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Combinatorics in the Rational Shuffle Theorem and the Delta Conjecture

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Combinatorics in the Rational Shuffle Theorem and the Delta Conjecture

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by

Dun Qiu

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2019

The Dissertation of Dun Qiu is approved, and it is acceptable in quality and form for publication on microfilm and electronically:

Chair

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2019

DEDICATION

For my mother.

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ABSTRACT OF THE DISSERTATION

Combinatorics in the Rational Shuffle Theorem and the Delta Conjecture

by

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The Classical Shuffle Conjecture proposed by Haglund, Haiman, Loehr, Remmel and Ulyanov gives a well-studied combinatorial expression for the bigraded Frobenius characteristic of $\mathcal{S}_n$-module of the ring of diagonal harmonics, which has been proved by Carlsson and Mellit as the Shuffle Theorem, stating that a symmetric function expression ∇e_n equals a generating function of combinatorial objects called parking functions. The Rational Shuffle Theorem of the expression $\mathrm{Q}_{m,n}(-1)^n$ of Mellit and the Delta Conjecture of the expression $\Delta'_{e_k} e_n$ proposed by Haglund, Remmel and Wilson are two natural generalizations of the Shuffle Theorem. The primary goal of this dissertation is to prove some special cases of the conjectures, and compute the Schur function expansions of the corresponding symmetric function expressions. We explore

several symmetries in the combinatorics of the coefficients that arise in the Schur function expansion of $\mathrm{Q}_{m,n}(-1)^n$ in the Rational Shuffle Theorem. Especially, we study the hook-shaped Schur function coefficients, and the Schur function expansion of $\mathrm{Q}_{m,n}(-1)^n$ in the case where m or n equals 3. We give a combinatorial proof that the coefficient of s_λ in the Delta expression $\Delta_{e_2}e_n$ has a non-negative expansion in terms of q,t-analogues. We propose a new *valley version* conjecture of the expression $\Delta'_{e_k}\Delta_{h_r}e_n$, and we give a proof of the *valley version* conjecture of $\Delta'_{e_k}\Delta_{h_r}e_n$ when t or q equals 0. Our work lead to many new results about the combinatorial objects in the conjectures, such as the Mahonian distribution in extended ordered multiset partitions and the straightening action in parking functions.

Chapter 1

Introduction

The field of algebraic combinatorics is interested in analyzing algebraic structures using combinatorial methods, such as bijections and generating functions. Many algebraic combinatorial problems are related to symmetric functions or Macdonald polynomials. The *Shuffle Theorem* of Carlsson and Mellit [CM18] is such an algebraic combinatorial problem which gives a well-studied combinatorial expression for the bigraded Frobenius characteristic of the ring of diagonal harmonics. The *Rational Shuffle Theorem* of Mellit [Mel16] and the *Delta Conjecture* are two widely studied generalizations of the Shuffle Theorem.

In this chapter, we give an introduction of symmetric functions, Macdonald polynomials and some combinatorial concepts related to the Rational Shuffle Theorem and the Delta Conjecture.

1.1 Background about symmetric functions and Macdonald polynomials

We shall define symmetric functions, Macdonald polynomials, and other combinatorial terminologies related to symmetric functions in this section. Material related to this section can be found in Chapter 7 of [Sta99] or in [Sag02].

1.1.1 Permutations, partitions, compositions and tableaux

For any integer n, a *permutation* $\sigma = \sigma_1 \cdots \sigma_n$ (in *one-line notation*) of size n is a rearrangement of the numbers $1,\dots,n$. The ith number in σ from left to right is denoted by σ_i. When viewed as a function from the set $[n] = \{1,\dots,n\}$ to itself, σ sends i to σ_i, i.e. $\sigma(i) = \sigma_i$. Using the function viewpoint, we can write a permutation in *cycle notation* that we use parenthesis to denote the cycles of the map σ.

The *symmetric group* $\mathcal{S}_n$ is the set of permutations of size n. For example, $\sigma = 3672451 \in \mathcal{S}_7$ is a permutation of size 7, and the cycle notation for σ is $\sigma = (1,3,7)(2,6,5,4)$.

Given any permutation $\sigma = \sigma_1 \cdots \sigma_n \in \mathcal{S}_n$, the *descent number* of σ is defined to be $\mathrm{des}(\sigma) := |\{i : \sigma_i > \sigma_{i+1}\}|$, and the *major index* of σ is $\mathrm{maj}(\sigma) := \sum_{\sigma_i > \sigma_{i+1}} i$.

For any integer n, a weakly decreasing sequence of positive integers $\lambda = (\lambda_1,\dots,\lambda_k)$ is a *partition* (or an *integer partition*) of n if $\sum_{i=1}^k \lambda_i = n$, written $\lambda \vdash n$. We let $|\lambda| = n$ and $\ell(\lambda) = k$ denote the size and length (number of parts) of the partition λ. We also write $\lambda = n^{m_n} \cdots 2^{m_2} 1^{m_1}$ for the partition $\lambda \vdash n$ with m_i parts of size i. For example, $\lambda = (4,2,1,1) \vdash 8$ is a partition of the integer 8 with $\ell(\lambda) = 4$, and we also write $\lambda = 421^2$.

The definition of compositions is similar to that of partitions. For an integer n, a *weak composition* of n is defined to be a sequence of *non-negative* integers $\alpha = (\alpha_1,\dots,\alpha_k)$ such that $\sum_{i=1}^k \alpha_i = n$, written $\alpha \vDash n$; and a *strong composition* of n is defined to be a sequence of *positive* integers $\alpha = (\alpha_1,\dots,\alpha_k)$ such that $\sum_{i=1}^k \alpha_i = n$, written $\alpha \vDash_{\text{strong}} n$. We let $|\alpha| = n$ denote the size of the composition α.

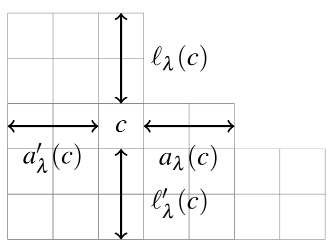
For any strong composition $\alpha = (\alpha_1,\dots,\alpha_k)$ of n with k parts, we associate to it a subset $S(\alpha) = \{\alpha_1, \alpha_1+\alpha_2, \dots, \alpha_1+\cdots+\alpha_{k-1}\}$ of $[n-1]$ with $k-1$ elements. This builds a bijective relation between the set of strong compositions of n and the set of subsets of $[n-1]$.

We still let $\ell(\alpha)$ denote the number of parts of α, and we let $\lambda(\alpha)$ be the partition obtained by organizing the parts in α in a decreasing order and deleting all the 0's. For example, $\alpha = (2,1,0,1,3) \vdash 7$ is a weak composition of 7 with $\ell(\alpha) = 5$, and $\lambda(\alpha) = (3,2,1,1)$.

For each partition $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash n$, we can associate to the partition a *Ferrers diagram* (or a *Young diagram*) in French notation, which is a diagram with n squares such that there are λ_i squares in the ith row, counting from bottom to top. We let λ' be the *conjugate* of λ (i.e. the Ferrers diagram of λ and λ' are symmetric about the main diagonal, turning the rows (columns) of λ into the columns (rows) of λ'). We say that λ is *self-conjugate* if $\lambda = \lambda'$.

Figure 1.1 shows an example of the Ferrers diagram of a partition $\lambda = (7,7,5,3,3) \vdash 25$ in French notation, and $\lambda' = (5,5,5,3,3,2,2)$. We shall also use λ to denote the Ferrers diagram associated to λ.

For each cell $c \in \lambda$, we let the *arm* of c, $a_\lambda(c)$, be the number of cells to the left of c; the *coarm* of c, $a'_\lambda(c)$, be the number of cells to the right of c; the *leg* of c, $\ell_\lambda(c)$, be the number of cells on top of c; the *coleg* of c, $\ell'_\lambda(c)$, be the number of cells below c, as shown in Figure 1.1. We usually let $(a'_\lambda(c), l'_\lambda(c))$ denote the coordinate of the cell c. We often abbreviate the notations to $a(c), a'(c), \ell(c), \ell'(c)$.



$a_\lambda(c) =$ arm of $c = 2$,
$a'_\lambda(c) =$ coarm of $c = 2$,
$\ell_\lambda(c) =$ leg of $c = 2$,
$\ell'_\lambda(c) =$ coleg of $c = 2$.
cell c is at $(2,2)$

Figure 1.1: The Ferrers diagram of the partition $\lambda = (7,7,5,3,3)$.

Now let λ be a partition of n. We can fill the cells of the Ferrers diagram of λ with integers to obtain a *tableau* T (where λ is called the shape of the tableau). We also use T to denote the multiset of the filled integers, and we write

$$X^T := \prod_{i \in T} x_i.$$

If we fill the cells with positive integers without other restrictions, then we obtain a *tableau*. The set of tableaux of shape λ is denoted by $\mathrm{Tab}(\lambda)$. Figure 1.2 shows an example of a tableau T with $X^T = x_1 x_2^2 x_4^2 x_5 x_6$.

4			
1	5		
2	6	2	4

Figure 1.2: A tableau T with shape $\lambda = (4,2,1)$.

If we fill the cells of λ with the integers $\{1,\dots,n\}$ such that the numbers in each row are increasing from left to right and the numbers in each column are increasing from bottom to top, then we get a *standard Young tableau*. The set of standard Young tableaux of shape λ is denoted by $\mathrm{SYT}(\lambda)$.

If we fill the cells of λ with positive integers such that the numbers in each row are weakly increasing from left to right and the numbers in each column are strictly increasing from bottom to top, then we get a *semi-standard Young tableau* (or a *column strict tableau*). The set of semi-standard Young tableaux of shape λ is denoted by $\mathrm{SSYT}(\lambda)$. Figure 1.3 shows examples of such tableaux.

5			
2	6		
1	3	4	7

6			
2	3		
1	1	3	4

Figure 1.3: A standard Young tableau and a semi-standard Young tableau.

1.1.2 Symmetric functions

Let $\mathbb{R}[[X]]$ be the ring formal power series with variables in $X = \{x_1, x_2, \dots\}$. For any formal power series $f[X] = f[x_1, x_2, \dots] \in \mathbb{R}[[X]]$ and any permutation $\sigma = \sigma_1 \cdots \sigma_n \in \mathcal{S}_n$, we define the action $\sigma \circ f[X]$ of σ on $f[X]$ by:

$$\sigma \circ f[x_1, x_2, \ldots] := f[x_{\sigma_1}, x_{\sigma_2}, \ldots],$$

sending x_i to x_{σ_i}, here we use the convention that $\sigma_i = i$ for $i > n$.

Given any $f[X] \in \mathbb{R}[[X]]$ with a finite degree in X, $f[X]$ is a *symmetric function* if and only if:

$$\forall n \in \mathbb{Z}_+ \text{ and } \forall \sigma \in \mathcal{S}_n, \;\; \sigma \circ f[X] = f[X].$$

The *ring of symmetric functions* Λ consists of all symmetric functions $f[X] \in \mathbb{R}[[X]]$. Let $\Lambda^{(n)}$ denote the set of symmetric functions $f[X] \in \mathbb{R}[[X]]$ that are homogeneous of degree n. We have $\Lambda = \bigoplus_{n \geq 0} \Lambda^{(n)}$.

When taken as a vector space, $\Lambda^{(n)}$ has dimension $p(n)$ which is the number of partitions of the integer n, and it has six classical and natural basis:

- The *monomial symmetric function* basis $\{m_\lambda[X]\}_{\lambda \vdash n}$, defined by

$$m_\lambda[X] := \sum_{i_1, \ldots, i_{\ell(\lambda)} \in \mathbb{Z}_+ \text{ distinct}} x_{i_1}^{\lambda_i} \cdots x_{i_{\ell(\lambda)}}^{\lambda_{\ell(\lambda)}}.$$

- The *elementary symmetric function* basis $\{e_\lambda[X]\}_{\lambda \vdash n}$, defined by

$$e_k[X] := \sum_{i_1 < \cdots < i_k} x_{i_1} \cdots x_{i_k}, \quad \text{and} \quad e_\lambda[X] := e_{\lambda_1} \cdots e_{\lambda_{\ell(\lambda)}}.$$

- The *homogeneous symmetric function* basis $\{h_\lambda[X]\}_{\lambda \vdash n}$, defined by

$$h_k[X] := \sum_{i_1 \leq \cdots \leq i_k} x_{i_1} \cdots x_{i_k}, \quad \text{and} \quad h_\lambda[X] := h_{\lambda_1} \cdots h_{\lambda_{\ell(\lambda)}}.$$

- The *power-sum symmetric function* basis $\{p_\lambda[X]\}_{\lambda\vdash n}$, defined by

$$p_k[X] := \sum_{i\geq 1} x_i^k, \quad \text{and} \quad p_\lambda[X] := p_{\lambda_1}\cdots p_{\lambda_{\ell(\lambda)}}.$$

- The *Schur symmetric function* basis $\{s_\lambda[X]\}_{\lambda\vdash n}$, defined by

$$s_\lambda[X] := \sum_{T\in \mathrm{SSYT}(\lambda)} X^T.$$

- The *forgotten symmetric function* basis $\{f_\lambda[X]\}_{\lambda\vdash n}$, defined by

$$f_\lambda[X] := (-1)^{|\lambda|-\ell(\lambda)} \sum_{c=(c_1,c_2,c_3,\ldots)\in CR(\lambda)} x_1^{c_1}x_2^{c_2}x_3^{c_3}\cdots,$$

  where $CR(\lambda)$ is the set of sequences of compositions $c = (c_1, c_2, c_3, \ldots)$ such that the parts of all c_i's rearrange to λ.

We also abbreviate the bases to $\{m_\lambda\}_{\lambda\vdash n}$, $\{e_\lambda\}_{\lambda\vdash n}$, $\{h_\lambda\}_{\lambda\vdash n}$, $\{p_\lambda\}_{\lambda\vdash n}$, $\{s_\lambda\}_{\lambda\vdash n}$ and $\{f_\lambda\}_{\lambda\vdash n}$. The first five bases are more famous, and more details about the forgotten basis can be found in [MR15] and [GHQR19] Proposition 1.2.

As a remark, an easier way of defining the set Λ is

$$\Lambda := \mathbb{C}\text{-span}\{m_\lambda : \lambda \text{ a partition}\}.$$

1.1.3 Operations on symmetric functions

We shall introduce three operations on symmetric functions: the omega involution, the Hall scalar product and plethysm.

The *omega involution* ω is an endomorphism of the symmetric function ring Λ defined by

$$\omega(e_\lambda) := h_\lambda,\ \forall\lambda.$$

It is a well-known result that $\omega(s_\lambda) = s_{\lambda'}$, and it follows immediately that ω is an involution (i.e. $\omega^2 = \mathrm{id}$) since $\omega^2(s_\lambda) = \omega(s'_\lambda) = s_\lambda$.

The *Hall scalar product* is a scalar product of the space $\Lambda^{(n)}$ defined by

$$\langle s_\lambda, s_\mu \rangle := \chi(\lambda = \mu)$$

for any $\lambda, \mu \vdash n$, where $\chi(x)$ is the function that takes value 1 if the statement x is true, and 0 otherwise. As a consequence, one can prove the following:

$$\begin{aligned} \langle m_\lambda, h_\mu \rangle &= \chi(\lambda = \mu), \\ \langle e_\lambda, f_\mu \rangle &= \chi(\lambda = \mu), \\ \langle p_\lambda, p_\mu \rangle &= z_\lambda \chi(\lambda = \mu), \end{aligned}$$

where for $\lambda = n^{m_n} \cdots 2^{m_2} 1^{m_1}$, $z_\lambda := \prod_{i=1}^{n} i^{m_i} m_i!$.

The most important part of this section is the definition of *plethysm*. From now on, we let $\mathbb{C}(q,t)$ denote the field of coefficients of the ring of symmetric functions Λ, i.e. we define

$$\Lambda := \mathbb{C}(q,t)\text{-span}\{m_\lambda : \lambda \text{ a partition}\}.$$

If $E = E(t_1, t_2, \ldots)$ is a rational function of the variables $t_1, t_2, \ldots$ and $F \in \Lambda$. We define the *plethysm* $F[E]$ by the following rules:

- $p_k[E] = E(t_1^k, t_2^k, \ldots)$.
- Given $F, G \in \Lambda$, $(F \cdot G)[E] = F[E] \cdot G[E]$.

- For any scalar α, $(\alpha F + G)[E] = \alpha F[E] + G[E]$.

Using the plethystic notation, we can state the *Cauchy formula* as follows: let $\{u_\lambda\}_{\lambda\vdash n}$ and $\{v_\lambda\}_{\lambda\vdash n}$ be a pair of *dual bases* of the space $\Lambda^{(n)}$ with respect to the Hall scalar product, and let X and Y be two sums of signed monomials. Then,

$$h_n[XY] = \sum_{\lambda\vdash n} u_\lambda[X]v_\lambda[Y].$$

1.1.4 The Frobenius map

For any group G, a representation M of G is uniquely associated to a function χ^M called the *character*, which is computed by taking traces of the matrices associated with the representation. Every character is *a class function* meaning it is constant on conjugacy classes.

Symmetric functions are closely related to the representation of the symmetric group $\mathcal{S}_n$. Due to Young's work, the set of *irreducible characters* $\{\chi^\lambda\}_{\lambda\vdash n}$ of $\mathcal{S}_n$ has cardinality $p(n)$ (the number of partitions of n), which is the same as the dimension of class functions of $\mathcal{S}_n$. The *Frobenius map* can be defined by

$$\mathrm{Frob}(\chi^\lambda) := s_\lambda \quad \forall\lambda\vdash n, \tag{1.1}$$

which builds a bridge between the world of representation theory and the world of symmetric functions. For a module M of $\mathcal{S}_n$, we also write

$$\mathrm{Frob}(M) = \mathrm{Frob}(\chi^M).$$

The symmetric function $\mathrm{Frob}(M)$ of an $\mathcal{S}_n$-module M is called the *Frobenius characteristic* of M. Other definitions and further details about the Frobenius map can be found in [Sag02] and [Sta99].

Thanks to the Frobenius map (1.1), one can simply expand the symmetric function $\mathrm{Frob}(M)$ in Schur function basis in order to study the irreducible decomposition of an $\mathcal{S}_n$-module M. On the other hand, if a symmetric function expanded in Schur basis has non-negative integer coefficients, then it must be the Frobenius characteristic of some $\mathcal{S}_n$-module.

In this dissertation, many of our $\mathcal{S}_n$-modules have natural gradings or bigradings. For such $\mathcal{S}_n$-modules, we can write $M = \bigoplus_{i\geq 0} M_i$ or $N = \bigoplus_{i,j\geq 0} N_{i,j}$, where M_i (or $N_{i,j}$) is the component of M (or N) of degree i (or (i,j)). The *graded (bigraded) Hilbert series* and the *graded (bigraded) Frobenius characteristic* of M (or N) is defined by

$$\mathrm{Hilb}(M;q) := \sum_{i\geq 0} q^i \dim(M_i), \qquad \mathrm{Frob}(M;q) := \sum_{i\geq 0} q^i \mathrm{Frob}(M_i),$$
$$\mathrm{Hilb}(N;q,t) := \sum_{i,j\geq 0} q^i t^j \dim(N_{i,j}), \qquad \mathrm{Frob}(N;q,t) := \sum_{i,j\geq 0} q^i t^j \mathrm{Frob}(N_{i,j}).$$

From the formulas above, it is not difficult to see that the Hilbert series of a (bigraded) $\mathcal{S}_n$-module M can be obtained from its Frobenius characteristic by taking the scalar product with the symmetric function p_1^n, i.e.

$$\mathrm{Hilb}(M;q,t) = \langle \mathrm{Frob}(M;q,t), p_1^n \rangle. \tag{1.2}$$

1.1.5 Quasi-symmetric functions

For any formal power series $f[X] \in \mathbb{R}[[X]]$ with a finite degree in X, $f[X]$ is said to be a *quasi-symmetric function* if for any composition $(\alpha_1, \ldots, \alpha_k)$, the coefficient of the monomial $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_k^{\alpha_k}$ is equal to the coefficient of the monomial $x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k}$ for any strictly increasing sequence of positive integers $i_1 < i_2 < \cdots < i_k$.

The *ring of quasi-symmetric functions* QSym consists of all quasi-symmetric functions $f[X] \in \mathbb{R}[[X]]$. The set of quasi-symmetric functions that are homogeneous of degree n is denoted by QSym_n, and clearly $\mathrm{QSym} = \bigoplus_{n\geq 0} \mathrm{QSym}_n$. Note that each symmetric function is also a quasi-symmetric function, thus Λ is a subring of QSym.

The set QSym_n can be viewed as a vector space, and its dimension is 2^{n-1} which equals the number of strong compositions of the integer n. QSym_n has two natural bases:

- The *monomial quasi-symmetric function* basis $\{M_\alpha[X]\}_{\alpha \vDash_{\text{strong}} n}$, defined by

$$M_\alpha[X] := \sum_{i_1 < \ldots < i_{\ell(\alpha)}} x_{i_1}^{\alpha_1} \cdots x_{i_{\ell(\alpha)}}^{\alpha_{\ell(\alpha)}}.$$

- Gessel's *fundamental quasi-symmetric function* basis $\{F_{S,n}[X]\}_{S \subseteq [n-1]}$, defined by

$$F_{S,n}[X] := \sum_{\substack{i_1 \le i_2 \le \ldots \le i_n, \\ i_j < i_{j+1} \text{ if } j \in S}} x_{i_1} x_{i_2} \ldots x_{i_n}.$$

Recall that for a strong composition $\alpha = (\alpha_1, \ldots, \alpha_k) \vDash_{\text{strong}} n$ with k parts, we associate to it a subset $S(\alpha)$ of $[n-1]$ where

$$S(\alpha) = \{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \cdots + \alpha_{k-1}\}.$$

For example, if $\alpha = (2,3,2,1)$, then $S(\alpha) = \{2,5,7\}$. Then, instead of indexing Gessel's fundamental quasi-symmetric function by subsets of $[n-1]$, we can associate one Gessel's fundamental quasi-symmetric function with each strong composition α by setting

$$F_{\alpha,n}[X] := \sum_{\substack{1 \le i_1 \le i_2 \le \cdots \le i_n \le n \\ i \in S(\alpha) \to i_i < i_{i+1}}} x_{i_1} x_{i_2} \cdots x_{i_n}. \tag{1.3}$$

Then the $F_{\alpha,n}[X]$'s as α ranges over all strong compositions of n also form a basis for the space of quasi-symmetric functions QSym_n.

We abbreviate $F_{S,n}[X]$ and $F_{\alpha,n}[X]$ to $F_S[X]$ and $F_\alpha[X]$ when there is no ambiguity. We will often omit the brackets to write M_α, F_S and F_α for these polynomials. Another way of defining

QSym is to set

$$\mathrm{QSym} := \mathbb{C}\text{-span}\{M_\alpha : \alpha \text{ a strong composition}\}.$$

1.1.6 Macdonald polynomials

In [Mac98], Macdonald introduced a family of orthogonal symmetric polynomials $\{P_\lambda[X;q,t]\}_{\lambda\vdash n}$ as a basis for the space $\Lambda^{(n)}$ that generalized many other existing families of symmetric functions, such as Schur symmetric functions, Jack polynomials and Hall-Littlewood polynomials. The polynomials $P_\lambda[X;q,t]$ are then known as *Macdonald polynomials*, which have nice mathematical and physical properties that interest people in many areas, such as physics, representation theory and algebraic geometry.

Macdonald polynomials have several transformations, and the form that we are using is called the *modified Macdonald polynomials* $\widetilde{H}_\mu[X;q,t]$ which are indexed by partitions $\mu \vdash n$. One combinatorial way to define $\widetilde{H}_\mu[X;q,t]$ is due to the work of Haglund, Haiman and Loehr [HHL05a]:

$$\widetilde{H}_\mu[X;q,t] := \sum_{T\in\mathrm{Tab}(\mu)} q^{\mathrm{inv}(T)} t^{\mathrm{maj}(T)} X^T,$$

where inv and maj are two statistics defined on the tableau T. We shall often abbreviate $\widetilde{H}_\mu[X;q,t]$ to $\widetilde{H}_\mu$.

The above definition gives a monomial quasi-symmetric function expansion of $\widetilde{H}_\mu[X;q,t]$. In fact, Macdonald polynomials are *Schur positive* (i.e. have non-negative coefficients in Schur function expansion), which is conjectured by Garsia and Haiman [GH93] and proved by Haiman [Hai01]. Further, in the expansion

$$\widetilde{H}_\mu[X;q,t] = \sum_{\lambda\vdash n} \widetilde{K}_{\lambda,\mu}(q,t) s_\lambda,$$

the coefficients $\widetilde{K}_{\lambda,\mu}(q,t) \in \mathbb{N}(q,t)$ are called *q,t-Kostka polynomials*.

1.1.7 The operators nabla ∇, Delta Δ and Delta prime Δ'

The symmetric function operators nabla ∇, Delta Δ and Delta prime Δ' are eigenoperators of Macdonald polynomials defined by Bergeron and Garsia [BG99] that will be frequently used in this dissertation.

For any partition $\mu \vdash n$, we let

$$B_\mu := \sum_{c\in\mu} q^{a'(c)}t^{l'(c)} \quad \text{and} \quad T_\mu := \prod_{c\in\mu} q^{a'(c)}t^{l'(c)}$$

be polynomials defined from the Ferrers diagram of μ. Given a modified Macdonald polynomial $\widetilde{H}_\mu[X;q,t]$, the operator *nabla* (∇) defines the operation:

$$\nabla\widetilde{H}_\mu := T_\mu\widetilde{H}_\mu.$$

Let f be a given symmetric function, then Δ_f and Δ'_f are the operators such that

$$\Delta_f\widetilde{H}_\mu := f[B_\mu]\widetilde{H}_\mu, \quad \Delta'_f\widetilde{H}_\mu := f[B_\mu - 1]\widetilde{H}_\mu,$$

where $f[B_\mu]$ and $f[B_\mu - 1]$ are plethystic expressions.

For example, for the partition $\mu = (3,1) \vdash 4$, we can first draw its Ferrers diagram, and fill in each cell $c \in \mu$ the weight $q^{a'(c)}t^{l'(c)}$. This process is pictured in Figure 1.4.

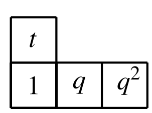


Figure 1.4: A partition $\mu = (3,1)$.

By definition, we have $B_{(3,1)} = 1+q+q^2+t$, $T_{(3,1)} = q^3t$, and

$$\nabla\widetilde{H}_{(3,1)} = q^3t\,\widetilde{H}_{(3,1)}.$$

Setting the symmetric function $f = e_2$, then

$$\begin{aligned}\Delta_{e_2}\widetilde{H}_{(3,1)} &= e_2[1+q+q^2+t]\,\widetilde{H}_{(3,1)} \\ &= (q+q^2+t+q^3+qt+q^2t)\,\widetilde{H}_{(3,1)},\end{aligned}$$

and

$$\begin{aligned}\Delta'_{e_2}\widetilde{H}_{(3,1)} &= e_2[q+q^2+t]\,\widetilde{H}_{(3,1)} \\ &= (q^3+qt+q^2t)\,\widetilde{H}_{(3,1)}.\end{aligned}$$

Note that for $\mu \vdash n$, $e_n[B_\mu] = e_{n-1}[B_\mu - 1] = T_\mu$, thus for a symmetric function $F \in \Lambda^{(n)}$,

$$\nabla F = \Delta_{e_n} F = \Delta'_{e_{n-1}} F. \tag{1.4}$$

Furthermore, since $e_k[X+1] = e_k[X] + e_{k-1}[X]$, we have the following relation between the operators Δ and Δ':

$$\Delta_{e_k} = \Delta'_{e_k} + \Delta'_{e_{k-1}}. \tag{1.5}$$

1.1.8 q-analogues and q,t-analogues

We shall list a couple of notations we use when writing polynomials of q or q,t.

For an integer n, we define the q-analogue of n by setting

$$[n]_q := \frac{1-q^n}{1-q},$$

and we define the q,t-analogue of n by setting

$$[n]_{q,t} := \frac{t^n - q^n}{t-q}.$$

When n is a non-negative integer, it follows from the definition that $[n]_q = 1+\cdots+q^{n-1}$ and $[n]_{q,t} = t^n + t^{n-1}q + \cdots + tq^{n-1} + q^n$, which both equal n when setting $t = q = 1$.

Similarly, we define the q-analogue and the q,t-analogue of $n!$ by setting

$$[n]_q! := [1]_q[2]_q \cdots [n]_q \quad \text{and} \quad [n]_{q,t}! := [1]_{q,t}[2]_{q,t} \cdots [n]_{q,t},$$

and we define the q-analogue and the q,t-analogue of binomial coefficients by setting

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q![n-k]_q!} \quad \text{and} \quad \begin{bmatrix} n \\ k \end{bmatrix}_{q,t} := \frac{[n]_{q,t}!}{[k]_{q,t}![n-k]_{q,t}!}.$$

As a remark, our q-analogue and q,t-analogue of integers works for negative numbers. If $n \in \mathbb{Z}_+$, then by definition,

$$[-n]_{q,t} = \frac{t^{-n} - q^{-n}}{t-q} = \frac{-[n]_{q,t}}{(qt)^n}.$$

Further, we also use the notation

$$[n \to m]_{q,t} := \sum_{i=n}^{m} [i]_{q,t} = \frac{\sum_{i=n}^{m} t^i - \sum_{i=n}^{m} q^i}{t-q} = \frac{t^n[m-n+1]_t - q^n[m-n+1]_q}{t-q},$$

or alternatively $\frac{(q-1)(t^{m+1}-t^n)-(t-1)(q^{m+1}-q^n)}{(t-1)(q-1)(t-q)}$.

1.2 Background about parking functions

In this section, we define several combinatorial objects that are related to the Shuffle Theorem, the Rational Shuffle Theorem and the Delta Conjecture. The material related to this section can be found in [Hag08].

1.2.1 Dyck paths and Catalan numbers

Let n be a positive integer. An (n,n)-*Dyck path* P is a lattice path from $(0,0)$ to (n,n) which always remains weakly above the main diagonal $y = x$. The number of Dyck paths of size n is given by the nth Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$. We let $\mathcal{D}_n$ denote the set of Dyck paths of size n. We shall always refer to (n,n)-Dyck paths as *classical Dyck paths* or Dyck paths in the classical case. Figure 1.5 shows a Dyck path of size 7.

For a Dyck path $P \in \mathcal{D}_n$, the cells that are cut through by the main diagonal are called *diagonal cells*, and the cells between the diagonal cells and the path are called *area cells*. We call the main diagonal the 0th diagonal; we call the line that parallel to and above the main diagonal with distance i the ith diagonal.

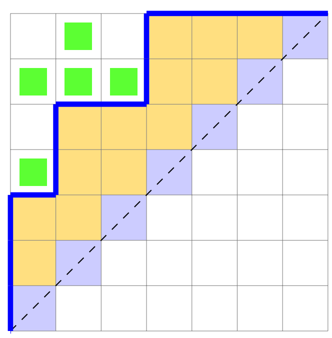
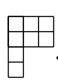


Figure 1.5: A $(7,7)$-Dyck path P.

We let $\mathrm{area}(P)$ be the number of area cells of path P. This is one of the most important statistic of Dyck paths. In the example in Figure 1.5, $\mathrm{area}(P) = 13$.

The collection of cells above a Dyck path P forms a Ferrers diagram (in English notation) of a partition $\lambda(P)$. For each cell $c \in \lambda(P)$, we can count its arm $a(c)$ and leg $\ell(c)$. In Figure 1.5, $\lambda(P) = (3,3,1,1)$ or .

Another important statistic is called *diagonal inversion*, or *dinv*. For an (n,n)-Dyck path

P, $\mathrm{dinv}(P)$ is given by

$$\mathrm{dinv}(P) := \sum_{c \in \lambda(P)} \chi\left(\frac{a(c)}{\ell(c)+1} \leq 1 < \frac{a(c)+1}{\ell(c)}\right).$$

The cells $c \in \lambda(P)$ such that $\frac{a(c)}{\ell(c)+1} \leq 1 < \frac{a(c)+1}{\ell(c)}$ are called *dinv cells*. Equivalently, $\mathrm{dinv}(P)$ is the number of pairs of north steps of P such that they are either on the same diagonal, or they are on consecutive diagonals and the north step on the left is one higher than the one on the right. The example in Figure 1.5 has dinv 5, and the corresponding dinv cells are marked in $\lambda(P)$.

The *bounce* statistic of a Dyck path is defined by the following steps: given a Dyck path P, we draw a *bounce path* that starts from the lattice point $(0,0)$ going north; then it bounces back to the diagonal each time it hits the horizontal boundary of P, and starts going north again from the place that it hits the diagonal until reaching the lattice point (n,n). Figure 1.6 shows a Dyck path and its bounce path. We label the diagonal lattice points between $(0,0)$ and (n,n) by $1,\ldots,n-1$ from top to bottom, and bounce is defined to be the sum of the labels at all hits of the bounce path on the diagonal. In Figure 1.6, $\mathrm{bounce}(P) = 4+2 = 6$.

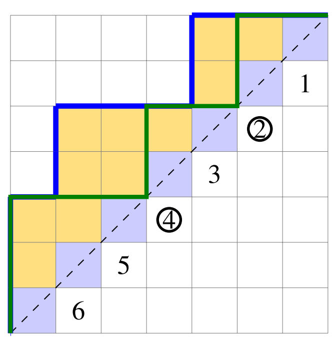


Figure 1.6: A Dyck path and its bounce path.

We shall mention the facts that the distribution of the statistics area, dinv and bounce are equal on $\mathcal{D}_n$; further, the pairs of statistics $(\mathrm{dinv}, \mathrm{area})$ and $(\mathrm{area}, \mathrm{bounce})$ are equi-distributed on $\mathcal{D}_n$, which implies that the following two definitions are well-defined.

The *q-Catalan number* $C_n(q)$ can be defined by

$$C_n(q) := \sum_{P\in\mathcal{D}_n} q^{\mathrm{area}(P)} = \sum_{P\in\mathcal{D}_n} q^{\mathrm{dinv}(P)} = \sum_{P\in\mathcal{D}_n} q^{\mathrm{bounce}(P)}.$$

The *q,t-Catalan number* $C_n(q,t)$ can be defined by

$$C_n(q,t) := \sum_{P\in\mathcal{D}_n} t^{\mathrm{area}(P)} q^{\mathrm{dinv}(P)} = \sum_{P\in\mathcal{D}_n} t^{\mathrm{bounce}(P)} q^{\mathrm{area}(P)}.$$

In fact, the polynomial $C_n(q,t)$ is symmetric in (q,t) for any positive integer n, i.e.

$$C_n(q,t) = C_n(t,q), \quad \forall n \in \mathbb{Z}_+,$$

and the q,t-symmetry of $C_n(q,t)$ has no combinatorial proof.

1.2.2 Parking functions

The original definition of a *parking function* in [KW66] is a sequence $\alpha = (a_1,\dots,a_n) \in \mathbb{Z}_+^n$ such that if $b_1 \le \cdots \le b_n$ is the increasing rearrangement of α, then $b_i \le i$. In this dissertation, we are going to use another equivalent definition of parking functions that is related to Dyck paths.

Given a Dyck path P, we can get an (n,n)*-parking function* PF by labeling the cells east of and adjacent to north steps of P with numbers $\{1,\dots,n\}$ such that the labels (called *cars*) are strictly increasing in each column. The set of parking functions of size n is denoted by $\mathcal{PF}_n$. The cardinality of the set $\mathcal{PF}_n$ is $(n+1)^{n-1}$. Figure 1.7 gives an example of a 5×5 Dyck path and a 5×5 parking function.

We let the *rank* of a car c in cell (x,y) be $\mathrm{rank}(c) := (n+1)y - nx$. We also define the *area* of PF to be the area of the underlying Dyck path, and the *dinv* of PF to be the number of pairs of cars $(i<j)$ such that $rank(i) < rank(j) \le rank(i)+n$.

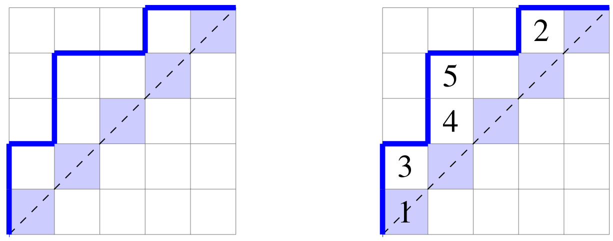
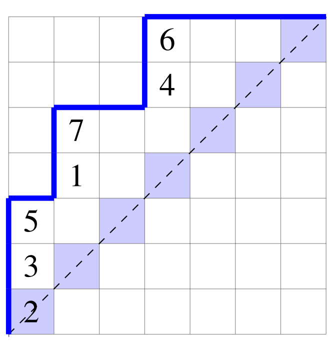


Figure 1.7: A $(5,5)$-Dyck path and a $(5,5)$-parking function.

To be more explicit, for a parking function $\mathrm{PF} \in \mathcal{PF}_n$, let $a_i(\mathrm{PF})$ be the number of full cells between the path and the diagonal in the ith row counting from bottom to top, let ℓ_i denote the car in the ith row, and let

$$d_i(\mathrm{PF}) := \big|\{(i,j) : i < j,\ a_i(\mathrm{PF}) = a_j(\mathrm{PF}) \text{ and } \ell_i < \ell_j\} \\ \cup \{(i,j) : i < j,\ a_i(\mathrm{PF}) = a_j(\mathrm{PF}) + 1 \text{ and } \ell_i > \ell_j\}\big|,$$

then $\mathrm{area}(\mathrm{PF}) := \sum_{i=1}^{n} a_i(\mathrm{PF})$ is the *area* of PF and $\mathrm{dinv}(\mathrm{PF}) := \sum_{i=1}^{n} d_i(\mathrm{PF})$ is the *dinv* of PF. Figure 1.8 gives an example of a $(7,7)$-parking function with area 13 and dinv 2.



i	$a_i(\mathrm{PF})$	$d_i(\mathrm{PF})$
7	3	0
6	2	0
5	3	1
4	2	1
3	2	0
2	1	0
1	0	0

Figure 1.8: A $(7,7)$-parking function with area 13 and dinv 2.

Let

$$P_n(q,t) := \sum_{\mathrm{PF} \in \mathcal{PF}_n} t^{\mathrm{area}(\mathrm{PF})} q^{\mathrm{dinv}(\mathrm{PF})}$$

be a polynomial associated to $\mathcal{PF}_n$ and set $P_0(q,t) := 1$. As a consequence of the Shuffle Theorem of Carlsson and Mellit [CM18], the polynomials $P_n(q,t)$ are symmetric in q,t. If we let

$P_n(q) = P_n(q,1) = P_n(1,q)$, then Kreweras in [Kre80] proved the following recursive formula:

$$P_{n+1}(q) = \sum_{k=0}^{n} \binom{n}{k} [k+1]_q P_k(q) P_{n-k}(q). \tag{1.6}$$

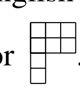
We let the *word* (or *diagonal word*), $\sigma(\text{PF})$, of PF be the permutation obtained by reading the cars in PF from the biggest rank to the smallest rank, and we let the *inverse descent*, $\text{ides}(\text{PF})$ of PF be the descent set of $\sigma(\text{PF})^{-1}$. $\text{ides}(\text{PF})$ is a subset of $[n-1]$, which corresponds to a strong composition of the integer n. We let $\text{pides}(\text{PF})$ be the composition corresponding to $\text{ides}(\text{PF})$. In the parking function in Figure 1.7, the word $\sigma(\text{PF}) = 52431$, $\text{ides}(\text{PF}) = \{1,3,4\}$ and $\text{pides}(\text{PF}) = (1,2,1,1)$. We shall give further details about these definitions when introducing rational parking functions.

1.2.3 Rational Dyck paths and rational Catalan numbers

We shall generalize Dyck paths and Catalan numbers to rational cases.

Let m and n be positive integers. An (m,n)-*Dyck path* is a lattice paths from $(0,0)$ to (m,n) which always remains weakly above the main diagonal $y = \frac{n}{m}x$. The set of (m,n)-Dyck paths is denoted by $\mathcal{D}_{m,n}$. The cells that are cut through by the main diagonal will be called *diagonal cells*. Figure 1.9(a) gives an example of a $(5,7)$-Dyck path, and Figure 1.9(b) gives an example of a $(4,6)$-Dyck path, where the diagonal cells are the light blue cells.

Similar to classical Dyck paths, we have the statistics area and dinv defined on rational Dyck paths. Given an (m,n)-Dyck path $P \in \mathcal{D}_{m,n}$, the cells between the path and its diagonal cells are called *area cells*, and we let $\text{area}(P)$ denote the number of area cells. The paths in Figure 1.9 have 4 and 3 area cells respectively.

We call the cells above the path P *coarea cells*, and we let $\text{coarea}(P)$ denote the number of coarea cells. The coarea cells of P form a partition (in English notation), and we still denote the partition by $\lambda(P)$. In Figure 1.9(a), $\lambda(P) = (3,3,1,1)$ or . Then for each cell $c \in \lambda(P)$,

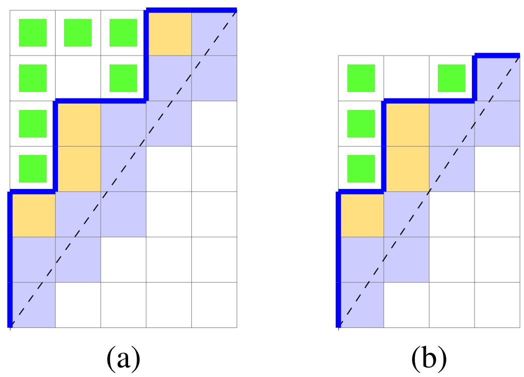


Figure 1.9: A $(5,7)$-Dyck path and a $(4,6)$-Dyck path.

we can count its arm $a(c)$ and leg $\ell(c)$.

In rational Dyck paths, we use the name *path dinv* (or *pdinv*) instead of dinv in order to distinguish several types of dinv statistics. For a path $P \in \mathcal{D}_{m,n}$, its path dinv is given by

$$\operatorname{pdinv}(P) := \sum_{c \in \lambda(P)} \chi\left(\frac{a(c)}{\ell(c)+1} \leq \frac{m}{n} < \frac{a(c)+1}{\ell(c)}\right).$$

The rational Catalan number $C_{m,n}$ is the number of rational Dyck paths in $\mathcal{D}_{m,n}$, and it is not difficult to prove the formula that $C_{m,n} = \frac{1}{m+n}\binom{m+n}{m,n}$, which is symmetric in m,n.

Similar to the classical case, there are q-analogues and q,t-analogues of rational Catalan numbers. It is a fact but not obvious that the statistics area and pdinv are equi-distributed in $\mathcal{D}_{m,n}$, and the *q-rational Catalan number* $C_{m,n}(q)$ can be defined by

$$C_{m,n}(q) := \sum_{P \in \mathcal{D}_{m,n}} q^{\operatorname{area}(P)} = \sum_{P \in \mathcal{D}_{m,n}} q^{\operatorname{pdinv}(P)}.$$

Further, the *q,t-rational Catalan number* $C_{m,n}(q,t)$ can be defined by

$$C_{m,n}(q,t) := \sum_{P \in \mathcal{D}_{m,n}} t^{\operatorname{area}(P)} q^{\operatorname{pdinv}(P)},$$

and $C_{m,n}(q,t)$ is symmetric in q,t as a consequence of the Shuffle Theorem of Carlsson and Mellit

[CM18].

1.2.4 Rational parking functions

Similar to the classical case, an (m,n)-*parking function* PF is obtained by labeling the cells east of and adjacent to north steps of an (m,n)-Dyck path with the integers $1,\ldots,n$ in such a way that the numbers increase in each column as we read from bottom to top. We will refer to these labels as *cars*. The underlying Dyck path is denoted by $\Pi(\mathrm{PF})$. The partition formed by the collection of cells above the path $\Pi(\mathrm{PF})$ is denoted by $\lambda(\mathrm{PF})$ (i.e. $\lambda(\mathrm{PF}) = \lambda(\Pi(\mathrm{PF}))$). The set of (m,n)-parking functions is denoted by $\mathcal{PF}_{m,n}$. Figure 1.10(a) pictures a $(5,7)$-parking function based on the $(5,7)$-Dyck path pictured in Figure 1.9(a).

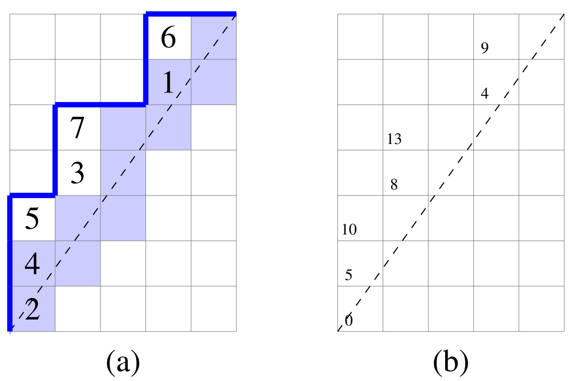


Figure 1.10: A $(5,7)$-parking function and its car ranks.

Next we define statistics ides(PF) and pides(PF) for any rational parking function PF. For any pair of coprime positive integers m and n, we define the *rank* of a cell (x,y) in the (m,n)-grid to be $rank(x,y) := my - nx$. If m and n are not coprime, we shall generalize the rank to be $rank(x,y) := my - nx + \lfloor \frac{x \cdot \gcd(m,n)}{m} \rfloor$. Figure 1.10(b) shows the ranks of the cars in Figure 1.10(a). $\sigma(\mathrm{PF})$, the *word* (or *diagonal word*) of PF, is obtained by reading cars from highest to lowest ranks. In our example in Figure 1.10(a), $\sigma(\mathrm{PF}) = 7563412$.

We define ides(PF) to be the descent set of $\sigma(\text{PF})^{-1}$. In other words,

$$\begin{aligned}\text{ides(PF)} &:= \{i \in \sigma(\text{PF}) : i+1 \text{ is to the left of } i \text{ in } \sigma(\text{PF})\} \\ &= \{i : \text{rank}(i) < \text{rank}(i+1)\}.\end{aligned}$$

Then we define pides(PF) to be the composition set of ides(PF). If ides(PF) $= \{i_1 < i_2 < \cdots < i_d\}$, then

$$\text{pides(PF)} := \{i_1, i_2 - i_1, \ldots, n - i_d\}.$$

In Figure 1.10(a), we have ides(PF) = ides(7563412) = $\{2,4,6\}$, and pides(PF) = $\{2,2,2,1\}$.

We have two remarks about the statistics word, ides and pides. If $i < j$ and rank$(i) >$ rank(j), then i and j cannot be in the same column, otherwise j lies on top of i, which lead to a contradiction that rank$(i) >$ rank(j). Thus,

Remark 1.1. *Let $i < j$ be two cars in the parking function* PF. *If i is to the left of j in σ*(PF), *then the cars i, j must be in different columns.*

If $M \in$ pides(PF) and $M > m$, then there exist M cars $k, k+1, \ldots, k+M-1$ with decreasing ranks. By Remark 1.1, the M cars are in different columns, which is impossible, thus the assumption $M \in$ pides(PF) is not true, and we have the following remark.

Remark 1.2. *The parts in the composition set* pides(PF) *of a parking function* PF $\in \mathcal{PF}_{m,n}$ *are less than or equal to m.*

In many papers, the statistic dinv of a parking function is defined by 3 components – path dinv (pdinv), max dinv (maxdinv) and temporary dinv (tdinv).

For a parking function PF $\in \mathcal{PF}_{m,n}$, the *path dinv* of PF is the path dinv of the underlying Dyck path, i.e.

$$\text{pdinv(PF)} := \text{pdinv}(\Pi(\text{PF})).$$

Then, the *temporary dinv* is defined by

$$\text{tdinv(PF)} := \sum_{\text{cars } i<j \in \text{PF}} \chi(\text{rank}(i) < \text{rank}(j) < \text{rank}(i)+m).$$

In Figure 1.10(a), $\text{tdinv(PF)} = 7$ since the pairs of cars contributing to tdinv are $(1,3)$, $(1,4)$, $(3,5)$, $(3,6)$, $(4,6)$, $(5,7)$ and $(6,7)$.

Then, the statistic *max dinv* is defined as the maximum of temporary dinvs of parking functions on the same path. Since max dinv is independent of the cars, it is also a statistic of rational Dyck paths like pdinv. For a parking function PF, we have

$$\text{maxdinv(PF)} := \max\{\text{tdinv(PF}') : \Pi(\text{PF}') = \Pi(\text{PF})\};$$

for a rational Dyck path P, we write

$$\text{maxdinv}(P) := \max\{\text{tdinv(PF)} : \Pi(\text{PF}) = P\}.$$

Finally, the statistic *dinv* is defined by setting

$$\text{dinv(PF)} := \text{tdinv(PF)} + \text{pdinv(PF)} - \text{maxdinv(PF)}.$$

We shall use this definition of dinv in some combinatorial proofs in Chapter 2.

Notice that the statistics pdinv and maxdinv of a parking function PF are determined by the underlying Dyck path Π(PF), thus the component (pdinv(PF) $-$ maxdinv(PF)) in the definition of dinv(PF) combines to a statistic of rational Dyck paths. Leven and Hicks in [HL15] gave a simplified formula for dinv(PF) by defining the statistic *dinvcorr* that satisfies $\text{dinvcorr}(\Pi(\text{PF})) = \text{pdinv(PF)} - \text{maxdinv(PF)}$.

To be more explicit, let P be an (m,n)-Dyck path and set $\frac{0}{0} = 0$ and $\frac{x}{0} = \infty$ for all $x \neq 0$, then we define

$$\text{dinvcorr}(P) \;:=\; \sum_{c\in\lambda(P)} \chi\left(\frac{a(c)+1}{\ell(c)+1} \le \frac{m}{n} < \frac{a(c)}{\ell(c)}\right) \;-\; \sum_{c\in\lambda(P)} \chi\left(\frac{a(c)}{\ell(c)} \le \frac{m}{n} < \frac{a(c)+1}{\ell(c)+1}\right).$$

An alternative definition of *dinv* is

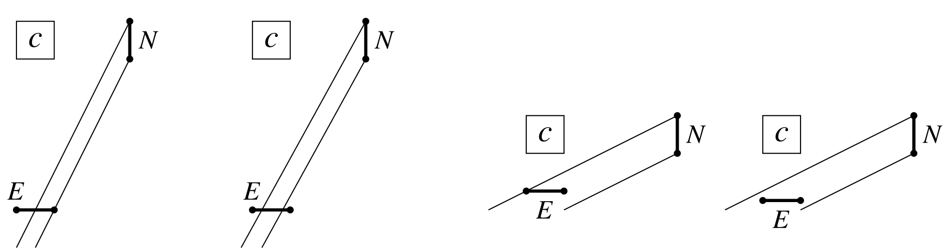
$$\text{dinv}(\text{PF}) := \text{tdinv}(\text{PF}) + \text{dinvcorr}(\Pi(\text{PF})).$$

Note that as a statistic of rational Dyck paths, dinvcorr only depends on the path P, and it is the difference of two sums $\sum_{c\in\lambda(P)} \chi\left(\frac{a(c)+1}{\ell(c)+1} \le \frac{m}{n} < \frac{a(c)}{\ell(c)}\right)$ and $\sum_{c\in\lambda(P)} \chi\left(\frac{a(c)}{\ell(c)} \le \frac{m}{n} < \frac{a(c)+1}{\ell(c)+1}\right)$, of which at most one is nonzero.

If $m = n$, then there is no dinvcorr. When $m \neq n$, we count dinvcorr by checking the cells c in the partition $\lambda(P)$. Given a cell $c \in \lambda(P)$, we high-light the vertical line segment N which is a north step of the path P to the east of c, and the horizontal line segment E which is a east step of P to the south of c. We draw two lines with slope $\frac{n}{m}$ from the north end and south end of N.

(1) If $n > m$, the cells of type (a) and (b) in Figure 1.11 contribute -1 to dinvcorr.

(2) If $m > n$, the cells of type (c) and (d) in Figure 1.11 contribute 1 to dinvcorr.



(a) (b) (c) (d)

Figure 1.11: Types of cells that contribute to dinvcorr.

Following Hikita [Hik14], we define *Hikita polynomials* $\mathrm{H}_{m,n}[X;q,t]$ where m and n are coprime by setting

$$\mathrm{H}_{m,n}[X;q,t] := \sum_{\mathrm{PF}\in\mathcal{PF}_{m,n}} t^{\mathrm{area(PF)}} q^{\mathrm{dinv(PF)}} F_{\mathrm{ides(PF)}}[X]. \tag{1.7}$$

For non-coprime case, we formulate Hikita polynomials as follows. Given m,n coprime and $k \geq 1$, we defined the *return*, ret(PF), of a (km,kn)-parking function PF to be the smallest positive integer i such that the supporting path of PF goes through the point (im,in). Then following the formulation of Garsia, Leven, Wallach, and Xin in [GLWX17], we define the *extended Hikita polynomial* to be

$$\mathrm{H}_{km,kn}[X;q,t] := \sum_{\mathrm{PF}\in\mathcal{PF}_{km,kn}} [\mathrm{ret(PF)}]_{\frac{1}{t}} t^{\mathrm{area(PF)}} q^{\mathrm{dinv(PF)}} F_{\mathrm{ides(PF)}}[X]. \tag{1.8}$$

The (extended) Hikita polynomials are proved to be symmetric functions in X in [Hik14].

Here we shall give a remark about Hikita polynomials in the Fuss-Catalan case. The number of (kn,n)-parking functions is equal to the number of $(kn+1,n)$-parking functions. For any (kn,n)-parking function PF, we let PF' be the $(kn+1,n)$-parking function obtained by adding a east step at the right end of the path $\Pi(\mathrm{PF})$. Then it follows immediately from the definition of the statistics that $\mathrm{area(PF)} = \mathrm{area(PF')}$, $\mathrm{dinv(PF)} = \mathrm{dinv(PF')}$ and $\mathrm{ides(PF)} = \mathrm{ides(PF')}$. Thus,

$$\mathrm{H}_{kn+1,n}[X;q,t] = \sum_{\mathrm{PF}\in\mathcal{PF}_{kn+1,n}} t^{\mathrm{area(PF)}} q^{\mathrm{dinv(PF)}} F_{\mathrm{ides(PF)}}[X] = \sum_{\mathrm{PF}\in\mathcal{PF}_{kn,n}} t^{\mathrm{area(PF)}} q^{\mathrm{dinv(PF)}} F_{\mathrm{ides(PF)}}[X]. \tag{1.9}$$

In particular, we have the following formula in the classical case:

$$\mathrm{H}_{n+1,n}[X;q,t] = \sum_{\mathrm{PF}\in\mathcal{PF}_{n+1,n}} t^{\mathrm{area(PF)}} q^{\mathrm{dinv(PF)}} F_{\mathrm{ides(PF)}}[X] = \sum_{\mathrm{PF}\in\mathcal{PF}_{n}} t^{\mathrm{area(PF)}} q^{\mathrm{dinv(PF)}} F_{\mathrm{ides(PF)}}[X]. \tag{1.10}$$

1.2.5 An alternative expression for Hikita polynomials

As we have mentioned, Hikita polynomials $\mathrm{H}_{m,n}[X;q,t]$ are symmetric (in X) for any pair of integers m,n. We shall introduce an alternative expression for Hikita polynomials from this fact.

Suppose that $\alpha = (\alpha_1,\ldots,\alpha_k)$ is a strong composition of n into k parts ($k \le n$), then we set $\alpha_j = 0$ for $j > k$. We let $X = \{x_1,\ldots,x_n\}$ be the set of n variables and

$$\Delta_\alpha[X] := \det||x_i^{\alpha_j+n-j}|| = \sum_{\sigma\in S_n} sgn(\sigma)\sigma(x_1^{\alpha_1+n-1}\cdots x_n^{\alpha_n+n-n}).$$

We let $\Delta[X] := \det||x_i^{n-j}||$ be the Vandermonde determinant. Then the Schur symmetric function $s_\alpha[X]$ associated to α can be defined by

$$s_\alpha[X] := \frac{\Delta_\alpha[X]}{\Delta[X]}. \tag{1.11}$$

It is well-known that for any such strong composition α, either we have $s_\alpha[X] = 0$ or there is a partition $\lambda \vdash n$ such that $s_\alpha[X] = \pm s_\lambda[X]$. In fact, there is a *straightening* relation which allows us to prove that fact. Namely, if $\alpha_{i+1} > \alpha_i$, then

$$s_{(\alpha_1,\ldots,\alpha_i,\alpha_{i+1},\ldots,\alpha_k)}[X] = -s_{(\alpha_1,\ldots,\alpha_{i+1}-1,\alpha_i+1,\ldots,\alpha_k)}[X]. \tag{1.12}$$

In a remarkable and important paper, Egge, Loehr and Warrington [ELW10] gave a combinatorial description of how to start with the a quasi-symmetric function expansion of a homogeneous symmetric function $P[X]$ of degree n,

$$P[X] = \sum_{\alpha\vDash_{\text{strong}} n} a_\alpha F_\alpha[X],$$

and transform it into an expansion in terms of Schur functions,

$$P[X] = \sum_{\lambda \vdash n} b_\lambda s_\lambda[X].$$

The following theorem due to Garsia and Remmel [GR18] is implicit in the work of [ELW10], but is not explicitly stated and it allows one to find the Schur function expansion by using the straightening laws.

Theorem 1.1 (Garsia-Remmel)**.** *Suppose that $P[X]$ is a symmetric function which is homogeneous of degree n and*

$$P[X] = \sum_{\alpha \vDash_{\text{strong}} n} a_\alpha F_\alpha[X]. \tag{1.13}$$

Then

$$P[X] = \sum_{\alpha \vDash_{\text{strong}} n} a_\alpha s_\alpha[X]. \tag{1.14}$$

Recall that $\text{pides}(\sigma)$ is the composition set of $\text{ides}(\sigma)$, then Theorem 1.1 and the straightening action allow us to transform $\mathrm{H}_{m,n}[X;q,t]$ into Schur function expansion that

$$\begin{aligned} \mathrm{H}_{m,n}[X;q,t] &= \sum_{\mathrm{PF} \in \mathcal{PF}_{m,n}} [\text{ret}(\mathrm{PF})]_{\frac{1}{t}} t^{\text{area}(\mathrm{PF})} q^{\text{dinv}(\mathrm{PF})} F_{\text{ides}(\mathrm{PF})} \\ &= \sum_{\mathrm{PF} \in \mathcal{PF}_{m,n}} [\text{ret}(\mathrm{PF})]_{\frac{1}{t}} t^{\text{area}(\mathrm{PF})} q^{\text{dinv}(\mathrm{PF})} s_{\text{pides}(\mathrm{PF})}. \end{aligned} \tag{1.15}$$

From Chapter 2, we shall use the expression (1.15) for $\mathrm{H}_{m,n}[X;q,t]$ to prove several facts about the coefficients in the Schur function expansions of the Rational Shuffle Theorem.

1.3 The Shuffle Theorem, the Rational Shuffle Theorem and the Delta Conjecture

1.3.1 The ring of diagonal harmonics and the Shuffle Theorem

The *Shuffle Theorem* comes from the study of the ring of diagonal harmonics. Let $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ be two sets of n variables. The *ring of diagonal harmonics* consists of those polynomials in $\mathbb{Q}[X,Y]$ which satisfy the following system of differential equations

$$\partial_{x_1}^a \partial_{y_1}^b f(X,Y) + \partial_{x_2}^a \partial_{y_2}^b f(X,Y) + \ldots + \partial_{x_n}^a \partial_{y_n}^b f(X,Y) = 0$$

for each pair of integers a and b such that $a+b>0$. Haiman in [Hai94] proved that the ring of diagonal harmonics has dimension $(n+1)^{n-1}$.

Further, Haiman [Hai94] proved that the *bigraded Frobenius characteristic* of the $\mathcal{S}_n$-module of diagonal harmonics, $DH_n(X;q,t)$, is given by

$$DH_n(X;q,t) = \nabla e_n. \tag{1.16}$$

The *Classical Shuffle Conjecture* proposed by Haglund, Haiman, Loehr, Remmel and Ulyanov [HHL$^+$05b] gives a well-studied combinatorial expression for the bigraded Frobenius characteristic of the ring of diagonal harmonics. The Shuffle Conjecture has been proved by Carlsson and Mellit [CM18] as the *Shuffle Theorem* as follows.

Theorem 1.2 (Carlsson and Mellit)**.** *For any integer $n \geq 0$,*

$$\nabla e_n = \mathrm{H}_{n+1,n}[X;q,t]. \tag{1.17}$$

By the identities (1.10) and (1.15), we also write the Shuffle Theorem as

$$\nabla e_n = \sum_{\mathrm{PF}\in\mathcal{PF}_n} t^{\mathrm{area(PF)}} q^{\mathrm{dinv(PF)}} s_{\mathrm{pides(PF)}}[X], \tag{1.18}$$

which is saying that the Frobenius characteristic of diagonal harmonics can be written as a generating function of some combinatorial objects (parking functions in $\mathcal{PF}_n$). We shall refer to the left hand side of the Shuffle Theorem as the *symmetric function side* or the *algebraic side*, and refer to the right hand side as the *parking function side* or the *combinatorial side*.

1.3.2 The Rational Shuffle Theorem

The Rational Shuffle Theorem is a rational generalization of the Shuffle Theorem. In the algebraic side of the conjecture, Gorsky and Negut [GN15] introduced the symmetric function operator $\mathrm{Q}_{m,n}$ and extended the algebraic side of the Shuffle Theorem from ∇e_n to $\mathrm{Q}_{m,n}$ applied to $(-1)^n$.

As shown in [BGLX15], the $\mathrm{Q}_{m,n}$ operators of the Gorsky-Negut conjecture can be defined in terms of the operators D_k which were introduced by Bergeron and Garsia in [BG99]. In plethystic notation, the action of D_k on a symmetric function $F[X]$ is defined as

$$D_k\, F[X] := F\left[X+\frac{M}{z}\right] \sum_{i\geq 0} (-z)^i e_i[X] \bigg|_{z^k}, \tag{1.19}$$

where $M=(1-t)(1-q)$.

Then one can construct a family of symmetric function operators $\mathrm{Q}_{m,n}$ for any pair of coprime positive integers (m,n) as follows. First for any $n \geq 0$, set $\mathrm{Q}_{1,n} = D_n$. Next, one can recursively define $\mathrm{Q}_{m,n}$ for $m>1$ as follows. Consider the $m\times n$ lattice with diagonal $y=\frac{n}{m}x$. Let (a,b) be the lattice point which is closest to and below the diagonal. Set $(c,d)=(m-a,n-b)$.

In such a case, we will write

$$\mathrm{Split}(m,n) = (a,b) + (c,d). \tag{1.20}$$

Note that the pairs (a,b) and (c,d) are coprime since any point of the form (kx,ky) is further from the diagonal than the point (x,y). Then we have the following recursive definition of the $\mathrm{Q}_{m,n}$ operators:

$$\mathrm{Q}_{m,n} = \frac{1}{M}[\mathrm{Q}_{c,d}, \mathrm{Q}_{a,b}] = \frac{1}{M}(\mathrm{Q}_{c,d}\mathrm{Q}_{a,b} - \mathrm{Q}_{a,b}\mathrm{Q}_{c,d}). \tag{1.21}$$

Figure 1.12 gives an example of $\mathrm{Split}(3,5) = (2,3) + (1,2)$, so that

$$\mathrm{Q}_{3,5} = \frac{1}{M}[\mathrm{Q}_{1,2}, \mathrm{Q}_{2,3}] = \frac{1}{M}[D_2, \mathrm{Q}_{2,3}]. \tag{1.22}$$

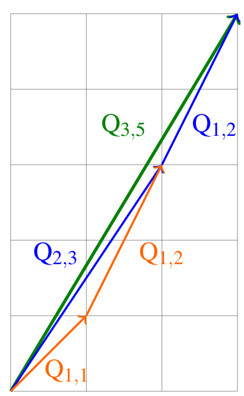


Figure 1.12: The geometry of $\mathrm{Split}(3,5)$.

The same procedure gives $\mathrm{Q}_{2,3} = \frac{1}{M}[\mathrm{Q}_{1,2}, \mathrm{Q}_{1,1}] = \frac{1}{M}[D_2, D_1]$. Therefore,

$$\mathrm{Q}_{3,5} = \frac{1}{M^2}[D_2, [D_2, D_1]] = \frac{1}{M^2}(D_2D_2D_1 - 2D_2D_1D_2 + D_1D_2D_2). \tag{1.23}$$

For the non-coprime case, we can define the $\mathrm{Q}_{km,kn}$ operator as follows. We choose one of the lattice points, (a,b), in the $km \times kn$ lattice that are strictly below and closest to the diagonal, then we set

$$\mathrm{Q}_{km,kn} = \frac{1}{M}[\mathrm{Q}_{km-a,kn-b}, \mathrm{Q}_{a,b}]. \tag{1.24}$$

This recursive definition is well-defined as it is proved in [BGLX15] that any choice of such point (a,b) defines the same operation.

In the combinatorial side, the generalization is straightforward from the generating function $\mathrm{H}_{n+1,n}[X;q,t]$ of classical parking functions to the generating function $\mathrm{H}_{m,n}[X;q,t]$ of rational parking functions (though the definition of dinv in the rational case is relatively complicated).

The *Extended Rational Shuffle Theorem* conjectured by Garsia, Leven, Wallach and Xin [GLWX17] and Gorsky and Negut [GN15] is the following:

Theorem 1.3 (Mellit-Garsia-Leven-Wallach-Xin)**.** *For all pairs of coprime positive integers (m,n) and any $k \in \mathbb{Z}^+$, we have*

$$\mathrm{Q}_{km,kn}(-1)^{kn} = \mathrm{H}_{km,kn}[X;q,t], \tag{1.25}$$

which was proved by Mellit [Mel16] and Garsia, Leven, Wallach and Xin [GLWX17].

The original *Rational Shuffle Theorem* proposed by Gorsky and Negut [GN15] and proved by Mellit [Mel16] in the case where m and n are relatively prime is the special case when $k = 1$ in Theorem 1.3. In this case, ret(PF) is always 1 and the theorem can be described as follows.

Theorem 1.4 (Mellit)**.** *For all pairs of coprime positive integers (m,n), we have*

$$\mathrm{Q}_{m,n}(-1)^{n} = \mathrm{H}_{m,n}[X;q,t]. \tag{1.26}$$

1.3.3 The Delta Conjecture

The *Delta Conjecture* of Haglund, Remmel and Wilson [HRW18] is another well studied extension of the Shuffle Theorem. The Delta Conjecture has two versions, the *rise version* and the *valley version*.

In order to introduce the Delta Conjecture, we need to define some more combinatorial terminology about parking functions. For a parking function $\mathrm{PF} \in \mathcal{PF}_n$, recall that we use $a_i(\mathrm{PF})$ and $d_i(\mathrm{PF}) := \left|\{(i,j)|i<j,\ a_i = a_j \text{ and } \ell_i < \ell_j\} \cup \{(i,j)|i<j,\ a_i = a_j+1 \text{ and } \ell_i > \ell_j\}\right|$ to denote the area and dinv component in the ith row counting from bottom to top (see Figure 1.8). We also let $\ell_i(\mathrm{PF})$ be the car in the ith row. We define

$$\begin{aligned}
\mathrm{valley}(\mathrm{PF}) &:= \{i : a_i(\mathrm{PF}) \leq a_{i-1}(\mathrm{PF})\}, \\
\mathrm{Rise}(\mathrm{PF}) &:= \{i : a_i(\mathrm{PF}) = a_{i-1}(\mathrm{PF})+1\}, \quad \text{and} \\
\mathrm{Val}(\mathrm{PF}) &:= \{i : a_i(\mathrm{PF}) < a_{i-1}(\mathrm{PF}) \text{ or } a_i(\mathrm{PF}) = a_{i-1}(\mathrm{PF}) \text{ and } \ell_i(\mathrm{PF}) > \ell_{i-1}(\mathrm{PF})\}
\end{aligned}$$

to be the sets of *valleys*, *double rises* and *contractible valleys* of PF.

We let

$$\mathrm{Rise}_{n,k}[X;q,t] := \sum_{\mathrm{PF}\in\mathcal{PF}_n} t^{\mathrm{area}(\mathrm{PF})} q^{\mathrm{dinv}(\mathrm{PF})} F_{\mathrm{ides}(\mathrm{PF})} \prod_{i\in\mathrm{Rise}(\mathrm{PF})} (1+\frac{z}{t^{a_i(\mathrm{PF})}})\Big|_{z^{n-k-1}}$$

and

$$\mathrm{Val}_{n,k}[X;q,t] := \sum_{\mathrm{PF}\in\mathcal{PF}_n} t^{\mathrm{area}(\mathrm{PF})} q^{\mathrm{dinv}(\mathrm{PF})} F_{\mathrm{ides}(\mathrm{PF})} \prod_{i\in\mathrm{Val}(\mathrm{PF})} (1+\frac{z}{q^{d_i(\mathrm{PF})+1}})\Big|_{z^{n-k-1}}$$

be two generating functions of $\mathcal{PF}_n$, then the rise and the valley version of the Delta Conjecture are the following.

Conjecture 1.1 (Haglund, Remmel and Wilson)**.** *For any integers $n > k \geq 0$,*

$$\Delta'_{e_k} e_n = \mathrm{Rise}_{n,k}[X;q,t] = \mathrm{Val}_{n,k}[X;q,t].$$

The Delta Conjecture is still open, but a lot of cases of the Delta Conjecture have been proved. The conjecture for $\Delta_{e_1} e_n$ is proved by Haglund, Remmel and Wilson [HRW18]; the rise version Delta Conjecture at $q = 1$ is proved by Romero [Rom17]; the Catalan case of the conjecture is proved by Zabrocki [Zab16]. The Delta Conjecture at the case when t or q equals 0 is proved by Garsia, Haglund, Remmel and Yoo [GHRY17]; Wilson [Wil16]; Rhoades [Rho18]; Haglund, Rhoades and Shimozono [HRS18].

Chapter 2

Schur function expansions of the Rational Shuffle Theorem

It is an important combinatorial question to find the Schur function expansion of ∇e_n since that would allow us to find the bigraded $\mathcal{S}_n$-isomorphism type of the ring of diagonal harmonics, see [Hai94]. More generally, we would like to find a combinatorial interpretation of the coefficients that arise in the Schur function expansion of $\mathrm{Q}_{m,n}(-1)^n$.

In this chapter, we study the combinatorics of the Schur function expansion of $\mathrm{Q}_{m,n}(-1)^n$. We explore several symmetries in the combinatorics of the coefficients that arise in the Schur function expansion of $\mathrm{Q}_{m,n}(-1)^n$. In particular, we study the hook-shaped Schur function coefficients, and the Schur function expansion of $\mathrm{Q}_{m,n}(-1)^n$ in the case where m or n equals 3.

2.1 Introduction

Mellit [Mel16] and Garsia, Leven, Wallach and Xin [GLWX17] in 2016 proved the *Extended Rational Shuffle Theorem* that for any pair of positive integers (m,n),

$$\mathrm{Q}_{m,n}(-1)^n = \mathrm{H}_{m,n}[X;q,t]. \tag{2.1}$$

Thus, we can find the Schur function expansion in one of two ways. That is, we can use the properties of the operator $\mathrm{Q}_{m,n}$ to find the Schur function expansion of $\mathrm{Q}_{m,n}(-1)^n$ which we will refer to as working on the *symmetric function side* of the Rational Shuffle Theorem. Second, one could start with the Hikita polynomial $\mathrm{H}_{m,n}[X;q,t]$ and expand that polynomial into Schur functions which we will call working on the *combinatorial side* of the Rational Shuffle Theorem. We let $[s_\lambda]_{m,n}$ be the coefficient of the Schur function s_λ in both polynomials $\mathrm{Q}_{m,n}(-1)^n$ and $\mathrm{H}_{m,n}[X;q,t]$.

The Schur function expansion of $\mathrm{Q}_{m,n}(-1)^n$ in the case where m and n are coprime and either m or n equals 2 was given by Leven [Lev14] summarized in the following theorem.

Theorem 2.1 (Leven). *For any integer $k \geq 0$,*

$$\mathrm{Q}_{2k+1,2}(1) = \mathrm{H}_{2k+1,2}[X;q,t] = [k]_{q,t}s_2 + [k+1]_{q,t}s_{1,1} \tag{2.2}$$

and

$$\mathrm{Q}_{2,2k+1}(-1) = \mathrm{H}_{2,2k+1}[X;q,t] = \sum_{r=0}^{k} [k+1-r]_{q,t}s_{2^r1^{2k+1-2r}}. \tag{2.3}$$

By the combinatorial side of the Extended Rational Shuffle Theorem formulated in [BGLX15], we can extend Leven's theorem to compute the Schur function expansion of $\mathrm{Q}_{m,n}(-1)^n$ where either m or n is equal to 2, but m and n are not coprime. That is, we can give a combinatorial proof of the following.

Theorem 2.2. *For any integer $k \geq 0$,*

$$\mathrm{Q}_{2k,2}(1) = \mathrm{H}_{2k,2}[X;q,t] = ([k]_{q,t} + [k-1]_{q,t})s_2 + ([k+1]_{q,t} + [k]_{q,t})s_{1,1} \tag{2.4}$$

and

$$\mathrm{Q}_{2,2k}(1) = \mathrm{H}_{2,2k}[X;q,t] = \sum_{r=0}^{k} ([k+1-r]_{q,t} + [k-r]_{q,t})s_{2^r 1^{2k+1-2r}}. \tag{2.5}$$

As we have introduced in Chapter 1, the coefficient at s_{1^n} in $\mathrm{Q}_{m,n}(-1)^n$ is known as the rational q,t-Catalan number, computed by Gorsky and Mazin [GM14] for the case $n = 3$ and studied by Lee, Li and Loehr [LLL14] for the case $n = 4$. The coefficients at the hook-shaped Schur functions were discussed by Armstrong, Loehr and Warrington [ALW16].

In this chapter, we explore the combinatorics of the Schur function expansion of $\mathrm{Q}_{m,n}(-1)^n$ in several special cases, and the organization of this chapter is as follows.

In Section 2.2, we prove a number of symmetries of the coefficients of Schur functions. We can combinatorially prove

Theorem 2.3. *For all $m,n > 0$ and $\lambda' \vdash (n-am)$,*

(a) $[s_{1^n}]_{m,n} = [s_n]_{m+n,n}$,

(b) $[s_{m^a\lambda'}]_{m,n} = [s_{\lambda'}]_{m,n-am}$,

(c) $[s_{k1^{n-k}}]_{m,n} = [s_{k1^{m-k}}]_{n,m}$.

In Section 2.3, we prove the following theorem to give an explicit formula for the Schur function expansion of $\mathrm{Q}_{m,3}(-1)$ from both symmetric function side and combinatorial side.

Theorem 2.4. *For any integer $k \geq 0$,*

$$\mathrm{Q}_{3k+1,3}(-1) = \mathrm{H}_{3k+1,3}[X;q,t] = \left(\sum_{i=0}^{k-1}(qt)^{k-1-i}[3i+1]_{q,t}\right)s_3 + \left(\sum_{i=0}^{k-1}(qt)^{k-1-i}([3i+2]_{q,t}+[3i+3]_{q,t})\right)s_{2,1} + \left(\sum_{i=0}^{k}(qt)^{k-i}[3i+1]_{q,t}\right)s_{1^3}, \quad (2.6)$$

$$\mathrm{Q}_{3k+2,3}(-1) = \mathrm{H}_{3k+2,3}[X;q,t] = \left(\sum_{i=0}^{k-1}(qt)^{k-1-i}[3i+2]_{q,t}\right)s_3 + \left(\sum_{i=-1}^{k-1}(qt)^{k-1-i}([3i+3]_{q,t}+[3i+4]_{q,t})\right)s_{2,1} + \left(\sum_{i=0}^{k}(qt)^{k-i}[3i+2]_{q,t}\right)s_{1^3}, \quad (2.7)$$

$$\mathrm{Q}_{3k,3}(-1) = \mathrm{H}_{3k,3}[X;q,t] = \left(\sum_{i=0}^{k-1}(qt)^{k-1-i}([3i-1]_{q,t}+[3i]_{q,t}+[3i+1]_{q,t})\right)s_3 + \Bigg((qt)^{k+1}([3]_{q,t}+2[2]_{q,t}+[1]_{q,t}) + \sum_{i=1}^{k-1}(qt)^{k-1-i}([3i]_{q,t}+2[3i+1]_{q,t}+2[3i+2]_{q,t} + [3i+3]_{q,t})\Bigg)s_{2,1} + \left(\sum_{i=0}^{k}(qt)^{k-i}([3i-1]_{q,t}+[3i]_{q,t}+[3i+1]_{q,t})\right)s_{1^3}. \quad (2.8)$$

Note that this independently proves the Rational Shuffle Theorem and the Shuffle Theorem in the case when $n \leq 3$.

In Section 2.4, we study several Schur function coefficient formulas and symmetries in $\mathrm{Q}_{3,n}(-1)^n$ (some of which are consequences of Theorem 2.3), and conjecture a concise recursive formula for Schur function coefficients $[s_\lambda]_{3,n}$ generally for any $\lambda \vdash n$. In particular, we study a new symmetry that

$$[s_{2^a 1^b}]_{3,n} = [s_{2^b 1^a}]_{3,3(a+b)-n}, \quad (2.9)$$

and a combinatorial action on parking functions called the switch map $\mathbb{S}$.

2.2 Combinatorial results about Schur function expansions of the (m,n) case

We shall work on the combinatorial side by studying the Hikita polynomials in this section. By Equation (1.15) in Section 1.2.5, we use the alternative expression for Hikita polynomials

$$\mathrm{H}_{m,n}[X;q,t] = \sum_{\mathrm{PF}\in\mathcal{PF}_{m,n}} [\mathrm{ret(PF)}]_{\frac{1}{t}} t^{\mathrm{area(PF)}} q^{\mathrm{dinv(PF)}} s_{\mathrm{pides(PF)}}.$$

In the rational (m,n) case, we have n cars, i.e. the word of an (m,n)-parking function is a permutation of $[n]=\{1,\ldots,n\}$. Recall that $[s_\lambda]_{m,n}$ is the coefficient of s_λ in $\mathrm{H}_{m,n}[X;q,t]$. By Remark 1.2 in Section 1.2.4, $[s_\lambda]_{m,n}\neq 0$ implies that λ must be of the form $m^{\alpha_m}\cdots 1^{\alpha_1}$ with $\sum_{i=1}^m i\alpha_i = n$, i.e. $[s_\lambda]_{m,n}\neq 0$ only if the partition λ only has parts of size less than or equal to m. In this section, we shall prove the 3 symmetries about $[s_\lambda]_{m,n}$ described in Theorem 2.3, stated as the following three results.

Result 1. $[s_{1^n}]_{m,n} = [s_n]_{m+n,n}$.

Note that a parking function with pides n must have word $12\cdots n$, and a parking function with pides 1^n must have word $n\cdots 21$.

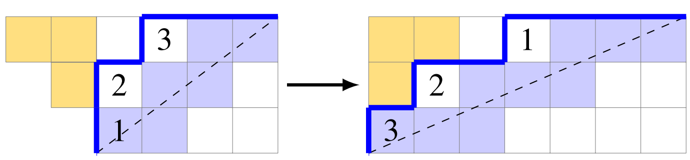


Figure 2.1: Bijection between $\mathcal{PF}_{m,3}$ with word 123 and $\mathcal{PF}_{m+3,3}$ with word 321.

A parking function in $\mathcal{PF}_{m,n}$ with word $n\cdots 21$ corresponds to a unique (m,n)-Dyck path, and a parking function in $\mathcal{PF}_{m+n,n}$ with word $12\cdots n$ corresponds to an $(m+n,n)$-Dyck path with no consecutive north steps. As shown is Figure 2.1, we can obtain a parking function $\mathcal{PF}_{m+n,n}$ with word $12\cdots n$ by pushing a staircase into a parking function $\mathrm{PF}\in\mathcal{PF}_{m,n}$ with word $n\cdots 21$. Given a parking function $\mathrm{PF} \in \mathcal{PF}_{m,n}$ with word $n\cdots 21$, let $\lambda = \lambda(\mathrm{PF})$, we define $hstr(\mathrm{PF}) \in \mathcal{PF}_{m+n,n}$, the *horizontal stretch* of PF, to be the parking function with word $12\cdots n$ and $\lambda(hstr(\mathrm{PF})) = (\lambda_1+n-1, \lambda_2+n-2, \ldots, \lambda_{n-1}+1)$, then

Theorem 2.5.

$$\begin{aligned} hstr: \{\mathrm{PF} \in \mathcal{PF}_{m,n} : \mathrm{word}(\mathrm{PF}) = n\cdots 21\} &\rightarrow \{\mathrm{PF} \in \mathcal{PF}_{m+n,n} : \mathrm{word}(\mathrm{PF}) = 12\cdots n\}, \\ \mathrm{PF} &\mapsto hstr(\mathrm{PF}) \end{aligned}$$

is a bijection, and

$$\mathrm{area}(hstr(\mathrm{PF})) = \mathrm{area}(\mathrm{PF}), \tag{2.10}$$

$$\mathrm{dinv}(hstr(\mathrm{PF})) = \mathrm{dinv}(\mathrm{PF}), \tag{2.11}$$

$$\mathrm{ret}(hstr(\mathrm{PF})) = \mathrm{ret}(\mathrm{PF}). \tag{2.12}$$

Proof. The bijectivity of the map *hstr* is clear since the map is invertible. Comparing the coarea of both parking functions immediately proves Equations (2.10) and (2.12). To prove Equation (2.11), recall that $\mathrm{dinv}(\mathrm{PF}) = \mathrm{tdinv}(\mathrm{PF}) + \mathrm{dinvcorr}(\mathrm{PF})$, we shall compare the two components of dinv, i.e. tdinv and dinvcorr.

For a parking function $\mathrm{PF} \in \mathcal{PF}_{m,n}$ with $\mathrm{word}(\mathrm{PF}) = n\cdots 21$, its temporary dinv statistic $\mathrm{tdinv}(\mathrm{PF})$ reaches the maximum possible value of the path $\Pi(\mathrm{PF})$, i.e. any two north step with rank difference less than m will contribute 1 to tdinv. For any two north steps, we fire two lines from the two end points of the upper north step, then rank difference less than m means that either the upper line or the lower line intersects the lower north step. The two cases are pictured in Figure 2.2.

On the other hand, the parking function $hstr(\mathrm{PF}) \in \mathcal{PF}_{m+n,n}$ always has no tdinv since $\mathrm{word}(hstr(\mathrm{PF})) = 12\cdots n$. We want to show that the increase of the dinvcorr statistic makes up for the missing tdinv.

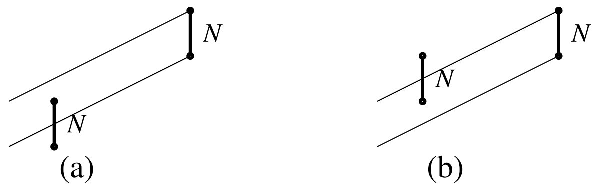


Figure 2.2: Pair of north steps contributing to tdinv.

For a parking function PF $\in \mathcal{PF}_{m,n}$, suppose that in the English partition $\lambda(\text{PF})$, there are j cells in row r of PF (counting from bottom to top of the n rows) with leg i, and their arms are $a, a+1, \ldots, a+j-1$, pictured in Figure 2.3 (a). We fire two lines with slope $\frac{n}{m}$ from the two end points of the north step (called N_1) in row r, then they intersect the east steps (called EEs) below the j cells at points A, B which have horizontal distances $\frac{mi}{n}$ and $\frac{m(i+1)}{n}$ to N_1.

Now consider the parking function $hstr(\text{PF}) \in \mathcal{PF}_{m+n,n}$. By definition of the map $hstr$, there are $j+1$ cells in row r with leg i in the partition $\lambda(hstr(\text{PF}))$, and their arms are $a+i, a+i+1, \ldots, a+i+j$, pictured in Figure 2.3 (b). We again fire two lines with slope $\frac{n}{m+n}$ from the two end points of the north step N_1 in row r, then they intersect the east steps below the $j+1$ cells at points A, B which have horizontal distances $\frac{mi}{n}+i$ and $\frac{m(i+1)}{n}+i+1$ to N_1.

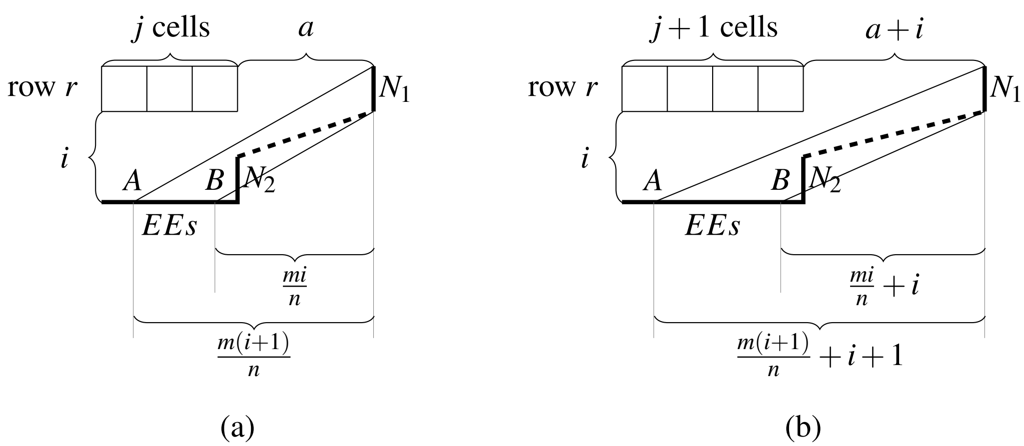


Figure 2.3: Cells in row r with leg i.

Now recall the definition of the dinv correction. The dinvcorr contribution of N_1 in each picture is equal to the whole east steps contained in line segment $\overline{AB}$. The line segment $\overline{AB}$ in *hstr*(PF) contains one more east step than $\overline{AB}$ in PF in the following 2 cases:

(1) In PF, A is not on EEs but B is on EEs.

(2) In PF, A is on EEs.

In case (1), the car in row r of PF produces a tdinv with the car in the row immediately below EEs; in case (2), the car in row r of PF produces a tdinv with the car in the row of the next north step that the upper line fired from N_1 intersects. Thus, the new dinvcorr in case (1) and case (2) matches the tdinv in the two cases in Figure 2.2, and the increase of dinv correction is equal to tdinv(PF), which proves the theorem. □

Since *hstr* is an (area,dinv,ret)-preserving bijection, $[s_{1^n}]_{m,n} = [s_n]_{m+n,n}$ follows immediately.

Result 2. $[s_{m^{\alpha_m}\cdots 1^{\alpha_1}}]_{m,n} = [s_{m^{\alpha_m+1}\cdots 1^{\alpha_1}}]_{m,n+m}$

This is a rewording of Theorem 2.3 (b). For a parking function PF $\in \mathcal{PF}_{m,n}$, we define a map *vstr*, vertical stretch, that we push a staircase down to PF, then replace the car i in PF by $i+m$, and fill the bottom of the m columns of the new parking function with cars $1,\ldots,m$ in a rank decreasing way to get *vstr*(PF), as shown in Figure 2.4.

Similar to Theorem 2.5, we have the following theorem about the vertical stretch action.

Theorem 2.6.

$$vstr: \ \{\text{PF} \in \mathcal{PF}_{m,n} : \text{pides(PF)} = m^{\alpha_m}\cdots 1^{\alpha_1}\} \to \{\text{PF} \in \mathcal{PF}_{m,n+m} : \text{pides(PF)} = m^{\alpha_m+1}\cdots 1^{\alpha_1}\},$$

$$\text{PF} \mapsto vstr(\text{PF})$$

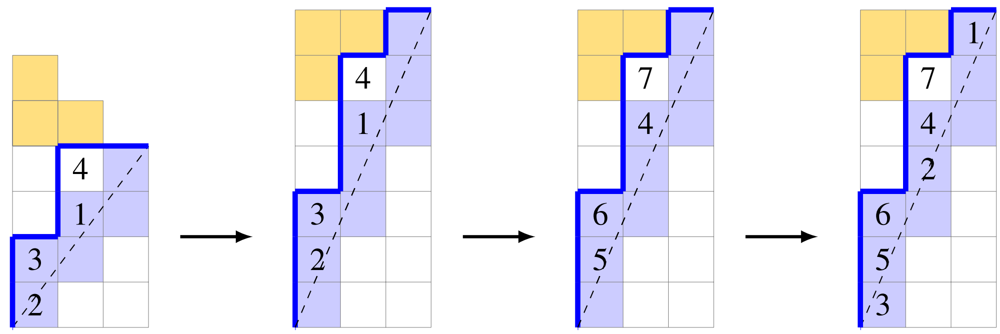


Figure 2.4: Bijection between $\mathcal{PF}_{3,n}$ with pides $3^a2^b1^c$ and $\mathcal{PF}_{3,n+3}$ with pides $3^{a+1}2^b1^c$.

is a bijection, and

$$\text{area}(vstr(\text{PF})) = \text{area}(\text{PF}), \tag{2.13}$$

$$\text{dinv}(vstr(\text{PF})) = \text{dinv}(\text{PF}), \tag{2.14}$$

$$\text{ret}(vstr(\text{PF})) = \text{ret}(\text{PF}). \tag{2.15}$$

Proof. The bijectivity is true since the map is invertible. Equations (2.13) and (2.15) are true for the same reason as Equations (2.10) and (2.12). The proof of Equation (2.14) is based on the same idea as the proof of (2.11): the action *vstr* changes each car i in PF into $i+m$, and the rank is also increased by m, thus the temporary dinv of PF is equal to the temporary dinv of the cars $m+1,\dots,m+n$ in $vstr(\text{PF})$. Since the dinv correction is negative, we can match each tdinv between cars $1,2,\dots,m$ and $m+1,\dots,m+n$ with a new negative dinv correction, showing that the change of dinv is zero. □

Result 3. $[s_{k1^{n-k}}]_{m,n} = [s_{k1^{m-k}}]_{n,m}$.

We shall prove the special case when $k=1$ first. That is, we first show $[s_{1^n}]_{m,n} = [s_{1^m}]_{n,m}$. The bijection for this identity is that we can transpose the path of $\text{PF} \in \mathcal{PF}_{m,n}$ and fill the word $(m, m-1, \dots, 1)$ to get $\text{PF}' \in \mathcal{PF}_{n,m}$.

It is obvious that PF′ has the same area as PF since their underlying Dyck paths are transposes of each other. For the statistic dinv, recall that the tdinv of a parking function with word $(n, n-1, \ldots, 1)$ is equal to the maxdinv of the path, thus

$$\begin{aligned} \text{dinv}(\text{PF}) &= \text{tdinv}(\text{PF}) + \text{pdinv}(\Pi(\text{PF})) - \text{maxdinv}(\text{PF}) \\ &= \text{pdinv}(\Pi(\text{PF})) \\ &= \sum_{c \in \lambda(\Pi)} \chi\left(\frac{arm(c)}{leg(c)+1} \leq \frac{m}{n} < \frac{arm(c)+1}{leg(c)}\right). \end{aligned} \tag{2.16}$$

From the Equation (2.16), we see that dinv is symmetric about m and n, and preserved by the transpose action. Figure 2.5 shows an example of this bijection.

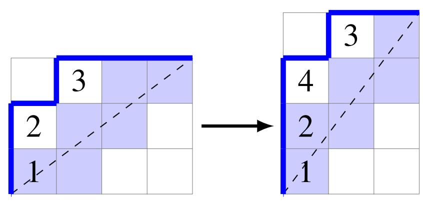


Figure 2.5: Bijection between $\mathcal{PF}_{4,3}$ with pides 1^3 and $\mathcal{PF}_{3,4}$ with pides 1^4.

Then we consider the equality $[s_{k1^{n-k}}]_{m,n} = [s_{k1^{m-k}}]_{n,m}$. This bijective proof is similar to that of $[s_{1^n}]_{m,n} = [s_{1^m}]_{n,m}$.

That is, given a parking function PF $\in \mathcal{PF}_{m,n}$ with pides $k1^{n-k}$, one transposes the path and labels the path to produce pides $k1^{m-k}$. If there are only k peaks (which means k different columns) in the Dyck paths, then the filling of cars in both (m,n) and (n,m) cases are unique since the cars $1, \ldots, k$ must be filled in a rank-decreasing way at bottom of each column in the two parking functions, while the remaining cars should be filled in a rank-increasing way in the remaining north steps. One can check that they have the same area and dinv values.

Otherwise, in any rational (m,n)-Dyck path Π with $j > k$ peaks, the car k must be in the first row since it has the smallest rank, and there are $\binom{j-1}{k-1}$ ways to choose columns to place the cars $1, \ldots, k-1$ in the north steps of both Π and its transpose, while the remaining cars should be filled in a rank-increasing way in the remaining north steps. Using the idea of analyzing dinv in the proof of Theorem 2.5, we can match the $\binom{j-1}{k-1}$ possible positions of cars $1,\ldots,k-1$ in both (m,n) and (n,m) cases by the dinv statistic, thus prove the result.

Note that Theorem 2.3 (c) is a result about the hook-shaped Schur functions. As we proved within this result, Theorem 2.3 (c) implies the following corollary.

Corollary 2.7. *For all $m,n>0$,*

$$[s_{1^n}]_{m,n} = [s_{1^m}]_{n,m}.$$

2.3 Schur function expansions of the $(m,3)$ Case

The Rational Shuffle Theorem when $n=3$ has a nice Schur function expansion, summarized in Theorem 2.4. For example, one can compute the Schur function expansion of $Q_{3k+1,3}(-1)$ by Maple to get Table 2.1.

In this section, we give two proofs of Theorem 2.4 by both working on the symmetric function side and the combinatorial side of the Rational Shuffle Theorem. Our proofs independently prove the Rational Shuffle Theorem and the Shuffle Theorem when $n \leq 3$.

2.3.1 Algebraic proof — $Q_{m,3}(-1)$

We shall use Leven's method in [Lev14] to prove the theorem by induction. We use the following lemma about q,t-analogue integers to simplify our computation.

Lemma 2.8. *Let $n,k \geq 0$ be two non-negative integers, we have*

$$[n]_{q,t}[k]_{q,t} = [n+k-1]_{q,t} + qt[k-1]_{q,t}[n-1]_{q,t}. \tag{2.17}$$

Table 2.1: Coefficients of s_λ in $Q_{3k+1,3}(-1)$.

$Q_{3k+1,3}(-1)$ \ s_λ	s_3	s_{21}	s_{1^3}
$Q_{1,3}(-1)$	0	0	$[1]_{q,t}$
$Q_{4,3}(-1)$	$[1]_{q,t}$	$[2]_{q,t}+[3]_{q,t}$	$[1]_{q,t}$ $+qt[4]_{q,t}$
$Q_{7,3}(-1)$	$[4]_{q,t}$ $+qt[1]_{q,t}$	$[5]_{q,t}+[6]_{q,t}$ $+qt([2]_{q,t}+[3]_{q,t})$	$[7]_{q,t}$ $+qt[4]_{q,t}$ $+(qt)^2[1]_{q,t}$
$Q_{10,3}(-1)$	$[7]_{q,t}$ $+qt[4]_{q,t}$ $+(qt)^2[1]_{q,t}$	$[8]_{q,t}+[9]_{q,t}$ $+qt([5]_{q,t}+[6]_{q,t})$ $+(qt)^2([2]_{q,t}+[3]_{q,t})$	$[10]_{q,t}$ $+qt[7]_{q,t}$ $+(qt)^2[4]_{q,t}$ $+(qt)^3[1]_{q,t}$
$Q_{13,3}(-1)$	$[10]_{q,t}$ $+qt[7]_{q,t}$ $+(qt)^2[4]_{q,t}$ $+(qt)^3[1]_{q,t}$	$[11]_{q,t}+[12]_{q,t}$ $+qt([8]_{q,t}+[9]_{q,t})$ $+(qt)^2([5]_{q,t}+[6]_{q,t})$ $+(qt)^3([2]_{q,t}+[3]_{q,t})$	$[13]_{q,t}$ $+qt[10]_{q,t}$ $+(qt)^2[7]_{q,t}$ $+(qt)^3[4]_{q,t}$ $+(qt)^4[1]_{q,t}$
…	…	…	…

Proof.

$$\begin{aligned}
[n]_{q,t}[k]_{q,t} &= (q^{n-1}+q^{n-2}t+\cdots+qt^{n-2}+t^{n-1})(q^{k-1}+q^{k-2}t+\cdots+qt^{k-2}+t^{k-1}) \\
&= q^{n-1}(q^{k-1}+q^{k-2}t+\cdots+qt^{k-2}+t^{k-1})+(q^{n-2}t+\cdots+qt^{n-2}+t^{n-1})t^{k-1} \\
&\quad +(q^{n-2}t+\cdots+qt^{n-2}+t^{n-1})(q^{k-1}+q^{k-2}t+\cdots+qt^{k-2}) \\
&= [n+k-1]_{q,t}+qt[k-1]_{q,t}[n-1]_{q,t}. \qquad \square
\end{aligned}$$

We need the following lemma from [BGLX15] to prove the symmetric function side of the theorem.

Lemma 2.9. *For any positive integers m,n,*

$$\nabla Q_{m,n}\nabla^{-1} = Q_{m+n,n}. \tag{2.18}$$

Since $\nabla a = a$ for any constant a, Lemma 2.9 allows us to write a recursion for $Q_{m,n}$ operator that

$$Q_{m+n,n}(-1)^n = \nabla Q_{m,n}\nabla^{-1}(-1)^n = \nabla Q_{m,n}(-1)^n. \tag{2.19}$$

Using the recursion, we can prove Theorem 2.4 by inducting on m. We shall give the complete algebraic proof of Equation (2.6) in Theorem 2.4 and omit the algebraic proof of Equations (2.7) and (2.8).

Proof of Equation (2.6). When $k = 0$, we can obtain by direct computation that

$$Q_{1,3}(-1) = s_{1^3}, \tag{2.20}$$

which satisfies Equation (2.6). Then we induct on k to prove Equation (2.6) that suppose the Schur function coefficients of $Q_{3k+1,3}(-1)$ are the following:

$$[s_3]_{3k+1,3} = \sum_{i=0}^{k-1}(qt)^{k-1-i}[3i+1]_{q,t}, \tag{2.21}$$

$$[s_{21}]_{3k+1,3} = \sum_{i=0}^{k-1}(qt)^{k-1-i}([3i+2]_{q,t}+[3i+3]_{q,t}), \tag{2.22}$$

$$[s_{1^3}]_{3k+1,3} = \sum_{i=0}^{k}(qt)^{k-1-i}[3i+1]_{q,t}, \tag{2.23}$$

we want to show that

$$[s_3]_{3(k+1)+1,3} = \sum_{i=0}^{k} (qt)^{k-1-i}[3i+1]_{q,t}, \tag{2.24}$$

$$[s_{21}]_{3(k+1)+1,3} = \sum_{i=0}^{k} (qt)^{k-1-i}([3i+2]_{q,t}+[3i+3]_{q,t}), \text{ and} \tag{2.25}$$

$$[s_{1^3}]_{3(k+1)+1,3} = \sum_{i=0}^{k+1} (qt)^{k-1-i}[3i+1]_{q,t}. \tag{2.26}$$

One can directly compute that

$$\nabla s_3 = (qt)^2 s_{21} + (qt)^2[2]_{q,t} s_{1^3}, \tag{2.27}$$

$$\nabla s_{21} = (qt)[2]_{q,t} s_{21} - (qt)[3]_{q,t} s_{1^3}, \tag{2.28}$$

$$\nabla s_{1^3} = s_3 + ([2]_{q,t}+[3]_{q,t}) s_{21} + (qt+[4]_{q,t}) s_{1^3}. \tag{2.29}$$

By Equation (2.19), we have

$$\begin{aligned}
\mathrm{Q}_{3(k+1)+1,3}(-1) &= [s_3]_{3k+4,3}s_3 + [s_{21}]_{3k+4,3}s_{21} + [s_{1^3}]_{3k+4,3}s_{1^3} \\
&= \nabla \mathrm{Q}_{3k+1,3}(-1) \\
&= \nabla([s_3]_{3k+1,3}s_3 + [s_{21}]_{3k+1,3}s_{21} + [s_{1^3}]_{3k+1,3}s_{1^3}) \\
&= [s_3]_{3k+1,3}\nabla s_3 + [s_{21}]_{3k+1,3}\nabla s_{21} + [s_{1^3}]_{3k+1,3}\nabla s_{1^3} \\
&= [s_{1^3}]_{3k+1,3}s_3 \\
&\quad + \big((qt)^2[s_3]_{3k+1,3} - qt[2]_{q,t}[s_{21}]_{3k+1,3} + ([2]_{q,t}+[3]_{q,t})[s_{1^3}]_{3k+1,3}\big)\, s_{21} \\
&\quad + \big((qt)^2[2]_{q,t}[s_3]_{3k+1,3} - qt[3]_{q,t}[s_{21}]_{3k+1,3} + (qt+[4]_{q,t})[s_{1^3}]_{3k+1,3}\big)\, s_{1^3},
\end{aligned}$$

which implies that

$$[s_3]_{3k+4,3} = [s_{1^3}]_{3k+1,3}, \tag{2.30}$$

$$[s_{21}]_{3k+4,3} = (qt)^2[s_3]_{3k+1,3} - qt[2]_{q,t}[s_{21}]_{3k+1,3} + ([2]_{q,t}+[3]_{q,t})[s_{1^3}]_{3k+1,3}, \tag{2.31}$$

$$[s_{1^3}]_{3k+4,3} = (qt)^2[2]_{q,t}[s_3]_{3k+1,3} - qt[3]_{q,t}[s_{21}]_{3k+1,3} + (qt+[4]_{q,t})[s_{1^3}]_{3k+1,3}. \tag{2.32}$$

By the recursions above, we can apply Lemma 2.8 and verify Equations (2.24), (2.25) and (2.26) inductively:

$$\begin{aligned}
[s_3]_{3k+4,3} &= [s_{1^3}]_{3k+1,3} = \sum_{i=0}^{k}(qt)^{k-i}[3i+1]_{q,t}, \\
[s_{21}]_{3k+4,3} &= (qt)^2\sum_{i=0}^{k-1}(qt)^{k-1-i}[3i+1]_{q,t} - qt[2]_{q,t}\sum_{i=0}^{k-1}(qt)^{k-1-i}([3i+2]_{q,t}+[3i+3]_{q,t}) \\
&\quad +([2]_{q,t}+[3]_{q,t})\sum_{i=0}^{k}(qt)^{k-i}[3i+1]_{q,t} \\
&= (qt)^2\sum_{i=0}^{k-1}(qt)^{k-1-i}[3i+1]_{q,t} - qt\sum_{i=0}^{k-1}(qt)^{k-1-i}([3i+3]_{q,t}+qt[3i+1]_{q,t}) \\
&\quad -qt[2]_{q,t}\sum_{i=0}^{k-1}(qt)^{k-1-i}[3i+3]_{q,t} \\
&\quad +\sum_{i=0}^{k}(qt)^{k-i}([3i+2]_{q,t}+[3i+3]_{q,t}+qt[3i]_{q,t}+qt[2]_{q,t}[3i]_{q,t}) \\
&= \sum_{i=0}^{k}(qt)^{k-i}([3i+2]_{q,t}+[3i+3]_{q,t}),
\end{aligned}$$

and

$$
\begin{aligned}
[s_{1^3}]_{3k+4,3} &= (qt)^2[2]_{q,t}[s_3]_{3k+1,3} - qt[3]_{q,t}[s_{21}]_{3k+1,3} + (qt+[4]_{q,t})[s_{1^3}]_{3k+1,3} \\
&= (qt)^2[2]_{q,t}\sum_{i=0}^{k-1}(qt)^{k-1-i}[3i+1]_{q,t} - qt\sum_{i=0}^{k-1}(qt)^{k-1-i}([3i+4]_{q,t}+qt[2]_{q,t}[3i+1]_{q,t}) \\
&\quad -qt[3]_{q,t}\sum_{i=0}^{k-1}(qt)^{k-1-i}[3i+3]_{q,t} \\
&\quad +\sum_{i=0}^{k}(qt)^{k-i}(qt[3i+1]_{q,t}+[3i+4]_{q,t}+qt[3]_{q,t}[3i]_{q,t}) \\
&= \sum_{i=0}^{k+1}(qt)^{k+1-i}[3i+1]_{q,t}. \qquad \square
\end{aligned}
$$

2.3.2 Combinatorial side — $\mathrm{H}_{m,3}[X;q,t]$

Now we consider the Hikita polynomial defined by

$$
\mathrm{H}_{m,n}[X;q,t] = \sum_{\mathrm{PF}\in\mathcal{PF}_{m,n}} [\mathrm{ret}(\mathrm{PF})]_{\frac{1}{t}} t^{\mathrm{area}(\mathrm{PF})} q^{\mathrm{dinv}(\mathrm{PF})} s_{\mathrm{pides}(\mathrm{PF})}. \tag{2.33}
$$

Any parking function $\mathrm{PF}\in\mathcal{PF}_{m,3}$ has 3 rows, thus only has 3 cars: $\{1,2,3\}$, and the word $\sigma(\mathrm{PF})$ can be any permutation $\sigma\in\mathcal{S}_3$. Table 2.2 shows the s_{pides} contribution of the 6 permutations in $\mathcal{S}_3$.

Table 2.2: s_{pides} contribution of permutations in $\mathcal{S}_3$.

$\sigma\in\mathcal{S}_3$	123	132	213	231	312	321
s_{pides}	s_3	s_{21}	$s_{12}=0$	s_{21}	$s_{12}=0$	s_{1^3}

By our notation, $\mathrm{H}_{m,3}[X;q,t] = [s_3]_{m,3}s_3 + [s_{21}]_{m,3}s_{21} + [s_{1^3}]_{m,3}s_{1^3}$. We can work out the combinatorial side of the Rational Shuffle Theorem in the case where $n=3$ by using (2.33).

2.3.2.1 Combinatorics of $\mathrm{H}_{3k+1,3}[X;q,t]$

We show the combinatorics of $\mathrm{H}_{3k+1,3}[X;q,t]$ by enumerating the parking functions on the $(3k+1)\times 3$ lattice to prove the following formulas of the coefficients of $\mathrm{H}_{3k+1,3}[X;q,t]$:

$$[s_3]_{3k+1,3} = \sum_{i=0}^{k-1}(qt)^{k-1-i}[3i+1]_{q,t}, \tag{2.34}$$

$$[s_{21}]_{3k+1,3} = \sum_{i=0}^{k-1}(qt)^{k-1-i}([3i+2]_{q,t}+[3i+3]_{q,t}), \tag{2.35}$$

$$[s_{1^3}]_{3k+1,3} = \sum_{i=0}^{k}(qt)^{k-1-i}[3i+1]_{q,t}. \tag{2.36}$$

Given a parking function $\mathrm{PF}\in\mathcal{PF}_{3k+1,3}$, we let $\Pi=\Pi(\mathrm{PF})$ be the path of PF. Since $3k+1>3$ for $k\geq 1$, dinv correction is non-negative by definition, and

$$\mathrm{dinvcorr}(\mathrm{PF}) = \sum_{c\in\lambda(\Pi)}\chi\left(\frac{arm(c)+1}{leg(c)+1}\leq\frac{m}{n}<\frac{arm(c)}{leg(c)}\right). \tag{2.37}$$

The partition corresponding to the Dyck path Π of PF has at most 2 parts, so $\mathrm{leg}(c)$ of a cell $c\in\lambda(\Pi)$ is either 0 or 1. Taking Figure 2.6 for reference, we have

(a) $c\in\lambda(\Pi)$ with $\mathrm{leg}(c)=0$ and $1\leq\mathrm{arm}(c)<k$ contributes 1 to dinv correction, marked $\bigcirc$ in Figure 2.6,

(b) $c\in\lambda(\Pi)$ with $\mathrm{leg}(c)=1$ and $k<\mathrm{arm}(c)\leq 2k-1$ contributes 1 to dinv correction, marked $\triangle$ in Figure 2.6.

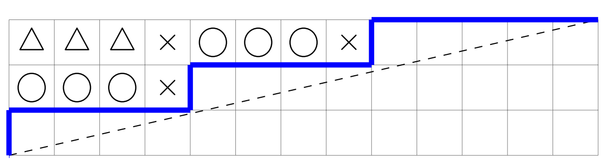


Figure 2.6: The dinv correction of a $(3k+1,3)$-Dyck path when $k=4$.

Further, we can directly count the statistics area and dinv correction (dinvcorr) from the partition $\lambda(\Pi)$ of the path Π. We write $\lambda = (\lambda_1, \lambda_2) = \lambda(\Pi)$, then $\lambda \subseteq \lambda_0 = (2k, k)$, i.e. $\lambda_1 \le 2k$ and $\lambda_2 \le k$. Clearly, the area of Π is counted by $|\lambda_0| - |\lambda|$, thus

$$\text{area}(\Pi) = 3k - \lambda_1 - \lambda_2. \tag{2.38}$$

We can also write the formula for dinv correction according to the partition λ:

$$\text{dinvcorr}(\Pi) = \begin{cases} \lambda_1 - 1 & \text{if } \lambda_2 = 0 \text{ and } \lambda_1 \le k, \\ k-1 & \text{if } \lambda_2 = 0 \text{ and } \lambda_1 > k, \\ \lambda_1 - 1 & \text{if } \lambda_2 = \lambda_1 \ge 1, \\ \lambda_1 - 2 & \text{if } \lambda_2 \ge 1, 1 \le \lambda_1 - \lambda_2 \le k, \text{ and } \lambda_1 \le k, \\ 2\lambda_1 - k - 3 & \text{if } \lambda_2 \ge 1, 1 \le \lambda_1 - \lambda_2 \le k, \text{ and } \lambda_1 \ge k+1, \\ 2\lambda_2 + k - 2 & \text{if } \lambda_2 \ge 1 \text{ and } \lambda_1 - \lambda_2 \ge k+1. \end{cases} \tag{2.39}$$

Note that the return statistic is always 1 since $3k+1$ and 3 are coprime. We shall compute $[s_3]_{3k+1,3}$ first.

From Table 2.2, we see that only the parking functions in $\mathcal{PF}_{3k+1,3}$ with word 123 contribute to the coefficient of s_3. We also notice that the 3 cars should be in different columns, otherwise there are cars $i < j$ with $\text{rank}(i) < \text{rank}(j)$, contradicting with the restriction that the word of the parking function is 123. Thus we have one $\text{PF} \in \mathcal{PF}_{3k+1,3}$ with word 123 on each $(3k+1,3)$ Dyck path which has no two consecutive north steps.

Let $\lambda(\text{PF}) = (\lambda_1, \lambda_2)$ be the partition associated to the Dyck path $\Pi(\text{PF})$ (see Figure 2.7), then area(PF) is counted by Equation (2.38). Since the ranks of cars $1,2,3$ are decreasing, there is always no tdinv, thus $\text{dinv(PF)} = \text{dinvcorr}(\Pi)$, which is counted by the latter 3 cases (since $\lambda_1 > \lambda_2 > 0$) of Equation (2.39).

For $[s_3]_{3k+1,3} = \sum_{i=0}^{k-1} (qt)^{k-1-i} [3i+1]_{q,t}$, we construct each term $(qt)^{k-1-i}[3i+1]_{q,t}$ as a

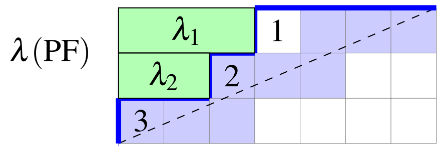


Figure 2.7: Example: a parking function PF $\in \mathcal{PF}_{7,3}$ with word 123.

sequence of parking functions. Since each parking function corresponds to a unique partition $\lambda \subset (2k,k)$ with 2 distinct parts, we shall use partitions to represent parking functions in $\mathcal{PF}_{3k+1,3}$ with diagonal word 123. For each i, we define the following 3 branches of partitions (parking functions with word 123):

$$\begin{aligned}
\Lambda_1 &= \{(k+i+1,k),(k+i,k-1),\ldots,(k+2,k-i+1)\},\\
\Lambda_2 &= \{(2k,i),(2k-1,i-1),\ldots,(2k+1-i,1)\},\\
\Lambda_3 &= \{(k+1,k-i),(k,i+1),\ldots,(k-i+1,k-i)\}.
\end{aligned}$$

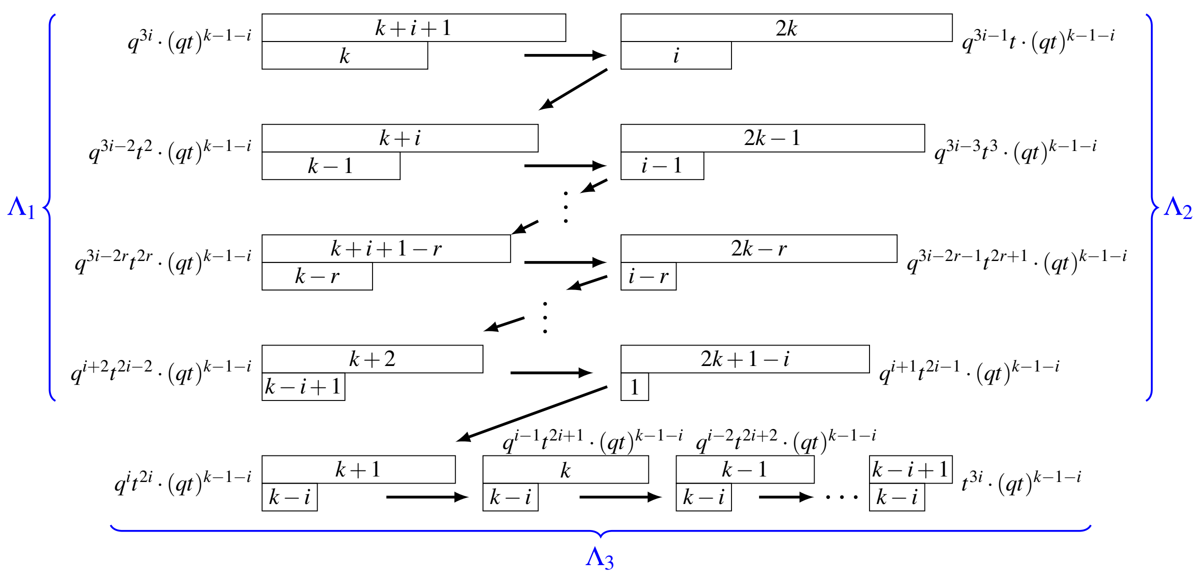


Figure 2.8: The construction of $(qt)^{k-1-i}[3i+1]_{q,t}$.

The branch Λ_1 contains all the partitions λ such that $\lambda_1-\lambda_2=i+1\le k$ with $\lambda_2>k-i$, the branch Λ_2 contains all the partitions λ such that $\lambda_1 - \lambda_2 = 2k - i > k$, and the branch Λ_3 contains all the partitions λ such that $\lambda_2 = i+1$ and $\lambda_1 - \lambda_2 \leq k - i$. Notice that $|\Lambda_1| = |\Lambda_2|$. As shown in Figure 2.8, the construction begins with *alternatively* taking partitions from Λ_1 and Λ_2, ending with the last partition of Λ_2. Then continue the chain by taking partitions in Λ_3 and end the chain with the last partition $(k-i+1, k-i)$ in Λ_3. The weights of the parking functions are $(qt)^{k-1-i}q^{3i}, (qt)^{k-1-i}q^{3i-1}t, \ldots, (qt)^{k-1-i}t^{3i}$ following the order of the chain.

To be more precise, it is not difficult to check that each parking function with diagonal word 123 is contained in $\Lambda_1 \cup \Lambda_2 \cup \Lambda_3$ for some i, and the parking function weights are

$$\sum_{\mathrm{PF}\in\Lambda_1} t^{\mathrm{area(PF)}} q^{\mathrm{dinv(PF)}} = (qt)^{k-i-1} q^{i+2} [i]_{q^2,t^2}, \tag{2.40}$$

$$\sum_{\mathrm{PF}\in\Lambda_2} t^{\mathrm{area(PF)}} q^{\mathrm{dinv(PF)}} = (qt)^{k-i-1} q^{i+1} t [i]_{q^2,t^2}, \quad \text{and} \tag{2.41}$$

$$\sum_{\mathrm{PF}\in\Lambda_3} t^{\mathrm{area(PF)}} q^{\mathrm{dinv(PF)}} = (qt)^{k-i-1} t^{2i} [i+1]_{q,t}, \tag{2.42}$$

which sum up to $(qt)^{k-1-i}[3i+1]_{q,t}$. This proves that $[s_3]_{3k+1,3} = \sum_{i=0}^{k-1}(qt)^{k-1-i}[3i+1]_{q,t}$. Figure 2.9 shows an example of the combinatorial construction of the coefficient $[s_3]_{10,3}$.

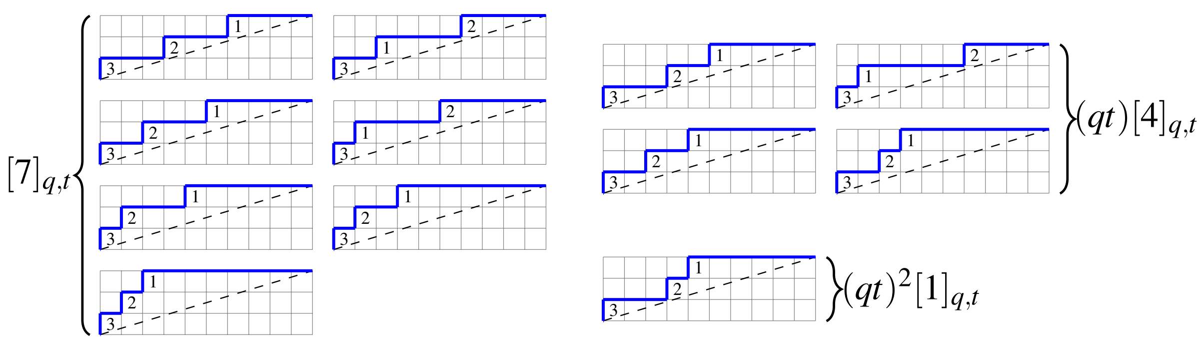


Figure 2.9: The construction of $[s_3]_{10,3} = [7]_{q,t} + (qt)[4]_{q,t} + (qt)^2[1]_{q,t}$.

We can combinatorially prove $[s_{21}]_{3k+1,3} = \sum_{i=0}^{k-1}(qt)^{k-1-i}([3i+2]_{q,t} + [3i+3]_{q,t})$ in a similar way. In this case, we have 2 possible diagonal words: 132 and 312. In both cases, the car 2 has the smallest rank, which means the label of the first (lowest) row must be 2. Thus, the pair of cars $(1,2)$ does not produce a tdinv. If we let $\lambda = (\lambda_1, \lambda_2)$ be the partition corresponding to the path Π, and let the labels of row 1, row 2, row 3 (counting from bottom to top) be ℓ_1, ℓ_2, ℓ_3, then we have the following formula for temporary dinv:

$$\mathrm{tdinv(PF)} = \begin{cases} \chi(\ell_3 > \ell_2) & \text{if } \lambda_2 = 0 \text{ and } \lambda_1 \leq k, \\ \chi(\ell_3 > \ell_1) + \chi(\ell_2 > \ell_3) & \text{if } \lambda_2 = 0 \text{ and } \lambda_1 > k, \\ \chi(\ell_2 > \ell_1) & \text{if } \lambda_2 = \lambda_1 \geq 1, \\ \chi(\ell_2 > \ell_1) + \chi(\ell_3 > \ell_2) & \text{if } \lambda_2 \geq 1, 1 \leq \lambda_1 - \lambda_2 \leq k,\ \lambda_1 \leq k \\ \chi(\ell_2 > \ell_1) + \chi(\ell_3 > \ell_1) + \chi(\ell_3 > \ell_2) & \text{if } \lambda_2 \geq 1, 1 \leq \lambda_1 - \lambda_2 \leq k,\ \lambda_1 \geq k+1 \\ \chi(\ell_2 > \ell_1) + \chi(\ell_3 > \ell_1) + \chi(\ell_2 > \ell_3) & \text{if } \lambda_2 \geq 1 \text{ and } \lambda_1 - \lambda_2 \geq k+1. \end{cases} \tag{2.43}$$

In the construction of the coefficient $[s_{21}]_{3k+1,3} = \sum_{i=0}^{k-1} (qt)^{k-1-i}([3i+2]_{q,t} + [3i+3]_{q,t})$, we construct each term $(qt)^{k-1-i}[3i+2]_{q,t}$ or $(qt)^{k-1-i}[3i+3]_{q,t}$ as a sequence of parking functions. First, we define the following 3 branches of parking functions to obtain the term $(qt)^{k-1-i}[3i+3]_{q,t}$ for each i:

$$\begin{aligned} \Lambda_1 &= \{\mathrm{PF} : \lambda(\mathrm{PF}) \in \{(2k, i+1), (2k-1, i), \ldots, (2k-i, 1)\}, (\ell_1, \ell_2, \ell_3) = (2,1,3)\}, \\ \Lambda_2 &= \{\mathrm{PF} : \lambda(\mathrm{PF}) \in \{(2k, i), (2k-1, i-1), \ldots, (2k-i, 0)\}, (\ell_1, \ell_2, \ell_3) = (2,3,1)\}, \\ \Lambda_3 &= \{\mathrm{PF} : \lambda(\mathrm{PF}) \in \{(k, k-i-1), \ldots, (k-i, k-i-1)\}, (\ell_1, \ell_2, \ell_3) = (2,3,1)\}. \end{aligned}$$

With the 3 branches defined, the construction is similar to the construction of $(qt)^{k-1-i}[3i+1]_{q,t}$ as a term of $[s_3]_{3k+1,3}$. We *alternatively* take parking functions from Λ_1 and Λ_2, ending with the last partition of Λ_2. Then continue the chain by taking partitions in Λ_3 and end the chain with the last parking function corresponding to the partition $(k-i, k-i-1)$ with labels $(\ell_1, \ell_2, \ell_3) = (2,3,1)$ in Λ_3. The weights of the parking functions are $(qt)^{k-1-i}q^{3i+2}, \ldots, (qt)^{k-1-i}t^{3i+2}$, which add up to $(qt)^{k-1-i}[3i+3]_{q,t}$.

Second, we define another three branches of parking functions to obtain the term $(qt)^{k-1-i}[3i+2]_{q,t}$ for each i:

$$\begin{aligned}
\Lambda_4 &= \{\mathrm{PF} : \lambda(\mathrm{PF}) \in \{(k+i+1,k),(k+i,k-1),\ldots,(k+1,k-i)\},(\ell_1,\ell_2,\ell_3)=(2,3,1)\},\\
\Lambda_5 &= \{\mathrm{PF} : \lambda(\mathrm{PF}) \in \{(k+i,k),(k+i-1,k-1),\ldots,(k,k-i)\},(\ell_1,\ell_2,\ell_3)=(2,1,3)\},\\
\Lambda_6 &= \{\mathrm{PF} : \lambda(\mathrm{PF}) \in \{(k-1,k-i),\ldots,(k-i,k-i)\},(\ell_1,\ell_2,\ell_3)=(2,1,3)\}.
\end{aligned}$$

The construction is the same as that of $(qt)^{k-1-i}[3i+3]_{q,t}$, and the weights of the parking functions are $(qt)^{k-1-i}q^{3i+1},\ldots,(qt)^{k-1-i}t^{3i+1}$ which add up to $(qt)^{k-1-i}[3i+2]_{q,t}$.

Thus we have proved that $[s_{21}]_{3k+1,3} = \sum_{i=0}^{k-1}(qt)^{k-1-i}([3i+2]_{q,t}+[3i+3]_{q,t})$. Figure 2.10 shows an example of the combinatorial construction of the coefficient $[s_{21}]_{7,3}$.

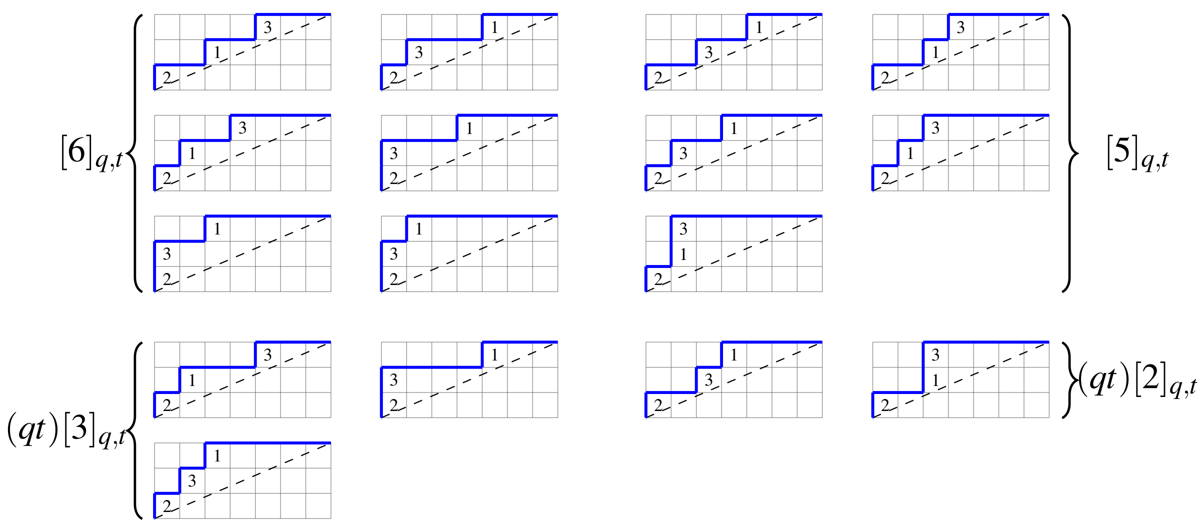


Figure 2.10: The construction of $[s_{21}]_{7,3} = [6]_{q,t}+[5]_{q,t}+(qt)([3]_{q,t}+[2]_{q,t})$.

The equality that $[s_{1^3}]_{3k+1,3} = \sum_{i=0}^{k}(qt)^{k-i}[3i+1]_{q,t} = [s_3]_{3k+4,3}$ follows immediately from the following corollary of Theorem 2.3 (a):

Corollary 2.10. *For any $m>0$, $[s_{1^3}]_{m,3} = [s_3]_{m+3,3}$.*

2.3.2.2 Combinatorics of $\mathrm{H}_{3k+2,3}[X;q,t]$

We study the combinatorics of $\mathrm{H}_{3k+2,3}[X;q,t]$ in a similar manner by enumerating the parking functions on the $(3k+2)\times 3$ lattice to prove that

$$[s_3]_{3k+2,3} = \sum_{i=0}^{k-1}(qt)^{k-1-i}[3i+2]_{q,t}, \tag{2.44}$$

$$[s_{21}]_{3k+2,3} = \sum_{i=-1}^{k-1}(qt)^{k-1-i}([3i+3]_{q,t}+[3i+4]_{q,t}), \quad \text{and} \tag{2.45}$$

$$[s_{1^3}]_{3k+2,3} = \sum_{i=0}^{k}(qt)^{k-i}[3i+2]_{q,t}. \tag{2.46}$$

Given a parking function $\mathrm{PF}\in\mathcal{PF}_{3k+2,3}$ with $\Pi(\mathrm{PF})=\Pi$, we can compute the dinv correction of PF by examining the cells $c\in\lambda(\Pi)$. Taking Figure 2.11 for reference,

(a) $c\in\lambda(\Pi)$ with $\mathrm{leg}(c)=0$ and $1\le \mathrm{arm}(c)<k$ contributes 1 to dinv correction, marked ◯ in Figure 2.11,

(b) $c\in\lambda(\Pi)$ with $\mathrm{leg}(c)=1$ and $k<\mathrm{arm}(c)\le 2k$ contributes 1 to dinv correction, marked △ in Figure 2.11.

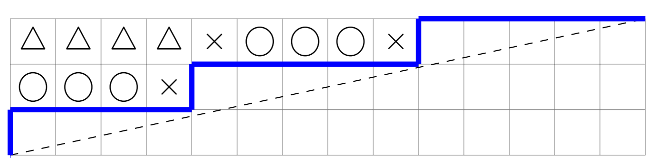


Figure 2.11: The dinv correction of a $(3k+2,3)$-Dyck path when $k=4$.

Further, we can directly count the statistics area and dinv correction (dinvcorr) from the partition $\lambda(\Pi)=(\lambda_1,\lambda_2)\subseteq(2k+1,k)$ of a $(3k+2,3)$-Dyck path Π. Similar to Equation (2.38), we have

$$\mathrm{area}(\Pi)=3k+1-\lambda_1-\lambda_2. \tag{2.47}$$

The dinv correction formula is the same as Equation (2.39), and the return statistic is still always equal to 1.

To prove $[s_3]_{3k+2,3} = \sum_{i=0}^{k-1}(qt)^{k-1-i}[3i+2]_{q,t}$, we shall construct the following 3 branches of partitions (parking functions with word 123) for each term $(qt)^{k-1-i}[3i+2]_{q,t}$:

$$
\begin{aligned}
\Lambda_1 &= \{(2k+1,i+1),(2k,i),\ldots,(2k+1-i,1)\}, \\
\Lambda_2 &= \{(k+i+1,k),(k+i,k-1),\ldots,(k+1,k-i)\}, \\
\Lambda_3 &= \{(k,k-i),(k-1,i+1),\ldots,(k-i+1,k-i)\}.
\end{aligned}
$$

Then, we can follow the same construction as the $(3k+1,3)$ case to get all parking functions with word 123 and weights $(qt)^{k-1-i}q^{3i+1},\ldots,(qt)^{k-1-i}t^{3i+1}$.

To prove $[s_{21}]_{3k+2,3} = \sum_{i=-1}^{k-1}(qt)^{k-1-i}([3i+3]_{q,t}+[3i+4]_{q,t})$, we have 6 branches of parking functions as follows (which are similar to the $(3k+1,3)$ case):

$$
\begin{aligned}
\Lambda_1 &= \{\mathrm{PF}:\lambda(\mathrm{PF})\in\{(k+i+2,k),\ldots,(k+2,k-i)\},(\ell_1,\ell_2,\ell_3)=(2,3,1)\}, \\
\Lambda_2 &= \{\mathrm{PF}:\lambda(\mathrm{PF})\in\{(k+i+1,k),\ldots,(k+1,k-i)\},(\ell_1,\ell_2,\ell_3)=(2,1,3)\}, \\
\Lambda_3 &= \{\mathrm{PF}:\lambda(\mathrm{PF})\in\{(k+1,k-i-1),\ldots,(k-i,k-i-1)\},(\ell_1,\ell_2,\ell_3)=(2,3,1)\}, \\
\Lambda_4 &= \{\mathrm{PF}:\lambda(\mathrm{PF})\in\{(2k+1,i+1),\ldots,(2k-i+1,1)\},(\ell_1,\ell_2,\ell_3)=(2,1,3)\}, \\
\Lambda_5 &= \{\mathrm{PF}:\lambda(\mathrm{PF})\in\{(2k+1,i),\ldots,(2k-i+1,0)\},(\ell_1,\ell_2,\ell_3)=(2,3,1)\}, \\
\Lambda_6 &= \{\mathrm{PF}:\lambda(\mathrm{PF})\in\{(k,k-i),\ldots,(k-i,k-i)\},(\ell_1,\ell_2,\ell_3)=(2,1,3)\}.
\end{aligned}
$$

Then, the total weight of parking functions in the first 3 branches is $(qt)^{k-1-i}[3i+4]_{q,t}$, and the total weight of parking functions in the last 3 branches is $(qt)^{k-1-i}[3i+3]_{q,t}$.

The proof of $[s_{1^3}]_{3k+2,3} = [s_3]_{3(k+1)+2,3} = \sum_{i=0}^{k}(qt)^{k-i}[3i+2]_{q,t}$ follows from Corollary 2.10.

2.3.2.3 Combinatorics of $\mathrm{H}_{3k,3}[X;q,t]$

Notice that the area and dinv of parking functions in $\mathcal{PF}_{3k,3}$ are equal to those of the parking functions in $\mathcal{PF}_{3k+1,3}$ (as we discussed in Section 1.2.4). Given a parking function $\mathrm{PF} \in \mathcal{PF}_{3k+1,3}$, let $\lambda = \lambda(\mathrm{PF})$, then the return statistic of PF is formulated as

$$\mathrm{ret}(\mathrm{PF}) = 2\chi(\lambda_1 = 2k) + \chi(\lambda_2 = k) - 2\chi(\lambda_1 = 2k) \cdot \chi(\lambda_2 = k).$$

By the Extended Rational Shuffle Theorem in the non-coprime case,

$$\begin{aligned} \mathrm{H}_{3k,3}[X;q,t] &= \sum_{\mathrm{PF} \in \mathcal{PF}_{3k,3}} [\mathrm{ret}(\mathrm{PF})]_{\frac{1}{t}} t^{\mathrm{area}(\mathrm{PF})} q^{\mathrm{dinv}(\mathrm{PF})} F_{\mathrm{ides}(\mathrm{PF})}[X] \\ &= \sum_{\mathrm{PF} \in \mathcal{PF}_{3k+1,3}} [\mathrm{ret}(\mathrm{PF})]_{\frac{1}{t}} t^{\mathrm{area}(\mathrm{PF})} q^{\mathrm{dinv}(\mathrm{PF})} s_{\mathrm{pides}(\mathrm{PF})}. \end{aligned} \tag{2.48}$$

To prove that

$$[s_3]_{3k,3} = \sum_{i=0}^{k-1} (qt)^{k-1-i}([3i-1]_{q,t} + [3i]_{q,t} + [3i+1]_{q,t}), \tag{2.49}$$

$$\begin{aligned} [s_{21}]_{3k,3} &= (qt)^{k+1}([3]_{q,t} + 2[2]_{q,t} + [1]_{q,t}) \\ &\quad + \sum_{i=1}^{k-1} (qt)^{k-1-i}([3i]_{q,t} + 2[3i+1]_{q,t} + 2[3i+2]_{q,t} + [3i+3]_{q,t}), \end{aligned} \tag{2.50}$$

$$[s_{1^3}]_{3k,3} = \sum_{i=0}^{k} (qt)^{k-i}([3i-1]_{q,t} + [3i]_{q,t} + [3i+1]_{q,t}), \tag{2.51}$$

we use the constructions of $[s_3]_{3k+1,3}, [s_{21}]_{3k+1,3}, [s_{1^3}]_{3k+1,3}$ and modify the weight of parking functions with nonzero returns. We use the set of partitions $\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4, \Lambda_5, \Lambda_6$ in Section 2.3.2.1.

For $[s_3]_{3k,3}$, the first parking function in each set Λ_1 and Λ_2 has return statistic 1, except that the first parking function in Λ_1 when $i = k-1$ has return 2. All the remaining parking functions have return 3. Then we prove Equation (2.49) by summing up the parking function weights.

For $[s_{21}]_{3k,3}$, the first parking function in each set $\Lambda_1, \Lambda_2, \Lambda_4, \Lambda_5$ has return statistic 1 since the second parts of these partitions are k, except that the first parking function in Λ_1 and Λ_4 when $i = k-1$ has return 2. All the remaining parking functions have return 3. Then again we obtain Equation (2.50) by direct computation.

The proof of $[s_{1^3}]_{3k,3} = [s_3]_{3(k+1),3}$ follows from Corollary 2.10.

2.4 Combinatorial results about Schur function expansions of the $(3,n)$ case

2.4.1 Recursive formula for $[s_\lambda]_{3,n}$

In $(3,n)$ case, we have n cars, i.e. the word of a $(3,n)$ parking function is a permutation of $[n]$. By Remark 1.2, $[s_\lambda]_{3,n} \neq 0$ implies that λ must be of the form $3^a2^b1^c$ with $3a+2b+c=n$, i.e. $[s_\lambda]_{3,n} \neq 0$ only if the partition λ only has parts of sizes less than or equal to 3.

We have the following corollary of Theorem 2.3 summarizing some symmetries about $[s_\lambda]_{3,n}$.

Corollary 2.11. *For all $m,n > 0$ and $a,b,c \geq 0$,*

(a) $[s_{3^a2^b1^c}]_{3,n} = [s_{2^b1^c}]_{3,n-3a}$,

(b) $[s_{1^n}]_{3,n} = [s_{1^3}]_{n,3}$,

(c) $[s_{21^{n-2}}]_{3,n} = [s_{21}]_{n,3}$,

Further, we conjecture another important symmetry.

Conjecture 2.1. *For all $a,b,n \geq 0$,*

$$[s_{2^a1^b}]_{3,n} = [s_{2^b1^a}]_{3,3(a+b)-n}.$$

We have found the straightening action in parking functions combinatorially from parking functions with pides $\{\cdots,1,3,\cdots\}$ to parking functions with pides $\{\cdots,2,2,\cdots\}$, which is an *involution* whose fixed points are the coefficients of $[s_{2^a1^b}]_{3,n}$. Further, we have conjectured a bijection between the fixed parking functions with pides 2^a1^b and the fixed parking functions with pides 2^b1^a, mapping the 2 cars (or 1 car) causing part 2 (or 1) in pides 2^a1^b to 1 car (or 2 cars) causing part 1 (or 2) in pides 2^b1^a. We will state the details later.

The results above show that the problem of computing the Schur function expansion of $Q_{3,n}(-1)^n$ can be reduced to the problem of finding the coefficients of Schur functions of the form $s_{2^a1^b}$ where $a<b$ in $Q_{3,n}(-1)^n$. Finally, we conjecture a recursive formula for such coefficients $[s_{2^a1^b}]_{3,n}$ where $a<b$.

Conjecture 2.2. *Let $a<b$, then*

$$[s_{2^a1^b}]_{3,n} = \sum_{i=0}^{a}[b+i]_{q,t} + (qt)[s_{2^a1^{b-3}}]_{3,n-3}.$$

We have verified this formula by Maple for $n<27$. If the conjectures are true, then we have solved the Schur function expansion in the $(3,n)$ case.

2.4.2 The symmetry $[s_{2^a1^b}]_{3,n} = [s_{2^b1^a}]_{3,3(a+b)-n}$

For this symmetry, we shall first introduce an involution on $(3,n)$-parking functions whose pides contain $1,3$ or $2,2$. The fixed points of the involution is a subset of parking functions whose pides do not contain $1,3$. Then, we give a bijection between the fixed parking functions with pides 2^a1^b and the fixed parking functions with pides 2^b1^a.

2.4.2.1 The involution Φ

An *involution* f of a set S is a bijection from S to itself, such that $f^2=\mathrm{id}$ is the identity map. An element $s\in S$ such that $f(s)=s$ is called a *fixed point* of the involution.

Suppose that there is a weight function $w(s)$ of the elements s in the set S. A *sign-reversing involution* f of the set S (with respect to the weight w) is an involution such that for all $s \in S$, if $f(s) \neq s$, then $w(f(s)) = -w(s)$. As a consequence of a sign-reversing involution f, we have

$$\sum_{s\in S} w(s) = \sum_{s\in S} w(f(s)) = \sum_{s\in S, f(s)=s} w(s), \tag{2.52}$$

i.e. we only need to consider the fixed points of f when computing the total weight of the set S.

Note that by the straightening action on Schur functions, we have

$$s_{\lambda 13\mu} = -s_{\lambda 22\mu}, \tag{2.53}$$

where λ and μ are two compositions, and $\lambda 13\mu$ (or $\lambda 22\mu$) is the composition obtained by first listing all the parts in λ, then adding two parts of sizes 1 and 3 (or 2 and 2), finally listing all the parts in μ.

Let $\mathcal{PF}_{3,n|\lambda 13\mu}$ be the set of all the parking functions in $\mathcal{PF}_{3,n}$ with pides $\lambda 13\mu$ and $\mathcal{PF}_{3,n|\lambda 22\mu}$ be the set of all the parking functions in $\mathcal{PF}_{3,n}$ with pides $\lambda 22\mu$, then we can give an involution Φ of the set $\mathcal{PF}_{3,n|\lambda 13\mu} \cup \mathcal{PF}_{3,n|\lambda 22\mu}$, such that

- all the fixed points of Φ are in the set $\mathcal{PF}_{3,n|\lambda 22\mu}$, and
- the set of non-fixed points in $\mathcal{PF}_{3,n|\lambda 22\mu}$ are in bijection with the set $\mathcal{PF}_{3,n|\lambda 13\mu}$.

Let PF be a parking function in $\mathcal{PF}_{3,n}$. If $\text{pides(PF)} = \lambda 13\mu$, then without loss of generality, we suppose that the cars that cause pides 13 are $1,2,3,4$, which means that $\text{rank}(1) < \text{rank}(2) > \text{rank}(3) > \text{rank}(4)$. Then, there are 3 possible subwords (subsequences of the words of PF) formed by the 4 cars, which are

$$2341, \ \ 2314, \ \ 2134.$$

On the other hand, the cars $2,3,4$ are in different columns since $\text{rank}(2) > \text{rank}(3) > \text{rank}(4)$. Since there are only 3 columns, we have three possible placement of the four cars:

(I) Cars 1 and 4 are in the same column.

(II) Cars 1 and 2 are in the same column.

(III) Cars 1 and 3 are in the same column.

If the four cars form a word 2341, then (I), (II), (III) are all possible; if the four cars form a word 2314, then only (II), (III) are possible; if the four cars form a word 2134, then only (II) is possible.

Next, we consider the case when $\text{pides}(\text{PF}) = \lambda 13\mu$, i.e. the cars $1,2,3,4$ cause pides 22, and $\text{rank}(1) > \text{rank}(2) < \text{rank}(3) > \text{rank}(4)$. The possible words are

$$3412,\ 3142,\ 3124,\ 1324,\ 1342.$$

The cars $1,2$ and the cars $3,4$ have to be in different columns since $\text{rank}(1) > \text{rank}(2)$ and $\text{rank}(3) > \text{rank}(4)$, thus we have the following 5 possible placement of the four cars:

(i) Both cars $1,3$ and $2,4$ are in the same column.

(ii) Only cars 1 and 4 are in the same column.

(iii) Only cars 2 and 4 are in the same column.

(iv) Only cars 1 and 3 are in the same column.

(v) Only cars 2 and 3 are in the same column.

If the four cars form a word 3412, then (i), (ii), (iii), (iv) and (v) are all possible; if the four cars form a word 3142, then only (i), (iii), (iv) and (v) are possible; if the four cars form a word 3124, then only (iv) and (v) are possible; if the four cars form a word 1324, then only (v) is possible; if the four cars form a word 1342, then only (iii) and (v) are possible.

For any permutation $\sigma \in \mathcal{S}_n$ and any PF $\in \mathcal{PF}_{3,n}$, we let $\sigma \cdot$PF be the parking function obtained by permuting the cars of PF by the permutation σ. We also let word(PF) be the word of cars $1,2,3,4$. Then we can define the map

$$\Phi|_{\mathcal{PF}_{3,n|\lambda 13\mu}}: \quad \mathcal{PF}_{3,n|\lambda 13\mu} \quad \to \quad \mathcal{PF}_{3,n|\lambda 22\mu}.$$

Following is the detailed definition, while the words and the placements of the images are recorded in each case:

$$\Phi(\mathrm{PF}) = \begin{cases} (1,2)\mathrm{PF} & \text{if word(PF)} = 2341 \text{ and placement is (I).} \quad \Phi(\mathrm{PF}) \text{ has word 1342 (iii).} \\ (1,2,3)\mathrm{PF} & \text{if word(PF)} = 2341 \text{ and placement is (II).} \quad \Phi(\mathrm{PF}) \text{ has word 3142 (v).} \\ (1,2)\mathrm{PF} & \text{if word(PF)} = 2341 \text{ and placement is (III).} \; \Phi(\mathrm{PF}) \text{ has word 1342 (v).} \\ (1,2,3)\mathrm{PF} & \text{if word(PF)} = 2314 \text{ and placement is (II).} \quad \Phi(\mathrm{PF}) \text{ has word 3124 (v).} \\ (1,2)\mathrm{PF} & \text{if word(PF)} = 2314 \text{ and placement is (III).} \; \Phi(\mathrm{PF}) \text{ has word 1324 (v).} \\ (2,3)\mathrm{PF} & \text{if word(PF)} = 2134 \text{ and placement is (II).} \quad \Phi(\mathrm{PF}) \text{ has word 3124 (iv).} \end{cases} \tag{2.54}$$

Then we shall define the map Φ on the set $\mathcal{PF}_{3,n|\lambda 22\mu}$ that

$$\Phi|_{\mathcal{PF}_{3,n|\lambda 22\mu}}: \quad \mathcal{PF}_{3,n|\lambda 22\mu} \quad \to \quad \mathcal{PF}_{3,n|\lambda 13\mu} \cup \mathcal{PF}_{3,n|\lambda 22\mu}.$$

Notice that other than the fixed points, all the parking functions in $\mathcal{PF}_{3,n|\lambda 22\mu}$ are mapped into the set $\mathcal{PF}_{3,n|\lambda 13\mu}$. We define

$$\Phi(\mathrm{PF}) = \begin{cases} \mathrm{PF} & \text{if word(PF)} = 3412 \text{ or word(PF)} = 3142 \text{ and placement is (i),(iii),(iv).} \\ (1,3,2)\mathrm{PF} & \text{if word(PF)} = 3142 \text{ and placement is (v). } \Phi(\mathrm{PF}) \text{ has word 2341 (II).} \\ (2,3)\mathrm{PF} & \text{if word(PF)} = 3124 \text{ and placement is (iv). } \Phi(\mathrm{PF}) \text{ has word 2134 (II).} \\ (1,3,2)\mathrm{PF} & \text{if word(PF)} = 3124 \text{ and placement is (v). } \Phi(\mathrm{PF}) \text{ has word 2314 (II).} \\ (1,2)\mathrm{PF} & \text{if word(PF)} = 1324 \text{ and placement is (v). } \Phi(\mathrm{PF}) \text{ has word 2314 (III).} \\ (1,2)\mathrm{PF} & \text{if word(PF)} = 1342 \text{ and placement is (iii). } \Phi(\mathrm{PF}) \text{ has word 2341 (I).} \\ (1,2)\mathrm{PF} & \text{if word(PF)} = 1342 \text{ and placement is (v). } \Phi(\mathrm{PF}) \text{ has word 2341 (III).} \end{cases} \tag{2.55}$$

The first case in the definition above defines the fixed points of Φ.

It is easy to check that the map Φ does not change the area and the dinv of PF since Φ does not change the Dyck path of PF, and it also preserves the cars other than $\{1,2,3,4\}$. Since Φ changes the sign of the non-fixed points, it follows immediately that Φ forms a sign-reversing involution of the set of parking functions in $\mathcal{PF}_{3,n}$ with pideses of either $\lambda 13\mu$ or $\lambda 22\mu$. As we mentioned, the set of fixed points of this involution is

$$\mathrm{fp}(\Phi) = \{\mathrm{PF} \in \mathcal{PF}_{3,n|\lambda 22\mu} \ : \ \text{word(PF)} = 3412, \text{ or word(PF)} = 3142 \text{ (i), (iii), (iv)}\}.$$

If we apply the involution Φ to all the parking functions $\mathrm{PF} \in \mathcal{PF}_{3,n}$ that we compute pides(PF) and scan from left to right to find the first occurrence of either $(1,3)$ or non-fixed $(2,2)$ and apply Φ at that position. Then, the fixed points in $\mathcal{PF}_{3,n}$ have weakly decreasing pides in the form $3^a 2^b 1^c$, and these parking functions contribute to the coefficients of the Schur function bases. Thus, we have the Schur positivity of the $m = 3$ case that

$$[s_{2^a 1^b}]_{3,n} = \sum_{\substack{\mathrm{PF} \in \mathcal{PF}_{3,n},\ \mathrm{pides(PF)}=2^a 1^b, \\ \mathrm{PF\ fixed\ by\ }\Phi}} t^{\mathrm{area(PF)}} q^{\mathrm{dinv(PF)}}. \tag{2.56}$$

We will write *fixed parking functions* for the *parking functions fixed by* Φ.

2.4.2.2 The conjectured bijection implying $[s_{2^a 1^b}]_{3,n} = [s_{2^b 1^a}]_{3,3(a+b)-n}$

For any parking function $\mathrm{PF} \in \mathcal{PF}_{3,n}$ with $\mathrm{pides(PF)} = 2^a 1^b$, we are interested in the placement of the a pairs of numbers

$$\{(1,2),(3,4),\ldots,(2a-1,2a)\}$$

and the b singletons

$$\{2a+1,\ldots,2a+b\}.$$

Note that the two cars in each pair cannot be placed in the same column since the rank of the smaller car is bigger than the rank of the bigger car.

Since there are 3 columns, we have $\binom{3}{2}$ ways to choose columns for each pair $(2i-1,2i)$. We name the 3 columns from left to right by ℓ, c, r. Once we determine the 2 columns of the pair, the filling of the two cars in the pair is fixed by their ranks since $\mathrm{rank}(2i-1) > \mathrm{rank}(2i)$. Now, we define the notation for the placement of a pair $(2i-1,2i)$:

1. L means $(2i-1,2i)$ are in the left 2 columns ℓ, c,
2. R means $(2i-1,2i)$ are in the right 2 columns c, r,
3. C means $(2i-1,2i)$ are in columns ℓ, r.

Similarly, we have $\binom{3}{1}$ ways to choose a column for each singleton. For a singleton j, we define the notation for the placement:

1. L means j is in the left column ℓ,

2. R means j is in the right column r,

3. C means j is in column c.

Now we state our conjectured bijection between the fixed parking functions in $\mathcal{PF}_{3,n}$ with $\text{pides}(\text{PF}) = 2^a 1^b$ and the fixed parking functions in $\mathcal{PF}_{3,n'}$ with $\text{pides}(\text{PF}) = 2^b 1^a$, where $n' = 3(a+b) - n$.

Given a parking function $\text{PF} \in \mathcal{PF}_{3,n}$ with $\text{pides}(\text{PF}) = 2^a 1^b$ fixed by Φ, we track the placements of the a pairs of cars $\{(1,2),\ldots,(2a-1,2a)\}$ and b singleton cars $\{2a+1,\ldots,2a+b\}$. Let the $a+b$ placements of these $a+b$ objects be $p_1,\ldots,p_a,p_{a+1},\ldots,p_{a+b}$ (here p_i is one of L, R or C).

Then we consider b pairs of cars $\{(1,2),\ldots,(2b-1,2b)\}$ and a singleton cars $\{2b+1,\ldots,2b+a\}$. We assign the $b+a$ placements $p_{a+b},\ldots,p_1$ to the $b+a$ objects, then we build a new parking function $\mathbb{S}(\text{PF})$ by first counting how many cars in each column, then constructing the path according to the numbers of cars of the columns. Finally, we fill from first pair $(1,2)$ to last singleton $2b+a$ based on the rule that $\text{rank}(2i-1) < \text{rank}(2i)$ for $i \le b$ about the b pairs of cars and the column placement choice $p_{a+b},\ldots,p_1$. We call this map the switch map $\mathbb{S}$. Figure 2.12 shows an example that we can construct a parking function in $\mathcal{PF}_{3,5}$ with pides 21^3 from a parking function in $\mathcal{PF}_{3,7}$ with pides $2^3 1$.

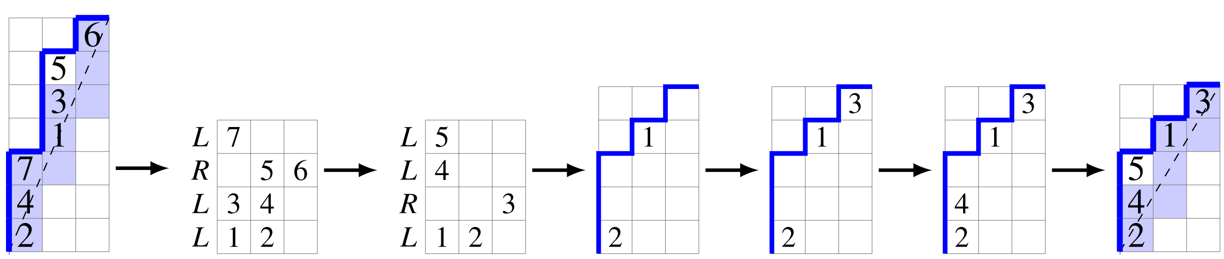


Figure 2.12: Bijection between $\mathcal{PF}_{3,7}$ with pides $2^3 1$ and $\mathcal{PF}_{3,5}$ with pides 21^3.

In order to show that the map $\mathbb{S}$ is a bijection, we need to prove several properties of this map. It is even not obvious that the image of a parking function is still above the diagonal, thus we shall show that

Theorem 2.12. *If* PF *is a* $(3,n)$*-parking function with pides* $2^a 1^b$*, then* $\mathbb{S}(\mathrm{PF})$ *is also a parking function.*

Proof. We still consider the a pairs of cars $\{(1,2),\ldots,(2a-1,2a)\}$ and b singleton cars $\{2a+1,\ldots,2a+b\}$ of PF. Suppose that there are ℓ_1, c_1, r_1 placements of the first a pairs of cars which are L, R and C, and ℓ_2, c_2, r_2 placements of the last b singleton cars which are L, R and C. Without loss of generality, we suppose that $n = 3k+1$. Then we have that

$$\ell_1 + c_1 + r_1 = a, \tag{2.57}$$

$$\ell_2 + c_2 + r_2 = b, \tag{2.58}$$

$$2a + b = 3k+1. \tag{2.59}$$

Since PF is a parking function, the path of the parking function should be above the diagonal, thus the number of cars in the left column is at least $k+1$ and the number of cars in the left two columns is at least $2k+1$.

Note that an L placement of a pair contribute 1 left car and 1 center car, a C placement of a pair contribute 1 left car and 1 right car, and an R placement of a pair contribute 1 right car and 1 center car. The contribution of the singleton cars are obvious. Thus the number of cars in the left column is $\ell_1 + c_1 + \ell_2$, and the number of cars in the left 2 columns is $2\ell_1 + r_1 + c_1 + \ell_2 + c_2 = a + \ell_1 + \ell_2 + c_2$, and we have that

$$\ell_1 + c_1 + \ell_2 \geq k+1, \tag{2.60}$$

$$a + \ell_1 + \ell_2 + c_2 \geq 2k+1. \tag{2.61}$$

Next, for $\mathbb{S}(\mathrm{PF})$, it has ℓ_2, c_2, r_2 placements of the first b pairs of cars, and ℓ_1, c_1, r_1

placements of the last a singleton cars. The total number of cars is equal to $2a+b = 3(a+b) - (2b+a) = 3(a+b) - 3k - 1 = 3(a+b-k-1)+2$, and the number of cars in the left column should be at least $a+b-k = (2a+b) - a - k = 3k+1-a-j = 2k+1-a$ and the number of cars in the left two columns should be at least $2a+2b-2k = b+(3k+1)-2k = b+k+1$. $\mathbb{S}(\mathrm{PF})$ is a parking function if the following is true:

$$\ell_1 + c_2 + \ell_2 \geq 2k+1-a, \tag{2.62}$$

$$b + \ell_1 + \ell_2 + c_1 \geq b+k+1. \tag{2.63}$$

Clearly, (2.60) implies (2.63), (2.61) implies (2.62). □

Next, we have the formula for area.

Theorem 2.13. *Let* PF *be a* $(3,n)$*-parking function with pides* $2^a 1^b$*. Using the definition of* $\ell_1, c_1, r_1, \ell_2, c_2, r_2$ *in the proof of Theorem 2.12. Let* $L = \ell_1 + \ell_2, R = r_1 + r_2, C = c_1 + c_2$*, then*

$$\mathrm{area}(\mathrm{PF}) = L - R - 1. \tag{2.64}$$

Proof. We want to compute the area of a parking function as the difference of its coarea and the maximum coarea of a $(3, 2a+b)$-parking function. The maximum coarea of a $(3, 2a+b)$-parking function is equal to $\frac{(2a+b-1)(3-1)}{2} = 2a+b-1$.

Notice that the cars in the right column contribute 2 to coarea, and the cars in the center column contribute 1 to coarea, thus the coarea of PF is

$$\ell_1 + 3r_1 + 2c_1 + 2r_2 + c_2 = a + 2(r_1 + r_2) + (c_1 + c_2) = a + 2R + C. \tag{2.65}$$

Then,

$$\mathrm{area}(\mathrm{PF}) = 2a+b-1-(a+2R+C) = a+(L+R+C)-1-(a+2R+C) = L-R-1. \tag{2.66}$$

□

It follows immediately from Theorem 2.13 that

Theorem 2.14. *For any* $\mathrm{PF} \in \mathcal{PF}_{3,n}$ *with* $\mathrm{pides(PF)} = 2^a 1^b$,

$$\mathrm{area(PF)} = \mathrm{area}(\mathbb{S}(\mathrm{PF})). \tag{2.67}$$

We have not yet proved, but verified all parking functions with less than or equal to 10 rows for the following conjecture:

Conjecture 2.3. *For any* $\mathrm{PF} \in \mathcal{PF}_{3,n}$ *with* $\mathrm{pides(PF)} = 2^a 1^b$,

(a) $\mathrm{dinv(PF)} = \mathrm{dinv}(\mathbb{S}(\mathrm{PF}))$.

(b) If PF *is a fixed point of the map* Φ*, then so is* $\mathbb{S}(\mathrm{PF})$*, and* $\mathrm{pides}(\mathbb{S}(\mathrm{PF})) = 2^b 1^a$.

By (2.56), it follows from Conjecture 2.3 immediately that

$$[s_{2^a 1^b}]_{3,n} = [s_{2^b 1^a}]_{3,3(a+b)-n}.$$

2.4.2.3 The switch map $\mathbb{S}$ in the m column case

We haven't completely understood how to use straightening to compute the coefficients of s_λ for general (m,n) case, but computations in Maple have led us to conjecture the following:

Conjecture 2.4. *For all* $m,n > 0$ *and* $\alpha_i \geq 0$,

$$[s_{(m-1)^{\alpha_{m-1}}(m-2)^{\alpha_{m-2}}\cdots 1^{\alpha_1}}]_{m,n} = [s_{(m-1)^{\alpha_1}(m-2)^{\alpha_2}\cdots 1^{\alpha_{m-1}}}]_{m,(m\sum_{i=1}^{m-1}\alpha_i - n)}. \tag{2.68}$$

On the other hand, the switch map $\mathbb{S}$ that we have defined for the three column case can be naturally generalized to the m column case, which conjecturally has many nice properties and is considered to be useful in proving Conjecture 2.4. The definition of an m columns switch map will need some new definitions.

Given any parking function PF $\in \mathcal{PF}_{m,n}$, we suppose that $s, s+1, \ldots, s+r-1$ is an increasing subsequence of the word $\sigma(\text{PF})$, then by Remark 1.1, the cars $s, s+1, \ldots, s+r-1$ must be placed in r different columns in a rank decreasing way.

There are $\binom{m}{r}$ possible choices to pick r columns for such cars $s, s+1, \ldots, s+r-1$. Let $p = \{s_1, s_2, \ldots, s_r\} \subset \{1, \ldots, m\}$ be a possible placement, then we define the *reverse complement* of p to be $p^{rc} = \{1, \ldots, m\} \backslash \{m+1-s_r, \ldots, m+1-s_1\}$, which is a placement for $m-r$ cars.

Given $\mu = \mu_1 \cdots \mu_k \models n$. Now suppose that $\sigma(\text{PF})$, the word of PF, is a shuffle of the increasing sequences $(1, \ldots, \mu_1), (\mu_1+1, \ldots, \mu_1+\mu_2), \ldots, (n-\mu_k+1, \ldots, n)$, and the placement (i.e. the choice of columns) of the sequence $(\mu_1+\ldots+\mu_{i-1}+1, \ldots, \mu_1+\ldots+\mu_i)$ is p_i, then we construct $\mathbb{S}(\text{PF})$ as follows:
let $\mu^\vee = \mu_1^\vee \cdots \mu_k^\vee$, where $\mu_i^\vee = m - \mu_{k+1-i}$. We make the word of $\mathbb{S}(\text{PF})$ to be a shuffle of $(1, \ldots, \mu_1^\vee), (\mu_1^\vee+1, \ldots, \mu_1^\vee+\mu_2^\vee), \ldots, (n-\mu_k^\vee+1, \ldots, n)$, and the placement of $(\mu_1^\vee+\ldots+\mu_{i-1}^\vee+1, \ldots, \mu_1^\vee+\ldots+\mu_i^\vee)$ is p_{k+1-i}^{rc}. This construction is well defined, and for each given composition μ, we can invert the the map easily.

For example, suppose that there are $m=4$ columns. Take $\mu = (2,2,3) \models n$ where $n=7$. For a $(4,7)$-parking function PF whose word $\sigma(\text{PF}) = 5613472$ is a shuffle of $(1,2), (3,4)$, $(5,6,7)$ with placements $\{1,3\}, \{1,2\}, \{1,2,4\}$, we construct $\mathbb{S}(\text{PF})$ such that its word is a shuffle of $(1), (2,3), (4,5)$ and the placements are $\{2\}, \{1,2\}, \{1,3\}$, shown in Figure 2.13.

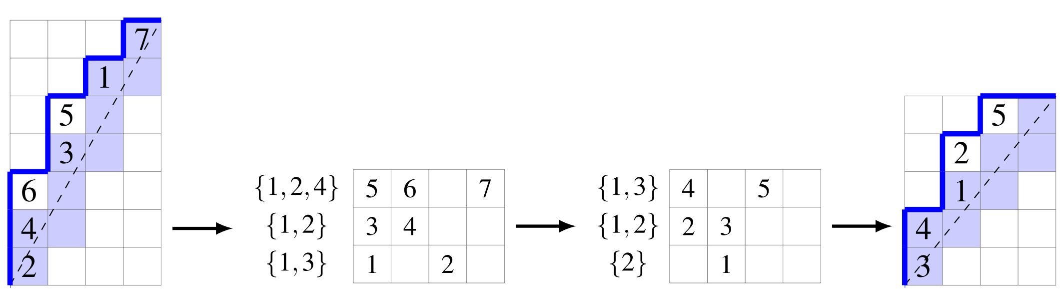


Figure 2.13: An example of PF and $\mathbb{S}(\text{PF})$.

Like the 3 column case, we have:

Theorem 2.15. PF *is an* (m,n)*-parking function if and only if* $\mathbb{S}(\mathrm{PF})$ *is a parking function. Further,*

$$\mathrm{area}(\mathrm{PF}) = \mathrm{area}(\mathbb{S}(\mathrm{PF})).$$

For a composition $\mu = \mu_1 \cdots \mu_k \models n$, we say a permutation σ is a shuffle of μ if σ is a shuffle of the increasing sequences $(1,\ldots,\mu_1),(\mu_1+1,\ldots,\mu_1+\mu_2),\ldots,(n-\mu_k+1,\ldots,n)$. Then we have:

Theorem 2.16. *The switch map* $\mathbb{S}$ *is a bijection between* (m,n)*-parking functions whose words are shuffle of* $\mu = \mu_1 \cdots \mu_k$ *and* $(m,mk-n)$*-parking functions whose words are shuffle of* $\mu^{\vee} = (m-\mu_k)\cdots(m-\mu_1)$.

The switch map of m column case still keeps the dinv statistic experimentally (summarized in the following conjecture), which we are not able to prove.

Conjecture 2.5. *For any* $\mathrm{PF} \in \mathcal{PF}_{m,n}$ *where* $\sigma(\mathrm{PF})$ *is a shuffle of* $\mu \models n$,

$$\mathrm{dinv}(\mathrm{PF}) = \mathrm{dinv}(\mathbb{S}(\mathrm{PF})).$$

Thus conjecturally, the switch map $\mathbb{S}$ is an *area,dinv-preserving bijective map* between (m,n)-parking functions whose words are shuffle of μ and $(m,mk-n)$-parking functions whose words are shuffle of $\mu^{\vee}$. In the end, we shall discuss a consequence of Conjecture 2.5 and the switch map $\mathbb{S}$.

Referring to Haglund's work in [Hag08], for any parking function whose word is a shuffle of $\mu = \mu_1 \cdots \mu_k \models n$, we can replace the cars $\mu_1 + \ldots + \mu_{i-1} + 1, \ldots, \mu_1 + \ldots + \mu_i$ with number i to obtain a parking function with cars $1^{\mu_1} \cdots k^{\mu_k}$ with the same area and dinv statistics. Further, it is a fact that

$$\mathrm{Q}_{m,n}(-1)^n\Big|_{m_\mu} = \mathrm{H}_{m,n}[X;q,t]\Big|_{m_\mu} = \sum_{\mathrm{PF}\in\mathcal{PF}_{m,n},\ \sigma(\mathrm{PF})\text{ is a shuffle of }\mu} t^{\mathrm{area(PF)}} q^{\mathrm{dinv(PF)}}. \tag{2.69}$$

By definition of the Hall scalar product, for any symmetric function f, we have

$$\langle f, h_\mu\rangle = f\big|_{m_\mu}. \tag{2.70}$$

Thus, the properties of the switch map $\mathbb{S}$ (Theorem 2.15, Theorem 2.16 and Conjecture 2.5) imply the following identities:

Conjecture 2.6. *For $m,n>0$, $\mu=\mu_1\cdots\mu_k\vdash n$ and $\mu^\vee=(m-\mu_k)\cdots(m-\mu_1)$,*

$$\langle \mathrm{Q}_{m,n}(-1)^n, h_\mu\rangle = \langle \mathrm{Q}_{m,mk-n}(-1)^{mk-n}, h_{\mu^\vee}\rangle, \tag{2.71}$$

$$\langle \mathrm{H}_{m,n}[X;q,t], h_\mu\rangle = \langle \mathrm{H}_{m,mk-n}[X;q,t], h_{\mu^\vee}\rangle. \tag{2.72}$$

Since the area-preserving property of $\mathbb{S}$ has been proved in Theorem 2.15, we have the following theorem which is a special case of Conjecture 2.6 at $q=1$:

Theorem 2.17. *For $m,n>0$, $\mu=\mu_1\cdots\mu_k\vdash n$ and $\mu^\vee=(m-\mu_k)\cdots(m-\mu_1)$,*

$$\langle \mathrm{Q}_{m,n}(-1)^n\big|_{q=1}, h_\mu\rangle = \langle \mathrm{Q}_{m,mk-n}(-1)^{mk-n}\big|_{q=1}, h_{\mu^\vee}\rangle, \tag{2.73}$$

$$\langle \mathrm{H}_{m,n}[X;q,t]\big|_{q=1}, h_\mu\rangle = \langle \mathrm{H}_{m,mk-n}[X;q,t]\big|_{q=1}, h_{\mu^\vee}\rangle. \tag{2.74}$$

The majority of Chapter 2 has been submitted for publication. Qiu, Dun; Remmel, Jeffrey Brian. “Schur function expansions and the Rational Shuffle Conjecture (full version)", available in Mathematics arXiv:1806.04348v2. An extended abstract of this work has been published in the Proceedings of Formal Power Series and Algebraic Combinatorics 2017. Qiu, Dun; Remmel, Jeffrey Brian. “Schur function expansions and the Rational Shuffle Conjecture",

Séminaire Lotharingien de Combinatoire, vol. 78B, 2017. The dissertation author was the primary investigator and author of this work.

Chapter 3

The Schur positivity of $\Delta_{e_2} e_n$

Haglund, Remmel and Wilson [HRW18] have conjectured that the coefficient of any Schur function s_λ in $\Delta_{e_k} e_n$ is a polynomial in $\mathbb{N}[q,t]$. In this section, we give a combinatorial proof in the case when $k = 2$ that the coefficient of s_λ in $\Delta_{e_2} e_n$ has a non-negative expansion in terms of q,t-analogues.

3.1 Introduction

We have introduced the Delta Conjecture of Haglund, Remmel and Wilson [HRW18] about the expression $\Delta'_{e_k} e_n$ in Section 1.3.3. There is another version of the Delta Conjecture, which is about the expression $\Delta_{e_k} e_n$ also due to the work of Haglund, Remmel and Wilson in [HRW18] that

Conjecture 3.1 (Haglund, Remmel and Wilson)**.** *For any integers $n \geq k \geq 0$,*

$$\begin{aligned} \Delta_{e_k} e_n &= \sum_{\mathrm{PF} \in \mathcal{PF}_n} t^{\mathrm{area(PF)}} q^{\mathrm{dinv(PF)}} F_{\mathrm{ides(PF)}} (1+z) \prod_{i \in \mathrm{Rise(PF)}} (1 + \frac{z}{t^{a_i(\mathrm{PF})}}) \Big|_{z^{n-k-1}} \\ &= \sum_{\mathrm{PF} \in \mathcal{PF}_n} t^{\mathrm{area(PF)}} q^{\mathrm{dinv(PF)}} F_{\mathrm{ides(PF)}} (1+z) \prod_{i \in \mathrm{Val(PF)}} (1 + \frac{z}{q^{d_i(\mathrm{PF})+1}}) \Big|_{z^{n-k-1}}. \end{aligned}$$

They also conjectured that coefficients in the Schur function expansion of $\Delta_{e_k} e_n$ are polynomials in q,t with non-negative integer coefficients. There are two cases that are known. Namely, when $k=n$, the expression $\Delta_{e_n} e_n = \nabla e_n$ is proved by Haiman [Hai02] to be the Frobenius characteristic of the ring of diagonal harmonics, as we have mentioned in Section 1.3.1. Thus in this case, representation theory tells us that the coefficient of the Schur function s_λ, $\langle \nabla e_n, s_\lambda \rangle$, is a polynomial in q,t with non-negative integer coefficients.

The other known case is when $k=1$. In [HRW18], Haglund, Remmel and Wilson proved that

$$\Delta_{e_1} e_n = \sum_{m=0}^{\lfloor n/2 \rfloor} s_{2^m,1^{n-2m}} \sum_{p=m}^{n-m} [p]_{q,t}. \tag{3.1}$$

The main goal of this chapter is to give a proof of the fact that $\Delta_{e_2} e_n$ is Schur positive, i.e. for all $\lambda \vdash n$, $\langle \Delta_{e_2} e_n, s_\lambda \rangle \in \mathbb{N}[q,t]$, in hopes that some of the ideas in the proof can be adapted to prove the Schur positivity of $\Delta_{e_k} e_n$ for $k \geq 3$.

Our proof starts with the following result of Haglund [Hag04].

Lemma 3.1. *For all integers $n,d>0$ and symmetric functions $F[X]$,*

$$\langle \Delta_{e_{d-1}} e_n,\ F \rangle = \langle \Delta_{\omega F}\, e_d,\ s_d \rangle. \tag{3.2}$$

Let λ be any partition of n. By setting $F = s_\lambda$, we have

$$\langle \Delta_{e_{d-1}} e_n,\ s_\lambda \rangle = \langle \Delta_{s_{\lambda'}} e_d,\ s_d \rangle. \tag{3.3}$$

The formula works nicely when d is small, since we can compute an explicit expansion of e_d in terms of modified Macdonald polynomials. In the case when $d=2$ we have

$$e_2 = \frac{1}{t-q} \widetilde{H}_{1,1}[X;q,t] - \frac{1}{t-q} \widetilde{H}_2[X;q,t].$$

This leads to

$$\begin{aligned}\langle \Delta_{e_1} e_n,\, s_\lambda \rangle &= \langle \Delta_{s_{\lambda'}} e_2,\, s_2 \rangle \\ &= \left\langle \frac{1}{t-q} s_{\lambda'}[1+t] \widetilde{H}_{1,1}[X;q,t] - \frac{1}{t-q} s_{\lambda'}[1+q] \widetilde{H}_2[X;q,t],\, s_2 \right\rangle \\ &= \frac{1}{t-q} s_{\lambda'}[1+t] - \frac{1}{t-q} s_{\lambda'}[1+q],\end{aligned}$$

which is easily seen to be an element of $\mathbb{N}[q,t]$.

In the case when $d=3$, the expansion of e_3 leads to the following formula:

$$\begin{aligned} g_\lambda &:= \langle \Delta_{e_2} e_n,\, s_\lambda \rangle \\ &= \frac{(t-q^2) s_{\lambda'}[1+t+t^2] - (q+t+1)(t-q) s_{\lambda'}[1+q+t] + (t^2-q) s_{\lambda'}[1+q+q^2]}{(t-q)(t^2-q)(t-q^2)}. \end{aligned} \tag{3.4}$$

At first glance, this formula does not seem to be useful. Indeed, it is not immediately obvious that this quotient is a polynomial. Our approach to proving that g_λ is in $\mathbb{N}[q,t]$ relies on the following alternative representation of g_λ.

Lemma 3.2. *Let τ be the operation which switches t and q. Then*

$$g_\lambda = \langle \Delta_{e_2} e_n,\, s_\lambda \rangle = \frac{F_{\lambda'} - \tau F_{\lambda'}}{t-q} = \frac{\mathrm{id} - \tau}{t-q} F_{\lambda'}, \tag{3.5}$$

where $\tau F = F\big|_{q=t,t=q}$ and

$$F_{\lambda'} = \frac{s_{\lambda'}[1+t+t^2] - s_{\lambda'}[1+t+q]}{t^2-q}. \tag{3.6}$$

Proof. By using the formula

$$(t-q)(1+q+t) = (t-q^2) - (q-t^2) = (\mathrm{id}-\tau)(t-q^2),$$

Equation (3.4) becomes

$$\begin{aligned}\langle \Delta_{e_2} e_n,\, s_\lambda \rangle &= \frac{(\mathrm{id}-\tau)(t-q^2)s_{\lambda'}[1+t+t^2] - (\mathrm{id}-\tau)(t-q^2)s_{\lambda'}[1+q+t]}{(t-q)(t^2-q)(t-q^2)} \\ &= \frac{1}{t-q}(\mathrm{id}-\tau)\left(\frac{s_{\lambda'}[1+t+t^2] - s_{\lambda'}[1+t+q]}{t^2-q}\right).\end{aligned}$$

This is just the desired Equation (3.5). □

From this formula, it is clear that g_λ is in $\mathbb{Z}[q,t]$ where $\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$ is the set of integers. In Section 3.2, we present our proof that g_λ is in $\mathbb{N}[q,t]$ by directly computing g_λ by breaking g_λ into a sum of terms where each term is easily seen to be a polynomial in q,t with non-negative coefficients. By this proof, we can recursively produce explicit formulas for g_λ.

In Section 3.3, we give a formula of $\Delta_{e_3} e_n$. However, it is not clear how we can split up this formula into polynomials in $\mathbb{N}[q,t]$. Thus, the general problem of establishing the Schur positivity of $\Delta_{e_k} e_n$ seems to require new ideas.

3.2 Proof of $g_\lambda \in \mathbb{N}[q,t]$ by direct computation

To show that $g_\lambda \in \mathbb{N}[q,t]$, it is sufficient to show that g_λ has a non-negative q,t-analogue expansion, which is a stronger condition than $g_\lambda \in \mathbb{N}[q,t]$. For instance, $q^2+t^2 \in \mathbb{N}[q,t]$, but $q^2+t^2 = [3]_{q,t} - qt\,[1]_{q,t}$ does not have non-negative q,t-analogue expansion.

In this section, we shall give a recursive formula for g_λ for any $\lambda \vdash n$ to show that g_λ has non-negative q,t-analogue expansion.

3.2.1 Preliminaries

First, we should mention the fact that

$$\frac{(\mathrm{id}-\tau)t^j q^i}{t-q} = -\frac{(\mathrm{id}-\tau)t^i q^j}{t-q} \text{ and } \frac{(\mathrm{id}-\tau)t^i q^j}{t-q} = (tq)^j [i-j]_{q,t}, \quad \text{if } i \geq j.$$

Recall that by plethystic notation, we have

$$s_{\lambda'}[x_0+x_1+x_2] := \sum_T x_0^{\omega_0(T)} x_1^{\omega_1(T)} x_2^{\omega_2(T)},$$

where the sum is over all semi-standard Young tableaux T of shape λ' filled with numbers $0,1,2$, and $\omega_i(T)$ is the number of i's in T. Generic semi-standard Young tableaux T of shape λ' are pictured in Figure 3.1. There is no semi-standard Young tableau with fillings $0,1,2$ of more than three rows, thus

$$s_{\lambda'}[x_0+x_1+x_2] = 0 \quad \text{if } \ell(\lambda') > 3.$$

Recall that by Lemma 3.2, we have the following formula for g_λ:

$$\begin{aligned} g_\lambda &= \frac{\mathrm{id}-\tau}{t-q}\,\frac{s_{\lambda'}[1+t+t^2]-s_{\lambda'}[1+t+q]}{t^2-q} \\ &= \frac{s_{\lambda'}[1+t+t^2]-s_{\lambda'}[1+t+q]}{(t-q)(t^2-q)}+\frac{s_{\lambda'}[1+q+q^2]-s_{\lambda'}[1+q+t]}{(t-q)(t-q^2)}. \end{aligned}$$

It follows that $g_\lambda = 0$ if λ' has more than 3 rows. Thus we only consider g_λ where λ' has 3 or fewer rows.

We let $\mathrm{SSYT}(\lambda',012)$ denote the set of all semi-standard Young tableaux T of shape λ' with cells filled by $\{0,1,2\}$. Given a semi-standard Young tableau $T \in \mathrm{SSYT}(\lambda',012)$, we suppose that T has ω_1 1's and ω_2 2's. Then we define the *weight* of T to be

$$\begin{aligned} g_T &:= \frac{t^{\omega_1+2\omega_2}-t^{\omega_1}q^{\omega_2}}{(t-q)(t^2-q)}+\frac{q^{\omega_1+2\omega_2}-q^{\omega_1}t^{\omega_2}}{(t-q)(t-q^2)} \\ &= \frac{t^{\omega_1}[\omega_2]_{t^2,q}-q^{\omega_1}[\omega_2]_{q^2,t}}{t-q}. \end{aligned}$$

Then it is clear that

$$g_\lambda = \sum_{T\in\mathrm{SSYT}(\lambda',012)} g_T.$$

We will use the weight g_T to deduce a formula for g_λ. Note that the weight of T only depends on the numbers of 1's and 2's it contains. We shall use the notation

$$w(\omega_1,\omega_2) := \frac{t^{\omega_1}[\omega_2]_{t^2,q} - q^{\omega_1}[\omega_2]_{q^2,t}}{t-q},$$

thus a tableau $T \in \mathrm{SSYT}(\lambda', 012)$ with ω_1 1's and ω_2 2's has weight

$$g_T = w(\omega_1,\omega_2).$$

Given a tableau $T \in \mathrm{SSYT}(\lambda', 012)$, we can write T in 4 parts as shown in Figure 3.1: a_1 is the part with 3 rows, k_1 is the part with two rows and the bottom row is filled with 0's, a_2 is the part with two rows and the bottom row is filled with 1's, k_2 is the part with one row and the fillings are not 0. If there is no a_2 part, there can be a part called a_0 at the same place as a_2 which consists of one row filled with 0's.

In our weighting scheme for $T \in \mathrm{SSYT}(\lambda', 012)$ given below, the weight of any 0 will be 1. Hence a_0 will not contribute anything to g_T and we will not consider a_0 in our formulas. We define the set $S_\lambda[a_1,k_1,a_2,k_2]$ to be the collections of T's having the *part composition* $[a_1,k_1,a_2,k_2]$. Since a_1 and a_2 have the same kind of contribution to the formula, we can define

$$g_\lambda[a_1+a_2,k_1,k_2] := \sum_{T\in S_\lambda[a_1,k_1,a_2,k_2]} g_T.$$

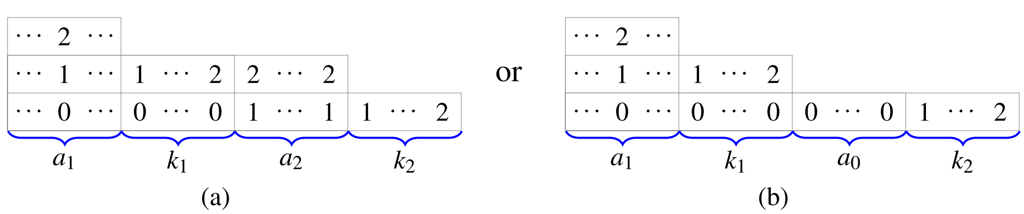


Figure 3.1: $T \in \mathrm{SSYT}(\lambda', 012)$.

Clearly for any $\lambda \vdash n$, g_λ can be expressed as a sum of $g_\lambda[a,k_1,k_2]$'s by classifying the tableaux in $\mathrm{SSYT}(\lambda',012)$ by part compositions $[a_1,k_1,a_2,k_2]$. We will deduce a formula for $g_\lambda[a,k_1,k_2]$, which will in turn allow us to compute an explicit formula for g_λ.

3.2.2 The computation of $g_\lambda[a,k_1,k_2]$

3.2.2.1 A formula for $g_\lambda[0,0,k]$

The set $S[0,0,0,k]$ contains the tableaux T of shape $\underbrace{\boxed{1\cdots 2}}_{k}$. If there are i 1's, then there will be $k-i$ 2's. Thus we have the following theorem:

Theorem 3.3. $g_\lambda[0,0,1]=0$. *For* $k\geq 2$,

$$g_\lambda[0,0,k] = \sum_{i=0}^{\lfloor (2k-2)/3\rfloor - \chi(k=1(\mathrm{mod}3))} (qt)^i \left[k-i-\lfloor \tfrac{i+1}{2}\rfloor \to 2k-2-3i\right]_{q,t}. \tag{3.7}$$

Proof. It is easy to see by direct computation that $g_\lambda[0,0,1]=0$. Next observe that for any $r\geq 1$, $\omega(r,0)=0$. Thus we only need to consider the cases where there is at least one 2 in the tableau. It follows that

$$\begin{aligned}
g_\lambda[0,0,k] &= \sum_{T\in S[0,0,0,k]} g_T \\
&= \sum_{i=0}^{k-1} w(i,k-i) \\
&= \sum_{i=0}^{k-1} \frac{t^i[k-i]_{t^2,q} - q^i[k-i]_{q^2,t}}{t-q} \\
&= \sum_{i=0}^{k-1} \frac{\sum_{j=0}^{k-1-i} t^{2k-2j-i-2}q^j - q^{2k-2j-i-2}t^j}{t-q} \\
&= \sum_{i=0}^{k-1}\sum_{j=0}^{k-1-i} (qt)^j [2k-3j-i-2]_{q,t}.
\end{aligned}$$

Now let $A^{(k)}_{i,j} = (qt)^j[2k-3j-i-2]_{q,t}$. We have the array $\{A^{(8)}_{i,j} : 0 \le i \le 7 \ \&\ 0 \le j \le i\}$ pictured in Figure 3.2 as an example. In general, if one looks at the first row of $A^{(k)}_{i,j}$ which is the sequence $((qt)^j[2k-3j-2]_{q,t})$, the terms will be non-negative if $2k-2 \ge 3j$, or equivalently, if $j \le \lfloor (2k-2)/3 \rfloor$. We shall show that for any negative terms in the first row of the form $(qt)^j[-m]$, the first $m+1$ terms along the anti-diagonal starting at that position will sum to 0. This will leave us only with positive terms corresponding to sum stated in the theorem.

For example, in Figure 3.2, one can easily compute that the sum of the first two terms of the anti-diagonal starting at the term $(qt)^5[-1]_{q,t}$ equals 0, the sum of the first five terms of the anti-diagonal starting at the term $(qt)^6[-4]_{q,t}$ equals 0, and the sum of the first eight terms of the anti-diagonal starting at the term $(qt)^7[-7]_{q,t}$ equals 0. These are the terms corresponding to the green, blue and red diagonals respectively. In this case, we see that $g_\lambda[0,0,8]$ equals

$$[8 \to 14]_{q,t} + qt[6 \to 11]_{q,t} + (qt)^2[5 \to 8] + (qt)^3[3 \to 5] + (qt)^4[2 \to 2],$$

which are exactly the terms predicted by the theorem.

$i \backslash j$	0	1	2	3	4	5	6	7
0	$[14]_{q,t}$	$(qt)^1[11]_{q,t}$	$(qt)^2[8]_{q,t}$	$(qt)^3[5]_{q,t}$	$(qt)^4[2]_{q,t}$	$(qt)^5[-1]_{q,t}$	$(qt)^6[-4]_{q,t}$	$(qt)^7[-7]_{q,t}$
1	$[13]_{q,t}$	$(qt)^1[10]_{q,t}$	$(qt)^2[7]_{q,t}$	$(qt)^3[4]_{q,t}$	$(qt)^4[1]_{q,t}$	$(qt)^5[-2]_{q,t}$	$(qt)^6[-5]_{q,t}$	
2	$[12]_{q,t}$	$(qt)^1[9]_{q,t}$	$(qt)^2[6]_{q,t}$	$(qt)^3[3]_{q,t}$	$(qt)^4[0]_{q,t}$	$(qt)^5[-3]_{q,t}$		
3	$[11]_{q,t}$	$(qt)^1[8]_{q,t}$	$(qt)^2[5]_{q,t}$	$(qt)^3[2]_{q,t}$	$(qt)^4[-1]_{q,t}$			
4	$[10]_{q,t}$	$(qt)^1[7]_{q,t}$	$(qt)^2[4]_{q,t}$	$(qt)^3[1]_{q,t}$				
5	$[9]_{q,t}$	$(qt)^1[6]_{q,t}$	$(qt)^2[3]_{q,t}$					
6	$[8]_{q,t}$	$(qt)^1[5]_{q,t}$						
7	$[7]_{q,t}$							

Figure 3.2: The table of $A^{(8)}_{i,j}$.

The proof requires a careful case by case analysis by considering the parity of k modulo 3. Note that

1. if $k=3t$, then $\lfloor(2k-2)/3\rfloor = 2t-1$,

2. if $k=3t+1$, then $\lfloor(2k-2)/3\rfloor = 2t$, and

3. if $k=3t+2$, then $\lfloor(2k-2)/3\rfloor = 2t$.

Case 1. $k=3t$.

The negative terms in the first row are

$$(qt)^{2t-1+s}[6t-2-3(2t-1+s)]_{q,t} = (qt)^{2t-1+s}[-3s+1]_{q,t}$$

for $s=1,\ldots,t$. In particular, the last term in the first row equals $(qt)^{3t-1}[-3t+1]$ and the first negative term is $A^{(3t)}_{0,2t} = (qt)^{2t}[-2]$.

Then we have two subcases depending on whether s is even or odd.

Subcase 1.1. $s=2r$.

In this case, $A^{(3t)}_{0,2t-1+2r} = q^{2t+2r-1}[-6r+1]_{q,t}$. We claim that $\sum_{a=0}^{6r-1} A^{(3t)}_{a,2t-1+2r-a} = 0$. We shall prove this by showing that for all $0 \le a \le 3r-1$,

$$A^{(3t)}_{a,2t-1+2r-a} = -A^{(3t)}_{6r-1-a,2t-1+2r-(6r-1-a)} = -A^{(3t)}_{6r-1-a,2t-4r+a}.$$

Note that

$$\begin{aligned} A^{(3t)}_{a,2t-1+2r-a} &= (qt)^{2t-1+2r-a}[6t-2-a-3(2t-1+2r-a)]_{q,t} \\ &= (qt)^{2t-1+2r-a}[-6r+1+2a]_{q,t} \\ &= -(qt)^{2t-1+2r-a-(6r-1-2a)}[6r-1+2a]_{q,t} \\ &= -(qt)^{2t-4r+a}[6r-1+2a]_{q,t}. \end{aligned}$$

On the other hand,

$$A^{(3t)}_{6r-1-a,2t-4r+a} = (qt)^{2t-4r+a}[6t-2-(6r-1-a)-3(2t-4r+a)]_{q,t} = (qt)^{2t-4r+a}[6t-1+2a]_{q,t}$$

as desired.

Subcase 1.2. $s = 2r+1$.

In this case, $A^{(3t)}_{0,2t-1+2r+1} = q^{2t+2r-1}[-6r-2]_{q,t}$. We claim that $\sum_{a=0}^{6r+2} A^{(3t)}_{a,2t+2r-a} = 0$. First note that

$$A^{(3t)}_{3r+1,2t+2r-(3r+1)} = A^{(3t)}_{3r+1,2t-r-1} = (qt)^{2t-r-1}[6t-2-(3r+1)-3(2t-r-1)]_{q,t} = (qt)^{2t-r-1}[0]_{q,t} = 0.$$

Thus we can prove our claim if we show that if $0 \leq a \leq 3r$,

$$A^{(3t)}_{a,2t+2r-a} = -A^{(3t)}_{6r+2-a,2t+2r-(6r+2-a)} = -A^{(3t)}_{6r+2-a,2t-4r-2+a}.$$

Note that

$$\begin{aligned} A^{(3t)}_{a,2t+2r-a} &= (qt)^{2t+2r-a}[6t-2-a-3(2t+2r-a)]_{q,t} \\ &= (qt)^{2t+2r-a}[-6r-2+2a]_{q,t} \\ &= -(qt)^{2t+2r-a-(6r+2-2a)}[6r+2-2a]_{q,t} \\ &= -(qt)^{2t-4r-2+a}[6r+2-2a]_{q,t}. \end{aligned}$$

On the other hand,

$$\begin{aligned} A^{(3t)}_{6r+2-a,2t-4r-2+a} &= (qt)^{2t-4r-2+a}[6t-2-(6r+2-a)-3(2t-4r-2+a)]_{q,t} \\ &= (qt)^{2t-4r+a}[6r+2-2a]_{q,t}. \end{aligned}$$

Observe that the bottom term of the r-th column of the array $\{A^{(3t)}_{i,j}\}_{i=0,\ldots,3t-1\,\&\,0\leq j\leq i}$ is $A^{(3t)}_{3t-1-r,r}$. Our computations above show that in the array $\{A^{(3t)}_{i,j}\}_{i=0,\ldots,3t-1\,\&\,0\leq j\leq i}$, the first $3s$ terms of any anti-diagonal starting at $A^{(3t)}_{0,2t-1+s}$ sum to 0 for $s = 1,\ldots,t$. This means that the corresponding terms in the array make no contribution to $g_\lambda[0,0,k]$. It follows that we can ignore all the terms in columns $2t,\ldots,3t-1$. Note that the first $3t$ terms of the anti-diagonal starting at $A^{(3t)}_{0,3t-1}$ cancel out the bottom term in each column. Next, the first $3t-3$ terms of the anti-diagonal starting at $A^{(3t)}_{0,3t-2}$ reach only to column 2 so they will cancel out the next to last terms in columns $2,\ldots,2t-1$. Then the first $3t-6$ terms of the anti-diagonal starting at $A^{(3t)}_{0,3t-3}$ reach only to column 4 so they will cancel out the second to last terms in columns $4,\ldots,2t-1$. Continuing on in this way, we finally see that the 3 anti-diagonal terms starting at $A^{(3t)}_{0,2t}$ will only cancel out terms in columns $2t-2$ and $2t-1$. It follows that for $r = 0,\ldots,t-1$, we can ignore that last $r+1$ terms in columns $2r$ and $2r+1$. This means that if $0 \leq r \leq t-1$, the lowest term that can contribute to $g_\lambda[0,0,k]$ in column $2r$ is

$$\begin{aligned} A^{(3t)}_{3t-1-2r-(r+1),2r} &= A^{(3t)}_{3t-3r-2,2r} = (qt)^{2r}[6t-2-(3t-3r-2)-3(2r)]_{q,t} \\ &= (qt)^{2r}[3t-3r]_{q,t} = [3t-(2r)-\lfloor 2r+1/2 \rfloor]_{q,t}. \end{aligned}$$

Note that the top element in column $2r$ is $A^{(3t)}_{0,2r} = (qt)^{2r}[3t-2-3(2r)]_{q,t}$. Since the q,t-numbers of the terms in column $2r$ increase by 1 as one moves up, it follows that the contribution of column $2r$ to $g_\lambda[0,0,k]$ is $(qt)^{2r}[k-(2r)-\lfloor 2r+1/2 \rfloor \to 2k-2-3(2r)]_{q,t}$ as predicted by our formula.

Similarly, if $0 \le r \le t-1$, the lowest term that can contribute to $g_\lambda[0,0,k]$ in column $2r+1$ is

$$\begin{aligned} A^{(3t)}_{3t-1-(2r+1)-(r+1),2r+1} &= A^{(3t)}_{3t-3r-3,2r+1} = (qt)^{2r+1}[6t-2-(3t-3r-3)-3(2r+1)]_{q,t} \\ &= (qt)^{2r+1}[3t-3r-2]_{q,t} = [3t-(2r+1)-\lfloor 2r+2/2 \rfloor]_{q,t}. \end{aligned}$$

Note that the top element in column $2r+1$ is $A^{(3t)}_{0,2r+1} = (qt)^{2r+1}[3t-2-3(2r+1)]_{q,t}$. Since the q,t-numbers in the terms in column $2r+1$ increase by 1 as one moves up, it follows that the contribution of column $2r+1$ to $g_\lambda[0,0,k]$ is $(qt)^{2r+1}[k-(2r+1)-\lfloor 2r+1/2 \rfloor \to 2k-2-3(2r+1)]_{q,t}$ as predicted by our formula.

Thus our formula holds in this case.

Case 2. $k = 3t+1$.

The negative terms in the first row are

$$(qt)^{2t+s}[6t+2-2-3(2t+s)]_{q,t} = (qt)^{2t+s}[-3s]_{q,t}$$

for $s = 1, \ldots, t$. In particular, the last term in the first row equals $(qt)^{3t}[-3t]$ and the first negative term is $A^{(3t+1)}_{0,2t+1} = (qt)^{2t+1}[-3]$.

Then as in Case 1, we have two subcases depending on whether s is even or odd.

Subcase 2.1. $s = 2r$.

In this case, $A^{(3t+1)}_{0,2t+2r} = q^{2t+2r}[-6r]_{q,t}$. We claim that $\sum_{a=0}^{6r} A^{(3t)}_{a,2t-1+2r-a} = 0$. First observe that

$$A^{(3t+1)}_{3r,2t+2r-(3r)} = q^{2t-r}[6t+2-2-3r-3(2t-r)]_{q,t} = q^{2t-r}[0]_{q,t}.$$

Thus we can prove our claim by showing that for $0 \leq a \leq 3r-1$,

$$A^{(3t+1)}_{a,2t+2r-a} = -A^{(3t+1)}_{6r-a,2t+2r-(6r-a)}.$$

This is a straightforward computation so we will not include the details here.

Subcase 2.2. $s = 2r+1$.

In this case, $A^{(3t+1)}_{0,2t+2r+1} = q^{2t+2r+1}[-6r-3]_{q,t}$. We claim that $\sum_{a=0}^{6r+3} A^{(3t+1)}_{a,2t+2r+1-a} = 0$. In this case, one can easily check that $0 \leq a \leq 3r+1$,

$$A^{(3t+1)}_{a,2t+2r+1-a} = -A^{(3t+1)}_{6r+3-a,2t+2r+1-(6r+3-a)}$$

so we shall not include the details here.

Next observe that the bottom term of the array $\{A^{(3t+1)}_{i,j}\}_{i=0,\ldots,3t\ \&\ 0\leq j\leq i}$ in the r-th column is $A^{(3t+1)}_{3t-r,r}$. Our computations above show that in the array $\{A^{(3t+1)}_{i,j}\}_{i=0,\ldots,3t\ \&\ 0\leq j\leq i}$, the first $3s+1$ terms of any anti-diagonal terms starting at $A^{(3t+1)}_{0,2t+s}$ sum to 0 for $s = 1,\ldots,t$. This means that the corresponding terms in the array make no contribution to $g_\lambda[0,0,k]$. It follows that we can ignore all the terms in columns $2t+1,\dots,3t$. One can use a similar reasoning as we used in Case 1 to show that for $r=0,\dots,t-1$, we can ignore the bottom $r+1$ terms in columns $2r$ and $2r+1$. Moreover, we can ignore the bottom t terms in column $2t$. This is because $A^{(3t+1)}_{0,2t+1}=[-3]$, which means that the first four terms of the anti-diagonal starting at $A^{(3t+1)}_{0,2t+1}$ will cancel terms in columns $2t-2$, $2t-1$, and $2t$. It follows that if $0\le r\le t-1$, the lowest term that can contribute to $g_\lambda[0,0,k]$ in column $2r$ is

$$\begin{aligned} A^{(3t+1)}_{3t-2r-(r+1),2r} &= A^{(3t+1)}_{3t-3r-1,2r}=(qt)^{2r}[6t+2-2-(3t-3r-1)-3(2r)]_{q,t} \\ &= (qt)^{2r}[3t-3r+1]_{q,t}=[3t+1-(2r)-\lfloor 2r+1/2\rfloor]_{q,t}. \end{aligned}$$

Note that the top element in column $2r$ is $A^{(3t+1)}_{0,2r}=(qt)^{2r}[2(3t+1)-2-3(2r)]_{q,t}$. Since the q,t-numbers in the terms in column $2r$ increase by 1 as one moves up, it follows that the contribution of column $2r$ to $g_\lambda[0,0,k]$ is $(qt)^{2r}[k-(2r)-\lfloor 2r+1/2\rfloor\rightarrow 2k-2-3(2r)]_{q,t}$ as predicted by our formula.

Similarly, if $0\le r\le t-1$, the lowest term that can contribute to $g_\lambda[0,0,k]$ in column $2r+1$ is

$$\begin{aligned} A^{(3t+1)}_{3t-(2r+1)-(r+1),2r+1} &= A^{(3t+1)}_{3t-3r-2,2r+1}=(qt)^{2r+1}[6t+2-2-(3t-3r-2)-3(2r+1)]_{q,t} \\ &= (qt)^{2r+1}[3t-3r-1]_{q,t}=[(3t+1)-(2r+1)-\lfloor 2r+2/2\rfloor]_{q,t}. \end{aligned}$$

Note that the top element in column $2r+1$ is $A^{(3t+1)}_{0,2r+1}=(qt)^{2r+1}[2k-2-3(2r+1)]_{q,t}$. Since the q,t-numbers in the terms in column $2r+1$ increase by 1 as one moves up, it follows that the contribution of column $2r+1$ to $g_\lambda[0,0,k]$ is $(qt)^{2r+1}[k-(2r+1)-\lfloor 2r+1/2\rfloor\rightarrow 2k-2-3(2r+1)]_{q,t}$ as predicted by our formula.

Finally in column $2t$, the lowest term that can contribute to $g_\lambda[0,0,k]$ is

$$\begin{aligned} A^{(3t+1)}_{3t-(2t)-(t),2r+1} &= A^{(3t+1)}_{0,2t} = (qt)^{2t}[6t+2-2-3(2t)]_{q,t} \\ &= (qt)^{2t}[0]_{q,t}. \end{aligned}$$

Thus this column makes no contribution which is why we exclude this term from the sum. Note that in this case, $3t+1-2t-\lfloor 2t+1 \rfloor = 1$ while $2k-2-3(2t) = 6t+2-2-6t = 0$ so that $[k-2t-\lfloor 2t+1 \rfloor \to 2k-2-3(6t)] = [1 \to 0]$ which is an empty sum.

Thus our formula holds in this case.

Case 3. $k = 3t+2$.

The negative terms in the first row are

$$(qt)^{2t+s}[6t+4-2-3(2t+s)]_{q,t} = (qt)^{2t+s}[-3s+2]_{q,t}$$

for $s = 1,\ldots,t+1$. In particular, the last term in the first row equals $(qt)^{3t+1}[-3t-1]$ and the first negative term is $A^{(3t+2)}_{0,2t+1} = (qt)^{2t+1}[-1]$.

Then as before, we have two subcases depending on whether s is even or odd.

Subcase 3.1. $s = 2r$.

In this case, $A^{(3t+2)}_{0,2t+2r} = q^{2t+2r}[-6r+2]_{q,t}$. We claim that $\sum_{a=0}^{6r-2} A^{(3t)}_{a,2t-1+2r-a} = 0$. First observe that

$$A^{(3t+2)}_{3r-1,2t+2r-(3r-1)} = q^{2t-r+1}[6t+4-2-(3r-1)-3(2t-r+1)]_{q,t} = q^{2t-r+1}[0]_{q,t}.$$

Thus we can prove our claim by showing that for $0 \leq a \leq 3r-2$,

$$A^{(3t+2)}_{a,2t+2r-a} = -A^{(3t+2)}_{6r-2-a,2t+2r-(6r-2-a)}.$$

This is a straightforward computation so we will not include the details here.

Subcase 3.2. $s = 2r+1$.

In this case, $A^{(3t+2)}_{0,2t+2r+1} = q^{2t+2r+1}[-6r-1]_{q,t}$. We claim that $\sum_{a=0}^{6r+1} A^{(3t+2)}_{a,2t+2r+1-a} = 0$. One can easily check that for $0 \leq a \leq 3r$,

$$A^{(3t+1)}_{a,2t+2r+1-a} = -A^{(3t+1)}_{6r+1-a,2t+2r+1-(6r+1-a)},$$

so we shall not include the details here.

Next we observe that the bottom term of the array $\{A^{(3t+2)}_{i,j}\}_{i=0,\ldots,3t \ \& \ 0\leq j\leq i}$ in the r-th column is $A^{(3t+2)}_{3t+1-r,r}$. Our computations above have shown that in the array $\{A^{(3t+2)}_{i,j}\}_{\substack{i=0,\ldots,3t+1,\\ 0\leq j\leq i}}$, the first $3s-1$ terms of any anti-diagonal starting at $A^{(3t+2)}_{0,2t+s}$ sum to 0 for $s = 1,\ldots,t+1$. This means that the corresponding terms in the array make no contribution to $g_\lambda[0,0,k]$. It follows that we can ignore all the terms in columns $2t+1,\ldots,3t+1$. One can use a similar reasoning as we used in Case 1 to show that for $r = 0,\ldots,t-1$, we can ignore the bottom $r+1$ terms in columns $2r$ and $2r+1$. We can also ignore the bottom $t+1$ terms in column $2t$. This is because $A^{(3t+2)}_{0,2t+1} = (qt)^{2t+1}[-1]$, so that the sum of the first two anti-diagonal terms starting at $A^{(3t+2)}_{0,2t+1}$ will only cancel elements in columns $2t$ and $2t+1$.

This means that if $0 \leq r \leq t-1$, the lowest term that can contribute to $g_\lambda[0,0,k]$ in column

$2r$ is

$$\begin{aligned} A^{(3t+2)}_{3t+1-2r-(r+1),2r} &= A^{(3t+2)}_{3t-3r,2r} = (qt)^{2r}[6t+4-2-(3t-3r)-3(2r)]_{q,t} \\ &= (qt)^{2r}[3t-3r+2]_{q,t} = [3t+2-(2r)-\lfloor 2r+1/2 \rfloor]_{q,t}. \end{aligned}$$

Note that the top element in column $2r$ is $A^{(3t+2)}_{0,2r} = (qt)^{2r}[2(3t+2)-2-3(2r)]_{q,t}$. Since the q,t-numbers in the terms in column $2r$ increase by 1 as one moves up, it follows that the contribution of column $2r$ to $g_\lambda[0,0,k]$ is $(qt)^{2r}[k-(2r)-\lfloor 2r+1/2 \rfloor \to 2k-2-3(2r)]_{q,t}$ as predicted by our formula.

Similarly, if $0 \leq r \leq t-1$, the lowest term that can contribute to $g_\lambda[0,0,k]$ in column $2r+1$ is

$$\begin{aligned} A^{(3t+2)}_{3t+1-(2r+1)-(r+1),2r+1} &= A^{(3t+2)}_{3t-3r-1,2r+1} = (qt)^{2r+1}[6t+4-2-(3t-3r-1)-3(2r+1)]_{q,t} \\ &= (qt)^{2r+1}[3t-3r]_{q,t} = [(3t+2)-(2r+1)-\lfloor 2r+2/2 \rfloor]_{q,t}. \end{aligned}$$

Note that the top element in column $2r+1$ is $A^{(3t+2)}_{0,2r+1} = (qt)^{2r+1}[2(3t+2)-2-3(2r+1)]_{q,t}$. Since the q,t-numbers in the terms in column $2r+1$ increase by 1 as one moves up, it follows that the contribution of column $2r+1$ to $g_\lambda[0,0,k]$ is

$(qt)^{2r+1}[k-(2r+1)-\lfloor 2r+1/2 \rfloor \to 2k-2-3(2r+1)]_{q,t}$ as predicted by our formula.

Finally for column $2t$, the lowest term that can contribute to $g_\lambda[0,0,k]$ in column $2t$ is

$$\begin{aligned} A^{(3t+2)}_{3t+1-(2t)-(t+1),2t} &= A^{(3t+2)}_{0,2t} = (qt)^{2t}[6t+4-2-3(2t)]_{q,t} \\ &= (qt)^{2t}[2]_{q,t} = [(3t+2)-(2t)-\lfloor 2t+1/2 \rfloor]_{q,t}. \end{aligned}$$

It follows that the contribution of column $2t$ to $g_\lambda[0,0,k]$ is

$(qt)^{2t}[k-(2t)-\lfloor 2t+1/2 \rfloor \to 2k-2-3(2t)]_{q,t} = (qt)^{2t}[2]$ as predicted by our formula.

Thus our formula holds in this case which completes our proof. □

For example, we have

$$\begin{aligned} g_\lambda[0,0,12] &= \sum_{i=0}^{7}(qt)^i\Big[12-i-\lfloor \tfrac{i+1}{2}\rfloor \to 22-3i\Big]_{q,t} \\ &= [12\to 22]_{q,t}+(qt)[10\to 19]_{q,t}+(qt)^2[9\to 16]_{q,t}+(qt)^3[7\to 13]_{q,t} \\ &\quad +(qt)^4[6\to 10]_{q,t}+(qt)^5[4\to 7]_{q,t}+(qt)^6[3\to 4]_{q,t}+(qt)^7[1]_{q,t}. \end{aligned}$$

3.2.2.2 A formula for $g_\lambda[a,0,k]$

We have the following theorem about $g_\lambda[a,0,k]$.

Theorem 3.4. *For any $a,k \geq 0$, we have*

$$g_\lambda[a,0,k] = (qt)^a g_\lambda[0,0,k] + \sum_{i=1}^{a}(qt)^{a-i}[k+3i \to 2k+3i]_{q,t}.$$

Proof. We have

$$\begin{aligned} g_\lambda[a,0,k] &= \sum_{T\in S[a,0,0,k]} g_T \\ &= \sum_{i=0}^{k} w(a+i, a+k-i) \\ &= \sum_{i=0}^{k} \frac{t^{a+i}[a+k-i]_{t^2,q} - q^{a+i}[a+k-i]_{q^2,t}}{t-q}. \end{aligned}$$

Notice that

$$[a+k-i]_{t^2,q} = q^a[k-i]_{t^2,q} + \sum_{j=0}^{a-1} t^{2(k-i+a-j-1)}q^j \tag{3.8}$$

and

$$[a+k-i]_{q^2,t} = t^a[k-i]_{q^2,t} + \sum_{j=0}^{a-1} q^{2(k-i+a-j-1)}t^j, \tag{3.9}$$

we can get the following equation by plugging in Equation (3.8) and Equation (3.9):

$$\begin{aligned}
g_\lambda[a,0,k] &= t^a q^a \sum_{i=0}^{k} \frac{t^i[k-i]_{t^2,q} - q^i[k-i]_{q^2,t}}{t-q} \\
&\quad + \sum_{i=0}^{k}\sum_{j=0}^{a-1}(qt)^j \frac{t^{2k-i+3a-3j-2} - q^{2k-i+3a-3j-2}}{t-q} \\
&= (qt)^a g_\lambda[0,0,k] + \sum_{j=0}^{a-1}(qt)^j \sum_{i=0}^{k}[2k-i+3a-3j-2]_{q,t} \\
&= (qt)^a g_\lambda[0,0,k] + \sum_{i=0}^{a-1}(qt)^i[k+3a-3i-2 \to 2k+3a-3i-2]_{q,t} \\
&= (qt)^a g_\lambda[0,0,k] + \sum_{i=1}^{a}(qt)^{a-i}[k+3i-2 \to 2k+3i-2]_{q,t}\ . \qquad \square
\end{aligned}$$

3.2.2.3 The computation of $g_\lambda[a,k_1,k_2]$

We shall add the component k_1 to complete the formula. Note that the function $g_\lambda[a,k_1,k_2] = g_\lambda[a,k_2,k_1]$. Without loss of generality, we suppose $k_1 \le k_2$.

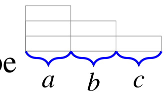
Theorem 3.5. *For non-negative integers $k_1 \le k_2$ and a, we have*

$$g_\lambda[a,k_1,k_2] = \sum_{i=0}^{k_1} g_\lambda[a+i,0,k_1+k_2-2i].$$

Proof.

$$\begin{aligned}
g_\lambda[a,k_1,k_2] &= \sum_{T\in S[a,k_1,0,k_2]} g_T \\
&= \sum_{j=0}^{k_1}\sum_{i=0}^{k_2} w(a+i+j,a+k_1+k_2-i-j) \\
&= \sum_{i=0}^{k_1}\sum_{j=0}^{k_1+k_2-2i} w(a+i+j,a+k_1+k_2-i-j) \\
&= \sum_{i=0}^{k_1} g_\lambda[a+i,0,k_1+k_2-2i]. \qquad \square
\end{aligned}$$

3.2.3 The formula for g_λ

For any $\lambda = (3^a 2^b 1^c)$, λ' has the shape  . We can then write the formula of g_λ in terms of $g_\lambda[x,y,z]$.

Theorem 3.6. *Let $\lambda = (3^a 2^b 1^c)$ where a,b,c are non-negative integers. Then*

$$g_\lambda = \sum_{i=0}^{b} g_\lambda[a+i, b-i, c] + \sum_{i=1}^{c} g_\lambda[a, b, c-i].$$

Proof. The first term $\sum_{i=0}^{b} g_\lambda[a+i, b-i, c]$ sums over all the cases in Figure 3.1(a) and the second term $\sum_{i=0}^{c} g_\lambda[a, b, c-i]$ sums over all the cases in Figure 3.1(b). □

Thus, we have a complete recursive formula for g_λ. The recursive formula for g_λ not only shows that g_λ is Schur positive in q,t-analogues, also gives us a way of writing g_λ into q,t-analogues and powers of (qt). For example, suppose $\lambda = 1^4$. Then $\lambda' = (4)$ so that taking into account the possible numbers of 0's in a tableau $T \in SSYT((4), 012)$, we see that

$$g_{(1^4)} = g_\lambda[0,0,0] + g_\lambda[0,0,1] + g_\lambda[0,0,2] + g_\lambda[0,0,3] + g_\lambda[0,0,4].$$

In the right hand side, $g_\lambda[0,0,0] = g_\lambda[0,0,1] = 0$, and we can apply Theorem 3.3 to compute

$$g_\lambda[0,0,2] = \sum_{i=0}^{0} (qt)^i [2-i-\lfloor (i+1)/2 \rfloor \to 4-2-3i]_{q,t} = [2]_{q,t},$$

$$\begin{aligned} g_\lambda[0,0,3] &= \sum_{i=0}^{1} (qt)^i [3-i-\lfloor (i+1)/2 \rfloor \to 6-2-3i]_{q,t} \\ &= [3 \to 4] + (qt)[1 \to 1] = [3]_{q,t} + [4]_{q,t} + (qt)[1]_{q,t}, \end{aligned}$$

and

$$\begin{aligned} g_\lambda[0,0,4] &= \sum_{i=0}^{1}(qt)^i[4-i-\lfloor(i+1)/2\rfloor \to 8-2-3i]_{q,t} \\ &= [4\to 6]+(qt)[2\to 3] \\ &= [4]_{q,t}+[5]_{q,t}+[6]_{q,t}+qt([2]_{q,t}+[3]_{q,t}). \end{aligned}$$

Thus

$$g_{(1^4)} = [2]_{q,t}+[3]_{q,t}+2[4]_{q,t}+[5]_{q,t}+[6]_{q,t}+qt([1]_{q,t}+[2]_{q,t}+[3]_{q,t}).$$

In general, we see that

$$\langle \Delta_{e_2} e_n, e_n\rangle = \sum_{s=2}^{n} g_\lambda[0,0,s].$$

We claim that $g_\lambda[0,0,n]$ is a q,t-analogue of $2\binom{n+1}{3}$. To see this, we shall use a formula of [HRW18] to show that

$$\langle \Delta_{e_2} e_n, e_n\rangle|_{q=t=1} = 2\binom{n+2}{4}$$

from which it follows that

$$\begin{aligned} g_\lambda[0,0,n]|_{q=t=1} &= \langle \Delta_{e_2} e_n, e_n\rangle|_{q=t=1} - \langle \Delta_{e_2} e_{n-1}, e_{n-1}\rangle|_{q=t=1} \\ &= 2\binom{n+2}{4} - 2\binom{n+1}{4} = 2\binom{n+1}{3}. \end{aligned}$$

It is proved in [HRW18] that

$$\Delta_{e_k} e_n|_{t=1/q} = \frac{q^{\binom{k}{2}-k(n-1)}}{[k+1]_q}\begin{bmatrix} n \\ k \end{bmatrix}_q e_n[X(1+q+\cdots+q^k)]. \tag{3.10}$$

Repeatedly applying the sum rule that

$$s_\lambda[X+Y] = \sum_{\mu\subseteq\lambda} s_\mu[X] s_{\lambda/\mu}[Y],$$

we see that

$$\begin{aligned}
\Delta_{e_k} e_n|_{t=1/q} &= \frac{q^{\binom{k}{2}-k(n-1)}}{[k+1]_q} \begin{bmatrix} n \\ k \end{bmatrix}_q \sum_{\substack{i_s\geq 0 \\ i_0+i_1+\cdots+i_k=n}} \prod_{s=0}^{k} e_{i_s}[q^s X] \\
&= \frac{q^{\binom{k}{2}-k(n-1)}}{[k+1]_q} \begin{bmatrix} n \\ k \end{bmatrix}_q \sum_{\substack{i_s\geq 0 \\ i_0+i_1+\cdots+i_k=n}} \prod_{s=0}^{k} q^{si_s} e_{i_s}.
\end{aligned}$$

It follows that

$$\langle \Delta_{e_k} e_n, e_n\rangle|_{t=1/q} = \frac{q^{\binom{k}{2}-k(n-1)}}{[k+1]_q} \begin{bmatrix} n \\ k \end{bmatrix}_q \sum_{\substack{i_s\geq 0 \\ i_0+i_1+\cdots+i_k=n}} \prod_{s=0}^{k} q^{si_s}.$$

It is easy to see that

$$\sum_{\substack{i_s\geq 0 \\ i_0+i_1+\cdots+i_k=n}} \prod_{s=0}^{k} q^{si_s} = \begin{bmatrix} n+k \\ k \end{bmatrix}_q$$

since the LHS is the sum of $q^{|\lambda|}$ over all partitions λ contained in the $n\times k$ rectangle. Thus,

$$\langle \Delta_{e_k} e_n, e_n\rangle|_{t=1/q} = \frac{q^{\binom{k}{2}-k(n-1)}}{[k+1]_q} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n+k \\ k \end{bmatrix}_q. \tag{3.11}$$

Setting $q=1$ and $k=2$ in (3.11), we see that

$$\begin{aligned}
\langle \Delta_{e_2} e_n, e_n\rangle|_{q=t=1} &= \frac{1}{3}\frac{n!}{2!(n-2)!}\frac{(n+2)!}{2!n!} \\
&= \frac{4}{2}\frac{(n+2)!}{4!(n-2)!} = 2\binom{n+2}{4}.
\end{aligned}$$

We shall apply our formula to compute another example at the end of this section. Consider $\lambda=(2,1^2)$ so that $\lambda'=(3,1)$. In this case, we can classify the tableaux

$T \in \mathrm{SSYT}((3,1),012)$ by whether the bottom left corner square contains a 1, in which case we get a term $g_\lambda[1,0,2]$, or the bottom corner square contains a 0, in which case we get a contribution of $g_\lambda[0,1,2]$, $g_\lambda[0,1,1]$, or $g_\lambda[0,1,0]$, depending on the number of 0's in the first row. Since $g_\lambda[0,1,0] = g_\lambda[0,0,1] = 0$, by Theorem 3.5, we have

$$\begin{aligned} g_\lambda[0,1,1] &= \sum_{i=0}^{1} g_\lambda[i,0,2-2i] \\ &= g_\lambda[0,0,2] + g_\lambda[1,0,0] \\ &= [2]_{q,t} + g_\lambda[1,0,0], \end{aligned}$$

and

$$\begin{aligned} g_\lambda[0,1,2] &= \sum_{i=0}^{1} g_\lambda[i,0,3-2i] \\ &= g_\lambda[0,0,3] + g_\lambda[1,0,1] \\ &= [3]_{q,t} + [4]_{q,t} + qt + g_\lambda[1,0,1]. \end{aligned}$$

By Theorem 3.4, we have

$$\begin{aligned} g_\lambda[1,0,0] &= (qt)g_\lambda[0,0,0] + [3 \to 3] = [1]_{q,t}, \\ g_\lambda[1,0,1] &= (qt)g_\lambda[0,0,1] + [4 \to 5] = [2]_{q,t} + [3]_{q,t}, \text{ and} \\ g_\lambda[1,0,2] &= (qt)g_\lambda[0,0,2] + [3 \to 5] \\ &= (qt)[2]_{q,t} + [3]_{q,t} + [4]_{q,t} + [5]_{q,t}. \end{aligned}$$

It follows that

$$g_{1^2,2} = [1]_{q,t} + 2[2]_{q,t} + 3[3]_{q,t} + 2[4]_{q,t} + [5]_{q,t} + (qt)([1]_{q,t} + [2]_{q,t}).$$

3.3 The $\Delta_{e_3} e_n$ case

For the case $\Delta_{e_3} e_n$, we shall do a similar computation. First we compute the modified Macdonald polynomial expansion of e_4:

$$e_4 = \frac{\widetilde{H}_4[X;q,t]}{(q-t)(q^2-t)(q^3-t)} - \frac{\left(q^2+q+t+1\right)\widetilde{H}_{3,1[X;q,t]}}{(q+t)(q^3-t)(q-t)^2} - \frac{(qt-1)\widetilde{H}_{2,2}[X;q,t]}{(-t^2+q)(q^2-t)(q-t)^2} + \frac{\left(t^2+q+t+1\right)\widetilde{H}_{2,1,1}[X;q,t]}{(q+t)(-t^3+q)(q-t)^2} - \frac{\widetilde{H}_{1,1,1,1}[X;q,t]}{(q-t)(-t^3+q)(-t^2+q)}.$$

By applying Lemma 3.1, we have

$$\begin{aligned}\langle \Delta_{e_3} e_n,\, s_\lambda \rangle &= \langle \Delta_{s_{\lambda'}} e_4,\, s_4 \rangle \\ &= \frac{s_{\lambda'}[B_4]}{(q-t)(q^2-t)(q^3-t)} - \frac{\left(q^2+q+t+1\right)s_{\lambda'}[B_{3,1}]}{(q+t)(q^3-t)(q-t)^2} - \frac{(qt-1)s_{\lambda'}[B_{2,2}]}{(-t^2+q)(q^2-t)(q-t)^2} \\ &\quad + \frac{\left(t^2+q+t+1\right)s_{\lambda'}[B_{2,1,1}]}{(q+t)(-t^3+q)(q-t)^2} - \frac{s_{\lambda'}[B_{1,1,1,1}]}{(q-t)(-t^3+q)(-t^2+q)}.\end{aligned}$$

By applying partial fraction decomposition, the best formula we obtain is

$$\langle \Delta_{e_3} e_n,\, s_\lambda \rangle = \frac{F_\lambda(q,t) - F_\lambda(t,q)}{q-t} - \frac{s_{\lambda'}[1+q+t+q^2]/q^2 - s_{\lambda'}[1+q+t+t^2]/t^2}{2(q^2-t^2)}, \tag{3.12}$$

where $F_\lambda = F_\lambda(q,t)$ is given by

$$F_\lambda = \frac{s_{\lambda'}[1+q+q^2+q^3] - s_{\lambda'}[1+q+t+qt]}{(q-1)q^2(q^2-t)} - \frac{s_{\lambda'}[1+q+q^2+q^3] - s_{\lambda'}[1+q+t+q^2]}{q^2(q-1)(q^3-t)} - \frac{(q+1)(s_{\lambda'}[1+q+t+q^2] - s_{\lambda'}[1+q+t+qt])}{2(q-t)q^2(q-1)} + \frac{s_{\lambda'}[1+q+t+qt]}{2q^2t}.$$

One can use this formula to prove that $\langle \Delta_{e_3} e_n,\, s_\lambda \rangle$ is a polynomial in $\mathbb{N}[q,t]$. Nevertheless, it is clear that this approach becomes more and more complicated so that the proof of the general $\Delta_{e_k} e_n$ case seems to require new ideas.

The majority of Chapter 3 has been published in Electronic Journal of Combinatorics. Qiu, Dun; Remmel, Jeffrey Brian; Sergel, Emily; Xin, Guoce. “On the Schur positivity of $\Delta_{e_2} e_n[X]$", Electronic Journal of Combinatorics, vol. 25 (4), 2018. The dissertation author was the primary investigator and author of this paper.

Chapter 4

Conjectures about the expression $\Delta'_{e_k}\Delta_{h_r}e_n$

We have introduced the Delta Conjecture of Haglund, Remmel and Wilson [HRW18] about the expressions $\Delta'_{e_k}e_n$ and $\Delta_{e_k}e_n$, which are important open problems in algebraic combinatorics. In the same paper, Haglund et al. gave a conjecture about the Delta operator expression $\Delta'_{e_k}\Delta_{h_r}e_n$, which is analogous to the *rise version* of the Delta Conjecture of $\Delta'_{e_k}e_n$. Very recently, D'Adderio, Iraci and Wyngaerd in [DIW19] proved the rise version conjecture of the expression $\Delta'_{e_k}\Delta_{h_r}e_n$ at the case when $t=0$.

In this chapter, we shall propose a new *valley version* conjecture of the expression $\Delta'_{e_k}\Delta_{h_r}e_n$. Then, we work on the combinatorial side on extended ordered multiset partitions to prove that the two conjectures about $\Delta'_{e_k}\Delta_{h_r}e_n$ are equivalent at the cases when t or q equals 0, thus give a proof of the *valley version* conjecture of $\Delta'_{e_k}\Delta_{h_r}e_n$ when t or q equals 0.

4.1 Introduction

In the origin Delta Conjecture paper of Haglund, Remmel and Wilson [HRW18], the authors used an alternative combinatorial object called *labeled Dyck path* (we shall also use the name *word parking function*), which makes an equivalent combinatorial formulation as normal parking functions.

Given an (n,n)-Dyck path P, an (n,n)-*word parking functions* PF is obtained by labeling the north steps of P with positive integers such that the labels (called *cars*) are strictly increasing along each column of P. We still let $\ell_i(\mathrm{PF})$ be the ith row label of PF. Notice that the only difference with a normal parking function is that we use any positive integers as cars (with repetitions allowed) rather than the distinct integers $\{1,\dots,n\}$. We let $\mathcal{WPF}_n$ denote the set of (n,n)-word parking functions. We shall call $\mathrm{PF}\in\mathcal{WPF}_n$ a parking function without causing ambiguity.

The statistics of parking functions can be naturally generalized to the set of word parking functions. Thus we have the statistics *area*, *dinv*, *word*, *rank*, *ides*, *pides* defined on $\mathcal{WPF}_n$ in the same way as in Section 1.2.2. Further, we have the sets valley(PF), Rise(PF) and Val(PF) defined in the same way as in Setion 1.3.3. For a word parking function $\mathrm{PF}\in\mathcal{WPF}_n$, we define the *label weight* (or *car weight*) of PF to be

$$X^{\mathrm{PF}} := \prod_{i=1}^{n} x_{\ell_i(\mathrm{PF})}.$$

Then the Delta Conjecture can also be stated as

Conjecture 4.1 (Haglund, Remmel and Wilson)**.** *For any integers $n>k\geq 0$,*

$$\Delta'_{e_k}e_n = \sum_{\mathrm{PF}\in\mathcal{WPF}_n} t^{\mathrm{area(PF)}}q^{\mathrm{dinv(PF)}}X^{\mathrm{PF}}\prod_{i\in\mathrm{Rise(PF)}}(1+\frac{z}{t^{a_i(\mathrm{PF})}})\Big|_{z^{n-k-1}} \tag{4.1}$$

$$= \sum_{\mathrm{PF}\in\mathcal{WPF}_n} t^{\mathrm{area(PF)}}q^{\mathrm{dinv(PF)}}X^{\mathrm{PF}}\prod_{i\in\mathrm{Val(PF)}}(1+\frac{z}{q^{d_i(\mathrm{PF})+1}})\Big|_{z^{n-k-1}}. \tag{4.2}$$

Notice that the component $F_{\mathrm{ides(PF)}}$ in the previous version of the Delta Conjecture is replaced by the car weight X^{PF}. Based on our previous definition, the right hand sides of Equations (4.1) and (4.2) are denoted by $\mathrm{Rise}_{n,k}[X;q,t]$ and $\mathrm{Val}_{n,k}[X;q,t]$.

Consider the factor $t^{\mathrm{area(PF)}}\prod_{i\in\mathrm{Rise(PF)}}(1+\frac{z}{t^{a_i(\mathrm{PF})}})\Big|_{z^{n-k-1}}$ in Equation (4.1). Each term in the expansion of this factor is a power of t, and the power is area(PF) minus $n-k-1$ row-areas

$a_i(\mathrm{PF})$ of the double rise rows. Similarly in the factor $q^{\mathrm{dinv}(\mathrm{PF})}\prod_{i\in\mathrm{Val}(\mathrm{PF})}(1+\frac{z}{q^{d_i(\mathrm{PF})+1}})\Big|_{z^{n-k-1}}$ in Equation (4.2), each term is a power of q, and the power is dinv(PF) minus $n-k-1$ row-dinvs $(d_i(\mathrm{PF})+1)$ of the contractible valley rows. Thus, if we define

$$\begin{aligned}\mathcal{WPF}_{n,k}^{\mathrm{Rise}} &:= \{(\mathrm{PF},R) : P\in\mathcal{WPF}_n, R\subseteq \mathrm{Rise}(\mathrm{PF}), |R|=k\},\\ \mathcal{WPF}_{n,k}^{\mathrm{Val}} &:= \{(\mathrm{PF},V) : P\in\mathcal{WPF}_n, V\subseteq \mathrm{Val}(\mathrm{PF}), |V|=k\}\end{aligned}$$

and let

$$\begin{aligned}\mathrm{area}^-(\mathrm{PF},R) &:= \sum_{i\in[n]\setminus R} a_i(\mathrm{PF}),\\ \mathrm{dinv}^-(\mathrm{PF},V) &:= \sum_{i\in[n]\setminus V} d_i(\mathrm{PF})-|V|,\end{aligned}$$

then

$$\begin{aligned}\mathrm{Rise}_{n,k}[X;q,t] &= \sum_{(\mathrm{PF},R)\in\mathcal{WPF}_{n,n-k-1}^{\mathrm{Rise}}} t^{\mathrm{area}^-(\mathrm{PF},R)}q^{\mathrm{dinv}(\mathrm{PF})}X^{\mathrm{PF}},\\ \mathrm{Val}_{n,k}[X;q,t] &= \sum_{(\mathrm{PF},V)\in\mathcal{WPF}_{n,n-k-1}^{\mathrm{Val}}} t^{\mathrm{area}(\mathrm{PF})}q^{\mathrm{dinv}^-(\mathrm{PF},V)}X^{\mathrm{PF}}.\end{aligned}$$

We call a pair $(\mathrm{PF},R)\in\mathcal{WPF}_{n,k}^{\mathrm{Rise}}$ (or $(\mathrm{PF},V)\in\mathcal{WPF}_{n,k}^{\mathrm{Val}}$) a *rise-decorated* (or *valley-decorated*) parking function, which can be seen as a parking function PF with k rows in Rise (or Val) marked with a star $*$. Figure 4.1 shows examples of rise-decorated and valley-decorated parking functions.

In [HRW18], the author also conjectured a combinatorial formula for the expression $\Delta'_{e_k}\Delta_{h_r}e_n$, and the combinatorial side is a generating function of the set of *extended word parking functions with blank valleys*.

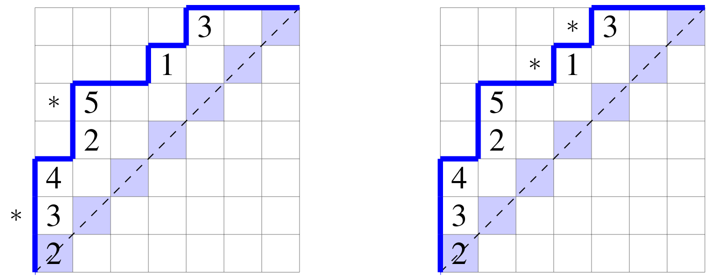
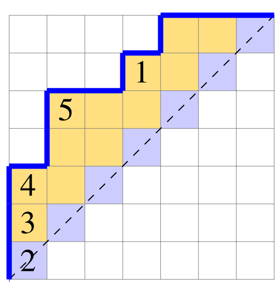


Figure 4.1: Examples: parking functions in $\mathcal{WPF}^{\text{Rise}}_{7,2}$ and $\mathcal{WPF}^{\text{Val}}_{7,2}$.

Given an (n,n)-Dyck path P, remind that the valley set of P is defined to be

$$\text{valley}(P) := \{i : a_i < a_{i-1}\}.$$

We say that a word-labeling of a Dyck path *has r blank valleys* if there are r valleys not receiving a label. Such labeled Dyck paths are called extended word parking functions. We let $\mathcal{WPF}_{n;r}$ denote the set of extended word parking functions of size $n+r$ with r blank valleys. Figure 4.2 shows an example of a parking function in the set $\mathcal{WPF}_{5;2}$.



i	$a_i(\text{PF})$	$d_i(\text{PF})$
7	2	0
6	2	0
5	3	2
4	2	1
3	2	0
2	1	0
1	0	0

Figure 4.2: A $(7,7)$-extended parking function with 2 blank valleys.

A more convenient way to draw an extended word parking function is that, we can fill the blank valleys with 0's, thus an extended word parking function is a parking function with labels in $\mathbb{Z}_{\geq 0}$ such that 0 does not appear in the first row (since the first row is not a valley).

With 0's labeled in the blank valley positions, we can define the area and dinv components $a_i(\text{PF})$ and $d_i(\text{PF})$ on each parking functions in $\mathcal{WPF}_{n;r}$ in the same way. We still let $\text{Rise}(\text{PF}) = \{i : a_i(\text{PF}) = a_{i-1}(\text{PF}) + 1\}$ denote the double rise set. For sake of labeling the blank valleys with

0's, we can define the contractible valley set Val(PF) in the same way as Section 1.3.3.

Further, we can define the set of *rise-decorated* (or *valley-decorated*) *parking functions with blank valleys*. The set of rise-decorated (or valley-decorated) parking functions with n cars, r blank valleys and k marked double rises (or contractible valleys) is denoted by $\mathcal{WPF}^{\mathrm{Rise}}_{r;n,k}$ (or $\mathcal{WPF}^{\mathrm{Val}}_{r;n,k}$).

The conjecture of Haglund, Remmel and Wilson [HRW18] is

Conjecture 4.2 (rise conjecture of $\Delta'_{e_k}\Delta_{h_r}e_n$ of Haglund, Remmel and Wilson)**.** *For any positive integers n, k, and r with $k < n$,*

$$\Delta'_{e_k}\Delta_{h_r}e_{n+r} = \sum_{\mathrm{PF}\in\mathcal{WPF}_{n;r}} t^{\mathrm{area(PF)}}q^{\mathrm{dinv(PF)}}x^{\mathrm{PF}}\prod_{i\in\mathrm{Rise(PF)}}\left(1+\frac{z}{t^{a_i(\mathrm{PF})}}\right)\bigg|_{z^{n-k-1}}.$$

By labeling the blank valleys with 0's, we are able to conjecture that

Conjecture 4.3 (valley conjecture of $\Delta'_{e_k}\Delta_{h_r}e_n$)**.** *For any positive integers n, k, and r with $k < n$,*

$$\Delta'_{e_k}\Delta_{h_r}e_{n+r} = \sum_{\mathrm{PF}\in\mathcal{WPF}_{n;r}} t^{\mathrm{area(PF)}}q^{\mathrm{dinv(PF)}}x^{\mathrm{PF}}\prod_{i\in\mathrm{Val(PF)}}\left(1+\frac{z}{q^{d_i(\mathrm{PF})+1}}\right)\bigg|_{z^{n-k-1}}.$$

We let $\mathrm{Rise}_{r;n,k}[X;q,t]$ denote the combinatorial side of Conjecture 4.2 and $\mathrm{Val}_{r;n,k}[X;q,t]$ denote the combinatorial side of Conjecture 4.3. Notice that the combinatorial sides of the two conjectures could also be written as generating functions of the sets $\mathcal{WPF}^{\mathrm{Rise}}_{r;n,n-k-1}$ and $\mathcal{WPF}^{\mathrm{Val}}_{r;n,n-k-1}$.

The rise version conjecture of $\Delta'_{e_k}\Delta_{h_r}e_n$ is well studied. Very recently, D'Adderio, Iraci and Wyngaerd [DIW19] proved this conjecture in the case $t = 0$. However, the valley version conjecture of $\Delta'_{e_k}\Delta_{h_r}e_n$ is new and has not appeared anywhere before. We believe that the valley version conjecture is true since we have verified the conjecture for $n \leq 10$ by Maple programs, and we have also proved the valley version conjecture at the case when t or q is zero.

In Section 4.2, we shall introduce ordered multiset partitions, extended ordered multiset partitions and their connections to the Delta expression conjectures. In Section 4.3, we shall prove that the statistics inv, maj and dinv are equi-distributed by three insertion algorithms. In Section 4.4, we shall prove that the statistics inv and minimaj are equi-distributed by generalizing a method of Rhoades [Rho18], which completes a proof of the valley version conjecture of $\Delta'_{e_k}\Delta_{h_r}e_n$ when t or q equals 0.

4.2 Extended ordered multiset partitions

4.2.1 Ordered set partitions and ordered multiset partitions

Let $n \geq 0$ be any integer. A *set partition* π of the set $[n] = \{1,\ldots,n\}$ is a family of nonempty, pairwise disjoint subsets $B_1, B_2, \ldots, B_k$ of $[n]$ called *parts* (or *blocks*) such that $\cup_{i=1}^{k} B_i = [n]$. We let $\ell(\pi)$ denote the number of parts in π and $|\pi| = n$ denote the size of π. We let $\min(B_i)$ and $\max(B_i)$ denote the minimum and maximum elements of B_i and we use the convention that we order the parts so that $\min(B_1) < \cdots < \min(B_k)$. To simplify notation, we shall write π as $B_1/\cdots/B_k$. Thus we would write $\pi = 134/268/57$ for the set partition π of [8] with parts $B_1 = \{1,3,4\}$, $B_2 = \{2,6,8\}$ and $B_3 = \{5,7\}$.

An *ordered set partition* with underlying set partition π is just a permutation of the parts of π, i.e. $\delta = B_{\sigma_1}/\cdots/B_{\sigma_k}$ for some permutation σ in the symmetric group $\mathcal{S}_k$. For example, $\delta = 57/134/268$ is an ordered set partition of the set [8] with underlying set partition $\pi = 134/268/57$.

Let $\pi = B_1/\cdots/B_k$ be an ordered set partition of $[n]$. The strong composition $\lambda(\pi) = (|B_1|,\ldots,|B_k|)$ is called the *shape* of π. We let $\mathcal{OP}_n$ denote the set of ordered set partitions of $[n]$, and $\mathcal{OP}_{n,k}$ denote the set of ordered set partitions of $[n]$ with k parts. Further, we let $\mathcal{OP}_{n,\alpha}$ denote the set of ordered set partitions of $[n]$ with shape α.

More generally, for a weak composition $\beta = \beta_1\cdots\beta_\ell \vDash n$, an *ordered multiset partition* with *content* β is defined to be a partition of the multiset $A(\beta) = \{i^{\beta_i} : 1 \leq i \leq \ell\}$ into several ordered sets called *blocks* where *repetition is not allowed* in each block. We denote the set of ordered multiset partitions with content β by $\mathcal{OP}_\beta$. Similar, we have $\mathcal{OP}_{\beta,k}$ and $\mathcal{OP}_{\beta,\alpha}$. For example, $\pi = 234/26/123$ is an ordered multiset partition in $\mathcal{OP}_{(1,3,2,1,0,1),(3,2,3)}$.

We shall define 4 statistics: inv, maj, dinv and minimaj on ordered multiset partitions.

Given $\pi = B_1/\cdots/B_k \in \mathcal{OP}_{\beta,k}$, the *inversion* statistic $\mathrm{inv}(\pi)$ is defined to be the number of pairs $a > b$ such that b is the minimum of its block, and a is in some block that is strictly left of b's block. Such pairs are called *inversion pairs*. For example, $\pi = 134/268/57$ has 4 inversions, and the inversion pairs are $(3,2),(4,2),(6,5),(8,5)$.

For an ordered partition $\pi = B_1/\cdots/B_k \in \mathcal{OP}_{\beta,k}$, let B_i^h denote the hth smallest element in part B_i, then the *diagonal inversion* of π is defined to be

$$\mathrm{dinv}(\pi) := |\{(h,i,j) : i < j, B_i^h > B_j^h\} \cup \{(h,i,j) : i < j, B_i^h > B_j^{h+1}\}|,$$

where the triples in the left set are called *primary dinvs*, and the triples in the right set are called *secondary dinvs*. For example, $\pi = 134/268/57$ has 4 dinvs, which are all secondary dinvs: $(1,1,2),(1,1,3),(1,2,3),(2,1,2)$.

We let $\sigma = \sigma(\pi)$ of a partition $\pi \in \mathcal{OP}_{\beta,k}$ be the word obtained by writing each block B_i in decreasing order for $i = 1\cdots k$. We also define the index word $\mathrm{index}(\pi) = 0^{|B_1|}1^{|B_2|}\cdots(k-1)^{|B_k|}$. Then the *major index* of π is

$$\mathrm{maj}(\pi) := \sum_{i:\sigma_i > \sigma_{i+1}} \mathrm{index}(\pi)_{i+1}.$$

For example, if $\pi = 134/268/57$, then $\sigma = 43186275$, $\mathrm{index}(\pi) = 00011122$ and $\mathrm{maj}(\pi) = 0+0+1+1+2 = 4$.

Given $\pi = B_1/\cdots/B_k \in \mathcal{OP}_{\beta,\alpha}$ where $\alpha = (\alpha_1,\ldots,\alpha_k)$, we first construct a word miniword(π) by organizing the elements in each block and list the organized blocks $B_1,\ldots,B_k$. We first organize the numbers in B_k in increasing order. Then suppose that we have processed block B_{i+1}, we shall organize the numbers in B_i by placing the numbers strictly bigger than the first number of B_{i+1} first in increasing order, followed by the remaining numbers also in increasing order, then we place the organized numbers on the left of the existing sequence. For example, if $\pi = 2/34/13/13/2$, then $\text{miniword}(\pi) = 23413312$. The *minimum major index* of π is defined by

$$\text{minimaj}(\pi) := \text{maj}(\text{miniword}(\pi)).$$

The four statistics are closely related to the Delta Conjecture. Let

$$D^{\text{stat}}_{\beta,k}(q) := \sum_{\pi \in \mathcal{OP}_{\beta,k}} q^{\text{stat}(\pi)}$$

where stat is one of the statistics *inv, maj, dinv, minimaj*, Haglund, Remmel and Wilson in [HRW18] proved that

Theorem 4.1 (Haglund, Remmel and Wilson)**.** *For any integers n,k and weak composition β,*

$$\text{Rise}_{n,k}[X;q,0]|_{M_\beta} = D^{\text{dinv}}_{\beta,k+1}(q), \tag{4.3}$$

$$\text{Rise}_{n,k}[X;0,q]|_{M_\beta} = D^{\text{maj}}_{\beta,k+1}(q), \tag{4.4}$$

$$\text{Val}_{n,k}[X;q,0]|_{M_\beta} = D^{\text{inv}}_{\beta,k+1}(q), \tag{4.5}$$

$$\text{Val}_{n,k}[X;0,q]|_{M_\beta} = D^{\text{minimaj}}_{\beta,k+1}(q). \tag{4.6}$$

They proved Theorem 4.1 by constructing 4 bijections of the form γ^{stat} for stat = dinv, maj, inv and minimaj between ordered multiset partitions and word parking functions. We present the four bijections in Appendix A. It is a **fact** that for any ordered multiset partition π, *each bijection γ^{stat} maps the the minimum element in the last part of π to the car in the first row in the parking function $\gamma^{\text{stat}}(\pi)$* mentioned in Appendix A. We are going to use the fact when we prove Theorem 4.3.

On the combinatorial side, Wilson [Wil16] and Rhoades [Rho18] proved the following theorem:

Theorem 4.2 (Rhoades and Wilson)**.** *For any integers n, k,*

$$\mathrm{Rise}_{n,k}(X;q,0) = \mathrm{Rise}_{n,k}(X;0,q) = \mathrm{Val}_{n,k}(X;q,0) = \mathrm{Val}_{n,k}(X;0,q). \tag{4.7}$$

4.2.2 Extended permutations, extended ordered set and multiset partitions

We shall generalize the definitions of permutations, ordered set partitions and ordered multiset partitions in the way that the number 0 is allowed to be an entry.

Let $\beta = \{\beta_1, \ldots, \beta_\ell\} \vDash n$ be a weak composition and $A(\beta) = \{i^{\beta_i} : 1 \le i \le \ell\}$ be its corresponding multiset. A permutation of $A(\beta)$ is an ordering of the entries in the multiset $A(\beta)$. We let $\mathcal{S}_\beta$ denote the set of permutations of $A(\beta)$.

Given a weak composition $\beta \vDash n$ and an integer $r \ge 0$, an *extended permutation* (or a *tail positive permutation*) is a permutation of the multiset $A(\beta) \cup \{0^r\}$ such that the last entry is not 0. We let $\mathcal{S}_{r;\beta}$ denote the set of extended permutations of $A(\beta) \cup \{0^r\}$. Clearly, $\mathcal{S}_{0;\beta} = \mathcal{S}_\beta$.

In a similar way, one can define extended ordered set and multiset partitions. We let $\mathcal{OP}_{1;n}$ denote the set of *extended ordered set partitions*, which are ordered set partitions of the set $\{0\} \cup \{1, \ldots, n\}$ such that the number 0 is not contained in the last block. Similar to the definition of $\mathcal{OP}_{n,k}$ and $\mathcal{OP}_{n,\alpha}$, we have $\mathcal{OP}_{1;n,k}$ and $\mathcal{OP}_{1;n,\alpha}$.

An *extended ordered multiset partition* with content $\beta \vDash n$ with r 0's is an ordered multiset partition of the set $A(\beta) \cup \{0^r\}$ such that 0 is not contained in the last block. We let $\mathcal{OP}_{r;\beta}$ denote the set of all such extended ordered multiset partitions. Similarly, we have $\mathcal{OP}_{r;\beta,k}$ and $\mathcal{OP}_{r;\beta,\alpha}$.

The above three new combinatorial objects are defined from the same idea that they do not end with 0, and extended ordered multiset partitions have nice combinatorial properties. It is easy to check that all the for statistics: inv, maj, dinv, minimaj are well defined on the set $\mathcal{OP}_{r;\beta,\alpha}$. Let

$$D^{\text{stat}}_{r;\beta,k}(q) := \sum_{\pi \in \mathcal{OP}_{r;\beta,k}} q^{\text{stat}(\pi)}$$

where stat is one of the statistics *inv, maj, dinv, minimaj*. Using the notation of decorated parking functions with blank valleys, we have

$$\begin{aligned}
\text{Rise}_{r;n,k}[X;q,t] &= \sum_{(\text{PF},R) \in \mathcal{WPF}^{\text{Rise}}_{r;n,n-k-1}} t^{\text{area}^-(\text{PF},R)} q^{\text{dinv}(\text{PF})} x^{\text{PF}}, \\
\text{Val}_{r;n,k}[X;q,t] &= \sum_{(\text{PF},V) \in \mathcal{WPF}^{\text{Val}}_{r;n,n-k-1}} t^{\text{area}(\text{PF})} q^{\text{dinv}^-(\text{PF},V)} x^{\text{PF}}.
\end{aligned}$$

We can prove the following theorem:

Theorem 4.3. *For any integers n,k,r and weak composition β,*

$$\text{Rise}_{r;n,k}[X;q,0]|_{M_\beta} = D^{\text{dinv}}_{r;\beta,k+1}(q), \tag{4.8}$$

$$\text{Rise}_{r;n,k}[X;0,q]|_{M_\beta} = D^{\text{maj}}_{r;\beta,k+1}(q), \tag{4.9}$$

$$\text{Val}_{r;n,k}[X;q,0]|_{M_\beta} = D^{\text{inv}}_{r;\beta,k+1}(q), \tag{4.10}$$

$$\text{Val}_{r;n,k}[X;0,q]|_{M_\beta} = D^{\text{minimaj}}_{r;\beta,k+1}(q). \tag{4.11}$$

Proof. Similar to the definition of $\mathcal{OP}_{r;\beta}$, we shall let $\mathcal{OP}^{\text{all}}_{r;\beta}$ denote the set of ordered multiset partitions of the set $A(\beta) \cup \{0^r\}$, but there is no restriction of the placement of 0 (i.e. 0 is allowed to be in the last block). Similarly, we have $\mathcal{OP}^{\text{all}}_{r;\beta,k}$ and $\mathcal{OP}^{\text{all}}_{r;\beta,\alpha}$.

Haglund et al. proved Theorem 4.1 by constructing 4 bijections $\gamma^{\text{dinv}}, \gamma^{\text{maj}}, \gamma^{\text{inv}}, \gamma^{\text{minimaj}}$

between ordered multiset partitions and decorated word parking functions:

$$\begin{aligned}
\gamma^{\mathrm{dinv}}:\mathcal{OP}_{\beta,k+1} &\rightarrow \{(\mathrm{PF},R)\in\mathcal{WPF}^{\mathrm{Rise}}_{n,n-k-1},\ X^{\mathrm{PF}}=\prod_{i=1}^{\ell(\beta)}x_i^{\beta_i},\ \mathrm{area}^-(\mathrm{PF},R)=0\},\\
\gamma^{\mathrm{maj}}:\mathcal{OP}_{\beta,k+1} &\rightarrow \{(\mathrm{PF},R)\in\mathcal{WPF}^{\mathrm{Rise}}_{n,n-k-1},\ X^{\mathrm{PF}}=\prod_{i=1}^{\ell(\beta)}x_i^{\beta_i},\ \mathrm{dinv}(\mathrm{PF})=0\},\\
\gamma^{\mathrm{inv}}:\mathcal{OP}_{\beta,k+1} &\rightarrow \{(\mathrm{PF},V)\in\mathcal{WPF}^{\mathrm{Val}}_{n,n-k-1},\ X^{\mathrm{PF}}=\prod_{i=1}^{\ell(\beta)}x_i^{\beta_i},\ \mathrm{area}(\mathrm{PF})=0\},\\
\gamma^{\mathrm{minimaj}}:\mathcal{OP}_{\beta,k+1} &\rightarrow \{(\mathrm{PF},V)\in\mathcal{WPF}^{\mathrm{Val}}_{n,n-k-1},\ X^{\mathrm{PF}}=\prod_{i=1}^{\ell(\beta)}x_i^{\beta_i},\ \mathrm{dinv}^-(\mathrm{PF},V)=0\}.
\end{aligned}$$

The details can be found in Appendix A. If we allow 0 as an element of an ordered multiset partition, then the four maps can be naturally generalized to the set $\mathcal{OP}^{\mathrm{all}}_{r;\beta,k}$, and the range of the maps are parking functions that allow 0 as a car, i.e. if we let $\mathcal{WPF}^{\mathrm{Rise}+}_{r;n,k}$ and $\mathcal{WPF}^{\mathrm{Val}+}_{r;n,k}$ be the set of rise and valley decorated word parking function with r 0's (car 0 is allowed in the first row), then we have bijections

$$\begin{aligned}
\gamma^{\mathrm{dinv}}:\mathcal{OP}^{\mathrm{all}}_{r;\beta,k+1} &\rightarrow \{(\mathrm{PF},R)\in\mathcal{WPF}^{\mathrm{Rise}+}_{r;n,n-k-1},\ X^{\mathrm{PF}}=\prod_{i=1}^{\ell(\beta)}x_i^{\beta_i},\ \mathrm{area}^-(\mathrm{PF},R)=0\},\\
\gamma^{\mathrm{maj}}:\mathcal{OP}^{\mathrm{all}}_{r;\beta,k+1} &\rightarrow \{(\mathrm{PF},R)\in\mathcal{WPF}^{\mathrm{Rise}+}_{r;n,n-k-1},\ X^{\mathrm{PF}}=\prod_{i=1}^{\ell(\beta)}x_i^{\beta_i},\ \mathrm{dinv}(\mathrm{PF})=0\},\\
\gamma^{\mathrm{inv}}:\mathcal{OP}^{\mathrm{all}}_{r;\beta,k+1} &\rightarrow \{(\mathrm{PF},V)\in\mathcal{WPF}^{\mathrm{Val}+}_{r;n,n-k-1},\ X^{\mathrm{PF}}=\prod_{i=1}^{\ell(\beta)}x_i^{\beta_i},\ \mathrm{area}(\mathrm{PF})=0\},\\
\gamma^{\mathrm{minimaj}}:\mathcal{OP}^{\mathrm{all}}_{r;\beta,k+1} &\rightarrow \{(\mathrm{PF},V)\in\mathcal{WPF}^{\mathrm{Val}+}_{r;n,n-k-1},\ X^{\mathrm{PF}}=\prod_{i=1}^{\ell(\beta)}x_i^{\beta_i},\ \mathrm{dinv}^-(\mathrm{PF},V)=0\}.
\end{aligned}$$

We have mentioned the fact below Theorem 4.1 and in Appendix A that each bijection γ^{stat} maps the minimum element in the last part of π into the car in the first row of $\gamma^{\mathrm{stat}}(\pi)$. Since the set $\mathcal{OP}_{r;\beta,k}$ contains ordered multiset partitions in $\mathcal{OP}^{\mathrm{all}}_{r;\beta,k}$ that 0 is not contained in the last block, the restriction of the maps γ^{stat} on the set $\mathcal{OP}_{r;\beta,k}\subseteq\mathcal{OP}^{\mathrm{all}}_{r;\beta,k}$ is a bijection between $\mathcal{OP}_{r;\beta,k}$ and the corresponding set of parking functions with r 0's but 0 is not allowed in the first row, which exactly matches the set $\mathcal{WPF}^{\mathrm{Rise}}_{r;n,n-k-1}$ or $\mathcal{WPF}^{\mathrm{Val}}_{r;n,n-k-1}$, and the restriction of the maps γ^{stat} on $\mathcal{OP}_{r;\beta,k} \subseteq \mathcal{OP}^{\mathrm{all}}_{r;\beta,k}$ are bijections:

$$\begin{aligned}
\gamma^{\mathrm{dinv}} : \mathcal{OP}_{r;\beta,k+1} &\to \{(\mathrm{PF},R) \in \mathcal{WPF}^{\mathrm{Rise}}_{r;n,n-k-1},\, X^{\mathrm{PF}} = \prod_{i=1}^{\ell(\beta)} x_i^{\beta_i},\ \mathrm{area}^-(\mathrm{PF},R) = 0\},\\
\gamma^{\mathrm{maj}} : \mathcal{OP}_{r;\beta,k+1} &\to \{(\mathrm{PF},R) \in \mathcal{WPF}^{\mathrm{Rise}}_{r;n,n-k-1},\, X^{\mathrm{PF}} = \prod_{i=1}^{\ell(\beta)} x_i^{\beta_i},\ \mathrm{dinv}(\mathrm{PF}) = 0\},\\
\gamma^{\mathrm{inv}} : \mathcal{OP}_{r;\beta,k+1} &\to \{(\mathrm{PF},V) \in \mathcal{WPF}^{\mathrm{Val}}_{r;n,n-k-1},\, X^{\mathrm{PF}} = \prod_{i=1}^{\ell(\beta)} x_i^{\beta_i},\ \mathrm{area}(\mathrm{PF}) = 0\},\\
\gamma^{\mathrm{minimaj}} : \mathcal{OP}_{r;\beta,k+1} &\to \{(\mathrm{PF},V) \in \mathcal{WPF}^{\mathrm{Val}}_{r;n,n-k-1},\, X^{\mathrm{PF}} = \prod_{i=1}^{\ell(\beta)} x_i^{\beta_i},\ \mathrm{dinv}^-(\mathrm{PF},V) = 0\}.
\end{aligned}$$

Theorem 4.3 follows from the fact that γ^{stat} maps the statistic stat into parking function statistics $\mathrm{dinv}, \mathrm{area}^-, \mathrm{dinv}^-, \mathrm{area}$. □

Thus, the combinatorial sides of the conjectures about the expression $\Delta'_{e_k}\Delta_{h_r}e_n$ at the case when q or t equals 0 become generating functions about generalized ordered multiset partitions. We shall show in the following two sections that the statistics inv, maj, dinv, minimaj are equi-distributed on $\mathcal{OP}_{r;\beta,k}$.

4.3 The identity $D^{\mathrm{dinv}}_{r;\beta,k}(q) = D^{\mathrm{maj}}_{r;\beta,k}(q) = D^{\mathrm{inv}}_{r;\beta,k}(q)$

Recall that we let $\mathcal{OP}^{\mathrm{all}}_{r;\beta}$ denote the set of ordered multiset partitions of the set $A(\beta)\cup\{0^r\}$ and 0 is allowed to be in the last block. We also have $\mathcal{OP}^{\mathrm{all}}_{r;\beta,k}$ and $\mathcal{OP}^{\mathrm{all}}_{r;\beta,\alpha}$.

In fact, $\mathcal{OP}^{\mathrm{all}}_{r;\beta,k}$ only enlarge the alphabet of $\mathcal{OP}_{\beta,k}$ from $\mathbb{Z}_+$ to $\mathbb{Z}_{\geq 0}$, and it will inherit all the properties of $\mathcal{OP}_{\beta,k}$. For a composition $\beta = (\beta_1,\ldots,\beta_n)$ and integers r,k, we let

$$D^{\mathrm{stat}+}_{r;\beta,k}(q) := \sum_{\pi\in\mathcal{OP}^{\mathrm{all}}_{r;\beta,k}} q^{\mathrm{stat}(\pi)}$$

where stat is one of the statistics *inv, maj, dinv, minimaj*, then clearly

$$D^{\text{stat}+}_{r;(\beta_1,\ldots,\beta_n),k}(q) = D^{\text{stat}}_{(r,\beta_1,\ldots,\beta_n),k}(q),$$

since we can add 1 to all the entries of a multiset partition in $\mathcal{OP}^{\text{all}}_{r;(\beta_1,\ldots,\beta_n),k}$ to get a multiset partition in $\mathcal{OP}_{(r,\beta_1,\ldots,\beta_n),k}$. It follows from Theorem 4.2 that,

Corollary 4.4. *For any integers n, r and composition β,*

$$D^{\text{inv}+}_{r;\beta,k}(q) = D^{\text{maj}+}_{r;\beta,k}(q) = D^{\text{dinv}+}_{r;\beta,k}(q) = D^{\text{minimaj}+}_{r;\beta,k}(q).$$

For a composition $\beta = (\beta_1,\ldots,\beta_n)$, we let $\beta^- = (\beta_1,\ldots,\beta_{n-1})$ be the composition obtained by removing the last part of β. We also let $[0,\ell]$ be the set $\{0,1,\ldots,\ell\}$. In order to prove the result about ordered multiset partition that $D^{\text{inv}}_{\beta,k}(q) = D^{\text{maj}}_{\beta,k}(q) = D^{\text{dinv}}_{\beta,k}(q)$, Wilson in [Wil16] constructed 3 *insertion* maps:

$$\phi^{\text{stat}}_{\beta,k,\ell} : \mathcal{OP}_{\beta^-,\ell} \times \binom{[0,\ell-1]}{\beta_n - k + \ell} \times \left(\!\!\binom{[0,\ell]}{k-\ell}\!\!\right) \to \mathcal{OP}_{\beta,k},$$

where stat is one of the statistics *inv, maj, dinv*, and he proved that

$$\text{stat}\left(\phi^{\text{stat}}_{\beta,k,\ell}(\pi,U,B)\right) = \text{stat}(\pi) + \sum_{u\in U} u + \sum_{b\in B} b$$

for all the three statistics. In this section, we shall generalize Wilson's insertion maps to extended ordered multiset partitions to prove the identity that

$$D^{\text{dinv}}_{r;\beta,k}(q) = D^{\text{maj}}_{r;\beta,k}(q) = D^{\text{inv}}_{r;\beta,k}(q).$$

This identity is also proved by D'Adderio, Iraci and Wyngaerd in [DIW19] independently.

4.3.1 The insertion map for inv

We shall generalize the map $\phi^{\text{inv}}_{\beta,k,\ell}$ of Wilson to the extended case as

$$\phi^{\text{inv}}_{r;\beta,k,\ell} : \mathcal{OP}_{r;\beta^-,\ell} \times \binom{[0,\ell-1]}{\beta_n - k + \ell} \times \left(\!\!\binom{[0,\ell]}{k-\ell}\!\!\right) + \left(\mathcal{OP}^{\text{all}}_{r;\beta^-,\ell} - \mathcal{OP}_{r;\beta^-,\ell}\right) \times \binom{[0,\ell-1]}{\beta_n - k + \ell} \times \left(\!\!\binom{[0,\ell]}{k-\ell-1}\!\!\right) \to \mathcal{OP}_{r;\beta,k}$$

such that

$$\operatorname{inv}\left(\phi^{\text{inv}}_{r;\beta,k,\ell}(\pi,U,B)\right) = \operatorname{inv}(\pi) + \sum_{u\in U} u + \sum_{b\in B} b. \tag{4.12}$$

Given $\beta = (\beta_1,\ldots,\beta_n)$ and $\pi \in \mathcal{OP}_{r;\beta^-,\ell}$, we label each block plus the space to the left of π from right to left with numbers $0,1,\ldots,\ell$. Then for any $U \in \binom{[0,\ell-1]}{\beta_n-k+\ell}$ and $B \in \left(\!\binom{[0,\ell]}{k-\ell}\!\right)$, we construct $\phi^{\text{inv}}_{r;\beta,k,\ell}(\pi,U,B)$ as follows.

We repeatedly remove the largest number i from the multiset $U \cup B$, taking from U first if the largest numbers are equal. If $i \in U$, then we place an n to the block with label i; if $i \in B$, then we add a new block of a singleton n to the right of the block with label i. This process constructs all the ordered multiset partitions in $\mathcal{OP}_{r;\beta,k}$ such that the last block that is not a singleton $\{n\}$ does not contain 0.

In order to construct the remaining ordered partitions in $\mathcal{OP}_{r;\beta,k}$, we take ordered multiset partitions π in the set $\left(\mathcal{OP}^{\text{all}}_{r;\beta^-,\ell} - \mathcal{OP}_{r;\beta^-,\ell}\right)$ (which means the last block of π contains 0). Then for any $U \in \binom{[0,\ell-1]}{\beta_n-k+\ell}$ and $B' \in \left(\!\binom{[0,\ell]}{k-\ell-1}\!\right)$, we set the multiset $B = B' \cup \{n\}$, and we construct $\phi^{\text{inv}}_{r;\beta,k,\ell}(\pi,U,B)$ by repeatedly inserting numbers in the multiset $U \cup B$ in the same way. One can check easily that this gives all the ordered multiset partitions in $\mathcal{OP}_{r;\beta,k}$, and the inv statistic increases by i each time we insert an i, thus Equation (4.12) follows.

4.3.2 The insertion map for maj

In order to define the map $\phi^{\mathrm{maj}}_{r;\beta,k,\ell}$, we shall introduce the *descent-starred permutation notation* of an ordered partition. For any ordered partition $\pi = B_1/\cdots/B_n$, we write the numbers of each block in decreasing order, remove the slashes and add stars at the descent positions that is entirely contained in some block of π. This permutation with stars is called the *descent-starred permutation notation* of π.

The set of the positions with stars is denoted by $S(\pi)$, and the permutation is denoted by $\sigma(\pi)$ introduced in Section 4.2.1. For example, if $\pi = 134/47/23$, then $\sigma(\pi) = 4317432$, $S(\pi) = \{1,2,4,6\}$ and $4_*3_*17_*43_*2$ is the corresponding descent-starred permutation.

The map

$$\phi^{\mathrm{maj}}_{r;\beta,k,\ell} : \mathcal{OP}_{r;\beta^-,\ell} \times \binom{[0,\ell-1]}{\beta_n - k + \ell} \times \left(\!\!\binom{[0,\ell]}{k-\ell}\!\!\right) + \left(\mathcal{OP}^{\mathrm{all}}_{r;\beta^-,\ell} - \mathcal{OP}_{r;\beta^-,\ell}\right) \times \binom{[0,\ell-1]}{\beta_n - k + \ell} \times \left(\!\!\binom{[0,\ell]}{k-\ell-1}\!\!\right) \to \mathcal{OP}_{r;\beta,k}$$

is defined as follows.

Given $\beta = (\beta_1,\ldots,\beta_n)$ and $\pi \in \mathcal{OP}_{r;\beta^-,\ell}$, we write π in descent-starred notation and let $\sigma = \sigma(\pi)$. We label the rightmost position firstly, and label the unstarred descent positions of π secondly, then label the unstarred non-descent positions (including the leftmost position) thirdly with labels $0,\ldots,\ell$.

For any $U \in \binom{[0,\ell-1]}{\beta_n-k+\ell}$ and $B \in \left(\!\binom{[0,\ell]}{k-\ell}\!\right)$, we construct $\phi^{\mathrm{maj}}_{r;\beta,k,\ell}(\pi,U,B)$ by setting $U^+ = \{u+1 : u \in U\}$, then repeatedly remove the largest number i from the multiset $U^+ \cup B$, taking from B first if the largest numbers are equal. The algorithm of inserting i is as follows:

1. Insert the number n at the position with label i.

2. Move each star that appears to the right of the new n one descent to the left.

3. If $i \in U^+$, then star the rightmost descent.

4. Relabel the starred permutation as before, stopping at i if $i \in B$ and $i-1$ if $i \in U^+$.

This process constructs all the ordered multiset partitions in $\mathcal{OP}_{r;\beta,k}$ such that the last block that is not a singleton $\{n\}$ does not contain 0.

In order to construct the remaining ordered partitions in $\mathcal{OP}_{r;\beta,k}$, we take ordered multiset partitions π in the set $\left(\mathcal{OP}^{\text{all}}_{r;\beta^-,\ell} - \mathcal{OP}_{r;\beta^-,\ell}\right)$ that the last block contains 0. Then for any $U \in \binom{[0,\ell-1]}{\beta_n-k+\ell}$ and $B' \in \left(\!\binom{[0,\ell]}{k-\ell-1}\!\right)$, we set $U^+ = \{u+1 : u \in U\}$ and $B = B' \cup \{n\}$, and we construct $\phi^{\text{maj}}_{r;\beta,k,\ell}(\pi,U,B)$ by repeatedly inserting numbers in the multiset $U^+ \cup B$ in the same way. One can check easily that this gives all the ordered multiset partitions in $\mathcal{OP}_{r;\beta,k}$. Wilson [Wil16] gave a proof that the maj statistic increases by i each time we insert an i in the non-extended case, which works naturally for the extended case, thus we have

$$\operatorname{maj}\left(\phi^{\text{maj}}_{r;\beta,k,\ell}(\pi,U,B)\right) = \operatorname{maj}(\pi) + \sum_{u\in U} u + \sum_{b\in B} b. \tag{4.13}$$

4.3.3 The insertion map for dinv

We define a map

$$\begin{aligned}\phi^{\text{dinv}}_{r;\beta,k,\ell} : \mathcal{OP}_{r;\beta^-,\ell} \times \binom{[0,\ell-1]}{\beta_n-k+\ell} \times \left(\!\binom{[0,\ell]}{k-\ell}\!\right) \\ + \left(\mathcal{OP}^{\text{all}}_{r;\beta^-,\ell} - \mathcal{OP}_{r;\beta^-,\ell}\right) \times \binom{[0,\ell-1]}{\beta_n-k+\ell} \times \left(\!\binom{[0,\ell]}{k-\ell-1}\!\right) \to \mathcal{OP}_{r;\beta,k}.\end{aligned}$$

Given $\beta = (\beta_1,\ldots,\beta_n)$ and $\pi \in \mathcal{OP}_{r;\beta^-,\ell}$, we label the $\ell+1$ spaces (the spaces between parts as well as the spaces in the two ends) of π from right to left with numbers $0,1,\ldots,\ell$ which we call the *gap labels*. Next, we label the blocks from highest to lowest length (from left to right for each length) with numbers $0,1,\ldots,\ell-1$ which we call the *block labels*.

For any $U \in \binom{[0,\ell-1]}{\beta_n-k+\ell}$ and $B \in \left(\!\binom{[0,\ell]}{k-\ell}\!\right)$, we can construct $\phi^{\text{dinv}}_{r;\beta,k,\ell}(\pi,U,B)$ by inserting an n into each block whose label is in U and inserting a singleton block $\{n\}$ at the gap b for each

$b \in B$. This process constructs all the ordered multiset partitions in $\mathcal{OP}_{r;\beta,k}$ such that the last block that is not a singleton $\{n\}$ does not contain 0.

In order to construct the remaining ordered partitions in $\mathcal{OP}_{r;\beta,k}$, we take ordered multiset partitions π in $\left(\mathcal{OP}^{\text{all}}_{r;\beta^-,\ell} - \mathcal{OP}_{r;\beta^-,\ell}\right)$. Then for any $U \in \binom{[0,\ell-1]}{\beta_n-k+\ell}$ and $B' \in \left(\!\binom{[0,\ell]}{k-\ell-1}\!\right)$, we set the multiset $B = B' \cup \{n\}$, and we construct $\phi^{\text{dinv}}_{r;\beta,k,\ell}(\pi,U,B)$ in the same way. One can check easily that this gives all the ordered multiset partitions in $\mathcal{OP}_{r;\beta,k}$, and the dinv statistic increases by i each time we insert an i, thus we have

$$\operatorname{dinv}\left(\phi^{\text{dinv}}_{r;\beta,k,\ell}(\pi,U,B)\right) = \operatorname{dinv}(\pi) + \sum_{u\in U} u + \sum_{b\in B} b. \tag{4.14}$$

According to the definitions of maps $\phi^{\text{inv}}_{r;\beta,k,\ell}$, $\phi^{\text{maj}}_{r;\beta,k,\ell}$, $\phi^{\text{dinv}}_{r;\beta,k,\ell}$ and Equations (4.12), (4.13) and (4.14), one can conclude that

Theorem 4.5. *For any integers n, r and composition β,*

$$D^{\text{inv}}_{r;\beta,k}(q) = D^{\text{maj}}_{r;\beta,k}(q) = D^{\text{dinv}}_{r;\beta,k}(q).$$

4.4 The identity $D^{\text{inv}}_{r;\beta,k}(q) = D^{\text{minimaj}}_{r;\beta,k}(q)$

The goal of this section is to generalize the $(\text{inv}, \text{minimaj})$ equi-distribution theorem of Rhoades [Rho18] from the set $\mathcal{OP}_{\beta,k}$ to the set $\mathcal{OP}_{r;\beta,k}$. For our convenience, we shall abbreviate D^{inv} and D^{minimaj} to I and M, i.e. we shall use the notations

$$I_{\beta,k}(q) = D^{\text{inv}}_{\beta,k}(q), \quad I_{\beta,\alpha}(q) = D^{\text{inv}}_{\beta,\alpha}(q), \quad I_{r;\beta,k}(q) = D^{\text{inv}}_{r;\beta,k}(q), \quad I_{r;\beta,\alpha}(q) = D^{\text{inv}}_{r;\beta,\alpha}(q),$$
$$M_{\beta,k}(q) = D^{\text{minimaj}}_{\beta,k}(q), \quad M_{\beta,\alpha}(q) = D^{\text{minimaj}}_{\beta,\alpha}(q),$$
$$M_{r;\beta,k}(q) = D^{\text{minimaj}}_{r;\beta,k}(q), \quad M_{r;\beta,\alpha}(q) = D^{\text{minimaj}}_{r;\beta,\alpha}(q).$$

Further, we let

$$I^{\text{all}}_{r;\beta,k}(q) = D^{\text{inv}+}_{r;\beta,k}(q), \quad I^{\text{all}}_{r;\beta,\alpha}(q) = D^{\text{inv}+}_{r;\beta,\alpha}(q)$$
$$M^{\text{all}}_{r;\beta,k}(q) = D^{\text{minimaj}+}_{r;\beta,k}(q), \quad \text{and} \quad M^{\text{all}}_{r;\beta,\alpha}(q) = D^{\text{minimaj}+}_{r;\beta,\alpha}(q)$$

denote the generating functions that 0 is allowed in the last block.

4.4.1 The recursion for inv

For any integer m and set $S \subseteq [m]$, we let $\chi_S = (\chi_S(1), \ldots, \chi_S(m))$ be the sequence such that $\chi_S(i) = \chi(i \in S)$. For two sequences γ_1 and γ_2 of the same length, we write $\gamma_1 \leq \gamma_2$ if each entry of γ_1 is less than or equal to the corresponding entry of γ_2.

Given an integer n, a weak composition $\beta = (\beta_1, \ldots, \beta_m) \vDash n$ and a strong composition $\alpha = (\alpha_1, \ldots, \alpha_k) \vDash_{\text{strong}} n$, we still use the notation $\alpha^- = (\alpha_1, \ldots, \alpha_{k-1})$ for the composition of $n - \alpha_k$ that the last part of α is removed.

Recall that by definition, $\mathcal{OP}_{r;\beta,\alpha}$ is the set of extended ordered multiset partition of the multiset $A(\beta) \cup \{0^r\}$ and shape α such that 0 is not contained in the last block, while $\mathcal{OP}^{\text{all}}_{r;\beta,\alpha}$ allows 0 in the last block. Their generating functions tracking the statistic inv are $I_{r;\beta,\alpha}(q)$ and $I^{\text{all}}_{r;\beta,\alpha}(q)$ respectively. Then we have the following theorem which is analogous to Lemma 3.2 in [Rho18].

Theorem 4.6. *The generating function $I_{r;\beta,\alpha}(q)$ satisfies the following equation:*

$$I_{r;\beta,\alpha}(q) = \sum_{\substack{S \subseteq [m],\ |S| = \alpha_k, \\ \chi_S \leq \beta}} q^{\sum_{i=min(S)+1}^{m} (\beta_i - \chi_S(i))} I^{\text{all}}_{r;\beta-\chi_S,\alpha^-}(q). \tag{4.15}$$

Proof. Consider an ordered multiset partition $\mu = B_1/\cdots/B_k \in \mathcal{OP}_{r;\beta,\alpha}$. Writing $S = B_k$, we have that $B_1/\cdots/B_{k-1} \in \mathcal{OP}^{\text{all}}_{r;\beta-\chi_S,\alpha^-}$. Since each element in the ordered partition $B_1/\cdots/B_{k-1}$

that is bigger than $\min(S)$ creates an inversion with the last block, Equation (4.15) follows immediately. □

Summing over all the strong compositions α of n with k parts, we have the following corollary.

Corollary 4.7. *The generating function $I_{r;\beta,k}(q)$ satisfies the following equation:*

$$I_{r;\beta,k}(q) = \sum_{S\subseteq[m],\ \chi_S\leq\beta} q^{\sum_{i=min(S)+1}^{m}(\beta_i-\chi_S(i))} I^{\text{all}}_{r;\beta-\chi_S,k-1}(q). \tag{4.16}$$

We shall prove a similar result about the statistic minimaj in the following subsection.

4.4.2 The recursion for minimaj

In our new notation, Corollary 4.4 shows that

$$I^{\text{all}}_{r;\beta,k}(q) = M^{\text{all}}_{r;\beta,k}(q). \tag{4.17}$$

We shall prove in this subsection that

Theorem 4.8. *The generating function $M_{r;\beta,k}(q)$ satisfies the following equation:*

$$M_{r;\beta,k}(q) = \sum_{S\subseteq[m],\ \chi_S\leq\beta} q^{\sum_{i=min(S)+1}^{m}(\beta_i-\chi_S(i))} M^{\text{all}}_{r;\beta-\chi_S,k-1}(q). \tag{4.18}$$

Then as a consequence of Corollary 4.7, Theorem 4.8 and Equation (4.17), we have

Theorem 4.9. *For any integers n, r and composition β,*

$$D^{\text{inv}}_{r;\beta,k}(q) = D^{\text{minimaj}}_{r;\beta,k}(q).$$

In order to prove Theorem 4.8, we need to state some combinatorial actions and properties about the statistic minimaj. We always use the setting that for any integers n, r, we consider ordered multiset partitions of the form $\mu = B_1/\cdots/B_k \in \mathcal{OP}_{r;\beta,\alpha}$, where $\beta = (\beta_1,\ldots,\beta_m) \vDash n$ is a weak composition and $\alpha = (\alpha_1,\ldots,\alpha_k) \vDash_{\text{strong}} n$ is a strong composition. We let $\alpha^- = (\alpha_1,\ldots,\alpha_{k-1})$.

A *k-segmented word* is a pair (w,α) such that $w = w_1\cdots w_n$ is a length n word and α is a strong composition of n. We write such k-segmented word in the form of a word w with dots after $w_{\alpha_1}, w_{\alpha_1+\alpha_2},\ldots,w_{\alpha_1+\cdots+\alpha_{k-1}}$. The components of the words separated by the dots are called *segments*. For example, the 3-segmented word $(3342412,(2,3,2))$ can be written as $33\cdot 424\cdot 12$.

For an ordered multiset partition $\mu = B_1/\cdots/B_k \in \mathcal{OP}_{r;\beta,\alpha}$ where $B_i = \{j_1^{(i)} < \ldots < j_{\alpha_i}^{(i)}\}$, we let $w(\mu) = \mu[1]\cdot\mu[2]\cdot\ \cdots\ \cdot\mu[k]$ denote the *k-segmented word* obtained in the following way: we let the last segment $\mu[k]$ be the increasing word $j_1^{(k)}\cdots j_{\alpha_k}^{(k)}$. For $1 \le i \le k-1$, assume that the $i+1^{\text{st}}$ segment $\mu[i+1]$ is defined and let r be the first letter of $\mu[i+1]$. Let $j_1^{(i)},\ldots,j_m^{(i)}$ be the numbers that are less than or equal to r, and let $j_{m+1}^{(i)},\ldots,j_{\alpha_i}^{(i)}$ be the numbers that are greater than r, then we define $\mu[i] = j_{m+1}^{(i)}\cdots j_{\alpha_i}^{(i)} j_1^{(i)}\cdots j_m^{(i)}$. We also refer to $w(\mu)$ as the permutation component of the segmented word without causing ambiguity. Note that $w(\mu)$ as a permutation coincides with our definition of miniword(μ). Thus we have the following lemma:

Lemma 4.10. *Let μ be an ordered multiset partition, then* $\text{minimaj}(\mu) = \text{maj}(w(\mu))$.

Rhoades in [Rho18] defined an action on ordered multiset partitions μ to interchange the number of i and $i+1$ in μ, called the *t_i-switch map*. Let s_i be the action on a sequence that interchange its ith and $i+1$st component, then Rhoades proved the following theorem:

Theorem 4.11 (Rhoades)**.** *There exists a bijective map*

$$t_i: \quad \mathcal{OP}_{\beta,k} \to \mathcal{OP}_{s_i\cdot\beta,k}$$

such that $\text{minimaj}(t_i(\mu)) = \text{minimaj}(\mu)$.

Recall that we can add 1 to all the entries of a multiset partition in $\mathcal{OP}^{\text{all}}_{r;(\beta_1,\ldots,\beta_n),k}$ to get a multiset partition in $\mathcal{OP}_{(r,\beta_1,\ldots,\beta_n),k}$, we can naturally generalize this result to the set $\mathcal{OP}^{\text{all}}_{r;\beta,k}$ that allows us to rearrange the component of β and the number r:

Corollary 4.12. *Let $(\gamma_0,\gamma_1,\ldots,\gamma_m)$ be any rearrangement of the sequence $(r,\beta_1,\ldots,\beta_m)$, then there is a minimaj-preserving bijection ψ between the sets $\mathcal{OP}^{\text{all}}_{\gamma_0;(\gamma_1,\ldots,\gamma_m),k}$ and $\mathcal{OP}^{\text{all}}_{r;(\beta_1,\ldots,\beta_m),k}$.*

It is obvious that for an ordered multiset partition, the contribution of the last block to minimaj only depends on the minimum element of the last block. Thus we have the following lemma.

Lemma 4.13. *Let $B_1/\cdots/B_k$ be an ordered multiset partition. Then*

$$\operatorname{minimaj}(B_1/\cdots/B_k) = \operatorname{minimaj}(B_1/\cdots/\min(B_k)). \tag{4.19}$$

Rhoades in [Rho18] defined an action of the group $\mathbb{Z}_m = \langle c\rangle$ on $\mathcal{OP}_{\beta,\alpha}$ by decrementing all the letters by 1 modulo m. Analogously, we define the group action of $\mathbb{Z}_{m+1} = \langle c\rangle$ on $\mathcal{OP}^{\text{all}}_{r;\beta,\alpha}$ by decrementing all the letters by 1 modulo $m+1$. Rhoades in [Rho18] proved that

Lemma 4.14 (Lemma 3.4 in [Rho18])**.** *If the last component of α is 1, then $w(c.\mu) = c.w(\mu)$ for any $\mu \in \mathcal{OP}_{\beta,\alpha}$.*

Recall that there is a bijective relation between $\mathcal{OP}^{\text{all}}_{r;(\beta_1,\ldots,\beta_m),\alpha}$ and $\mathcal{OP}_{(r,\beta_1,\ldots,\beta_m),\alpha}$. It follows from Lemma 4.14 and our new group action of $\mathbb{Z}_{m+1}$ that

Lemma 4.15. *If the last component of α is 1, then $w(c.\mu) = c.w(\mu)$ for any $\mu \in \mathcal{OP}^{\text{all}}_{r;\beta,\alpha}$.*

Another property about the action c is summarized in the following lemma:

Lemma 4.16. *For any word $w = w_1\cdots w_n$ with content $\{0^r, 1^{\beta_1},\ldots,m^{\beta_m}\}$ such that $w_n \neq 0$, we have* $\operatorname{maj}(c.w) = \operatorname{maj}(w) + r$.

Proof. The map c moves every descent occurring before a maximal contiguous run of 0's in w to the position at the end of this run. □

Now we can prove the following lemma.

Lemma 4.17. *Given integers n,r. Let $\alpha = (\alpha_1,\ldots,\alpha_k) \vDash_{\text{strong}} n$ be a strong composition with $\alpha_k = 1$ and let $\beta = (\beta_1,\ldots,\beta_m) \vDash n$ be a weak composition. We have*

$$M_{r;\beta,\alpha}(q) = \sum_{\beta_i>0} q^{\beta_{i+1}+\cdots+\beta_m} M^{\text{all}}_{r;(\beta_{i+1},\ldots,\beta_m,\beta_1,\ldots,\beta_i-1),\alpha^-}(q). \tag{4.20}$$

Proof. We shall prove the recursion above about the generating function $M_{r;\beta,\alpha}(q)$ where $\alpha_k = 1$. Without loss of generality, we assume that β is a strong composition. Consider an ordered multiset partition $\mu \in \mathcal{OP}_{r;\beta,\alpha}$. If the last block of μ is a singleton $\{m\}$, then clearly it does not contribute anything to minimaj(μ). Writing $\mu = \mu'/m$, then minimaj$(\mu) =$ minimaj(μ').

Next consider the case when $\mu = \mu'/m-i$ end with $m-i$ for some $i \in \{1,\ldots,m-1\}$, then $\mu' \in \mathcal{OP}^{\text{all}}_{r;(\beta_1,\ldots,\beta_{m-i}-1,\ldots,\beta_m),\alpha^-}$. It follows that we have the following consequence of Lemma 4.15 and Lemma 4.16:

$$\begin{aligned}
\text{minimaj}(\mu'/m-i) &= \text{minimaj}(c^i.(c^{-i}.\mu'|m)) \\
&= \text{minimaj}(c^{-i}.\mu'|m) + \beta_{m-i+1} + \cdots + \beta_m
\end{aligned}$$

where $c^{-i}.\mu' \in \mathcal{OP}^{\text{all}}_{\beta_{m-i+1},(\beta_{m-i+2},\ldots,\beta_m,r,\beta_1,\ldots,\beta_{m-i}-1),\alpha^-}$, and we have

$$M_{r;\beta,\alpha}(q) = \sum_{\beta_{m-i}>0} q^{\beta_{m-i+1}+\cdots+\beta_m} M^{\text{all}}_{\beta_{m-i+1};(\beta_{m-i+2},\ldots,\beta_m,r,\beta_1,\ldots,\beta_{m-i}-1),\alpha^-}(q). \tag{4.21}$$

Equation (4.20) follows immediately from Equation (4.21) and Corollary 4.12 since we can permute r and entries of β. □

Now we are ready to prove Theorem 4.8.

Proof of Theorem 4.8. Let $\mu = B_1/\cdots/B_k \in \mathcal{OP}_{r;\beta,\alpha}$. For the case when $\alpha = (\alpha_1,\ldots,\alpha_{k-1},1)$, we have the following recursion as a consequence of Lemma 4.17:

$$\begin{aligned}
\sum_{\alpha^-} M_{r;\beta,\alpha}(q) &= \sum_{\alpha^-}\sum_{\beta_i>0} q^{\beta_{i+1}+\cdots+\beta_m} M^{\text{all}}_{r;(\beta_{i+1},\ldots,\beta_m,\beta_1,\ldots,\beta_i-1),\alpha^-}(q) \\
&= \sum_{\beta_i>0}\sum_{\alpha^-} q^{\beta_{i+1}+\cdots+\beta_m} M^{\text{all}}_{r;(\beta_{i+1},\ldots,\beta_m,\beta_1,\ldots,\beta_i-1),\alpha^-}(q) \\
&= \sum_{\beta_i>0} q^{\beta_{i+1}+\cdots+\beta_m} M^{\text{all}}_{r;(\beta_{i+1},\ldots,\beta_m,\beta_1,\ldots,\beta_i-1),k-1}(q) \\
&= \sum_{\beta_i>0} q^{\beta_{i+1}+\cdots+\beta_m} M^{\text{all}}_{r;(\beta_1,\ldots,\beta_i-1,\ldots,\beta_m),k-1}(q).
\end{aligned} \tag{4.22}$$

The first line is Equation (4.20) summed over all compositions $\alpha^- \models_{\text{strong}} (n-1)$ with $k-1$ parts; the second line interchanges the order of the two summations; the third line evaluates the inner sum over all possible α^-'s; the last line is an application of Corollary 4.12.

More generally, if the last block is of size $\alpha_k \geq 1$, then the following equation follows as a consequence of Equation (4.22):

$$M_{r;\beta,k}(q) = \sum_{B_k\subseteq[m],\ \chi_{B_k}\leq\beta} q^{\sum_{i=min(B_k)+1}^{m}(\beta_i-\chi_{B_k}(i))} M^{\text{all}}_{r;\beta-\chi_{B_k},k-1}(q), \tag{4.23}$$

which proves Theorem 4.8. □

4.5 The Mahonian distribution on $\mathcal{OP}_{r;\beta,k}$

Following from Theorem 4.5 and Theorem 4.9, we have

Corollary 4.18. *For any integers n, r and composition β,*

$$D^{\text{inv}}_{r;\beta,k}(q) = D^{\text{maj}}_{r;\beta,k}(q) = D^{\text{dinv}}_{r;\beta,k}(q) = D^{\text{minimaj}}_{r;\beta,k}(q).$$

Benkart et al. [BCH+18] proved the equi-distributivity of the statistics minimaj and maj on ordered multiset partitions using a crystal structure. This provides another idea of proving Corollary 4.18.

Given that D'Adderio, Iraci and Wyngaerd [DIW19] have proved the following,

Theorem 4.19 (D'Adderio, Iraci and Wyngaerd)**.** *For any integers $n,k \geq 0$, we have the equality*

$$\mathrm{Rise}_{r;n,k}[X;q,0] = \mathrm{Rise}_{r;n,k}[X;0,q] = \Delta'_{e_k}\Delta_{h_r}e_n|_{t=0} = \Delta'_{e_k}\Delta_{h_r}e_n|_{q=0,t=q}. \tag{4.24}$$

We have the following corollary as a consequence of Corollary 4.18 and Theorem 4.19 which gives a proof of the valley version Delta Conjecture about the expression $\Delta'_{e_k}\Delta_{h_r}e_n$ at the case when t or q is zero.

Corollary 4.20. *For any integers $n,k \geq 0$, we have the equality*

$$\begin{aligned}\mathrm{Rise}_{r;n,k}[X;q,0] = \mathrm{Rise}_{r;n,k}[X;0,q] = \mathrm{Val}_{r;n,k}[X;q,0] = \mathrm{Val}_{r;n,k}[X;0,q] \\ = \Delta'_{e_k}\Delta_{h_r}e_n|_{t=0} = \Delta'_{e_k}\Delta_{h_r}e_n|_{q=0,t=q}.\end{aligned} \tag{4.25}$$

Define the *Mahonian distribution* on $\mathcal{OP}_{r;\beta,k}$ to be the polynomial

$$D_{r;\beta,k}(q) := D^{\mathrm{inv}}_{r;\beta,k}(q) = D^{\mathrm{maj}}_{r;\beta,k}(q) = D^{\mathrm{dinv}}_{r;\beta,k}(q) = D^{\mathrm{minimaj}}_{r;\beta,k}(q)$$

and let $D^{+}_{r;\beta,k}(q) := D^{\mathrm{inv}+}_{r;\beta,k}(q) = D^{\mathrm{maj}+}_{r;\beta,k}(q) = D^{\mathrm{dinv}+}_{r;\beta,k}(q) = D^{\mathrm{minimaj}+}_{r;\beta,k}(q)$, then $D_{r;\beta,k}(q)$ generalizes the *Mahonian distribution on ordered multiset partitions $D_{\beta,k}(q)$* of Wilson [Wil16] that

$$D_{\beta,k}(q) = D_{0;\beta,k}(q).$$

By either of the Equations (4.12), (4.13) and (4.14), we have the base case that $D_{0;\emptyset,0}(q) = 1$,

$D_{r;\emptyset,k}(q) = 0$ for $r+k > 0$ and the recursion:

$$
\begin{aligned}
D_{r;\beta,k}(q) &= \sum_{\ell=0}^{k} q^{\binom{\beta_n-k+\ell}{2}} \begin{bmatrix} \ell \\ \beta_n - k + \ell \end{bmatrix}_q \begin{bmatrix} k \\ \ell \end{bmatrix}_q D_{r;\beta^-,\ell}(q) \\
&\qquad + q^{\binom{\beta_n-k+\ell}{2}} \begin{bmatrix} \ell \\ \beta_n - k + \ell \end{bmatrix}_q \begin{bmatrix} k-1 \\ \ell \end{bmatrix}_q \left(D^+_{r;\beta^-,\ell}(q) - D_{r;\beta^-,\ell}(q) \right) \\
&= \sum_{\ell=0}^{k} q^{\binom{\beta_n-k+\ell}{2}} \begin{bmatrix} \ell \\ \beta_n - k + \ell \end{bmatrix}_q \left(\begin{bmatrix} k-1 \\ \ell-1 \end{bmatrix}_q D_{r;\beta^-,\ell}(q) + \begin{bmatrix} k-1 \\ \ell \end{bmatrix}_q D^+_{r;\beta^-,\ell}(q) \right).
\end{aligned}
$$

Note that $D_{r;\beta,k}(q)$ is a generalization of the *q-Stirling number* $S_{n,k}(q)$ defined by

$$
S_{n,k}(q) = S_{n-1,k-1}(q) + [k]_q S_{n-1,k}(q) \tag{4.26}
$$

as a consequence of the following equation due to the work of Wilson [Wil16]:

$$
D_{0;1^n,k}(q) = S_{n,k}(q). \tag{4.27}
$$

The content of Chapter 4 is currently being prepared for submission for publication of the material. Qiu, Dun; Wilson, Andrew Timothy. “Conjectures about the expression $\Delta'_{e_k}\Delta_{h_r}e_n$". The dissertation author was the primary investigator and author of this material.

Chapter 5

Conclusion and future directions

In this chapter, we give a brief summary of our research projects mentioned in this dissertation. In the meanwhile, we discuss several directions that our work can be extended in. We also give some other open problems which are closely related to our work for research in the future.

5.1 The Rational Shuffle Theorem in more general cases

As a generalization of the Shuffle Theorem, the Rational Shuffle Theorem gives rise to a great number of combinatorial problems about Macdonald polynomials and rational parking functions, which will generalize the rational q,t-Catalan theory.

Our work presented in Chapter 2 gives a proof of the Rational Shuffle Theorem at the 3-row case, and conjecturally gives the Schur function expansion of the 3 column case parking function generating function.

Recall that our method on the algebraic side about the 3-row case uses a recursion that

$$\mathrm{Q}_{m+n,n}(-1)^n = \nabla \mathrm{Q}_{m,n}(-1)^n \tag{5.1}$$

where $n = 3$. Using the same recursion for bigger n, the algebraic side results can be easily generalized to the n-row case, and the proof can be fulfilled with help of mathematical software such as Maple when n is relatively small. We have run the data for the 4-row case, and the Schur function expansions with q,t-analogue coefficients are as nice as the 3-row case. Thus we can compute the Schur function expansion of the 4-row case (or even 5, 6-row cases) using the recursion in Equation (5.1).

On the 3 column case, we have more open problems to be solved in the future. By checking a large volume of program data, we are able to discover the *straightening action* on 3-column parking functions, which in fact proves the Schur positivity of the 3-column case. The data of parking functions with more than 3 columns will be hard to be looked into, but it will be a meaningful research problem to generalize our result about the straightening action on parking functions.

Problem 5.1. Define the straightening action of the m column rational parking functions for any positive integer m. This will prove the Schur positivity of the parking function side of the Rational Shuffle Conjecture.

Lascoux, Leclerc and Thibon in [LLT97] defined an important class of symmetric functions called *LLT polynomials*. The Shuffle Theorem and its generalizations are closely related to a subclass of it called *column LLT polynomials*. In fact, the following generating function of parking functions on a particular rational Dyck path $\Pi \in \mathcal{PF}_{m,n}$,

$$\sum_{\Pi(\mathrm{PF})=\Pi} t^{\mathrm{area}(\mathrm{PF})} q^{\mathrm{dinv}(\mathrm{PF})} F_{\mathrm{ides}(\mathrm{PF})} = t^{\mathrm{area}(\Pi)} \sum_{\Pi(\mathrm{PF})=\Pi} q^{\mathrm{dinv}(\mathrm{PF})} F_{\mathrm{ides}(\mathrm{PF})}$$

is a column LLT polynomial (up to a power of t). Notice that our straitening action on 3-column parking functions does not change the path, thus it would be useful to prove the Schur positivity of (column) LLT polynomials combinatorially if it is generalizable to n-column case.

Problem 5.2. Prove the Schur positivity of LLT polynomials combinatorially using parking functions.

About the other action $\mathbb{S}$ called the *switch map*, we hit an iceberg when proving that $\mathbb{S}$ preserves the statistic dinv, from which Conjecture 2.5 and Conjecture 2.6 follows immediately.

Problem 5.3. Prove that the action $\mathbb{S}$ preserves dinv for any rational parking function.

5.2 The Delta expression conjectures

As another generalization of the Shuffle Theorem, the Delta Conjecture still remains open, though a number of special cases have been proved.

In Chapter 3, we prove that the expression $\Delta_{e_k} e_n$ when $k = 2$ is Schur positive. The Schur positivity of the case when $k \geq 3$ is still open. Note that the Schur positivity of $\Delta_{e_k} e_n$ follows if Problem 5.2 or Problem 5.3 is proved.

Problem 5.4. Prove the Schur positivity of $\Delta_{e_k} e_n$ when $k \geq 3$.

Haglund, Remmel and Wilson gave a conjecture (called the Extended Delta Conjecture) about the Delta operator expression $\Delta'_{e_k} \Delta_{h_r} e_n$ which is analogous to the *rise version* of the Delta Conjecture about $\Delta'_{e_k} e_n$. In Chapter 4, we propose a new *valley version* conjecture of the expression $\Delta'_{e_k} \Delta_{h_r} e_n$, making the Extended Delta Conjecture completely analogues to the two-versioned Delta Conjecture. Further, we give a proof of the *valley version* conjecture of $\Delta'_{e_k} \Delta_{h_r} e_n$ when t or q equals 0. The main goal of this study is:

Problem 5.5. Prove the Extended Delta Conjecture in general.

This includes the origin Delta Conjecture.

It is proved that the two versions of the Delta Conjecture and the Extended Delta Conjecture are equivalent at the case when q or t is 0. However, there is no proof that the combinatorial side of the two versions are equivalent in general.

Problem 5.6. Prove that

$$\mathrm{Rise}_{r;n,k}[X;q,t] = \mathrm{Rise}_{r;n,k}[X;t,q] = \mathrm{Val}_{r;n,k}[X;q,t]. \tag{5.2}$$

This includes the problem that $\mathrm{Rise}_{n,k}[X;q,t] = \mathrm{Rise}_{n,k}[X;t,q] = \mathrm{Val}_{n,k}[X;q,t]$.

Finally, the Delta operator satisfies $\Delta_{h_r e_k} = \Delta_{s_{r,1^k}} + \Delta_{s_{r+1,1^{k-1}}}$ and $\Delta_{h_r e_k} = \Delta'_{e_k}\Delta_{h_r} + \Delta'_{e_{k-1}}\Delta_{h_r}$, which is saying that the sum of two expressions in the Extended Delta Conjecture is also a sum of two Delta hook-Schur functions. We want to explore a conjecture for Δ_{s_λ} for hook shape partition λ.

Problem 5.7. Give a combinatorial conjecture about the expression $\Delta_{s_\lambda} e_n$, where $\lambda \vdash n$ is of hook shape.

Appendix A

Four bijections between ordered multiset partitions and parking functions

In this appendix, we present four bijections, $\gamma^{\text{dinv}}, \gamma^{\text{maj}}, \gamma^{\text{inv}}, \gamma^{\text{minimaj}}$, of Haglund, Remmel and Wilson [HRW18] when they were proving the following equations appear in Theorem 4.1:

$$\begin{aligned} \text{Rise}_{n,k}[X;q,0]|_{M_\beta} &= D^{\text{dinv}}_{\beta,k+1}(q), && \text{(A.1)} \\ \text{Rise}_{n,k}[X;0,q]|_{M_\beta} &= D^{\text{maj}}_{\beta,k+1}(q), && \text{(A.2)} \\ \text{Val}_{n,k}[X;q,0]|_{M_\beta} &= D^{\text{inv}}_{\beta,k+1}(q), && \text{(A.3)} \\ \text{Val}_{n,k}[X;0,q]|_{M_\beta} &= D^{\text{minimaj}}_{\beta,k+1}(q). && \text{(A.4)} \end{aligned}$$

We shall omit the proof of bijectivity which can be found in [HRW18].

A.1 The bijection γ^{dinv} of $\text{Rise}_{n,k}[X;q,0]|_{M_\beta} = D^{\text{dinv}}_{\beta,k+1}(q)$

Recall that

$$D^{\text{dinv}}_{\beta,k+1}(q) = \sum_{\pi \in \mathcal{OP}_{\beta,k+1}} q^{\text{dinv}(\pi)}, \text{ and}$$

$$\mathrm{Rise}_{n,k}[X;q,0]|_{M_\beta} = \sum_{(\mathrm{PF},R)\in\mathcal{WPF}^{\mathrm{Rise}}_{n,n-k-1},\ X^{\mathrm{PF}}=\prod_{i=1}^{\ell(\beta)} x_i^{\beta_i},\ \mathrm{area}^-(\mathrm{PF},R)=0} q^{\mathrm{dinv(PF)}}.$$

The map

$$\gamma^{\mathrm{dinv}} : \mathcal{OP}_{\beta,k+1} \to \{(\mathrm{PF},R)\in\mathcal{WPF}^{\mathrm{Rise}}_{n,n-k-1},\ X^{\mathrm{PF}}=\prod_{i=1}^{\ell(\beta)} x_i^{\beta_i},\ \mathrm{area}^-(\mathrm{PF},R)=0\}$$

that satisfies $\mathrm{dinv}(\gamma^{\mathrm{dinv}}(\pi)) = \mathrm{dinv}(\pi)$ is defined as follows.

Given $\pi = \pi_1/\cdots/\pi_{k+1} \in \mathcal{OP}_{\beta,k+1}$ where $|\pi_i| = \alpha_i$, we construct a Dyck path $N^{\alpha_{k+1}}E^{\alpha_{k+1}}N^{\alpha_k}E^{\alpha_k}\cdots N^{\alpha_1}E^{\alpha_1}$ which is of size n. Then, the rise-decorated parking function $\gamma^{\mathrm{dinv}}(\pi)$ is obtained by labeling the north steps N^{α_i} with entries in the block π_i, and mark all the $n-k-1$ double rises. Clearly, the resulting parking function has area$^-$ 0, and the map γ^{dinv} is invertible.

For example, for an ordered multiset partition $\pi = 24/13/235$ with $\mathrm{dinv}(\pi) = 8$, its image under the map γ^{dinv} is given in Figure A.1 which also has dinv 8.

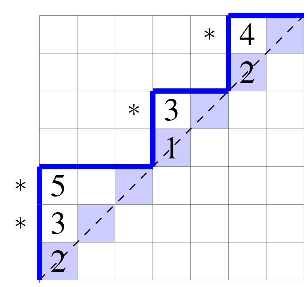


Figure A.1: The image $\gamma^{\mathrm{dinv}}(\pi)$ for $\pi = 24/13/235$.

A.2 The bijection γ^{maj} of $\mathrm{Rise}_{n,k}[X;0,q]|_{M_\beta} = D^{\mathrm{maj}}_{\beta,k+1}(q)$

In this section, we construct the map

$$\gamma^{\mathrm{maj}} : \mathcal{OP}_{\beta,k+1} \to \{(\mathrm{PF},R)\in\mathcal{WPF}^{\mathrm{Rise}}_{n,n-k-1},\ X^{\mathrm{PF}}=\prod_{i=1}^{\ell(\beta)} x_i^{\beta_i},\ \mathrm{dinv(PF)}=0\}$$

that satisfies $\text{area}^-(\gamma^{\text{maj}}(\pi)) = \text{maj}(\pi)$.

Given $\pi = \pi_1/\cdots/\pi_{k+1} \in \mathcal{OP}_{\beta,k+1}$ where $|\pi_i| = \alpha_i$, we shall write the descent-starred permutation notation of π: we first write π as a permutation $\sigma(\pi) = \sigma_1 \cdots \sigma_n$ of the multiset $A(\beta) = \{i^{\beta_i} : 1 \leq i \leq \ell(\beta)\}$ by organizing the elements in each block π_i in decreasing order. We mark a star $*$ at the lower-right corner of each number in $\sigma(\pi)$ that is in the same block with the next number. Now we are ready to construct the rise-decorated parking function $\gamma^{\text{maj}}(\pi)$.

We read $\sigma(\pi)$ from right to left. We start with drawing a north step and labeling it with σ_n when reading the rightmost number σ_n (notice that σ_n cannot have a star mark). Inductively, suppose that the next number we read is σ_i. If $\sigma_i \leq \sigma_{i+1}$, we add 2 steps EN at the end of the previous path, and label the new north step with σ_i. Otherwise when $\sigma_i > \sigma_{i+1}$, we add another north step and label it with σ_i (this must be a double rise). We decorate the new north step with a star if σ_i has a star $*$. Then we proceed to the next number σ_{i-1}.

In this way, we construct a parking function with no dinv. For example, for an ordered multiset partition $\pi = 24/13/35/2$ with $\text{maj}(\pi) = 6$, we have $\sigma(\pi) = 4_* 2 3_* 1 5_* 3 2$, and its image under the map γ^{maj} is given in Figure A.2 which has area^- 6.

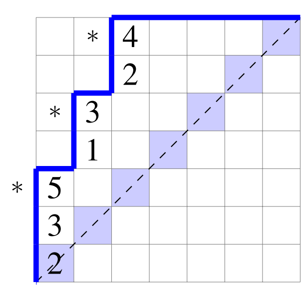


Figure A.2: The image $\gamma^{\text{maj}}(\pi)$ for $\pi = 24/13/35/2$.

A.3 The bijection γ^{inv} of $\mathrm{Val}_{n,k}[X;q,0]|_{M_\beta} = D^{\mathrm{inv}}_{\beta,k+1}(q)$

In this section, we construct the map

$$\gamma^{\mathrm{inv}} : \mathcal{OP}_{\beta,k+1} \to \{(\mathrm{PF},V) \in \mathcal{WPF}^{\mathrm{Val}}_{n,n-k-1},\ X^{\mathrm{PF}} = \prod_{i=1}^{\ell(\beta)} x_i^{\beta_i},\ \mathrm{area}(\mathrm{PF}) = 0\}$$

that satisfies $\mathrm{dinv}^-(\gamma^{\mathrm{inv}}(\pi)) = \mathrm{inv}(\pi)$.

Given $\pi = \pi_1/\cdots/\pi_{k+1} \in \mathcal{OP}_{\beta,k+1}$ where $|\pi_i| = \alpha_i$, we construct a diagonal (n,n)-Dyck path $(NE)^n$. Then we proceed from the lowest to the highest north step and from the last to the first block of π. We label the first α_{k+1} north steps increasingly with numbers in π_{k+1}, and add stars to the north steps from the second row to the α_{k+1}th row. Suppose that we have completed the procedure for block π_{i+1}. For block π_i, we label the next α_i north steps increasingly with numbers in π_i while adding stars to all except the first step in the α_i steps. Then we proceed to the next block π_{i-1}.

In this way, we construct a valley-decorated parking function with no area. For example, for an ordered multiset partition $\pi = 24/13/235$ with $\mathrm{inv}(\pi) = 4$, its image under the map γ^{inv} is given in Figure A.3 which has dinv^- 4.

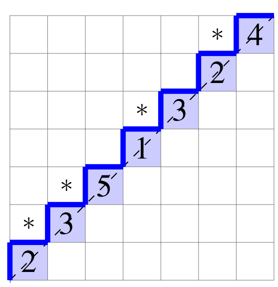


Figure A.3: The image $\gamma^{\mathrm{inv}}(\pi)$ for $\pi = 24/13/235$.

A.4 The bijection γ^{minimaj} of $\text{Val}_{n,k}[X;0,q]|_{M_\beta} = D^{\text{minimaj}}_{\beta,k+1}(q)$

In this section, we construct the map

$$\gamma^{\text{minimaj}} : \mathcal{OP}_{\beta,k+1} \to \{(\text{PF},V) \in \mathcal{WPF}^{\text{Val}}_{n,n-k-1},\ X^{\text{PF}} = \prod_{i=1}^{\ell(\beta)} x_i^{\beta_i},\ \text{dinv}^-(\text{PF},V) = 0\}$$

that satisfies $\text{area}(\gamma^{\text{minimaj}}(\pi)) = \text{minimaj}(\pi)$. γ^{minimaj} is most technical among the four maps.

Given $\pi = \pi_1/\cdots/\pi_{k+1} \in \mathcal{OP}_{\beta,k+1}$ where $|\pi_i| = \alpha_i$, we construct $\tau = \text{miniword}(\pi)$ as in the definition of minimaj. We define the runs of τ as its maximal, contiguous, weakly increasing subsequences. Suppose that τ has s runs, then we label the runs with $0,\dots,s-1$ from right to left. We shall construct the parking function $\gamma^{\text{minimaj}}(\pi)$ inductively by reading from the 0th to the $(s-1)$st run of τ, such that the row has number in the ith run has area i (this is sufficient for showing $\text{area}(\gamma^{\text{minimaj}}(\pi)) = \text{minimaj}(\pi)$).

Suppose that $\tau_a, \tau_{a+1}, \dots, \tau_n$ is the 0th run, and the numbers from τ_b to τ_n are contained in blocks $\pi_p,\dots,\pi_{k+1}$ that only consist of numbers in the 0th run (for some $b \geq a$). Suppose that the numbers $\tau_c,\dots,\tau_{b-1}$ form the first block from right to left containing elements in run 1. Starting from the empty path, we first construct steps $(NE)^{n-b+1}$, filling the north steps with entries in $\pi_{k+1},\dots,\pi_p$ increasingly for each block from bottom to top. We add star mark on the north steps where its label is in the same block as the label in the row immediately below it. Then we find the biggest number τ_d among τ_b to τ_n that is smaller than τ_c (which must exist by definition of miniword). We insert steps $(NE)^{a-c}$ above the north step of τ_d, label the steps with $\tau_c,\dots,\tau_{a-1}$ from bottom to top, and add stars to the rows of $\tau_{c+1},\dots,\tau_{a-1}$. Then we insert steps $(NE)^{b-a}$ after the east step after $(NE)^{a-c}$ that we just inserted, and label the steps with $\tau_a,\dots,\tau_{b-1}$ from bottom to top, adding stars to all these rows. We let A denote the north step of τ_a.

For greater value $i \in \{1,\dots,s-1\}$, we suppose that the procedures for runs $0,\dots,i-1$ have been completed and we proceed the algorithm inductively as follows. Suppose that $\tau_a, \tau_{a+1}, \dots, \tau_{n'}$ is the ith run that has not been read, and the numbers from τ_b to $\tau_{n'}$ are contained in blocks $\pi_p, \dots, \pi_{k'}$ that only consist of numbers in the ith run (for some $b \geq a$). Suppose that the numbers $\tau_c, \dots, \tau_{b-1}$ form the first block from right to left containing elements in run $i+1$.

Starting from the top of A in the previous procedure, we first insert steps $(NE)^{n'-b+1}$, filling the north steps with entries in $\pi_{k'}, \dots, \pi_p$ increasingly for each block from bottom to top. We add star mark on the north steps where its label is in the same block as the label in the row immediately below it.

Then we find the biggest number τ_d among τ_b to $\tau_{n'}$ that is smaller than τ_c. We insert steps $(NE)^{a-c}$ above the north step of τ_d, label the steps with $\tau_c, \dots, \tau_{a-1}$ from bottom to top, and add stars to the rows of $\tau_{c+1}, \dots, \tau_{a-1}$. Then we insert steps $(NE)^{b-a}$ after the east step after $(NE)^{a-c}$ that we just inserted, and label the steps with $\tau_a, \dots, \tau_{b-1}$ from bottom to top, adding stars to all these rows. We renew A to be the north step of the new τ_a in this procedure.

For example, for $\pi = 13/23/14/234$, its miniword is $\tau = 312341234$ which has 3 runs: 1234, 1234, 3 from right to left. The procedure of computing $\gamma^{\text{minimaj}}(\pi)$ is given in Figure A.4.

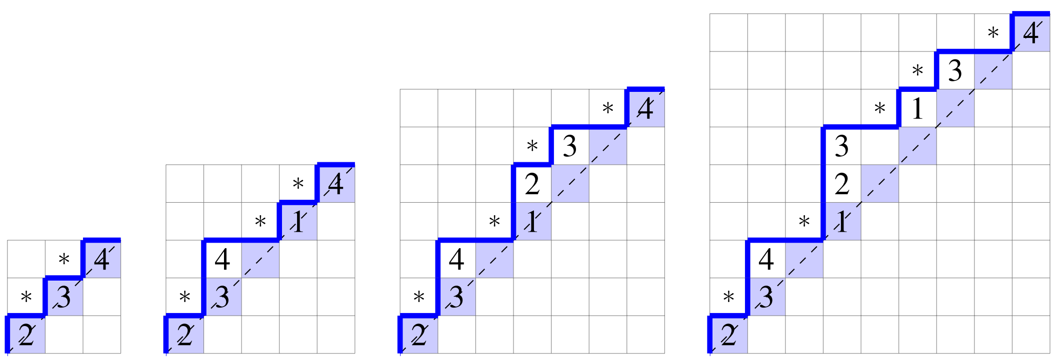


Figure A.4: The procedure of computing $\gamma^{\text{minimaj}}(\pi)$ for $\pi = 13/23/14/234$.

A.5 Summary

We have presented the four bijective maps of the form γ^{stat} for stat $=$ dinv, maj, inv and minimaj. In [HRW18], Haglund, Remmel and Wilson proved that the maps γ^{stat} are bijective, and they map the statistic stat into some parking function statistic (stated in each section of this appendix).

Further, by checking the four bijections, we notice that each bijection γ^{stat} maps the minimum element in the last part of π into the car in the first row of $\gamma^{\text{stat}}(\pi)$. We use this fact to prove Theorem 4.3.

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