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Publication Date

1967-05-05

UCRL-17552

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DISCLAIMER

Submitted to The Physical Review

UCRL-17552
Preprint

UNIVERSITY OF CALIFORNIA

Lawrence Radiation Laboratory

Berkeley, California

AEC Contract No. W-7405-eng-48

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ABSTRACT

We have shown that the assumption of maximal analyticity of first degree and fixed-t power behavior of the scattering amplitudes in general imply a lower bound at a fixed angle. The fixed angle lower bound takes the form $\exp[-c_\gamma(z_s)\, s^\gamma \log s]$ where $c_\gamma(z_s)$ and γ are positive. The precise value of γ depends on the specific assumptions on the fixed-t bound of the scattering amplitude. In particular, the assumptions made by Cerulus and Martin correspond to $\gamma = 1/2$, and for the case of a linearly rising trajectory, $\gamma = 1$. Furthermore, we obtain a finite lower bound at $z_s = 0$, which heretofore was given to be zero.

I. INTRODUCTION

Cerulus and Martin[1] have shown that the Mandelstam representation, together with a weak unitarity condition, implies that the scattering amplitude has a lower bound as the energy s increases at fixed center-of-mass scattering angle z_s. They used the finite range of the interaction and the assumption of a polynomial bound to show that $|f(s,z_s)| \geq \exp[-c(z_s)\, s^{\frac{1}{2}} \log s]$. Subsequently, Martin[2] rederived this result under a weaker assumption. The rapid decrease of the differential pp scattering cross section at large momentum transfer led Kinoshita[3] to postulate the principle of "minimal interaction"--that the physical amplitude takes the minimum value consistent with the general requirements of analyticity and unitarity.

Doubts concerning the uniform polynomial bound assumption have been expressed by Cerulus and Martin,[1] and by Mandelstam himself.[4] Martin[2] was able to include the possibility of Regge cuts when the leading branch point in the J-plane does not increase faster than $t^{\frac{1}{2}}$ as the momentum transfer t increases. Recent experiments,[5] however, suggest that Regge trajectories could increase linearly with t. If this trend persits, then the assumptions made by Cerulus and Martin,[1] and by Martin[2] could be too strong.

We shall show that the assumptions of maximal analyticity of the first degree and fixed-t power behavior do indeed imply a lower bound at fixed angle. The fixed-angle-lower bound (FALB) takes the form $\exp[-c_\gamma(z_s)\, s^\gamma \log s]$, where γ is a positive constant and

$c_\gamma(z_s)$ is a positive definite function. The precise value of the constant γ depends on the specific assumptions on the fixed-t bound of the scattering amplitude. In terms of the Regge pole hypothesis, the FALB depends on the behavior of the leading singularity in the J-plane as a function of t. We show that the result of Cerulus and Martin (C-M) is a special case of this general result. Furthermore, we improve the lower bound at $z_s = 0$, which heretofore was given to be zero.[1-3]

In Section II, we set up our mathematical problem and prove a theorem which will be used repeatedly. We shall see that the FALB is closely connected to an angle θ_{max} which determines a domain in the t-plane within which the fixed-t polynomial bound is assumed. We shall reduce the problem formally to that of a theory satisfying uniform fixed-t polynomial bound assumption, where the range of interaction depends on the energy. In Section III, we shall exhibit a potential model with its domain for polynomial bound enlarged as compared to the domain assumed by C-M; consequently that the FALB obtained is higher than that of C-M. In Section IV, we consider relativistic scattering amplitudes. We discuss the additional constraint of simultaneous unitarity in all channels. We show that the C-M result is consistent with this constraint, but is by no means the most general one. In particular, we show that for a linear Regge trajectory, the best FALB we can obtain is $\exp[-c_1(z_s)\ s \log s]$. We shall make some concluding remarks in Section V.

II. A MATHEMATICAL THEOREM

We now show how the assumption of fixed-t power behavior implies a FALB for a scattering amplitude. Consider an analytic function $T(s,t)$ of two complex variables. Let $T(s,t)$ be analytic in t with a branch point at t_0 on the positive real axis. If $T(s,t)$ has fixed-t power behavior in s, we can define a function

$$N(t) = \lim_{s\to\infty} (\log |T(s,t)| / \log s). \qquad \text{(II.1)}$$

The function $N(t)$ is a real-valued continuous function on the complex t-plane. We assume $|T(s,t)| < c\, s^{N(t)}$ at large s, where c is a large positive constant. We shall show that the specific behavior of this function $N(t)$ will determine a FALB for the function $f(s,z_s) \equiv T(s,t)$, with $z_s = 1 + 2t/(s-4)$.

First, we need a lemma, which is a generalized version of a theorem proved by Cerulus and Martin.[1]

Lemma (Cerulus-Martin Theorem):

Let $g(s,w)$ be an analytic function of the real variable s and the complex variable w, such that, at large s,

1) $g(s,w)$ is analytic in w with a branch point at $w_0 = 1 + \alpha_1/(s-4)^{2\gamma}$, $\alpha_1 > 0$, $\gamma > 0$;
2) $|g(s,w)| < s^N$ for $w \in D$, where D is a finite domain in the w-plane (to be specified in the proof);
3) $\log|g(s,1)| = 0\,(\alpha_2 \log s)$, $\alpha_2 < N$ and finite.

Then, $|g(s,w)| \geq \exp[-c_\gamma(w)s^\gamma \log s]$, for s sufficiently large, and $-\infty < w < 1$, where the real function $c_\gamma(w)$ is positive definite in this interval. (See the appendix for the proof.)

We cannot apply this lemma to our function $f(s,z_s)$ directly, by identifying the variable w with z_s, because condition 2) of the lemma is in general not satisfied. We can, however, make an appropriate change of variable so that this lemma becomes applicable. Let us first define a new variable $\xi \equiv r \exp[i(\pi - \theta)] \equiv t - t_0$, and write $N(t)$ as $\bar{N}(\xi)$. Denote by θ_{max} the biggest angle for which $\bar{N}(\xi)$ is bounded by some constant N_0, whenever ξ lies to the left of two lines $\xi_\pm$ in the ξ-plane, $\xi_\pm = r \exp[i(\pi \pm \theta_{max})]$, $0 \leq r < \infty$. In this sector of the ξ-plane, subtended by an angle of $2\theta_{max}$, the function $T(s,t)$ is bounded by s^{N_0}. The desired transformation is the one which maps this sector onto a plane. On this new complex plane, we can apply the above lemma and obtain a FALB.

We define

$$w = -(\rho - z_s)^{2\gamma} + (\rho - 1)^{2\gamma} + 1, \qquad \text{(II.2)}$$

where $\gamma = \pi/2\theta_{max}$, and $\rho = 1 + 2t_0/(s - 4)$; and we use $f(s,w)$ to represent $f[s,z_s(s,w)]$. In the w-plane, $f(s,w)$ is bounded by s^{N_0}, and it is analytic except for a branch point at

$$w_0 = 1 + (\rho - 1)^{2\gamma} = 1 + (2t_0)^{2\gamma}/(s - 4)^{2\gamma}. \qquad \text{(II.3)}$$

Now we apply the lemma, assuming the condition 3) is satisfied by $f(s,w)$, and obtain

$$|f(s,w)| \geqslant \exp[-c_\gamma(w)\, s^\gamma \log s]. \qquad \text{(II.4)}$$

Since $w(s,z_s) = 1 - (1 - z_s)^{2\gamma} + O(s^{-2\gamma})$, we immediately obtain a FALB for $f(s,z_s)$.

Theorem:

If $|T(s,t)| \leqslant C\, s^{N(t)}$ and $T(s,t)$ has the singularity structures described above, then $f(s,z_s)$ has a FALB, i.e.,

$$|f(s,z_s)| \geqslant c' \exp[-c''_\gamma(z_s)\, s^\gamma \log s], \quad \text{for} \quad -\infty < z_s < 1, \qquad \text{(II.5)}$$

where $\gamma = \pi/2\theta_{max}$.

Thus the asymptotic lower bound at a fixed angle derived here is connected to assumptions on the domain specified by θ_{max}. The usually assumed polynomial bound corresponds to $\theta_{max} = \pi$. In that case, $\alpha = \frac{1}{2}$, and we obtain the C-M result. But in general, the angle θ_{max} will not be equal to π. For a theory with a finite range of interaction, $t_0^{\frac{1}{2}}$ is a constant (it is the inverse of the range.) It corresponds to a singularity in z_s-plane, located at the point $\rho = 1 + [2t_0/(s - 4)]$. This point ρ approaches $z_s = 1$ as $1/s$ when s is large. If we make an analogy with a problem in which w is the cosine of the scattering angle, we see from Eq. (II.3) that the range of the interaction has an energy dependence $(s-4)^{\gamma-1/2}$. In this new theory, the scattering amplitude satisifes a uniform fixed-t polynomial bound.

Martin[2] has subsequently abandoned the assumption of a uniform polynomial bound, and rederived the C-M result under a somewhat different assumption. His "new" assumption requires that the function $T(s,t)$ be bounded by $s^{N(t)}$, $N(t) = O(|t|^{\frac{1}{2}})$. This weaker assumption might still be too strong. Recent experiments seem to suggest the power $N(t)$ grows linearly with t. However, we can make use of his mathematical method and derive our theorem with a relaxed condition. The angle θ_{max} can now be the biggest angle so that $\bar{N}(\xi) = O(|\xi|^{\frac{1}{2}})$ in the sector defined by this angle θ_{max}.

In the next section, we shall exhibit a potential model where $\theta_{max} > \pi$ and the corresponding FALB derived is higher than that of C-M. In Section IV, we shall see that for a linear trajectory the angle θ_{max} will be $\pi/2$. Hence the FALB will be lower than that of C-M.

III. A POTENTIAL MODEL

The existence of the Mandelstam representation has been proved for a certain class of potentials.[6] Through the work of Regge,[7] we also know that the scattering amplitude has fixed-t power behavior due to the existence of t-channel Regge poles. The uniform polynomial bound, required by the Mandelstam representation, puts a restriction on the possible position of a t-channel Regge pole. Let the scattering amplitude $A(s,t)$ be bounded by $|s|^N$. The power N has the physical interpretation of being the "maximum" angular momentum a t-channel Regge pole can have, as can be seen easily from the Froissart-Gribov amplitude

$$a(t,\ell) = \int_{s_0}^{\infty} D_s(s,t)\, Q_\ell \left(1 + \frac{2s}{t-4}\right) ds \,. \qquad \text{(III.1)}$$

Since the discontinuity function $D_s(s,t)$ is also bounded by the same power, Eq. (III.1) defines an analytic function of ℓ, regular for $\mathrm{Re}(\ell) > N$, for all t on the t-channel physical sheet. Since a Regge pole arises from the divergence of the integral in Eq. (III.1), it follows that no Regge pole can move to the right of the line $\mathrm{Re}(\ell) = N$ for all t on the first sheet. The angle θ_{max} defined in the last section will then be at least as large as π. Using the Theorem of Section II, we see that the FALB of the scattering amplitude is $\exp[-c_{\frac{1}{2}}(z_s)\, s^{\frac{1}{2}} \log s]$. This is exactly the Cerulus-Martin result.

If the angle θ_{max} is bigger than π, the corresponding FALB will be bigger than that of C-M. If $a(t,\ell)$ is also regular on the

second sheet of the t-plane, for $\mathrm{Re}(\ell) > N$, the double spectral function $\rho(s,t)$ will also be bounded by s^N, for all t. This follows from the fact that the discontinuity of $a(t,\ell)$ across its elastic cut is given by

$$(2i) \int_{s(t)}^{\infty} \rho(s,t)\, Q_\ell \left(1 + \frac{2s}{t-4}\right) ds \,. \qquad \text{(III.3)}$$

Now we can use a dispersion relation to find $D_t(s,t)$, and $D_t(s,t)$ is also bounded by s^N in the same region where $\rho(s,t)$ is bounded. Since $(2i)\, D_t(s,t)$ is the difference of the scattering amplitude $A(s,t)$ between the first and the second t-sheets, the angle θ_{max} is now possibly bigger than π.

For potentials sufficiently "analytic,"[8] all poles of $a(t,\ell)$ move towards the negative q_t^2 axis on the second sheet where $\mathrm{Re}(\ell)$ is large. For this class of potentials, θ_{max} can be as large as $2\pi - \epsilon$. We apply our theorem of Section II and obtain a FALB

$$\exp[-c_{1/4}(z_s)\, s^{1/(4-\epsilon')}\log s]. \qquad \text{(III.3)}$$

We can generalize this method to include any finite number of channels, and obtain a FALB higher than that of C-M.

Additional considerations have to be made for relativistic problems. As we shall see, that there are infinitely many multiparticle channels opening as t increases, and their existence is probably what is responsible for the ever-rising Regge trajectory.[9] If the leading Regge trajectory increases too fast, the FALB will be considerably lower than that of C-M.

IV. POWER BEHAVIOR AT FIXED-MOMENTUM TRANSFER AND FIXED-ANGLE LOWER BOUND

In a relativistic theory, the scattering amplitude $A(s,t)$ is the sum of two analytic functions $A^{(t)}(s,t)$ and $A^{(u)}[s,u(s,t)]$, defined by

$$A^{(t)}(s,t) = \frac{1}{\pi} \int_{t_0}^{\infty} \frac{D_t(s,t')}{t' - t} \, dt$$

$$A^{(u)}[s,u(s,t)] = \frac{1}{\pi} \int_{u_0}^{\infty} \frac{D_u(s,u')}{u' - u(s,t)} \, du' \qquad \text{(IV.1)}$$

with

$$A(s,t) = A^{(t)}(s,t) + A^{(u)}[s,u(s,t)] \quad \text{and} \quad u + s + t = 4.$$

Both $A^{(t)}(s,t)$ and $A^{(u)}(s,u)$ have the same singularity structure in s as does $A(s,t)$, and both have right hand cuts in the t and u planes respectively. For potential scattering, the complete scattering amplitude $A(s,t)$ is just $A^{(t)}(s,t)$. Here we are interested in the behavior of $A(s,t)$ when s goes to infinity while keeping z_s fixed, where z_s is the cosine of the s-channel center of mass scattering angle. Let us define

$$f(s,z_s) \equiv A(s,t), \quad f^{(t)}(s,z_s) \equiv A^{(t)}(s,z_s), \text{ and } f^{(u)}(s,z_s) \equiv A[s,u(s,t)]$$

$$\text{where } z_s = 1 + \frac{2t}{s - 4} = -1 - \frac{2u}{s - 4}. \qquad \text{(IV.2)}$$

We shall find lower bounds for $f^{(t)}(s,z_s)$ and $f^{(u)}(s,z_s)$ as s becomes large (while keeping z_s fixed) from our knowledge of the asymptotic behaviors of $A^{(t)}$ and $A^{(u)}$.

It does not follow immediately that the lower bounds for $f^{(t)}$ and $f^{(u)}$ should also be lower bounds for the full amplitude $f(s,z_s)$, because cancellation[10] might occur between $f^{(t)}$ and $f^{(u)}$. We think such cancellations are unlikely and shall assume that they occur only at isolated points of z_s. It then follows that the bigger of these two lower bounds for $f^{(t)}$ and $f^{(u)}$ is the best lower bound obtained for averaged functions. For instance, define

$$\bar{f}(s,z_s) \equiv \frac{1}{\Delta}\int_{z_s}^{z_s+\Delta} f(s,x)\,dx, \quad \Delta > 0. \tag{IV.3}$$

If Δ is not too small, we immediately have a lower bound for $\bar{f}(s,z_s)$ in both energy and z_s dependence. From this point on, we shall discuss only $f^{(t)}(s,z_s)$, (and drop the subscript) for simplicity.

The FALB of $A(s,t)$ can now be found in exactly the same fashion as that for potential models. In the Regge theory, the function $N(t)$ would be the value of the real coordinate of the rightmost singularity in the j_t-plane at a given t. Because of s-channel unitarity, the Froissart bound[11] tells us that $N(t) \leqslant 1$, for $-\infty \leqslant t \leqslant 0$. It also follows from unitarity that[12] $N(0) \leqslant N(t) \leqslant 2$ for $0 \leqslant t \leqslant t_0$, where t_0 is the lowest t-channel singularity. The value of $N(t)$ for an arbitrary t is, in general, not known. In most model

theories,[13] $N(t)$ is bounded for the whole complex t-plane, as was shown in the example in the preceding section. However, all these models could not incorporate simultaneous unitarity in all channels. It is a well-known fact that a finite number of normal threshold singularities in one channel is inconsistent with unitarity in the cross-channel. As discussed in Ref. 9, the existence of infinitely many multiparticle channels may have the effect of producing ever-rising Regge trajectories. If this is the case, $N(t)$ will not be bounded as $|t| \to \infty$.

Applying the theorem of Section II, we obtain a lower bound for $f(s,z_s)$,

$$|f(s,z_s)| \geq \exp[-c_\gamma(z_s)\, s^\gamma \text{ logs}], \quad \text{where} \quad \gamma = \pi/2\theta_{max}. \tag{IV.4}$$

The existence of infinitely many multiparticle channels also limits the angle θ_{max} to less than π.[14] When $\theta_{max} = \pi$, we obtain the C-M lower bound. In this sense, the C-M lower bound is the best possible lower bound for $f(s,z_s)$ consistent with s-channel unitarity.[15]

We can easily see that the FALB of $f(s,z_s)$ can be considerably lower than that of C-M. In particular, if a Regge trajectory increases linearly with t, θ_{max} will be $\pi/2$. It then follows that the FALB of $f(s,z_s)$ is

$$\exp[-c_1(z_s)\, s \text{ logs}]. \tag{IV.5}$$

In general, the FALB will be

$$\exp[-c_\gamma(z_s)\, s^\gamma \text{ logs}], \ \tfrac{1}{2} \leq \gamma < \infty. \tag{IV.6}$$

We note[16] that this bound is true for all z_s between 1 and -1, including the point $z_s = 0$. (Numerical form for C_γ is given in Ref. 17.)

V. CONCLUDING REMARKS

We have seen that there is a connection between the asymptotic behaviors of the scattering amplitude $A(s,t)$ in different asymptotic regions. It is not clear at the present if the asymptotic conditions of the S-matrix are determined by axioms of maximal analyticity of the first degree. The postulate of second degree maximal analyticity is intended to fill this gap. It is nevertheless encouraging to know that the knowledge of fixed-t asymptotic behavior, which can be obtained from the second degree maximal analyticity, implies certain constraints on the fixed-angle behavior, whether the former is simply a consequence of basic S-matrix axioms or not.

The conjecture of ever-rising Regge trajectory might sound alarming at first. It will not allow us to write down a double-spectral representation in the form originally proposed by Mandelstam. This worry is really an ill-founded one. Unitarity in all channels and simultaneous analyticity in s and t can give definite meaning to the Mandelstam representation through analytic continuation.[18] All singularities of scattering amplitudes are dynamically determined. The ever-rising-Regge-trajectory model is definitely a qualified candidate to satisfy all these requirements.

It is interesting that both the energy dependence and the angular dependence of the pp cross section data at large angle are compatible[17] with the form of our lower bound for either $\gamma = \frac{1}{2}$ or $\gamma = 1$. Unfortunately, the data are not sufficient to make a meaningful test on

the minimal interaction hypothesis. Should the scattering amplitude at large angle indeed coincide with the lower bound amplitude, the situation will be quite puzzling, since there are other general requirements not used in our derivation of the results. So it is still possible that the fixed-angle behavior is of the form proposed by Martin and by Kinoshita, with $\gamma = \frac{1}{2}$, whereas the lower bound will decrease faster. Consideration of analyticity in the s-channel might, however, raise the lower bound obtained here.

ACKNOWLEDGMENTS

We thank Professor Geoffrey Chew for his interest and his invaluable discussions on the present work. We also thank Dr. J. Finkelstein and Mr. J. Dash for their critical reading of the manuscript.

APPENDIX

Proof of the Lemma (Cerulus-Martin Theorem):

The original C-M theorem was given in Ref. 1, and was subsequently clarified by T. Kinoshita in Ref. 3. We shall not repeat their results and refer interested readers to Ref. 3 for details. We shall only point out the major modifications necessary for our proof.

(1) $g(s,w)$ has a branch point at $w_0 = 1 + \dfrac{a_1}{(s-4)^{2\gamma}}$, $0 < \gamma < \infty$, whereas the actual scattering amplitude has a branch point in z_s-plane at $1 + \dfrac{2t_0}{(s-4)}$.

(2) $g(s,w)$ has no left-hand cut in w. Consequently, we do not have to restrict ourselves to mappings centered at $w = 0$.

(3) For any point w_b on the real axis to the left of $w = 1$, define $\bar{w} \equiv \dfrac{w - w_b}{1 - w_b}$. If we consider $g(s,w)$ as a function of s and $\bar{w}$, we see that $g(s,\bar{w})$ has a branch point at

$$\bar{w} = 1 + \frac{[a_1/(1 - w_b)]}{(s-4)^{2\gamma}} \equiv 1 + \frac{\bar{a}_1}{(s-4)^{2\gamma}} . \qquad (A.1)$$

(4) Now we do precisely the same thing as C-M did. Using the mappings of Refs. 1 and 3, centered at $\bar{w} = 0$, and applying Hadamard's three-circle theorem,[19] one obtains:

$$M_1 \leqslant M_0^{(1 - \ell n\, r/\ell n\, R)} \; M^{(\ell n\, r/\ell n\, R)} \qquad (A.2)$$

where

$$M_1 = s^{a_2} \tag{A.3}$$

$$M_0 = s^N \tag{A.4}$$

$$M = \exp\,[-\varphi(w,s)] = g(s,w) \tag{A.5}$$

$$\frac{\ell n\, r}{\ell n\, R} = 1 - \frac{\bar{c}(\bar{w})(2\bar{a}_1)^{\frac{1}{2}}}{(s-4)^{\gamma}} + 0\left(\frac{1}{(s-4)^{2\gamma}}\right) \tag{A.6}$$

with

$$\bar{c}(\bar{w})^{-1} = \frac{\lambda}{2}\,\ell n\,\left(\frac{1+\lambda}{1-\lambda}\right), \quad \lambda = \left(\frac{2\sin\bar{\theta}}{1+\sin\bar{\theta}}\right)^{1/2};$$

and

$$\sin\bar{\theta} = (1-\bar{w}^2)^{1/2}.$$

Substituting Eqs. (A.3), (A.4), (A.5), and (A.6) into (A.2), one has

$$\varphi(w,s) \leqslant F(w,\, w_b)\,(s-4)^{\gamma}\,\ell n\, s, w_b \leqslant w \leqslant 1, \tag{A.7}$$

where we have introduced $F(w,\, w_b) = \dfrac{(N - a_2)}{\bar{c}(w,w_b)(2\bar{a}_1)^{\frac{1}{2}}}$.

(5) Define $c_\gamma(w) \equiv \min\,\{F(w,w_b)\,|\,-\infty < w_b < w\}$. Putting this back into Eq. (A.5), we obtain

$$|g(s,w)| \geqslant \exp\,[-c_\gamma(w)\,(s-4)^{\gamma}\,\ell n\, s].$$

(6) One can check that $c_\gamma(w)$ is finite for all w, $-\infty < w < 1$.

FOOTNOTES AND REFERENCES

* Work done under the auspices of the U. S. Atomic Energy Commission.

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13. Page 54 of Ref. 8.

14. In order to apply our theorem, we need a function analytic in whole w-plane except branch points on positive axis. A mapping with $\theta_{max} > \pi$ will bring singularities onto both upper and lower w-plane.

15. This is implicitly mentioned in Ref. 1.

16. Equation (IV.6) actually holds for all z_s, $-\infty < z_s < 1$. We have a similiar expression for $f^{(u)}(s,z_s)$, true for $-1 < z_s < \infty$. The physical region is, of course, $-1 \leqslant z_s \leqslant 1$. In obtaining this result, we have also assumed that $f^{(t)}(s,z_s)$ decreases monotonically along the real axis as z_s decreases from 1.

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