Numbers vs. Variables: The Effect of Symbols on Students' Math Problem-Solving

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Abstract

Numbers and variables often follow the same principles of arithmetic operations, yet numbers can be computed to a value whereas variables cannot. We examined the effect of symbols—numbers versus variables—on middle school students' problem-solving behaviors in a dynamic algebra notation system by presenting problems in numbers (e.g., 3+5−3) or variables (e.g., $x+y-x$). We found that compared to problems presented in numbers, students attempted the problems more times and took more total steps when the problems were presented in variables. We did not find differences in pre-solving pause time or strategy efficiency on the two types of problems, indicating that students might notice problem structure in both types of problems. The results have implications for research on cognitive processes of symbols as well as the design of educational technologies.

Keywords: variables, algebra, arithmetic, problem-solving, mathematical structure, online learning

Introduction

Abstract symbols are cognitive tools that support reasoning, problem-solving, and higher-level thinking (Koedinger et al., 2008; Vygotsky, 1978). However, children's struggles with abstract symbols are documented across development and domains. In mathematics, infants as young as six months of age can perceive differences in the quantity of sets of objects (e.g., 4 vs. 8; Xu & Spelke, 2000), yet the process of mapping sets of concrete objects (e.g., 4 cookies) to abstract number words or symbols (e.g., "four", "4") is protracted. Although two-year-olds can recite the count list, children may take another three years to reliably map number words to sets of objects (Gelman & Meck, 1983). In science, providing undergraduate students physical experimentations with concrete models before transitioning to virtual simulations better promote learning compared to the physical experimentations alone (Zacharia, 2007). These findings suggest the importance of abstract symbols yet students continually struggle with them.

Fyfe et al. (2014) propose the concreteness fading theory, and recommend "beginning with concrete materials and then explicitly and gradually fading to the more abstract (p. 9)". Doing so may help students interpret abstract symbols, ground them in physical experiences, and distill the generalizable properties. Building on this theory, we investigate the effects of numbers and variables on students' algebra problem-solving. Focusing on students' problem-solving processes, we move beyond the traditional approaches that focus on the answer correctness to explore how symbols impact the microstructure of problem-solving.

Algebra is an important building block for success in higher-level mathematics, yet many students struggle to learn algebra (Matthews & Farmer, 2008). One defining challenge is the shift from numbers to variables. Variables are letters that represent a range of unknown values and denote a systematic relation within expressions (Küchemann, 1981). While this shift from numbers to variables can be challenging for students, much research has found that using algebraic symbols in earlier years can promote a smooth transition (Blanton & Kaput, 2005; Carpenter et al., 2005). For instance, prior work has highlighted the instructional and developmental sequence transitioning students from arithmetic to algebra, such as using concrete numbers to support thinking about abstract unknown variables (Fyfe et al., 2014; Koedinger & Anderson, 1998). Although students are introduced to some concepts of algebra at a young age, including equivalence, rarely are they asked to work with variables before sixth grade (CCSS, 2010). Consequently, less is known about how variables may help students reason about algebraic structures and arithmetic principles.

We hypothesize that variables may help middle schoolers avoid the impulse to calculate and provide them with opportunities to notice structures within algebraic equations. Although variables have typically been obstacles for most students (Kaput, 1998; Koedinger et al., 2008), this approach could potentially help bridge the gap between arithmetic and algebra by guiding students' attention to the structure of the mathematical equations.

Students' Struggle with Variables

Arithmetic is a foundation for algebra. Arithmetic relies on numbers that represent specific, known, and concrete values. Building on arithmetic, algebra introduces the use of variables, representing unknowns that cannot be simplified to a numerical value. Students often struggle to solve problems involving variables because solving these problems requires them to understand the meaning of variables, operate with unknowns, and make explicit relations between the unknown and the numbers (Malisani & Spagnolo, 2009; Philipp, 1992).

Although the transition from arithmetic to algebra can be challenging, arithmetic and algebra share many underlying principles that are consistent across the use of symbols in mathematical notation (Booth, 1981, 1984). In arithmetic, students operate with numbers and apply mathematical properties (e.g., commutativity, associativity, distributivity) to transform or simplify expressions. Students vary in their conceptual understanding of these properties when operating with numbers (Robinson et al., 2018), and this understanding is an important indicator of flexible problem-solving in arithmetic. Although students can benefit from leveraging these properties and noticing the structures of the algebraic equations (Kieran, 1989), this conceptual understanding often do not transfer (Robinson et al., 2006). Further, whereas students can simply compute using numbers, they have to rely on their understanding of these properties as the option to compute becomes more restricted with variables.

Mathematical Structures and Strategy

Noticing structures is an important foundation for algebra learning (Kaput, 1998; Venkat et al., 2019). There are two kinds of "structure", surface and systemic structures (Kieran, 1989). Surface structure refers to how the terms and operands are presented within an expression to create mathematically valid options for computing. For instance, "$3 + 5 - 3$" has the surface structure of 3 on the left, +5 in the middle, and -3 on the right. Systemic structure refers to the underlying properties, such as commutativity, associativity, and distributivity, within the expression. The systemic structure of "$3 + 5 - 3$" involves recognizing the inverse relation between 3 and -3, and applying the commutative property to simplify the expression to 5. Recognizing systemic structures can help students apply efficient problem-solving strategies.

Understanding and noticing the systemic structure is an important aspect of flexible and efficient equation-solving (Rittle-Johnson & Star, 2007; Schneider et al., 2011), and a primary goal in mathematics education (CCSS, 2010). However, in numerical expressions, students can rely on the surface structure instead of leveraging systemic structure (Newton et al., 2020). For instance, in "$3 + 5 - 3$", students may apply the computation from left to right in response to the surface structure ($3 + 5 - 3 = 8 - 3$, $8 - 3 = 5$), and simplify the expression in two steps. Alternatively, leveraging the systemic structure, students can combine 3 and -3 to efficiently reach 5 in one step. Although students can use the surface structure to compute and simplify numbers in numerical expressions, the same approach does not apply to algebraic expressions involving variables. Instead, in "$x + y - x$", students need to notice the systemic structure and combine x with $-x$ in order to isolate y. As students progress from arithmetic to algebra, the need to notice systemic structure becomes more poignant.

Prior work suggested that the ability to use efficient equation-solving strategies require relevant content knowledge and attention to structures (Rittle-Johnson & Star, 2007; Schneider et al., 2011; Xu et al., 2017). Further, the amount of time that students pause before enacting their first step positively predicts the strategy efficiency, suggesting that pause time may provide a window for students to notice the systemic structure and in turn use more efficient strategies (Chan et al., 2020). Here, we examine the effect of symbols on problem-solving by compare middle schoolers' behaviors on problems presented in numbers versus variables.

The Current Study

To examine how numbers and variables impact students' problem-solving, we designed problems that were similar in systemic structure but varied in whether they were presented with numbers (e.g., $3 + 5 - 3$) or variables (e.g., $x + y - x$). We present these problems in From Here to There! (FH2T), a web-based interactive game that builds on a dynamic algebra notation system (graspablemath.com; Weitnauer et al., 2016) to allow real-time manipulation of algebraic symbols on the screen. Using the log data within this system, we examine the microstructure of students' problem-solving behavior. Specifically, we test whether presenting expressions in variables as opposed to numbers helps students suppress the impulse of performing computations leading to a longer pause time before their first action (RQ1). Further, we examine whether students struggle with expressions presented in variables as opposed to numbers by comparing the number of problem-solving attempts (RQ2) and steps students made prior to completing the problems (RQ3). Finally, we explore whether students' strategy efficiency on their final solution, as measured by their solution steps, vary when expressions were presented in variables as opposed to numbers (RQ4). Investigating how students' pause times, attempts, steps, and strategy efficiency vary by problem features extends prior work on the relation between pause time and strategy efficiency (Chan et al., 2020) and demonstrates how symbols impact aspects of student behaviors during problem-solving.

Based on the prior literature, we hypothesize that when expressions are presented in variables as opposed to numbers, students may pause longer prior to taking their first action (H1). They may also struggle more and consequently take more attempts (H2) and steps (H3) to complete the problem presented in variables. We do not have a directional hypothesis on students' strategy efficiency (H4). While students may struggle less and be more efficient at solving problems presented in numbers, the variables may force students to notice the systemic structure of the expression leading to a decrease of unnecessary computations and a more efficient solution strategy that involve fewer steps.

Methods

Participants

The sample was drawn from a larger study conducted in Fall 2019. The aim of the larger study was to improve students' algebra performance through educational technologies. Students were randomly assigned to one of two technology conditions at the student level within their classrooms. All students worked on their assigned technology at their own pace on their own device as a part of their regular instruction.

Here, we focused on the 125 middle schoolers who received FH2T as their technology intervention and completed the four focal problems within FH2T. We designed the four focal problems to address our research questions regarding the effects of variables and numbers on students' problem-solving.

These 125 students (57 girls) were recruited from four schools in a large urban district in the Southeastern United States. The majority of students were in sixth grade (98.4%), and the remaining were in seventh grade (1.6%). Most students were in advanced mathematics class (96%), and the remaining were in on-level (2.4%) or support (1.6%) classes. In terms of race, 64.8% were Asian, 26.2% were White, 4.1% were Hispanic or Latino, 2.5% were multiracial, 1.6% were American Indian or Alaska Native, and 0.8% were Black.

Procedure

All students began by completing a pretest on algebra knowledge. After, students completed four 30-minute sessions using their assigned technology; they had two weeks to complete each intervention session. Then, they completed a posttest on algebra knowledge that mirrored the pretest. All study assessments and assignments were administered online in mathematics classrooms during instructional periods.

We used the log data within FH2T and the algebra pretest for our research questions. The results on the efficacy of the learning technologies are reported in Chan et al. (2021), thus we only describe the tasks relevant to the current study.

From Here to There! (FH2T)

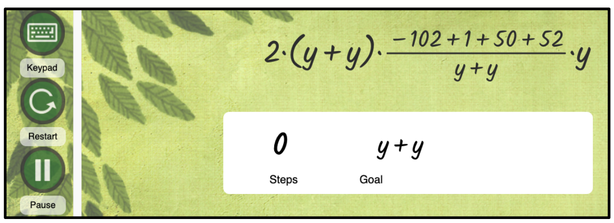

Figure 1: A sample problem in From Here to There!.

In FH2T (https://graspablemath.com/projects/fh2t), students were presented with an initial expression at the top and a mathematically equivalent goal at the bottom in the white box (Figure 1). The objective was to transform the expression into the goal using a series of gesture-actions (e.g., tapping or dragging) that applying mathematical operations and properties. As an example (Figure 1), a student could drag and distribute the 2 inside the parentheses to transform the expression into $2y + 2y \cdot \frac{-102+1+50+52}{y+y} \cdot y$. All student actions were counted and reflected as the number of steps in the white box, informing students of how many steps they have taken to solve the problem. If students were stuck, they could reset the expression to the starting state and reattempt problems as many times as needed.

Because all student actions and the corresponding transformations were time-stamped and recorded, we could systematically and quantitatively compare students' equation-solving processes in ways not accessible in answer-based learning systems or paper-and-pencil tasks (Figure 2). Furthermore, students could take any series of mathematically valid steps that link the initial expression and the goal. The activity thus provided an ideal context for examining variation in algebra problem-solving processes.

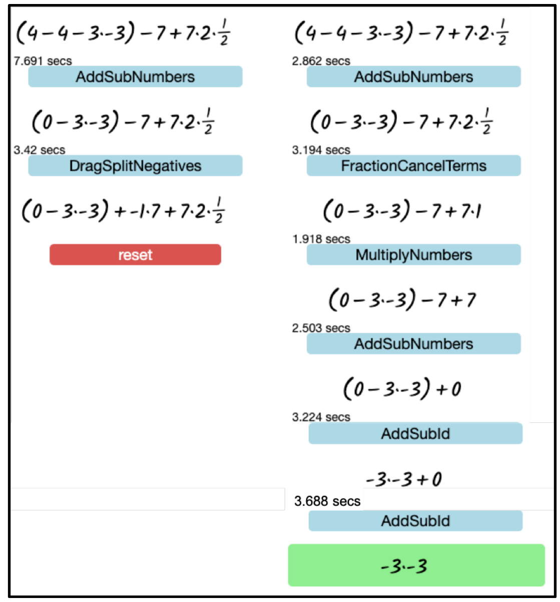

Figure 2: A visualization of the data on a student's problem-solving process for problem 2-Number.

All students who received FH2T as their intervention in the larger study worked on problems in the same order starting from basic arithmetic operations to more complex topics, such as fractions, distributions, and algebraic equations. We designed and embedded two pairs of problems to directly compare student behaviors on problems with variables versus numbers. Within each pair, problems shared a similar systemic structure of the starting expression and the goal; the paired problems varied on whether the symbols were numbers or variables (Table 1). Both pairs of problems required six steps as the optimal solution strategy—cancelling the opposite, cancelling the inverse, and simplifying the 0 and 1. For the primary analyses, we dummy coded the numerical problems as 0 and variable problems as 1 in order to test the effect of variables as opposed to numbers on students' problem-solving.

Table 1: The paired problems with a similar structure presented in either numbers or variables.

Pair	Initial Expression	Goal
1-Number	$2+2\left(\frac{2+2\cdot 2-2}{2}\right)-2$	$2\cdot 2$
1-Variable	$-z+y\left(\frac{x-a\cdot -a-x}{y}\right)+z$	$-a\cdot -a$
2-Number	$(4-4-3\cdot -3)-7+7\cdot 2\cdot \frac{1}{2}$	$-3\cdot -3$
2-Variable	$\frac{1}{b}\cdot b\cdot(-b+b+b\cdot b)+b-b$	$b\cdot b$

Measures

Algebra Knowledge Assessment at Pretest Students' algebra knowledge was assessed with 10 items selected from a validated measure (Star et al., 2014; Cronbach's $\alpha = .89$). The 10 items were selected because they measured aspects of students' algebra knowledge that were relevant to the intervention. Two sample items were as follows: solve for y in $5(y - 2) = 3(y - 2) + 8$, and identify the expressions that are equivalent to $4(n + 3)$. Each item was scored as correct (1) or incorrect (0). The total score on this assessment was included as a covariate in all primary analyses.

Measures in FH2T All student actions were recorded for each FH2T problem. The five variables described below were computed based on the log data.

Pause Time We computed the number of seconds students spent before taking their first step—a valid transformation—on their first attempt of each problem. This value represented the number of seconds from when the problem first appeared on the screen to when students took their very first step. As an example, the pause time in Figure 2 was 7.691 seconds. We used the pause time as the dependent variable for RQ1 in our analyses.

Attempt Count We computed the attempt count using the number of times students reset the problem. The attempts represented the number of times students tried to solve the problem. For example (Figure 2), the student reset the problem once, thus attempted the problem twice. We used the attempt count as the dependent variable for RQ2 in our analyses.

Total Step Count We computed the total number of steps students took on the problems. The total step count consisted of all the steps students took from when the problem first appeared on the screen to when the expression matched the goal. As an example, the student in Figure 2 took a total of eight steps on the problem. We used the total step count as the dependent variable for RQ3 in our analyses.

Final Solution Step Count We computed the number of steps students took to reach the goal on their final, successful attempt. If students completed the problem on their first attempt, the final solution step count included all the steps they took on the problem. If students reset and re-attempted the problem, the final solution step count only included the number of steps they took from when they last reset the problem to reaching the goal. As an example, the final solution step count in Figure 2 was six steps. We used the final solution step count as the dependent variable for RQ4 in our analyses.

Exploratory Step Count Prior to the Final Attempt On the problems where students did not reach the goal on their first attempt, we computed the number of steps they took from when the problem first appeared on the screen to when they last reset the problems. We classified these actions leading up to their final attempt as exploratory steps. Exploratory step count represents the amount of actions students took prior to their final successful attempt of solving the problem. For example (Figure 2), the student took two exploratory steps before they reset the problem and solved it on their next attempt. We used the exploratory step count as a covariate for RQ4. Because students who took more exploratory steps had more opportunities to practice with the problems and might use more efficient strategies involving fewer steps on their final solution, we included exploratory steps as a covariate to account for the potential practice effects on students' final strategy efficiency.

Results

Prior to addressing the research questions, we conducted descriptive analyses to examine the distribution of the algebra pretest scores and the measures in FH2T for each focal problem. Because students could take as many attempts and steps as needed to reach the goal, we used the interquartile-range methods to replace outliers (Walfish, 2006). This method extracted the top and bottom 25% values from the data. Within these two quartiles, the values that were beyond 1.5 times the interquartile range were considered as outliers. These values were then replaced with either the fifth or ninety-fifth percentile observation value. This method allowed us to retain all participants in the analyses while avoiding the results being distorted by the influential cases.

To address our research questions, we conducted a series of mixed-effect linear regression models on the two pairs of problems using the *lme4* package (Bates et al., 2015) with maximum likelihood estimation in R. In each model, we included students as a random effect to account for the repeated measures (i.e., all students completed all four focal problems), the algebra knowledge score as a covariate, and the symbol of the problem (variables vs. numbers) as the focal predictor. To address our first research question, we conducted a mixed-effect linear model with pause time as the dependent variable. Next, we replaced pause time with attempt count or total step count as the dependent variable to address our second and third research question, respectively. Finally, we used the final solution step count as the dependent variable for the fourth research question. We included exploratory step count as an additional covariate to examine whether symbols (number vs. variable) impacted students' final strategy efficiency above and beyond the practices they had with the problem. To aid the interpretation of the results, all covariates (i.e., algebra knowledge score, exploratory step counts) were mean-centered in all models.

Descriptive Analysis

On average, students scored 7.18 points ($SD = 2.22$) on the algebra knowledge assessment, indicating that the scores were not subject to ceiling or floor effects. A total of 500 problems were completed by the 125 students (each student completed four focal problems). See the descriptive statistics (Table 2) and the data distribution (Figure 3) below.

Table 2: Means and (standard deviations) of the FH2T measures on each problem.

	Pair1		Pair 2	
	Number	Variable	Number	Variable
Pause time	9.08 (6.47)	8.90 (6.65)	8.96 (6.81)	8.67 (6.10)
Attempts	1.51 (1.15)	3.36 (1.75)	1.57 (1.04)	2.62 (2.00)
Total Steps	8.92 (4.94)	19.41 (8.71)	11.78 (6.15)	15.45 (10.00)
Solution Steps	6.94 (1.57)	7.36 (1.51)	8.46 (1.75)	8.17 (2.00)
Exploratory Steps	1.87 (4.52)	11.96 (8.56)	3.29 (6.31)	7.42 (10.24)

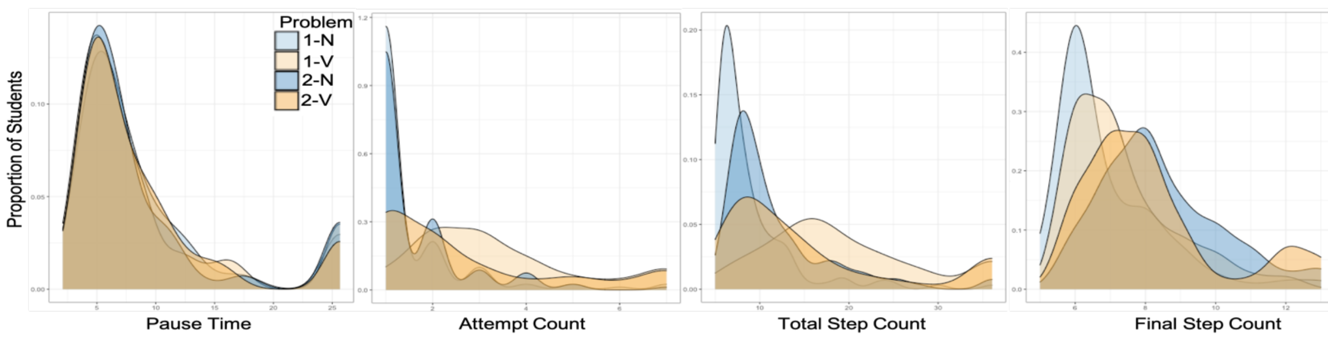

Figure 3: Density plots of the dependent variables by problem.

Table 3: Fixed effect estimates (standard errors) in the mixed-effect linear regression models.

Dependent variable		Model 1: Pause	Model 2: Attempts	Model 3: Total steps	Model 4: Final steps
Pair 1	Intercept	9.49 (0.62)***	1.52 (0.14)***	8.74 (0.66)***	6.91 (0.15)***
	Algebra	0.33 (0.21)	-0.02 (0.05)	-0.26 (0.21)	-0.14 (0.05)**
	Exp. Steps	--	--	--	0.002 (0.01)
	Variable	-0.15 (0.72)	1.85 (0.17)***	10.49 (0.87)***	0.40 (0.23)
Pair 2	Intercept	8.98 (0.60)***	1.59 (0.15)***	11.92 (0.77)***	8.45 (0.18)***
	Algebra	-0.07 (0.19)	-0.02 (0.05)	-0.25 (0.25)	-0.14 (0.06)*
	Exp. Steps	--	--	--	-0.27 (0.20)
	Variable	-0.29 (0.80)	1.06 (0.19)***	3.67 (1.00)***	-0.01 (0.01)

Note. Algebra = Algebra knowledge score; Exp. steps = Exploratory step count, Variable = Symbol (variable = 1, number = 0).* indicates $p < .05$; ** indicates $p < .01$; *** indicates $p < .001$

Primary Analyses

All primary analyses were first conducted with the first pair of problems, then repeated on the second pair of problems. Because the pattern of the results was consistent between the two pairs of problems, we summarized the findings below and presented the results in Table 3.

A generalized mixed-effect model revealed that students' pause time before first action did not differ whether the problems were presented in variables or numbers (Model 1). Next, we found that students attempted the problems 1.85 (Pair 1) or 1.06 (Pair 2) more times when they were presented in variables as opposed to numbers (Model 2). Students took 10.49 (Pair 1) or 3.67 (Pair 2) more steps on problems presented in variables as opposed to numbers (Model 3). Finally, students' final solution steps did not differ by symbols, but students with higher algebra knowledge tended to take fewer steps on their final solution. Specifically, for a student with the average score on the algebra knowledge assessment, a one-point increase in the assessment was associated with 0.14 steps decrease in final solution steps (Model 4).

Discussion

The aim of this study is to examine how variables and numbers impact students' algebra problem-solving behaviors. We address the aim by testing the effect of presenting problems in variables as opposed to numbers on aspects of students' behaviors. Our findings provide insights into ways in which symbols—specifically variables and numbers—influence students' problem-solving. The results have implications for future research on algebra problem-solving, and instructional practices on guiding students' attention to the systemic structures of expressions within learning technologies.

First, contrary to our hypothesis, students do not pause longer on problems presented in variables as opposed to numbers. On average, they pause for approximately nine seconds prior to taking their first step. The pause time is longer than that of guessing (five seconds; Kong et al., 2007), suggesting that students may be thinking and planning during pause time (Chan et al., 2020; Welsh et al., 1995). The comparable pause time suggests that presenting problems in variables does not force students to pause longer, inhibit the impulse to calculate, or notice the systemic structures.

Next, students made more attempts and steps on problems presented in variables as opposed to numbers. Although these actions may indicate explorations within the context of an intervention, they may also indicate student struggles. Students can explore all problems as much as they want in FH2T, but they tend to solve problems presented in numbers in one attempt, suggesting that they may be solving problems efficiently rather than exploring. The higher attempt and step counts in variable problems thus may be signs of student struggles, aligning with previous research documenting the difficulties of understanding and solving problems involving variables (Heffernan & Koedinger, 1998; Malisani & Spagnolo, 2009; Philipp, 1992). The findings further suggest some convergence across contexts, and student behaviors in FH2T may providing some insights into their reasonings about algebraic symbols beyond the game.

Extending beyond prior research, we find that students solve problems presented in variables just as efficiently (using a similar number of final solution steps) as problems presented in numbers on their final solution, albeit they take more attempts and steps on variable problems prior to their final solution. Considering the relatively long pause time, low attempt count, and high strategy efficiency, as measured by the final solution step count, we posit that students may notice the systemic structure within the problems presented in numbers. Although students pause for a similar amount of time on variable problems, they took more attempts and steps prior to using an efficient final solution strategy, indicating that presenting problems in variables do not help students notice the systemic structures but interacting with problems may help. The findings suggest that presenting problems in non-calculatable variables may not be sufficient in guiding students' attention to systemic structures of problems, but providing them with opportunities to dynamically manipulate algebraic symbols may help. This aligns with the theory of concreteness fading (Fyfe et al., 2014), and demonstrates the importance of explicitly drawing connections between concrete materials and abstract symbols as well as grounding abstract symbols in concrete, perceptual, and physical experiences. Prior work has demonstrated the efficacy of FH2T in improving students' mathematical performance (e.g., Chan et al., 2021; Ottmar et al., 2015). FH2T may be one way to provide students the concrete experiences with the abstract algebraic symbols. Future research should explore *how* FH2T support algebraic reasoning, and ways to explicitly draw connections between numbers and variables.

Several limitations warrant mention. First, given the nature of the intervention, the paired problems are not identical in the problem structure. As an example, the first number problem starts with 2 and ends with -2 whereas the first variable problem starts with $-z$ and ends with $+z$. Further, the former involves all 2s whereas the latter involves other variables. Second, the order in which the problems are presented was not counterbalanced. In both pairs of problems, students received the variable problems followed by the number problems. Third, the majority of the sample was Asian students in advanced mathematics classes, and is not representative of the US population. We are currently conducting a larger randomized controlled trial with middle school students from a large, public school district. When the data become available, we will replicate the current findings with a larger, more diverse sample and with more problems that matched exactly on the systemic structure and counterbalanced on the presentation order. Furthermore, we plan to conduct additional analyses on the visualization of students' problem-solving (e.g., Figure 2) to reveal the sequence of steps that students take. Visualizing students' problem-solving steps will provide insights into whether students leverage the systemic structure of the problem to efficiently reach the goal.

Overall, the study reveals ways in which symbols influence students' problem-solving processes, and contribute novel findings to the literature on the transition between arithmetic and algebra. Going beyond correctness, we investigate students' behaviors in problems presented in variables versus numbers to demonstrate how symbols may impact students' behavior at a microlevel while problem-solving. The findings provide important information for future research examining students' conceptualization of numbers and variables, and how the conceptualization impacts aspects of students' mathematical thinking, learning, and problem-solving. The findings also have implications for designing educational technologies that support algebra problem-solving through drawing explicit connections between symbols and ground abstract variables in concrete, embodied experiences.

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