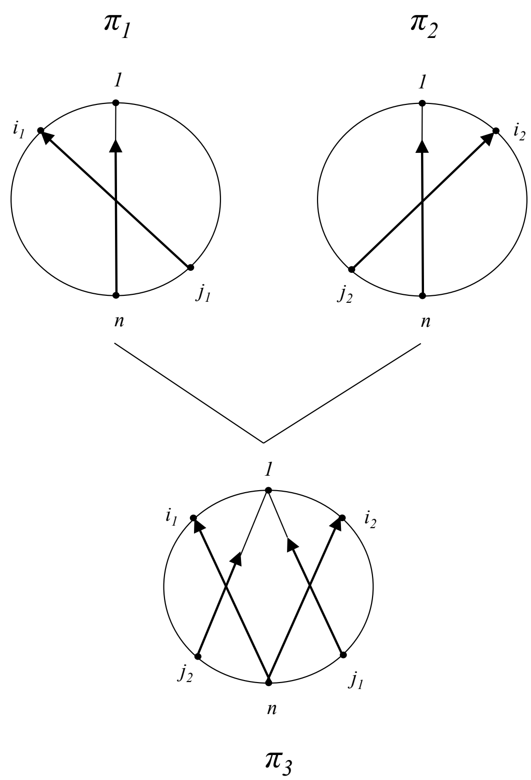


Figure 2.13: Chord diagrams π_1 and π_2 fail to be adjacent because their crossings are not simple.

the fact that the decorated permutation representation of a unit interval positroid is a Dyck path. Since the $2n-1$ cycle of π_3 can be generated from either π_1 or π_2 in the same way, we have that the cycles resulting from removing k from π_1 and from π_2 are equal, as desired. $\square$

See Figure 2.14 for an example of two adjacent unit interval positroid cells and the cells they simultaneously cover. Figure 2.15 is a complete diagram of all adjacencies for unit interval positroids of rank 3.

Typically chord diagram (equivalently decorated permutation) representations of positroid cells are illustrated via their Hasse diagram. Given the adjacencies among unit interval positroid cells can be described so nicely, we believe the complex structure formed by this subset of positroid cells may have an interesting description. A proposed problem stemming from Theorem 2.7.1 explores this structure.

Problem 2.7.2. *Describe the subcomplex of the totally nonnegative Grassmannian formed by unit interval positroids and their closures, where a k-face represents a set of unit interval positroid cells that intersect in codimension k.*

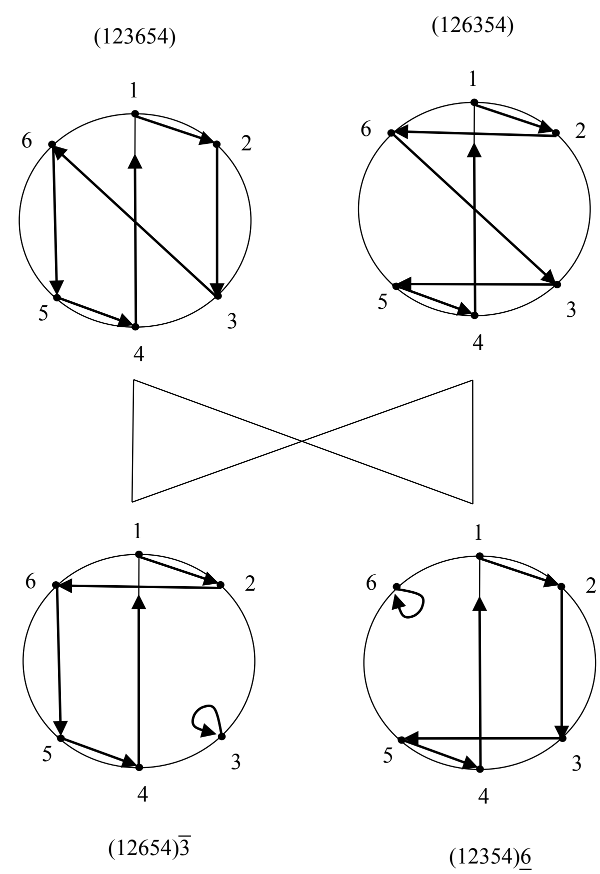


Figure 2.14: Two adjacent unit interval positroid cells in $Gr_{3,6}(\mathbb{R})$ and the positroid cells they simultaneously cover.

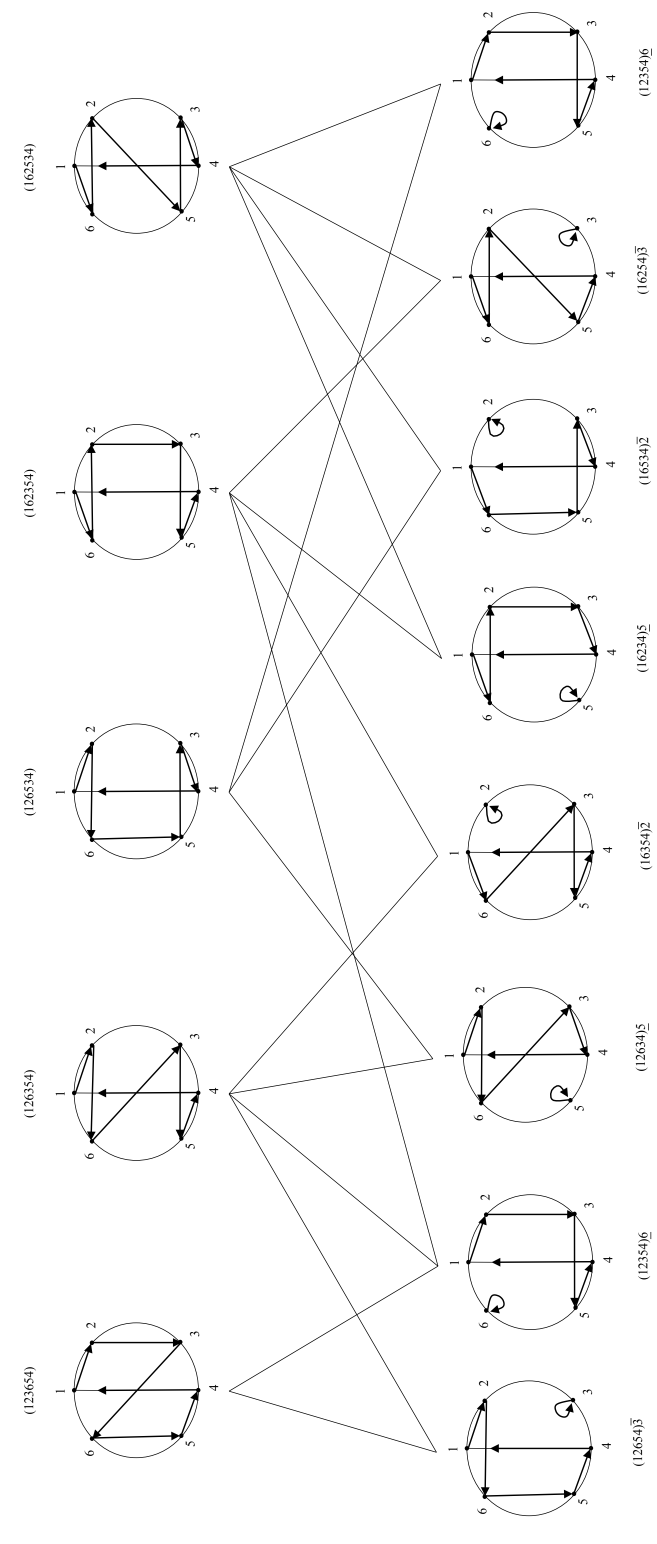


Figure 2.15: Unit interval positroid cells in $Gr_{3,6}(\mathbb{R})$ and their adjacent cell relations.

2.8 An Interpretation of the f-vector of a Poset

In hopes of a more thorough understanding of the *f-vectors* of $(3+1)$-free posets, Skandera and Reed in [47] posed the following open problem: characterize the f-vectors of unit interval orders. To this aim, we provide a combinatorial interpretation for the f-vector of any naturally labeled poset in terms of its antiadjacency matrix. Throughout this section, P is assumed to be a naturally labeled poset of cardinality n with antiadjacency matrix $A_P = (a_{i,j})$.

Definition 2.8.1. The *f-vector* of P is the sequence $f = (f_0, f_1, \dots, f_{n-1})$, where f_k is the number of k-element chains of P.

We wish to interpret the k-element chains of P in terms of some special Dyck paths inside A_P. To do this, define a *bounce Dyck path* of A_P to be a Dyck path drawn inside A_P that has its endpoints on the main diagonal, all its peaks in positions (i,j) such that $a_{i,j} = 0$, and it must return to the main diagonal between peaks. Figure 2.16 illustrates an example of a bounce Dyck path with three peaks.

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Figure 2.16: A bounce Dyck path with three peaks inside the antiadjacency matrix of the poset displayed in Figure 2.17.

Theorem 2.8.2. *The entries of the f-vector of a naturally labeled poset P are*

$$f_k = \begin{cases} n & \text{for } k = 0 \\ \#\text{ of bounce Dyck paths of } A_P \text{ with } k \text{ peaks} & \text{for } k \geq 1 \end{cases},$$

where A_P is the associated antiadjacency matrix.

Proof. To each k-element chain $\mathsf{c} : i_1 <_P \cdots <_P i_{k+1}$ we can assign a bounce Dyck path v_{c} with k peaks as follows: the j-th peak begins at (i_j, i_j), heads east to (i_j, i_{j+1}), and then heads south to (i_{j+1}, i_{j+1}). To see that v_{c} is, indeed, a bounce Dyck path, it suffices to notice that as $i_j <_P i_{j+1}$ for each $j = 1, \dots, k$, every peak of v_{c} occurs at a zero entry of A_P. On the other hand, suppose that v is a bounce Dyck path with k peaks, namely $(i_1, i'_1), \dots, (i_k, i'_k)$. As v is a bounce Dyck path, every valley of v is supported on the main diagonal, which means that $i'_j = i_{j+1}$ for each $j = 1, \dots, k$. Setting $i_{k+1} = i'_k$, we obtain that $v = v_{\mathsf{c}}$, where c

is the k-element chain $i_1 <_P \cdots <_P i_{k+1}$. Thus, we have established a bijection that yields the theorem. $\square$

Remark: Theorem 2.8.2 holds for any naturally labeled poset P, so in particular, it provides an interpretation of the f-vector of any unit interval order.

Example 2.8.3. Consider the naturally labeled poset P, which is not a unit interval order, depicted in Figure 2.17. Then the f-vector of P is

$$f = (7, 12, 8, 2, 0, 0, 0).$$

Examples of bounce Dyck paths realized on A_P are shown in Figure 2.18. Note that since P is not a unit interval order then A_P is not a Dyck matrix.

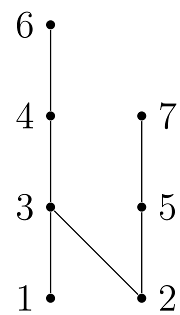


Figure 2.17: Naturally 7-labeled poset.

$$\underset{\text{one peak}}{\begin{pmatrix} 1&1&0&0&0&0&0\\ 1&1&0&1&1&0&0\\ 1&1&1&1&1&0&0\\ 1&1&1&1&0&1&1\\ 1&1&1&1&1&1&1\\ 1&1&1&1&1&1&0\\ 1&1&1&1&1&1&1 \end{pmatrix}} \quad \underset{\text{two peaks}}{\begin{pmatrix} 1&1&0&0&0&0&0\\ 1&1&0&1&1&0&0\\ 1&1&1&1&1&0&0\\ 1&1&1&1&0&1&1\\ 1&1&1&1&1&1&1\\ 1&1&1&1&1&1&0\\ 1&1&1&1&1&1&1 \end{pmatrix}} \quad \underset{\text{three peaks}}{\begin{pmatrix} 1&1&0&0&0&0&0\\ 1&1&0&1&1&0&0\\ 1&1&1&1&1&0&0\\ 1&1&1&1&0&1&1\\ 1&1&1&1&1&1&1\\ 1&1&1&1&1&1&0\\ 1&1&1&1&1&1&1 \end{pmatrix}}$$

Figure 2.18: Bounce Dyck paths inside A_P with one, two, and three peaks.

Chapter 3

Dehn–Sommerville Relations and the Catalan Matroid

3.1 Introduction

One of the most important achievements of the theory of polytopes is the g-theorem, conjectured by McMullen [35] and subsequently proven by Billera–Lee, and Stanley [10, 9, 51], which provides a complete characterization of the f-vectors of simplicial polytopes. The goal of this chapter is to further contribute to our understanding of those f-vectors.

The Dehn–Sommerville relations condense the f-vector of a simplicial d-dimensional polytope into the g-vector, which has length $\lceil \frac{d+1}{2} \rceil$. This raises the question: Which $(\lceil \frac{d+1}{2} \rceil)$-subsets of the f-vector of a general simplicial polytope are sufficient to determine all d entries of the f-vector? Define a *Dehn–Sommerville basis* to be a minimal subset S such that $\{f_{i-2} \mid i \in S\}$ determines the entire f-vector for any simplicial polytope.

For example, via the g-theorem one can check that $f(P_1) = (1, 8, 27, 38, 19)$ and $f(P_2) = (1, 9, 28, 38, 19)$ are f-vectors of two different simplicial 4-polytopes. Therefore the entries $f(P) = (1, *, *, 38, 19)$ do not determine a simplicial f-vector uniquely, and $\{1, 4, 5\}$ is not a Dehn–Sommerville basis in dimension 4.

In this chapter we prove the following theorem.

Theorem 3.3.1. *The Dehn–Sommerville bases of dimension $2n$ are precisely the upstep sets of the Dyck paths of length $2(n + 1)$.*

Dehn–Sommerville bases of dimension $2n - 1$ have a similar description.

The chapter is organized as follows. In Section 3.2 we define the Dehn–Sommerville matrix, and the Catalan matroid defined by Ardila [3] and Bonin–de Mier–Noy [16]. In Section 3.3 we prove our main theorem. In Section 3.4 we establish several positroid representations of the Dehn–Sommerville matroid. This chapter is joint work with Nicole Yamzon [19].

3.2 Matroids and the Dehn–Sommerville matrix

Matroids

Recall the definition of a matroid given in Section 2.2. A key example is the matroid $M(A)$ of a matrix A. Let A be a $d \times n$ matrix of rank d over a field K. Denote the columns of A by $a_1, a_2, \ldots, a_n \in K^d$. Then $B \subset [n]$ is a basis of $M(A)$ on the ground set $[n]$ if $\{a_i \mid i \in B\}$ forms a linear basis for K^d.

The Dehn-Sommerville relations

Definition 3.2.1. Let P be a d-dimensional simplicial polytope, that is, a polytope whose facets are simplices. Define $f(P) = (f_{-1}, f_0, f_1, \ldots, f_{d-1})$ to be the *f-vector of P* where f_i is the number of i-dimensional faces of P. It is convention that $f_{-1} = 1$.

Definition 3.2.2. The *h-vector* of a simplicial polytope is the sequence with elements

$$h_k = \sum_{i=0}^{k} (-1)^{k-i} \binom{d-i}{k-i} f_{i-1},$$

for $k \in [0, d]$. The *g-vector* of a simplicial polytope is the sequence where $g_0 = 1$ and $g_i = h_i - h_{i-1}$ for $i \in [1, \lfloor \frac{d}{2} \rfloor]$.

The Dehn–Sommerville relations can be stated most simply in terms of the h-vector.

Theorem 3.2.3. *[34] The h-vector of a simplicial d-polytope satisfies*

$$h_k = h_{d-k}$$

for $k = 0, 1, \ldots, d$.

We now discuss a matrix reformation of Theorem 3.2.3.

Definition 3.2.4. The *Dehn–Sommerville matrix*, M_d, is defined by

$$(M_d)_{\substack{0 \le i \le \lfloor \frac{d}{2} \rfloor \\ 0 \le j \le d}} := \binom{d+1-i}{d+1-j} - \binom{i}{d+1-j},$$

for $d \in \mathbb{N}$.

Theorem 3.2.5. *[14] Let P be a simplicial d-polytope, and let f and g denote its f and g-vectors. Then*

$$g \cdot M_d = f.$$

Label the columns of M_d from 1 to $d+1$. We define the *Dehn–Sommerville matroid* DS_d of rank d to be the pair $([d+1], \mathcal{B})$ where $B \in \mathcal{B}$ if B is a collection of columns associated with a non-zero maximal minor of M_d. Its bases are the *Dehn–Sommerville bases*, the minimal sets S such that $\{f_{i-2} \mid i \in S\}$ determines the whole f-vector for any simplicial polytope.

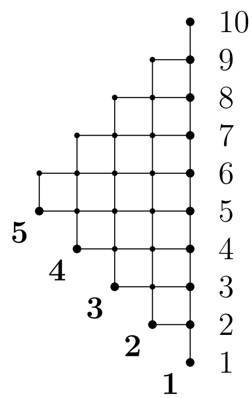
For example, let $d = 4$. Then

$$M_4 = \begin{pmatrix} 1 & 5 & 10 & 10 & 5 \\ 0 & 1 & 4 & 6 & 3 \\ 0 & 0 & 1 & 2 & 1 \end{pmatrix}$$

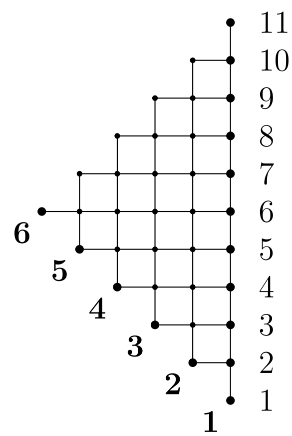
and the Dehn–Sommerville bases of DS_4 are

$$\mathcal{B} = \{123, 124, 125, 134, 135\}.$$

Definition 3.2.6. Define the *Dehn–Sommerville graph* $\mathbf{DS}_d$ to be a directed graph (or *digraph*), as shown in Figure 3.1, where all horizontal edges are directed east and all vertical edges are directed north. Number the nodes along the southwest border $\mathbf{1}, \ldots, \lceil \frac{\mathbf{d+1}}{\mathbf{2}} \rceil$ in bold, and call them the *sources*. Number the nodes along the east border $1, \ldots, d+1$, and call them the *sinks*.



(a) $\mathbf{DS}_9$



(b) $\mathbf{DS}_{10}$

Figure 3.1: Dehn–Sommerville graphs for odd and even d.

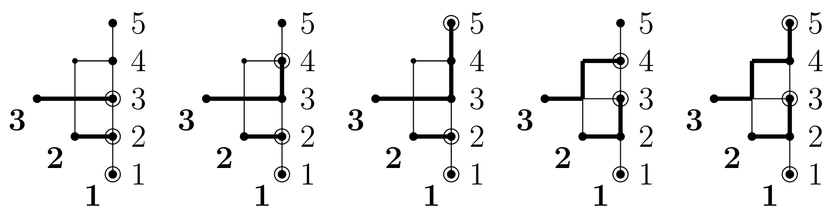
Definition 3.2.7. A *routing* is a set of vertex-disjoint paths in a digraph.

Considering the collection of routings on the Dehn–Sommerville graph $\mathbf{DS}_d$ produces the following theorem.

Theorem 3.2.8. *A subset $B \subset [d+1]$ is a basis of DS_d if and only if there is a routing from the source set $[\lceil \frac{\mathbf{d+1}}{\mathbf{2}} \rceil]$ to the sink set B in the graph $\boldsymbol{DS}_d$.*

Proof. Björklund and Engstöm [11] found a way to give positive weights to the edges of $\mathbf{DS}_d$ so that M_d is the path matrix of $\mathbf{DS}_d$; that is, entry $(M_d)_{ij}$ equals the sum of the product of the weights of the paths from source $\mathbf{i}$ to sink j. Then, by the Lindström–Gessel–Viennot lemma [24, 31], the determinant of columns $j_1, \ldots, j_{\lceil \frac{d+1}{2} \rceil}$ is the sum of the products of weights of the routing from the source nodes $\mathbf{1} < \cdots < \lceil \frac{\mathbf{d+1}}{\mathbf{2}} \rceil$ to the sink nodes $j_1, \ldots, j_{\lceil \frac{d+1}{2} \rceil}$. It follows that the determinant is non-zero if and only if there is a routing. □

Continuing with our example for $d = 4$, we see the possible destination sets of the routings on digraph $\mathbf{DS}_4$ match with the bases $\{123, 124, 125, 134, 135\}$ of DS_4:



Dyck Paths and the Catalan Matroid

Definition 3.2.9. For $n \in \mathbb{N}$, a *D*yck path of length $2n$ is a path in the plane from $(0, 0)$ to $(2n, 0)$ with upsteps, $(1, 1)$, and downsteps, $(1, -1)$, that never falls below the x-axis. The number of Dyck paths of length $2n$ is the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$.

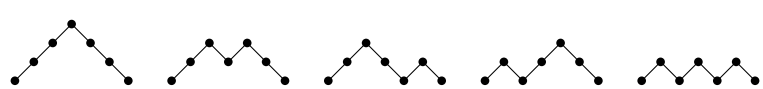
Dyck paths can be characterized in the following way.

Lemma 3.2.10. *[49, Problem 6.19 (i,t)] Let $a_1 < a_2 < \cdots < a_n$ be the upsteps of a lattice path P of length $2n$. Then P is a Dyck path if and only if*

$$a_1 = 1, a_2 \leq 3, a_3 \leq 5, \ldots, a_n \leq 2n - 1.$$

Definition 3.2.11. [3, 16] The *Catalan matroid*, $\mathbf{C}_n$, is a matroid with a ground set of $[2n]$ whose bases are the upstep sets of the Dyck paths of length $2n$.

Let $n = 3$. Then the bases of the Catalan matroid are the upstep sets of the Dyck paths of length 6:



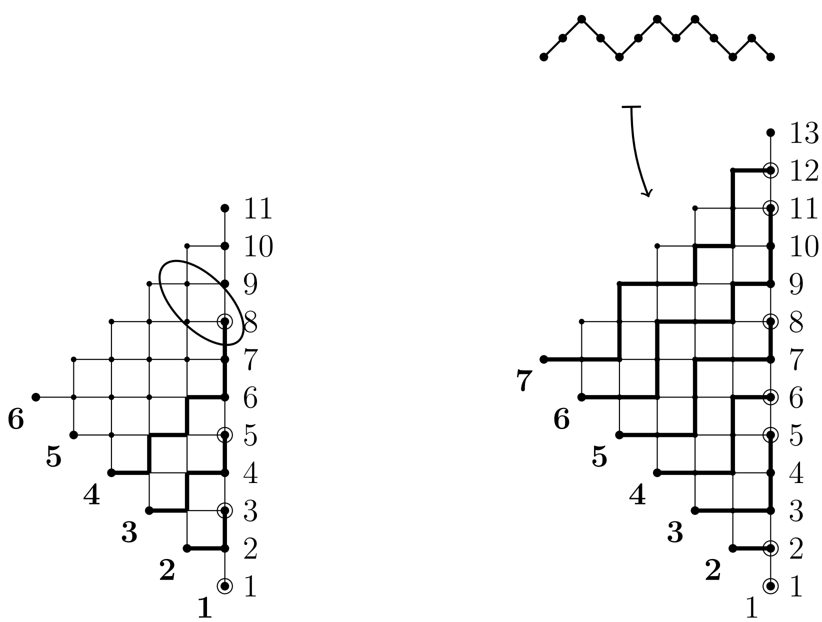
$$\mathcal{B} = \{123, 124, 125, 134, 135\}.$$

3.3 Main Result

Theorem 3.3.1. *The Dehn–Sommerville bases of dimension $2n$ are precisely the upstep sets of the Dyck paths of length $2(n+1)$.*

Proof. To prove the forward direction, assume for the sake of contradiction that $B = \{b_1 < b_2 < \cdots < b_{n+1}\}$ is a basis of DS_{2n} that is not the upstep set of a Dyck path. By Lemma 3.2.10 then $b_i \geq 2i$ for some i. Notice that the northwest diagonal starting at sink $2i$ has $n - i$ vertices (excluding sink $2i$). Since there is a routing from $\mathbf{1}, \ldots, \mathbf{n+1}$ to B, the $n+1-i$ paths starting at source nodes $\mathbf{i+1}, \ldots, \mathbf{n+1}$ must pass through this diagonal of width $n-i$. By the Pigeonhole Principle, this means two paths must use the same vertex, a contradiction. This argument is illustrated in Figure 3.2a.

For the backward direction, assume P is a Dyck path with up step set $\{b_1 < b_2 < \cdots < b_{n+1}\}$. We define a path P_i from the source $\mathbf{i}$ to the sink b_i by truncating the Dyck path P immediately before its ith up step, and converting its up steps to East steps and its down steps to North steps. The resulting paths $P_1, \ldots, P_{n+1}$ form a routing. Therefore the set $\{b_1 < b_2 < \cdots < b_{n+1}\}$ is a Dehn–Sommerville basis, as desired. See Figure 3.2b for an example of this construction. □



(a) (b)

Figure 3.2: (a) Sink $b_4 = 8$ prevents a routing in $\mathbf{DS}_{10}$, forcing the paths starting at sources 5 and 6 to collide. (b) A routing constructed from a Dyck path with upsteps $B = \{1, 2, 5, 6, 8, 11, 12\}$ in DS_{12}.

Having settled the even dimensional case, we now address the odd dimensional case.

Lemma 3.3.2. *The bases of the Dehn–Sommerville matroid DS_{2n} are in bijection with the bases of the Dehn–Sommerville matroid DS_{2n-1}. More precisely, $DS_{2n} \cong DS_{2n-1} \oplus C$, where $C = 2n$ is a coloop.*

Proof. For the graph $\mathbf{DS}_{2n}$, any path leaving the top source node $\lceil \frac{\mathbf{2n+1}}{\mathbf{2}} \rceil = n+1$ must first travel east along the only edge leaving it. To satisfy the condition that all paths in a routing are vertex-disjoint, any path leaving all other source nodes, excluding **1**, must also first travel east. Source node **1** is forced to have the trivial path. Thus, the horizontal edges from every source node, excluding **1**, can be contracted without affecting the potential destinations of the routings. This results in a graph equal to the graph $\mathbf{DS}_{2n-1}$ with an added vertical edge extending from the bottom.

It follows that the number of bases of DS_{2n} is equal to the number of bases of DS_{2n-1} and there exists a bijection between the bases, namely,

$$\{1, b_2, b_3, \ldots, b_n\} \mapsto \{b_2 - 1, b_3 - 1, \ldots, b_n - 1\},$$

where $\{1, b_2, b_3, \ldots, b_n\}$ is a basis of DS_{2n}. □

We may restate Theorem 3.3.1 and Lemma 3.3.2 in the language of matroids in the following way.

Theorem 3.3.3. *The Dehn–Sommerville matroids are obtained from the Catalan matroids by removing trivial elements:*

$$DS_{2n} \cong \mathbf{C}_{n+1} \backslash (2n+2) \quad \textit{and} \quad DS_{2n-1} \cong \mathbf{C}_{n+1} \backslash 1 \backslash (2n+2) \quad \textit{for } n \in \mathbb{N}.$$

Note that 1 is a loop and $2n+2$ is a loop in $\mathbf{C}_{n+1}$.

3.4 Positroid Representations

Björner [12] conjectured and Björklund and Engstöm [11] proved that M_d is a totally nonnegative matrix, which implies DS_d is a positroid. As seen in Chapter 2, there are several positroid representations introduced by Postnikov [39] that can be used to further represent the Dehn–Sommerville matroid. We give the Grassmann necklace description and the simple decorated permutation representation, which may be used to establish the remaining representations (see Chapter 2 for relevant definitions).

Theorem 3.4.1. *The Grassmann necklace $\mathcal{I}_{2n}$ corresponding to the Dehn–Sommerville matroid of rank $2n$ has as its entries*

- $I_1 = \{1, 2, 3, \ldots, n+1\}$,
- $I_{2i} = \{2i, 2i+1, \ldots, i+n, 1, 2, \ldots, i\}$ *for* $1 \leq i \leq n$, *and*
- $I_{2i+1} = \{2i+1, 2i+2, \ldots, i+n+1, 1, 2, \ldots, i\}$ *for* $1 \leq i \leq n$.

Similarly, the Grassmann necklace $\mathcal{I}_{2n-1}$ corresponding to the Dehn–Sommerville matroid of rank $2n-1$ has as its entries

- $I_1 = \{1, 2, 3, \ldots, n\}$,
- $I_2 = \{2, 3, 4, \ldots, n+1\}$,
- $I_{2i} = \{2i, 2i+1, \ldots, i+n, 1, 2, \ldots, i-1\}$ *for* $2 \leq i \leq n$, *and*
- $I_{2i+1} = \{2i+1, 2i+2, \ldots, i+n, 1, 2, \ldots, i\}$ *for* $1 \leq i \leq n-1$.

Proof. We first prove $\mathcal{I}_{2n}$ has the above description. Consider the sets $I_1 = \{1, 2, 3, \ldots, n{+}1\}$, $I_{2i} = \{2i, \ldots, i+n, 1, 2, \ldots, i\}$ and $I_{2i+1} = \{2i+1, \ldots, i+n+1, 1, 2, \ldots, i\}$. We need only to show that these are the Gale-minimal bases for DS_{2n}.

By Theorem 3.3.1 each basis is a Dyck path. The upstep set of a Dyck path containing $2i$ that is Gale-minimal must have the maximum number of upsteps possible between $2i$ and $2(n+1)$. By definition of a Dyck path, the number of upsteps in $[2i, 2(n+1)]$ is equal to the number of downsteps in $[0, 2i-1]$. Moreover, there must be $n+1$ total upsteps. Thus, if there are i upsteps beginning at $2i$, then there are also i upsteps less than $2i$, which implies $I_{2i} = \{2i, \ldots, i+n, 1, 2, \ldots, i\}$ is the $2i$-Gale-minimal basis.

Entry $I_{2i+1} = \{2i+1, \ldots, i+n+1, 1, 2, \ldots, i\}$ follows by a similar argument with only a shift needed in the beginning upsteps.

By Lemma 3.3.2, the Grassmann necklace for DS_{2n-1} can be achieved from the Grassmann necklace of DS_{2n} by removing 1 from every basis and decreasing each remaining value by 1. □

We may now describe the decorated permutation for the Dehn–Sommerville matroids.

Theorem 3.4.2. *The decorated permutation corresponding to the Dehn–Sommerville matroid is*

$$\begin{aligned} &\bar{1}35\cdots(2n+1)246\cdots(2n) \quad \textit{for} \quad DS_{2n} \\ &246\cdots(2n)135\cdots(2n-1) \quad \textit{for} \quad DS_{2n-1}. \end{aligned}$$

Proof. We prove this for rank $2n$, from which case $2n-1$ follows by Lemma 3.3.2 and Theorem 3.4.1. To determine the decorated permutation π for DS_{2n} we will compute its inverse directly from the associated Grassmann necklace.

By Theorem 3.4.1, we know I_1 and I_2 are equal. Since 1 is a loop in DS_{2n}, it follows $\pi^{-1}(1) = 1$ is a counterclock wise fixed point and $\pi(1) = \bar{1}$. For the remaining entries, recall $1 \leq i \leq n$. To compute $\pi^{-1}(2i)$, compare the entries $I_{2i} = \{2i, \ldots, i+n, 1, 2, \ldots, i\}$ and $I_{2i+1} = \{2i+1, \ldots, i+n+1, 1, 2, \ldots, i\}$ of $\mathcal{I}$. Then $\pi^{-1}(2i) = (i+n+1)$. To compute $\pi^{-1}(2i+1)$, compare the entries $I_{2i+1} = \{2i+1, \ldots, i+n+1, 1, 2, \ldots, i\}$ and $I_{2(i+1)} = \{2(i+1), \ldots, (i+1)+n, 1, 2, \ldots, (i+1)\} = \{2i+2, \ldots, i+n+1, 1, 2, \ldots, i+1\}$. Then $\pi^{-1}(2i+1) = (i+1)$. Thus, $\pi(1) = \bar{1}$, $\pi(i+1) = 2i+1$, and $\pi(i+n+1) = 2i$, as desired. □

Using this description we can obtain the Le diagram, plabic graph and juggling pattern. Another corollary to our main theorem is that the Catalan matroid is a positroid, a result also discovered by Pawlowski [38].

Chapter 4

Knuth Equivalence Graphs and CAT(0) Combinatorics

4.1 Introduction

A useful equivalence relation on S_n is defined in terms of the *Knuth relations* [28], defined below. Equivalently, the well-known Robinson–Schensted algorithm bijectively assigns to each permutation π a pair of standard Young tableaux $(P(\pi), Q(\pi))$ of the same shape λ, and π and σ are Knuth equivalent if and only if $P(\pi) = P(\sigma)$ [28, 40, 42]. This equivalence relation gives rise to a graph G_λ where the nodes are the permutations π such that $P(\pi)$ has shape λ, and the edges correspond to Knuth relations between them. We call such graphs *Knuth equivalence graphs*. Similar graphs have been studied and characterized by Assaf [8].

In this chapter we take a new perspective regarding G_λ as the 1-skeleton of a cubical complex, C_λ, and analyzing whether C_λ has the CAT(0) property, a metric property studied by geometric group theorists. This is useful combinatorially because CAT(0) cubical complexes are in bijection with posets of inconsistent pairs [5], so these results shed light on the combinatorics of Knuth equivalent graphs. This chapter is joint work with John Guo [18].

4.2 Knuth Equivalence, the Robinson–Schensted Algorithm and Young Tableaux

Permutations $\pi, \sigma \in S_n$ differ by a *Knuth relation of the first kind*, denoted $\pi \overset{1}{\cong} \sigma$, if

1. $\pi = x_1 \dots yxz \dots x_n$ and $\sigma = x_1 \dots yzx \dots x_n$ for some positive integers $x < y < z$.

They differ by a *Knuth relation of the second kind*, denoted $\pi \overset{2}{\cong} \sigma$, if

2. $\pi = x_1 \dots xzy \dots x_n$ and $\sigma = x_1 \dots zxy \dots x_n$ for some positive integers $x < y < z$.

We say π and σ are Knuth equivalent, $\pi \overset{K}{\cong} \sigma$, if there exists permutations such that

$$\pi = \pi_1 \overset{i}{\cong} \pi_2 \overset{j}{\cong} \cdots \overset{l}{\cong} \pi_k = \sigma,$$

where $i, j, \ldots, l \in \{1, 2\}$.

Example 4.2.1. Here are the non-trivial Knuth equivalence classes for S_4.

$$1243 \overset{2}{\cong} 1423 \overset{2}{\cong} 4123 \qquad 3124 \overset{2}{\cong} 1324 \overset{2}{\cong} 1342 \qquad 1432 \overset{2}{\cong} 4132 \overset{2}{\cong} 4312$$
$$4213 \overset{1}{\cong} 4231 \overset{2}{\cong} 2431 \qquad 2134 \overset{1}{\cong} 2314 \overset{1}{\cong} 2341 \qquad 3214 \overset{1}{\cong} 3241$$
$$3142 \overset{1,2}{\cong} 3412 \qquad 2143 \overset{1,2}{\cong} 2413$$

As mentioned before, the Robinson–Schensted algorithm bijectively assigns to each permutation π a pair of standard Young tableaux $(P(\pi), Q(\pi))$ of the same shape λ, and Knuth equivalence can be described in terms of tableaux [28, 40, 42]. We now clarify this relationship.

Denote the partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of n as $\lambda \vdash n$, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ and $|\lambda| = \sum_{i=1}^{k} \lambda_i = n$. Then the *Ferrers diagram*, or *shape*, of λ is an array of n boxes having k left-justified rows with row i containing λ_i boxes for $1 \leq i \leq k$. From a Ferrers diagram of boxes one can form a *standard Young tableau*, SYT, where the boxes are filled with elements from $[n]$ such that no element is repeated and rows and columns are strictly increasing.

Example 4.2.2. Let $\lambda \vdash 8$ be the partition $\lambda = (4, 2, 1, 1)$. Then a SYT of shape λ is

1	3	4	6
2	7		
5			
8			

We will now describe the *Robinson–Schensted algorithm* denoted $\pi \xrightarrow{\text{RS}} (P, Q)$ where $\pi \in S_n$ and P and Q are standard λ-tableaux, for $\lambda \vdash n$. Given

$$\pi = \begin{matrix} 1 & 2 & \ldots & n \\ \pi(1) & \pi(2) & \ldots & \pi(n) \end{matrix},$$

we construct a sequence of tableaux pairs $(P_1, Q_1), (P_2, Q_2), \ldots, (P_n, Q_n) = (P, Q)$, where $\pi(1), \ldots, \pi(n)$ are inserted into P and $1, 2, \ldots, n$ are inserted into Q in that order. Entry $x = \pi(i)$ is inserted into P_{i-1} in the following way:

RS1 Let R be the first row of P_{i-1}.

RS2 **While** x is less than some element of row R, do the following

RSa Let y be the smallest element of R greater than x and replace y by x in R.

RSb Set $x := y$ and let R be the next row down.

RS3 Now x is greater than every element of R, so place x at the end of R and **s**top.

Tableau Q_i is created by recording i in the box corresponding to the final box inserted into P_i so that P_i and Q_i have the same shape. We call $P = P_n$ the *insertion tableau*, denoted $P(\pi)$, and $Q = Q_n$ the *recording tableau*, denoted $Q(\pi)$. Notice that both P and Q have the same shape by construction.

Example 4.2.3. Let $\pi = 58273461$. We will apply the Robinson–Schensted algorithm to π and show each step of the process in the table below.

π	5	8	2	7	3	4	6	1
P_i	$\begin{ytableau} 5 \end{ytableau}$	$\begin{ytableau} 5 & 8 \end{ytableau}$	$\begin{ytableau} 2 & 8 \\ 5 \end{ytableau}$	$\begin{ytableau} 2 & 7 \\ 5 & 8 \end{ytableau}$	$\begin{ytableau} 2 & 3 \\ 5 & 7 \\ 8 \end{ytableau}$	$\begin{ytableau} 2 & 3 & 4 \\ 5 & 7 \\ 8 \end{ytableau}$	$\begin{ytableau} 2 & 3 & 4 & 6 \\ 5 & 7 \\ 8 \end{ytableau}$	$\begin{ytableau} 1 & 3 & 4 & 6 \\ 2 & 7 \\ 5 \\ 8 \end{ytableau}$
Q_i	$\begin{ytableau} 1 \end{ytableau}$	$\begin{ytableau} 1 & 2 \end{ytableau}$	$\begin{ytableau} 1 & 2 \\ 3 \end{ytableau}$	$\begin{ytableau} 1 & 2 \\ 3 & 4 \end{ytableau}$	$\begin{ytableau} 1 & 2 \\ 3 & 4 \\ 5 \end{ytableau}$	$\begin{ytableau} 1 & 2 & 6 \\ 3 & 4 \\ 5 \end{ytableau}$	$\begin{ytableau} 1 & 2 & 6 & 7 \\ 3 & 4 \\ 5 \end{ytableau}$	$\begin{ytableau} 1 & 2 & 6 & 7 \\ 3 & 4 \\ 5 \\ 8 \end{ytableau}$

Thus, $P(\pi) = \begin{ytableau} 1 & 3 & 4 & 6 \\ 2 & 7 \\ 5 \\ 8 \end{ytableau}$ and $Q(\pi) = \begin{ytableau} 1 & 2 & 6 & 7 \\ 3 & 4 \\ 5 \\ 8 \end{ytableau}$.

Once the algorithm is complete, pair (P, Q) is uniquely determined by π. Furthermore, π can be recovered from P and Q.

Theorem 4.2.4. *[40, 42] The map*

$$\pi \xrightarrow{\mathsf{RS}} (P(\pi), Q(\pi))$$

is a bijection between elements of S_n and pairs of standard tableaux of the same shape $\lambda \vdash n$.

The Robinson–Schensted algorithm produces Knuth equivalence classes as follows.

Theorem 4.2.5. *[28] If $\pi, \sigma \in S_n$, then*

$$\pi \overset{K}{\cong} \sigma \Leftrightarrow P(\pi) = P(\sigma).$$

There is a very natural permutation π_P such that $P(\pi_P) = P$. Define the *row word of P* to be the permutation π_P formed by reading the entries of P from bottom to top and left to right. The row word of Example 4.2.3 is $\pi_P = 85271346$. It is easy to verify that π_P indeed gives the desired P tableau.

Proposition 4.2.6. *[8] If π and π' have insertion tableaux $P(\pi)$ and $P(\pi')$ of the same shape λ then their Knuth equivalence graphs are isomorphic, and denoted G_λ.*

Our goal is to apply the theory in Section 4.3 and examine the structure of Knuth equivalence graphs.

4.3 CAT(0) Cubical Complexes and Posets with Inconsistent Pairs

Informally, a *reconfigurable system* is a collection of states with a set of reversible moves that are used to navigate from one state to another. These moves are tethered to particular states and can only be used to traverse back and forth between them. Moves are *commutative* if they are physically independent of one another, and thus can be done simultaneously. A more complete formalization of a reconfigurable system can be found in [1, 25].

Definition 4.3.1. [1, 25] A *cubical complex* X is a polyhedral complex formed by joining cubes of various dimensions such that the intersection of any two cubes is a face of both.

Definition 4.3.2. The *state complex* $\mathcal{S}(\mathcal{R})$ of a reconfigurable system $\mathcal{R}$ is a cubical complex whose vertices correspond to the states of $\mathcal{R}$. There is an edge between two states if they differ by an application of a single move. The k-cubes are associated to k-tuples of commutative moves.

Remark 4.3.3. The 1-skeleton of $\mathcal{S}(\mathcal{R})$ is the *transition graph* $\mathcal{T}(\mathcal{R})$, a graph whose vertices are the states of the system and whose edges correspond to the permissible moves between them.

Definition 4.3.4. A metric space X is said to be CAT(0) if:

- there is a unique geodesic (shortest) path between any two points in X, and
- X has non-positive global curvature.

The second property of being CAT(0) can be described as follows. Let X be a metric space with a unique geodesic (shortest) path between any two points. Consider a triangle T in X with side lengths a, b, and c, and construct a comparison triangle T' with the same lengths in Euclidean space. If every chord in the comparison triangle T' is of equal or greater length than the corresponding chord in T (in Figure 4.1, $|xy| \leq |x'y'|$), for every triangle T in X, then we say that X is CAT(0).

There are several characterizations of being CAT(0). We utilize the characterization given by Ardila–Owen–Sullivant [5] given in terms of partially ordered sets with inconsistent pairs (PIPs).

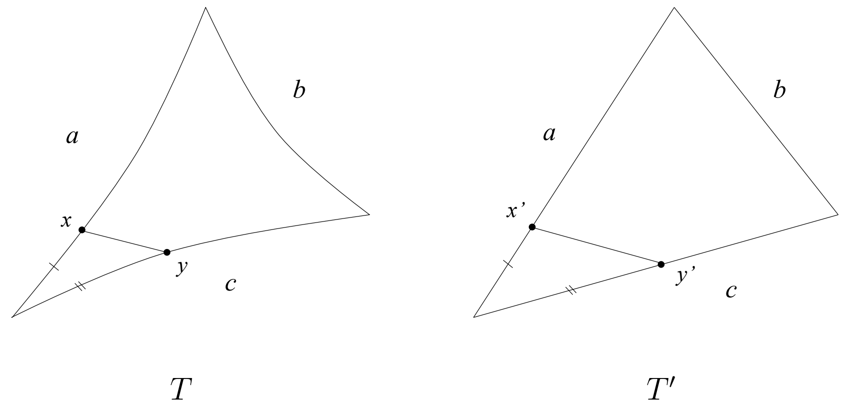


Figure 4.1: The CAT(0) property: X has non-positive global curvature.

Definition 4.3.5. If X is a CAT(0) cubical complex and v is any vertex of X, then (X, v) is a *rooted CAT(0) cubical complex* rooted at v. This can be thought of as identifying a home state if the cubical complex is a *state complex*.

Definition 4.3.6. A *poset with inconsistent pairs (PIP)* is a locally finite poset P of finite width, together with a collection of inconsistent pairs $\{p, q\}$, such that:

- If p and q are inconsistent, then there is no r such that $r \geq p$ and $r \geq q$.
- If p and q are inconsistent and $p' \geq p$ and $q' \geq q$, then p' and q' are inconsistent.

The corresponding *Hasse diagram* of a PIP is constructed by taking the poset and appending a dotted line between minimal inconsistent pairs. An *order ideal* of P is a subset I of P such that if $a \leq b$ and $b \in I$ then $a \in I$. A *consistent order ideal* is an order ideal that contains no inconsistent pairs. An *antichain* is a collection of elements in the poset such that any pair of these elements is incomparable.

We provide the following definition which describes the relationship between PIPs and cubical complexes.

Definition 4.3.7. If P is a poset with inconsistent pairs, we construct the *cube complex of* P, which we denote $X(P)$. The vertices of $X(P)$ are identified with the consistent order ideals of P. There will be a cube $C(I, M)$ for each pair (I, M) of a consistent order ideal I and a subset $M \subset I_{max}$, where I_{max} is the set of maximal elements of I. This cube has dimension $|M|$, and its vertices are obtained by removing from I the $2^{|M|}$ possible subsets of M. The cubes are naturally glued along their faces according to their labels.

Remark 4.3.8. When P has no inconsistent pairs, this is precisely the bijection between posets P and distributive lattices $J(P) = L$. To recover P from $L = J(P)$, we consider the poset of join-irreducibles of L.

See Figure 4.2 for an example of a tableau, its cubical complex $X(P)$, and the associated PIP P.

Theorem 4.3.9. *[5] The map $P \to X(P)$ is a bijection between posets with inconsistent pairs and rooted CAT(0) cube complexes.*

Theorem 4.3.9 provides a method of proving a cubical complex has the desirable CAT(0) property, namely, by constructing the associated PIP after choosing a root for the cubical complex.

4.4 Knuth Equivalence Graphs

In this section we will prove that the only tableau whose equivalence graph G_λ is the 1-skeleton of a CAT(0) cubical complex is the hook, namely for $\lambda = (n-k, 1^k)$. Equivalently, we will show that when λ contains $(2,2)$ then the 1-skeleton of G_λ is not CAT(0).

Definition 4.4.1. Define a *hook* to be a partition of the form $(n-k, 1^k)$.

Note that we need only consider hooks $\lambda = (n-k, 1^k)$ for $k \leq \lfloor \frac{n}{2} \rfloor$ since conjugate partitions produce isomorphic equivalence graphs, as stated in the following proposition.

Proposition 4.4.2. *[8] Given partition λ and its conjugate λ', then*

$$G_\lambda \cong G_{\lambda'}.$$

For completeness, we provide a proof of Proposition 4.4.2, which utilizes the following theorem.

Theorem 4.4.3. *[42] If $P(\pi) = P$, then $P(\pi^r) = P^t$, where t denotes transposition and π^r is the reverse reading word of π.*

Proof of Proposition 4.4.2. Let π produce tableau P of shape λ. By Theorem 4.4.3, the conjugate λ^t is the shape of the transposed tableau P^t for π^r. Moreover, $\pi \overset{1,2}{\cong} \omega$ if and only if $\pi^r \overset{1,2}{\cong} \omega^r$. Thus, relabeling the nodes π of G_λ with π^r preserves its graphical structure. $\square$

We first establish that if λ contains shape $(2,2)$ then the 1-skeleton of G_λ is not CAT(0).

Theorem 4.4.4. *Let the shape λ contain $(2,2)$. Then C_λ is not a CAT(0) cubical complex.*

Proof. Assume the shape λ contains $(2,2)$ and is canonically filled so that the first row contains numbers 1 through k_1 in increasing order, the second row is filled with numbers k_1+1 to k_2 in increasing order, etc. Consider the row word π of λ, which ends by adjoining row 2 and then row 1 in that order. Then π has the following entries in this order: $k_2-1, k_2, 1, 2$. This means G_λ will have a double edge connecting the vertices $\pi = \ldots k_2-1, k_2, 1, 2 \ldots$

and $\pi' = k_2 - 1, 1, k_2, 2 \ldots$ because both Knuth moves of type 1 and 2 can be performed on π to produce the same adjacent permutation π'. Note these moves are dependent and therefore do not commute. This implies the edges form a hole and so there are two shortest geodesics between π and π'. Since C_λ is not a space with unique geodesics, then it cannot be CAT(0). $\square$

We will now prove that when λ is a hook then C_λ is a CAT(0) cubical complex.

Theorem 4.4.5. *When λ is a hook, the graph G_λ is the 1-skeleton of a CAT(0) cubical complex.*

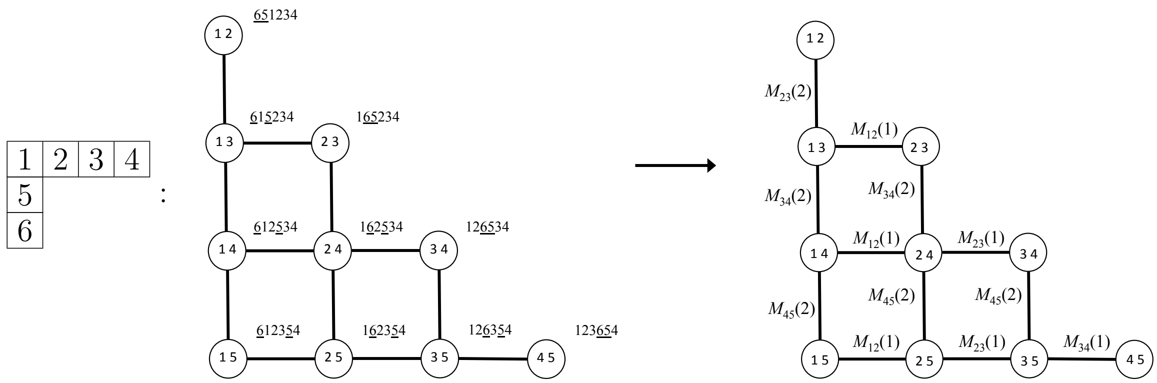
Proof. We begin by describing G_λ in terms of permutations. Consider the permutation $\pi = n, n-1, \ldots, n-k+1, 1, 2, ..., n-k$ which, after applying the Robinson–Schensted algorithm, has insertion tableau $P(\pi)$ of shape $\lambda = (n-k, 1^k)$, a hook. Notice a series of Knuth moves beginning with π effectively pushes entries $\{n, n-1, \ldots, n-k+1\}$ to the right. As consecutive integers will never be swapped by a Knuth move, it follows that the Knuth equivalence class generated by π consists of the permutations that have the integers $\{n, n-1, \ldots, n-k+1\}$ in decreasing order and the integers $\{1, 2, \ldots, n-k\}$ in increasing order. Thus, these determine the vertices of G_λ where there is an edge between vertices π and σ when $\pi \overset{1,2}{\cong} \sigma$.

We now will describe G_λ in terms of sequences. From above it follows that the vertices of G_λ can be uniquely determined by the positions of the integers $n, n-1, \ldots, n-k+1$ in the associated permutation. With this observation we introduce a new labeling of each vertex of G_λ to be the index sequence of the positions of the integers $n, n-1, \ldots, n-k+1$ in the corresponding permutation. See Figure 4.2(a) for an example of this labeling. Recall an edge between vertices means a Knuth swap occurred; that is, an integer from $n, n-1, \ldots, n-k+1$ moved from position i to $i+1$. In terms of this new labeling, this means two vertices share an edge when the corresponding sequences have a difference of 1 in only one position.

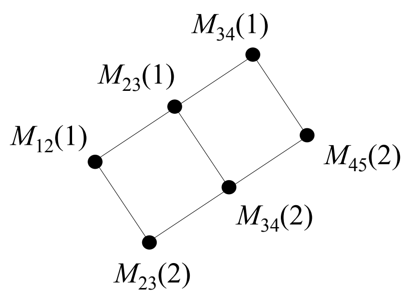
Next we define a poset structure L_λ on the vertices of G_λ and prove it is a distributive lattice. The sequence description of G_λ produces a natural component-wise ordering on its vertices. For sequences $s = (i_1, i_2, \ldots, i_k)$ and $s' = (j_1, j_2, \ldots, j_k)$ in L_λ, we say $s \leq_{L_\lambda} s'$ if $i_r \leq j_r$ for $1 \leq r \leq k$. Define the function max: $L_\lambda^t \to L_\lambda$ to be the component-wise maximum. Define the function min similarly. Since L_λ is finite, then max and min are well-defined. Moreover, since max and min are distributive on each component, it follows that they are distributive on L_λ as well. Thus, $(L_\lambda, \leq_{L_\lambda})$ is a distributive lattice.

By Birkhoff's representation [50] theorem there exists a poset of order ideals isomorphic to L_λ. We now construct the poset P_λ such that $L_\lambda \cong J(P_\lambda)$. We construct P_λ by describing the join-irreducible elements of L_λ. The order on P_λ will follow from the component-wise order on L_λ.

Consider the following edge labeling of G_λ. The edge between s' and s is labeled $M_{i,i+1}(l)$ to indicate that position l is the location where s' and s differ by 1. As these edges correspond to Knuth moves in G_λ, we will refer to the label $M_{i,i+1}(l)$ as a *move*. See Figure 4.2(b) for an



(a) A vertex labeling by index sets and an edge labeling by Knuth moves for C_λ where $\lambda = (4, 1^2)$



(b) The PIP P_λ constructed from edge labels of C_λ for $\lambda = (4, 1^2)$, such that the order ideals of P_λ correspond to vertex labels of C_λ.

Figure 4.2: Given $\lambda = (4, 1, 1)$, we construct C_λ and the associated PIP P.

example of this labeling. Since λ is a hook, there are no double edges in G_λ so this labeling is well-defined.

The join-irreducible elements of L_λ are those that have a unique cover relation. For a sequence this means there is only one entry that can be decreased by 1 and produce a sequence still in L_λ. In particular, they are of the form

$$s = (1, 2, \ldots, l, j, j+1, \ldots, j+m) \text{ such that } l \geq 0,\ j+m < n,\ j > l+1 \text{ and } \ l+m = k-1.$$

When s has a unique cover relation, we identify it with the move $M_{j-1,j}(l+1)$. Define P_λ to be the set of moves $\{M_{j-1,j}(l+1)\}$ associated with the join-irreducible elements $s \in L_\lambda$ with the order induced by L_λ. By construction,

$$M_{j-1,j}(l+1) \in P_\lambda \text{ for } 0 \leq l \leq k-1 \text{ and } l+2 \leq j \leq l+k+2.$$

It follows from the component-wise order on L_λ that the order on P_λ is

$$M_{i-1,i}(l+1) \leq_{P_\lambda} M_{j-1,j}(m+1) \text{ if } l \geq m \text{ and } i-l \leq j.$$

Thus P_λ is just the product of two chains $(\mathbf{k}) \times (\mathbf{n-k-1})$. Therefore

$$L_\lambda \cong J((\mathbf{k}) \times (\mathbf{n-k-1})).$$

We will now regard P_λ as a PIP with no inconsistent pairs and show that the CAT(0) cubical complex $X(P_\lambda)$ from Definition 4.3.7 is isomorphic to the cubical complex C_λ of G_λ rooted at π. To do this we will first describe explicitly the bijection between the vertices of $X(P_\lambda)$ and those of C_λ which follows directly from Birkhoff's theorem. In particular, the order ideal I generated by the set of moves $\{M_{i,i+1}(l)\}$ corresponds to the following k-sequence $s(I)$. The l-th entry of $s(I)$ is determined by the move that is maximal among all moves in position l in I. For example, consider P_λ as in Figure 4.2(b). The ideal generated by $\{M_{23}(1)\}$ is $I = \{M_{23}(2), M_{34}(2), M_{12}(1), M_{23}(1)\}$. Move $M_{23}(1)$ is maximal among all $M_{i,i+1}(1) \in I$ and $M_{34}(2)$ is maximal among all $M_{i,i+1}(2) \in I$. Thus, the corresponding vertex in C_λ is the sequence $s(I) = (3, 4)$. Similarly, $I = \{M_{12}(1), M_{23}(2), M_{34}(2)\}$ gives $s(I) = (2, 4)$.

To complete the proof, we give an explicit bijection between d-dimensional cubes of $X(P_\lambda)$ and d-dimensional cubes of C_λ. By Definition 4.3.7, this equates to giving a bijection between a set of subideals of a consistent ideal of P_λ and a set of vertices that form a d-cube in C_λ. Since all ideals of P_λ are consistent, consider any $I \in X(P_\lambda)$. We know I corresponds to a vertex $s(I) = (i_1, i_2, \ldots, i_k)$ of C_λ. Let M be the set of moves determined by the entries of $s(I)$. For example, for $I = \{M_{12}(1), M_{23}(2), M_{34}(2)\}$, we have $s(I) = (2, 4)$ and $M = \{M_{12}(1), M_{34}(2)\}$. Then I is equal to the ideal generated by M. Let $M_{max} = I_{max}$ and $d = |M_{max}|$. Then vertices of a d-cube in $X(P_\lambda)$ are obtained by removing from I the 2^d possible subsets of M_{max}. For $M' \subset M_{max}$, removing M' from I corresponds to decreasing the entries of $s(I)$ in position l for every $M_{j,j+1}(l) \in M'$. So the set of 2^d subsets of I obtained by removing subsets M' corresponds to the set of 2^d vertices of C_λ achieved by decreasing $s(I)$ in all positions determined by M'. This completes the bijection and concludes the proof. $\square$

Theorems 4.4.4 and 4.4.5 can be combined and restated in the following theorem.

Theorem 4.4.6. *The graph G_λ is the 1-skeleton of a CAT(0) cubical complex if an only if λ is a hook.*

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