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Fragments of Frege's *Grundgesetze* and Gödel's Constructible Universe

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Abstract

Frege's *Grundgesetze* was one of the 19th century forerunners to contemporary set theory which was plagued by the Russell paradox. In recent years, it has been shown that subsystems of the *Grundgesetze* formed by restricting the comprehension schema are consistent. One aim of this paper is to ascertain how much set theory can be developed within these consistent fragments of the *Grundgesetze*, and our main theorem (Theorem 2.9) shows that there is a model of a fragment of the *Grundgesetze* which defines a model of all the axioms of Zermelo-Fraenkel set theory with the exception of the power set axiom. The proof of this result appeals to Gödel's constructible universe of sets and to Kripke and Platek's idea of the projectum, as well as to a weak version of uniformization (which does not involve knowledge of Jensen's fine structure theory). The axioms of the *Grundgesetze* are examples of *abstraction principles*, and the other primary aim of this paper is to articulate a sufficient condition for the consistency of abstraction principles with limited amounts of comprehension (Theorem 3.5). As an application, we resolve an analogue of the joint consistency problem in the predicative setting.

1 Introduction

There has been a recent renewed interest in the technical facets of Frege's *Grundgesetze* ([6], [8]) paralleling the long-standing interest in Frege's philosophy of mathematics and logic ([11], [2]). This interest has been engendered by the consistency proofs, due to Parsons [31], Heck [21], and Ferreira-Wehmeier [14], of this system with limited amounts of comprehension. The broader intellectual interest in Frege's *Grundgesetze* stems in part from the two related ways in which it was a predecessor of contemporary set theory: first, the system was originally designed to be able to reconstruct much of ordinary mathematics, and second it comes equipped with the resources needed to define a membership relation. It is thus natural to ask how much set theory can be consistently developed within these fragments of the *Grundgesetze*. Our main theorem (Theorem 2.9) shows it is possible within some models of these fragments to recover all the axioms of Zermelo-Fraenkel set theory with the exception

of the power set axiom. To make this precise, one needs to carefully set out the primitives of the consistent fragments of the *Grundgesetze* and indicate what precisely it means to recover a fragment of set theory. This is the primary goal of §2 of the paper.

Following Wright and Hale ([20], cf. [8]), the system of the *Grundgesetze* has been studied in recent decades as a special case of so-called *abstraction principles*. These are principles that postulate lower-order representatives for equivalence relations on higher-order entities. Many of these principles are inconsistent with full comprehension, which intuitively says that every formula determines a concept or higher-order entity. So as with the *Grundgesetze*, the idea has been to look for consistency with respect to the so-called predicative instances of the comprehension schema, in which the presence of higher-order quantifiers within formulas is highly restricted. Of course, while predicativity in connection with the *Grundgesetze* is a fairly new topic, predicativity has a long tradition within mathematical logic, beginning with Poincaré, Russell, and Weyl ([22], [43]), and found in our day in the work of Feferman ([13, 12]) and in ACA_0 and related systems of Friedman and Simpson’s project of reverse mathematics ([18], [39]).

The other chief theorem of this paper (Theorem 3.5) shows that an abstraction principle associated to an equivalence relation is consistent with predicative comprehension so long as the equivalence relation is provably an equivalence relation in a limited theory of pure second-order logic and is expressible in the signature of that pure second-order logic. One application of this result is a resolution to the joint consistency problem in the predicative setting. For, in the setting of full comprehension, it has been known for some time that there are abstraction principles which are individually but not jointly consistent. In §3, we define the notion of an abstraction principle and further contextualize our results within the extant literature on abstraction principles.

The methods used in all these results draw on considerations related to Gödel’s constructible universe of sets. Whereas in the cumulative hierarchy of sets V_α , one proceeds by iterating the operation of the powerset into the transfinite, in the constructible hierarchy of sets L_α , one proceeds by iterating the operation of taking definable subsets into the transfinite. Gödel showed that just like the universe of sets $V = \bigcup_\alpha V_\alpha$ is a model of the axioms of set theory, so the constructible universe of sets $L = \bigcup_\alpha L_\alpha$ is a model of the axioms of set theory, along with a strong form of the axiom of choice according to which the elements of L are well-ordered by a relation $<_L$ (cf. [25] Chapter 13, [29] Chapter 6, [30] II.6, [10]).

Our present understanding of the more “local” or “micro” properties of the constructible sets was furthered by the work of Kripke ([28]), Platek ([32]) and Jensen ([26]), in whose results we find the key ideas of the projectum and uniformization. Roughly, a level L_α of the constructible hierarchy satisfies uniformization if whenever it satisfies $\forall x \exists y R(x, y)$ then there is a definable function f of the same level of complexity as R which satisfies $\forall x R(x, f(x))$. The projectum, on the other hand, is related to the idea that certain initial segments L_α of the constructible universe can be shrunk via a definable injection $\iota : L_\alpha \rightarrow \rho$ to a smaller ordinal $\rho < \alpha$. The formal definitions of the projectum and uniformization are given in §4. It bears emphasizing that we only employ a weak version of uniformization which has an elementary proof, and so this paper does not presuppose

knowledge of Jensen’s fine structure theory (cf. Proposition 4.3).

It’s actually rather natural to think that uniformization and the projectum would be useful in producing models of abstraction principles. On the one hand, given an equivalence relation E on the set $P(\rho) \cap L_\alpha$, we can conceive of the elements of this set as higher-order entities, and then we can take the lower-order representative in ρ of an E -equivalence class to be the injection ι applied to the $<_L$ -least element of E ’s equivalence class. On the other hand, uniformization allows one to secure further instances of the comprehension schema in which there are some controlled occurrences of higher-order quantifiers, in essence because one can use uniformization to choose one particular higher-order entity with which to work.

This, in any case, is the intuitive idea behind the proof of our theorem on the consistency of abstraction principles (Theorem 3.5) which we prove in §5. However, this does not itself deliver our result on how much set theory one can recover in the consistent fragments of the *Grundgesetze*. For this, we need to additionally show that if we start from a level of the constructible hierarchy which satisfies certain axioms of set theory, and if we perform the construction of a model of the fragment of the *Grundgesetze* in the manner intimated in the above paragraph and made precise in §5, then we can recover these original constructible sets definably within the model of the fragment of the *Grundgesetze*. The details of this argument are carried out in §6 where our Main Theorem 2.9 is finally established.

This paper is the first in a series of three papers – the other two being [42], [41]– which collectively constitute a sequel to the Basic Law V components of our paper [40]. In that earlier paper [40], we gave a proof of the consistency of Frege’s *Grundgesetze* system with limited amounts of comprehension using tools from hyperarithmetic theory. However, we were unable to use these models to ascertain how much Zermelo-Fraenkel set theory could be consistently done in the Fregean setting. The work in this paper explains why this was the case. The key to this was an axiom known as Axiom Beta (cf. Definition 4.2), which effectively ensures that the Mostowski Collapse Theorem holds in a structure. As one can see by inspection of the proofs in §6, it is being able to invoke this theorem in a model which allows us to obtain finally the Main Theorem 2.9. It turns out that the usual models associated to hyperarithmetic theory simply are not models of Axiom Beta.

This present paper does not depend on results from our earlier paper [40], nor does it depend on its two thematically-linked companion papers, [41], [42]. In the companion paper [41], we use the constructible hierarchy to develop models of an intensional type theory, roughly analogous to how one can use the cumulative hierarchy to build models of an extensional type theory. This intensional type theory can in turn interpret fragments of the *Grundgesetze* system, and so stands to the predicative *Grundgesetze* system as the stage axioms of Shoenfield [35, 36, 37] and Boolos [3] stands to the Zermelo-Fraenkel system. In the other companion paper [42], we examine the deductive strength of the theory consisting of all the predicative abstraction principles whose consistency we establish here.

2 The *Grundgesetze* and Its Set Theory

Basic Law V is the crucial fifth axiom of Frege’s *Grundgesetze* ([15], [17]), and it axiomatizes the behavior of a certain type-lowering operator from second-order entities to first-order entities, called the “extension operator.” In Frege’s type-theory, the second-order entities are called “concepts” while the first-order entities are called “objects,” so that the extension operator ∂ takes a concept X and returns an object $\partial(X)$. (There is no standard notation for the extension operator, and so some authors write $\S(X)$ in lieu of $\partial(X)$). Basic Law V then simply postulates that the extension operator is injective:

$$(2.1) \quad \text{Basic Law V:} \quad \forall X, Y (\partial(X) = \partial(Y) \leftrightarrow X = Y)$$

Here the identity of concepts is regarded as extensional in character, so that two concepts X, Y are said to be identical precisely when they are coextensive, i.e. $X = Y$ if and only if for all objects z we have that Xz if and only if Yz .

Models of Basic Law V have the following form:

$$(2.2) \quad \mathcal{M} = (M, S_1(M), S_2(M), \dots, \partial)$$

wherein M is a non-empty set that serves as the interpretation of the objects, and the set $S_n(M) \subseteq P(M^n)$ serves as the interpretation of the n -ary concepts, and wherein the function $\partial : S_1(M) \rightarrow M$ is an injection. Further, we assume that in the object-language of the structure from equation (2.2) we have the resources to describe when an n -tuple (a_1, \dots, a_n) from M^n is in an n -ary concept R from $S_n(M)$, and we write this in the object-language alternatively as $R(a_1, \dots, a_n)$ or $(a_1, \dots, a_n) \in R$, and we refer to this relation as the predication relation.

As is well-known, Basic Law V is inconsistent with the full second-order comprehension schema:

Definition 2.1. *The Full Comprehension Schema consists of the all axioms of the form $\exists R \forall \bar{a} (R\bar{a} \leftrightarrow \varphi(\bar{a}))$, wherein $\varphi(\bar{x})$ is allowed to be any formula, perhaps with parameters, and \bar{x} abbreviates (x_1, \dots, x_n) and R is an n -ary concept variable for $n \geq 1$ that does not appear free in $\varphi(\bar{x})$.*

In spite of this inconsistency, Parsons and Heck ([31], [21]) showed that Basic Law V is *consistent* with the version of the comprehension schema in which $\varphi(x)$ contains no second-order quantifiers:

Definition 2.2. *The First-Order Comprehension Schema consists of all axioms of the form $\exists R \forall \bar{a} (R\bar{a} \leftrightarrow \varphi(\bar{a}))$, wherein $\varphi(\bar{x})$ is allowed to be any formula with no second-order quantifiers but perhaps with parameters, and \bar{x} abbreviates (x_1, \dots, x_n) and R is an n -ary concept variable for $n \geq 1$ that does not appear free in $\varphi(\bar{x})$.*

Ferreira and Wehmeier extended the Parsons-Heck result by showing that there are models $\mathcal{M} = (M, D(M), D(M^2), \dots, \partial)$ of Basic Law V which also model stronger forms of comprehension, namely the Δ_1^1 -comprehension schema and the Σ_1^1 -choice schema ([14] §4). These schemata are defined as follows:

Definition 2.3. The Δ_1^1 -Comprehension Schema consists of all axioms of the form

$$(2.3) \quad \forall \bar{x} (\varphi(\bar{x}) \leftrightarrow \psi(\bar{x})) \rightarrow \exists R \forall \bar{a} (R\bar{a} \leftrightarrow \varphi(\bar{a}))$$

wherein $\varphi(\bar{x})$ is a Σ_1^1 -formula and $\psi(\bar{x})$ is a Π_1^1 -formula that may contain parameters, and \bar{x} abbreviates (x_1, \dots, x_n) , and R is an n -ary concept variable for $n \geq 1$ that does not appear free in $\varphi(\bar{x})$ or $\psi(\bar{x})$.

Definition 2.4. The Σ_1^1 -Choice Schema consists of all axioms of the form

$$(2.4) \quad [\forall \bar{x} \exists R' \varphi(\bar{x}, R')] \rightarrow \exists R [\forall \bar{x} \forall R' [(\forall \bar{y} (R'\bar{y} \leftrightarrow R\bar{x}\bar{y})) \rightarrow \varphi(\bar{x}, R')]]$$

wherein the formula $\varphi(\bar{x}, R')$ is Σ_1^1 , perhaps with parameters, and \bar{x} abbreviates (x_1, \dots, x_n) and \bar{y} abbreviates (y_1, \dots, y_m) and R is an $(n + m)$ -ary concept variable for $n, m \geq 1$ that does not appear free in $\varphi(\bar{x}, R')$ where R' is an m -ary concept variable.

Here, as is usual, a Σ_1^1 -formula (resp. Π_1^1 -formula) is one which begins with a block of existential quantifiers (resp. universal quantifiers) over n -ary concepts for various $n \geq 1$ and which contains no further second-order quantifiers. Given this variety of comprehension schemata, it becomes expedient to explicitly distinguish between different formal theories that combine these schemata with the axiom Basic Law V from equation (2.1). In particular, one defines the following systems (cf. [40] Definition 5 p. 1683):

Definition 2.5. The theory ABL_0 is Basic Law V together with the First-Order Comprehension Schema (cf. Definition 2.2). The theory $\Delta_1^1\text{-BL}_0$ is Basic Law V together with the Δ_1^1 -Comprehension Schema (cf. Definition 2.3). The theory $\Sigma_1^1\text{-LB}_0$ is Basic Law V together with the Σ_1^1 -Choice Schema (cf. Definition 2.4) and the First-Order Comprehension Schema (cf. Definition 2.2).

We opt to designate the subsystem formed with Σ_1^1 -Choice by inverting the letters “BL” to “LB”, since this convention saves us from needing to write out the word “choice” when referring to a theory, and since it is compatible with the convention in subsystems of second-order arithmetic ([39]), wherein the Δ_1^1 -comprehension fragment is called $\Delta_1^1\text{-CA}_0$ and the Σ_1^1 -choice fragment is called $\Sigma_1^1\text{-AC}_0$.

In the companion paper [42], we work deductively in theories containing limited amounts of comprehension. In these situations, it will prove expedient to consider an enrichment of the above theories by the addition of certain function symbols. In particular, we assume that for every $m, n > 0$ we have a $(n + 1)$ -ary function symbol in the language for the map $(R, a_1, \dots, a_n) \mapsto R[a_1, \dots, a_n]$ from a single $(n + m)$ -ary relation R and an n -tuple of objects (a_1, \dots, a_n) to the m -ary relation

$$(2.5) \quad R[a_1, \dots, a_n] = \{(y_1, \dots, y_m) : R(a_1, \dots, a_n, y_1, \dots, y_m)\}$$

One benefit of the addition of these symbols is that it allows for a compact formalization of the key clause (2.4) of the Σ_1^1 -choice schema, namely:

$$(2.6) \quad [\forall \bar{x} \exists R' \varphi(\bar{x}, R')] \rightarrow [\exists R \forall \bar{x} \varphi(\bar{x}, R[\bar{x}])]$$

The addition of these function symbols to the signature impacts the axiom system because we continue to assume that we have Σ_1^1 -choice and first-order comprehension. In particular, the inclusion of the function symbols $(R, a_1, \dots, a_n) \mapsto R[a_1, \dots, a_n]$ in the signature then adds to the collection of terms of the signature, which in turn adds to the collection of quantifier-free and hence first-order formulas of the signature.

Let us call this expansion of $\Sigma_1^1\text{-LB}_0$ the system $\Sigma_1^1\text{-LB}$, i.e. we drop the “zero” subscript; and likewise for the other systems from Definition 2.2. For ease of future reference, let’s explicitly record this in the following definition:

Definition 2.6. *The theory ABL is Basic Law V together with the First-Order Comprehension Schema (cf. Definition 2.2) in the signature including the function symbols $(R, a_1, \dots, a_n) \mapsto R[a_1, \dots, a_n]$. The theory $\Delta_1^1\text{-BL}$ is Basic Law V together with the Δ_1^1 -Comprehension Schema (cf. Definition 2.3) in the signature with these function symbols. The theory $\Sigma_1^1\text{-LB}$ is Basic Law V together with the Σ_1^1 -Choice Schema (cf. Definition 2.4) and the First-Order Comprehension Schema (cf. Definition 2.2) in the signature including these function symbols.*

In building models of these consistent fragments of Frege’s system, one of our chief aims is to understand how much set theory can be thereby recovered. The crucial idea is to define an ersatz membership-relation η in terms of the extension operator and predication:

$$(2.7) \quad a\eta b \iff \exists B (\partial(B) = b \ \& \ Ba)$$

Since the *extensions* are precisely the objects in the range of the extension operator ∂ , we write the collection of extensions as $\text{rng}(\partial)$. Now it follows from considerations related to the Russell paradox that $\text{rng}(\partial)$ is not a concept in the presence of Δ_1^1 -comprehension (cf. [40] Proposition 29 p. 1692). In contrast to $\text{rng}(\partial)$, the collections $V = \{x : x = x\}$ and $\emptyset = \{x : x \neq x\}$ do form concepts since they are first-order definable. The following elementary proposition, provable in $\Sigma_1^1\text{-LB}$, says that for subconcepts of $\text{rng}(\partial)$, the η -relation restricted to this concept exists as a binary concept:

Proposition 2.7. *(Existence of Restricted η -relation) ($\Sigma_1^1\text{-LB}$) For every concept $X \subseteq \text{rng}(\partial)$ there is a binary concept R such that for all a , we have that Xa implies $\partial(R[a]) = a$. So for all concepts $X \subseteq \text{rng}(\partial)$ there is a binary relation $E_X \subseteq V \times X$ such that Xa implies: $E_X(b, a)$ iff $b\eta a$.*

It will also be helpful in what follows to have some fixed notation for subset and successor. So similar to equation (2.7) we define the associated Fregean subset relation \subseteq_η as follows:

$$(2.8) \quad a \subseteq_\eta b \iff \forall c (c\eta a \rightarrow c\eta b)$$

However, note that if a is *not* an extension, then $c\eta a$ is always false and so $(c\eta a \rightarrow \psi)$ is always true, regardless of what ψ is. Hence, if a is *not* an extension, then $a \subseteq_\eta b$ is always true. So the expressions $a\eta b$ and $a \subseteq_\eta b$ will behave like membership and subset only if one restricts attention to a, b , that are extensions. In what follows, it will also be useful to introduce some notation for a successor-like operation on extensions. So let us say that

$$(2.9) \quad \sigma(x) = y \iff \exists F \exists G [\partial(F) = x \ \& \ \partial(G) = y \ \& \ \forall z (Gz \leftrightarrow (Fz \vee z = x))]$$

However, this function is not total, and in particular it should be emphasized that $\sigma(x)$ is only well-defined when x is an extension. Accordingly, the graph of the function $x \mapsto \sigma(x)$ does not exist as a binary concept, since if it did, then its domain would likewise exist, and its domain is precisely $\text{rng}(\partial)$. However, when $\sigma(x)$ is defined, note that it satisfies $z\eta(\sigma(x))$ iff either $z\eta x$ or $z = x$. This of course reminds us of the usual set-theoretic successor operation $x \mapsto (x \cup \{x\})$.

In the axiomatic development of systems related to Σ_1^1 -LB, the crucially important concept is the notion of transitive closure. If F is a concept, then let us say that F is η -transitive or η -closed if $(Fx \ \& \ y\eta x)$ implies Fy , for all x, y . Then we define transitive closure as follows:

$$(2.10) \quad (\text{Trcl}_\eta(x))(y) \equiv \forall F [F \text{ is } \eta\text{-transitive} \ \& \ x \subseteq_\eta \partial(F)] \rightarrow Fy$$

It is easily provable that $\text{Trcl}_\eta(x)$ also has the following properties:

Proposition 2.8. (*Elementary Facts about Transitive Closure*)

1. Transitive Closure is η -transitive: $[(\text{Trcl}_\eta(x))(y) \ \& \ z\eta y]$ implies $(\text{Trcl}_\eta(x))(z)$.
2. Transitive Closure is an η -superclass: $w\eta x$ implies $(\text{Trcl}_\eta(x))(w)$.

So now we may describe the procedure for carving out a model of a fragment of classical set theory ZFC from a model of \mathcal{M} of Σ_1^1 -LB. Since the foundation axiom is a traditional part of ZFC, we want to ensure that our fragments always include this axiom, and for this purpose it is important that we avoid infinite descending η -chains. Since \mathcal{M} has second-order resources, this can be effected in a straightforward manner. In particular, if $\mathcal{X} \subseteq M$ and $\mathcal{R} \subseteq \mathcal{X} \times \mathcal{X}$ are \mathcal{M} -definable (but not necessarily elements of $S_k(M)$), then let us say that “ $(\mathcal{X}, \mathcal{R})$ is well-founded in \mathcal{M} ” if \mathcal{M} models that every non-empty *subconcept* of \mathcal{X} has an \mathcal{R} -least member, i.e. \mathcal{M} models $\forall F [\exists x Fx \ \& \ \forall x (Fx \rightarrow \mathcal{X}(x))] \rightarrow [\exists y Fy \ \& \ \forall z (Fz \rightarrow \neg \mathcal{R}(z, y))]$. A special case of this is when X is a concept and R is a binary concept, in which case we likewise define “ (X, R) is well-founded in \mathcal{M} ” to mean that \mathcal{M} models that every non-empty subconcept of X has an R -least element, i.e. that \mathcal{M} models

$$(2.11) \quad \forall F [\exists x Fx \ \& \ \forall x (Fx \rightarrow Xx)] \rightarrow [\exists y Fy \ \& \ \forall z (Fz \rightarrow \neg R(z, y))]$$

Since $S_1(M)$ is in general a small subset of $P(M)$, we of course need to be wary of inferring from “ (X, R) is well-founded in \mathcal{M} ” to (X, R) having no infinite descending R -chains, or to (X, R) having no infinite \mathcal{M} -definable descending R -chains.

Finally, putting this all together, let us define the notion of a “well-founded extension”:

$$(2.12) \quad \begin{aligned} \text{wfExt}(x) \equiv & \ x \text{ is an extension} \ \& \ (\text{Trcl}_\eta(\sigma(x)), \eta) \text{ is well-founded} \\ & \ \& \ (\text{Trcl}_\eta(\sigma(x)) \subseteq \text{rng}(\partial)) \end{aligned}$$

Given a model \mathcal{M} of Σ_1^1 -LB, let us define its collection of well-founded extensions as follows:

$$(2.13) \quad \text{wfExt}(\mathcal{M}) = \{x \in M : \mathcal{M} \models \text{wfExt}(x)\}$$

In broad analogy with its usage in set theory, we shall sometimes refer to this as the *inner model* of well-founded extensions relative to a model of $\Sigma_1^1\text{-LB}$.

The other definition that we need in order to state and prove our results is a global choice principle. Suppose that T is a theory in one of our signatures. Then we let $T + \text{GC}$ be the expansion of T by a new binary relation symbol $<$ on objects in the signature, with axioms saying that $<$ is a linear order of the first-order objects, and we additionally have a schema in the expanded signature saying that any instantiated formula $\varphi(x)$ in the expanded signature, perhaps containing parameters, that holds of some first-order object x will hold of a $<$ -least element:

$$(2.14) \quad [\exists x \varphi(x)] \rightarrow [\exists x \varphi(x) \ \& \ \forall y < x \neg \varphi(y)]$$

Since all our theories T contain first-order comprehension (cf. Definition 2.2), and since instances of $<$ are quantifier-free and hence first-order, we have that the graph of $<$ forms a binary concept in $T + \text{GC}$. Of course the postulated binary relation $<$ does not necessarily have anything to do with the usual “less than” relation on the natural numbers.

With this all notation in place, our main theorem can be expressed as follows, wherein P denotes the power set axiom:

Theorem 2.9. (*Main Theorem*) *There is a model \mathcal{M} of $\Sigma_1^1\text{-LB} + \text{GC}$ such that $(\text{wfExt}(\mathcal{M}), \eta)$ satisfies the axioms of ZFC-P .*

This result is proven at the close of §6. It is significant primarily because it shows us what kind of set theory may be consistently developed if one takes Basic Law V as a primitive. Now, one subtlety should be mentioned here at the outset: in the absence of power set, it is not entirely obvious which form of replacement and which form of choice is optimal. The discussion in Gitman-Hamkins-Johnstone ([19]) suggests that instead of the replacement schema one should use the collection schema, and as for the axiom of choice one should use the principle that every set can be well-ordered; the reason in each case being that these are the deductively stronger principles in the absence of powerset. (For a formal statement of the collection schema, cf. equation (4.2)). As we will note when establishing our main theorem in §6, our models satisfy these principles as well. Hence, for the sake of concreteness, in this paper we may define ZFC-P as follows:

Definition 2.10. *ZFC-P is the theory consisting of extensionality, pairing, union, infinity, separation, collection, foundation, and the statement that every set can be well-ordered.*

For precise definitions of these axioms, one may consult any standard set theory textbook ([29, 30], [25]; and for the collection schema see again equation (4.2)).

The Main Theorem 2.9 is a natural analogue of the work of Boolos, Hodes, and Cook’s on the axiom “New V” ([4], [23], [7]). This is the axiom in the signature of Basic Law V, but where, for the sake of disambiguation, we write the type-lowering operator with the symbol ∂' as opposed to ∂ . The axiom *New V* then says that

$$(2.15) \quad \text{New V: } \forall X, Y (\partial'(X) = \partial'(Y) \leftrightarrow ((\text{Small}(X) \vee \text{Small}(Y)) \rightarrow X = Y))$$

Here $\text{Small}(X)$ is an abbreviation for the statement that X is not bijective with the universe of first-order objects $\{x : x = x\}$. So if $\mathcal{M} = (M, S_1(M), S_2(M), \dots, \partial')$ is a model of New V, then $\mathcal{M} \models \text{Small}(X)$ if and only if there's no bijection $f : X \rightarrow M$ whose graph is in $S_2(M)$.

To see the connection between New V and ZFC, recall that for a cardinal κ , the set H_κ is defined as $H_\kappa = \{x : |\text{trcl}(x)| < \kappa\}$ (cf. [29] §IV.6 pp. 130 ff, [30] p. 78, [25] p. 171). Suppose that $\kappa > \omega$ is regular and satisfies $|H_\kappa| = \kappa$. In this circumstance, let us define:

$$(2.16) \quad \mathbb{H}_\kappa = (H_\kappa, P(H_\kappa), P(H_\kappa \times H_\kappa), \dots, \partial')$$

where $\partial'(X) = \langle 1, X \rangle$ if $|X| < \kappa$ and $\partial'(X) = \langle 0, 0 \rangle$ otherwise (wherein $\langle \cdot, \cdot \rangle$ is the usual set-theoretic pairing function). Then in analogue to Frege's definition of membership in equation (2.7), we can define a quasi-membership relation η' in models of New V as follows:

$$(2.17) \quad a\eta'b \iff \exists B (\text{Small}(B) \ \& \ \partial'(B) = b \ \& \ Ba)$$

Likewise, we can define wfExt' using the relation η' just as wfExt is defined in equation (2.12) using the relation η . Then one may prove that \mathbb{H}_κ is a model of New V and $(\text{wfExt}'(\mathbb{H}_\kappa), \eta')$ is isomorphic to (H_κ, \in) , which is known to model ZFC-P when $\kappa > \omega$ is regular (cf. [29] Theorem IV.6.5 p. 132, [30] Theorem II.2.1 p. 109, [25] p. 171). Hence one has following:

Proposition 2.11. *There is a model \mathcal{M} of New V and the Full Comprehension Schema (cf. Definition 2.1) such that $(\text{wfExt}'(\mathcal{M}), \eta')$ satisfies the axioms of ZFC-P (cf. Definition 2.10).*

The Main Theorem 2.9 establishes an analogous result for Basic Law V in the setting of limited amounts of comprehension.

3 Predicative Abstraction Principles

The axioms Basic Law V and New V are examples of what are now called *abstraction principles*. If $E(R, S)$ is a formula of second-order logic with exactly two free n -ary relation variables for some $n \geq 1$ then the *abstraction principle* $A[E]$ associated to E is the following axiom in a signature expanded by a new function symbol ∂_E from n -ary relations to objects:

$$(3.1) \quad A[E] : \quad \forall R, S, [\partial_E(R) = \partial_E(S) \leftrightarrow E(R, S)]$$

Abstraction principles have been studied extensively for many decades. For an introduction to this subject, see Burgess [6], and for many important papers, see the collections edited by Demopoulos [9] and Cook [8].

The first thing that one observes in this subject is that some abstraction principles are consistent with the Full Comprehension Schema in their signature (cf. Definition 2.1) while others are not. For instance, we saw above that New V is consistent with the Full Comprehension Schema in its signature, while Basic Law V (2.1) itself is not. Given that Basic Law V is consistent with weaker forms of comprehension, one may ask whether there is any general method for determining whether the abstraction principle $A[E]$ is consistent with these weaker forms of comprehension. In answering this question, it's helpful to have specific names for the theories consisting of combinations of the abstraction principle $A[E]$ with the weaker forms of comprehension:

Definition 3.1. For each formula $E(R, S)$ with exactly two free n_E -ary variables R, S for a specific $n_E \geq 1$, let the theory $\Delta_1^1\text{-A}[E]$ (resp. $\Sigma_1^1\text{-[E]A}$) consist of $A[E]$ from equation (3.1) plus the Δ_1^1 -Comprehension Schema (cf. Definition 2.3) in the signature containing the function symbol ∂_E (resp. $A[E]$ from equation (3.1) plus the Σ_1^1 -Choice Schema (cf. Definition 2.4) and the First-Order Comprehension Schema (cf. Definition 2.2) in the signature containing the function symbol ∂_E).

Further, let us define a theory of pure second-order logic in a very limited signature:

Definition 3.2. The theory S0 is the second-order theory consisting of the Full Comprehension Schema (2.1) in the signature of pure second-order logic bereft of all type-lowering function symbols.

Here the abbreviation “ S0 ” is chosen because it reminds us of “second-order logic.” It’s worth emphasizing that while the theory S0 has full comprehension in its signature, its signature is very impoverished and does not include any of the type-lowering function symbols featuring in abstraction principles. But as with the fragments of Basic Law V discussed in the previous sections, we’re assuming that we have the function symbols $(\bar{a}, R) \mapsto R[\bar{a}]$ from equation (2.5) in the signature of all our theories, including S0 . So just to be clear: the signature of S0 consists merely of the predication relations $R\bar{x}$ and the maps $(\bar{a}, R) \mapsto R[\bar{a}]$, and its axioms consist solely of the extensionality axioms and the instances of the Full Comprehension Schema (2.1) in its signature. Sometimes in what follows we consider the extension $\text{S0} + \text{GC}$, which per the discussion of global choice in the previous section adds to the signature of S0 a binary relation on objects and posits that it is a well-order. In the theory $\text{S0} + \text{GC}$, we adopt the convention that instances of the Full Comprehension Schema (2.1) may include the global well-order.

One of our chief results on the consistency of abstraction principles with predicative levels of comprehension is the following:

Theorem 3.3. Suppose that $n \geq 1$ and that $E(R, S)$ is a formula in the signature of $\text{S0} + \text{GC}$ which is provably an equivalence relation on n -ary concepts in $\text{S0} + \text{GC}$. Then $\Sigma_1^1\text{-[E]A} + \text{S0} + \text{GC}$ is consistent.

This result is proven in §5 below. This result indicates that the fact that Basic Law V is consistent with the Δ_1^1 -comprehension schema and Σ_1^1 -choice schema is not an isolated phenomena, but follows from the fact that $E(X, Y) \equiv X = Y$ is provably an equivalence relation in $\text{S0} + \text{GC}$. It’s worth stressing that the theory $T = \Sigma_1^1\text{-[E]A} + \text{S0} + \text{GC}$ has full comprehension for formulas in the signature of $\text{S0} + \text{GC}$ but only has predicative comprehension for formulas in T ’s full signature which includes the type-lowering function ∂_E .

A related problem of long-standing interest has been the “joint consistency problem.” This is the problem of determining natural conditions on E_1, E_2 so that if $A[E_1]$ and $A[E_2]$ has a standard model then $A[E_1] \wedge A[E_2]$ has a standard model. A second-order theory is said to have a *standard model* if it has a model \mathcal{M} satisfying $S_n(M) = P(M^n)$ for all $n \geq 1$, where we here employ the notation introduced in the previous section in equation (2.2) for models. This is a non-trivial problem: for, some $A[E_1]$ have standard models \mathcal{M} only when

the underlying first-order domain M is finite, such as when $E_1(X, Y)$ is expressive of the symmetric difference of X and Y being Dedekind-finite (cf. [5] p. 215, [20] pp. 289 ff). However, other $A[E_2]$ have a standard model \mathcal{M} with underlying first-order domain M only when M is infinite, such as when $E_2(X, Y)$ is expressive of X, Y being bijective.

In the setting of limited amounts of comprehension, the most obvious analogue of the joint consistency problem is to ask about the extent to which it is consistent that $A[E_1] \wedge A[E_2]$ has a model satisfying e.g. the Δ_1^1 -comprehension schema when each $A[E_i]$ -individually does. Formally, let us introduce the following theories:

Definition 3.4. *The theory $\Delta_1^1\text{-A}[\mathbf{E}_1, \dots, \mathbf{E}_k]$ (resp. $\Sigma_1^1\text{-}[\mathbf{E}_1, \dots, \mathbf{E}_k]\mathbf{A}$) consists both of the abstraction principles $A[E_1] \wedge \dots \wedge A[E_k]$ (3.1) and the Δ_1^1 -Comprehension Schema (cf. Definition 2.3) (resp. plus the Σ_1^1 -Choice Schema (cf. Definition 2.4) and the First-Order Comprehension Schema (cf. Definition 2.2)) in the signature containing all the function symbols $\partial_{E_1}, \dots, \partial_{E_k}$.*

Our result Theorem 3.3 from above is a direct consequence of the following theorem, which indicates that the joint consistency problem does not arise in the setting with limited amounts of comprehension, assuming that we can prove the formulas are equivalence relations in $\mathbf{SO} + \mathbf{GC}$, and assuming that the equivalence relations are expressible in the signature of $\mathbf{SO} + \mathbf{GC}$:

Theorem 3.5. *(Joint Consistency Theorem) Suppose $n_1, \dots, n_k \geq 1$ and that the formulas $E_1(R, S), \dots, E_k(R, S)$ in the signature of $\mathbf{SO} + \mathbf{GC}$ are provably equivalence relations on m_i -ary concepts in $\mathbf{SO} + \mathbf{GC}$. Then the theory $\Sigma_1^1\text{-}[\mathbf{E}_1, \dots, \mathbf{E}_k]\mathbf{A} + \mathbf{SO} + \mathbf{GC}$ is consistent.*

This result is proven in §5 below. It's worth again underscoring that the theory $T = \Sigma_1^1\text{-}[\mathbf{E}_1, \dots, \mathbf{E}_k]\mathbf{A} + \mathbf{SO} + \mathbf{GC}$ has full comprehension for formulas in the signature of $\mathbf{SO} + \mathbf{GC}$ but only has predicative comprehension for formulas in T 's full signature which includes the type-lowering function $\partial_{E_1}, \dots, \partial_{E_k}$. By compactness, this theorem establishes the consistency of a theory which includes abstraction principles associated to each formula in the signature of $\mathbf{SO} + \mathbf{GC}$ which one can prove to be an equivalence relation in $\mathbf{SO} + \mathbf{GC}$. In our companion paper [42], we study the deductive strength of this theory.

4 Constructibility and Generalized Admissibility

The aim of this section is to briefly review several of the tools from constructibility that we use in the below proofs. Hence, it might be advisable to skip this section on a first read-through and refer back to this section as needed. In this section, we work entirely with fragments and extensions of the standard ZFC-set theory, so that all structures M are structures in the signature of set-theory. The tools which we review and describe in this section come from constructibility, the study of Gödel's universe L (cf. [25] Chapter 13, [29] Chapter 6, [30] II.6, [10]). This is the union of the sets L_α that are defined recursively as follows, wherein $\text{Defn}(M)$ refers to the subsets of M which are definable with parameters

(when M is conceived of as having, as its only primitive, the membership relation restricted to its elements):

$$(4.1) \quad L_0 = \emptyset, \quad L_{\alpha+1} = \text{Defn}(L_\alpha), \quad L_\alpha = \bigcup_{\beta < \alpha} L_\beta \text{ for } \alpha \text{ a limit}$$

One tool which we shall use frequently in this paper is the following natural generalization of the notion of an admissible ordinal:

Definition 4.1. *For $n \geq 1$, an ordinal α is Σ_n -admissible if α a limit and $\alpha > \omega$ and L_α models Σ_n -collection and Σ_{n-1} -separation.*

Recall that the collection schema is the following schema:

$$(4.2) \quad \forall \bar{p} [\forall x \exists y \varphi(x, y, \bar{p})] \rightarrow [\forall u \exists v (\forall x \in u \exists y \in v \varphi(x, y, \bar{p}))]$$

By abuse of notation, we also say that L_α is Σ_n -admissible iff α is Σ_n -admissible; and we write “admissible” in lieu of “ Σ_1 -admissible.” The notion of Σ_n -admissibility can be described axiomatically as well. In particular, it is not difficult to see that L_α is Σ_n -admissible if and only if L_α satisfies extensionality, pairing, union, infinity, the foundation schema, Σ_n -collection, and Σ_{n-1} -separation. In the case $n = 1$, this set of axioms provides an equivalent axiomatization of Kripke-Platek set theory ([28], [32], [10] p. 48, p. 36). Further, the union of this set of axioms for all $n \geq 1$, along with the axiom choice (in the form that every set can be well-ordered), is deductively equivalent to ZFC-P (cf. Definition 2.10). Finally, an equivalent definition of Σ_n -admissibility is as follows: α is Σ_n -admissible if and only if α is a limit and $\alpha > \omega$ and L_α models Kripke-Platek set theory and Σ_n -replacement in the strong form that both the graph and range of Σ_n -definable functions on sets exists (cf. [38] p. 368, [33] p. 174).

Several of the classical results about Kripke-Platek set theory easily generalize to Σ_n -admissibles. In particular, if L_α is Σ_n -admissible, then (i) L_α satisfies Δ_n -separation, (ii) L_α models that the Σ_n - and Π_n -formulas are uniformly closed under bounded quantification, and (iii) L_α satisfies Σ_n -transfinite recursion. For the proofs of these results for the case $n = 1$, see Chapters I-II of Devlin’s book [10]; the proofs for the results $n > 1$ carry over word-for-word. An idea closely related to transfinite recursion is Mostowski Collapse. Since admissible L_α don’t necessarily model the Mostowski Collapse Lemma, it is natural to formulate axioms pertaining directly to the Mostowski Collapse Lemma. In particular, we define:

Definition 4.2. *Axiom Beta says that for all sets X, R such that (X, R) is well-founded, there is a set π such that π is a function with domain X satisfying, for each y from X , the equation $\pi(y) = \{\pi(y') : y' \in X \ \& \ y'Ry\}$ (cf. Barwise [1] Definition I.9.5 p. 39).*

The set-version of the Mostowski Collapse Lemma holds in admissible L_α which satisfy Axiom Beta. The set-version of this lemma states that for all sets X, E such that (X, E) is well-founded and extensional, there is a transitive set M and an isomorphism $\pi : (X, E) \rightarrow (M, \in)$ (cf. [29] pp. 105-106, [30], [30] p. 56 ff, [25] p. 69). The traditional ZFC-proof

of Axiom Beta uses Σ_1 -replacement and Σ_1 -separation, and so L_α models Axiom Beta for all Σ_2 -admissible α .

Other basic properties of the structures L_α relate to its canonical well-ordering $<_L$. The well-order $<_L$ may be taken to be given by a canonical formula that is uniformly Δ_1 in admissible L_α . Further, since $<_L$ is uniformly Δ_1 , we have that this well-order is absolute between various admissible L_α . Moreover, one has that the function $x \mapsto \text{pred}_{<_L}(x)$ is uniformly Δ_1 in admissibles where we define $y \in \text{pred}_{<_L}(x)$ iff $y <_L x$ (cf. Devlin [10] pp. 74-75). Finally, just as the Σ_m - and Π_m -formulas are closed under bounded quantification for $0 \leq m \leq n$ in Σ_n -admissibles, so for $0 < m \leq n$ they are closed under $<_L$ -bounding in Σ_n -admissibles.

Other important properties of Σ_n -admissibles that we shall use are related to uniformization. A structure M satisfies Σ_n -uniformization if for every Σ_n^M -definable relation $R \subseteq M \times M$ there is a Σ_n^M -definable relation $R' \subseteq R$ such that $M \models \forall x [(\exists y R(x, y)) \rightarrow (\exists! y R'(x, y))]$. In this case, R' is called a Σ_n^M -definable uniformization of R . In his famous paper, Jensen showed that admissible L_α are models of Σ_n -uniformization for all $n \geq 1$ (cf. [26] Theorem 3.1 p. 256 and Lemma 2.15 p. 255; [10] Theorem 4.5 p. 269). The proof of this theorem is very difficult, and in fact holds for all members J_α of Jensen's alternate hierarchy. However, in what follows we can avoid direct appeal to Jensen's Theorem by appealing to the following weak version, whose elementary proof proceeds by choosing $<_L$ -least witnesses:

Proposition 4.3. (Weak Uniformization) *Suppose $n \geq 1$. If L_α is Σ_n -admissible then L_α satisfies Σ_m -uniformization for every $1 \leq m \leq n$. Moreover, the parameters in the Σ_m -definition of the uniformization R' can be taken to be the same as the parameters in the Σ_m -definition of R .*

Let's finally state a simple consequence of uniformization that we shall appeal to repeatedly in what follows:

Proposition 4.4. (Proposition on Right-Inverting a Surjection) *Suppose that $n \geq 1$ and that L_α is Σ_n -admissible. Suppose that Y is a $\Sigma_n^{L_\alpha}$ -definable subset of L_α and X is a subset of L_α . Suppose there is a $\Sigma_n^{L_\alpha}$ -definable surjection $\pi : Y \rightarrow X$. Then X is a $\Sigma_n^{L_\alpha}$ -definable subset of L_α and there is a $\Sigma_n^{L_\alpha}$ -definable injection $\iota : X \rightarrow Y$ satisfying $\pi \circ \iota = \text{id}_X$.*

An important concept in what follows is the n -th projectum of the structure L_α . This was introduced by Kripke ([28]) and Platek ([32]), and it records how small one can possibly make α under a Σ_n -definable injection:

Definition 4.5. *Suppose that $n > 0$ and $\alpha > \omega$. Then the n -th projectum $\rho_n(\alpha) = \rho_n$ of α is the least $\rho \leq \alpha$ such that there is a $\Sigma_n^{L_\alpha}$ -definable injection $\iota : \alpha \rightarrow \rho$.*

There are several different equivalent characterizations of the n -th projectum (cf. [33] p. 157, [1] Definition V.6.1 p. 174, [26] pp. 256-257, [34] Definition 2.1 p. 619). In particular, for admissible α , the n -th projectum may be equivalently defined as the smallest $\rho \leq \alpha$ such that there is a $\Sigma_n^{L_\alpha}$ -definable injection $\iota : L_\alpha \rightarrow \rho$.

Another basic tool that we employ is the notion of a Σ_n -elementary substructure. Recall that if M and N are structures in the signature of ZFC, then $M \prec_n N$ is said to hold, and M is said to be a Σ_n -elementary substructure of N , if $M \subseteq N$ and for every Σ_n -formula $\varphi(\bar{x})$ and every tuple of parameters \bar{a} from M , it is the case that $M \models \varphi(\bar{a})$ if and only if $N \models \varphi(\bar{a})$. Here are some basic facts about Σ_n -elementary substructures and the constructible hierarchy that we shall use:

Proposition 4.6.

1. The Σ_n -Definable Closure is a Σ_n -Elementary Substructure: *Suppose that L_α is Σ_n -admissible and $A \subseteq L_\alpha$. Let the Σ_n -definable closure of L_α with parameters A , written $\text{dcl}_{\Sigma_n}^{L_\alpha}(A)$, denote the set of elements a of L_α such that there is a Σ_n -formula $\varphi(x, \bar{y})$ with all free variables displayed and parameters $\bar{p} \in A$ such that $L_\alpha \models \varphi(a, \bar{p}) \wedge \forall a' (\varphi(a', \bar{p}) \rightarrow a = a')$. Then $\text{dcl}_{\Sigma_n}^{L_\alpha}(A) \prec_n L_\alpha$.*
2. *If κ is an uncountable regular cardinal, then L_κ is a model of ZFC-P (cf. Definition 2.10).*
3. Admissibility and Axiom Beta Preserved Under Elementary Substructure: *Suppose that $n \geq 1$ and that $L_\alpha \prec_n L_\beta$ where β is Σ_n -admissible. Then α is Σ_n -admissible. Further, if $L_\beta \models$ Axiom Beta then $L_\alpha \models$ Axiom Beta.*
4. Consequence of V=L for Σ_1 -Substructures of L up to a Successor Cardinal: *Suppose that V=L and λ is an infinite cardinal and $\lambda \cup \{\lambda\} \subseteq M$, $M \prec_1 L_{\lambda^+}$, $|M| = \lambda$. Then $M = L_\gamma$ for some γ with $|\gamma| = \lambda$.*

Proof. For the first item, see Devlin [10] Lemma II.5.3 p. 83 which proves the result for $n = \omega$; the same proof works for $1 \leq n < \omega$. For the next item on ZFC-P, see [29] p. 177, [30] Lemma II.6.22 p. 139, [25] p. 198, noting that these same proofs also give one the collection schema from the official definition of ZFC-P (cf. Definition 2.10). The proof of the third item follows from a routine induction. For the fourth and final item, see Devlin [10] Lemma II.5.10 p. 85 for the special case $\lambda = \omega$; the same proof works for the general case. \square

5 Construction and Existence Theorems, and Joint Consistency Problem

The aim of this section is to build models of abstraction principles with predicative amounts of comprehension, and these yield our solution to the joint consistency problem described at the close of §3. The first step is the following construction. This construction is also an important part of the proof of our Main Theorem 2.9, whose proof is presented in the next section §6. In the statement of this construction theorem, the key concepts of Σ_n -admissible and n -th projectum ρ_n were defined in the previous section §4. Likewise, recall that the theories $\mathbf{S0}$ and $\Sigma_1^1\text{-}[E_1, \dots, E_k]\mathbf{A}$ were defined respectively in Definition 3.2 and Definition 3.4 from §3.

Theorem 5.1. (Construction Theorem). *Suppose that $n \geq 1$ and that α is Σ_n -admissible with $\rho_n(\alpha) = \rho < \alpha$ and let $\iota : L_\alpha \rightarrow \rho$ be a witnessing $\Sigma_n^{L_\alpha}$ -definable injection. Then consider the following structure \mathcal{M} in the signature of **SO** (cf. Definition 3.2):*

$$(5.1) \quad \mathcal{M} = (\rho, P(\rho) \cap L_\alpha, P(\rho \times \rho) \cap L_\alpha, \dots)$$

Further, suppose for each $i \in [1, k]$, the relation E_i is a $\Sigma_{n-1}^{1, \mathcal{M}}$ -definable equivalence relation on $(P(\rho^{m_i}) \cap L_\alpha)$.

Then consider the $\Sigma_n^{L_\alpha}$ -definable maps $\partial_{E_i} : (P(\rho^{m_i}) \cap L_\alpha) \rightarrow \rho$ defined by $\partial_{E_i}(X) = \iota(\ell_i(X))$ where $\ell_i(X)$ is the $<_L$ -least member of X 's E_i -equivalence class. Then the following expansion of \mathcal{M} is a model of the theory $\Sigma_1^1\text{-}[E_1, \dots, E_k]\mathbf{A} + \mathbf{GC}$, where the global well-order $<$ on objects is given by the membership relation on the ordinal ρ :

$$(5.2) \quad \mathcal{N} = (\rho, P(\rho) \cap L_\alpha, P(\rho \times \rho) \cap L_\alpha, \dots, \partial_{E_1}, \dots, \partial_{E_k}, <)$$

Proof. For each $i \in [1, k]$, define

$$(5.3) \quad \widehat{E}_i = \{(X, Y) \in (P(\rho^{m_i}) \cap L_\alpha) \times (P(\rho^{m_i}) \cap L_\alpha) : \mathcal{M} \models E_i(X, Y)\}$$

Then since E_i is $\Sigma_{n-1}^{1, \mathcal{M}}$ -definable, it follows that \widehat{E}_i is $\Sigma_{n-1}^{L_\alpha}$ -definable, so that \widehat{E}_i is a $\Sigma_{n-1}^{L_\alpha}$ -definable equivalence relation on $(P(\rho^{m_i}) \cap L_\alpha)$. For each element X of $(P(\rho^{m_i}) \cap L_\alpha)$, let $[X]_{\widehat{E}_i} \subseteq (P(\rho^{m_i}) \cap L_\alpha)$ denote the \widehat{E}_i -equivalence class of X . Then $\ell_i : (P(\rho^{m_i}) \cap L_\alpha) \rightarrow (P(\rho^{m_i}) \cap L_\alpha)$ is defined by $\ell_i(X) = \min_{<_L}([X]_{\widehat{E}_i})$, and its graph has the following definition:

$$(5.4) \quad \begin{aligned} \ell_i(X) = Y \iff & X, Y \in L_\alpha \ \& \ X, Y \subseteq \rho^{m_i} \ \& \ \widehat{E}_i(X, Y) \\ & \ \& \ \forall Z <_L Y [Z \subseteq \rho^{m_i} \rightarrow \neg \widehat{E}_i(X, Z)] \end{aligned}$$

Since adding quantifiers bounded by $<_L$ to Σ_m - or Π_m -formulas for $m \leq n$ does not increase their complexity, we have that the graph of ℓ_i is defined by the conjunction of a $\Sigma_{n-1}^{L_\alpha}$ -formula with a $\Pi_{n-1}^{L_\alpha}$ -formula and so is $\Sigma_n^{L_\alpha}$ -definable. Then the map $\partial_{E_i} : (P(\rho^{m_i}) \cap L_\alpha) \rightarrow \rho$ is defined by $\partial_{E_i}(X) = \iota(\ell_i(X))$, which is likewise $\Sigma_n^{L_\alpha}$ -definable since it is the composition of two $\Sigma_n^{L_\alpha}$ -definable functions. (Note that in the case $n = 1$, the function ℓ_i is defined by the conjunction of a $\Sigma_0^{L_\alpha}$ -formula with a $\Sigma_1^{L_\alpha}$ -formula and so is $\Sigma_1^{L_\alpha}$ -definable. For, the formula $\forall Z <_L Y \theta(Z, Y)$ for any $\Sigma_0^{L_\alpha}$ -definable $\theta(Z, Y)$ is equivalent to the formula $\exists Y' Y' = \text{pred}_{<_L}(Y) \ \& \ \forall Z \in Y' \theta(Z, Y)$ which is $\Sigma_1^{L_\alpha}$ -definable because the map $Y \mapsto \text{pred}_{<_L}(Y)$ is $\Delta_1^{L_\alpha}$ -definable).

Now let us argue that the so-defined structure \mathcal{N} from equation (5.2) satisfies the abstraction principle $A[E_i]$ (3.1). First suppose that $\mathcal{N} \models \partial_{E_i}(X) = \partial_{E_i}(Y)$ for some $X, Y \in (P(\rho^{m_i}) \cap L_\alpha)$. Then since $\iota : L_\alpha \rightarrow \rho$ is an injection we have that $\ell_i(X) = \ell_i(Y)$, so that $\min_{<_L}([X]_{\widehat{E}_i}) = \min_{<_L}([Y]_{\widehat{E}_i})$. Hence $\widehat{E}_i(X, Y)$ so that $\mathcal{M} \models E_i(X, Y)$ and hence its expansion \mathcal{N} also models this. Conversely, suppose that $\mathcal{N} \models E_i(X, Y)$, so that its reduct \mathcal{M} also models this. Then $\widehat{E}_i(X, Y)$ and hence $[X]_{\widehat{E}_i} = [Y]_{\widehat{E}_i}$ and $\min_{<_L}([X]_{\widehat{E}_i}) = \min_{<_L}([Y]_{\widehat{E}_i})$,

so that $\ell_i(X) = \ell_i(Y)$ and hence $\partial_{E_i}(X) = \partial_{E_i}(Y)$. Hence in fact the structure \mathcal{N} from equation (5.2) satisfies the abstraction principle $A[E_i]$.

So now it remains to show that the structure \mathcal{N} from equation (5.2) satisfies the First-Order Comprehension Schema (cf. Definition 2.2) and the Σ_1^1 -Choice Schema (cf. Definition 2.4) in the expanded signature containing the function symbols $\partial_{E_1}, \dots, \partial_{E_k}$. First let us establish a preliminary result that any $\Sigma_1^{1, \mathcal{N}}$ -definable subset of \mathcal{N} is $\Sigma_n^{L_\alpha}$ -definable (indeed, in the same parameters). This result is proven by induction on the complexity of the formula defining the subset of \mathcal{N} . By a subset of the many-sorted structure \mathcal{N} , we mean any subset of any finite product $S_1 \times \dots \times S_n$, wherein S_i is one of the sorts $\rho, P(\rho) \cap L_\alpha, P(\rho \times \rho) \cap L_\alpha, \dots$ of the structure \mathcal{N} as displayed in equation (5.2). So our preliminary result establishes not only that $\Sigma_1^{1, \mathcal{N}}$ -definable subsets of the first-order part ρ are $\Sigma_n^{L_\alpha}$ -definable, but also that e.g. $\Sigma_1^{1, \mathcal{N}}$ -definable subsets of $\rho \times (P(\rho) \cap L_\alpha)$ are $\Sigma_n^{L_\alpha}$ -definable.

As a base case, we show that any subset of \mathcal{N} defined by an atomic formula is $\Delta_n^{L_\alpha}$ -definable. Recall that an *unnested* atomic formula in a signature is one of the form $x = y$, $c = y$, $f(\bar{x}) = y$ or $P\bar{x}$, where c, f, P are respectively constant symbols, function symbols, and relation symbols of the signature (cf. [24] p. 58). Then in \mathcal{N} , any atomic formula φ is equivalent to both a $\Sigma_1^{1, \mathcal{N}}$ -formula $\varphi^\exists \equiv \exists \bar{R} \exists \bar{y} \varphi_0^\exists$ and a $\Pi_1^{1, \mathcal{N}}$ -formula $\varphi^\forall \equiv \forall \bar{R} \forall \bar{y} \varphi_0^\forall$ in which φ_0^\exists and φ_0^\forall are quantifier-free and in which any atomic subformula of them is unnested (cf. [24] Theorem 2.6.1 p. 58). Now, the unnested atomics of the signature of \mathcal{N} are $\partial_{E_i}(R) = x$, $R[\bar{x}] = S$, $R\bar{y}$, and $x = y$. The first is by construction $\Sigma_n^{L_\alpha}$ -definable and the last three are trivially $\Sigma_n^{L_\alpha}$ -definable. Further, their negations are likewise $\Sigma_n^{L_\alpha}$ -definable: for instance, $\partial_{E_i}(R) \neq x$ iff $\exists y \in \rho (\partial_{E_i}(R) = y \ \& \ y \neq x)$, and Σ_n -formulas are closed under bounded quantification in L_α . Now, without loss of generality, we may assume that the negations in φ_0^\exists and φ_0^\forall are all pushed to the inside, so that they apply only to unnested atomics. Then since the Σ_n -formulas are closed under finite union and intersection in L_α , we have that each subformula of φ_0^\exists and φ_0^\forall is $\Sigma_n^{L_\alpha}$ -definable and thus so are they themselves. By the same reasoning, this holds true for their negations $\neg\varphi_0^\exists$ and $\neg\varphi_0^\forall$ as well. Since φ^\exists is formed from φ_0^\exists by adding a single block of existential quantifiers, we have that φ^\exists is $\Sigma_n^{L_\alpha}$ -definable. Since $\neg(\varphi^\forall)$ is formed from $\neg(\varphi_0^\forall)$ by adding a single block of existential quantifiers, we have that $\neg(\varphi^\forall)$ is $\Sigma_n^{L_\alpha}$ -definable, so that φ^\forall is $\Pi_n^{L_\alpha}$ -definable. Since the original atomic formula φ is equivalent to both φ^\exists and φ^\forall , we have that the original atomic formula φ is indeed $\Delta_n^{L_\alpha}$ -definable.

Then we show that any $\Sigma_1^{1, \mathcal{N}}$ -definable subset of \mathcal{N} is $\Sigma_n^{L_\alpha}$ -definable by a straightforward induction. We may again assume that all the negations are pushed to the inside and apply only to atomics. And the atomics and negated atomics are $\Sigma_n^{L_\alpha}$ -definable by the previous paragraph. Since $\Sigma_n^{L_\alpha}$ -definability is closed under finite union and intersection, the induction steps for conjunction and disjunction are trivial. Likewise, the induction steps for first-order quantification hold because first-order quantification in \mathcal{N} corresponds to bounded quantification over elements of ρ in L_α , and the Σ_n -formulas are closed under bounded quantification in L_α . So the inductive argument up this point establishes the result for formulas with no higher-order quantifiers. But since we're restricting to the case of $\Sigma_1^{1, \mathcal{N}}$ -definable subsets, the addition of a block of higher-order existential quantifiers ranging over

$P(\rho^k) \cap L(\alpha)$ does not bring us out of the complexity class $\Sigma_n^{L_\alpha}$.

Hence indeed any $\Sigma_1^{1, \mathcal{N}}$ -definable subset of \mathcal{N} is $\Sigma_n^{L_\alpha}$ -definable. From this, the First-Order Comprehension Schema (cf. Definition 2.2) follows directly from Δ_n -separation in L_α . As for the Σ_1^1 -choice schema, suppose that $\mathcal{N} \models \forall x \exists R \varphi(x, R)$, wherein φ is Σ_1^1 . Choose a Σ_n -formula ψ such that for all $x \in \rho$ and $R \in (P(\rho) \cap L_\alpha)$ one has $\mathcal{N} \models \varphi(x, R)$ iff $L_\alpha \models \psi(x, R)$. Then one has that $L_\alpha \models \forall x \in \rho \exists R \subseteq \rho \psi(x, R)$. Define $\Gamma(x, R) \equiv [x \in \rho \ \& \ R \in (P(\rho) \cap L_\alpha) \ \& \ L_\alpha \models \psi(x, R)]$. By weak uniformization (4.3), choose a $\Sigma_n^{L_\alpha}$ -definable uniformization Γ' of Γ . Since $\Gamma' : \rho \rightarrow (P(\rho) \cap L_\alpha)$, it follows from Σ_n -replacement that its graph is an element of L_α , and obviously we have $L_\alpha \models \forall x \in \rho \psi(x, \Gamma'(x))$. Then define $R'xy$ if and only if $y \in \Gamma'(x)$, so that $R' \in (P(\rho \times \rho) \cap L_\alpha)$ and $R'[x] = \Gamma'(x)$. Then one has $L_\alpha \models \forall x \in \rho \psi(x, R'[x])$ and hence $\mathcal{N} \models \varphi(x, R'[x])$.

As for the global choice principle GC, we may briefly note that \mathcal{N} obviously satisfies it when we use the ordinary ordering $<$ on the ordinal ρ as the witness. For, since the ordering $<$ on ρ is Δ_0 -definable, it exists in $P(\rho \times \rho) \cap L_\alpha$ by Δ_0 -separation on the set $\rho \times \rho$ in L_α . In the previous paragraphs, we have verified that various forms of comprehension hold on \mathcal{N} , in which parameters are allowed to occur. Hence these forms of comprehension continue to hold when $<$ is permitted to occur within the formulas because we can view this as simply yet another parameter. \square

Theorem 5.2. (Existence Theorem). *Let $\gamma \geq 0$ and let $\lambda = \omega_\gamma^L$ and $\kappa = \omega_{\gamma+1}^L$. Then for each $n \geq 1$ there is an Σ_n -admissible α_n such that*

$$(5.5) \quad \lambda < \alpha_n < \kappa, \quad \rho_n(\alpha_n) = \lambda, \quad L_{\alpha_n} \prec_n L_\kappa, \quad L_{\alpha_n} \models \text{Axiom Beta}$$

More specifically, we can choose α_n so that $L_{\alpha_n} = \text{dcl}_{\Sigma_n}^{L_\kappa}(\lambda \cup \{\lambda\})$. Further, the following set $\mathcal{F}_n \subseteq \lambda$ is $\Sigma_1^{L_{\alpha_n}}$ -definable, wherein $\langle \cdot, \cdot \rangle : \lambda \times \lambda \rightarrow \lambda$ is Gödel's Σ_1 -definable pairing function and $\text{Form}(\Sigma_n)$ is the set of Gödel numbers of Σ_n -formulas:

$$(5.6) \quad \mathcal{F}_n = \{ \langle \ulcorner \varphi(x, \bar{y}, z) \urcorner, \bar{\beta} \rangle : \varphi(x, \bar{y}, z) \in \text{Form}(\Sigma_n) \ \& \ \bar{\beta} < \lambda \}$$

Moreover, there is a $\Sigma_n^{L_{\alpha_n}}$ surjective partial map $\theta_n : \mathcal{F}_n \dashrightarrow L_{\alpha_n}$ such that

$$(5.7) \quad \begin{aligned} \theta_n(\langle \ulcorner \varphi(x, \bar{y}, z) \urcorner, \bar{\beta} \rangle) = a &\implies L_{\alpha_n} \models \varphi(a, \bar{\beta}, \lambda) \\ (L_{\alpha_n} \models \exists ! x \varphi(x, \bar{\beta}, \lambda)) &\implies \langle \ulcorner \varphi(x, \bar{y}, z) \urcorner, \bar{\beta} \rangle \in \text{dom}(\theta_n) \end{aligned}$$

and a $\Sigma_n^{L_{\alpha_n}}$ -definable injection $\iota_n : L_{\alpha_n} \rightarrow \text{dom}(\theta_n)$ such that $\theta_n \circ \iota_n$ is the identity on L_{α_n} . Further, the sequence α_n is strictly increasing. Finally, for each $n \geq 1$ there is an injection $\chi_n : \lambda \rightarrow \theta_n^{-1}(\{0, 1\})$ whose graph is in L_{α_n} .

Proof. Since the result is absolute, we may assume $\mathbf{V}=\mathbf{L}$, and hence we may assume that λ and κ are cardinals. Since κ is regular uncountable one has that $L_\kappa \models \text{ZFC-P}$ (cf. Proposition 4.6, item 2). Let $M = \text{dcl}_{\Sigma_n}^{L_\kappa}(\lambda \cup \{\lambda\})$. Since the Σ_n -definable closure is a Σ_n -elementary substructure (cf. Proposition 4.6, item 1), we have $M \prec_n L_\kappa$. Since $\kappa = \lambda^+$ and $\lambda \cup \{\lambda\} \subseteq M$

and $M \prec_1 L_{\lambda^+}$, it follows from the consequence of $V=L$ for Σ_1 -substructures of L up to a successor cardinal (Proposition 4.6 item 4) that $M = L_{\alpha_n}$ where $|\alpha_n| = \lambda$. Then $\lambda \leq \alpha_n < \kappa$. But since $\lambda \in M = L_{\alpha_n}$ we have $\lambda < \alpha_n < \kappa$. By Proposition 4.6 item 3, we also have that L_{α_n} is Σ_n -admissible and satisfies Axiom Beta.

Then define the following relation $R_n \subseteq \mathcal{F}_n \times L_{\alpha_n}$ by

$$(5.8) \quad R_n(\langle \ulcorner \varphi(x, \bar{y}, z) \urcorner, \bar{\beta} \rangle, a) \iff \langle \ulcorner \varphi(x, \bar{y}, z) \urcorner, \bar{\beta} \rangle \in \mathcal{F}_n \ \& \ L_{\alpha_n} \models \varphi(a, \bar{\beta}, \lambda)$$

Then by the definability of partial satisfaction predicates, R_n is $\Sigma_n^{L_{\alpha_n}}$ -definable. Then by weak uniformization (Proposition 4.3), choose a $\Sigma_n^{L_{\alpha_n}}$ -definable uniformization $\theta_n : \mathcal{F}_n \dashrightarrow L_{\alpha_n}$ of R_n . Then θ_n is a surjective partial function. For, suppose that $a \in L_{\alpha_n}$. Since $L_{\alpha_n} = \text{dcl}_{\Sigma_n}^{L_{\kappa}}(\lambda \cup \{\lambda\})$, there is Σ_n -formula $\varphi(x, \bar{y}, z)$ and $\bar{\beta} < \lambda$ such that

$$(5.9) \quad L_{\alpha_n} \models \varphi(a, \bar{\beta}, \lambda) \ \& \ [\forall x (\varphi(x, \bar{\beta}, \lambda) \rightarrow x = a)]$$

Then $L_{\alpha_n} \models \exists x \varphi(x, \bar{\beta}, \lambda)$ and $\langle \ulcorner \varphi(x, \bar{y}, z) \urcorner, \bar{\beta} \rangle$ is in \mathcal{F}_n . Then on the input $u = \langle \ulcorner \varphi(x, \bar{y}, z) \urcorner, \bar{\beta} \rangle$, we have that $\theta_n(u)$ is defined and if $\theta_n(u) = a'$ then $L_{\alpha_n} \models \varphi(a', \bar{\beta}, \lambda)$. But in conjunction with equation (5.9), it thus follows that $a = a' = \theta_n(u)$. Hence, indeed $\theta_n : \mathcal{F}_n \dashrightarrow L_{\alpha_n}$ is a surjective partial function. Let $\mathcal{F}'_n \subseteq \mathcal{F}_n$ be the domain of θ_n , which is likewise $\Sigma_n^{L_{\alpha_n}}$ -definable. By the Proposition on Right-Inverting a Surjection (Proposition 4.4), it follows that there is a $\Sigma_n^{L_{\alpha_n}}$ -definable injection $\iota_n : L_{\alpha_n} \rightarrow \mathcal{F}'_n$ such that $\theta_n \circ \iota_n = \text{id}_{L_{\alpha_n}}$. Since $\mathcal{F}_n \subseteq \lambda$, we then have that $\rho_n(\alpha_n) \leq \lambda$. Since λ is a cardinal and L_{α_n} has cardinality λ , we must have then that $\rho_n(\alpha_n) = \lambda$.

Now we argue that $\alpha_1 < \alpha_2 < \alpha_3 < \dots$. Since $L_{\alpha_n} = \text{dcl}_{\Sigma_n}^{L_{\kappa}}(\lambda \cup \{\lambda\})$, we have that $L_{\alpha_n} \subseteq L_{\alpha_{n+1}}$ and hence that $\alpha_n \leq \alpha_{n+1}$. Suppose that it was not always that case that $\alpha_n < \alpha_{n+1}$ for all $n \geq 1$. Then $\alpha_n = \alpha_{n+1}$ for some $n \geq 1$. Since $L_{\alpha_{n+1}}$ is Σ_{n+1} -admissible and $L_{\alpha_n} = L_{\alpha_{n+1}}$, we have that L_{α_n} is Σ_{n+1} -admissible and so satisfies Σ_n -separation. Hence $\mathcal{F}'_n \in L_{\alpha_n}$ and hence by Σ_n -replacement, the $\Sigma_n^{L_{\alpha_n}}$ -definable map $\theta_n : \mathcal{F}'_n \rightarrow L_{\alpha_n}$ would be bounded and thus not surjective.

Finally, we verify that for each $n \geq 1$ there is injection $\chi_n : \lambda \rightarrow \theta_n^{-1}(\{0, 1\})$. Let $\varphi(x, y, z)$ say “ $x = 0$ and y is an ordinal.” Then for each $\beta < \lambda$ there is exactly one x in L_{α_n} such that $L_{\alpha_n} \models \varphi(x, \beta, \lambda)$. Then by equation (5.7) we have that $\langle \ulcorner \varphi(x, y, z) \urcorner, \beta \rangle \in \text{dom}(\theta_n)$ and we have by equation (5.7) that $\theta_n(\langle \ulcorner \varphi(x, y, z) \urcorner, \beta \rangle) = 0$. Then define the function $\chi_n : \lambda \rightarrow \theta_n^{-1}(\{0, 1\})$ by $\chi_n(\beta) = \langle \ulcorner \varphi(x, y, z) \urcorner, \beta \rangle$, which is clearly injective; further clearly the graph of χ is in L_{α_n} . \square

The extra information about the injection $\chi_n : \lambda \rightarrow \theta_n^{-1}(\{0, 1\})$ in Theorem 5.2 will be primarily useful for our companion paper [41], where we use constructible sets to build models of an intensional type theory (cf. §5 of [41]). The reason for the focus on $\{0, 1\}$ is that in the tradition of type-theory these are used as ersatzes for the truth-values $\{F, T\}$. The map χ_n then allows us to inject ordinals $\beta < \lambda$ into intensional entities $\chi_n(\beta)$ which determine truth-values $\theta_n(\chi_n(\beta))$, roughly after the manner in which we inject natural numbers e into algorithms P_e which determine computable number-theoretic functions φ_e .

The proofs of these results can be seen as a generalization of our earlier constructions of models of $\Sigma_1^1\text{-LB}_0$ of the form $\mathcal{N} = (\omega, \text{HYP}, \dots, \partial)$ ([40] Theorem 53 p. 1695). Here HYP denotes the hyperarithmetic subsets of natural numbers and $\partial(Y) = \langle b, e \rangle$ only if b is a code for a computable ordinal β and Y is computable from b 's canonical coding H_b of the β -th Turing jump by the program e . This earlier result can be seen as a special case of these results by virtue of the fact that if $\alpha = \omega_1^{\text{CK}}$ then $P(\omega) \cap L_\alpha = \text{HYP}$ (cf. Sacks [33] § III.9 Exercise 9.12 p. 87). The primary difference between the proofs here and our earlier constructions of models of $\Sigma_1^1\text{-LB}_0$ ([40] Theorem 53 p. 1695) was that the latter used Kondô's Uniformization Theorem ([39] p. 224, [27] p. 306), while the proof here used uniformization results in the constructible hierarchy like weak uniformization 4.3. Further, our results here can cover not just Basic Law V, but the abstraction principles described in §3.

Finally, we can now prove the main results on the consistency of abstraction principles in the predicative setting, which were first stated and motivated in §3. As for Theorem 3.3, this is a limiting case of the Joint Consistency Theorem 3.5. So it remains to establish this latter theorem:

Proof. (of Joint Consistency Theorem 3.5): So suppose that the formulas $E_1(R, S), \dots, E_k(R, S)$ in the signature of $\text{SO} + \text{GC}$ are provably equivalence relations on m_i -ary concepts in $\text{SO} + \text{GC}$.

The theory SO from Definition 3.2 can be naturally written as the union of theories SO_m , where SO_m restricts the instances of the Full Comprehension Schema (2.1) in the signature of $\text{SO} + \text{GC}$ to its Σ_m^1 -instances, where this is the standard notion for formulas which begin with m -alternating blocks of quantifiers, the first of which is a block of second-order existential quantifiers. By compactness, it suffices to show, for each $m \geq 1$, that the theory $\Sigma_1^1\text{-}[E_1, \dots, E_k]\text{A} + \text{SO}_m + \text{GC}$ is consistent.

Let us then fix, for the remainder of this proof, an $m \geq 1$. Choose $n > m$ sufficiently large so that (i) the formulas $E_1(R, S), \dots, E_k(R, S)$ are provably equivalence relations in $\text{SO}_{n-1} + \text{GC}$, and (ii) each of the formulas $E_1(R, S), \dots, E_k(R, S)$ is a Σ_{n-1}^1 -formula.

By the Existence Theorem 5.2, choose Σ_n -admissible α such that $\rho = \rho(\alpha) < \alpha$ and L_α is a model of Axiom Beta. Then consider the following structure \mathcal{M} in the signature of SO , as was also featured in equation (5.1) of the hypothesis of the Construction Theorem 5.1:

$$(5.10) \quad \mathcal{M} = (\rho, P(\rho) \cap L_\alpha, P(\rho \times \rho) \cap L_\alpha, \dots)$$

Since α is Σ_n -admissible, one has that L_α satisfies Σ_{n-1} -separation. Thus the structure \mathcal{M} from equation (5.10) satisfies the theory SO_{n-1} since instances of the Full-Comprehension Schema (2.1) in the signature of SO associated to Σ_{n-1}^1 -formulas correspond naturally to Σ_{n-1} -instances of separation in L_α on the set ρ . When expanded by the well-order $<$ on objects given by the membership relation on ρ , it likewise satisfies the theory $\text{SO}_{n-1} + \text{GC}$. Since the formulas $E_1(R, S), \dots, E_k(R, S)$ are provably equivalence relations on m_i -ary concepts in $\text{SO}_{n-1} + \text{GC}$, it follows that they are likewise a $\Sigma_{n-1}^{1, \mathcal{M}}$ -definable equivalence relation on $P(\rho^{m_i}) \cap L_\alpha$. Then by the Construction Theorem, one can build a model of $\Sigma_1^1\text{-}[E_1, \dots, E_k]\text{A} + \text{GC}$ of the following form, wherein again the global well-order $<$ is given by the membership relation on ρ :

$$(5.11) \quad \mathcal{N} = (\rho, P(\rho) \cap L_\alpha, P(\rho \times \rho) \cap L_\alpha, \dots, \partial_{E_1}, \dots, \partial_{E_k}, <)$$

Since satisfaction of formulas in the signature of \mathcal{M} is invariant between it and its expansion \mathcal{N} , this model \mathcal{N} is the witness to the consistency of $\Sigma_1^1\text{-}[\mathbf{E}_1, \dots, \mathbf{E}_k]\mathbf{A} + \mathbf{SO}_m + \mathbf{GC}$. \square

6 Identifying the Well-Founded Extensions

The goal of this section is to establish the Main Theorem 2.9. This is done in two steps: (i) first by identifying in Theorem 6.1 the well-founded extensions within models induced via the Construction Theorem 5.1 from L_α , and (ii) second in Theorem 6.2 by an identification within models satisfying Axiom Beta (cf. Definition 4.2). The basic idea of these proofs is to relate the notion $\text{Trcl}_\eta(x)$ from §2 equation (2.10) defined in the object-language of a model of $\Sigma_1^1\text{-LB}$ to the notion $\text{trcl}_\eta(x)$ defined in the meta-language. In particular, given an arbitrary relation R , the notion $\text{trcl}_R(x)$ is defined to be the set of all y such that there is a finite sequence x_1, \dots, x_n such that $x_1 = y$ and $x_n = x$ and $x_m R x_{m+1}$ for all $m < n$. So a model \mathcal{N} of $\Sigma_1^1\text{-LB}$ induces a specific relation η via the definition of the Fregean membership relation from equation (2.7), and then $\text{trcl}_\eta(x)$ is defined to be $\text{trcl}_R(x)$ with $R = \eta$. Finally, recall that the well-founded extensions wfExt were defined in (2.12).

Theorem 6.1. (First Identification of Well-Founded Extensions) *Suppose $n \geq 1$. Suppose that L_α is Σ_n -admissible. Let $\rho = \rho_n(\alpha)$ and let $\partial : L_\alpha \rightarrow \rho$ be a witnessing $\underline{\Sigma}_n^{L_\alpha}$ -definable injection. Suppose also that $\rho < \alpha$. Then the structure*

$$(6.1) \quad \mathcal{N} = (\rho, P(\rho) \cap L_\alpha, P(\rho \times \rho) \cap L_\alpha, \dots, \partial \upharpoonright (P(\rho) \cap L_\alpha))$$

is a model of $\Sigma_1^1\text{-LB} + \mathbf{GC}$, where the global well-order on objects is given by the membership relation on ρ . Further:

$$(6.2) \quad \begin{aligned} \text{wfExt}(\mathcal{N}) = \{x \in \rho : (\text{trcl}_\eta(x) \cup \{x\}, \eta) \text{ is} \\ \underline{\Delta}_n^{L_\alpha}\text{-well-founded} \ \& \ (\text{trcl}_\eta(x) \cup \{x\}) \subseteq \text{rng}(\partial)\} \end{aligned}$$

Moreover, there is a $\underline{\Sigma}_n^{L_\alpha}$ -definable embedding $j : (L_\alpha, \in) \rightarrow (\text{wfExt}(\mathcal{N}), \eta)$, and its image is:

$$(6.3) \quad \begin{aligned} \text{wfExt}_*(\mathcal{N}) = \{x \in \rho : (\text{trcl}_\eta(x) \cup \{x\}, \eta) \text{ is} \\ \text{well-founded} \ \& \ (\text{trcl}_\eta(x) \cup \{x\}) \subseteq \text{rng}(\partial)\} \end{aligned}$$

Finally, the isomorphism $j : (L_\alpha, \in) \rightarrow (\text{wfExt}_(\mathcal{N}), \eta)$ is the inverse of the Mostowski collapse $\pi : (\text{wfExt}_*(\mathcal{N}), \eta) \rightarrow (L_\alpha, \in)$.*

For the statement of the Mostowski collapse theorem, see the discussion immediately following the definition of Axiom Beta (Definition 4.2).

Proof. By the Construction Theorem 5.1, the structure \mathcal{N} is a model of $\Sigma_1^1\text{-LB} + \mathbf{GC}$. Now we argue for the identity in equation (6.2). To see this identity, let us first show both of the following, wherein x is an arbitrary element of ρ :

$$(6.4) \quad w \in \text{trcl}_\eta(x) \implies \mathcal{N} \models (\text{Trcl}_\eta(x))(w)$$

$$(6.5) \quad (\text{trcl}_\eta(x) \cup \{x\}) \subseteq \text{rng}(\partial) \implies \text{trcl}_\eta(x) \in L_\alpha$$

For equation (6.4), suppose that $w \in \text{trcl}_\eta(x)$ and suppose that $F \in (P(\rho) \cap L_\alpha)$ is such that $\mathcal{N} \models [\forall z (z\eta x \rightarrow Fz) \ \& \ \forall u, v ((Fv \ \& \ u\eta v) \rightarrow Fu)]$. We must show that $w \in F$. Since $w \in \text{trcl}_\eta(x)$, choose a sequence $y_1, \dots, y_n \in \rho$ such that $y_1 = w$ and $y_n = x$ and $y_i\eta y_{i+1}$ for all $i < n$. Then we may show by induction on $0 < k \leq n-1$ that $y_{n-k} \in F$.

For equation (6.5), first define a map $\tau : P(\rho) \rightarrow P(\rho)$ by $\tau(U) = \{v \in \rho : \exists w \in U \ v\eta w\}$. Now, it follows from the proposition on the existence of restricted η -relation (Proposition 2.7) that the map τ has the property:

$$(6.6) \quad [U \in L_\alpha \ \& \ U \subseteq \text{rng}(\partial)] \Rightarrow \exists S \in (P(\rho \times \rho) \cap L_\alpha) [\forall w \in U \ \partial(S[w]) = w \\ \& \ \tau(U) = \{v \in \rho : \exists w \in U \ v \in S[w]\} \in L_\alpha]$$

Let us note one further property of the map τ , namely its connection to transitive closure:

$$(6.7) \quad U \in (P(\rho) \cap L_\alpha) \Rightarrow \text{trcl}_\eta(\partial(U)) = \bigcup_{n=0}^{\infty} \tau^{(n)}(U)$$

To see this, suppose that $U \in (P(\rho) \cap L_\alpha)$. First consider the left-to-right direction of the identity. Suppose that $y \in \text{trcl}_\eta(\partial(U))$. Then there are y_1, \dots, y_n where $y_1 = y$ and $y_n = \partial(U)$ and $y_i\eta y_{i+1}$ for $i < n$. By induction on $0 < k \leq n-1$ we may then show that $y_{n-k} \in \tau^{(k-1)}(U)$. Second, consider the right-to-left direction of the identity in equation (6.7). For this one simply shows by induction on $n \geq 0$, that $\tau^{(n)}(U) \subseteq \text{trcl}_\eta(\partial(U))$.

Turning now to the verification of equation (6.5), suppose that $(\text{trcl}_\eta(x) \cup \{x\}) \subseteq \text{rng}(\partial)$. Then $\partial(X) = x$ for some $X \in (P(\rho) \cap L_\alpha)$. Now we argue that $\tau^{(n)}(X) \in L_\alpha$ for all $n \geq 0$. Clearly this holds for $n = 0$, since by hypothesis one has that $\tau^{(0)}(X) = X \in L_\alpha$. Suppose, for the induction step, that $\tau^{(n)}(X) \in L_\alpha$. Then by equation (6.7) we can collect together the following information: $\tau^{(n)}(X) \in L_\alpha$ and $\tau^{(n)}(X) \subseteq \text{trcl}_\eta(x) \subseteq \text{rng}(\partial)$. Then we can deduce immediately from equation (6.6) that $\tau^{(n+1)}(X) = \tau(\tau^{(n)}(X)) \in L_\alpha$. So now we have finished arguing that $\tau^{(n)}(X) \in L_\alpha$ for all $n \geq 0$. By appealing repeatedly to the proposition on the existence of restricted η -relation (Proposition 2.7), one has that L_α models that for all $n < \omega$ there is a sequence $\langle U_0, S_0, \dots, U_n, S_n \rangle$ of elements of $U_i \in P(\rho) \cap L_\alpha$, $S_i \in P(\rho \times \rho) \cap L_\alpha$ such that $U_0 = X$ and

$$(6.8) \quad \forall m \leq n \ \forall w \in U_m \ \partial(S_m[w]) = w$$

$$(6.9) \quad \forall m < n \ U_{m+1} = \{v \in \rho : \exists w \in U_m \ v \in S_m[w]\}$$

Let $n < \omega$ and let $\langle U_0, S_0, \dots, U_n, S_n \rangle$ be such a sequence. We argue by induction on $m \leq n$ that $U_m = \tau^{(m)}(X)$. Clearly this holds for $m = 0$ since $U_0 = X$. Suppose it holds for $m < n$. To see it holds for $m + 1$, note that equation (6.8) and equation (6.9) and the induction hypothesis imply

$$(6.10) \quad \forall w \in \tau^{(m)}(X) \ \partial(S_m[w]) = w$$

$$(6.11) \quad U_{m+1} = \{v \in \rho : \exists w \in \tau^{(m)}(X) \ v\eta w\} = \tau^{(m+1)}(X)$$

So consider the following function $f : \omega \rightarrow L_\alpha$ defined as follows: $f(m) = U$ iff there is a sequence $\langle U_0, S_0, \dots, U_m, S_m \rangle$ satisfying (6.8)-(6.9) such that $U = U_m$. Then the graph of f

is $\Sigma_n^{L_\alpha}$ -definable and so by Σ_n -replacement, its graph exists as a set in L_α . Hence the infinite sequence $\langle \tau^{(0)}(X), \tau^{(1)}(X), \dots, \tau^{(n)}(X), \dots \rangle$ is an element of L_α and so by equation (6.7), one also has that $\text{trcl}_\eta(x) = \text{trcl}_\eta(\partial(X)) \in L_\alpha$. So we have finished now the verification of equation (6.5).

Now we proceed to the verification of equation (6.2). Suppose first that we have an extension $x \in \text{wfExt}(\mathcal{N})$. Recall that the membership conditions of $\text{wfExt}(\mathcal{N})$ are defined in equation (2.13), so that

$$(6.12) \quad \mathcal{N} \models (\text{Trcl}_\eta(\sigma(x)), \eta) \text{ is well-founded } \& (\text{Trcl}_\eta(\sigma(x))) \subseteq \text{rng}(\partial)$$

By equation (6.4), we automatically have that

$$(6.13) \quad (\text{trcl}_\eta(x) \cup \{x\}) \subseteq \{w \in \rho : \mathcal{N} \models \text{Trcl}_\eta(x)(w) \vee w = x\} \subseteq \text{rng}(\partial)$$

Hence from equation (6.5), we may conclude that $\text{trcl}_\eta(x) \in L_\alpha$. Note that if we set $F = \text{trcl}_\eta(x)$ then F satisfies the following condition:

$$(6.14) \quad [\forall z (z\eta x \rightarrow Fz) \& \forall u, v ((Fv \& u\eta v) \rightarrow Fu)]$$

Since $F \in (P(\rho) \cap L_\alpha)$, it follows that the converse to equation (6.4) holds as well, so that we may conclude $\text{trcl}_\eta(x) = \{w \in \rho : \mathcal{N} \models \text{Trcl}_\eta(x)(w)\}$. So now suppose that $(\text{trcl}_\eta(x) \cup \{x\}, \eta)$ is not $\Delta_n^{L_\alpha}$ -well-founded. Then there is some non-empty $\Delta_n^{L_\alpha}$ -definable subset Z of $(\text{trcl}_\eta(x) \cup \{x\}, \eta)$ which has no η -least member. By Δ_n -separation in L_α on the set $(\text{trcl}_\eta(x) \cup \{x\}) \in L_\alpha$, we have $Z \in P(\rho) \cap L_\alpha$, which is a contradiction. So we just completed the left-to-right direction of equation (6.2). For the other direction, suppose that $x \in \rho$ and

$$(6.15) \quad (\text{trcl}_\eta(x) \cup \{x\}, \eta) \text{ is } \Delta_n^{L_\alpha}\text{-well-founded } \& (\text{trcl}_\eta(x) \cup \{x\}) \subseteq \text{rng}(\partial)$$

Then equation (6.5) implies that $\text{trcl}_\eta(x) \in L_\alpha$. By a similar argument, we have that $x \in \text{wfExt}(\mathcal{N})$. So we have now finished verifying equation (6.2).

Now we turn to constructing an embedding $j : L_\alpha \rightarrow \rho$. By transfinite recursion, there is $\Sigma_n^{L_\alpha}$ -definable $j : L_\alpha \rightarrow \rho$ which satisfies $j(x) = \partial(\{j(y) : y \in x\})$. Then one has that $y \in x$ implies $j(y)\eta j(x)$. Further, since $\partial : L_\alpha \rightarrow \rho$ is an injection, we may argue by induction that $j : L_\alpha \rightarrow \rho$ is an injection. Since $j : L_\alpha \rightarrow \rho$ is an injection, $y \in x$ iff $j(y)\eta j(x)$. Hence, $j : L_\alpha \rightarrow \rho$ is indeed an embedding.

Now we argue that $j : L_\alpha \rightarrow \text{wfExt}_*(\mathcal{N})$. First let us show:

$$(6.16) \quad x \in L_\alpha \implies (\text{trcl}_\eta(j(x)) \cup \{j(x)\}) \subseteq \text{rng}(j) \subseteq \text{rng}(\partial)$$

Let $x \in L_\alpha$ and let $y \in \text{trcl}_\eta(j(x))$. Then there are y_1, \dots, y_n in ρ with $y_1 = y$ and $y_n = j(x)$ and $y_1\eta y_2, \dots, y_{n-1}\eta y_n$. Then using the definition of j we may argue by induction that $y_i = j(x_i)$ for $x_i \in L_\alpha$. Let us now argue that

$$(6.17) \quad x \in L_\alpha \implies (\text{trcl}_\eta(j(x)) \cup \{j(x)\}, \eta) \text{ is well-founded}$$

For, suppose that there was an infinite descending η -sequence y_n in the set $(\text{trcl}_\eta(j(x)) \cup \{j(x)\}) \subseteq \text{rng}(j)$. Then since j is an embedding this would lead to an infinite descending ϵ -sequence.

Before proceeding, let's note that η is well-founded on $\text{wfExt}_*(\mathcal{N})$. For, suppose that $\emptyset \neq X \subseteq \text{wfExt}_*(\mathcal{N})$. Choose x with Xx , so that of course x is in $\text{wfExt}_*(\mathcal{N})$. Then consider $X' = X \cap (\text{trcl}_\eta(x) \cup \{x\})$, which is a non-empty subset of $\text{trcl}_\eta(x) \cup \{x\}$. So there is some x_0 with $X'x_0$ such that $y\eta x_0$ implies $\neg X'y$. Suppose that $y\eta x_0$ with Xy . Since x_0 is in $\text{trcl}_\eta(x) \cup \{x\}$ and $y\eta x_0$, we have that y is in $(\text{trcl}_\eta(x) \cup \{x\})$. Then of course y is in $X' = X \cap (\text{trcl}_\eta(x) \cup \{x\})$, which is a contradiction. So indeed η is well-founded on $\text{wfExt}_*(\mathcal{N})$.

Now let us argue that $j : L_\alpha \rightarrow \text{wfExt}_*(\mathcal{N})$ is surjective. First note that it follows from the definitions that the class $\text{wfExt}_*(\mathcal{N})$ is transitive in the following sense:

$$(6.18) \quad [y, z \in \rho \ \& \ y \in \text{wfExt}_*(\mathcal{N}) \ \& \ z\eta y] \implies z \in \text{wfExt}_*(\mathcal{N})$$

So let's proceed in establishing surjectivity by reductio: suppose that $j : L_\alpha \rightarrow \text{wfExt}_*(\mathcal{N})$ is not surjective. So there is some $y \in \text{wfExt}_*(\mathcal{N}) \setminus j''L_\alpha$. Since η is well-founded on $\text{wfExt}_*(\mathcal{N})$ and since $\text{wfExt}_*(\mathcal{N})$ is transitive (6.18), there is $y \in \text{wfExt}_*(\mathcal{N}) \setminus j''L_\alpha$ such that

$$(6.19) \quad z\eta y \implies z \in (\text{wfExt}_*(\mathcal{N}) \cap j''L_\alpha)$$

Since $y \in \text{wfExt}_*(\mathcal{N}) \subseteq \text{rng}(\partial)$, choose $Y \in (P(\rho) \cap L_\alpha)$ such that $\partial(Y) = y$. Then by the previous equation, we may conclude that $L_\alpha \models \forall z \in Y \exists x j(x) = z$. By Σ_n -collection, choose $X \in L_\alpha$ such that $L_\alpha \models \forall z \in Y \exists x \in X j(x) = z$. Then set $X' = X \cap j^{-1}(Y) = \{x \in X : j(x) \in Y\}$ which is in L_α by Δ_n -separation since in addition to its natural $\Sigma_n^{L_\alpha}$ -definition it has the following $\Pi_n^{L_\alpha}$ -definition: $X' = \{x \in X : \forall y \in (L_\alpha \setminus Y) j(x) \neq y\}$. Also $\{j(x) : x \in X'\} = Y$, so that we have $j(X') = \partial(\{j(x) : x \in X'\}) = \partial(Y) = y$ which contradicts the hypothesis that y was not in the image of j .

Finally note that the isomorphism $j : (L_\alpha, \in) \rightarrow (\text{wfExt}_*(\mathcal{N}), \eta)$ is the inverse of the Mostowski collapse $\pi : (\text{wfExt}_*(\mathcal{N}), \eta) \rightarrow (L_\alpha, \in)$ due to the uniqueness of the latter isomorphism. \square

Theorem 6.2. (Second Identification of the Well-Founded Extensions) *Suppose that $n \geq 1$ and L_α is Σ_n -admissible and satisfies Axiom Beta. Let $\rho = \rho_n(L_\alpha) < \alpha$ and let $\partial : L_\alpha \rightarrow \rho$ be a witnessing $\Sigma_n^{L_\alpha}$ -definable injection. Then the structure*

$$(6.20) \quad \mathcal{N} = (\rho, P(\rho) \cap L_\alpha, P(\rho \times \rho) \cap L_\alpha, \dots, \partial \upharpoonright P(\rho) \cap L_\alpha)$$

is a model of $\Sigma_1^1\text{-LB} + \text{GC}$, where the global well-order on objects is given by the membership relation on ρ . Further, (L_α, \in) is isomorphic to $(\text{wfExt}(\mathcal{N}), \eta)$.

Proof. By the previous theorem, it suffices to show that $\text{wfExt}(\mathcal{N}) \subseteq \text{wfExt}_*(\mathcal{N})$. For this, it suffices to show that for all $x \in \rho$ we have

$$(6.21) \quad \begin{aligned} & [(\text{trcl}_\eta(x) \cup \{x\}, \eta) \text{ is } \Delta_n^{L_\alpha}\text{-well-founded} \ \& \ (\text{trcl}_\eta(x) \cup \{x\}) \subseteq \text{rng}(\partial)] \\ & \implies (\text{trcl}_\eta(x) \cup \{x\}, \eta) \text{ is well-founded} \end{aligned}$$

So suppose that $x \in \rho$ satisfies the hypothesis of this conditional. Then define the set $X = (\text{trcl}_\eta(x) \cup \{x\})$, which is in L_α by (6.5) of the previous proof. Then by the proposition on the existence of restricted η -relation (Proposition 2.7), choose binary relation $E_X \in L_\alpha$ such that $E_X \subseteq \rho \times X$ and such that Xa implies $E_X(b, a)$ iff $b\eta a$. Since X is η -transitive, we have that $E_X \subseteq X \times X$. Then the hypothesis that $(\text{trcl}_\eta(x) \cup \{x\}, \eta)$ is $\Delta_n^{L_\alpha}$ -well-founded and the η -transitivity of X implies

$$(6.22) \quad L_\alpha \models (X, E_X) \text{ is well-founded and extensional}$$

Since the structure L_α satisfies Axiom Beta, the structure L_α satisfies the Mostowski Collapse Theorem (cf. discussion following Definition 4.2). Then there is a transitive set M in L_α and a map π in L_α such that $\pi : (X, E_X) \rightarrow (M, \in)$ is an isomorphism. Suppose that (X, E_X) is not well-founded. Then there is an infinite decreasing η -sequence x_i in $X \subseteq L_\alpha$. Then $\pi(x_i)$ is an infinite decreasing \in -sequence. \square

This allows us to now establish the Main Theorem 2.9:

Proof. (of Theorem 2.9): By compactness, this follows from the Existence Theorem 5.2 and the Second Identification of the Well-Founded Sets Theorem 6.2. Here we're also appealing to the connection between the union of the axiomatic characterizations of Σ_n -admissibility and ZFC-P, which we noted immediately after the definition of Σ_n -admissibility (cf. Definition 4.1). \square

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