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Linearization: Towards a Planar Algebra Analog of the Free Skew Field

A dissertation submitted in partial satisfaction
of the requirements for the degree
Doctor of Philosophy in Mathematics

by

Juniper Bahr

2024

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ABSTRACT OF THE DISSERTATION

Linearization: Towards a Planar Algebra
Analog of the Free Skew Field

by

Juniper Bahr
Doctor of Philosophy in Mathematics
University of California, Los Angeles, 2024
Professor Dimitri Y. Shlyakhtenko, Chair

The set of non-commutative rational functions on n indeterminates, called the free skew field and denoted $\mathbb{C}\langle\!\langle x_1, \ldots, x_n \rangle\!\rangle$ [Ami66], can sometimes be evaluated on a tuple of random variables in a tracial von Neumann algebra $X_1, \ldots, X_n \in (M, \tau)$, resulting in a set in $\mathrm{Aff}(M)$ of non-commutative rational expressions in the random variables $X_1, \ldots, X_n$, where $\mathrm{Aff}(M) \supseteq M$ is the algebra of affiliated operators [MSY23].

Constructing the free skew field can be done with linearization, a technique for representing non-commutative rational functions as products of a row vector, the inverse of a matrix that is *linear* in the x_i's, and a column vector [CR99].

The graded algebra construction $\mathrm{Gr}_k P$ associated to a planar algebra can be thought of as generalizing the set non-commutative polynomials and matrices over those polynomials [GJS10]. We prove an analog of a linearization result in the context of planar algebras, a step on the path towards the construction of a planar algebra analog of the free skew field in n indeterminates.

The dissertation of Juniper Bahr is approved.

Wilfrid Dossou Gangbo

Georg Menz

Sorin Popa

Dimitri Y. Shlyakhtenko, Committee Chair

University of California, Los Angeles

2024

To the kindest people I've ever known,

who make me want to be gentler and tougher all at once,

and who taught me how connected our liberation is;

to my boyfriend, Meagan, I love the home we've made together;

to my friends, to Becky, Patrick, xan, Merlin, Tesla, Erin,

and everyone who has shared love and friendship with me;

to my sibling, I hope the years bring us closer still,

to my parents,

to all my family,

to my teachers, especially Mrs. Chesebro, Mr. McCarthy, Dr. Tice,

and everyone who encouraged my curiosity;

to everyone who's given me patience as we figure this out together,

to everyone who couldn't give me patience, too,

to the future,

to the present,

to the past,

to the little yellow rumped warbler that visited my window this spring,

to everyone resisting the ways you're asked to hurt yourselves and others,

and the ways others hurt you;

to Leslie Feinberg, for writing "I got what I needed. You have no idea how much I want."

and to transsexuals, leather dykes, and freaks and faggots across the world,

I love you.

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CHAPTER 1

Motivation

1.1 Rational numbers and the field of fractions

Undergraduate real analysis courses typically begin at the natural numbers, $\mathbb{N} = \{0, 1, 2, \ldots\}$, then work their way up to constructing the integers, $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$, the rationals, $\mathbb{Q} = \{1/2, -3, 13/12, \ldots\}$, and then finally the real numbers $\mathbb{R} = \{1.3, \sqrt{7}, -\pi, \ldots\}$.

Once we have the integers and want to construct the rationals, we usually do this by formally representing rational numbers as pairs of integers (a, b) and then define operations on these pairs so that they act like we'd expect a/b to behave. That is, $(3, 2) \times (5, 3)$ should equal $(5, 2)$ (representing the equation $3/2 \times 5/3 = 5/2$). A small concern is that multiple pairs of integers "should be" the same rational number: because $4/2 = 2/1 = 6/3$, we expect the pairs $(4, 2), (2, 1), (6, 3)$ to be "the same" when viewed as rational numbers.

The standard way of addressing this concern is by defining an equivalence relation between pairs. Formally, $(a, b) \sim (c, d)$ if and only if $ad = bc$. This way, $(4, 2) \sim (2, 1) \sim (6, 3)$ just as we wanted. They are "the same" in the sense that they "are $\sim$".

We can do one step better though, and consider the set of equivalence classes of these pairs of integers, for instance $[(2, 3)] = \{(a, b) \mid (a, b) \sim (2, 3)\}$ will be the collection of all pairs that are $\sim$ to $(2, 3)$. This way $[(2, 3)] = [(4, 6)]$, since everything equivalent to $(4, 6)$ is equivalent to $(2, 3)$, and vice versa. It's the classic way of upgrading an equivalence relation to an *equality*. Finally, we say that $\mathbb{Q}$ is the set of all of these equivalence classes [DF09].

This idea lets us move from integers to all the possible fractions we can make out of them, the set of rational numbers.

This technique generalizes to the classic field of fractions construction, a staple of undergraduate abstract algebra courses. Start with a commutative ring R with a (commutative) multiplication and addition operation on it. Assume that R is an integral domain, that is, if $xy = 0$, then $x = 0$ or $y = 0$ (if we don't assume this, we'll have issues defining division). From R, one can build the *field of fractions* of R, denoted $\mathrm{Frac}(R)$. It consists of all the possible quotients of elements in R defined as pairs modded out by an equivalence relation, just like what's done to construct $\mathbb{Q}$ from $\mathbb{Z}$ [DF09].

This construction comes with an inclusion $R \hookrightarrow \mathrm{Frac}(R)$, and every non-zero element of R is invertible in $\mathrm{Frac}(R)$ (in fact, every non-zero element of $\mathrm{Frac}(R)$ is invertible).

Keep this example in mind throughout the rest of the work. It's a common theme: look at pairs, consider operations on those pairs, define an equivalence relation, then take the quotient.

1.2 The free skew field: rational functions in non-commutative variables

The field of fractions construction works great when the ring is commutative, but for non-commutative rings, things get a bit messier. Whenever I write "ring" without qualifying whether it's commutative or not, assume it's a not necessarily commutative ring.

Suppose R is a non-commutative ring. It would be nice to embed R into a new ring S such that every non-zero element of S is invertible. Such a ring is called a *division ring* (think: ring + division) or a *skew field* (think: field minus commutativity of multiplication). However, we have a few problems. First, it's not obvious how to do this, and second, it's sometimes downright impossible.

If we naïvely attempt to redo the field of fractions construction, we'll consider the set of all words (strings of symbols connected with addition and multiplication) involving a and the formal symbol a^{-1} for $a \neq 0$ when $a \in R$ (expressions like $xy^{-1}x + z$ or $x^{-1} + xyz^2y$, where $x, y, z \in R$).

Unfortunately, it's not obvious which of these words are equivalent. For example, when $R = \mathbb{C}\langle x, y\rangle$, the set of non-commutative polynomials in two variables, do we have that the two expressions

$$(x+y)^{-1} \stackrel{?}{=} y^{-1}(x^{-1}+y^{-1})^{-1}x^{-1}$$

are equivalent?[1] Or are these two expressions,

$$x^{-1}+y^{-1} \stackrel{?}{=} 4(x+y-(x-y)^2((x-y)(x+y)(x-y))^{-1}(x-y)^2)^{-1},$$

equivalent?[2] Or how about

$$y^{-1}+y^{-1}(z^{-1}x^{-1}-y^{-1})^{-1}y^{-1} \stackrel{?}{=} (y-zx)^{-1}$$

these?[3] Or maybe

$$(x-y^{-1})^{-1}-x^{-1}-(xyx-x)^{-1} \stackrel{?}{=} 0 \text{ ?}$$

these?[4]

If we want to take this constructive approach, we need a good system for understanding when two of these expressions are equivalent. Hold this thought for now. *Linearization* and the construction of the free skew field as outlined in [CR99] will be how we understand when two expressions are equivalent.

---

[1] Yes, these are equivalent, see [Ami66]

[2] These are equivalent too, but it takes a good deal of work. See [KPP20].

[3] Yes, see [CR99, p. 13].

[4] Yes, see [MSY20, p. 31].

Before we elaborate on this, we have a bigger problem to confront. As I mentioned, sometimes there is *no* skew field S containing R. Let's now discuss this occasional impossibility in more depth.

While it is true that for any non-commutative ring R and multiplicatively closed *subset* V (containing 1 but not 0), there is a homomorphism (satisfying some universal property) from R into a new ring R_V that sends V into the group of units (invertible elements) of R_V. However, this R_V is not guaranteed to be a skew field, nor is the homomorphism guaranteed to be an embedding. In fact, R_V can even be zero! [Lam99]. Worse still, some non-commutative rings without zero divisors simply cannot be embedded in *any* skew field. See [Mal37] for an example and [Lam99] for a helpful exposition.

So if we have no hope for a general ring R to be embedded in some skew field S, we narrow our line of questioning to easier rings like $\mathbb{C}\langle x_1, \ldots, x_n\rangle$, the ring of non-commutative polynomials in the indeterminates $x_1, \ldots, x_n$.

Reframing the question now, we wonder: Is there an embedding from $\mathbb{C}\langle x_1, \ldots, x_n\rangle$ into some ring S that takes every non-zero non-commutative polynomial into a unit of S? Or to ask a stronger sounding question, can we also require that S is a skew field, in which *every* non-zero element is a unit?

Yes! To both! Actually, in general there can be multiple non-isomorphic embeddings of $\mathbb{C}\langle x_1, \ldots, x_n\rangle$ into skew fields [KV12, p. 54].

But we can impose an additional condition on S by requiring it to be a *universal skew field of fractions* [KV12], and we find our many choices for S become just one: *the* **free skew field**. The free skew field is also called the *skew field of non-commutative rational functions*, is denoted $\mathbb{C}\langle\!\!\langle x_1, \ldots, x_n\rangle\!\!\rangle$, and was first constructed by Amitsur in [Ami66].

We'll be relying heavily on Cohn's later study of this object in [CR99] and [Coh06, Chap. 7], because Cohn's construction relies on linearization, a method for expressing rational functions in terms of inverses of matrices that are linear in the variables.

1.3 Linearization

One way of constructing the free skew field on n indeterminates is to define the set of triples (u, A, v) where $A \in M_{n\times n}(\mathbb{C}\langle x_1, \ldots, x_n\rangle)$ is a matrix, and u, v are row and column vectors respectively over $\mathbb{C}\langle x_1, \ldots, x_n\rangle$. Each triple should be thought of as representing the term $uA^{-1}v$ in $\mathbb{C}\langle\!\langle x_1, \ldots, x_n\rangle\!\rangle$, just like how in the field of fractions construction, (a, b) represented the rational number a/b.

This is where we come to that thought I asked you to hold in Section 1.1: how do we tell when two expressions are equivalent? Cohn defines an equivalence relation on these triples, and it's now a lot easier to deal with than directly looking at the equivalence of terms like $(x - y^{-1})^{-1} - x^{-1} - (xyx - x)^{-1}$.

Taking a quotient by this equivalence relation yields the free skew field [CR99].

Why is this possible though? Why should every non-commutative rational function be able to be written as $uA^{-1}v$ when we're specifically requiring that A is linear in $x_1, \ldots, x_n$? Without knowing this, we can't proceed with this construction at all.

The argument that explains why just looking at terms like $uA^{-1}v$ where A is linear is called (unsurprisingly) *linearization*. Depending on your perspective, it's either a result you prove after you've built the free skew field some different way, or it's a consequence of proving that the construction relying on linearization works.

This is a very useful tool. Informally, the simplicity we lose by going from expressions in $\mathbb{C}\langle x_1, \ldots, x_n\rangle$ to *matrices* over $\mathbb{C}\langle x_1, \ldots, x_n\rangle$, is more than made up for by the simplicity we gain by restricting our attention to only the matrices with *linear* entries and by the easier time we have describing the equivalence relation.

1.4 Division closure and evaluating the free skew field on non-commutative random variables

Let's shift perspectives somewhat. So far, we've been looking at $\mathbb{C}\langle x_1, \ldots, x_n \rangle$, from which we construct the free skew field, $\mathbb{C}\langle\!\langle x_1, \ldots, x_n \rangle\!\rangle$, which we think of as the set of non-commutative rational functions.

Rather than the *abstract* indeterminates $x_1, \ldots, x_n$, let's imagine we have random variables $X_1, \ldots, X_n$ and consider what rational functions of them might be.

As we're fully invested in non-commutativity at this point (I hope you are, I know I am), we look at non-commutative probability spaces.

In particular, we'll consider (M, τ) where M is a finite von Neumann algebra, and τ a faithful normal trace. Elements of M are referred to as non-commutative random variables. See [Spe11] for an introductory survey of free probabilty theory.

So, given a tuple of non-commutative random variables $X_1, \ldots, X_n \in M$, what are non-commutative rational functions of the X_i?

We could either frame this as a question of *evaluating* the free skew field that's already been constructed, extending the evaluation map $\mathbb{C}\langle x_1, \ldots, x_n \rangle \to M$ to $\mathbb{C}\langle\!\langle x_1, \ldots, x_n \rangle\!\rangle \to \operatorname{Aff}(M)$. Here $\operatorname{Aff}(M)$ is the algebra of operators affiliated to M, inside which M is a subalgebra.

Or, we could try to look at sums and products and quotients of the X_i and take a *division closure*: the smallest subalgebra of $\operatorname{Aff}(M)$ containing each X_i and closed under inverses.

To take as many reciprocals as we can, in both cases we're passing to $\operatorname{Aff}(M) \supseteq M$, the algebra of affiliated operators. This way, any element of M will be invertible in $\operatorname{Aff}(M)$ so long as it has trivial kernel.

Actually, the first idea (extending the evaluation map) doesn't work that well in general, as the evaluation map from $\mathbb{C}\langle\!\langle x_1, \ldots, x_n \rangle\!\rangle$ may simply not be defined, depending on the

$X_1, \ldots, X_n$ chosen! As [MSY23] point out: consider ℓ, the left shift operator on $\mathcal{B}(\mathcal{H})$ with $\mathcal{H}$ a separable infinite dimensional Hilbert space. Observe that $\ell\ell^* = 1$ while $\ell^*\ell \neq 1$. And while $y(xy)^{-1}x = 1$ in the free skew field, we do *not* have the corresponding equality when we set $x = \ell$ and $y = \ell^*$, since $\ell^*(\ell\ell^*)^{-1}\ell \neq 1$.

In fact, there are random variables $X_1, \ldots, X_n$ in the tracial von Neumann algebra M which have trivial kernel (and hence are invertible in $\mathrm{Aff}(M)$), and non-constant polynomials p such that $p(X_1, \ldots, X_n)$ has non-trivial kernel (and hence is non-invertible in $\mathrm{Aff}(M)$). In [ACS24, Thm. 7.1], the polynomial $(xy - yx)$ is shown to have a non-trivial kernel (and thus be non-invertible in $\mathrm{Aff}(M)$) whenever either of the random variables x or y in a tracial von Neumann algebra has an atom anywhere in its distribution with measure more than $1/2$.

To make sense of evaluation, we need to understand when rational expressions have an atom in their distribution at zero, and we need some assurance that no expressions which are identical in $\mathbb{C}\langle\!\langle x_1, \ldots, x_n \rangle\!\rangle$ are distinguished in the affiliated operators. We restrict to *stably finite* algebras, where left and right invertibility are equivalent and where this equivalence holds for even matrix amplifications of the algebra. On stably finite algebras $\mathcal{A}$, each rational function $r \in \mathbb{C}\langle\!\langle x1, \ldots, x_n \rangle\!\rangle$ has an associated domain $dom_{\mathcal{A}}(r) \subseteq \mathcal{A}^n$ on which evaluation is well-defined [MSY23, p. 17].

The second idea, of taking a division closure of the algebra generated by the X_i inside $\mathrm{Aff}(M)$, leads to a more thorough understanding of non-commutative rational functions of the X_i. Mai et al. find necessary and sufficient conditions for the division closure of $(X_1, \ldots, X_n)$ to be isomorphic to the free skew field, in which case the evaluation map is well-defined and injective. They also find necessary and sufficient conditions for the division closure to be *some* skew field (i.e., when all non-zero elements are invertible), in which case evaluation is still well-defined and injective [MSY23].

At this point, we could continue to study polynomials and rational functions of random variables, or we could attempt to generalize by studying a stranger setting: planar algebras.

1.5 Introduction to planar algebras

Broadly speaking, a planar algebra is a sequence of vector spaces $(P_{2n}^{\pm})$ that comes equipped with a function that turns certain kinds of diagrams, *planar tangles*, into multilinear maps on these vector spaces. These planar tangles come with an operad structure that corresponds to composition between the maps on the vector spaces.

Let's be a bit more specific but still relatively informal in describing planar tangles now, saving the technicalities for Section 2.

A *shaded planar tangle* is a diagram with:

1. finitely many labelled "input" discs
2. one "output" disc (containing the input discs)
3. an even (possibly zero) number of marked points on each disc
4. one distinguished interval per disc in between the marked points (usually marked with a $\star$ when drawn)
5. non-crossing paths called strings whose endpoints lie on the marked points or are loops
6. a shading of the regions in between the strings so that adjacent regions are shaded oppositely.

These diagrams are considered up to isotopy that preserves the $\star$'s, shading, and disc numberings. As an example, consider the following planar tangle with three input discs:

To compose planar tangles, informally speaking, you place the outer disc of one into the inner disc of another, lining up distinguished points, strings, and shadings, then erase the border between them. As there may be multiple input discs, there are multiple compositions. As a concrete example, we may compose the following two tangles, but only with the second tangle placed in the input disc labelled 1. Placing the second tangle in the other input isn't

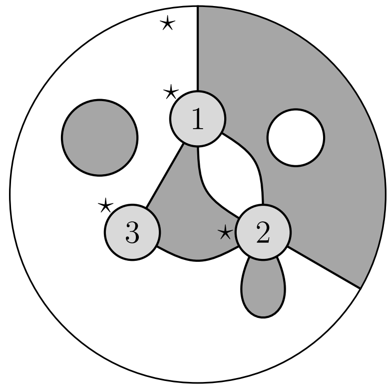


Figure 1.1: An example of a shaded planar tangle.

valid, as the number of marked points doesn't line up: the output disc of the second tangle has four marked points, while the input disc labeled 2 has two marked points.

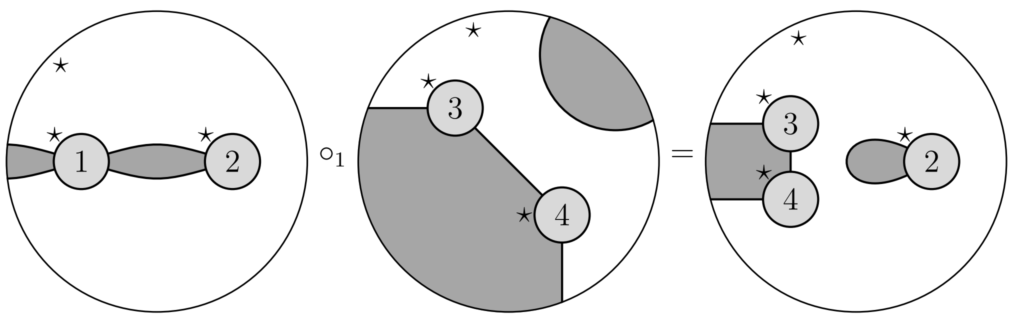


Figure 1.2: An example of planar tangle composition.

Very roughly, the vectors in a planar algebra can be thought of as objects that can be plugged into the input discs of a planar tangle, producing a new object in the planar algebra. The space P_{2k}^{+} in particular, should be thought of as consisting of objects with $2k$ marked points and the distinguished interval shaded white. In this way, each planar tangle will correspond to a multilinear function on the planar algebra.

In particular, a planar tangle T with input discs $D_1, \ldots, D_k$ with $k_1, \ldots, k_n$ marked points on each, and distinguished intervals shaded $\varepsilon_1, \ldots, \varepsilon_k \in \{+, -\}$, and output disc D_0 with k_0 marked points and distinguished interval shaded ε_0 induces a map

$$Z_T : P_{2k_1}^{\varepsilon_1} \otimes \cdots \otimes P_{2k_n}^{\varepsilon_n} \to P_{2k_0}^{\varepsilon_0}.$$

We'll generally restrict to a certain class of planar algebras called *subfactor planar algebras* which arise when studying subfactors, and which we'll define formally in Section 2.3. These subfactor planar algebras come with several additional conditions, including finite dimensionality of the $P^{\pm}_{2n}$, equivalence of two particular tangles (the left and right trace), and the fact that any string which forms a closed loop in a planar tangle may be replaced by a simple factor of δ, a constant called the (square root of) the *index* of the planar algebra.

Let's talk now about the connection between planar algebras and non-commutative polynomials.

1.6 Polynomials, the GJS Construction, and Matrix-like Behavior

Guionnet, Jones, and Shlyakhtenko construct from a subfactor planar algebra P a sequence of algberas $(\mathrm{Gr}_k(P))_{k=0}^{\infty}$ with multiplication operations $\wedge_k$ [GJS10]. The construction involves reinterpreting the planar algebra's vectors as rectangles with the marked points on the top and sides, and with distinguished intervals in the top-left corner of the rectangle.

Figure 1.3 shows the planar tangle that induces the operation $\wedge_k$, with the shadings removed.

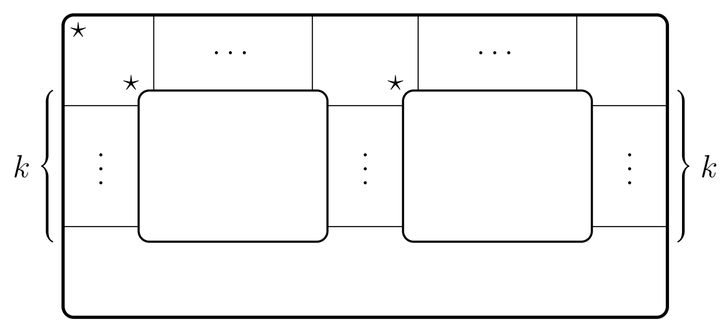
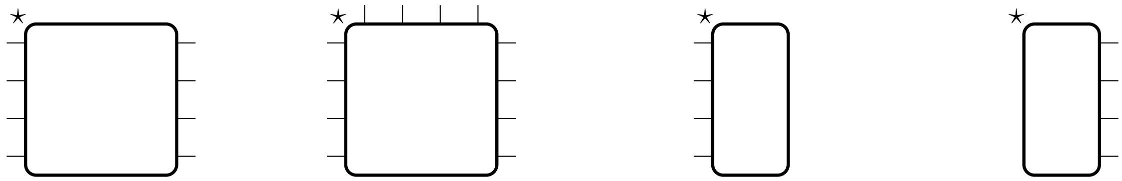


Figure 1.3: The planar tangle that induces $\wedge_k$.

These $\mathrm{Gr}_k(P)$ spaces behave in some ways like sets of square matrices of a particular size over a set of non-commutative polynomials.

Informally, the size of the matrix is determined by the number of marked points on the sides, and the degree of the polynomial is determined by the number of marked points on the top. Under this interpretation, $\wedge_k$ represents matrix multiplication.



(a) An element of Gr_4, to be thought of like a $\delta^4 \times \delta^4$ scalar valued matrix.

(b) An element of Gr_4, to be thought of like a $\delta^4 \times \delta^4$ matrix with values in degree 4 polynomials.

(c) Not an element of Gr_4, but in a related algebra; to be thought of like a vector.

(d) Not an element of Gr_4, but in a related algebra; to be thought of like a covector.

Figure 1.4: Interpretations of diagrams depending on where the marked points are located.

What I've said here is very imprecise. We'll see in Section 2.3.2 that there's a very nice case when these genuinely *are* matrices, and there's a very reasonable sense in which they're *like* matrices for every subfactor planar algebra.

The nice case when this interpretation is true, is the (shaded) polynomial planar algebra on k variables. The vectors of $P_{2n}^{\pm}$ consist of linear combinations of non-commuting polynomials in $x_1, \ldots, x_k, x_1^*, \ldots, x_k^*$ that alternate between non-adjoint and adjointed variables. The $\pm$ grading determines whether they start on a x_i or an x_i^*. Each of these polynomials is then thought of like a disc where the marked points correspond to the variables in the monomials.

Planar tangles act on these polynomials by contracting indices: a string joining some x_i to an x_i^* contributes a factor of 1 (removing the pair $x_i x_i^*$), and strings joining x_i to x_j^* for $j \neq i$ result in zero. In this example, $\mathrm{Gr}_k(P)$ is precisely the set of square matrices of a fixed size over the even-degree polynomials. We'll cover this more carefully in Section 2.3.2.

More broadly, the matrix interpretation comes from the combination of a few facts. First, every subfactor planar algebra is a sub-planar algebra of a *graph planar algebra*. A graph planar algebra is associated to a particular bipartite graph, and the vectors should be thought of as loops in the graph of a particular length. A loop of length $2n$ is thought of as like an element of the vector space P_{2n}.

Roughly speaking, planar tangles act on these loops by gluing them together to form new loops, where the strings indicate how the gluing occurs. Gluing arbitrary loops of course isn't guaranteed to produce a path, let alone a loop. Only strings that join the marked point corresponding to an edge e and its oppositely oriented version e^o will contribute. The point is, the structure of a graph planar algebra is a generalization of the structure of the polynomial planar algebra. So interpreting the structure of Gr_k as somewhat like matrices over non-commutative polynomials is still meaningful intuition.

Section 2.3.2 is intended to clarify this interpretation of $\mathrm{Gr}_k(P)$ being like a set of square matrices over polynomials.

1.7 Connecting planar algebras and the free skew field

Let's connect the ideas discussed so far. We have the free skew field $\mathbb{C}\langle\!\langle x_1, \ldots, x_n \rangle\!\rangle$, which is the space of non-commutative rational functions and is constructed as equivalence classes of triples (u, A, v) where u is a covector, v a vector, and A a linear matrix over $\mathbb{C}\langle x_1, \ldots, x_n\rangle$. Each triple (u, A, v) represents the non-commutative rational function $uA^{-1}v$.

We also have, under some conditions, the realization of the free skew field in $\mathrm{Aff}(M)$, the algebra of operators affiliated to a finite tracial von Neumann algebra (M, τ). In particular, the free group factor $M = L(\mathbb{F}_n)$ meets those conditions.

And given a subfactor planar algebra P, the $\mathrm{Gr}_k(P)$ construction has structure analogous to matrices over a set of non-commutative polynomials.

Combining these ideas, we come to the question: what happens if we consider equivalence classes of triples (u, A, v) where u, A, v are in a planar algebra and are a “covector”, “linear matrix”, and “vector” respectively? Can we construct something like the free skew field over the set of non-commutative polynomials? We expect this object to depend on the planar algebra considered.

The first step of such a task is to understand linearization in this new context. In this dissertation, we produce for each $p \in \mathrm{Gr}_0$, a planar algebra triple (u, A, v) that satisfies $p = uA^{-1}v$ in $\mathrm{Aff}(M_k)$, where M_k is a von Neumann algebra containing Gr_k. We also provide formulas for addition and multiplication of these triples, as well as show that A may be chosen to be self-adjoint.

CHAPTER 2

Background

Let's now go through the background material carefully so we can formally state and prove our results.

2.1 von Neumann Algebras

2.1.1 C^* and von Neumann algebras

The background we assume here is familiarity with vector spaces, norms, completeness, Hilbert spaces, the spectral theorem, and other material in line with a first or second year graduate level course in functional analysis.

Let's go over some notation.

I'll usually denote Hilbert spaces by $\mathcal{H}$, denote their associated inner product by $\langle \cdot, \cdot \rangle$ or by $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ when more than one inner product is in use. Inner products are assumed to be conjugate-linear in their first variable and linear in their second. Similarly I'll denote the associated norm $\|\cdot\| = \sqrt{\langle \cdot, \cdot \rangle}$ or $\|\cdot\|_{\mathcal{H}}$.

Next, the set of bounded linear operators on $\mathcal{H}$ (that is, from $\mathcal{H}$ to itself) is $\mathcal{B}(\mathcal{H})$. For an operator $T \in \mathcal{B}(\mathcal{H})$, we denote the operator norm by $\|T\|$, $\|T\|_\infty$, or $\|T\|_{\text{op}}$, and it's given by $\sup_{\|\xi\| \le 1} \|T(\xi)\|$. Let's recall some topologies on $\mathcal{B}(\mathcal{H})$, following [Bla06].

Definition 2.1.1. The *norm topology* is the topology where convergence of a net $T_\alpha \to T$ is equivalent to $\|T_\alpha - T\| \to 0$.

Definition 2.1.2. The *strong operator topology (SOT)* is given by the seminorms $T \to \|T(\xi)\|$ for each $\xi \in \mathcal{H}$. Convergence of a net $T_\alpha \to T$ is thus equivalent to $\|(T_\alpha - T)(\xi)\| \to 0$ for each $\xi \in \mathcal{H}$

Definition 2.1.3. The *weak operator topology (WOT)* is given by the seminorms $T \to |\langle T(\xi), \zeta\rangle|$ for each $\xi, \zeta \in \mathcal{H}$. Thus, convergence of a net $T_\alpha \to T$ is equivalent to $\langle T(\xi), \zeta\rangle \to 0$ for each $\xi, \zeta \in \mathcal{H}$.

Let's also mention a few definitions we'll need for the abstract characterization of C^* algebras.

Definition 2.1.4. A *Banach algebra* is a Banach space $(A, \|\cdot\|)$ equipped with a multiplication such that $\|xy\| \leq \|x\|\|y\|$ for all $x, y \in A$.

A Banach algebra with a conjugate linear operator $x \mapsto x^*$ that squares to the identity, preserves the norm, and reverses multiplication (so $(xy)^* = y^*x^*$) is called *involutive*.

We can now recall the definitions of concrete and abstract C^* and W^* (or von Neumann) algebras.

Definition 2.1.5. A (concrete) C^* algebra is a $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ closed under the operator norm. It is called unital when it contains the identity $\mathrm{id}_{\mathcal{H}}$.

An (abstract) C^* algebra is an involutive Banach algebra with the property that $\|x^*x\| = \|x\|^2$. It is called unital if it contains an identity.

Definition 2.1.6. A (concrete) W^* algebra, or von Neumann algebra is a WOT (or equivalently SOT) closed $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ containing the identity $\mathrm{id}_{\mathcal{H}}$. In particular, it is a concrete unital C^* algebra.

An (abstract) W^* algebra is an (abstract) C^* algebra M such that there is some Banach space X with dual X^* isometrically isomorphic to M. That is, abstract von Neumann algebras have a predual [Bla06, p. 275].

The above concrete definition of von Neumann algebras is rather "analytic" in flavor, as it deals with topology and limits. Thanks to von Neumann's bicommutant theorem, we have an equivalent concrete defintion of von Neumann algebras with a more algebraic feel (in the sense of "algebraic relations").

Definition 2.1.7. The commutant of a subset $S \subseteq \mathcal{B}(\mathcal{H})$ is the set $S' = \{y \in \mathcal{B}(\mathcal{H}) \mid yx = xy \text{ for all } x \in S\}$. The commutant of a commutant S'' is called the *bicommutant*.

Theorem 2.1.8 (von Neumann's Bicommutant Theorem)**.** *For a unital $*$-subalgebra M of $\mathcal{B}(\mathcal{H})$, the following are equivalent:*

1. $M = M''$
2. *M is WOT closed*
3. *M is SOT closed.*

See [AP], [Bla06] for thorough discussions of these definitions and results.

In order to connect concrete and abstract C^* algebras, we'll discuss states and the GNS construction. The abstract characterization of von Neumann algebras, while presented here for completeness's sake, isn't of much use to us. However, the algebraic concrete form ($M = M''$) will be incredibly useful by allowing us to consider von Neumann algebras *generated* by operators.

2.1.2 Quick Interlude on Positivity and Spectra

Definition 2.1.9. For A a unital C^* algebra and $x \in A$, the spectrum of x is

$$\sigma_A(x) = \{\lambda \in \mathbb{C} \mid x - \lambda \operatorname{id}_A \text{ is not invertible}\}$$

There are quite a few facts about spectra, but we'll just need to mention that the spectrum is *intrinsic* for unital C^* algebras: a larger "ambient" unital C^* algebra doesn't change the spectrum, in the following precise sense.

Proposition 2.1.10. *If $B \subseteq A$ is a unital inclusion of unital C^* algebras, and $x \in B$, then $\sigma_B(x) = \sigma_A(x)$ [Bla06, p. 58].*

As a consequence, the notions of spectrum for abstract unital C^* algebras and for concrete ones (with invertibility considered in $\mathcal{B}(\mathcal{H})$) are equivalent.

Definition 2.1.11. For A an (abstract or concrete) C^* algebra and $x \in A$, we say x is positive and write $x \geq 0$ if $\sigma_A(x) \subseteq [0, \infty)$.

The positive elements in A form a *cone*, i.e., they are closed under addition and scalar multiplication by non-negative reals. I usually visualize the first quadrant $\{(x, y) \mid x, y \geq 0\}$ in $\mathbb{R}^2$, which is also a cone. You could also visualize an actual cone (like the shape); that works fine too.

2.1.3 States and the GNS construction

Given an abstract C^* algebra A, we'd like to view A as a concrete C^* algebra somehow. This means we need to construct some Hilbert space $\mathcal{H}$ and some embedding $\pi : A \to \mathcal{B}(\mathcal{H})$ such that $\pi(A)$ is a concrete C^* algebra on $\mathcal{B}(\mathcal{H})$. Assume A is unital for simplicity.

The first step on this journey is to describe *states* on C^* algebras.

Definition 2.1.12. A *state* on a C^* algebra A is a linear functional $\phi : A \to \mathbb{C}$ satisfying

1. $\phi(x) \geq 0$ if $x \geq 0$ (positivity)

2. $\phi(1) = 1$

A state is called *faithful* if $\phi(x) = 0 \implies x = 0$ on positive elements x. A state is called *tracial* (or simply called *a trace*) if $\phi(xy) = \phi(yx)$ for all $x, y \in A$. A state on a von Neumann algebra is *normal* if for every increasing net $x_\iota \nearrow x$ of positive elements, $\phi(x) = \sup_\iota \phi(x)$ [Bla06] [AP].

The supremum is well-defined here, since the set of projections on a von Neumann algebra with the operations $\wedge$ and $\vee$ (given by projections onto the intersections and sums of the images of projetions, respectively) forms a *complete* lattice [Bla06].

We quickly mention some terminology we'll use later before returning to the GNS construction.

Definition 2.1.13. A tracial von Neumann algebra (M, τ) is a von Neumann algebra M equipped with a normal faithful tracial state.

The GNS construction takes a C^* algebra A with a state $\phi : A \to \mathbb{C}$ and produces a Hilbert space $\mathcal{H}$ and a representation of A on $\mathcal{B}(\mathcal{H})$. The construction itself will be useful to keep in mind.

Theorem 2.1.14 (GNS Representation - [GN43], [Seg47])**.** *Let A be a C^* algebra and $\phi : A \to \mathbb{C}$ be a state on A.*

Define a sesquilinear function $\langle x, y\rangle_\phi = \phi(x^* y)$ on A. This may not be an inner product, as $\langle x, x\rangle = 0$ does not necessarily imply $x = 0$ (or equivalently, ϕ may not be faithful).

Set $N_\phi = \{x \in A \mid \phi(x^* x) = 0\}$. This is a closed left-ideal of A, and the sesquilinear function $\langle \cdot, \cdot\rangle_\phi$ descends to an inner product $\langle \cdot, \cdot\rangle$ on the quotient A/N_ϕ.

Let $\mathcal{H}$ be the Hilbert space given by the completion of A/N_ϕ equipped with the norm induced by $\langle \cdot, \cdot\rangle$.

Left multiplication $A \curvearrowright A$ descends to a well-defined operation $A \curvearrowright A/N_\phi$, which can be extended by continuity to a well-defined representation $\pi_A : A \to \mathcal{B}(\mathcal{H})$ called the GNS representation*.*

I like the presentation of this result in [AP], where it's noted that $\mathcal{H}$ is often denoted $L^2(A, \phi)$ in analogy to the case of the abelian tracial von Neumann algebra $L^\infty([0, 1])$, where $\mathcal{H} \simeq L^2([0, 1])$.

Remark 2.1.15. For $x \in A$, we write $\hat{x} \in \mathcal{H}$ for the element of $\mathcal{H}$ corresponding to $x + N_\phi \in A/N_\phi$. Using this notation, π_A is defined by the continuous extension of $\pi_A(a)(\hat{b}) = \widehat{(ab)}$.

Definition 2.1.16. Given a tracial von Neumann algebra (M, τ) acting by the GNS representation on $L^2(M)$, consider the map $\hat{x} \mapsto \hat{x^*}$ on $\hat{M} \subseteq L^2(M, \tau)$.

This map is a conjugate linear isometry which extends to Tomita's conjugation operator, $J : L^2(M) \to L^2(M)$. [AP], [Tak79]

We'll need J later. Let's move on to more topological results now.

2.1.4 Useful Topological Results

Definition 2.1.17. The von Neumann algebra *generated* by some $S \subseteq \mathcal{B}(\mathcal{H})$ is the WOT closure of the unital $*$-algebra generated by S.

By Theorem 2.1.8, the bicommutant theorem, and by the fact that commutants are always WOT closed, we may equivalently define it as S''.

Although the WOT and SOT are distinct topologies, their closures of convex subsets of $\mathcal{B}(\mathcal{H})$ agree.

Proposition 2.1.18. *A linear functional $\phi : \mathcal{H}(\mathcal{H}) \to \mathbb{C}$ is continuous with respect to the strong operator topology if and only if it's continuous with respect to the weak operator topology.*

As a consequence of the Hahn-Banach theorem, if $S \subseteq \mathcal{B}(\mathcal{H})$, the closures of S in SOT and WOT agree: $\overline{S}^{SOT} = \overline{S}^{WOT}$ [Dix81, p. 79].

The Kaplansky Density theorem is another topological result which we'll find useful.

Theorem 2.1.19 (Kaplansky)**.** *Let A be a unital $*$-subalgebra of $\mathcal{B}(\mathcal{H})$.*

Then $(A)_1 \subseteq (A'')_1$ is SOT dense, where $(B)_1 = \{x \in B \mid \|x\| \leq 1\}$ denotes the (closed) unit ball.

Furthermore, $A_{sa} \subseteq (A'')_{sa}$ is SOT dense, where B_{sa} refers to the self-adjoint elements of B.

See [AP] or [Bla06, 48] which mentions the σ-strong topology instead of SOT but is an equivalent result.

2.1.5 Type Classification

There are a few more things we need to mention before we move on to subfactors. Let's talk about *factors* and the *type classification* of von Neumann algebras, as we'll have much to say about type II_1 factors.

Definition 2.1.20. A factor is a von Neumann algebra with trivial center (recall the *center* of M is the set of elements of M that commute with all of M, where trivial means equal to $\mathbb{C}1$)

All finite dimensional von Neumann algebras have a pretty nice structure, they're finite direct sums of matrix algebras, like this

$$M_{k_1 \times k_1}(\mathbb{C}) \oplus M_{k_2 \times k_2}(\mathbb{C}) \oplus \cdots \oplus M_{k_n \times k_n}(\mathbb{C}),$$

see [Bla06, II.8.3.2.(iv)].

The center of such a von Neumann algebra is easy to calculate: it's $\mathbb{C}I_{k_1} \oplus \mathbb{C}I_{k_2} \oplus \cdots \oplus \mathbb{C}I_{k_n}$. Naturally then we see that finite dimensional *factors* are just matrix algebras $M_{k \times k}$ and general finite dimensional von Neumanna algebras are (finite) direct sums of factors.

This idea generalizes, and von Neumann algebras in general are direct *integrals* of factors. The *central decomposition* is

$$M = \int_X^{\oplus} M_x \, d\mu(x)$$

where almost every M_x is a factor [von49]. See also [Bla06, III.1.6.4].

The details of direct integrals or the central decomposition aren't important for us right now, but it helps explain why we spend time on factors rather than always looking at more general von Neumann algebras.

Next, let's say a few things about the famous type classification of von Neumann algebras by Murray and von Neumann [Mv36], which relies on the study of projections (idempotent and self-adjoint elements).

Definition 2.1.21. Two projections p, q in a von Neumann algebra $M \subseteq \mathcal{B}(\mathcal{H})$ are *Murray-von Neumann equivalent*, denoted $p \sim q$, if there is a partial isometry $u \in M$ such that $uu^* = p$ and $u^*u = q$.

We say p is *subordinate* to q (denoted $\partial \lesssim q$) if $p \sim p' \leq q$, where $\leq$ is the ordering on projections induced by inclusion of their images (so $\mathrm{im}(p') \subseteq \mathrm{im}(q) \subseteq \mathcal{H}$) [Bla06].

We say p is orthogonal to q, denoted $p \perp q$, if $\mathrm{im}(p) \perp \mathrm{im}(q) \subseteq \mathcal{H}$.

The $\lesssim$ relation behaves nicely with regards to addition.

Proposition 2.1.22. *If $\{p_\iota\}$ are mutually orthogonal projections and $\{q_\iota\}$ are mutually orthogonal projections, with $p_\iota \lesssim q_\iota$ for each ι, then $\sum p_\iota \lesssim \sum q_\iota$, where convergence is in SOT when the sum is infinite, see [Bla06] and [AP, Thm. 2.4.1].*

Proposition 2.1.23. *For two projections $p, q \in M$, we have $p \lesssim q$ or $q \lesssim p$, and if both are true, then $p \sim q$ [AP, Thm. 2.4.4 and Cor. 2.4.9].*

Definition 2.1.24 ([Bla06])**.** A projection p in a von Neumann algebra M is

1. *finite* if $p \sim q \leq p$ implies $p = q$.

2. *infinite* if it's not finite.

3. *properly infinite* if $p \sim q_1$ and $p \sim q_2$ with $q_1 \leq p$, $q_2 \leq p$ and $q_1 \perp q_2$. In other words, p is equivalent to two orthogonal projections that live under p.

4. *abelian* if pMp is commutative.

5. *continuous* if $0 \neq q \leq p$ implies q is not abelian.

6. *discrete* if $q \leq p$ implies q is not continuous

7. *semifinite* if $0 \neq q \leq p$ implies there exists some finite $0 \neq q' \leq q$.

8. *purely infinite* if $0 \neq q \leq p$ implies q is not finite.

This leads us to the type classification:

Theorem 2.1.25 ([Mv36]). *Let M be a factor on a separable Hilbert space $\mathcal{H}$. Then M is exactly one of the following types:*

1. *Type I: 1 is discrete*

   Type I_n: isomorphic to $\mathcal{B}(\ell^2(n))$, or 1 is discrete and finite

   Type I_∞: isomorphic to $\mathcal{B}(\ell^2(\mathbb{N}))$, or 1 is discrete and infinite

2. *Type II: 1 is continuous and semifinite*

   Type II_1: 1 is continuous and finite

   Type II_∞: 1 is continuous and properly infinite.

3. *Type III: 1 is purely infinite.*

See also [Bla06].

Theorem 2.1.26. *Let M be a type II_1 factor. Then there is a unique faithful normal trace $\tau : M \to \mathbb{C}$. The trace τ on projections is surjective onto $[0, 1]$ [Bla06, III.2.5.7].*

2.1.6 Subfactors

We don't need to discuss much about subfactors, but being aware of them is useful. Let's being with subfactors and the subfactor index. We'll be describing some things following [Jon91].

First, there's a notion of the *dimension* or *coupling constant* of a module of a type II_1 factor.

Definition 2.1.27. Let M be a II_1 factor acting via $\pi : M \to \mathcal{B}(\mathcal{H})$ where $\pi(M)$ is unital and WOT closed in $\mathcal{B}(\mathcal{H})$.

The *dimension* or *coupling constant* $\dim_M(\mathcal{H})$ is ∞ if M' is not a II_1 factor.

Otherwise, pick a vector $\xi \in \mathcal{H}$, and note that the projections p and q onto $\overline{M\xi}$ and $\overline{M'\xi}$ are in M' and M respectively. Then, given the unique normalized traces τ_M and $\tau_{M'}$ on $\pi(M)$ and M' respectively, the *dimension* or *coupling constant* is $\dim_M(\mathcal{H}) = \operatorname{tr}_M(q)/\operatorname{tr}_{M'}(p)$ [Mv36]. See [Jon91] for exposition and context.

Definition 2.1.28. Given $N \subseteq M$ a (unital) inclusion of II_1 factors, the *Jones index* or *subfactor index* is

$$\begin{aligned} [M : N] &= \dim_N(L^2M) \\ &= \dim_N(\mathcal{H})/\dim_M(\mathcal{H}) \end{aligned}$$

whenever M acts on $\mathcal{H}$ with finite coupling constant. Here N is called a *subfactor* of M.

Theorem 2.1.29. *The Jones index takes values in* $\{4\cos^2(\pi/n) \mid n \geq 3\} \cup [4, \infty)$*, and all values are realized by some subfactor [Jon83].*

Remark 2.1.30. As Jones notes in [Jon83], the name *index* comes from the group theoretic notion of index. Given a subgroup $H \subseteq G$, $[G : H]$ counts the number of cosets of H in G. Denoting the left-regular representation of G on $\ell^2(G)$ by λ, we have $[\lambda(G)'' : \lambda(H)''] = [G : H]$ when G and H are ICC (infinite conjugacy class) groups. Here, these notions of index agree.

Remark 2.1.31. It's not essential for our presentation here, but I'd encourage anyone interested to read up on Bratteli diagrams, as they're useful in constructing subfactors of a particular index, and naturally provide more context for why Perron-Frobenius eigenvectors will be popping up in a few sections. I rather like the presentation in [Jon91] and recommend it.

2.1.7 Affiliated Operators

Let's now cover a few facts about unbounded operators. Unbounded operators are formalized by discussing functions defined on subspaces of a Hilbert space.

Definition 2.1.32. A *densely defined operator* on a Hilbert space $\mathcal{H}$ is a dense subspace called the domain of the operator $D(T) \subseteq \mathcal{H}$ along with a linear function $T : D(T) \to \mathcal{H}$.

Such an operator is called *closed* if its graph

$$\Gamma(T) = \{(x, T(x)) \in \mathcal{H} \times \mathcal{H} \mid x \in D(T)\}$$

is closed in $\mathcal{H} \times \mathcal{H}$ [Bla06].

Note that bounded densely defined operators can be extended to the entire space, so most of the study of densely defined operators is concerned with the unbounded ones.

Equality of operators requires their domains be equal, which is a somewhat strong requirement.

Definition 2.1.33. If T and S are densely defined operators, we say $T \subseteq S$ if $D(T) \subseteq D(S)$ and on $D(T)$, $T = S\restriction_{D(T)}$.

If T and S have the same domain and are equal on that shared domain, we write $T = S$ [Bla06].

Definition 2.1.34. A densely defined operator T is *permutable* with a bounded operator S if $ST \subseteq TS$.

For (M, τ) a tracial von Neumann algebra acting via its GNS representation on $L^2(M)$, and for T a closed densely defined operator on $L^2(M)$, we say that T *is affiliated with* M if T is *permutable* with every element of $M' \subseteq \mathcal{B}(L^2M)$ (the commutant of M) [Bla06].

It's not clear why two densely defined operators should be comparable in any way. Their domains may or may not overlap, so defining composition or addition may be tricky.

The set of affiliated operators to a von Neumann algebra is rather nice when the von Neumann algebra is tracial.

Theorem 2.1.35. *Let (M, τ) be a tracial von Neumann algebra, then the set of closed densely defined operators affiliated to M, denoted $\text{Aff}(M)$ or $L^0(M)$, is a $*$-algebra [AP, 7.2.8].*

Remark 2.1.36. The notation $L^0(M)$ is a nod to the commutative case. The algebra of operators affiliated to $L^\infty([0,1])$ acting via multiplication on $L^2([0,1])$ is $L^0([0,1])$, the set of measurable functions [AP, Ex. 7.2.9].

Affiliated operators on a tracial von Neumann algebra may be well understood by their action on the image of a projection of trace close to one.

Proposition 2.1.37. *Let (M, τ) be a tracial von Neumann algebra, and let $T \in \text{Aff}(M)$. Then for every $\varepsilon > 0$, there exists p a projetion in M with trace $\tau(p) > 1 - \varepsilon$ and $pL^2(M) \subseteq D(T)$. That is, T is defined everywhere on pL^2M, and thus by the closed graph theorem, is bounded on that subspace. [AP, 7.2.5]*

Note also that it's enough to consider these kinds of subspaces restricted to the images of projections with trace close to one.

Proposition 2.1.38. *Let (M, τ) be a tracial von Neumann algebra. If $W \subseteq L^2M$ and for every $\varepsilon > 0$ there exists a projection $p \in M$ with $\tau(p) > 1 - \varepsilon$ and $W \supseteq pL^2M$, then W is dense in L^2M.*

2.2 Free Skew Field

2.2.1 Abstract Construction

Let's now discuss the free skew field on the ring of non-commutative polynomials $\mathbb{C}\langle x_1, \ldots, x_n \rangle$. Much of this will follow [CR99], although the order of some propositions will differ.

Definition 2.2.1. Let R be a ring. A matrix $A \in M_{n\times n}(R)$ is *full* if it cannot be written as the product $BC = A$ with $B \in M_{n\times(n-1)}(R), C \in M_{(n-1)\times n}(R)$ [CR99].

Definition 2.2.2. A *representation* is a quadruple written as $c + (u, A, v)$ where A is a full matrix over $\mathbb{C}\langle x_1, \ldots, x_n\rangle$, u and v are row and column vectors over $\mathbb{C}\langle x_1, \ldots, x_n\rangle$, and $c \in \mathbb{C}\langle x_1, \ldots, x_n\rangle$. A representation $c + (u, A, b)$ is *pure* if $c = 0$. A representation is *linear* if the entries of A are linear in $x_1, \ldots, x_n$ (i.e., are polynomials of degree at most one) [CR99].

Remark 2.2.3. As a preview of what will follow, a matrix A over $\mathbb{C}\langle x_1, \ldots, x_n\rangle$ which is full is invertibile over the free skew field $\mathbb{C}\langle\!\langle x_1, \ldots, x_n\rangle\!\rangle$. [CR99]

We will think of $\rho = c + (u, A, v)$ as representing the element of the free skew field $c + uA^{-1}v$. Keep this interpretation in mind, as doing formal calculations with inverses provides some insight into why the following constructions work.

Remark 2.2.4. We will be able to restrict our attention to the linear representations because of "linearization by enlargement" (also called "Higman's trick" [Hig40]) [CR99, p. 309].

Definition 2.2.5. A *morphism* between pure linear representations from (u, A, v) to (u', A', v') is a pair (P, Q) of matrices over $\mathbb{C}$ satisfying $u' = uQ$, $Pv' = v$, and $PA' = AQ$ [CR99].

Proposition 2.2.6. *There is a category with pure linear representations as the objects and the morphisms as described above [CR99, p. 309].*

Definition 2.2.7. We say two pure linear representations (u, A, v) and (u', A', v') are *equivalent* if there exists a finite sequence of morphisms and inverse morphisms that goes from (u, A, v) to (u', A', v').

Remark 2.2.8. Actually, equivalence is defined in [CR99] between (not necessarily pure or linear) representations and it's remarked that each representation has an equivalent pure and linear representation. To simplify the discussion, we're going to focus on the pure and linear representations here though.

Remark 2.2.9. To see why this is the right notion, we look at the corresponding elements in the free skew field they are supposed to represent: $uA^{-1}v$ and $u'(A')^{-1}v'$, and do a purely formal calculation.

If there's a morphism (P, Q) from the former to the latter, then we just calculate the following as matrices over the free skew field:

$$uA^{-1}v = uA^{-1}Pv' = uQ(A')^{-1}v' = u'(A')^{-1}v'.$$

Theorem 2.2.10 ([CR99, Lem. 1.2 and Thm. 1])**.** *A representation (u, A, v) is equivalent to 0 if and only if for some invertible scalar matrices P, Q, we have the block decomposition:*

$$uQ = \begin{pmatrix} * & 0 \end{pmatrix}, PAQ = \begin{pmatrix} B & 0 \\ * & C \end{pmatrix}, Pv = \begin{pmatrix} 0 \\ * \end{pmatrix}$$

where B and C are square.

By 0 we mean a representation that represents the number 0. Cohn uses $(x, B, y) = (, ,)$ which is a representation of order 0, but if zero by zero matrices are too uncomfortable for you, feel free to consider $\big((1, 0), I_{2\times 2}, (0, 1)^\top\big)$ as a representation of zero, noting that $(1, 0)I_{2\times 2}^{-1}(0, 1)^\top = 0$.

Corollary 2.2.11 ([CR99])**.** *If (u, A, v) is equivalent to (the representation for) 0, the matrix*

$$\bar{A} = \begin{pmatrix} A & v \\ u & 0 \end{pmatrix}$$

is not full.

Proof. In particular, apply Theorem 2.2.10 to (u, A, v) to get P, Q such that

$$PAQ = \begin{pmatrix} B & 0 \\ * & C \end{pmatrix},$$

where B is $k \times k$ and C is $\ell \times \ell$. Now notice that

$$\begin{pmatrix} P & 0 \\ 0 & I \end{pmatrix} \bar{A} \begin{pmatrix} Q & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} B & 0 & 0 \\ * & C & * \\ * & 0 & 0 \end{pmatrix}.$$

Thus, up to multiplying by some invertible scalar matrices, $\bar{A}$ is seen to have a $(k+1) \times (\ell+1)$ sub-marix of zeroes.

If a matrix D has a sub-matrix of all zeroes where the number of rows and columns of the sub-matrix sums to greater than the number of rows or columns of D, we say that D is *hollow*.

The above calculation shows that, up to multiplying on either side by an invertible matrix, $\bar{A}$ is hollow.

Hollow matrices are not full. See the factorization in [Coh06, Prop. 3.1.2]. $\square$

Definition 2.2.12 ([CR99], [Coh95])**.** The free field $\mathbb{C}\langle\!\langle x_1, \ldots, x_n \rangle\!\rangle$ is the set of equivalence classes of pure linear representations with $0 = [((1,0), I_{2\times 2}, (0,1)^\top)]$, $1 = [(1, I_{1\times 1}, 1)]$, and the following field operations:

$$[(u, A, v)] + [(u', A', v')] = \left[\left(\begin{pmatrix} u & u' \end{pmatrix}, \begin{pmatrix} A & 0 \\ 0 & A' \end{pmatrix}, \begin{pmatrix} v \\ v' \end{pmatrix} \right) \right]$$

$$[(u, A, v)] \cdot [(u', A', v')] = \left[\left(\begin{pmatrix} u & 0 \end{pmatrix}, \begin{pmatrix} A & -vu' \\ 0 & A' \end{pmatrix}, \begin{pmatrix} 0 \\ v' \end{pmatrix} \right) \right]$$

$$[(u, A, v)]^{-1} = \left[\left(\begin{pmatrix} 0 & 1 \end{pmatrix}, \begin{pmatrix} A & -v \\ u & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) \right]$$

These are well-defined by [Coh95, Sect. 4.3] (and discussed in [CR99]), noting that the matrix $\begin{pmatrix} A & v \\ u & 0 \end{pmatrix}$ is non-full precisely when (u, A, v) is equivalent to zero.

Each linear element of $p \in \mathbb{C}\langle x_1, \ldots, x_n \rangle$ may be realized as the triple

$$\left(\begin{pmatrix} 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -p \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right),$$

and by a similar construction on higher degree terms, combined with the reduction steps outlined in [Coh85, Sect. 5.8], one obtains a *linear* representation for each element of $\mathbb{C}\langle x_1, \ldots, x_n \rangle$. This yields a map $\mathbb{C}\langle x_1, \ldots, x_n \rangle \to \mathbb{C}(\!\langle x_1, \ldots, x_n \rangle\!)$.

As a consequence of [Coh85, Theorem 2.9.15], the map $\phi : \mathbb{C}\langle x_1, \ldots, x_n \rangle \hookrightarrow \mathbb{C}(\!\langle x_1, \ldots, x_n \rangle\!)$ is an embedding. See also the discussion in [CR99].

Let's discuss now a few properties of the free skew field.

The free skew field $\mathbb{C}(\!\langle x_1, \ldots, x_n \rangle\!)$ is *free* in the sense that every relation in it holds for "algebraic reasons" as shown by Cohn in [CR99]. Let's think back to Section 1.2, where we talked about the idea of constructing rational expressions by directly looking at words in $x_1, \ldots, x_n$ using the operations of addition, multiplication, scalar multiplication, and reciprocation. From this perspective, we ask that expressions are only equivalent if their difference can be reduced algebraically to 0.

Let's elaborate now on the argument in [CR99]. Let R_k be defined inductively as follows.

Recall the embedding $\phi : \mathbb{C}\langle x_1, \ldots, x_n \rangle \hookrightarrow \mathbb{C}(\!\langle x_1, \ldots, x_n \rangle\!)$, and let

$$(R_0, u_0) = (\mathbb{C}\langle x_1, \ldots, x_n \rangle, \phi).$$

Now inductively define

$$R_{k+1} = R_k * \mathbb{C}\langle y_f \mid f \in R_k, u_k(f) \neq 0 \rangle \text{ modulo the relations } y_f f = f y_f = 1,$$

where the $\{y_f\}$ are symbols indexed by $f \in R_k$. In other words, we're adjoining to R_k the inverses of elements that have non-zero image under the map u_k. Think of the term y_f as the symbol in R_{k+1} that represents the inverse of the term $f \in R_k \subseteq R_{k+1}$.

The map u_{k+1} is the extension of u_k to R_{k+1} that sends $f \in R_k$ to $u_k(f) \in \mathbb{C}(\!\langle x_1, \ldots, x_n \rangle\!)$ and sends y_f to $u_k(f)^{-1} \in \mathbb{C}(\!\langle x_1, \ldots, x_n \rangle\!)$.

There's a nice homomorphism from each $R_k \to R_{k+1}$ given that the latter is a free product. This homomorphism respects the map u_k, and we can take an inductive limit of the (R_k, u_k) to get (R, u), where R is a ring and $u : R \to \mathbb{C}\langle\!\langle x_1, \dots, x_n \rangle\!\rangle$ is a homomorphism.

Theorem 2.2.13 ([CR99, Thm. 3.2])**.** *The map u is an isomorphism.*

Remark 2.2.14. In other words, the free skew field $\mathbb{C}\langle\!\langle x_1, \dots, x_n \rangle\!\rangle$ is given by repeatedly adjoining to $\mathbb{C}\langle x_1, \dots, x_n \rangle$ the inverses of elements, with no additional relations imposed. The lack of additional relations is one justification for the term "free" being used here.

2.2.2 Concrete Realization

We can now discuss concrete "realizations" of the free skew field. Given $X_1, \dots, X_n \in (M, \tau)$ a tracial von Neumann algebra, there's an evaluation map $\mathbb{C}\langle x_1, \dots, x_n \rangle \to M$ sending $x_i \mapsto X_i$. Depending on what properties the tuple $(X_1, \dots, X_n)$ satisfies, there might be a map extending the evaluation map from $\mathbb{C}\langle\!\langle x_1, \dots, x_n \rangle\!\rangle$ into $\operatorname{Aff}(M) \supseteq M$. Understanding when this kind of map is defined is the work of [MSY23].

Definition 2.2.15. Given B a subalgebra of A, the *division closure* of B in A, denoted $\mathbb{C}\langle\!\langle B \rangle\!\rangle$, is the smallest subalgebra of A containing B which is closed under A-inverses, i.e., if $x \in \mathbb{C}\langle\!\langle B \rangle\!\rangle$ and x has an inverse in A, then $x^{-1} \in \mathbb{C}\langle\!\langle B \rangle\!\rangle$ as well [MSY23].

Remark 2.2.16. Keep in mind, the division closure of an algebra may well *not* be a division ring. As a reminder, a division ring is another name for a skew field, and may be thought of as either a (not-necessarily-commutative!) ring in which every non-zero element is invertible, or equivalently as a field with the assumption that multiplication is commutative dropped.

The division closure is also not to be confused with the *rational closure* which concerns the entries of inverses of matrices.

Definition 2.2.17 ([Coh06])**.** Given a ring R, a homomorphism $\phi : R \to S$ into another ring S, and a set Σ of square matrices over R (not necessarily all the same size), we say that ϕ is Σ-*inverting* if its image under ϕ consists entirely of invertible matrices.

We use the convention that for a matrix $A \in M_{N\times N}(R)$, $\phi(A) = (\phi(A_{ij}))_{ij}$ is ϕ applied entrywise.

Definition 2.2.18 ([Coh06])**.** Given rings R, S, a set Σ of square matrices over R, and a Σ-inverting homomorphism $\phi : R \to S$, the Σ-*rational closure* $R_\Sigma(S)$ is the set

$$\left\{ \left(\phi(A)^{-1}\right)_{ij} \mid N \in \mathbb{N}, A \in \Sigma \cap M_{N\times N}(R), 1 \le i, j \le N \right\},$$

of all entries of inverses of matrices.

When Σ is taken to be the set of all matrices over R with invertible image under ϕ, the Σ-rational closure is called *the rational closure*, or ϕ-rational closure.

In [CS05, Eqn. 3.9], for a self-adjoint tuple of operators $X_1, \ldots, X_n \in M$ in a tracial von Neumann algebra (M, τ), the quantity $\Delta(X_1, \ldots, X_n) \le n$ is defined as follows.

Definition 2.2.19. For $X_1, \ldots, X_n \in M$ a tracial von Neumann algebra (M, τ) the quantity $\Delta(X_1, \ldots, X_n)$ is given by

$$\Delta(X_1, \ldots, X_n) = n - \dim_{M\otimes M^{\mathrm{op}}} \overline{\left\{ (T_1, \ldots, T_n) \in F(L^2(M, \tau))^n \mid \sum_{i=1}^{n} [T_i, JX_iJ] = 0 \right\}}^{L^2},$$

where $F(L^2(M, \tau))$ is the set of finite rank operators on the GNS Hilbert space $L^2(M, \tau)$, $[\cdot, \cdot]$ is the commutator, J is Tomita's conjugation operator, the conjugate-linear isometry on $L^2(M, \tau)$ induced by the map $x \mapsto x^*$ on M, and $\overline{\cdot}^{L^2}$ is the closure with respect to the L^2 norm $x \mapsto \mathrm{Tr}(x^*x)^{1/2}$, also known as the Hilbert-Schmidt norm [MSY23].

With the above terms defined, we may now discuss one of the main theorems of [MSY23].

Theorem 2.2.20 ([MSY23, Thm. 1.1])**.** *Let M be a von Neumann algebra with faithful normal trace $\tau : M \to \mathbb{C}$, and let $\mathrm{Aff}(M)$ denote the algebra of (closed, densely defined) operators affiliated with M.*

Consider $X_1, \dots, X_n \in M$ which are not necessarily self-adjoint operators, and denote the tuple $X = (X_1, \dots, X_n)$. Define the evaluation map

$$\mathrm{ev}_X : \mathbb{C}\langle x_1, \dots, x_n \rangle \to \mathrm{Aff}(M)$$

by extending the map sending $x_i \to X_i$ linearly. Then the following are equivalent:

1. *The evaluation map ev_X extends to an injective homomorphism*

$$\mathrm{Ev}_X : \mathbb{C}\langle\!\langle x_1, \dots, x_n \rangle\!\rangle \to \mathrm{Aff}(M)$$

   whose image is the division closure $\mathbb{C}\langle\!\langle X_1, \dots, X_n \rangle\!\rangle$.

2. *For any $N \in \mathbb{N}$ and $P \in M_N(\mathbb{C}\langle x_1, \dots, x_n \rangle)$, if P is* linear *and full then $P(X) \in M_N(\mathrm{Aff}(M))$ is invertible.*

3. *For any $N \in \mathbb{N}$ and $P \in M_N(\mathbb{C}\langle x_1, \dots, x_n \rangle)$, if P is full then $P(X) \in M_N(\mathrm{Aff}(M))$ is invertible.*

4. $\Delta(X_1, \dots, X_n) = n$

Remark 2.2.21. Let's relay some insight into these conditions found in [MSY23].

The equivalence between (1) and (2) comes from linearization: rational functions may be expressed in terms of inverses of linear full matrices, so we expect evaluation of rational functions to be well-defined when all linear and full matrices over $\mathbb{C}\langle X_1, \dots, X_n \rangle$ are indeed invertible in $M_{N\times N}(\mathrm{Aff}(M))$.

The equivalence between (2) and (3) is also from linearization, here specifically the idea that it suffices to consider the invertibility of only the linear full matrices, rather than all full matrices.

Condition (4) and (2) are connected through a theorem about Δ we'll now discuss.

Recall that an $k \times k$ matrix A is full if it cannot be expressed as a product BC with B being $k \times (k-1)$ and C $(k-1) \times k$.

Theorem 2.2.22 (Part of Thm 3.1 in [MSY23]). *With (M, τ) a tracial von Neumann algebra and $X_1, \ldots, X_n \in M$, the following are equivalent:*

1. $\Delta(X_1, \ldots, X_n) = n$
2. *For any $N \in \mathbb{N}$ and $A \in M_{N\times N}(\mathbb{C}\langle x_1, \ldots, x_n\rangle)$, if A is linear and full then* $\ker A(X) = \{0\}$.

The above theorem connects maximal Δ to a condition that looks more like invertiblity, as invertibility in $\mathrm{Aff}(M)$ is equivalent to having trivial kernel.

Proposition 2.2.23 ([MSY23, Lem. 5.5]). *For (M, τ) as above, $N \in \mathbb{N}$, and $A \in M_{N\times N}(\mathrm{Aff}(M))$,*

1. *A is invertible in $M_{N\times N}(\mathrm{Aff}(M))$*
2. *The rank of A is N, where rank is given by $\mathrm{Tr}_N \circ \tau(p_{\overline{\mathrm{im}\,A}})$ for $p_{\overline{\mathrm{im}\,A}}$ the projection onto $\overline{\mathrm{im}(A)}$, τ evaluated entrywise, and Tr_N the non-normalized trace on $M_{N\times N}(\mathbb{C})$.*

The rank may equivalently be given by $N - \mathrm{Tr}_N \circ \tau(p_{\overline{\ker A}})$, so $\mathrm{rank}(A) = N$ *is equivalent to* $\ker(A) = \{0\}$.

As a consequence, condition (2) of Theorem 2.2.22 [MSY23, Thm. 3.1] is equivalent to condition (2) of Theorem 2.2.20.

Questions of invertibility and evaluation are also somewhat clarified by Corollary 5.12 of [MSY23]. To do this, we'll need a quick definition of an algebraic kind of rank.

Definition 2.2.24. For $A \in M_{N\times N}(\mathbb{C}\langle x_1, \ldots, x_n\rangle)$, the *inner rank of A*, denoted $\rho(A)$, is the minimum r such that $A = BC$ where B is $N \times r$ and C is $r \times N$. The inner rank of the zero matrix is zero.

Proposition 2.2.25 ([MSY23, Cor. 5.12]). *For $X_1, \ldots, X_n \in M$, with (M, τ) a tracial von Neumann algebra and $\Delta(X_1, \ldots, X_n) = n$, the following are equivalent:*

1. *For all* $0 \neq r \in \mathbb{C}\langle\!\langle x_1, \ldots, x_n \rangle\!\rangle$, *the evaluation* $r(X) \in \mathrm{Aff}(M)$ *is well-defined and invertible*

2. *For all* $N \in \mathbb{N}$ *and* $P \in M_{N\times N}(\mathbb{C}\langle x_1, \ldots, x_n\rangle)$, *the rank (in* $\mathrm{Aff}(M)$*) of* $P(X)$ *and the inner rank (in* $M_{N\times N}(\mathbb{C}\langle x_1, \ldots, x_n\rangle)$*) of* P *are equal, that is,* $\mathrm{rank}(P(X)) = \rho(P)$.

Note that (2) implies that for linear P, P *is full in* $M_{N\times N}(\mathbb{C}\langle x_1, \ldots, x_n\rangle)$ *if and only if* $P(X)$ *is invertible in* $\mathrm{Aff}(M)$.

Remark 2.2.26. If we hope for a planar algebra analog of a free skew field, it seems like we may need:

1. a technique like linearization

2. a condition on the law of the planar algebra like $\Delta(X_1, \ldots, X_n) = n$ under which we hope for an equivalence between invertibility in affiliated operators and an algebraic condition like fullness.

After some background on planar algebras, we'll provide a version of linearization for planar algebras that works on the "polynomial-like" terms.

2.3 Planar Algebras

2.3.1 Basic Definitions

2.3.1.1 Planar Tangles and Planar Algebras

Definition 2.3.1. A *(shaded) planar tangle* is a disc (called the *output disc*) with finitely many (maybe zero) numbered non-overlapping discs in the interior (called the *input discs*).

There are also finitely many smooth non-intersecting curves (called *strings*) with end-points (if any) on the boundaries of the output disc or input discs, and remaining in the region outside of the input discs but inside the output disc. The strings intersect each disc at an even number of points (called *marked points*), and each region in between the discs and strings is shaded either black or white.

On each disc is a *distinguished interval* between the marked points, labelled in diagrams with an adjacent $\star$.

A shaded planar tangle in which the output disc intersets the strings at k points is called a *(shaded) planar k-tangle* [Jon99]. See also [GJS10]. We only consider planar tangles up to isotopy.

See Figure 1.1 from Section 1.5 as an example of a shaded planar tangle.

Definition 2.3.2. Given two planar tangles T and S, we say that T *can be composed with S in input disc k of T* if

1. If S is a j tangle, then input disc k of T must also have j strings intersecting it
2. The shadings are compatible: the region of T adjacent to the distinguished interval of input disc k has the same shading as the region of S adjacent to the distinguished interval of the output disc.

Definition 2.3.3. Given planar tangle T that can be composed with planar tangle S in input disc k of T, the *composition* $T \circ_k S$ is given by the planar tangle as follows:

Place S inside the input disc k of T, possibly with some isotopy so that the distinguished intervals and marked points line up, and the joined curves of the strings coming out of input disc k of T and the strings inside S that intersect the output disc are smooth.

Remove the boundary circle of S (and forget its distinguish interval) and consider the strings joined.

See Figure 1.2 from Section 1.5 for an example.

Definition 2.3.4. A *(shaded) planar algebra* is a vector space $P = \bigoplus_{n\geq 0,\ \varepsilon\in\{+,-\}} P_{2n}^{\varepsilon}$ along with a map Z from the set of planar tangles (up to isotopy) into multilinear operators on the $\{P_{2n}^{\pm}\}$ satisfying:

1. If T is a tangle with k input discs with $\ell_1, \ldots, \ell_k$ marked points, and m marked points on the output disc, and the shadings on the distinguished intervals of the input discs are $\varepsilon_1, \ldots, \varepsilon_k$, and the shading on the output interval is η, then

$$Z_T : P_{\ell_1}^{\varepsilon_1} \otimes \cdots \otimes P_{\ell_k}^{\varepsilon_k} \to P_m^{\eta}$$

where using the convention of [Jon99], regions shaded black are considered to have $-$ shading and regions shaded white are considered to having $+$ shading.

2. (Naturality): $Z_{T \circ_i S} = Z_T \circ_i Z_S$ where the latter $\circ_i$ refers to function composition of Z_T with Z_S in the ith argument of Z_T.

3. (Involutive): There is an involution $*$ on each $P_{2n}^{\pm}$ compatible with orientation-reversal of planar tangles, i.e., $Z_T(x^*) = Z_{\Phi(T)}(x)$, when Φ is an orientation-reversing diffeomorphism of the plane [GJS10].

Note we may occasionally express the involution of an element $R \in P_{2n}^{\pm}$ with the mirror image of the variable used.

Remark 2.3.5. Note that several sources (eg: [Jon99], [Jon00], [KS08]) use the term "(shaded) planar algebra" to denote the spaces $P_0^{\pm}$ and $(P_{2n}^{+})_{n\geq 1}$ equipped with a map Z from only those planar tangles whose distinguished intervals are all adjacent to a white-shaded region. That is a less restrictive definition than the one we use as that includes fewer planar tangles.

However, several examples of shaded planar algebras with only $P_0^{\pm}, (P_{2n}^{+})_{n\geq 1}$ given can be expanded readily into shaded planar algebras with $(P_{2n}^{\pm})_{n\geq 0}$ and a map from planar tangles with distinguished intervals of all shadings. Usually, this is easy when the relevant constructions don't do much with shading in the first place.

Warning 2.3.6. We caution the reader that planar algebra literature has somewhat inconsistent notation, especially in matters of shading and parity.

A shaded planar algebra with the shadings omitted for simplicity should not be confused with an *unshaded* planar algebra. Unshaded planar algebras consist of vector spaces $(P_n)_{n\geq 0}$

with an action of *unshaded* planar tangles, which are permitted to have an odd number of marked points connected to strings around discs (rendering them unshadeable) [Bro12].

In addition, what we call $(P_{2n}^{+})_{n\geq 0}$, is referred to as $(P_n^{+})_{n\geq 0}$ sometimes, as in [GJS10].

Definition 2.3.7. A *(shaded)* subfactor *planar algebra* is a (shaded) planar algebra $\{P_{2n}^{\pm} \mid n \geq 0\}$ satisfying the additional constraints:

1. $\dim(P_{2n}^{\pm}) < \infty$ for each n, and $\dim(P_0^{\pm}) = 1$.
2. On each $P_{2n}^{\pm}$, we have an inner product $\langle x, y\rangle$ given by the following (with shadings omitted):

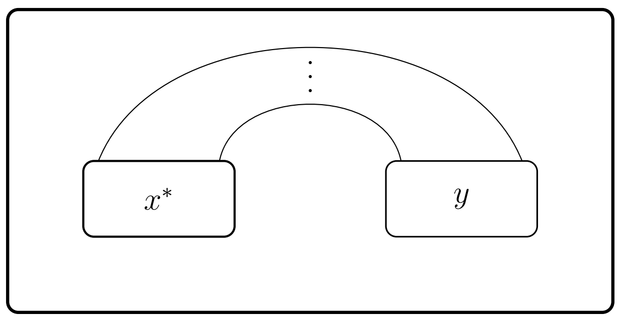


Figure 2.1: The inner product on a subfactor planar algebra.

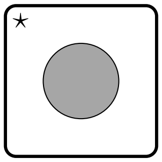
3. (Sphericality): The left and right trace are tangles that induce identical operators on P_2^{+} as well as on P_2^{-} (when shaded accordingly). See Figure 2.2 below.

(a) The left trace. (b) The right trace.

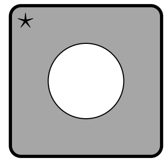
Figure 2.2: The left and right trace.

Since a subfactor planar algebra has $\dim P_0^{\pm} = 1$, we identify them with $\mathbb{C}$ such the unshaded disc and shaded disc (with no input discs or strings) both correspond to 1.

For the following planar tangles T_1 and T_2 in Figure 2.3, we see that $Z_{T_1} \in P_0^+$ and $Z_{T_2} \in P_0^-$ must correspond to some scalars δ_1 and δ_2. Sphericality (when composed with the tangle consisting of a single cup) implies these constants are equal.



(a) The tangle T_1 with shading inside a single loop.



(b) The tangle T_2 with shading outside of a single loop.

Figure 2.3: Two planar tangles T_1, T_2, both of which have zero input discs and a single loop of string.

Definition 2.3.8. The *index* of a shaded subfactor planar algebra is δ^2 where $\delta = \delta_1 = \delta_2$.

The index takes values in $\{4\cos^2(\pi/n)\} \cup [4, \infty)$ just like the subfactor index. See [Jon99] as well as Remark 2 of [GJS10], noting that the latter paper calls δ the "index parameter".

Remark 2.3.9. We've already omitted shadings on a few planar tangles, and we'll do it again. It makes creating diagrams easier and the shading usually doesn't affect much. Do still consider them shaded planar tangles though.

Also, as planar algebras consist of vectors which, roughly speaking, "fit in the discs of a planar tangle", we will also often represent these vectors diagramatically.

In addition, to represent that a diagram has been rotated or reflected, we will sometimes rotate or reflect a letter inside that represents it. When we do this, we'll pick a letter that has minimal symmetries, like j or R, but not x or A, so that it's clearer to see what's happened to it.

2.3.1.2 Temperley-Lieb Diagrams

A planar tangle is not required to have any input discs. Applying Z to such a tangle will produce an element of the planar algebra (or if you want to be technical, a map from $\mathbb{C}$ into the planar algebra, but just apply this map to the number 1). Thus, any planar alegbra contains a copy of these tangles. Keep in mind this may not be an embedding.

As we'll deal with mostly subfactor planar algebras, any strings that form loops simply contribute a factor of δ, and so it's enough to consider here the planar tangles without them.

Definition 2.3.10. Planar tangles without input discs and whose strings all connect to the output disc are called *Temperley Lieb diagrams.*

The set of Temperley-Lieb with $2n$ boundary points will be denoted $TL_n^{\pm}$, with the $\pm$ corresponding to the shading of the distinguished interval. When the $\pm$ is omitted, we will assume $+$.

These diagrams were introduced in [H 87], but the algebra they're named after comes from [TL71], and a thorough discussion occurs in [Jon99]

2.3.1.3 A construction of Guionnet, Jones, and Shlyakhtenko: $\mathrm{Gr}_k P$

Definition 2.3.11 ([GJS10])**.** For a planar algebra P, define $\mathrm{Gr}_k P = \bigoplus_{n\geq 0} P^+_{2n+2k}$ as a graded algebra with graded multiplication $\wedge_k$ given by the tangle in Figure 1.3

We equip $\mathrm{Gr}_k P$ with the involution $\dagger$ given by a horizontal reflection. By this we mean a planar tangle T acts on $P^\dagger$ the same way T composed with a horizontal reflection acts on P.

As in [GJS10], we will often omit the $\star$ from diagrams of planar tangles, assuming it to be in the top left corner of the rectangle. Keep in mind some sources depict this algebra sideways or assume the $\star$ to be in a different location.

In this context, we say that a vector in $P^+_{2n+2k} \subseteq \mathrm{Gr}_k(P)$ has k leftwards strings, k

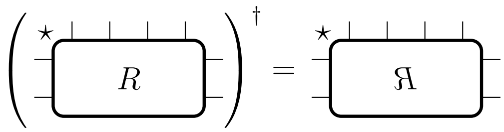


Figure 2.4: The involution † is given by a reflection along the horizontal axis.

rightwards strings, and $2n$ upwards strings [GJS10].

The algebra $\mathrm{Gr}_k P$ may be equipped with one of several traces. We'll make use of the Voiculescu trace, defined as follows

Definition 2.3.12. Let $\sum TL_n$ denote the sum of all positively shaded Temperley-Lieb diagrams with $2n$ marked points.

The Voiculescu trace of a term in $\mathrm{Gr}_k P$ is given by the tangle $\sum TL_n$ on the top-strings and pairing off the side strings, as shown in Figure 2.5, noting the output disc is omitted from the figure for clarity [GJS10].

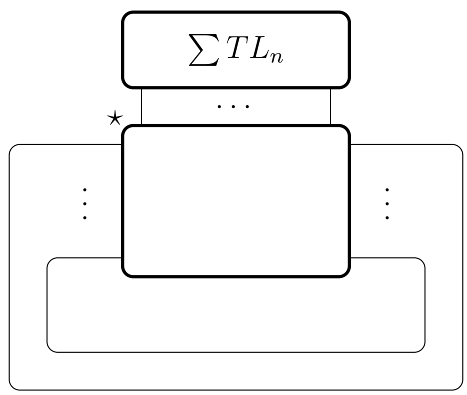


Figure 2.5: The Voiculescu trace, given by summing Temperley-Lieb diagrams on top and pairing off side strings.

Lemma 2.3.13 ([GJS10, Lem. 1]). *For a planar algebra P,* Tr_k *is a (not necessarily faithful) trace on* $(\mathrm{Gr}_k P, \wedge_k)$*.*

Theorem 2.3.14 ([GJS10, Thm. 6])**.** *If P is a* subfactor *planar algebra,* Tr_k *is in fact positive definite (faithful) on* $\mathrm{Gr}_k P$*.*

With the Tr_k faithful, it's natural to consider the inner products they induce.

Definition 2.3.15. For P a subfactor planar algebra, let $L^2(\mathrm{Gr}_k P)$ denote the closure of $\mathrm{Gr}_k P$ with respect to the norm induced by $\langle x, y\rangle = \mathrm{Tr}_k(x^\dagger \wedge_k y)$.

We associate to each $a \in \mathrm{Gr}_k P$ an operator L_a acting on $\mathrm{Gr}_k P \subseteq L^2 \mathrm{Gr}_k P$ by left multiplication:

$$L_a(x) = a \wedge_k x \in \mathrm{Gr}_k P \subseteq L^2 \mathrm{Gr}_k P$$

If $\mathrm{Gr}_k P$ acts by *bounded* operators, then we may carry out what is essentially the GNS construction to get a representation $\pi : \mathrm{Gr}_k P \to \mathcal{B}(L^2(\mathrm{Gr}_k P))$ and form von Neumann algebras $M_k = (\pi(\mathrm{Gr}_k P))''$.

Luckily, $\mathrm{Gr}_k P$ does in fact act by bounded operators on $L^2 \mathrm{Gr}_k P$. This is seen in a few ways. First, [JSW10, Sect. 5] defines an isometric isomorphism between $(\mathrm{Gr}_k P, \wedge_k, \dagger)$ and another space, and shows that the multiplication $\wedge_k$ in $\mathrm{Gr}_k P$ corresponds to a multiplication which is shown to be bounded using a nice diagrammatic proof. Keep in mind the notation switch if you read this paper: what we call Gr_k they call Hr_k, and they call the "other space" Gr_k. See also the independent and simultaneous discovery of this result in [KS08].

Alternatively, this is done in [GJS10] by embedding $\mathrm{Gr}_k P$ in a graph planar algebra. We will not discuss the details of the embedding here, but we will mention graph planar algebras later, as they're a great source of planar algebras and intuition about them.

Now that the von Neumann algebras M_k are defined, recall that given a von Neumann algebra M we may consider the algebra of affiliated operators $\mathrm{Aff}(M)$. These algebras $\mathrm{Aff}(M_k)$ will be useful later when we construct linearizations of terms in Gr_0.

Theorem 2.3.16 ([GJS10, Thm. 5])**.** *For a subfactor planar algebra with $\delta > 1$, the M_k are II_1 factors.*

2.3.2 Examples

The following examples of planar algebras should give a sense that planar algebras capture some notion of symmetry and that we can sometimes think of vectors in the algebra as representing polynomials.

This is well phrased in [Shl10]:

> Planar algebras can be thought of as families of linear spaces consisting of vectors "obeying a symmetry", where the word symmetry is taken in a very generalized sense (such "symmetries" include group actions as well as quantum group actions).

The elements in $\mathrm{Gr}_k(P)$ may be similarly interpreted as being like matrices over polynomials obeying a symmetry, with the side-strings standing in for the matrix-like behavior and the upwards-strings for the polynomial-like behavior.

We begin with one of the simplest planar algebras.

2.3.2.1 Temperley-Lieb planar algebra

Definition 2.3.17. Let $\delta > 0$. The Temperley-Lieb planar algebra with index δ^2 has as its graded vector space, the span of Temperley-Lieb *diagrams* of appropriate shadings $\{\mathrm{span}(TL_n^{\pm}) \mid n \geq 0\}$.

To make this graded vector space into a shaded planar algebra, we must define the operator Z.

Luckily, there's (almost) only one thing it can be. Suppose T is a planar tangle with $2m$ marked points on the output disc, and $2n_1, \ldots, 2n_k$ marked points on the input discs (with k potentially zero). Suppose the shading on the distinguished interval of the output disc is η and the shadings on the inputs discs are $\varepsilon_1, \ldots, \varepsilon_k$.

Let $S_1, \ldots, S_k$ be Temperley-Lieb diagrams (with appropriate shadings and numbers of boundary points).

Then T may be composed with $S_1, \ldots, S_k$ as planar tangles, producing a new tangle we'll call $T_{S_1,\ldots,S_k}$ with zero input discs. Every planar tangle with no input discs is a Temperley-Lieb diagram modified by potentially adding loops.

Let $\tilde{T}_{S_1,\ldots,S_k}$ be that Temperley-Lieb diagram after removing j-many loops from $T_{S_1,\ldots,S_k}$. We then define Z by its action on a basis via:

$$Z_T(S_1, \ldots, S_k) = \delta^j \tilde{T}_{S_1,\ldots,S_k},$$

and extend linearly [H 87] [Jon99].

Note, sometimes the Temperley-Lieb planar algebra is denoted TL_n, which we're using to denote only the diagrams which form the basis for the planar algebra.

Remark 2.3.18. The purpose behind mentioning this example (besides the fact that we will use it) is that the Temperley-Lieb planar algebra shows us that sometimes it's useful to think of a planar algebra as consisting of diagrams.

We keep this intuition close to our hearts and in general planar algebras we'll think of the vectors like diagrams, and represent them with pictures.

Remark 2.3.19. The Temperley-Lieb planar algebra as defined is not a subfactor planar algebra when $\delta^2 \in \{4\cos^2(\pi/n) \mid n \geq 3\}$, but is when $\delta^2 \in [4, \infty)$.

For the $\delta^2 < 4$ cases, one may quotient by an appropriate subspace to recover a subfactor planar algebra.

See the exposition in [Jon83, Sect. 5] or [Spe16, Chap. 7].

2.3.2.2 Polynomial planar algebra

Another planar algebra we can construct, and perhaps the most important example to keep in mind throughout this text, is the polynomial planar algebra.

This example comes from [Jon99, p. 38] as a special case of the tensor planar algebra, where we're viewing alternating tensors of vectors and covectors as noncommutative polynomials of basis vectors. This case is described more specifically in [Shl10, Sect. 3.6.1], and that's the definition we're writing below, with the modification that we'll be explicit about the shading here.

Definition 2.3.20. The polynomial planar algebra on N indeterminates (and their adjoints), $x_1, \dots, x_N$, $x_1^*, \dots, x_N^*$, is $(\mathcal{P}_{2n}^{\pm})_{n=0}^{\infty}$ where:

1. The vector spaces $\mathcal{P}_{2n}^{+}$ are spanned by degree $2n$ alternating monomials of the form

$$x_{i_1} x_{j_1}^* x_{i_2} x_{j_2}^* \dots x_{i_n} x_{j_n}^*$$

   where $i_1, j_1, \dots \in \{1, \dots, N\}$ and the $\mathcal{P}_{2n}^{-}$ are spanned by the same, but with the adjoints on the other variables.

2. Let T be a planar tangle wih input discs having $k_1, \dots, k_n$ marked points each and distinguished interval shadings $\varepsilon_1, \dots, \varepsilon_n$, and with an output disc of ℓ marked points and distinguished interval shading η. Given alternating monomials $p_1, \dots, p_n$ of appropriate degree and starting on non-adjointed or adjointed indetermintes according to the ε_i, we determine $Z_T(p_1, \dots, p_n)$ by the following process.

   View each monomial $x_{\alpha_1} x_{\beta_1}^* \dots x_{\alpha_r} x_{\beta_r}^*$ as a positively shaded diagram with $2r$ marked points on the boundary equipped with labels $\alpha_1, \beta_1, \dots, \alpha_r, \beta_r$. The monomials that begin with adjointed indeterminates are viewed as having distinguished intervals negatively shaded instead.

   On T we will be considering *states*, assignments σ of strings of T to integers $\{1, \dots, N\}$.

   We will say that a state σ on T is compatible with the monomials $p_1, \dots, p_n, q$ if: when the diagrams corresponding to the $p_1, \dots, p_n$ are glued into the input discs of T (matching distinguished intervals, shadings, and marked points), the labels on the p_i

and σ match; and the labels, marked points, and shadings on the diagram for q agrees with the output disc of T.

Finally, set

$$Z_T(p_1, \ldots, p_n) = \sum_q \sum_\sigma 1 \cdot q \tag{2.1}$$

where the sum is taken over states on T compatible with $p_1, \ldots, p_n$ and q.

Mentioning states and explicitly writing a multiple of 1 may look strange, but it will be clearer when we see the polynomial planar algebra as a special case of a graph planar algebra, where we will have a more general notion of state and the number we multiply by will be different.

The polynomial planar algebras are subfactor planar algebras, with $\delta = N$. See [Shl10] for more discussion and context.

Let's see an example of a tangle applied to two polynomials.

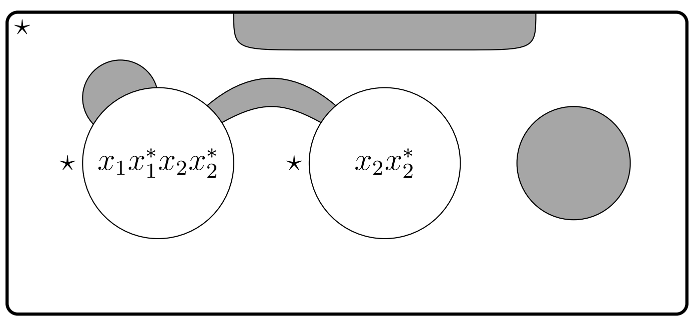


Figure 2.6: A example in the polynomial planar algebra of a tangle applied to two polynomials.

Working in the polynomial planar algebra over n indeterminates, we get $N \sum_{i=1}^{N} x_i x_i^*$.

Which states contribute? In order to contribute to the sum, a state must assign the leftmost string the label "1", and the two strings between the input discs must be assigned "2". This leaves the number of admissible states at N^2: assigning a number between 1 and N to the upper string and to the loop on the right.

Each label on the loop does not affect the output disc at all, so the loop contributes a factor of N (observe that $\delta = N$ here). Finally, if the upper string is labelled k, the output disc is compatible with only $x_k x_k^*$. This gives us $N \sum_{i=1}^{N} x_i x_i^*$.

There is in fact a construction of a planar algebra that represents not just the even-degree polynomials but all non-commutative polynomials. However, rather than being a shaded planar algebra like we've discussed, it's an *unshaded* planar algebra as described in [Bro12]. Unshaded planar algebras allow for a larger class of planar tangles by including those that are not necessarily able to be shaded. We won't need them, but it's worth knowing they exist.

2.3.2.3 Invariant polynomials

Another planar algebra we can discuss is the planar algebra of $U(N)$-invariant non-commutative polynomials.

Definition 2.3.21 ([Shl10])**.** Consider the following action of the unitary group $U(N)$ on $\mathbb{C}\langle x_1, \ldots, x_N, x_1^*, \ldots, x_N^* \rangle$ defined as follows, where $U \in U(N)$ is expressed as a matrix $(U_{ij})_{i,j=1}^{N}$.

1. For each variable x_k, we have $U \cdot x_k = \sum U_{ik} x_i$.
2. For each adjoint x_k^*, we have $U \cdot x_k^* = \sum \overline{U_{ik}} x_i^*$.

With the above action of $U(N)$ on the vector spaces $\mathcal{P}_{2n}^{\pm}$ of the polynomial planar algebra, we may consider the vector spaces $\left(\mathcal{P}_{2n}^{\pm}\right)^{U(N)}$ of those polynomials which are invariant under the action of $U(N)$.

Restricting the planar algebra structure on the polynomial planar algebra results in a planar algebra of the $U(N)$ invariant polynomials.

Remark 2.3.22. This action comes from identifying $\{x_1, \ldots, x_N\}$ with $\{e_1, \ldots, e_N\} \subseteq \mathbb{C}^N$ and $\{x_1^*, \ldots, x_N^*\}$ with a basis $\{\overline{e_1}, \ldots, \overline{e_N}\}$ for $\overline{\mathbb{C}^N}$, and then considering each U to act via the basic representation on $\mathbb{C}^N$ (where U acts via matrix multiplication) and by the conjugate representation on $\overline{\mathbb{C}^N}$ (via $U \cdot \overline{v} = \overline{Uv}$).

2.3.2.4 Graph planar algebra

We now consider the planar algebra associated to a finite connected bipartite graph permitting multiple edges between the same two vertices, introduced by Jones in [Jon00]. We're restricting to finite rather than locally finite graphs for ease of presentation.

First, we recall a relevant theorem

Theorem 2.3.23 (Perron, Frobenius)**.** *Let $A \neq 0$ be an $n \times n$* irreducible *real matrix with non-negative integer entries.*

The largest eigenvalue λ is multiplicity one and has an associated eigenvector with strictly positive entries, called a Perron-Frobenius *eigenvector [Gan59, Thm. III.2.2].*

Here a matrix is *reducible* if and only if its indices may be permuted to turn it into the form

$$\begin{pmatrix} A & 0 \\ C & D \end{pmatrix}.$$

Proposition 2.3.24. *The adjacency matrix of a strongly connected digraph is irreducible [BR91, Sect. 8.1].*

Let's now build the vector spaces for the graph planar algebra.

Definition 2.3.25. Let $\Gamma = (V, E)$ be a connected undirected bipartite graph with vertex set $V = V_+ \sqcup V_-$ and edge set E, where there are no edges between vertices strictly within V_+, or within V_-. Write $|V_+| = n_1$, $|V_-| = n_2$, and $n = n_1 + n_2$. For each undirected edge, consider an orientation on it positive if it goes from V_+ to V_-, and consider an orientation on it negative if it goes the other way. Denote the set of edges all equipped with positive orientations by E_+, the set of edges equipped with negative orientations as E_-, and for an oriented edge e, we denote the oppositely oriented edge e^o.

The the adjacency matrix A_Γ of Γ has a Perron-Frobenius eigenvector $\mu = (\mu_v)_{v \in V}$, where $\mu_v > 0$ for each $v \in V$, with eigenvalue δ so that $A_\Gamma \mu = \delta \mu$. The vector μ is called the *vector of spins* of Γ, with $\sqrt{\mu_v}$ called the *spin at* v [Jon00].

Set $P_{2n}^{\pm}$ to be the vector space generated by the set of loops of length $2n$ starting and ending on $V_\pm$, where the two $\pm$ signs match. [GJS10]

Remark 2.3.26. Regarding notation, keep in mind that the definition in [Jon00] denotes μ^2 the vector we call μ, and does not require it to be the Perron-Frobenius eigenvector.

We now define spin states and describe how planar tangles will act on the $P_{2n}^{\pm}$.

Definition 2.3.27. Given a planar tangle T, a *spin state* on T is a function σ on the regions and strings of T which

1. maps positively shaded regions to vertices in V_+,
2. maps negatively shaded regions to vertices in V_-, and
3. maps strings to (unoriented) edges of the graph such that a string adjacent to regions R_1 and R_2 is mapped to an edge connecting vertices $\sigma(R_1)$ and $\sigma(R_2)$.

Note that a spin state σ induces a loop $\partial\sigma$ associated to the output disc by reading off the assignment of regions and strings to vertices and edges clockwise starting at the distinguished interval.

Definition 2.3.28. Using the convention of [GJS10], for a planar tangle T, isotope each input disc of T to a rectangle with distinguished interval in the top left corner, each marked point on the top of each rectangle, and the output disc to a rectangle with all marked points on top.

Next, isotope each string so that any critical points in the y-value of the string occur as either local minima or maxima (where the y axis is up/down). See Figure 2.7 for an example of such a drawing.

Given a spin state σ, we assign each of these critical points a number:

$$\sqrt{\frac{\mu(\sigma(\textsc{convex}))}{\mu(\sigma(\textsc{concave}))}}.$$

where CONVEX denotes the region of T adjacent to the critical point that is convex near that critical point. It's the region above the critical point when it's a local minimum, and below when it's a local maximum. The region denoted CONCAVE is the other one. Recall that σ maps regions to vertices.

The *curvature factor* $c(\sigma)$ of a spin state σ (for a given planar tangle T) is the product of each of the above numbers over every critical point in the tangle.

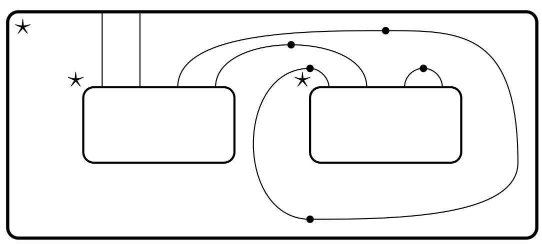


Figure 2.7: A planar tangle with strings in position for calculating the curvature factor and dots placed at the critical points where numbers would be assigned.

Definition 2.3.29. Given a planar tangle T, the map Z_T is defined as follows on simple tensors

$$Z_T(\ell_1 \otimes \cdots \otimes \ell_n) = \sum_{\ell} \sum_{\sigma} c(\sigma) \cdot \ell \tag{2.2}$$

where the sum is taken over σ compatible with $\ell_1, \ldots, \ell_n$ and ℓ. Here ℓ is

The spaces $(P^{\pm}_{2n})_{n\geq 0}$ equipped with the map Z define the planar algebra associated to a finite bipartite graph Γ with spin vector μ. This planar algebra is equipped with an adjoint given by reversing all loops. This object is called the graph planar algebra P^{Γ} [GJS10].

Remark 2.3.30. The index of the planar algebra associated to the bipartite graph Γ is δ^2 where δ is the Perron-Frobenius eigenvalue δ.

Let's calculate a few examples.

Example 2.3.31. First, consider the graph A_3 in Figure 2.8. This graph has Perron-Frobenius eigenvector $(1, \sqrt{2}, 1)$ with eigenvalue $\sqrt{2}$. This means the corresponding graph planar algebra has $\delta = \sqrt{2}$. Let's confirm this fact!

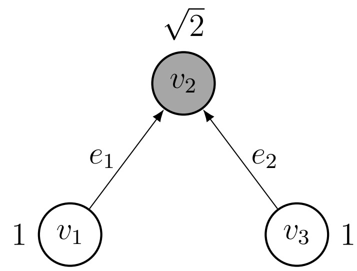


Figure 2.8: The bipartite graph A_3 with the value of the normalized Perron-Frobenius eigenvector at each vertex.

Let's look at the loop that's shaded inside, depicted in Figure 2.3a. Recall that this planar tangle acts on $P_0^+ = \text{span}\{v_1, v_3\}$.

Let's calculate its action on v_1. We examine the valid spin states. The inner region is shaded and thus gets assigned v_2, the outer region must be assigned v_1 to be compatible with the input loop of length zero (v_1). There are two critical values of the height function on the string, each gets assigned $\sqrt{\sqrt{2}/\sqrt{\mu(v_1)}}$, resulting in a curvature factor of $\sqrt{2}$. Thus Z_{T_1} acting on the trivial loop v_1 yields $\sqrt{2}v_1$.

Similarly, it acts by multiplication by $\sqrt{2}$ on v_3.

Let's now consider the other tangle with shading on the outside in Figure 2.3b. The space $P_0^- = \text{span}\{v_2\}$ is one dimensional, so we now only have to consider the states. There are two compatible spin states here: the outer region is assigned v_2, and the inner region is assigned either v_1 or v_3.

In both cases, the two critical points are assigned spin factors of $\sqrt{1/\sqrt{2}}$ resulting in a curvature factor of $1/\sqrt{2}$. Adding the two results together, the tangle T_2 on the trivial loop v_2 produces $2v_2/\sqrt{2} = \sqrt{2}v_2$. This is δv_2.

Note that the calculation of the first tangle where there were no input discs, and a zero-box space P_0^+ that's more than one dimensional, is a slightly unusual calculation. After all, normally the inputs aren't considered compatible with the boundary disc. Let's do a more normal looking, more complex calculation.

Example 2.3.32. Let's consider a slightly more complex example. Let's look at the graph A_5 in Figure 2.9.

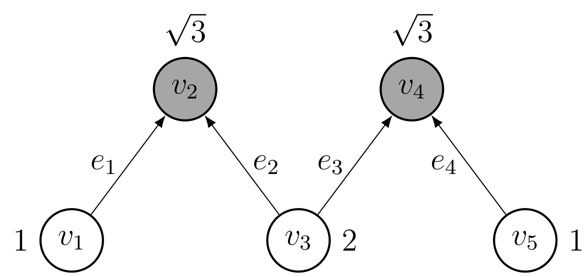


Figure 2.9: The bipartite graph A_5 with the value of the normalized Perron-Frobenius eigenvector at each vertex.

Note that for this planar algebra, $\delta = \sqrt{3}$. Let's look at a more interesting planar tangle this time.

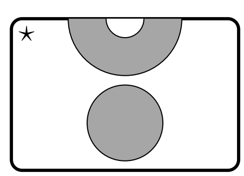


Figure 2.10: A planar tangle T.

We'll apply this planar tangle to the element $v_1 \in P_0^+ = \text{span}\{v_1, v_3, v_5\}$. Let's consider the valid states. The outer region is assigned v_1, the anular and upper grey region must be assigned v_2, same with the circular lower grey region. However, the semicircle shaded white region may be assigned either v_1 or v_3.

This means there are two states, let's call them σ_1 and σ_3 that assign v_1 and v_3 to the white semicircle respectively.

First, the boundary loop corresponding to the state σ_1 is the path $e_1 e_1^o e_1 e_1^o$. Next we look at the curvature factor. There are four critical points. The spin factors they contribute, listing from top to bottom, are $1/3^{1/4}$, $3^{1/4}/1$, $3^{1/4}/1$, and $3^{1/4}/1$. The combined curvature factor is $\sqrt{3}$. This results in the value $\sqrt{3} e_1 e_1^o e_1 e_1^o$ for the first term.

Next, we consider σ_3 which has compatible boundary loop $e_1 e_2^o e_2 e_1^o$. The four spin factors are, from top to bottom, $\sqrt{2}/3^{1/4}$, $3^{1/4}/1$, $3^{1/4}/1$, and $3^{1/4}/1$. This term works out to $\sqrt{6} e_1 e_2^o e_2 e_1^o$.

We get a final result of

$$Z_T(v_1) = \sqrt{3} e_1 e_1^o e_1 e_1^o + \sqrt{6} e_1 e_2^o e_2 e_1^o.$$

Note how in the above examples, we have not concerned ourselves with the assigning of strings to edges. This is because the above graphs do not have edges with interesting multiplicity. We'll see in the next section an example of a graph planar algebra on a graph with multiple edges between two vertices. It should look familiar, too.

Remark 2.3.33. Note that the planar algebra associated to a bipartite graph is usually *not* a subfactor planar algebra. This can be seen by noting that $P_0^{\pm} = \operatorname{span} V_{\pm}$, which is not necessarily one dimensional.

2.3.3 Motivating Connections

We can now show that the polynomial planar algebra is the planar algebra associated to the following bipartite graph.

Every loop on the graph in Figure 2.11 starting and ending on the positive vertex is an

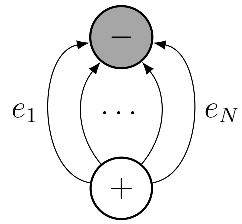


Figure 2.11: A graph with two nodes and N edges between them, drawn with arrows in the positive orientation. The node labelled "−" is shaded the same dark gray as regions in a shaded planar tangle.

even-length list of edges whose orientations alternate starting on positive: $e_{i_1}, e^o_{j_1}, \ldots, e_{i_n}, e^o_{j_n}$.

This is identified with the polynomial $x_{i_1} x^*_{j_1} \cdots x_{i_n} x^*_{j_n}$. A spin state on the graph planar algebra is an assignment of regions to vertices respecting shading (there's only one assignment!) and strings to edges, and so corresponds precisely to the state on a polynomial planar algebra which assigns strings to integers $\{1, \ldots, N\}$.

The adjacency matrix of the graph Γ is $\begin{bmatrix} 0 & N \\ N & 0 \end{bmatrix}$ which has Perron-Frobenius eigenvalue N and eigenvector $(1, 1)$.

We observe that the curvature factor $\sqrt{\mu(\sigma(\text{convex}))/\mu(\sigma(\text{concave}))}$ is always just 1. Putting it all together, we see that equation (2.1) is just a special case of equation (2.2).

The purpose of this connection here is for us to think of graph planar algebras as a generalization of polynomial planar algebras.

In fact, we should be thinking of every subfactor planar algebra as a sub-planar algebra of a graph planar algebra.

Corollary 2.3.34 ([GJS10], [PS03])**.** *Every subfactor planar algebra P is a sub-planar algebra of a graph planar algebra P^Γ.*

The full theorem notes that there is a canonical choice of graph Γ as the principal graph of the λ-lattice realizing P, but the details aren't important to us right now, although they are interesting.

Remark 2.3.35. Recalling how the planar algebra of unitarily invariant polynomials was a sub-planar algebra of the polynomial planar algebra, we combine these ideas to form the intuition that subfactor planar algebras, in some sense, are some kind of generalization of "invariant polynomials".

For more intuition with planar algebras, we now consider the example of $\mathrm{Gr}_k P$ when P is a polynomial planar algebra. Recall the multiplication $\wedge_k$ given by the tangle in Figure 1.3.

Consider the polynomial planar algebra P on N variables (and their adjoints). To a matrix $A \in M_{N\times N}$, associate the polynomial $\sum_{i,j=1}^{N} A_{ij} x_j x_i^*$.

With this identification, the $\wedge_1$ product on $\mathrm{Gr}_1(P)$ is simply matrix multiplication.

We can also identify $M_{N^k \times N^k}$ with polynomials of degree $2k$. Note that $M_{N^k\times N^k} \simeq M_{N\times N}^{\otimes k}$, and identify $A^{(1)} \otimes \cdots \otimes A^{(k)}$ with

$$\sum_{i_1,\ldots,i_k,j_1,\ldots,j_k=1}^{N} \left(\left(\prod_{\ell=1}^{k} A^{(\ell)}_{i_\ell j_\ell} \right) x_{j_1} \ldots x_{j_k}^{*(k-1)} x_{i_k}^{*(k)} \ldots x_{i_1}^* \right)$$

where $*m$ indicates an adjoint if m is odd and no adjoint otherwise.

Under this identification, matrix multiplication is readily seen to be equivalent to $\wedge_k$ on $\mathrm{Gr}_k P$.

Remark 2.3.36. With this interpretation, we can broadly state our motivating intuition: we think of Gr_k like matrices over polynomials where both the matrix part and polynomial part are invariant under some kind of "symmetry".

2.3.4 Other Planar Algebra Tools

We'll need to interpret elements of a planar algebra as diagrams quite often, and we'll need to stretch this interpretation somewhat, too. In particular, it's quite handy for us to consider viewing not just terms in Gr_k as diagrams, but also terms in M_k as well as $\mathrm{Aff}(M_k)$. Note that, as defined below, calculations involving tangles applied to the up-strings of an element of $L^2(\mathrm{Gr}_k)$ or M_k are not necessarily well-defined.

Proposition 2.3.37. *Let P be a subfactor planar algebra with $\delta > 1$ (so that M_k is a II_1 factor).*

Let $\xi \in L^2(\mathrm{Gr}_k)$ and $g \in \mathrm{Gr}_\ell$ with zero marked points on top. The diagram below in Figure 2.12 can be interpreted as an element of $L^2(\mathrm{Gr}_{k+\ell})$, and will be denoted $\xi \otimes g$. The dashed line is placed to separate which strings correspond to ξ and which correspond to g. The strings on top are not numbered, as tangles that act on individual strings may not necessarily be well-defined.

In particular, write xi as an L^2 limit of $x_n \in \mathrm{Gr}_k$, then the L^2 limit $\lim_n x_n \otimes g$ exists.

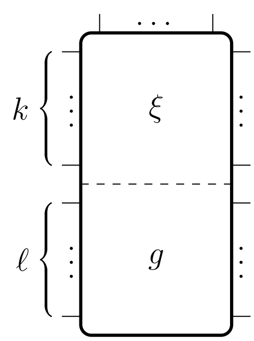


Figure 2.12: A diagram of $\xi \otimes g$ (ξ stacked on top of g).

Proof. It suffices to prove that the map $T : \mathrm{Gr}_k \to \mathrm{Gr}_{k+\ell}$ via $T : x \mapsto x \otimes g$ is bounded in operator norm, as then T can be extended to act on $L^2\,\mathrm{Gr}_k$.

A simple calculation with the Voiculescu trace shows that $\|T(x)\|_2 = \|g\|_2 \|x\|_2$. □

In fact, we can do this just fine with unbounded operators in $\mathrm{Aff}(M_k)$, which will be useful later when we express the inverses of of some of these operators diagramatically.

Proposition 2.3.38. *Let P be a subfactor planar algebra with $\delta > 1$. Let $A \in \mathrm{Aff}(M_k)$.*

Then there is an operator $\tilde{A} \in \mathrm{Aff}(M_{k+\ell})$ *given diagrammatically by* $A \otimes \mathrm{id}_\ell$*. The identity element* $\mathrm{id}_\ell \in \mathrm{Gr}_\ell$ *is given by the Temperley-Lieb diagram with* ℓ *horizontal lines.*

Proof. Note that the map $i : \mathrm{Gr}_k \hookrightarrow \mathrm{Gr}_{k+\ell}$ given by appending ℓ horizontal lines at the bottom of a diagram is an injective trace-preserving unital $*$-homomorphism (using the normalized traces). Let $\{p_n\}$ be a sequence of increasing projections with trace $\tau(p_n) \to 1$. Defining $\tilde{A}$ on $i(p_n)L^2(\mathrm{Gr}_{k+\ell})$ by the inclusion of A acting on $p_n L^2 \mathrm{Gr}_k$, and applying 2.1.38 (Lemma 7.2.6 of [AP]), we have that $\tilde{A}$ is densely defined. Also note that $\tilde{A}$ permutes with everything in $M'_{k+\ell}$ because on each of $i(p_n)L^2(\mathrm{Gr}_{k+\ell})$, it's a bounded map, by Lemma 7.2.5 of [AP].

It's closeable as well, which is easiest to see by noting that it's adjointable. This is because $D(\tilde{A}^*)$ is dense as it contains subspaces of trace tending to one.

In addition, it's clear that this $\tilde{A}$ really does act like the diagram we represent it as, given that on a sequence of subspaces $p_n L^2(\mathrm{Gr}_{k+\ell})$, it acts precisely like that diagram. □

Proposition 2.3.39. *Let* P *be a subfactor planar alegbra with* $\delta > 1$*,* $A \in \mathrm{Aff}(M_k)$*, and* $g \in \mathrm{Gr}_\ell$ *with no upwards-strings.*

There's an operator T *in* $\mathrm{Aff}(M_{k+\ell})$ *given diagrammatically by* $A \otimes g$*.*

Proof. Simply choose the product of the affiliated operator $\tilde{A}$ from above with the bounded operator $\mathrm{id}_k \otimes g$. □

Remark 2.3.40. Similarly, one may stack affiliated operators over diagrams with unequal numbers of strings on the left and right, and while these are now no longer affiliated operators on a von Neumann algebra (since they map between different spaces), they are still densely defined operators, and when their products yield an element that has equal numbers of side-strings, it is an affiliated operator. One just needs to look at any product with an unequal number of side-strings, fill in the gaps with $\supset$ or $\subset$, and we're back in the case with equal numbers of side-strings.

CHAPTER 3

Linearization

We now prove it is possile to express elements of Gr_0 as products $uA^{-1}v$, a product that happens in $\mathrm{Aff}(M_k)$. This is the planar algebra analog to linearization for the polynomial-like terms of the planar algebra. Note, the term A produced in this chapter is invertible in Gr_k.

3.1 Definitions

Let $P = (P_{2n})_{n=0}^{\infty}$ be a (shaded) subfactor planar algebra with index $\delta^2 > 1$.

Recall the construction of the graded $*$-algebras $\mathrm{Gr}_k(P)$, in which each algebra $\mathrm{Gr}_k(P) = \bigoplus_{n=0}^{\infty} P_{2n+2k}$ as vector spaces, equipped with the multiplication $\wedge_k$, and equipped with the Voiculescu trace Tr_k.

Each vector in $\mathrm{Gr}_k(P)$ is thought of as consisting of rectangular diagrams with at least $2k$ marked points, k of which are on the left side of the rectangle, k of which are on the right, and the remaining on the top of the rectangle.

Definition 3.1.1. Let G_k^{2i} denote $\bigoplus_{j\leq i} P_{2j+2k}^{+}$, specifically viewed as a subset of $\mathrm{Gr}_k(P)$. These are the terms with $2i$ (or fewer) upwards strings, k leftwards strings, and k rightwards strings.

As a reminder, the (non-normalized) Voiculescu trace sums all Temperley-Lieb diagrams over the top strings, and connects all the left and right strings to each other, paired off evenly. The normalized trace on Gr_k contains a factor of δ^{-k}, see Figure 3.1.

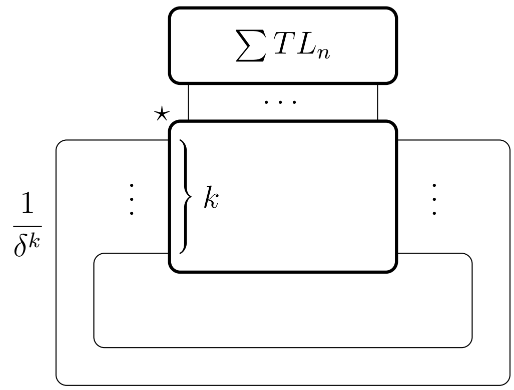


Figure 3.1: The normalized Voiculescu trace on Gr_k.

Recall also the von Neumann algebras M_k given by the bicommutant of $\pi(\mathrm{Gr}_k) \subseteq \mathcal{B}(L^2(\mathrm{Gr}_k))$, where π is the left multiplication action, and that the M_k are II_1 factors.

We also need to refer to "non-square" (in the sense of non-square matrix) elements of the planar algebra, i.e., elements of $P^{2n+k+\ell}$ viewed as diagrams with $2n$ upwards strings, k leftwards strings, and ℓ rightwards strings.

Definition 3.1.2. We use $G^{2i}_{k,\ell}$ to denote the space $\bigoplus_{j\leq i} P^{+}_{2j+k+\ell}$ in the planar algebra with the assumed multiplication $G^{2j}_{k,\ell} \times G^{2\alpha}_{\ell,\beta} \to G^{2j+2\alpha}_{k,\beta}$ for any $j \leq i$ and α, β which is given by the planar tangle

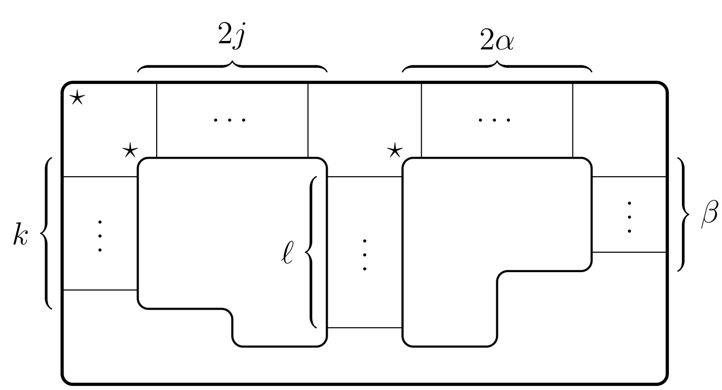


Figure 3.2: The tangle that induces multiplication on $G^{2j}_{k,\ell} \times g^{2\alpha}_{\ell,\beta}$.

Remark 3.1.3. Recalling the interpretation of $\mathrm{Gr}_k(P)$ as consisting of square matrices over non-commutative polynomials from section 2.3.3, we may think of G_k^{2i} as like a square matrix of size $\delta^k \times \delta^k$ over non-commutative polynomials of degree at most $2i$, and $G_{k,\ell}^{2i}$ as *like* a rectangular matrix of size $\delta^k \times \delta^\ell$ over non-commutative polynomials of degree at most $2i$.

One might object to the term "linear" being used for things we just said should be considered to have degree 2. As we're restricted to an even number of strings, 2 really is the smallest non-zero number we can use. In this sense it's "minimal degree".

Definition 3.1.4. Guided by this interpretaion, we call the elements of G_k^0 *purely matricial*, the elements of G_k^2 *linear* and *matricial*, and elements of G_0^{2i} *non-matricial* or *polynomial-like*.

3.2 Linearization for a single term in G_0^{2k}

Theorem 3.2.1. *Let P and G_n^{2i} be as above. Let $p \in G_0^{2k} \setminus G_0^{2k-2}$ be a non-matricial element of the planar algebra.*

Then there exists $n \in \mathbb{N}$, $u \in G_{0,n}^0$, $A \in G_{n,n}^2$, and $v \in G_{n,0}^0$ such that $uA^{-1}v = p$, with A^{-1} interpreted as an affiliated operator, and the products as densely defined operators among the relevant spaces: $L^2(\mathrm{Gr}_n)$ and $L^2(\mathrm{Gr}_0)$.

The proof will be constructive. We will build u, A, v in several steps. We will define j, t_k, a_k, c_k, and π_k and assemble the u, A, and v from these terms. We call (u, A, v) where $u \in G_{0,n}^0$, $A \in G_{n,n}^2$ is invertible in $\mathrm{Aff}(M_n)$, and $v \in G_{n,0}^0$ a *planar algebra triple of order n*, or when the context is hopefully clear, simply a *triple*.

Remark 3.2.2. The notion of triple is meant to mimic the construction of the free skew field in [CR99], the idea being that one might define an equivalence relation on the triples, define addition, multiplication, and division on those triples such that they are well-defined on equivalence classes, and finally consider the set of equivalence classes as a new object.

We present linearization of terms in Gr_0, as well as addition and multiplication operations on triples, but as of yet do not know how to define an equivalence relation between these triples, how to express the reciprocal of a triple, or what potential object may be created out of all this.

3.2.0.1 The term j

Recall that there's a planar algebra morphism from the Temperley-Lieb planar algebra into P. We'll use this to view Temperley-Lieb as sitting (not necessarily injectively) inside of P.

Let $j \in G_2^0$ be the linear combination of Temperley-Lieb diagrams in Figure 3.3 below. Observe that applying any tangle that pairs off the left two or right two strings of j results in zero.

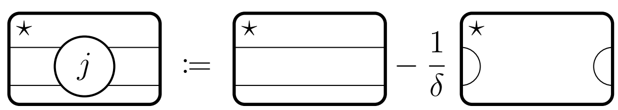


Figure 3.3: The Jones-Wenzl projection $j \in \mathrm{Gr}_2$.

This is the *Jones-Wenzl projection* for the Temperley-Lieb algebra with 2 strings on either side. We will denote it by a disc with a j in it, and two strings coming out the left and right.

See a general formula for Jones-Wenzl projections in [Mor15], or read [Spe16] for more exposition. These projections were first described in [Wen87].

As a subfactor-themed tangent and to specify some notation, we also note that the second term (ignoring the minus sign but keeping the $\frac{1}{\delta}$) is the *Jones projection* e_2 for the inclusion of M_0 into M_1 [GJS10, Thm. 7].

3.2.0.2 The terms t_k and the tangle a_k

We will define elements t_k and tangles a_k such that

$$Z_{a_k}(t_k^i) = \begin{cases} \delta^k & \text{if } i = 0 \\ 0 & \text{if } i < k \text{ or } i > k \\ \delta^{2k}(\delta^2 - 1)^k, & \end{cases}$$

where Z_T is the multilinear map induced by the tangle T.

We build $t_k \in G_{2k}^0$ as a Temperley-Lieb diagram. The top right pair of strings are capped off (i.e., connected by a string). The first $k-1$ pairs on the left are paired with the 2nd through kth pairs on the right hand side.

The bottom left pair is capped off. The bottom right $k-1$ pairs are lined up with the $(k+1)$th through $(2k-1)$th pairs on the left. The remaining middle pair on the left and right are paired with a j term.

This is better illustrated by Figure 3.4.

Note that t_k^i for $i > k$ is zero as there will be a copy of j that has its two left or two right strings paired off. Figure 3.5 depicts t_k^i for $i < k$.

See also Figure 3.6 for a diagram representing the planar tangle a_k.

Observe now that with a_k and t_k defined, we have the following equalities.

1. $Z_{a_k}(t_k^0) = \delta^k$

2. $Z_{a_k}(t_k^i) = 0$ for $0 < i < k$ and for $i > k$

3. $Z_{a_k}(t_k^k) = \delta^{2k}(\delta^2 - 1)^k$.

Also, even though a_k is described as a tangle, we could easily describe it as the product of two elements of the planar algebra, $a_k^{(L)} \wedge_{2k} a_k^{(R)}$, by dividing the diagram vertically in

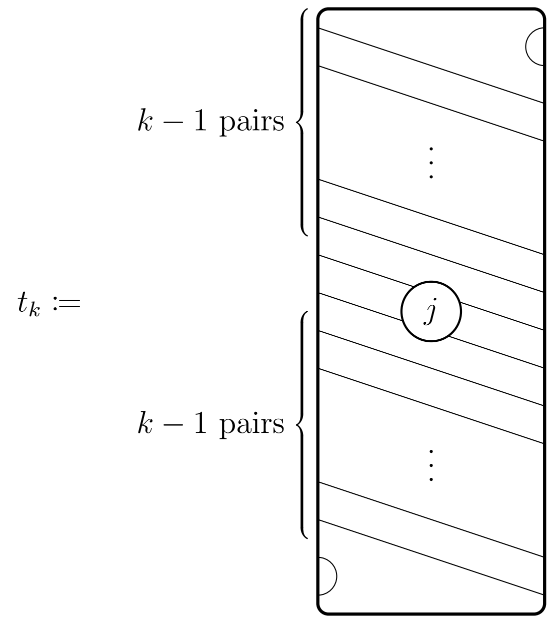


Figure 3.4: The term t_k.

half, so that $a_k^{(L)}$ has $3k$ pairs of strings, with the top and bottom set paired off together and the middle set "capped off".

Observe that $a_k^{(L)} \in G^0_{0,6k}$ and $a^{(R)}_{6k,0}$. Define $\tilde{t}_k = t_k \otimes \mathrm{id}_k$. Thus instead of looking at $Z_{a_k}(t_k^i)$ we may instead consider the equivalent expression $a_k^{(L)} \wedge_{3k} \tilde{t}_k^i \wedge_{3k} a_k^{(R)}$, depicted in Figure 3.7.

3.2.0.3 The terms c_k and π_k

The element $c_k \in G^2_{2k+4}$ has $k+2$ pairs of strings on either side and a single pair on top.

The very bottom pairs are matched off horizontally. Then the next-lowest left pair is capped off. The very top pair is matched to the very top right pair. The remainder are matched mostly-horizontally (with a slight offset). See Figure 3.8 below.

We will only care about c_k^1 and c_k^k.

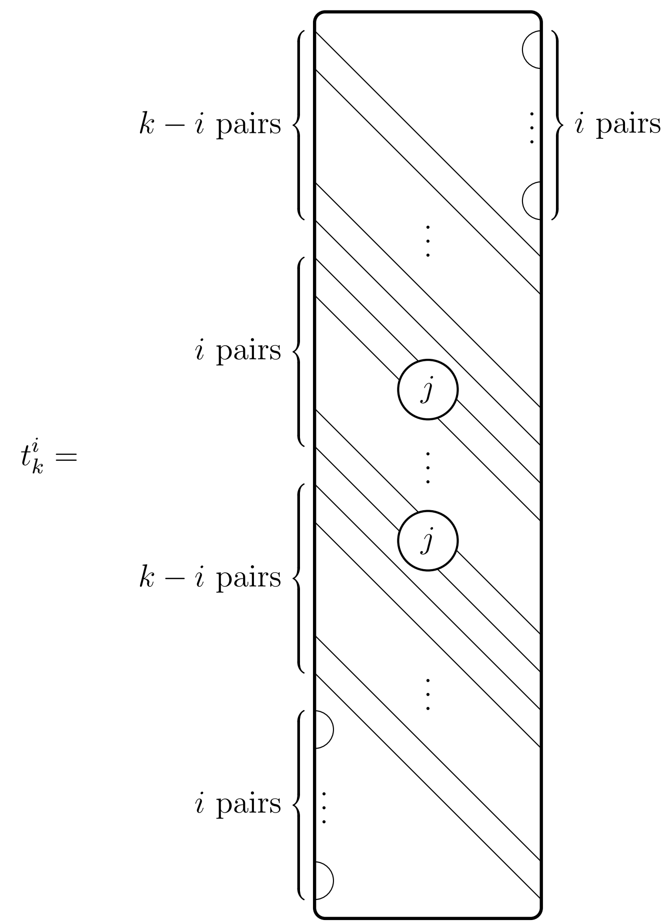


Figure 3.5: The term t_k^i.

Now define $\pi_k^{(L)} \in G^0_{0,2k+4}$ and $\pi_k^{(R)} \in G^0_{2k+4,0}$ as in Figure 3.10, noting that while $\pi_k^{(L)}$ is a Temperley-Lieb diagram, $\pi_k^{(R)}$ might not be, as p is an arbitrary element of G_0^{2i}.

Note that $\pi_k^{(L)} \wedge_{k+2} \pi_k^{(R)} = 0$ as the j term has two of its side strings paired off. Also, $\pi_k^{(L)} c_k^k \pi_k^{(R)} = (\delta^2 - 1)\delta^k p$.

3.2.0.4 Constructing u, A, and v

Define $B = c_k \otimes \tilde{t}_k$ where, as before, $\otimes$ is the operation induced by the tangle which stacks diagrams. Set $n = 8k + 4$

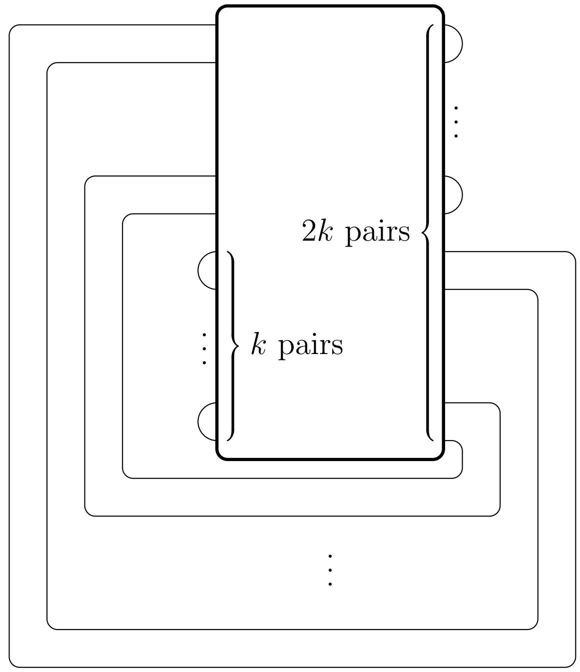


Figure 3.6: The planar tangle a_k.

Let $\varepsilon = \frac{1}{2\|B\|_{\mathrm{Gr}_n(P)}}$. Set $\alpha = \frac{1}{(\delta^2-1)^{k+1}\delta^{3k}}$, and define $u = \varepsilon^{-k}\alpha\pi_k^{(L)} \otimes a_k^{(L)}$ and $v = \pi_k^{(R)} \otimes a_k^{(R)}$.

Finally set $A = 1 + \varepsilon B$ where 1 is the identity in Gr_n, the Temperley-Lieb diagram with $n = 8k + 4$ horizontal through-strings.

As $\|\varepsilon B\| = 1/2$, we have that A is invertible in M_n and given by a power series. We can thus calculate (recalling Remark 2.3.40 that diagrammatic calculations of these products work just fine even with uneven numbers of strings on the left and right),

$$\begin{aligned} uA^{-1}v &= \sum_{j=0}^{\infty} uB^j v \\ &= u\,\mathrm{id}_n\, v + uB^k v \end{aligned}$$

because for $j \notin \{0, k\}$ the second tensor component is equivalent to $Z_{a_k}(t_k^i) = 0$, which makes the entire quantity zero.

Then for $j = 0$, the first component is $\pi_k^{(L)}(\mathrm{id}_k)\pi_k^{(R)} = 0$, making the entire quantity zero.

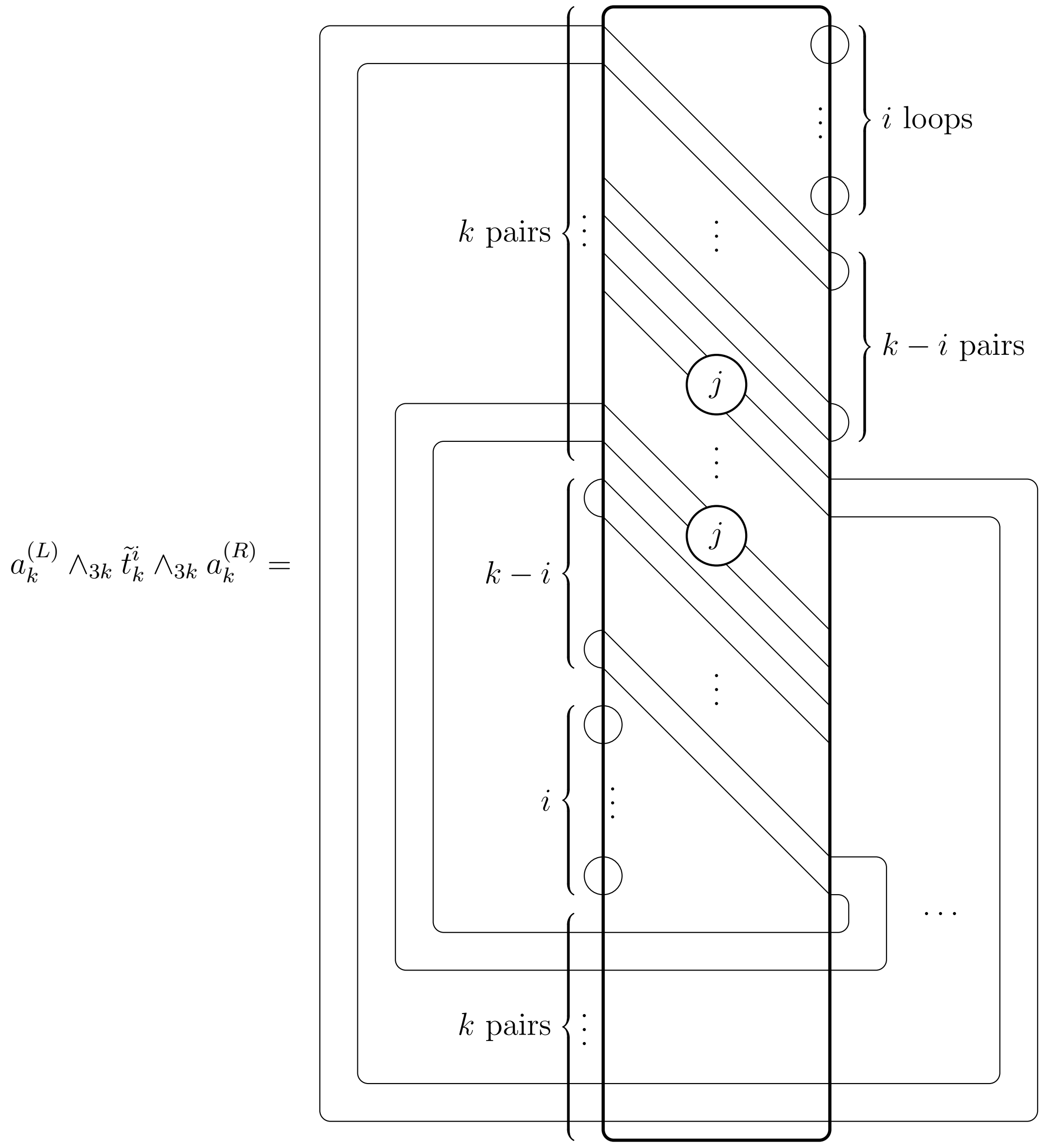


Figure 3.7: The product $a_k^{(L)} \wedge_{3k} \tilde{t}_k^i \wedge_{3k} a_k^{(R)}$ where $0 < i < k$.

Thus we can calculate

$$\begin{aligned} uA^{-1}v = uB^k v &= \left(\varepsilon^{-k}\alpha\right)\left(\pi_k^{(L)}\varepsilon^k c_k^k \pi_k^{(R)}\right)\left(a_k^{(L)}\tilde{t}_k^k a_k^{(R)}\right) \\ &= \left(\varepsilon^{-k}\alpha\right)\left(\varepsilon^k(\delta^2-1)\delta^k p\right)\left(\delta^{2k}(\delta^2-1)^k\right) \\ &= p. \end{aligned}$$

$c_k :=$

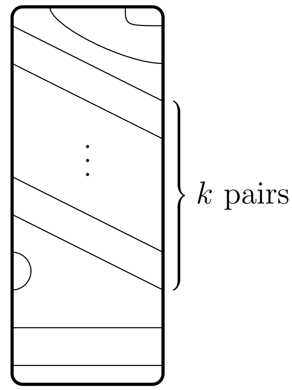


Figure 3.8: The term c_k.

$c_k^k =$

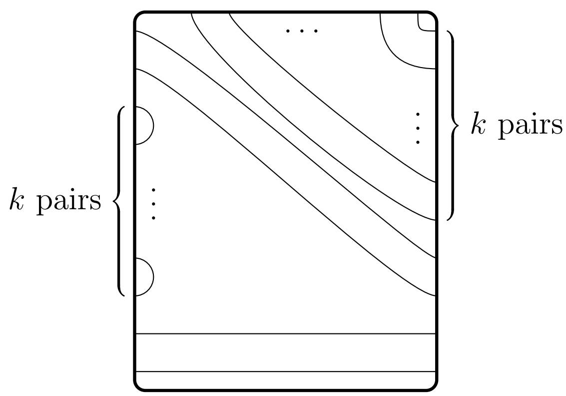


Figure 3.9: The term c_k^k.

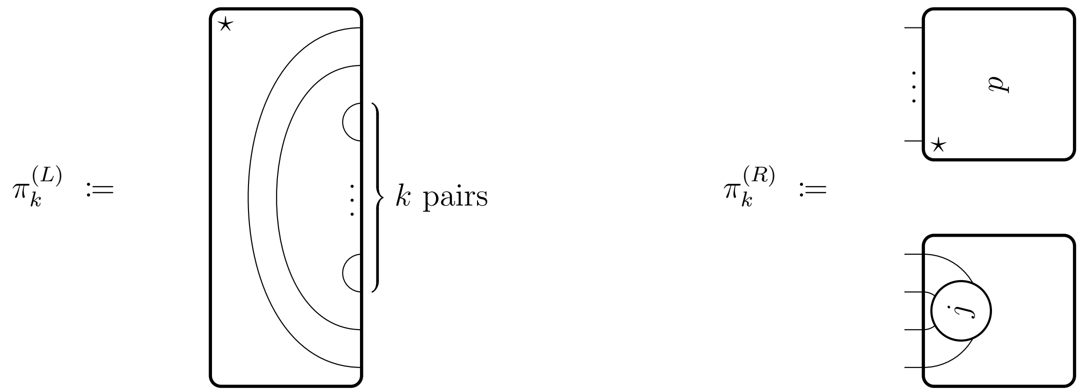
□

Remark 3.2.3. To compare this result to the usual linearization technique, one may think of A as a linear matrix, u as a row vector, and v as a column vector.

3.2.1 Extending to Gr_0

Recall that $\mathrm{Gr}_0 = \bigoplus_{k=0}^{\infty} G_0^{2k}$.

Corollary 3.2.4. *Suppose* $\mathrm{Gr}_0 \ni p = p_1 + \cdots + p_\ell$ *where* $p_i \in G_0^{2k_i} \setminus G_0^{2k_i-2}$*, that is, each* p_i



(a) $\pi_k^{(L)}$

(b) $\pi_k^{(R)}$

Figure 3.10: The planar algebra terms for $\pi_k^{(L)}$ and $\pi_k^{(R)}$.

can be thought of as a box shaped diagram with $2k_i$ strings on top and none on the left or right.

Then there exists elements $u \in G^0_{0,n}$, $v \in G^0_{n,0}$, and $A \in G^2_n$ such that when these are viewed in M_n, we have that A is invertible and $uA^{-1}v = p$.

Proof. For each p_i, find $u_i \in G^0_{0,n_i}$, $A_i = (I + B_i) \in G^2_{n_i}$, and $v_i \in G^0_{n_i,0}$ as in Theorem 3.2.1. When viewed as elements of M_{n_i}, we have $p_i = u_i A_i^{-1} v_i$.

Recall the inclusion of $\mathrm{Gr}_n \subset \mathrm{Gr}_{n+1}$ given by $x \mapsto x \otimes \mathrm{id}_1$, the Temperley-Lieb diagram with 1 through-string, see [GJS10] for more discussion of this inclusion. Up to this inclusion, we can assume all the n_i are equal to some (even) number m, by adding a number of $\subset$ or $\supset$ shaped diagrams to the u_i and v_i respectively and by juggling the appropriate factors of δ.

We'd like to do something like $(\sum u_i)(1+\sum B_i)^{-1}(\sum v_i)$, but there are many cross terms that are undesirable.

Consider the following elements of $G^0_{2\ell}$, writing j for the Jones-Wenzl projection of G^0_2, and writing e_2 for the Jones projection of M_0 into M_1 as mentioned in Section 3.2.0.1.

$$G^0_{2\ell+2} \ni s_i = \frac{1}{\delta^{\ell-1}(\delta^2-1)^2} e_2 \otimes \cdots \otimes e_2 \otimes j \otimes e_2 \otimes \cdots \otimes e_2 \otimes j$$

where the two j's are in the i-th and $\ell+1$th position, and $\otimes$ refers again to the stacking of these diagrams.

Observe that

$$s_i^2 = s_i = s_i^* \tag{3.1}$$

$$s_i s_j = 0 \text{ if } i \neq j \tag{3.2}$$

$$\|s_i\|^2 = 1. \tag{3.3}$$

Next define the terms λ_i as in Figure 3.11, and define $\rho_i \in G^0_{2\ell+2,0}$ to just be the mirror image of λ_i.

We calculate to obtain the following facts:

$$\lambda_i s_j = s_j \rho_i = 0 \text{ for } i \neq j$$

$$\lambda_i s_i \rho_i = \delta^{\ell-1}.$$

Now we construct u, A, and v as follows, setting $n = m + 2\ell + 2$.

$$u = \sum_{i=1}^{\ell} \sqrt{\ell}(u_i \otimes \lambda_i)$$

$$v = \sum_{i=1}^{\ell} v_i \otimes \rho_i$$

$$A = 1 + B$$

$$= 1 + \sum_{i=1}^{\ell} \frac{1}{\sqrt{\ell}} B_i \otimes s_i,$$

noting that $\|B\| < 1$, and so A^{-1} exists (viewing A as an element of M_n) and is given by a power series, and so we calculate:

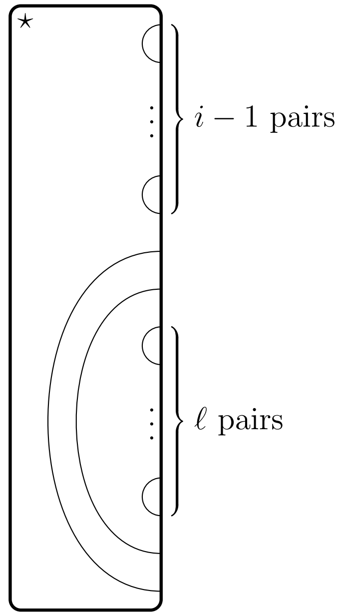
$\lambda_i \coloneqq$ 

Figure 3.11: The term λ_i.

$$
\begin{aligned}
uA^{-1}v &= \sum_{k=0}^{\infty} uB^k v \\
&= \frac{1}{\sqrt{\ell}} \sum_{i,j,h=1}^{\ell} \left[(u_i \otimes \lambda_i) \left(\sum_{k=0}^{\infty} B_j^k \otimes s_j \right) (v_h \otimes \rho_h) \right],
\end{aligned}
$$

but expanding using Equations 3.1–3.3, all terms of the form $(u_i \otimes \lambda_i)(B_j^k \otimes s_j)(v_h \otimes \rho_h)$ are zero except when $i = j = h$. When $k \neq k_i$, the product $u_i B_i^k v_i$ yields 0, so the only remaining terms are when $k = k_i$ and $i = j = h$, in which case, we have

$$
(\sqrt{\ell} u_i \otimes \lambda_i)(\tfrac{1}{\sqrt{\ell}} B_i^{k_i} \otimes s_i)(v_i \otimes \rho_i) = u_i B^{k_i} v_i = p_i
$$

Thus the series yields $uA^{-v}v = \sum p_i = p$.

□

3.3 Self-adjoint A

Proposition 3.3.1. *Suppose (u, A, v) is a planar algebra triple of order n with $uA^{-1}v = p$.*

Then there is a planar algebra triple $(\tilde{u}, \tilde{A}, \tilde{v})$ of order $n + 4$ with $\tilde{u}\tilde{A}^{-1}\tilde{v} = p$ and $\tilde{A}$ self-adjoint.

Proof. Let's think about the matrix case where we could take something like

$$\begin{pmatrix} 0 & u \end{pmatrix} \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix}^{-1} \begin{pmatrix} v \\ 0 \end{pmatrix}.$$

We do something quite similar. Pick $\tilde{u}, \tilde{A}, \tilde{v}$ as in Figure 3.12.

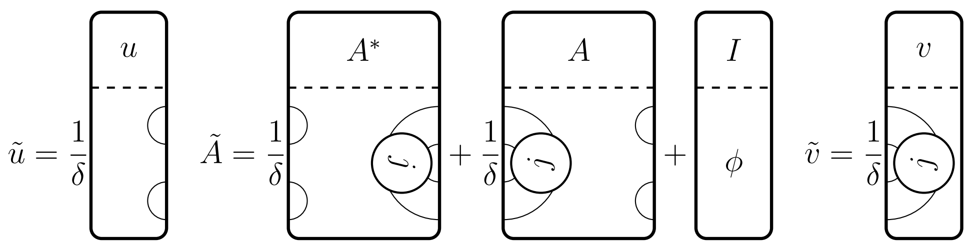
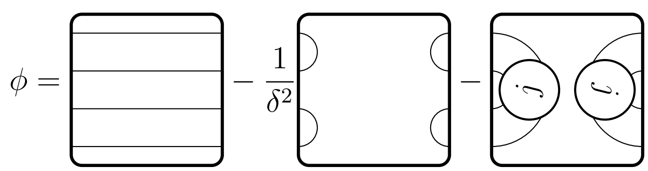


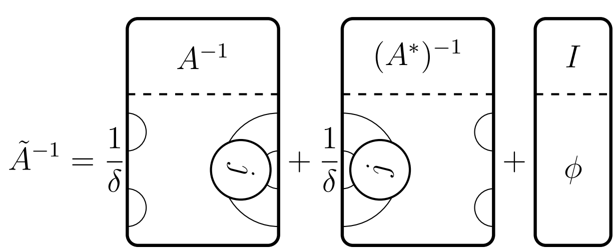
Figure 3.12: The choice of $\tilde{u}, \tilde{A}, \tilde{v}$ where $\tilde{A}$ is self-adjoint.

Define the projection $\phi \in \mathrm{Gr}_2\, P$ as follows.



(3.4)

Then $\tilde{A}^{-1}$ is given by



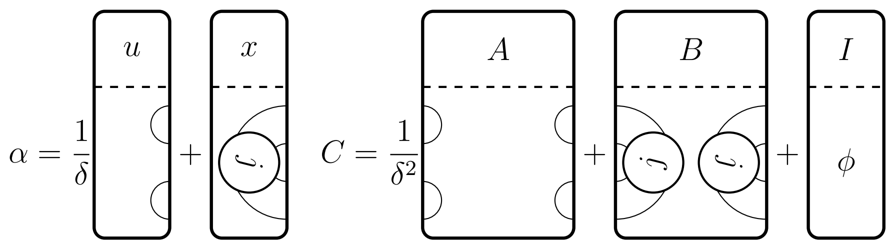
which is verified diagrammatically, and a quick calculation shows $\tilde{u}\tilde{A}^{-1}\tilde{v} = p$ as desired. □

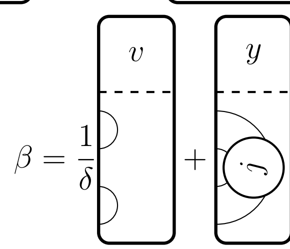
3.4 Sums of Triples

We can define the sum of terms in Gr_0 directly by considering the triples that represent them.

Theorem 3.4.1. *With P and M_k as before, let (u, A, v) and (x, B, y) be triples of order k respectively, and $p, q \in \mathrm{Gr}_0$ such that A and B are invertible in the $\mathrm{Aff}(M_k)$, and $uA^{-1}v = p$ and $xB^{-1}y = q$ where these two equations are taken in $\mathrm{Aff}(M_k)$.*

Now, define the triple (α, C, β) as follows:

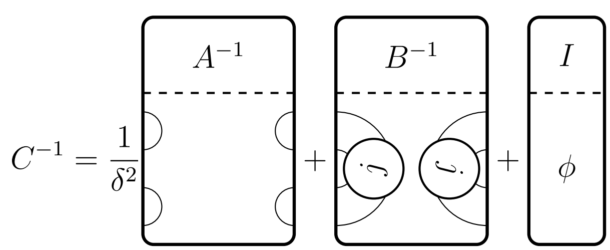




where I is the identity in Gr_k and ϕ is as in Equation 3.4.

Then C is invertible in $\mathrm{Aff}(M_{k+4})$ and $\alpha C^{-1}\beta = p + q$.

Proof. Keeping in mind Proposition 2.3.39, we may consider the following diagram, and by a simple calculation, we verify that it is the inverse of C.



Multiplying out $\alpha C^{-1}\beta$ now yields $p+q$ as desired, noting that the Jones-Wenzl j yields zero when either end is paired off with a $\subset$ or $\supset$, and so all terms cancel except $uA^{-1}v + xB^{-1}y$ $\square$

Remark 3.4.2. This construction should be thought of as an analog of the construction of addition in [CR99]. Instead of

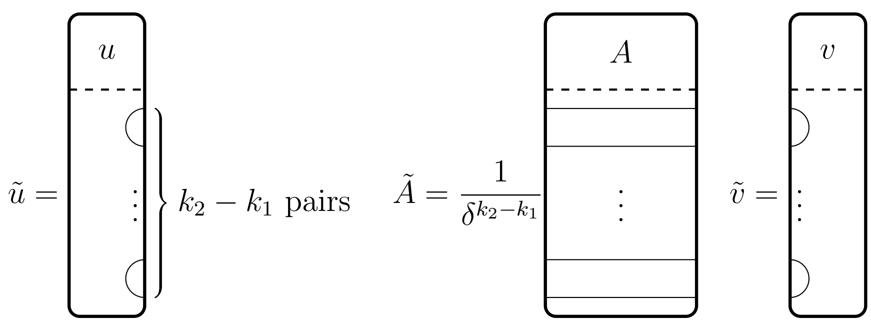
$$\left(\begin{pmatrix} u & x \end{pmatrix}, \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \begin{pmatrix} v \\ y \end{pmatrix} \right),$$

it's more correct to say that this construction is an analog of

$$\left(\begin{pmatrix} u & x & 0 \end{pmatrix}, \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & I \end{pmatrix}, \begin{pmatrix} v \\ y \\ 0 \end{pmatrix} \right)$$

for an appropriate block matrix decomposition.

Remark 3.4.3. With planar algebra triples (u, A, v) and (x, B, y) of different order, $k_1 < k_2$ respectively, we may replace (u, A, v) with



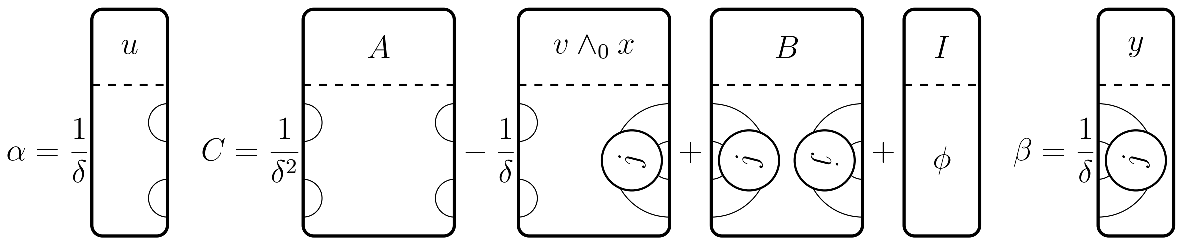
This is the same observation as before, that up to the inclusion $\mathrm{Gr}_{k_1} \hookrightarrow \mathrm{Gr}_{k_2}$, we may assume different terms have the same order.

Remark 3.4.4. This formula for the addition of triples of different orders could also work as a construction for the sum of terms in Gr_0.

3.5 Products of Triples

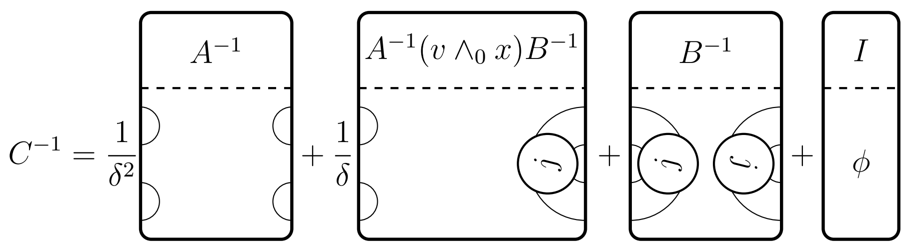
Theorem 3.5.1. *With P and M_k as before, let (u, A, v) and (x, B, y) be triples of order k, $p, q \in \mathrm{Gr}_0$ such that A and B are invertible in the* $\mathrm{Aff}(M_k)$*, and $uA^{-1}v = p$ and $xB^{-1}y = q$ where these two equations are taken in* $\mathrm{Aff}(M_k)$*.*

Define the triple (α, C, β) as follows:



Then, C is invertible in $\mathrm{Aff}(M_{k+4})$, $\alpha C^{-1}\beta = p \wedge q$.

Proof. It suffices to diagrammatically construct C^{-1}, which we do as follows.



The calculation that this is in the fact the inverse is fairly simple, recalling the fact that the projections listed below A, B and I sum to the identity, just as in the summation case.

The calculation is analogous to the following application of the Schur complement to finding the inverse of a matrix in block form:

$$\begin{pmatrix} A & t & 0 \\ 0 & B & 0 \\ 0 & 0 & I \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} & -A^{-1}tB^{-1} & 0 \\ 0 & B^{-1} & 0 \\ 0 & 0 & I \end{pmatrix}$$

□

CHAPTER 4

Open Questions

4.1 Quotients and Admissible Triples

With linearization of terms in Gr_0 complete, as well as addition and multiplication defined on triples representing terms in Gr_0, it's natural to consider quotients.

Note that [CR99] constructs the reciprocal of the non-commtuative rational function ρ represented by (u, A, v) with the representation

$$\left(\begin{pmatrix} 0 & 1 \end{pmatrix}, \begin{pmatrix} A & -v \\ u & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right),$$

where a key step in the proof is the fact that $\begin{pmatrix} A & -v \\ u & 0 \end{pmatrix}$ is full precisely when $\rho \neq 0$.

The "admissible" triples of Cohn's construction are precisely those triples where A is full. And recall that for a matrix over $\mathbb{C}\langle x_1, \ldots, x_n \rangle$, fullness characterizes invertibility over the free skew field.

By [MSY23], this is the same as the matrix being invertible over the affiliated operators for a particular choice of $X_1, \ldots, X_n$ with maximal Δ.

It makes sense then that the triples we should consider in the planar algebra case are where A is "linear" (at most 2 upwards strings) and invertible in $\mathrm{Aff}(M_k)$.

The issue we've come across is that we don't know how to produce an analog of the construction of reciprocals in [CR99] without losing invertibility.

Instead, relaxing the notion of admissible planar algebra triples to (u, A, v) where A is not necessarily invertible in $\mathrm{Aff}(M_k)$, but instead:

1. the right support of u is contained in the right range of A, and

2. the left support of v is contained in the left range of A.

may work, because in this case, $uA^{-1}v$ is still well-defined.

For simplicity, it might be valuable to restrict to A self-adjoint, at which point we just require that $\ker(A)$ is contained in the right kernel of u and the left kernel of v.

With this restriction, and for A invertible in the affiliated operators, something like Figure 4.1 may work, however defining this properly and ensuring it's well-defined with whatever equivalence relation is chosen will take some more work.

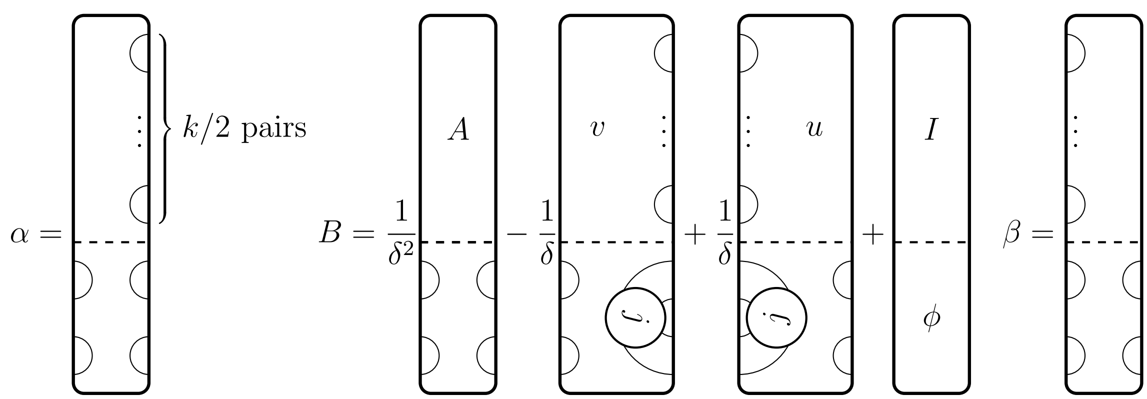


Figure 4.1: An idea for the reciprocal.

4.2 A Notion of Fullness for Planar Algebras

In [MSY23], it's shown that when $X_1, \ldots, X_n \in (M, \tau)$ have maximal Δ, the fullness of a matrix over the abstract set of polynomials $\mathbb{C}\langle x_1, \ldots, x_n\rangle$ is equivalent to the invertibility of its evaluation on $X_1, \ldots, X_n$ in the affiliated operators.

Is there some simple characterization of $A \in \mathrm{Gr}_k$ that describes invertibility in $\mathrm{Aff}(M_k)$? Matrix fullness is based on the notion of inner rank which depends on factorizations. It's unclear what the planar algebra analog of this might be.

Given a particular trace on a subfactor planar algebra satisfying some condition like maximal Δ, having an equivalence between the "analytic" condition of invertibility in $\mathrm{Aff}(M_k)$ and an "algebraic" condition in Gr_k seems useful, both for calculating examples and likely proving that the desired division operation on triples is well-defined.

4.3 Equivalence Between Triples

The equivalence relation in [CR99] between representations of elements of the free skew field is given by chains morphisms or inverse morphisms, where a morphism from (u, A, v) to (u', A', v') is given by a pair of matrices (P, Q) of appropriate order with $u' = uQ$, $Pv' = v$, and $PA' = AQ$.

Should morphisms in the planar algebra case be given by pairs of elements in $G^0_{k,\ell}$? Or perhaps a pair of maps between the relevant spaces that are not necessarily expressible as elements of the planar algebra? Restating the question more vaguely: do the morphisms obey the "symmetry" of the planar algebra?

4.4 Further applications?

If something like a planar algebra analog of the free skew field exists, denoting it for the moment as $\mathbb{C}(\!\langle \mathrm{Gr}_0 \rangle\!)$, do we get a commutative diagram like this?

$$\begin{array}{ccc} \mathrm{Gr}_0 & \overset{}{\underset{\text{linearization}}{\dashrightarrow}} & \mathbb{C}(\!\langle \mathrm{Gr}_0 \rangle\!) \\ \big\downarrow & & \big\downarrow ? \\ M_0 = \overline{\mathrm{Gr}_0} & \hookrightarrow & \mathrm{Aff}(M_0) \end{array}$$

This is in hopeful analogy to the case of maximal Δ, where we have the following.

$$\begin{array}{ccc} \mathbb{C}\langle x_1, \ldots, x_n \rangle & \hookrightarrow & \mathbb{C}\langle\!\!\langle x_1, \ldots, x_n \rangle\!\!\rangle \\ {\scriptstyle \mathrm{Ev}_X}\downarrow & & \downarrow \\ (M, \tau) & \hookrightarrow & \mathrm{Aff}(M) \end{array}$$

Beyond the above questions, I wonder what interesting things can be done with this hypothetical planar algebra analog of the free skew field if it can be defined. I suspect it will act like a division closure on Gr_0 inside $\mathrm{Aff}(M_0)$, and wonder if having something algebraic between Gr_0 and $\mathrm{Aff}(M_0)$ may yield structural insight. Keep in mind we do not expect this object to be a skew field in general, even with nice choices of trace.

There is a calculation in [GJS11, Sect 5.2] of the laws of the Jones-Wenzl projections in the Temperley-Lieb planar algebra. The law is calculated with the Voiculescu trace on the algebra Gr_0, so rather than the Jones-Wenzl $j \in \mathrm{Gr}_2$ used in this dissertation, this is something that is "purely polynomial".

The law of JW_n is the free multiplicative convolution of a free Poisson law and another measure which has a Dirac mass at 0. By [Bel03, Thm. 4.1], we have $\mu \boxtimes \nu(\{0\}) = \max(\mu(\{0\}), \nu(\{0\}))$. Thus the "purely polynomial" Jones-Wenzl in Gr_0 is not invertible in $\mathrm{Aff}(M_0)$. Does something similar occur for the two-cabled Voiculescu trace, where the $\cup$ elements has semicircular law? How does the choice of trace affect which members of Gr_0 are guaranteed to be invertible in $\mathrm{Aff}(M_0)$?

I also wonder if some planar algebra analog of the free skew field could be constructed possible for not just Gr_0 but Gr_k as well.

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