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ABSTRACT

A general study of the space-time symmetries of the scattering amplitude is made. The scattering amplitude in the c.m. frame has O(2,1) symmetry in the physical regions of the crossed channels. By working in this frame, we can use the same definition of the helicity as that used in the usual angular momentum analysis. For the pairwise equal-mass case the helicity amplitude in the forward scattering region of the crossed channels has O(2,2) symmetry in the c.m. frame, and O(3,1) symmetry in the brick-wall frame. We apply an O(2,2) expansion to a multiparticle system. We also continue the O(4) expansion in the brick wall frame into the O(3,1) region, and show the equivalence between the O(4) expansion and the O(3,1) expansion. Finally, we point out the difference between the implications of the three-dimensional and four-dimensional symmetries.

I. INTRODUCTION

Recently, Toller has studied the forward scattering amplitude of the crossed channel (s channel) in the pairwise equal-mass case,[1] and has expanded it in terms of the irreducible unitary representations (i.u. reps.) of the symmetry group, the Lorentz group or O(3,1). He and Sciarrino[2] have examined a model in which the Regge trajectory functions and the residue functions are assumed to be analytic functions of a coupling constant, which they call a transition-free model theory.[3] They show that a Lorentz pole generates an infinite number of integrally spaced Regge poles. Freedman and Wang[4] have investigated the corresponding helicity amplitude of the direct channel (t channel) at vanishing energy, $t = 0$, but outside the physical regions of both crossed channels, and expanded the amplitude in terms of the i.u. reps of the O(4) group, a symmetry group in the region considered. They considered particularly the daughter phenomena for the nucleon-nucleon scattering, and obtained results similar to those in Toller theory. So far no one has extended the continuation of the O(4) expansion to the general pairwise equal-mass case, and no one has removed the restrictions imposed by the assumptions of the transition free model.

In this paper, we wish to make a general study of the three-dimensional (3-dim.) and four-dimensional (4-dim.) symmetries of the scattering amplitude, and to study the relation between the Toller and Freedman-Wang amplitudes. The problem will involve continuing the expansion corresponding to a compact 4-dim. group to the expansion of a noncompact 4-dim. group, or vice versa, as in the 3-dim. symmetry case in which the O(3) partial-wave expansion is continued into an O(2,1) expansion plus some pole terms. The O(4) group is the only compact 4-dim. group, and we may take it as a starting point for continuation. However, certain questions arise: Are there continuable expressions for the O(4) representation function? What is the counterpart of the continued O(4) expansion? The O(3,1) expansion or others? One of the purposes of this paper is to study such problems. Some generalizations[5-8] for the unequal-mass cases and for the cases in which the energy t is nonvanishing have been made. Here, we are interested for the 4-dim. symmetries only in the pairwise equal-mass case at $t = 0$.

The usual parameterization of the total four-momentum for the treatment of the symmetries of the scattering amplitude is not analytic; the amplitude has, for $t > 0$, the symmetry of the rotation group; for $t < 0$, the symmetry of the 3-dim. Lorentz group. These two subgroups span different subspaces in the space-time continuum. Boyce showed that the helicities in two cases have different geometrical interpretations;[9] one is the c.m. helicity, the other is the brick wall (b.w.) helicity. We find that if the parameterizations of all the four momenta for any s and t are defined by continuation in s and t from the physical region of the direct channel, only one kind of the helicity is enough to describe the scattering system at any s and t. We investigate these problems in the c.m. frame and in the b.w. frame separately.

In Sec. IIA, conventions and notations are introduced. Then, we introduce and discuss the criteria for the space-time symmetries of the helicity amplitude. In Sec. IIB, we obtain the boundaries of the regions of the symmetries of the c.m. and the b.w. helicity amplitudes on the Mandelstam diagram. We find that the boundaries are exactly the same for these two amplitudes except for the 4-dim. symmetry regions, if any. The c.m. (or alternatively b.w.) helicity amplitude at $t = 0$ for the pairwise equal-mass case has O(4) symmetry for $(m - m')^2 < s < (m + m')^2$, and O(2,2) symmetry [or alternatively O(3,1)] for $s < (m - m')^2$ or $s > (m + m')^2$, where s is the momentum transfer.

In Sec. III we discuss the 3-dim. symmetries of the scattering amplitude. We introduce a parity-conserving amplitude which has the SU(2) or SU(1,1) representation function as its geometrical factor, and thus is suitable to compare with the SU(1,1) expansion obtained from the group-theoretic method with the consideration of parity conservation.

In Sec. IVA we derive the O(2,2) expansion for particular cases, and apply it to the multiparticle system. It might be the only application for the O(2,2) expansion. In Sec. IVB we continue the signatured O(4) expansion into the physical region of the crossed channel. In Sec. IVC we show the equivalence between the continued O(4) and the O(3,1) expansions. In Sec. V, we summarize the difference between the implications of the 3-dim. and 4-dim. symmetries.

II. SPACE-TIME SYMMETRIES OF THE SCATTERING AMPLITUDE

A. Criteria for the Space-Time Symmetries

Consider a two-body-to-two-body scattering:

$$(p_1 s_1 \lambda_1) + (p_2 s_2 \lambda_2) \to (p_3 s_3 \lambda_3) + (p_4 s_4 \lambda_4) \quad . \tag{1}$$

The triplet (p_i, s_i, λ_i) describes a particle of the kind i; i.e. of four-momentum p_i, spin s_i, and helicity λ_i.

The masses of the four particles could be equal or unequal. Only when we discuss 4-dim. symmetries, we shall restrict ourselves to the pairwise equal-mass case, i.e., $m_1 = m_2 = m$, and $m_3 = m_4 = m'$.

The helicity amplitude[10] $H^t[p]$ is defined in the physical region of the direct channel (which we hereafter call t channel):

$$H^t[p] \equiv H^t_{\lambda_3\lambda_4;\lambda_1\lambda_2}(p_3,p_4;\, p_1 p_2) \equiv \langle p_3 s_3 \lambda_3;\, p_4 s_4 \lambda_4 | T | p_1 s_1 \lambda_1;\, p_2 s_2 \lambda_2 \rangle \, . \tag{2}$$

It transforms[10,11] covariantly under the proper, orthochronous Lorentz group:

$$H^t_{\lambda_3\lambda_4;\lambda_1\lambda_2}[p_3,p_4;p_1 p_2]$$

$$= \sum D^{s_3}_{\lambda_3\lambda_3'}[R_{\overline{W}}^{-1}(\Lambda,p_3)]\, D^{s_4}_{-\lambda_4-\lambda_4'}[R_{\overline{W}}^{-1}(\Lambda,p_4)]\, H^t_{\lambda_3'\lambda_4';\lambda_1'\lambda_2'}(\Lambda p_3,\Lambda p_4;\, \Lambda p_1,\Lambda p_2)$$

$$\times\; D^{s_1}_{\lambda_1'\lambda_1}[R_{\overline{W}}(\Lambda,p_1)]\, D^{s_2}_{-\lambda_2'-\lambda_2}[R_{\overline{W}}(\Lambda,p_2)] \, , \tag{3}$$

where the Wigner rotations $R_{\overline{W}}(\Lambda, p_i)$ are given[10-12] by

$$R_{\overline{W}}(\Lambda, p_i) = L^{-1}(\Lambda p_i)\, \Lambda L(p_i) \,,$$

and (4)

$$L(p_i) = e^{-i\phi_i J_3}\, e^{-i\theta_i J_2}\, e^{-i\alpha_i K_3} \equiv u_z(\phi_i)\, u_y(\theta_i)\, a_z(\alpha_i) \,.$$

The operators J_i and K_i are the rotation and the boost generators of the Lorentz group. The parameters $(\alpha_i, \theta_i, \phi_i)$ are determined by the four-momentum p_i:

$$p_i = m_i(\cosh \alpha_i, \sinh\alpha_i \cos \theta_i, \sinh\alpha_i \sin\theta_i \cos \phi_i, \sin\alpha_i \sin\theta_i \sin \phi_i). \quad (5)$$

The Hall-Wightman theorem[13] states that Eq. (3) can be extended analytically to any transformation Λ of the complex Lorentz group.

In Regge theory, the scattering amplitude is usually defined in the c.m. frame, and denoted by

$$H^t(s,t) \equiv H^t_{\lambda_3\lambda_4;\lambda_1\lambda_2}(s,t) \equiv H^t_{\lambda_3\lambda_4;\lambda_1\lambda_2}[p] \,. \quad (6)$$

Conventionally, the incoming particles 1 and 2 are assigned to be along the z axis and the outgoing ones 3 and 4 are in the xz plane. In terms of group parameters, the four-momenta p_i can be expressed by

$$p_1 = m_1(\cosh\alpha_1,\ 0,\ 0,\ \sinh\alpha_1)\ ,$$

$$p_2 = m_2(\cosh\alpha_2,\ \sinh\alpha_2 \sin\pi,\ 0,\ \sinh\alpha_2 \cos\pi)\ ,$$

$$p_3 = m_3(\cosh\alpha_3,\ \sinh\alpha_3 \sin\theta,\ 0,\ \sinh\alpha_3 \cos\theta)\ ,$$

and

$$p_4 = m_4\Big(\cosh\alpha_4,\ \sinh\alpha_4 \sin(\pi-\theta)\cos\pi,\ 0,\ \sinh\alpha_4 \cos(\pi-\theta)\Big), \tag{7}$$

with the restrictions

$$m_1\cosh\alpha_1 + m_2\cosh\alpha_2 = m_3\cosh\alpha_3 + m_4\cosh\alpha_4\ ,$$

$$m_1\sinh\alpha_1 - m_2\sinh\alpha_2 = 0$$

and

$$m_3\sinh\alpha_3 - m_4\sinh\alpha_4 = 0\ . \tag{8}$$

The group parameters $(\alpha_i, \theta_i, \phi_i)$ are related to the Mandelstam variables:[14]

$$\cosh\alpha_1 = \frac{t + m_1^2 - m_2^2}{2m_1 t^{\frac{1}{2}}}\ , \qquad \cosh\alpha_2 = \frac{t - m_1^2 + m_2^2}{2m_2 t^{\frac{1}{2}}}\ ,$$

and

$$\cosh\theta = z$$

$$= \frac{t(s-u) + (m_3^2 - m_4^2)(m_1^2 - m_2^2)}{\{[t-(m_1+m_2)^2][t-(m_1-m_2)^2][t-(m_3+m_4)^2][t-(m_3-m_4)^2]\}^{\frac{1}{2}}}, \tag{9}$$

where $t = (p_1 + p_2)^2$, $s = (p_1 - p_3)^2$, and $u = \sum m_i^2 - s - t$. Similar expressions can be obtained for $\cosh \alpha_3$ and $\cosh \alpha_4$.

Suppose g is an element of the rotation group O(3), and is parameterized as $e^{-i\gamma J_3} e^{-i\beta J_2} e^{-i\delta J_3}$. The Wigner rotations $D^{s_i}_{\lambda_i \lambda_i'}[\Lambda(g), p_i]$ are diagonalized, and give some phase factors. From (3), one obtains

$$H^t_{\lambda_3\lambda_4;\lambda_1\lambda_2}[p] = e^{-i\Sigma} H^t_{\lambda_3\lambda_4;\lambda_1\lambda_2}[\Lambda(g)\, p] \quad , \tag{10}$$

where the total phase Σ is defined by

$$\Sigma = \lambda_1\zeta_1 - \lambda_2\zeta_2 - \lambda_3\zeta_3 + \lambda_4\zeta_4 \quad . \tag{11}$$

The phase angles ζ_i are given[10] by

$$\begin{aligned} \cosh \zeta_{1(2)} &= 1 \ , \\ \cos \zeta_{3(4)} &= (\cos\beta - \cos\Theta_{3(4)} \cos\Theta'_{3(4)}/\sin\Theta_{3(4)} \sin\Theta'_{3(4)} \ , \end{aligned} \tag{12}$$

with

$$\cos\Theta'_{3(4)} = -\sin\beta \cos\delta \sin\Theta_{3(4)} + \cos\beta \cos\Theta_{3(4)} \ .$$

One sees from (9) and (12) that the ζ_i are functions of s and t. In the physical region of the t channel, one has $-1 \leqslant \cos\zeta_i \leqslant 1$ for all i, and thus Σ is real. The amplitude $H^t[p]$ is invariant except for a phase factor under the transformation of the rotation group. Since we want to include the cases in which the total angular momentum is half-integral, we shall consider SU(2), the covering group of O(3).

Before studying the scattering amplitude away from the physical region, we set up two criteria for the space-time symmetries:

(i) The symmetry group must be a subgroup of the complex little group,[12] which leaves invariant the total four-momentum $p(= p_1 + p_2 = p_3 + p_4)$.

(ii) Each component of the four-momenta p_i of the external particle must be kept either pure real or pure imaginary when it is continued in s and t to be real or imaginary in the region considered.

Toller determines the symmetry group by the first criterion only.[8] In his earlier papers,[1] Toller considers the real little group. As mentioned in Sec. I, the helicities corresponding to the cases $t > 0$ and $t < 0$ are defined differently in Toller theory, even though we can show that they are mathematically equivalent.[9] The introducing of the complex group allows one to keep to one particular frame. Therefore, the parameterization of the four-momentum p remains unaltered when we pass through $t = 0$. The same definition of the helicities can be used for all t. We shall discuss the amplitude in the c.m. frame, except where we explicitly specify the b.w. frame.

The first criterion guarantees that the amplitude $H^t[p]$ can be expressed as a function of the group element of the symmetry group. The last criterion restricts the complex little group to its smallest subgroup which contains the group element $u_y(\Theta)$ and $a_z(\alpha_i)$ for the 4-dim. symmetry case, and thus allows one to avoid the superfluous new quantum numbers. For an instance, in the physical region of the t channel, the amplitude in the c.m. frame has O(3C) symmetry by the first criterion only, and so one has the new group indices in addition to the angular momentum. By the last criterion, the symmetry group is O(3), and the new group indices are avoided. For the 3-dim. symmetry case, these two criteria are equivalent to stating that the amplitude is invariant under the symmetry group except for a phase factor with a real phase angle Σ [see Eq. (10)]. For the 4-dim. symmetry case, we do not have equivalent statement. Whether this difference between the 3-dim. and the 4-dim. symmetry is essential is not clear.

B. Boundaries of the Regions of the Symmetries on the Mandelstam Diagram

We now continue the helicity amplitude $H^t(s,t)$ away from the physical region of the t channel. During the continuation, the amplitude is always kept in the c.m. frame, and the parameterizations (7) and the relations (9) still hold. However, the group parameters α_i and Θ may be complex for some s and t. For $t \neq 0$, the total three-momentum $\underset{\sim}{p}$ vanishes. From the first criterion, the complex little group is a complex rotation group O(3C). We have to restrict it further by the second criterion.

We are particularly interested in the physical regions of the two crossed channels (s channel and u channel). Hence it is sufficient to consider the boundaries of the regions of the symmetries on the Mandelstam diagram.[14]

The symmetry groups at $t \neq 0$ are determined by the value of Θ:

(i) Real Θ, i.e. $-1 \leqslant \cos\Theta \leqslant 1$: From (7), one has the general form of the three-momentum

$$\underset{\sim}{p}_i = (p_{i3}, p_{i1}, p_{i2})$$

$$= m_i \sinh\alpha_i(\cosh\Theta_i,\ \sin\Theta_i\cos\phi_i,\ \sin\Theta_i\sin\phi_i)\ . \tag{13}$$

By the last criterion, the symmetry group is the group which keeps invariant the form $-p_{i3}^{\ 2} - p_{i1}^{\ 2} - p_{i2}^{\ 2}$. Hence it is O(3) or its covering group SU(2). As we showed above, the amplitude $H^t[p]$ is invariant to within a phase factor under the transformation of the symmetry group SU(2). In this case, the α_i can be imaginary or real.

(ii) Imaginary Θ, i.e., $\cos\Theta \geqslant 1$ or $\cos\Theta < -1$: Suppose $\Theta_i' = i\Theta_i$. One has, from (7),

$$\underset{\sim}{p}_i = m_i \sinh\alpha_i(i\sinh\Theta_i'\cos\phi_i,\ i\sinh\Theta_i'\sin\phi_i,\ \cosh\Theta_i')$$

$$\equiv (ip_{i1}',\ ip_{i2}',\ p_{i3}')\ .$$

The symmetry group is determined by the quadratic form $-p_{i3}'^{2} + p_{i2}'^{2} + p_{i1}'^{2}$, and it is thus O(2,1) or its covering group SU(1,1). One can obtain the same result by using the fact that a one-parameter group having a real Lie algebra and imaginary group parameter is equivalent to a group of imaginary Lie algebra and real group parameter. Replacing J_2 by iJ_2 in the commutation relations of O(3) or SU(2), one can show that (1), J_1 must be replaced by iJ_1, and (2), iJ_1, iJ_2, and J_3 form the Lie algebra of O(2,1) or SU(1,1).

The amplitude $H^t[p]$ in this case is invariant to within a phase angle under the transformation of O(2,1) or SU(1,1). By replacing β and Θ_i by $i\beta$ and $i\Theta_i$ in (12), one can easily show that the phase angles ζ_i are real and so is the total phase angle Σ in (10). Again, the α_i can be real or pure imaginary.

(iii) Complex Θ, i.e., $\cos\Theta$ is complex or imaginary: The symmetry group is no longer a 3-dim. group, but a two-dimensional rotation along the z axis. This case does not cover the physical regions of the three related channels. We shall not discuss this further.

From the above discussions, one sees that the boundaries of the regions of the symmetries on the Mandelstam diagram are determined by the conditions $\cos\Theta = \pm 1$. From (9), one can obtain the equations of the boundary curves

$$t = 0 \quad \text{and} \quad \Phi(s,t,u) = 0 , \tag{14}$$

where the Kibble[15] boundary function $\Phi(s,t,u)$ is given by

$$\Phi(s,t,u) = stu - s(m_1^2 m_3^2 + m_2^2 m_4^2) - t(m_1^2 m_2^2 + m_3^2 m_4^2)$$

$$- u(m_1^2 m_4^2 + m_2^2 m_3^2) + 2m_1^2 m_2^2 m_3^2 m_4^2 (m_1^{-2} + m_2^{-2} + m_3^{-2} + m_4^{-2}) \quad . \tag{15}$$

For the general mass case, the amplitude $H^t[p]$ has O(3) or SU(2) symmetry in the physical region of the direct channel (t channel) and in the $t > 0$ parts, if any, of the physical region of the two crossed (s and t channels), whereas it has O(2,1) or SU(1,1) symmetry in the $t < 0$ parts of the physical regions of the two crossed channels. In the regions other than the three physical regions, the symmetries of the amplitude depend on the masses of four external particles. We summarize our results in Figs. 1a and 1b. In Fig. 1a, we take general masses for the particles. The symmetry regions for the pairwise equal-mass case are shown in Fig. 1b. For $t = 0$, we are interested in the pairwise equal-mass case ($m_1 = m_2 = m$ and $m_3 = m_4 = m'$). In this case, the total four-momentum p vanishes and thus the little group is complex Lorentz group or O(4C). From (9), one has

$$\cosh \alpha_i = \frac{t^{\frac{1}{2}}}{2m_i} , \tag{16}$$

which implies that $\alpha_i = i\,\frac{\pi}{2}$ at $t = 0$. For $(m - m')^2 \leqslant s \leqslant (m + m')^2$, one has the general form of the four-momentum

$$p_i = m_i(\cos \frac{\pi}{2},\ i \sin \frac{\pi}{2} \sin \Theta_i \cos \phi_i,\ i \sin \frac{\pi}{2} \sin \Theta_i \sin \phi_i,\ i \sin \frac{\pi}{2} \cos \Theta_i)$$

$$\equiv (p'_{i0},\ ip'_{i1},\ ip'_{i2},\ ip'_{i3}) \quad . \tag{17}$$

Hence the symmetry group is O(4), which keeps invariant the quadratic form $p'^2_{i0} + p'^2_{i3} + p'^2_{i1} + p'^2_{i2}$. In a similar manner, one can show that the symmetry group is O(2,2) for $s \geqslant (m + m')^2$ or $s \leqslant (m - m')^2$. This result leads us to study whether the O(4) expansion of the amplitude in the c.m. frame can be continued into the O(2,2) region. We shall discuss it later. If the amplitude is defined in the b.w. frame, the boundaries of the regions of the symmetries in the Mandelstam diagram are exactly the same as for the helicity amplitude in the c.m. frame, except at $t = 0$ for the pairwise equal-mass case. In the latter case, the amplitude has O(3,1) or Lorentz symmetry for $s > (m + m')^2$ or $s < (m - m')^2$, whereas it has O(4) symmetry for $(m - m')^2 \leqslant s \leqslant (m + m')^2$.

That the helicity amplitude in the c.m. and the b.w. frame have different 4-dim. symmetries is another one of the differences between the 4-dim. and the 3-dim. symmetries of the scattering amplitude.

III. THREE-DIMENSIONAL SYMMETRIES

The expansion of the helicity amplitude $H^t(s,t)$ with respect to the symmetry group O(3) or SU(2) in the physical region of the direct channel is well known.[10] This technique of expanding the helicity amplitude in terms of the u.i. rep. of O(3) or SU(2) is applicable to the other regions where the helicity amplitude has the O(3) or SU(2) symmetry. The O(2,1) or SU(1,1) partial-wave expansions of a square-integrable function are obtained by Bargmann[16] and Andrews and Gunson.[17] Toller[18] and collaborators extended the O(2,1) or SU(1,1) partial-wave expansion to the cases of non-square-integrable amplitude. The continuation from the SU(2) expansion to the SU(1,1) expansion has been performed by many people.[19] In this subsection, we emphasize introduction of a parity-conserving amplitude which has the function $D_{\lambda\mu}{}^j(g)$ as its geometrical factor, where the group element g belongs to the symmetry group of the amplitude. The function $D_{\lambda\mu}{}^j(g)$ denotes the relative orientation of the incoming and the outgoing particles.

We can define a trajectory of definite parity and signature by the equation

$$H^{t\eta\xi}_{\lambda_3\lambda_4,\lambda_1\lambda_2}(s,t) = \sum_{j=m} (2j+1)\, h^{\eta\xi}_{\lambda_3\lambda_4,\lambda_1\lambda_2}(t,j)$$

$$d_{\lambda\mu}{}^j(z)\, (1 + \xi e^{i\pi(j-v)})/2 \quad , \tag{18}$$

where $h^{\eta\xi}(t,j)$ is the partial-wave amplitude of definite η and definite parity $\eta\xi$, and m is the larger of $\mu(= \lambda_3 - \lambda_4)$ and

$\lambda(=\lambda_1 - \lambda_2)$. These parity-conserving amplitudes $H^{t\eta\xi}(s,t)$ have a geometrical factor $d_{\lambda\mu}^{\ j}(z)$, but are not free from the kinematical singularities.[20] The amplitude $H^{t\eta\xi}(s,t)$ has a meaning only if the scattering process conserves parity. Hence its asymptotic behavior in the j plane is known:

$$h^{\eta\xi}_{\lambda_3\lambda_4\lambda_1\lambda_2}(t,j) \sim j^{-\frac{1}{2}} \tag{19}$$

for large $|j|$ and $\mathrm{Re}(j+\frac{1}{2}) \geqslant 0$.

By a method similar to Boyce's,[21] one can perform a Sommerfeld-Watson transform by checking the asymptotic behavior, deforming the contour, and picking up the dynamical and kinematical pole terms, and obtain

$$\begin{aligned} H_{\mu\lambda}^{\ t\eta\xi}(s,t) = &-\frac{i}{4}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{2j+1}{\sin\pi(j-v)}\,[e^{-i\pi(j-v)} + \xi]\, d_{\lambda\mu}^{\ j}(z) \\ &+ \sum_i \frac{(\alpha_i+\frac{1}{2})}{\sin\pi(\alpha_i - v)}\,[e^{-i\pi(\alpha_i - v)} + \xi]\, d_{\lambda\mu}^{\ \alpha_i}(z)\, \beta^i_{\lambda_3\lambda_4,\lambda_1\lambda_2}(t) \\ &+ [\text{terms corresponding to the discrete series of } SU(1,1)], \end{aligned} \tag{20}$$

where α_i is an abbreviation for $\alpha_i^{\ \eta\xi}(t)$. Equations (18) and (20) are suitable for comparing with the partial-wave expansions of the

SU(2) and the SU(1,1) respectively, obtained by the group-theoretic method.

A similar discussion can be presented for the helicity amplitude in the b.w. frame.

IV. FOUR-DIMENSIONAL SYMMETRIES

For $t = 0$, the scattering amplitude has 4-dim. symmetries, the types of which depend on which frames it is defined in. In this section, we discuss the relationships between the expansions of the different symmetry groups, in the c.m. frame and in the b.w. frame. We also derive an O(2,2) expansion for the multiparticle scattering process.

A. The O(2,2) Expansion for the Multiparticle System

As we have shown in Sec. IIA, if we discuss the symmetry of the helicity amplitude in the c.m. frame, the amplitude at $t = 0$ has the O(4) or O(2,2) symmetry[22] depending on whether the four-momentum transfer s is inside or outside the region $(m - m')^2 \leqslant s \leqslant (m + m')^2$. We shall investigate whether the O(4) partial-wave expansion of the c.m. helicity amplitude can be continued into the O(2,2) expansion when the s is continued from the O(4) region to the O(2,2) region.

We first review the algebraic structures and the u.i. reps.[4,23] of the groups O(4) and O(2,2). Let J_i and K_i be the rotation and the boost generators of the two groups. We may define new Lie algebra

$$A_i = \frac{1}{2}(J_i + K_i)$$

and (21)

$$B_i = \frac{1}{2}(J_i - K_i)$$

so that the A_i and the B_i commute with each other, and form a Lie algebra of O(3) and O(2,1) for the group O(4) and O(2,2) respectively. Thus one has

$$O(4) = O(3) \otimes O(3)$$

and (22)

$$O(2,2) = O(2,1) \otimes O(2,1) .$$

The Casimir operators $\underset{\sim}{A}^2$ and $\underset{\sim}{B}^2$ have the eigenvalues

$$\underset{\sim}{A}^2 = A_1^{\,2} + A_2^{\,2} + \epsilon A_3^{\,2} = \epsilon\, a(a+1) ,$$

(23)

$$\underset{\sim}{B}^2 = B_1^{\,2} + B_2^{\,2} + \epsilon B_3^{\,2} = \epsilon\, b(b+1) ,$$

where $\epsilon = +1$ for O(4) and -1 for O(2,2). Another set (n, M) is defined[24] by

$$M^2 + n^2 - 1 = \underset{\sim}{J}^2 - \underset{\sim}{K}^2$$

and (24)

$$nM = -i(J_1K_1 + J_2K_2 + \epsilon J_3K_3) .$$

From (21)-(24) one can relate[25] the quantum numbers a and b to n and M by

$$n = a + b + 1$$

and (25)

$$M = -a + b .$$

From (2) and (17), the helicity amplitude at $t = 0$ for the pairwise equal-mass case can be expressed in terms of the group element of the symmetry groups,

$$H^t_{\lambda_3\lambda_4,\lambda_1\lambda_2}(s,t=0) = \langle \bar{p}'\lambda_3, \bar{p}'\lambda_4 | U(g^{-1})\, T | (\bar{p}\lambda;\, -\bar{p}\lambda_2\rangle \ , \tag{26}$$

where the state vector $|\bar{p}\lambda_1, -\bar{p}, \lambda_2\rangle$ is defined by

$$|\bar{p}\lambda_1; \bar{p}, \lambda_2\rangle = |\bar{p}, \lambda\rangle \otimes e^{i\pi K_3} |\bar{p}, -\lambda_2\rangle \tag{27}$$

and $\bar{p} = (m, 0, 0, 0)$. The group element g is given by

$$g = e^{-i(-\frac{\pi}{2})K_3}\, e^{-i\theta J_2}\, e^{-i\frac{\pi}{2} K_3} \equiv a_z(-\tfrac{\pi}{2})\, u_y(\theta)\, a_z(\tfrac{\pi}{2}) \ . \tag{28}$$

It is important to note that $a_z(\pm \frac{\pi}{2})$ in (28) is an element of the symmetry groups.

In the O(4) region, the helicity amplitude can be expanded[4] in the form

$$H^t_{\lambda_3\lambda_4;\lambda_1\lambda_2}(s,0) = \sum_{ss'nMj} (M^2 - n^2)\, d^{nM}_{s'\mu j}(-\tfrac{\pi}{2})\, d_{\mu\lambda}{}^{j}(\theta)\, d_{j\lambda s}{}^{nM}(\tfrac{\pi}{2})$$

$$\times\ T_{s's}{}^{nM}\, (-1)^{s_2+s_4-\lambda_2-\lambda_4}\, C(s_3,s_4,s';\lambda_3,-\lambda_4)\, C(s_1,s_2,s;\lambda_1,-\lambda_2) \ , \tag{29}$$

where $\mu = \lambda_3 - \lambda_4$ and $\lambda = \lambda_1 - \lambda_2$. The importance of this expansion comes from the result that the 4-dim. partial-wave amplitude

$T_{s's}^{nM}$ is independent of the helicities of the external particles; this result stems from the fact that the state vector sandwiching the scattering operator T is a basis vector of the u.i. rep. of O(3) corresponding to the rotations in the 3-dim. space.

In the O(2,2) region, the corresponding vector is not a basis vector of the u.i. rep. of O(2,1), and thus if one directly expands the helicity amplitude $H^t_{s'\mu s\lambda}(s,0)$ in terms of the representation functions of O(2,2) the 4-dim. partial-wave amplitude may depend on the helicities of the external particles. Hence we shall not continue the O(4) expansion into the O(2,2) regions.

One simple argument to show that the O(4) expansion can probably not be continued into an O(2,2) expansion is that the asymptotic expression in s of the scattering amplitude would depend on the poles in the M plane if such a continuation were possible.

The result of the O(3,1) expansion shows that the asymptotic behavior in the s plane does not depend on the poles in M. Hence the two expansions would contradict each other. Since we know from Toller's work that the O(3,1) expansion is correct, and since the results have been confirmed by the analyticity-factorization method,[26] we have to give up any attempt to continue the O(4) expansion into an O(2,2) expansion.

The i.u. reps. of the SU(1,1) group are given by Bargmann. The representation function has the same form as the SU(2) group. Their basis vectors have the following spectra in j and λ: For continuous series, we have

$$\text{Re } j = -\frac{1}{2}, \quad \lambda = 0, \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \cdots .$$

There are two discrete series, positive and negative. The spectra for the positive discrete series are

$$j = \frac{1}{2}, 1, \frac{3}{2}, \cdots$$

and

$$\lambda = j + 1, j + 2, \cdots,$$

and for the negative discrete series they are

$$j = \frac{1}{2}, 1, \frac{3}{2}, \cdots$$

and

$$\lambda = -j - 1, -j - 2, \cdots .$$

The particle states of the scattering system are not the basis vectors in the carrier spaces of the i.u. reps. of the SU(1,1) group, since either the spin or the helicity does not belong to the categories above for the i.u. reps. of the SU(1,1) group. Hence we cannot expand it in the usual sense. If we expand it by treating the u.i. reps. of the O(2,2) group as a complete set of the orthonormal function, we have

$$H^{t}_{\lambda_3\lambda_4;\lambda_1\lambda_2}(s,0) = \sum_{\substack{abjkk' \\ \lambda_1'\lambda_3'}} (M^2 - n^2)\, c(k, k', j; \lambda_1', \mu - \lambda_1)$$

$$\times\ c(k,k',j; \lambda_3', \lambda - \lambda_3')\, d^{nM}_{k'\mu j}(-\tfrac{\pi}{2})\, d_{\mu\lambda}{}^{j}(\Theta)\, d_{j\lambda k}{}^{nM}(\tfrac{\pi}{2})\, T^{nM}_{kk's_i\lambda_i\lambda_1'\lambda_3'} ,$$

where the summations[23] sum or integrate the dummy indices according to the SU(1,1) group. The coefficient gives no restrictions on the scattering amplitude. However, we can apply the O(2,2) expansion to a multiparticle system.

For a multiparticle system, the external particles can be collected into four groups (see Figs. 3a and 3b). Each group is characterized by the four-momentum p_i, the little group[1,12,27] g_i, and the helicity λ_i. The total mass t_i $(=p_i^2)$ of the particles in the ith group can be negative or positive, real or complex. Suppose that for some t_i there are no Regge poles in the right s_i half-plane corresponding to the little groups g_i. The total amplitude can be expressed in terms of the four submatrices T_i and a reduced matrix $\bar{T}$:

$$T(t_{12}, s_{13}; p_j s_j \lambda_j) = \sum_{\lambda_i' \lambda_i} \prod_{i=1}^{4} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} ds_i \; D^{s_i}_{\lambda_i' \lambda_i}(g_i)$$

$$\times \; T^{i}_{\lambda_i'} \frac{\zeta(s_i)}{\sin \pi(s_i - v)} \bar{T}_{\lambda_3\lambda_4,\lambda_1\lambda_2}(p_i, s_i)$$

$$+ \text{ [terms including at least one discrete series of } O(2,1)] \; , \qquad (30)$$

where the signature factors and the reduced amplitude are given by

$$\xi(s_i) = 1 + \xi\, e^{i\pi(s_i - v)}$$

and (31)

$$\overline{T}_{\lambda_3\lambda_4,\lambda_1\lambda_2}(p_i, s_i) = \langle p_3 s_3 \lambda_3;\ p_4 s_4 \lambda_4 | \overline{T} | p_1 s_1 \lambda_1, p_2 s_2 \lambda_2 \rangle .$$

The reduced amplitude $\overline{T}(p_i, s_i)$ has all the four legs as basis vectors of the u.i. reps. of the SU(1,1) group. By means of the techniques such as used in the O(4) and the O(3,1) cases,[1,4] we have in the O(2,2) symmetry region the partial-wave amplitude which is independent of the helicities of the external legs. It is probably the only application we know of so far. In the c.m. frame of the t_{12} channel, the four-momenta of the four external legs in the reduced matrix can be parameterized as in Eq. (7) and the relationships (9) between the group parameters and the invariant variables still hold, except that we replace m_i, s, and t by t_i, s_{13}, and t_{12}, where $s_{13} = (p_1 - p_2)^2$ and $t_{12} = (p_1 + p_2)^2$. When $t_1 = t_2 = t$, $t_3 = t_4 = t'$ and $t = 0$, the amplitude $\overline{T}$ has O(2,2) symmetry if $s_{13} > -[(-t_1)^{\frac{1}{2}} - (-t_2)^{\frac{1}{2}}]^2$ or $s_{13} < -[(-t_1)^{\frac{1}{2}} + (-t_2)^{\frac{1}{2}}]^2$ for negative t and t'; similar conditions can be obtained if $t_1, t_2 > 0$. Following the procedures in deriving Eq. (29) for the O(4) group, the reduced matrix can be reexpressed by

$$\overline{T}_{\lambda_3\lambda_4,\lambda_1\lambda_2}(p_i,s_i) = \sum_{kk'abj} C^*(s_3,s_4,k';\lambda_3\lambda_4)\,\eta(a)\,\eta(b)$$

$$d^{ab}_{k'\mu j}(\tfrac{\pi}{2})\; d_{\mu\lambda}{}^{j}(\Theta)\; d_{j\lambda k}{}^{ab}(\tfrac{\pi}{2})\; C(s_1,s_2,k;\lambda_1,\lambda_2)\;\overline{T}_{kk',ab}\quad, \qquad (32)$$

where the summations[23] sum or integrate over the indices according to the O(2,1) group, and $\eta(a)$ and $\eta(b)$ are the Plancherel measures of SU(1,1) corresponding to the u.i. reps.[23] a and b respectively. In deriving the Eq. (32), we have assumed that the amplitude $\overline{T}$ is square integrable. The amplitude may not be square integrable, since the scattering operators T may not be unitary outside the physical region of the t_{12} channel. If not, we may take some models such as Toller's transition-free model.[3]

In the physical region of some channels corresponding to the multiparticle system, the total scattering amplitude can be expressed in terms of the t_{12} channel amplitude through a crossing matrix.[28,29] Thus the O(2,2) expansion in these cases may be useful for the phenomenological analysis.

B. The Continuation of the O(4) Expansion Into the O(3,1) Region

In the b.w. frame, we have shown that the b.w. helicity amplitude at $t = 0$ for the pairwise equal-mass case has O(3,1) symmetry in the physical region of the crossed channel, outside which it has the O(4) symmetry. Hence these b.w. helicity amplitudes at $t = 0$ can be shown to be a function of the element of the symmetry groups,

$$H^t_{\lambda_3\lambda_4,\lambda_1\lambda_2}[p] = \langle -\bar{p}'\lambda_3, \bar{p}'\lambda_4 | U(g^{-1}) \, T | \bar{p}\,\lambda_1, -\bar{p}\,\lambda_4 \rangle \quad , \tag{33}$$

where the group element g is the boost along the x axis:

$$g = e^{-i\xi K_1} = a_x(\xi) \quad ,$$

and

$$\cosh \xi = \cos \Theta = z \, . \tag{34}$$

In the O(3,1) region, $|z| > 1$; in the O(4) region $|z| < 1$.

We define two new b.w. helicity amplitudes:

$$H^t_{s'\mu s\lambda}(s,0) = \sum_{\lambda_3\lambda_1} (-1)^{s_2+s_4-\lambda_2-\lambda_4} C(s_3,s_4,s';\lambda_3,\mu - \lambda_3)$$

$$\times \; C(s_1,s_2,s;\lambda_1,\lambda - \lambda_1) \, H^t_{\lambda_3\lambda_4;\lambda_1\lambda_2}(s,0) \tag{35}$$

and

$$H^{s}_{s'\mu s\lambda}(s,0)$$

$$= \sum_{\lambda_3\lambda_1} (-1)^{s_2+s_4-\lambda_2-\lambda_4} \; C(s_3,s_4,s';\lambda_3,\mu-\lambda_3)\; C(s_1,s_2,s;\lambda_1,\lambda-\lambda_1)$$

$$\times \; H^{s}_{-\lambda_1\lambda_4;-\lambda_3\lambda_2}(s,0) \; . \qquad (36)$$

They are related[28] by

$$H^{s}_{s\mu s'\lambda}(s,0) = \sum_{\mu\mu'} d_{\mu\mu}{}^{s'}(-\tfrac{\pi}{2})\; H^{t}_{s'\mu' s\lambda'}(s,0)\; d_{\lambda'\lambda}{}^{s}(\tfrac{\pi}{2}) \; . \qquad (37)$$

From (34)-(37), we have

$$\begin{aligned} H^{s}_{s'\mu s\lambda}(s,0) &= \langle \bar{p}'s'\mu | U\Big(e^{-i(-\frac{\pi}{2})J_2}\; a_x(\xi)\; e^{-i\frac{\pi}{2}J_2}\Big) T | \bar{p}s\lambda\rangle \\ &= \langle \bar{p}s'\mu | U\big(a_z(\xi)\big) T | \bar{p}s\lambda\rangle \; . \end{aligned} \qquad (38)$$

Hence we can expand the amplitude $H^{s}_{s'\mu s\lambda}(s,0)$ in terms of the representation function of its symmetry groups Thus one has, in the O(3,1) region,

$$H^{s}_{s'\mu s\lambda}(s,0) = -\delta_{\mu\lambda} \sum_{M=-q}^{q} \int_{-i\infty}^{i\infty} i dn (M^2 - n^2)\; T_{ss'}{}^{nM}\; d^{nM}_{s'\lambda s}(z) \; , \qquad (39)$$

and in the O(4) region,

$$H^{s}_{s'\mu s\lambda}(s,0) = \delta_{\mu\lambda} \sum_{M=-q}^{q} \sum_{n=q+1}^{\infty} (M^2 - n^2)\, T_{ss'}{}^{nM}\, d^{nM}_{s'\lambda s}(z) , \qquad (40)$$

where q is the smaller of the s and s'. Akyeampong, Boyce, and Rashid[30] have shown that, by introducing a signature, these two expansions of the s-channel amplitude can be continued into each other. We shall discuss the continuation of the Freedman-Wang expansion (40) into the O(3,1) region.

Using an identity

$$a_x(\theta) = a_z(-\tfrac{\pi}{2})\, u_y(\theta)\, a_z(\tfrac{\pi}{2}) \qquad (41)$$

for the O(4) group, one has, from (33)-(35),

$$H^{t}_{s'\mu s\lambda}(s,0)$$

$$= \sum_{M=-q}^{q} \sum_{n=q+1}^{\infty} \sum_{j=|M|}^{n} (M^2 - n^2)\, d^{nM}_{s'\mu j}(-\tfrac{\pi}{2})\, d_{\mu\lambda}{}^{j}(\theta)\, d_{j\lambda s}{}^{nM}(\tfrac{\pi}{2}) . \qquad (42)$$

From (42), we change the dummy variable j to r by the relation $r = n - j$, and obtain

$$H^{t}_{s'\mu s\lambda}(s,0)$$

$$= \sum_{M=-q}^{q} \sum_{n=q+1}^{\infty} \sum_{r=0}^{n-|M|} (M^2 - n^2)\, d^{nM}_{s'\mu n-r}(-\tfrac{\pi}{2})\, d_{\mu\lambda}{}^{n-r}(\theta)\, d^{nM}_{n-r\lambda s}(\tfrac{\pi}{2})$$

$$\times\; T_{ss'}{}^{nM} . \qquad (43)$$

The leading term corresponding to a particular r in the above equation is z^{n-r}. We are interested in a few leading terms which control the asymptotic behaviors in z. We may consider a component of Eq. (43) corresponding to definite M and r, and define

$$H^{tMr}_{s'\mu s\lambda}(s,0) = \sum_{n=q+1}^{\infty} (M^2 - n^2)\, d^{nM}_{s'\mu n-r}(-\tfrac{\pi}{2})\, d_{\mu\lambda}^{\,n-r}(\theta)\, d^{nM}_{n-r\ \lambda s}(\tfrac{\pi}{2}) \times T_{s's}^{nM} \quad (44)$$

Let us examine the asymptotic behavior of the right-hand side of (44) in the n plane, with the aim of converting the summation to a Sommerfeld-Watson integral. Freedman and Wang show[4] that the signatured scattering amplitude has the following asymptotic behavior in the n plane:

$$T_{s's}^{nM\pm} \sim O(1) \quad \text{as} \quad |n| \to \infty .$$

The matrix functions $d^{nM}_{00n-r}(\pm\frac{\pi}{2})$ can be expressed[4] in a simple form:

$$d^{nM}_{00n-r}(\pm\tfrac{\pi}{2}) = \left[\frac{(2n - 2r + 1)\Gamma(r + 1)}{(n + 1)\,\Gamma(2n - r + 2)}\right]^{\frac{1}{2}} (\pm 2i)^{n-r} \frac{2^r\,\Gamma(\frac{1}{2})\,\Gamma(n - \frac{1}{2}r + 1)}{\Gamma(r + 1)\,\Gamma(\frac{1}{2} - \frac{1}{2}r)} .$$

Thus the product $d^{nM}_{00n-r}(-\frac{\pi}{2})\, d^{nM}_{n-r00}(\frac{\pi}{2})$ falls down exponentially when $\mathrm{Re}\, n > 0$:

$$d^{nM}_{00n-r}(-\tfrac{\pi}{2})\, d^{nM}_{n-r00}(\tfrac{\pi}{2}) \sim O[n^{-\frac{1}{2}} \exp(-n\, \ell n\ 2)]$$

since $d^{nM}_{s\lambda j}(\delta) = d^{nM}_{j\lambda s}(\delta)$ for the O(4) group. In deriving this, we have used the relation[31]

$$\Gamma(2n - r + 2) = 2^{2n-r+2}\, \pi^{-\frac{1}{2}}\, \Gamma(n - \tfrac{1}{2}r + 1)\, \Gamma(n - \tfrac{1}{2}r + \tfrac{3}{2}) .$$

The asymptotic behavior in the n plane of the general function $d^{nM}_{s'\lambda' n-r}(\pm\frac{\pi}{2})$ differs from that of $d^{nM}_{00n-r}(\pm\frac{\pi}{2})$ only by a polynomial in n, since s, s', λ, and μ are finite, and the coefficients in the recursion relations between $d^{nM}_{j\lambda s}(\delta)$ behave at worst like polynomials in n and j.

We have thus shown that the asymptotic behavior of our function allows us to perform a Sommerfeld-Watson transform. If $T_{ss'}{}^{nM}$ has a pole at $n = \alpha_i$, we have

$$H^{t\ Mr\pm}_{s'\mu s\lambda}[p] \sim \frac{1 + \xi e^{i\pi(\alpha-r-v)}}{\sin \pi(\alpha - r - v)}\, d^{\alpha M}_{s'\mu\ \alpha-r}(-\tfrac{\pi}{2})\, d_{\mu r}{}^{\alpha-r}(\theta)\, d^{\alpha M}_{\alpha-r\ \lambda s}(\tfrac{\pi}{2})$$

$$\times\ \beta_s{}^{\alpha M}\, \beta_{s'}{}^{\alpha M} \tag{45}$$

for $z = |\cos \Theta| \gg 1$. This is the rth daughter contribution from a Lorentz family with Toller quantum number M.

Substituting (45) into (43), we have

$$H^{t}_{s'\mu s\lambda}[p] \sim \sum_{m=-q}^{q} \sum_{r=0}^{N} \frac{1 + \xi e^{+i\pi(\alpha-r-v)}}{\sin \pi(\alpha - v)}\, d^{\alpha M}_{s'\mu\ \alpha-r}(-\tfrac{\pi}{2})\, d_{\mu\lambda}{}^{\alpha-r}(\theta)$$

$$\times\ d^{\alpha M}_{\alpha-r\ \lambda s}(\tfrac{\pi}{2})\, \beta_s{}^{\alpha M}\, \beta_{s'}{}^{\alpha M} , \tag{46}$$

where N is a finite integer. The signature in (46) is the same as that at $t \neq 0$. The analyticity-factorization method[28] shows that only one value of M and α is allowed. The group-theoretic method shows the factorization property of the residue functions of the daughter Reggeons.

C. Equivalence Between the Continued O(4) Expansion and the O(3,1) Expansion

At first sight, Eq. (46) differs from Eq. (7.2) of Sciarrino and Toller.[2] However, we shall show that they are equivalent.

Since Toller did not introduce the signature in his O(3,1) expansion, we shall continue Eq. (46) without introducing a signature. The partial-wave amplitude can be reexpressed as

$$T_{ss'}^{nM} = A_{ss'}^{nM} + (-1)^n B_{ss'}^{nM} , \tag{47}$$

where $A_{ss'}^{nM}$ and $B_{ss'}^{nM}$ are bounded by a constant separately. Substituting (47) into (44), and replacing $(-1)^n d_{\mu\lambda}^{n-r}(\theta)$ by $(-1)^\mu d_{\mu-\lambda}^{n-r}(\pi - \theta)$, we have

$$H^{t\ Mr}_{s'\mu s\lambda}(s,0) = \sum_{n=q+1}^{\infty} (M^2 - n^2)\, d^{nM}_{s'\mu\ n-r}(-\tfrac{\pi}{2})\, d^{nM}_{n-r\ \lambda s}(\tfrac{\pi}{2})$$

$$\times\ [A_{ss'}^{nM}\, d_{\mu\lambda}^{n-r}(z) + (-1)^\mu B_{ss'}^{nM}\, d_{\mu\lambda}^{n-r}(-z)] \ . \tag{48}$$

Continuing the two terms in Eq. (48) separately, we obtain

$$H^{t\ Mr}_{s'\mu s\lambda}(s,0) \sim \frac{\beta_s^{\alpha M}\ \beta_{s'}^{\alpha M}}{\sin \pi(\alpha - v)}\ d^{\alpha M}_{s'\mu\ \alpha-r}(-\tfrac{\pi}{2})\ d_{\mu\lambda}^{\ \alpha-r}(z)\ d^{\alpha M}_{\alpha-r\ \lambda s}(\tfrac{\pi}{2}) \tag{49}$$

for $|z| \gg 1$.

We now continue in the n plane an identity

$$D^{nM}_{s'\mu s\lambda}\left(a_x(\theta)\right) = \sum_j d^{nM}_{s'\mu j}(-\tfrac{\pi}{2})\ d_{\mu\lambda}^{\ j}(\theta)\ d_{j\lambda s}^{\ nM}(\tfrac{\pi}{2}) \ , \tag{50}$$

which holds true for the O(4) group, by taking a few leading terms in z, and by replacing j by n - r. We then obtain

$$D^{\alpha M}_{s'\mu s\lambda}\left(a_x(\theta)\right) \sim \sum_{r=0}^{N} d^{\alpha M}_{s'\mu\ \alpha-r}(-\tfrac{\pi}{2})\ d_{\mu\lambda}^{\ \alpha-r}(z)\ d^{\alpha M}_{\alpha-r\ \lambda s}(\tfrac{\pi}{2}) \ , \tag{51}$$

where N is a finite integer. From Eqs. (42), (49), and (51), we have

$$H^{t}_{s'\mu s\lambda}\left(a_x(\theta)\right) \sim \sum_{M=-q}^{q} \frac{\beta_{s'}^{\alpha M}\ \beta_s^{\alpha M}}{\sin \pi(\alpha - v)}\ D^{\alpha M}_{s'\mu s\lambda}\left(a_x(\theta)\right) \tag{52}$$

for $|\cos \theta| \gg 1$. From Eq. (52) and the inverse relation of Eq. (35), one has

$$H^{t}_{\lambda_3\lambda_4,\lambda_1\lambda_2}\left(a_x(\theta)\right) \sim \sum_{M=-q}^{q} \sum_{s's} (-1)^{s_2+s_4-\lambda_2-\lambda_4}\ C(s_3,s_4,s';\ \lambda_3,-\lambda_4)$$
$$\times\ C(s_1,s_2,s;\ \lambda_1,-\lambda_2)\ \beta_{s'}^{\alpha M}\ \beta_s^{\alpha M}\ D^{\alpha M}_{s'\mu s\lambda}[a_x(\theta)] \ . \tag{53}$$

This expression is equivalent to Eq. (7.1) in Sciarrino and Toller's paper.[2] Thus the vertex function $V^{M\alpha}$ and $W^{M\alpha}$ can be related to that in Eq. (53); i.e.,

$$V_{s\lambda}^{M\alpha v} \propto d^{\alpha M}_{\alpha-r\ \lambda s}(\tfrac{\pi}{2})\ C(s_1,s_2,s;\ \lambda_1,-\lambda_2)\ (-1)^{s_2-\lambda_2} ,$$

$$W_{s'\mu}^{M\alpha r} \propto d^{\alpha M}_{\alpha-r\ \mu s}(-\tfrac{\pi}{2})\ C(s_3,s_4,s';\ \lambda_3,-\lambda_4)\ (-1)^{s_4-\lambda_4} . \tag{54}$$

These relationships show the equivalence between the O(3,1) and the O(4) expansions.

We may also reformulate the Toller asymptotic expansion in terms of s-channel helicity states. From an identity

$$a_x(\theta) = u_y(-\tfrac{\pi}{2})\ a_z(\theta)\ u_y(\tfrac{\pi}{2}) = a_z(-\tfrac{\pi}{2})\ u_y(\theta)\ a_z(\tfrac{\pi}{2}) ,$$

which holds true for the O(4) group, we have the formula

$$d^{nM}_{s'\lambda s}(\theta) = \sum_{\mu'\lambda' j} d_{\lambda\mu'}{}^{s'}(-\tfrac{\pi}{2})\ d^{nM}_{s'\mu' j}(-\tfrac{\pi}{2})\ d^{j}_{\mu'\lambda'}(\theta)\ d^{nM}_{j\lambda' s}(\tfrac{\pi}{2})\ d_{\lambda'\lambda}{}^{s}(\tfrac{\pi}{2}) , \tag{54}$$

which can be continued into the n plane by taking $j = n - r$ and by considering only a few leading terms in z:

$$d^{\alpha M}_{s'\lambda s}(\theta) \sim \sum_{\mu'\lambda' r} d_{\lambda\mu'}{}^{s'}(-\tfrac{\pi}{2})\ d^{\alpha M}_{s'\mu'\ \alpha-r}(-\tfrac{\pi}{2})\ d^{\alpha-r}_{\mu'\lambda'}(z)\ d^{nM}_{\alpha-r\ \lambda' s}(\tfrac{\pi}{2}) \times d^{s}_{\lambda'\lambda}(\tfrac{\pi}{2}) \tag{55}$$

for $|z| \gg 1$. From Eqs. (37), (49), and (55), we obtain

$$H^{S}_{s'\mu s\lambda}(s,0) \sim \delta_{\mu\lambda} \sum_{M=-q}^{q} \frac{\beta_s^{\alpha M} \beta_{s'}^{\alpha M}}{\sin \pi(\alpha - v)} d^{\alpha M}_{s'\lambda s}(\theta) \qquad (56)$$

for $|\cos \theta| = |z| \gg 1$. This result is the same as that from the analyticity-factorization method.[26]

V. CONCLUSION

The 3-dim. symmetries of a scattering amplitude provide a guide for continuation through the Sommerfeld-Watson transform. For the 4-dim. symmetry, we were not able to continue the O(4) expansion to the O(2,2) region in the c.m. frame. In the b.w. frame, we have to replace j by n - r in the continuation. Here we point out the differences between the 3-dim. and the 4-dim. symmetries, which may add to the understanding of this situation.

With the criteria stated in Sec. IIA, we found that the amplitude has the same kind of 3-dim. symmetries in a particular region, independently of whether the amplitude is defined in the c.m. frame or in the b.w. frame (see Sec. IIB). For the 4-dim. symmetries, the type of symmetry depends on the frames in which the amplitude is defined. At $t = 0$, the amplitude always has the symmetry associated with the complex Lorentz group.[8] However, this introduces many new quantum numbers, and complicates the problems. One may have difficulty in identifying the new quantum numbers with physical quantities.

As stated in Sec. IIA, the helicity amplitude is invariant except for a real phase factor under its symmetry groups in the 3-dim. regions, but it is covariant only in the 4-dim. symmetry regions.

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FOOTNOTES AND REFERENCES

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22. If there are some fermions which have the same mass as the bosons so that the reaction $B + F \rightarrow B' + F'$ occurs as an example of the pairwise equal-mass case, we have to deal with the covering groups of the O(4) and O(2,2) group. However, the results of this section will not change.

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FIGURE LEGENDS

Fig. 1a. The boundaries of the regions of the symmetries on the Mandelstam diagram for the general-mass case. The lines (1), (2), (3), and (4) are given by the equations $t = (m_1 + m_2)^2$, $t = (m_3 + m_4)^2$, $t = (m_1 - m_2)^2$, and $t = (m_3 - m_4)^2$.

Fig. 1b. The boundaries of the regions of symmetries of a c.m. helicity amplitude on the Mandelstam diagram for the pairwise equal-mass case, i.e., $m_1 = m_2 = m$, $m_3 = m_4 = m'$. For the b.w. helicity amplitude, the 4-dim. symmetry $O(2,2)$ must be replaced by $O(3,1)$.

Fig. 2 (a) A multiparticle system.

(b) A decomposition of the multiparticle system.

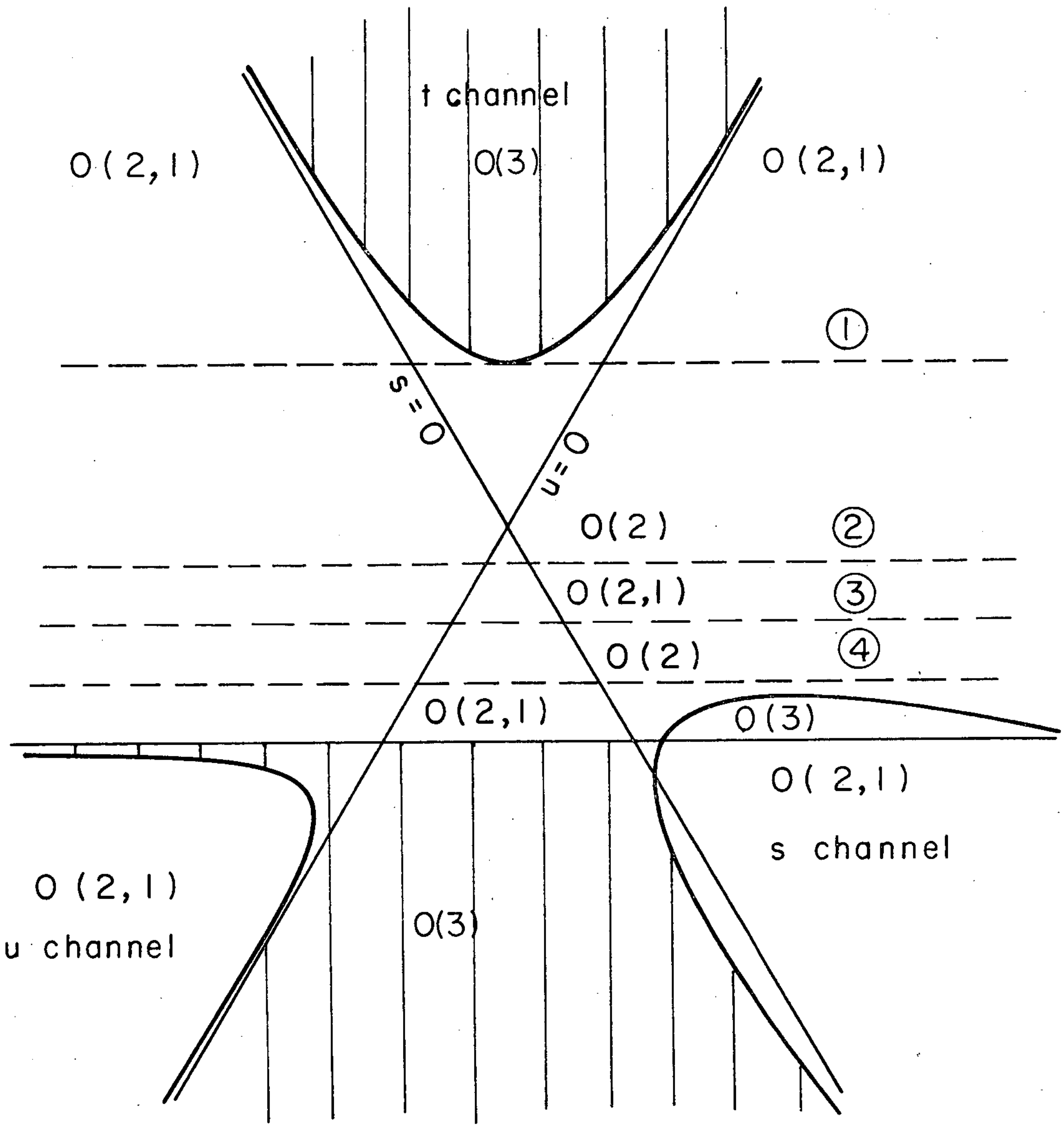
t channel
O(3)
O(2,1)
O(2,1)
①
s=0
u=0
O(2)
②
O(2,1)
③
O(2)
④
O(2,1)
O(3)
O(2,1)
s channel
O(2,1)
u channel
O(3)
XBL699-3743

Fig. 1a

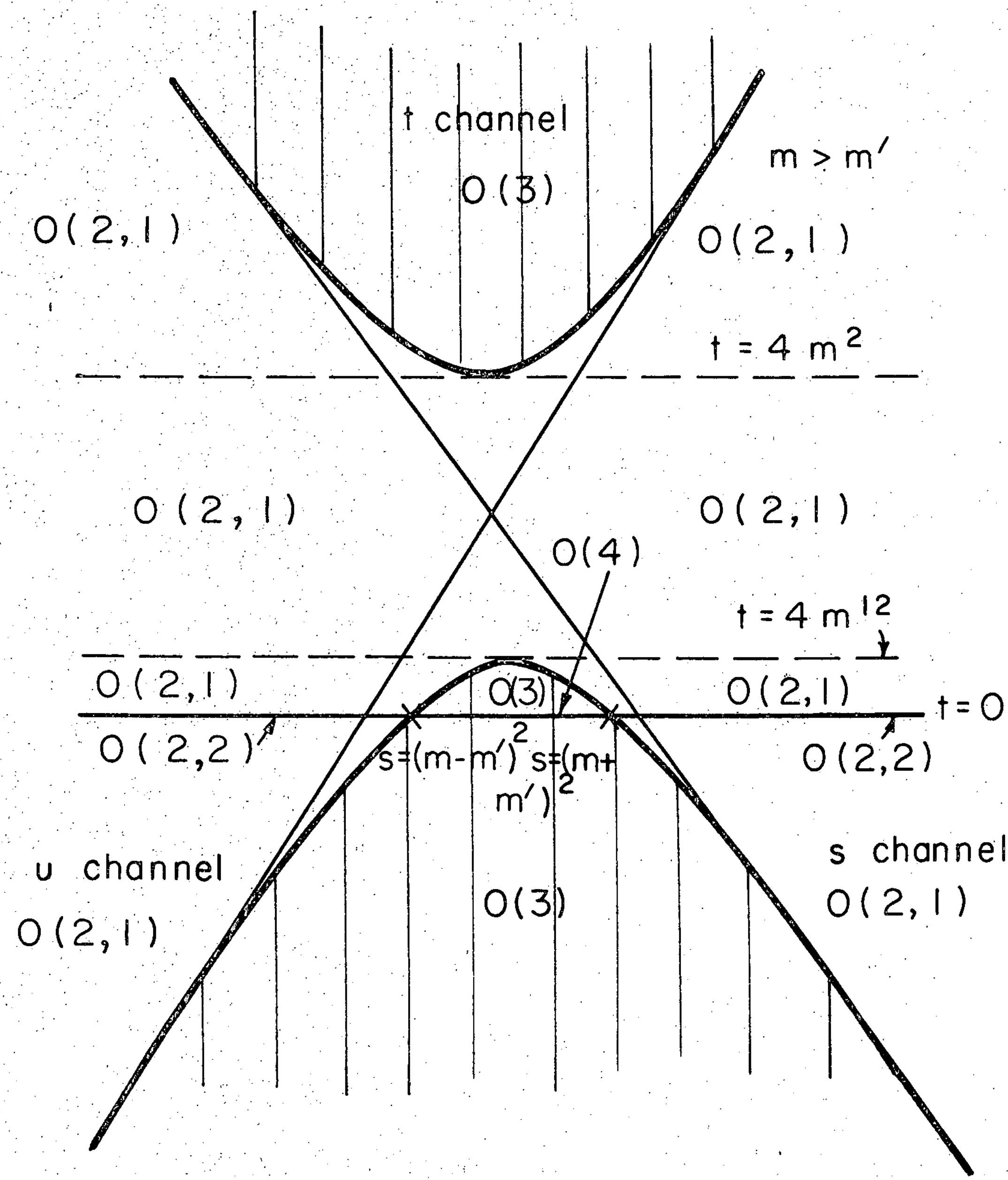
t channel
O(3)
m > m′
O(2,1)
O(2,1)
t = 4 m²
O(2,1)
O(2,1)
O(4)
t = 4 m′²
O(2,1)
O(3)
O(2,1)
t = 0
O(2,2)
O(2,2)
s=(m-m′)²
s=(m+m′)²
u channel
O(2,1)
O(3)
s channel
O(2,1)
XBL699-3742

Fig. 1b

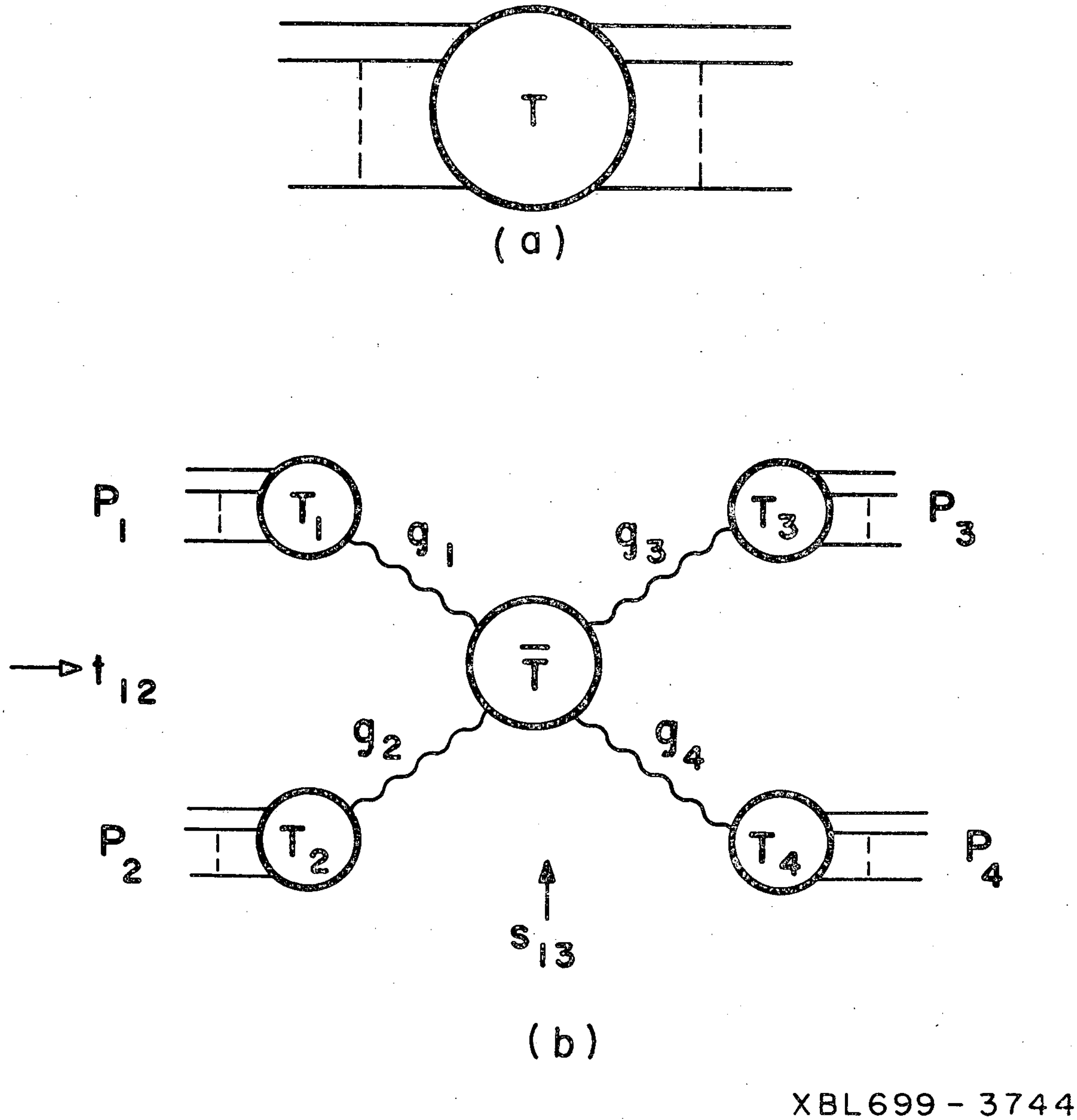

Fig. 2