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An Introduction To The Theory Of Heavy Mesons And Baryons*

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Introductory lectures on heavy quarks and heavy quark effective field theory. Applications to inclusive semileptonic decays and to interactions with light mesons are covered in detail.

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Contents

AN INTRODUCTION TO THE THEORY OF HEAVY MESONS AND BARYONS

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ABSTRACT

Introductory lectures on heavy quarks and heavy quark effective field theory. Applications to inclusive semileptonic decays and to interactions with light mesons are covered in detail.

1. Introduction

Several lectures in this series deal with the phenomenology of weak decays. Two central topics have been CP violation in decays of mesons and the determination of Cabibbo-Kobayashi-Maskawa (CKM) mixing angles. This set of lectures intends to serve a support role, by describing some of the theoretical techniques that are needed in order to do the computations that allow for the phenomenological analysis. For example, the extraction of the CKM element $|V_{cb}|$ requires knowledge of the matrix element for the $\bar{B}$-meson to D-meson transition; we will describe how this is calculated.

The lectures are not intended to be encyclopedic in any one subject. I have decided to try to convey the principal ideas as clearly as I can, and to give some sample applications here and there. Occasionally I may describe "the state of the art" in a given field, without necessarily entering into details. I hope to have included enough references that the reader may follow up on any of these subjects if so inclined.

Exercises are scattered throughout the lectures. In the age of electronic typography it was easy enough to display the exercises in smaller print and separate them with clearly visible horizontal lines. I hope that the material given in the lectures is sufficient for obtaining the solution to the problems.

My generation of high energy physicists learned about the standard model, QCD, charm, beauty and top *before* being introduced to the experimental foundations for these ideas. I, for one, was cogniscent of the problems of the day, and was able to invent models or calculate, but I did not know —nor did it seem necessary to know— what the evidence for QCD or beauty was, nor what the "November Revolution" was about. It was not until years after I graduated that I started filling in this gap. It is with students with this type of background in mind that I have prepared a short historical introduction.

Chapter 2 contains a brief description of what Effective Field Theories are, at least in the very specific context of weak interactions. The presentation is somewhat telegraphic. I expect the student to know something about effective lagrangians. The intention is to show you how I think about the subject so that the presentation of the effective lagrangian of heavy quarks, in Chapter 3, goes down more easily.

Chapter 3 describes the Heavy Quark Effective Field Theory (HQET) and its symmetries, to leading order in the large mass. Some examples and applications are given. The corrections of order $1/M$ are described in Chapter 4.

The last two Chapters concentrate on applications of the HQET that have received a lot of attention over the last year. This is where these lectures deviate substantially from my Mexican lectures. Chapter 5 gives the proof that the rate for inclusive semileptonic B decay is given by the parton decay rate, while Chapter 6 introduces an effective lagrangian of heavy mesons and pion, interacting in a chirally invariant way, and respecting heavy quark symmetries.

1.1. The November Revolution

In November of 1994 two experimental collaborations announced the discovery of a new very narrow resonance with mass 3.1 GeV. They had unearthed evidence for the charm quark, and for the validity of an asymptotically free theory, like QCD, for strong interactions. These events had extraordinary consequences, affecting the way we think today about particle physics. They are often referred to as the "November Revolution".

A MIT–Brookhaven collaboration, led by S. Ting, found evidence for the new resonance by measuring the e^+e^- mass spectrum in $p + \mathrm{Be} \to e^+ + e^- + X$ with a precise pair spectrometer at Brookhaven Natl. Lab.'s 30 GeV AGS[1]. Fig. 1a shows the spectrum in the e^+e^- invariant mass variable, $m_{e^+e^-} = \sqrt{(p_{e^+} + p_{e^-})^2}$, as reported by the MIT-BNL collaboration.

The Mark I collaboration, from SLAC and LBL, led by B. Richter was conducting experiments at the newly constructed e^+e^- ring, SPEAR, at SLAC. Their detector consisted of a spark chamber embedded in a solenoidal magnetic field, and surrounded by time-of-flight counters, shower counters and proportional counters embedded in slabs of iron for muon identification. They[2] "observed a very sharp peak in the cross sections for $e^+e^- \to$hadrons, e^+e^-, and $\mu^+\mu^-$ at a center of mass energy of 3.105 ± 0.003 GeV" and found an upper bound on the width of 1.3 MeV. Fig. 1b is reproduced from Ref. [2], and shows the cross sections measured by Mark I.

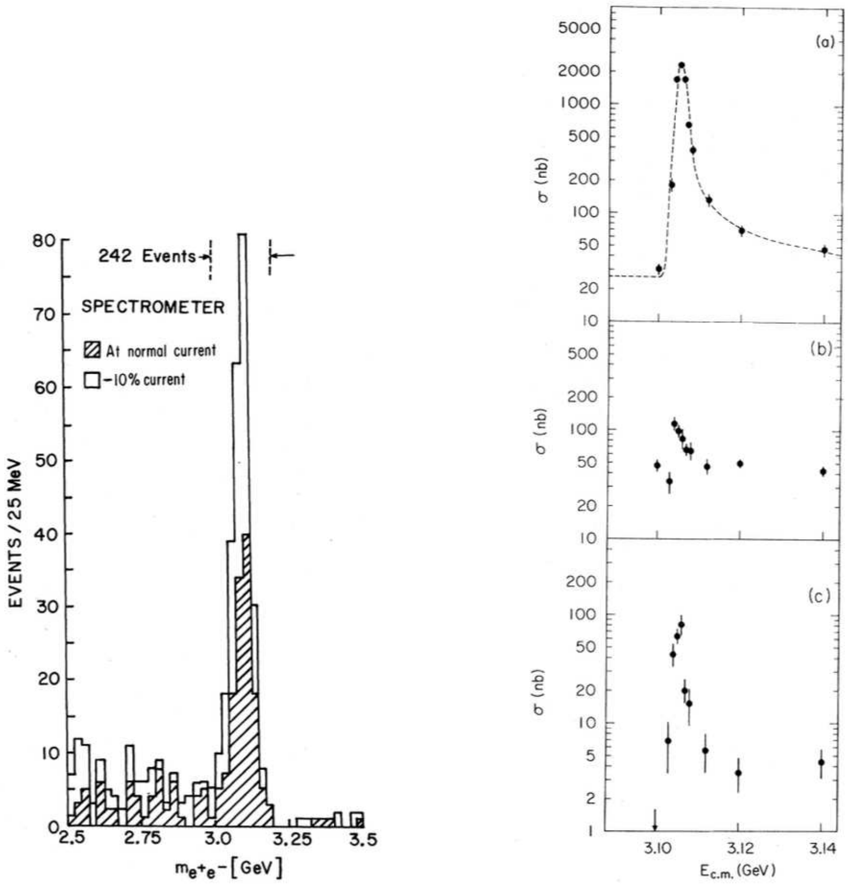

Figure 1 *(a)Mass spectrum showing the existence of a narrow resonance in the MIT-BNL collaboration. From Ref. [1]. (b)Cross sections vs. energy for* $e^+e^- \to$ *hadrons,* $\mu^+\mu^-$*,* $\mu^+\mu^- + \pi^+\pi^- + K^+K^-$*, reported by the Mark I collaboration in Ref. [2], with evidence for the new resonance at about* 3.1 *GeV.*

The MIT-BNL collaboration called the new resonance "J", Mark I called it "ψ", so it's now known as the J/ψ. The Mark I collaboration soon found a second narrow resonance[3], the ψ', at a mass of 3.695 ± 0.004 GeV. No other narrow resonances were found in the total e^+e^- cross sections at SPEAR, but broader structures did appear at energies above the ψ'[4]. Other narrow structures that could not be directly produced in e^+e^- collisions were found through cascade decays of the ψ'. The DASP collaboration working at DESY's e^+e^- storage ring DORIS found[5] the first χ state in $\psi' \to \chi + \gamma \to \psi + \gamma + \gamma$. The Crystal Ball collaboration detector provided the high spatial and energy resolution needed to finally unravel the spectroscopic levels of charmonium. Fig. 2 shows the inclusive photon spectrum from ψ' decays, from Ref. [6], which reports on the discovery of the η_c.

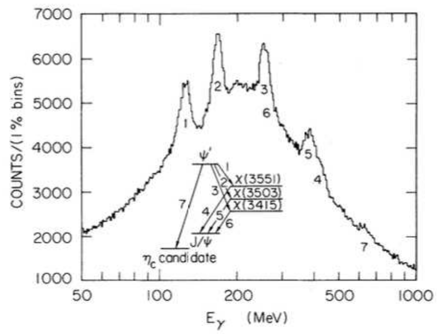

Figure 2 *Inclusive photon spectrum from ψ' decay from Ref. [6].*

The interpretation of these resonance soon became clear: they are atom-like bound states of a charm quark-antiquark pair. The interaction between the rather heavy quarks is coulomb like, since at short distances QCD becomes weak and a single gluon exchange gives an attractive coulomb potential between the quarks. The potential is not really coulomb: for one thing, it must be confining so it must grow without bound at long distances. But the physics of the spectrum of bound states is dominated by the short distance interaction and is not dissimilar from the physics of the hydrogen atom. The η_c and ψ families are in a quark-spin singlet and triplet state, respectively, with orbital angular momentum 0, giving $J^{PC} = 0^{-+}$ and 1^{--}, respectively. The χ_{c0}, χ_{c1} and χ_{c2} are in a spin triplet, with $J^{PC} = 0^{++}$, 1^{++} and 2^{++}, respectively.

Exercise 1.1

(i) The η_c is lighter than the ψ. In fact ψ decays into η_c radiatively. Why is ψ, but not η_c, copiously produced in e^+e^- collisions?

(ii) What are the allowed values of J^{PC} for charmonium?

Charmonium states have zero charm number, C. States with $|C| = 1$, with so called "naked charm", were first convincingly observed by Mark I at SPEAR.[7] They observed narrow peaks in the invariant mass spectra for neutral combinations of charged particles in $K\pi$ and $K3\pi$. They inferred the existence of an object of mass 1865 ± 15 MeV and put an upper limit on its width of 40 MeV. The invariant-mass spectra from Ref. [7] is reproduced in fig. 3. The new state, with $C = 1(-1)$, was the $D(\bar{D})$ pseudoscalar meson. They found "it significant that the threshold energy for pair-producing this state lies in the small interval between the very narrow ψ' and the broader ..." ψ''. That is, it became clear that the ψ'' was much broader because it decayed strongly into a D–$\bar{D}$ pair.

1.2. The b-quark

The discovery of "naked bottom" (or "naked beauty", outside the Americas) paralleled in many ways that of charm. Although a new sequential heavy lepton, the τ, had been discovered, and therefore the existence of beauty and top expected, the masses of these quarks were unknown.

L. Lederman led a collaboration at Fermilab that used a two arm spectrometer to search for muon pairs in 400 GeV proton-nucleus collisions. They had some experience. Years earlier the group conducted a similar experiment at BNL's AGS. Because their apparatus had smaller resolution than that of the MIT-BNL group, they did not report any evidence for a resonance. They had seen a cross section that, except for a small plateau in the 3 GeV region, fell with invariant mass as expected. After missing the J/ψ, they were ready for the discovery of bottomonium. They observed[8] a similar effect in the new experiment, and correctly interpreted it as a dimuon resonance at about 9.5 GeV; see Fig. 4a. A refined analysis of the experiment revealed actually two peaks, at 9.44 and 10.17 GeV. The states were named "Υ" and "Υ'".

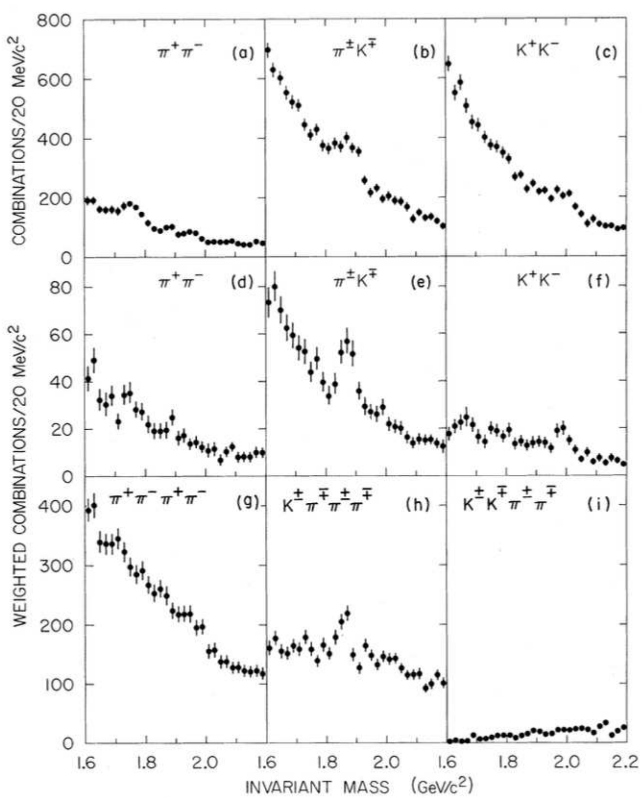

Figure 3 *Invariant mass spectra for neutral combinations of charged particles in $K\pi$ and $K3\pi$ channels, from Ref. [7].*

An upgrade of the energy of DORIS made it possible for the PLUTO and DASP II collaborations to observe the Υ in e^+e^- annihilation[9-10]; see Fig. 4b. A further energy upgrade made the Υ' accessible too[11,12]; see Fig. 4c.

After the Cornell Electron Storage Ring (CESR) was commissioned, the CUSB and CLEO collaborations successfully observed the Υ, Υ' and Υ''. All three resonances, with masses 9.460, 10.023 and 10.355 GeV are narrow. Shortly afterwards the two collaborations established the existence of a broader resonance, the Υ''', at a mass of ~ 10.55 GeV and a width of about 12.6 MeV. This is significant because, following the charm experience, it suggests looking for naked beauty in the decay of Υ'''. B-mesons were first found and reported by the CLEO collaboration in a paper which for once is straight and to the point in its title ("Observation of Exclusive Decay Modes of b-Flavored Mesons) and in its abstract (see Ref. [13]). To be sure, B-mesons had been inferred from the observation of high momentum leptons in Υ''' decays, but it was the reconstruction of a few exclusive decays that demonstrated their existence conclusively.

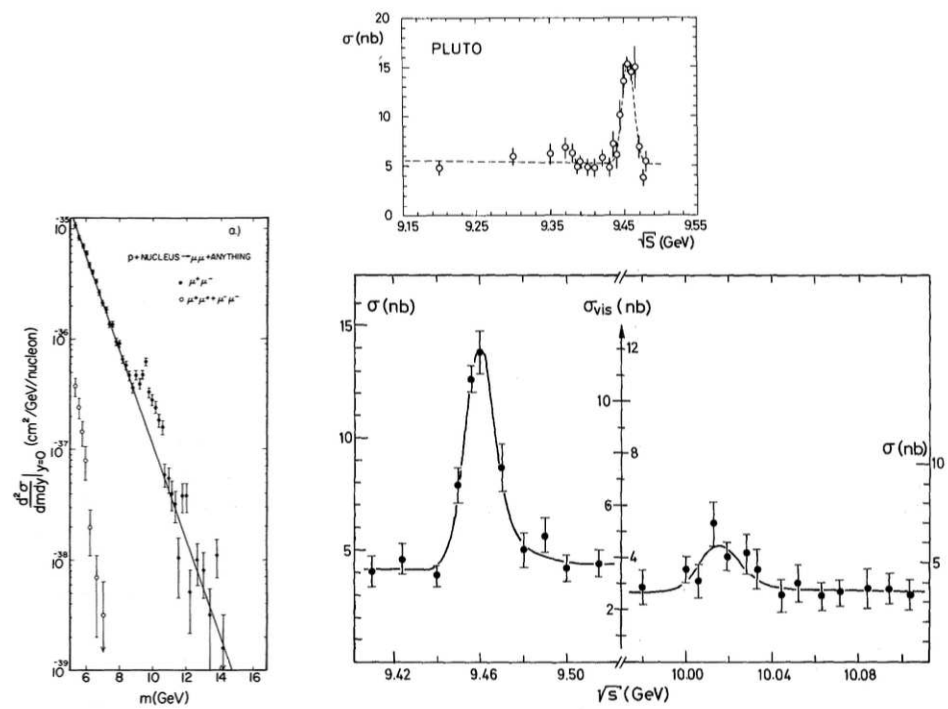

Figure 4 *(a)Inclusive cross section versus invariant mass for production of $\mu^+\mu^-$ pairs in collisions of 400 GeV protons on nuclei, from Ref. [8]. (b)Observation of the Υ by the PLUTO collaboration. From Ref. [9]. (c)Observation of the Υ' by the DESY-Heidelberg NaI and lead glass detector collaboration. From Ref. [12].*

Ever since, the ARGUS and CLEO collaboration have been competing to unravel the mysteries of beauty. As we shall see, measurement of B^0–$\bar{B}^0$ mixing and of charmless semileptonic decay rates are of utmost importance in the determination of the parameters of the Cabibbo-Kobayashi-Maskawa (CKM) matrix.

Today D and B mesons are established as universally accepted established resonances. They are the closest we can get to having naked charm and beauty. Their masses have been established to high accuracy[14]:

$$\begin{aligned} m_{D^\pm} &= 1869.4 \pm 0.4 \text{ MeV} \\ m_{D^0} &= 1864.6 \pm 0.5 \text{ MeV} \\ m_{B^\pm} &= 5278.7 \pm 2.0 \text{ MeV} \\ m_{B^0} &= 5279.0 \pm 2.0 \text{ MeV} \end{aligned} \tag{1.1}$$

2. Preliminaries

2.1. Conventions and Notation

The metric is $(+---)$. Gamma matrices satisfy $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$, γ^0 is hermitian, γ^i antihermitian, $\gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$ and $\sigma_{\mu\nu} \equiv \frac{i}{2}[\gamma^\mu, \gamma^\nu]$. In the Dirac convention $\gamma^0 = \text{diag}(1, 1, -1, -1)$. $\epsilon^{0123} = +1$.

Left and right handed fields are denoted by subscripts,

$$\psi_L = \tfrac{1}{2}(1 - \gamma_5)\psi \qquad \psi_R = \tfrac{1}{2}(1 + \gamma_5)\psi$$

States have the relativistic normalization,

$$\langle \vec{p} | \vec{p}\,' \rangle = 2E\delta(\vec{p} - \vec{p}\,') \tag{2.1}$$

unless otherwise noted.

The charged current interaction in the standard model is given by the lagrangian

$$\mathcal{L}_{\text{int}} = \frac{g_2}{\sqrt{2}} W^{+\mu} \begin{pmatrix} \bar{u} & \bar{c} & \bar{t} \end{pmatrix} \gamma_\mu (1 - \gamma_5) V \begin{pmatrix} d \\ s \\ b \end{pmatrix} + \text{h.c.}\ , \tag{2.2}$$

where V is the 3×3 unitary CKM matrix. Elsewhere in this volume you may find a description of the present status of the determination of the elements of V.

2.2. Effective Lagrangians

The study of decays of heavy mesons involves two very different types of physics. On the one hand there is the underlying interactions that are responsible for the decay. These may be the ordinary weak interactions of the standard model or some new, presumably weaker, interactions, *e.g.,* extended technicolor or exchange of scalar quarks in supersymmetric theories. On the other hand there are strong interactions which modulate the rates of the decays. Ultimately we would like to uncover the former, which requires some level of understanding of the latter.

Since the two physical aspects are conceptually different, it seems natural to separate them in our computations. This is where effective lagrangians come in handy. An effective lagrangian for meson decays will only involve the dynamical degrees of freedom that are relevant. For example, light quarks (u, d, s) and gluons for K meson decays. The interaction responsible for the decay, a W exchange, is represented as a $\Delta S = 1$ four-quark operator. The fact that the W-boson is no longer in the theory does not impair our ability to predict the decay rate, up to an accuracy or order m_K^2/M_W^2.

Is this a major step backwards? After all, you may argue, this is just the Fermi theory of weak interactions. Effective lagrangians prove useful because,

(i) The computation of long distance physics is decoupled from the short distance physics. Thus, one can make a catalog of matrix elements of operators between meson states. This could then be used to compute the effects of any fundamental theory, after reducing its lagrangian to an effective one.

(ii) They provide a method for the computation of amplitudes when disparate scales are present. In K-meson decays the ratio m_K/M_W is a small number. Logarithms of this ratio invalidate even the perturbative computation of the decay. Effective lagrangians provide a method for resummation of these large logs.

(iii) One can characterize, *à la* Fermi, all possible interactions in terms of operators. In other words, one can make *model independent* analysis of the possible effects of new physics.

(iv) It's the right way to *think* about the low energy effects of very heavy particles.

2.3. Formulating Effective Lagrangians

The precise meaning of effective lagrangians is best formulated in terms of relations between Green functions. Take again the example of weak interactions at low energies, that is, when all the momenta involved are much smaller than the

W–boson mass. Everyone knows that we can account for the effects of the W–boson by adding to the Lagrangian terms of the form

$$\Delta\mathcal{L}_{\rm eff} = \frac{1}{M_W^2}\kappa\mathcal{O} \ , \tag{2.3}$$

where $\mathcal{O}$ is a 4-fermion operator and κ contains mixing angles and factors of the weak coupling constant. This is simply the statement that a Green function G of the original theory (the standard model including QCD) can be approximated by a Green function $\widetilde{G}_{\mathcal{O}}$ of the effective theory (a gauge theory of QCD and electromagnetism) with an insertion of the effective Lagrangian:

$$G = \frac{1}{M_W^2}\kappa\widetilde{G}_{\mathcal{O}} + \dots \quad . \tag{2.4}$$

The ellipses stand for terms suppressed by additional powers of $(M_W)^{-2}$. This equation replaces the task of computing the more complicated left side, which depends on M_W, by the computation in the effective theory which is independent of M_W, and indeed, completely free of the W–boson dynamical degrees of freedom. On the right hand side, the factor of $1/M_W^2$ gives the dependence on the W–boson mass simply and explicitly.

The full theory has logarithmic dependence on M_W which has not been made explicit. Eq. (2.4) is not quite correct. The correct version is[15]

$$G = \frac{1}{M_W^2}\kappa C(M_W/\mu, g_s)\tilde{G}_{\mathcal{O}} + \dots \quad . \tag{2.5}$$

The function C is, in this case, also known as the 'short distance QCD effect' first calculated for the process $s \to u\bar{u}d$ by Altarelli and Parisi[16], and Gaillard and Lee[17].

Summing up, an effective theory (of either the 'normal' or the HQ type) is a method for extracting explicitly the leading large mass dependence of amplitudes. Moreover, the rules of computation of the effective theory are completely independent of the large mass.

2.4. Computing Effective Lagrangians

At tree level the computation of effective lagrangians is straightforward since the short distance QCD effects can be neglected. This is most easily explained through an example. Consider transitions with $\Delta C = -\Delta U = \Delta S = -\Delta D = 1$. In the full theory these take place by the exchange of a W-boson coupling to the charged currents $(\bar{c}_L\gamma^\mu s_L)$ and $(\bar{d}_L\gamma^\mu u_L)$. Take the Green function for, say, $\bar{c} \to \bar{s}\bar{u}d$ and expand in powers of momentum over the W-boson mass. This amount to expanding the W-boson propagator; in 'tHooft–Feynman gauge,

$$-i\frac{g_{\mu\nu}}{p^2 - M_W^2} = i\frac{g_{\mu\nu}}{M_W^2} + \cdots$$

The effective hamiltonian for the decay is then

$$\mathcal{H}_{\text{eff}} = \frac{4G_F}{\sqrt{2}} V_{ud}^* V_{cs} (\bar{c}_L\gamma^\mu s_L)(\bar{d}_L\gamma^\mu u_L) + \cdots \tag{2.6}$$

where we have introduced Fermi's constant $G_F = \sqrt{2}g_2^2/8M_W^2$. The ellipsis stand for operators of higher dimension which come from the expansion of the propagator, replacing $p \to i\partial$, as in

$$(\bar{c}_L\gamma^\mu s_L)\partial^2(\bar{d}_L\gamma^\mu u_L)$$

Exercise 2.1 Show that the effective hamiltonian for $\Delta B = 1$, $\Delta C = \Delta U = 0$, is given at tree level by

$$\mathcal{H}_{\text{eff}} = \frac{4G_F}{\sqrt{2}} \sum_{q=u,c}\sum_{q'=d,s} V_{qb}^* V_{qq'} (\bar{b}_{L\alpha}\gamma^\mu q_{L\alpha})(\bar{q}_{L\beta}\gamma_\mu q'_{L\beta}) \tag{2.7}$$

where α and β are color indices.

Beyond tree level the simple effective hamiltonian of Eq. (2.6) is replaced by a sum over operators of dimension six and with the same quantum numbers. Neglecting the masses of the light quarks there is only one more operator:

$$\mathcal{H}_{\text{eff}} = \frac{4G_F}{\sqrt{2}} V_{ud}^* V_{cs} \sum_i c_i \mathcal{O}_i + \cdots \tag{2.8}$$

where

$$\mathcal{O}_1 = (\bar{c}_L\gamma^\mu s_L)(\bar{d}_L\gamma^\mu u_L)$$
$$\mathcal{O}_2 = (\bar{c}_L\gamma^\mu u_L)(\bar{d}_L\gamma^\mu s_L)$$

and the ellipsis stand again for higher dimension operators suppressed by additional powers of G_F. The coefficients c_i encode all the information about the dependence on the W-mass (beyond the trivial factor of G_F). We already know that at tree level $c_1 = 1$ and $c_2 = 0$. They may be computed as an expansion in α_s. It is best, however to reorganize the expansion to account for the large ratio of scales m_c/M_W by assuming $\alpha_s \ll 1$ and $\alpha_s \log(m_c/M_W) \sim 1$, thus

$$\begin{aligned} c_1 &= 1 + \frac{\alpha_s}{2\pi}\log(m_c/M_W) + \cdots \\ &= \frac{1}{2}\left[\left(\frac{\alpha(m_c)}{\alpha(M_W)}\right)^{1/2b_0} + \left(\frac{\alpha(m_c)}{\alpha(M_W)}\right)^{-1/b_0}\right] + \mathcal{O}(\alpha_s^2\log(\frac{m_c}{M_W})) \\ c_2 &= 0 - \frac{3\alpha_s}{2\pi}\log(m_c/M_W) + \cdots \\ &= \frac{1}{2}\left[\left(\frac{\alpha(m_c)}{\alpha(M_W)}\right)^{1/2b_0} - \left(\frac{\alpha(m_c)}{\alpha(M_W)}\right)^{-1/b_0}\right] + \mathcal{O}(\alpha_s^2\log(\frac{m_c}{M_W})) \ . \end{aligned}$$

Here b_0 stands for the coefficient of the first term in the perturbative expansion of the beta function in QCD, $\beta(g) = -b_0\frac{g^3}{16\pi^2} + \ldots$. This is the so-called "leading-log" approximation to the coefficients. We have displayed order of the "sub-leading-log" or "next-to-leading-log" corrections.

Exercise 2.2 Write down the complete list of operators that contributes to the effective hamiltonian for $\Delta B = 1$, $\Delta C = \Delta U = 0$. Neglect the masses of u, d, s and c-quarks, but not of the b-quark. (*Hint:* What is the symmetry group of QCD in the massless limit? How do the operators in Eq. (2.7) transform under this symmetry?)

The computation of the coefficients is itself quite simple but it detracts from our main focus. A rather similar computation is presented below for the Heavy Quark Effective Theory; see Section 3.9. For a beautiful exposition of the method, see [15].

Exercise 2.3 Show that the effective hamiltonian for $\Delta B = -\Delta D = 1$, $\Delta C = \Delta U = 0$ of the previous exercises can be written as the sum of exactly two terms,

$$\mathcal{H}_w = \frac{4G_F}{\sqrt{2}}(\xi_c O_c + \xi_t O_t) \tag{2.9}$$

where $\xi_q \equiv V_{qb}^* V_{qd}$, and O_q are linear combinations of composite operators that may depend on M_W but not on CKM angles. Use phenomenological information to show that in the effective hamiltonian for $\Delta B = -\Delta S = 1$, $\Delta C = \Delta U = 0$, depends only on the single mixing angle $V_{cb}^* V_{cs}$.

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3. Heavy Quark Effective Field Theory

3.1. Intuitive Introduction

The central idea of the HQET is so simple, it can be described without reference to a single equation. And it should prove useful to refer back to the simple intuitive notion, to be presented below, wherever the formalism and corresponding equations become abstruse.

The HQET is useful when dealing with hadrons composed of one heavy quark and any number of light quarks. More precisely, the quantum numbers of the hadrons are unrestricted as far as isospin and strangeness, but are ± 1 for either B- or C-number. In what follows we shall (imprecisely) refer to these as 'heavy hadrons'.

The successes of the constituent quark model is indicative of the fact that, inside hadrons, strongly bound quarks exchange momentum of magnitude a few hundred MeV. We can think of the typical amount Λ by which the quarks are off–shell in the nucleon as $\Lambda \approx m_p/3 \approx 330\text{MeV}$. In a heavy hadron the same intuition can be imported, and again the light quark(s) is(are) very far off–shell, by an amount of order Λ. But, if the mass M_Q of the heavy quark Q is large, $M_Q \gg \Lambda$, then, in fact, this quark is almost on–shell. Moreover, interactions with the light quark(s) typically change the momentum of Q by Λ, but change the *velocity* of Q by a negligible amount, of the order of $\Lambda/M_Q \ll 1$. It therefore makes sense to think of Q as moving with constant velocity, and this velocity is, of course, the velocity of the heavy hadron.

In the rest frame of the heavy hadron, the heavy quark is practically at rest. The heavy quark effectively acts as a static source of gluons. It is characterized by its flavor and color–$SU(3)$ quantum numbers, but not by its mass. In fact, since spin–flip interactions with Q are of the type of magnetic moment transitions, and these involve an explicit factor of g_s/M_Q, where g_s is the strong interactions coupling constant, the spin quantum number itself decouples in the large M_Q case. Therefore, *the properties of heavy hadrons are independent of the spin and mass of the heavy source of color.*

The HQET is nothing more than a method for giving these observations a formal basis. It is useful because it gives a procedure for making explicit calculations. But more importantly, it turns the statement 'M_Q is large' into a *systematic* perturbative expansion in powers of Λ/M_Q. Each order in this expansion involves QCD to all orders in the strong coupling, g_s. Also, the statement of mass and spin independence of properties of heavy hadrons appears in the HQET as approximate internal symmetries of the Lagrangian.

Before closing this Section, we point out that these statements apply just as well to a very familiar and quite different system: the atom. The rôle of the heavy quark is played by the nucleus, and that of the light degrees of freedom by the electrons (and the electromagnetic field)[1]. That different isotopes have the same chemical properties simply reflects the nuclear mass independence of the atomic wave-function. Atoms with nuclear spin s are $2s+1$ degenerate; this degeneracy is broken when the finite nuclear mass is accounted for, and the resulting hyperfine splitting is small because the nuclear mass is so much larger than the binding energy (playing the rôle of Λ). It is not surprising that, using M_Q independence, the properties of B and D mesons are related, and using spin independence, those of B and B^* mesons are related, too.

3.2. The Effective Lagrangian and its Feynman Rules

We shall focus our attention on the calculation of Green functions in QCD, with a heavy quark line, its external momentum almost on–shell. The external momentum of gluons or light quarks can be far off–shell, but not much larger than the hadronic scale Λ. This region of momentum space is interesting because physical quantities —S–matrix elements— live there. And, as stated in the introduction, we expect to see approximate symmetries of Green functions in that region which are not symmetries away from it. That is, these are approximate symmetries of a sector of the S–matrix, but not of the lagrangian.

The effective Lagrangian $\mathcal{L}_{\rm eff}$ is constructed so that it will reproduce these Green functions, to leading order in Λ/M_Q. It is given, for a heavy quark of velocity v_μ ($v^2=1$), by[18],

$$\mathcal{L}_{\rm eff}^{(v)} = \bar{Q}_v iv{\cdot}DQ_v \ , \tag{3.1}$$

where the covariant derivative is

$$D_\mu = \partial_\mu + ig_s A_\mu^a T^a \ , \tag{3.2}$$

and the heavy quark field Q_v is a Dirac spinor that satisfies the constraint

$$\left(\frac{1+\not{v}}{2}\right) Q_v = Q_v \ . \tag{3.3}$$

[1] An obvious distinction between the atomic and hadronic systems is that in the latter the configuration of the light degrees of freedom is non–computable, due to the difficulties afforded by the non–perturbative nature of strong interactions. The methods that we are describing circumvent the need for a detailed knowledge of the configuration of light degrees of freedom. The price paid is that the range of predictions is restricted. To emphasize the non-computable aspect of the configuration of light degrees of freedom, Nathan Isgur informally referred to it as "brown muck", and the term has somewhat made it into the literature.

In addition, it is understood that the usual Lagrangian $\mathcal{L}_{\rm light}$ for gluons and light quarks is added to $\mathcal{L}_{\rm eff}^{(v)}$.

We can see how this arises at tree level, as follows[19]. Consider first the tree level 2-point function for the heavy quark

$$G^{(2)}(p) = \frac{i}{\not p - M_Q} \ . \tag{3.4}$$

We are interested in momentum representing a quark of velocity v_μ slightly off–shell:

$$p_\mu = M_Q v_\mu + k_\mu \ . \tag{3.5}$$

Here, 'slightly off–shell' means k_μ is of order Λ, and independent of M_Q. Substituting in Eq. (3.4), and expanding in powers of Λ/M_Q, we obtain, to leading order,

$$G^{(2)}(p) = i\left(\frac{1+\not v}{2}\right)\frac{1}{v{\cdot}k} + \mathcal{O}\left(\frac{\Lambda}{M_Q}\right) \ . \tag{3.6}$$

We recognize the projection operator of Eq. (3.3), and the propagator of the lagrangian in (3.1).

Similarly, the 3-point function (a heavy quark and a gluon) is given by

$$G^{(2,1)a}_\mu(p,q) = \frac{i}{\not p - M_Q}(-ig_s T^a\gamma^\nu)\frac{i}{\not p + \not q - M_Q}\Delta_{\nu\mu}(q), \tag{3.7}$$

where $\Delta_{\nu\mu}(q)$ is the gluon propagator. Expanding as above, we have

$$G^{a(2,1)}_\mu(p,q) = \left(\frac{1+\not v}{2}\right)\frac{i}{v{\cdot}k}(-ig_s T^a v^\nu)\frac{i}{v{\cdot}(k+q)}\Delta_{\mu\nu}(q) + \mathcal{O}\left(\frac{\Lambda}{M_Q}\right), \tag{3.8}$$

where we have used

$$\left(\frac{1+\not v}{2}\right)\gamma_\nu\left(\frac{1+\not v}{2}\right) = \left(\frac{1+\not v}{2}\right)v_\nu \ . \tag{3.9}$$

Again, this corresponds to the vertex obtained from the effective Lagrangian in Eq. (3.1).

Exercise 3.1 Extend these results to arbitrary tree-level Green functions (but only those with one heavy quark and all other (light) particles carrying momentum of order Λ).

The effective Lagrangian in (3.1) is appropriate for the description of a heavy quark, and indeed a heavy hadron, of velocity v_μ. It does, however, break Lorentz covariance. This is not a surprise, since we have expanded the Green functions about one particular velocity: in boosted frames, the expansion in powers of Λ/M_Q becomes invalid, since the boosted momentum k_μ can become arbitrarily large. Lorentz covariance is recovered, however, if we boost the velocity

$$v_\mu \to \Lambda_{\mu\nu} v_\nu \tag{3.10}$$

along with everything else. It will prove useful to keep this simple observation in mind[2].

Just as in the case of an effective theory for weak interactions, when one goes beyond tree level one must be careful to make explicit any anomalous mass dependence. When the W is integrated out the correction was encrypted in some function $C(M_W)$ in Eq. (2.5). The situation is entirely analogous in the HQET. We have introduced an effective Lagrangian $\mathcal{L}_{\text{eff}}^{(v)}$ such that Green functions $\widetilde{G}_v(k;q)$ calculated from it agree, at tree level, with corresponding Green functions $G(p;q)$, in the QCD to leading order in the large mass

$$G(p;q) = \widetilde{G}_v(k;q) + \mathcal{O}\left(\Lambda/M_Q\right) \qquad \text{(tree level)}\ . \tag{3.11}$$

Here, Λ stands for any component of k_μ or of the q's, or for a light quark mass, and $p = M_Q v + k$. We will come back to the study of the HQET beyond tree level and will make explicit the anomalous mass dependence in Section 3.8.

3.3. Symmetries

Flavor – $SU(N)$ The Lagrangian for N species of heavy quarks, all with velocity v, is

$$\mathcal{L}_{\text{eff}}^{(v)} = \sum_{j=1}^{N} \bar{Q}_v^{(j)}\ iv{\cdot}D\ Q_v^{(j)}\ . \tag{3.12}$$

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[2] In an alternative method, championed by Georgi[20], the effective Lagrangian $\mathcal{L}_{\text{eff}}$ consists of a sum over the different velocity Lagrangians, $\mathcal{L}_{\text{eff}}^{(v)}$, of Eq. (3.1). Lorentz invariance is recovered at the price of "integrating in" the heavy degrees of freedom. This does not lead to overcounting of states, because the sectors of different velocity do not couple to each other, a fact that Georgi refers to as a "velocity superselection rule". See also [21].

This Lagrangian has a $U(N)$ symmetry[22,23,24]. The subgroup $U(1)^N$ corresponds to flavor conservation of the strong interactions, and was a good symmetry in the original theory. The novelty in the HQET is then the nonabelian nature of the symmetry group. This leads to relations between properties of heavy hadrons with different quantum numbers. Please note that these will be relations between hadrons of a given velocity, even if of different momentum (since typically $M_{Q_i} \neq M_{Q_j}$ for $i \neq j$). Including the b and c quarks in the HQET, so that $N = 2$, we see that the B and D mesons form a doublet under flavor–$SU(2)$.

This flavor–$SU(2)$ is an approximate symmetry of QCD. It is a good symmetry to the extent that

$$m_c \gg \Lambda \qquad \text{and} \qquad m_b \gg \Lambda\ . \tag{3.13}$$

These conditions can be met even if $m_b - m_c \gg \Lambda$. This is in contrast to isospin symmetry, which holds because $m_d - m_u \ll \Lambda$.

In the atomic physics analogy of Section 3.1, this symmetry implies the equality of chemical properties of different isotopes of an element.

Spin – $SU(2)$ The HQET Lagrangian involves only two components of the spinor Q_v. Recall that

$$\left(\frac{1-\not{v}}{2}\right) Q_v = 0. \tag{3.14}$$

The two surviving components enter the Lagrangian diagonally, *i.e.*, there are no Dirac matrices in

$$\mathcal{L}^{(v)}_{\text{eff}} = \bar{Q}_v\ iv{\cdot}D\ Q_v. \tag{3.15}$$

Therefore, there is an $SU(2)$ symmetry of this Lagrangian which rotates the two components of Q_v among themselves[22,25–26].

Please note that this "spin"–symmetry is actually an *internal* symmetry. That is, for the symmetry to hold no transformation on the coordinates is needed, when a rotation among components of Q_v is made. On the other hand, to recover Lorentz covariance, one does the usual transformation on the light–sector, including a Lorentz transformation of coordinates and in addition a Lorentz transformation on the velocity v_μ. A spin–$SU(2)$ transformation can be added to this procedure, to mimic the original action of Lorentz transformations.

Exercise 3.2 To make it plain that this symmetry has nothing to do with "spin" in the usual sense, consider the large mass limit for a vector particle[27]. Use the massive vector propagator

$$-i\,\frac{g_{\mu\nu}-p_\mu p_\nu/m^2}{p^2-m^2} \tag{3.16}$$

to obtain the Lagrangian for the HVET (Heavy Vector Effective Theory)

$$\mathcal{L}^{(v)}_{\rm eff} = A^\dagger_{v\mu}\; iv{\cdot}DA_{v\mu}\ , \tag{3.17}$$

with the constraint

$$(v_\mu v_\nu - g_{\mu\nu})A_{v\nu} = A_{v\mu}\ . \tag{3.18}$$

What is the dimension of this effective vector field? Why? Show that the effective Lagrangian is invariant under an $SU(3)$ group of transformations, rotating the three components of the vector field among themselves. Note that the "spin" symmetry is not associated with $SU(2)$ in this case.

The symmetry of the theory is larger than the product of the flavor and spin symmetries. If there are N_S, N_F, and N_V species of heavy scalars, fermions, and vectors, respectively, all transforming the same way under color–$SU(3)$, the symmetry of the effective theory is $SU(N_S+2N_F+3N_V)$.

Exercise 3.3 What is the symmetry group for a theory with N_S, N_F, and N_V species of heavy scalars, fermions, and vectors, respectively, all transforming the same way under color–$SU(3)$, in D space-time dimensions? (If you can't handle arbitrary D, try $D=2$ and $D=3$).

3.4. Spectrum

The internal symmetries of the effective Lagrangian are explicitly realized as degeneracies in the spectrum and as relations between transition amplitudes. In this Section we will consider the spectrum of the theory[28].

Keep in mind that momenta, and therefore energies and masses, are measured in the HQET relative to $M_Q v_\mu$. Therefore, when we state that in the HQET the B and D mesons are degenerate, the implication is that the physical mesons differ in their masses by $m_b - m_c$.

For now let us specialize to the rest frame $v=(1,\mathbf{0})$. The total angular momentum operator $\mathbf{J}$, *i.e.,* the generator of rotations, can be written as

$$\mathbf{J} = \mathbf{L} + \mathbf{S}\ , \tag{3.19}$$

where $\mathbf{L}$ is the angular momentum operator of the light degrees of freedom, and $\mathbf{S}$, the angular momentum operator for the heavy quark, agrees with the generator of spin–$SU(2)$. Since $\mathbf{J}$ and $\mathbf{S}$ are separately conserved, $\mathbf{L}$ is also separately conserved. Therefore, the states of the theory can be labeled by their $\mathbf{L}$ and $\mathbf{S}$ quantum numbers $(l, m_l; s, m_s)$. Of course, $s = 1/2$, so m_s is $1/2$ or $-1/2$ only.

The simplest state has $l = 0$ and, therefore, $J = 1/2$. We will refer to it as the Λ_Q, by analogy with the nonrelativistic potential constituent quark model of the Λ–baryon, where the strange quark combines with a $l = 0$, $I = 0$, combination of the two light quarks.

Next is the state with $l = 1/2$. It leads to $J = 0$ and $J = 1$. We deduce that there is a meson and a vector meson that are degenerate. For the b-quark, the B and B^* fit the bill. They are the lowest lying $B = -1$ states. The lowest lying $C = 1$ states are the D and D^* mesons. These again can very well be assigned to our $J = 0$ and $J = 1$ multiplet. The difference $M_{D^*} - M_D = 145$ MeV is reasonably smaller than the splitting between the D^* and the next state, the D_1, with $M_{D_1} - M_{D^*} = 410$ MeV.

The splittings of B and B^* and of D and D^* result from spin-$SU(2)$ symmetry breaking effects. These must be corrections of order Λ/M_Q to the HQET predictions. Therefore, one must have $M_{B^*} - M_B = \Lambda^2/m_b$ and analogously for the D–D^* pair. Therefore

$$\frac{M_{B^*} - M_B}{M_{D^*} - M_D} = \frac{m_c}{m_b} \; . \tag{3.20}$$

Approximating m_c and m_b by M_D and M_B, respectively, we get $\sim 1/3$ on the right side, in remarkable agreement with the left side. Although these results also follow from potential models of constituent quarks, it is important that they can be derived in this generality, and this simply.

The states with $l = 3/2$ have $J = 1$ and 2. The D_1 and D_2^*, with $M_{D_2^*} - M_{D_1} = 40$ MeV, are remarkably closely spaced (and of course, have the appropriate quantum numbers to form a spin multiplet).

Exercise 3.4 A more complete classification of the spectrum would include parity. What modifications are needed, if any, to include parity as a quantum number? You should find that there are two possible $\ell = 1/2$ multiplets. One corresponds to the D and D^* (or B and B^*) mesons. It is usually argued that the multiplet with opposite parity would contain very broad resonances that could not be identified as stable states. Why?

While in the infinite mass limit states $|l, m_l; s, m_s\rangle$ have sharp L^2, L_z, S^2 and S_z, these are not good quantum numbers for physical states. Regardless of how small spin-symmetry breaking effects may be, they force states into linear combinations of sharp J^2, J_z, L^2 and S^2, $|J, m_J; l, s\rangle$. $SU(2)$-spin transformations connect states of $J = l + 1/2$ with those of $J = l - 1/2$. Now

$$|J, m_J; l, s\rangle = \sum |l, m_l; s, m_s\rangle C^{Jm_J}_{lm_l,sm_s}$$

where $C^{Jm_J}_{lm_l,sm_s} = C(lm_l;sm_s|Jm_J)$ are Clebsch-Gordan coefficients. The decomposition is useful because we know how the states on the right transform under spin-$SU(2)$. The inverse expression,

$$|l,m_l;s,m_s\rangle = \sum |J,m_J;l,s\rangle (C^{Jm_J}_{lm_l,sm_s})^*$$

gives the linear combinations of physical states with definite spin-$SU(2)$ numbers.

For example, for the B and B^* multiplet, the $m_l = 1/2$ and $m_l = -1/2$ states that form spin-$SU(2)$ doublets are, respectively

$$\psi_{1/2} = \begin{pmatrix} B^*(+) \\ \frac{B^*(0)+B}{\sqrt{2}} \end{pmatrix} \qquad \text{and} \qquad \psi_{-1/2} = \begin{pmatrix} \frac{B^*(0)-B}{\sqrt{2}} \\ B^*(-) \end{pmatrix} . \tag{3.21}$$

Rotations mix components among these doublets. We can combine them into a matrix $\Psi_{\alpha a} \equiv (\psi_a)_\alpha$. If $\mathcal{D}^{(l)}(R)$ stands for a $2l+1$ dimensional representation of the rotation R, then the action of spin-$SU(2)$ alone is $\Psi \to \mathcal{D}^{(1/2)}(R)\Psi$, while a rotation is $\Psi \to \mathcal{D}^{(1/2)}(R)\Psi\mathcal{D}^{(1/2)}(R)^\dagger$.

This is easily generalized. For arbitrary l there are $2l+1$ doublets of spin-$SU(2)$, ϕ_a, $a = -l, \ldots, l$. They can be assembled into a $2 \times (2l+1)$ matrix $\Phi_{\alpha a} = (\phi_a)_\alpha$, which transforms as $\Phi \to \mathcal{D}^{(1/2)}(R)\Phi\mathcal{D}^{(l)}(R)^\dagger$ under rotations. The linear combination of physical states in $\Phi_{\alpha a}$ can be written as a sum of at most two terms:

$$\Phi_{\alpha a} = \chi^{(+)A}_{\alpha a} + \chi^{(-)A}_{\alpha a} ,$$

where $\chi^{(+)A}_{\alpha a}$ ($\chi^{(-)A}_{\alpha a}$) is the state with $J = l + 1/2$ ($J = l - 1/2$), $A = m_J = \alpha + a$, weighted by the corresponding Clebsch-Gordan coefficient, $C^{l\pm1/2A}_{la,1/2\alpha}$.

If Q_0 is the two component heavy quark field for $v = (1, \vec{0})$ then, according to the Wigner-Eckart theorem the matrix elements of $\bar{Q}_0\Gamma Q_0$, for any Pauli matrix Γ, between B and B^* states are all given in terms of a single reduced matrix element, χ, times appropriate symmetry factors constructed out of Clebsch-Gordan coefficients. This can be summarized as follows,

$$\langle\psi'|\bar{Q}_0\Gamma Q_0|\psi\rangle = \chi \mathrm{Tr}\bar{\Psi}'\Gamma\Psi . \tag{3.22}$$

By this we mean that if the state ψ is, say, a B-meson, then the corresponding matrix Ψ on the right hand side is obtained from the matrix $\Psi_{a\alpha}$ of Eq. (3.21) by setting $B^* = 0$ and $B = 1$, and analogously for the other possible choices of the states ψ and ψ'. In the next Section we generalize this result to the case were the states ψ and ψ' may have different velocities.

In the Chapters that follow we will be interested in extracting the consequences of the spin and flavor symmetries of the HQET to a variety of processes. These processes may involve transitions between heavy hadrons of different velocities. It is convenient to develop a formalism that automatically extracts the information encoded in the symmetries[29]. I follow the simple presentation of Ref. [30].

A prototypical example of an application is the computation of relations between form factors in semileptonic $\bar{B}$ to D and D^* decays. There one needs to study the matrix elements

$$\langle D(v)|\bar{c}_v \Gamma b_{v'}|B(v')\rangle \quad \text{and} \quad \langle D^*(v),\epsilon|\bar{c}_v \Gamma b_{v'}|B(v')\rangle \ . \tag{3.23}$$

We would like to represent these $l = 1/2$ mesons as the product

$$u_Q \bar{v}_q \ , \tag{3.24}$$

where u_Q is a spinor representing the heavy quark, $\not{v} u_Q = u_Q$, and v_q is an antispinor representing the light stuff with $l = 1/2$, satisfying $\bar{v}_q \not{v} = \bar{v}_q$. The product in (3.24) is a superposition of states with $J = 0$ and 1. To identify the pseudoscalar meson P and the vector meson $V(\epsilon)$ with polarization ϵ, $\epsilon{\cdot}v = 0$, we must form appropriate linear combinations of the spin up and down spinors. This is most easily done in the rest frame $v = (1, \mathbf{0})$; the result will be generalized to arbitrary v by boosting. In the Dirac representation the spin operator is $\mathbf{S} = \gamma^5\gamma^0\boldsymbol{\gamma}/2$ so that the spinor basis $u^{(1)}_\alpha = \delta_{1\alpha}$ and $u^{(2)}_\alpha = \delta_{2\alpha}$ corresponds to spin up and spin down, and the antispinor basis $v^{(1)}_\alpha = -\delta_{3\alpha}$ and $v^{(2)}_\alpha = -\delta_{4\alpha}$ corresponds to spin down and spin up. With $\mathbf{S}(u\bar{v}) = (\mathbf{S}u)\bar{v} + u(\mathbf{S}\bar{v})$ it is easy to check that the combination

$$u^{(1)}_Q \bar{v}^{(1)}_q + u^{(2)}_Q \bar{v}^{(2)}_q = \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix} = \left(\frac{1+\gamma^0}{2}\right)\gamma^5 \tag{3.25}$$

has zero spin, while

$$\begin{aligned} u^{(1)}_Q \bar{v}^{(2)}_q &= \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & \sigma_1 + i\sigma_2 \\ 0 & 0 \end{pmatrix} = \left(\frac{1+\gamma^0}{2}\right)\not{\epsilon}^{(+)} \\ u^{(1)}_Q \bar{v}^{(1)}_q - u^{(2)}_Q \bar{v}^{(2)}_q &= \begin{pmatrix} 0 & \sigma_3 \\ 0 & 0 \end{pmatrix} = \left(\frac{1+\gamma^0}{2}\right)\not{\epsilon}^{(0)} \\ u^{(2)}_Q \bar{v}^{(1)}_q &= \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & \sigma_1 - i\sigma_2 \\ 0 & 0 \end{pmatrix} = \left(\frac{1+\gamma^0}{2}\right)\not{\epsilon}^{(-)} \end{aligned} \tag{3.26}$$

with $\epsilon^{(\pm)} = (0, 1, \pm i, 0)$ and $\epsilon^{(0)} = (0, 0, 0, 1)$, have total spin 1, with third component 1, 0 and -1, respectively. Thus, for arbitrary velocity v one obtains the representation for pseudoscalar and vector mesons:

$$\widetilde{M}(v) = \left(\frac{1+\not{v}}{2}\right)\gamma^5 \qquad\qquad \widetilde{M}^*(v,\epsilon) = \left(\frac{1+\not{v}}{2}\right)\not{\epsilon}\ . \tag{3.27}$$

By construction, the spin symmetry acts on this representation only on the first index of the matrices $\widetilde{M}(v)$ and $\widetilde{M}^*(v,\epsilon)$.

The power of this machinery can now be displayed. Consider the matrix elements (3.23). Using the above representation of states and noting that the result should transform under the spin symmetry just as the matrix Γ, we have

$$\langle D(v)|\bar{c}_v\Gamma b_{v'}|\bar{B}(v')\rangle = -\xi(v{\cdot}v')\mathrm{Tr}\,\overline{\widetilde{D}}(v)\Gamma\widetilde{B}(v') \tag{3.28a}$$

$$\langle D^*(v)\varepsilon|\bar{c}_v\Gamma b_{v'}|\bar{B}(v')\rangle = -\xi(v{\cdot}v')\mathrm{Tr}\,\overline{\widetilde{D}}^*(v,\varepsilon)\Gamma\widetilde{B}(v')\ , \tag{3.28b}$$

where $\overline{X} = \gamma^0 X^\dagger\gamma^0$. The common factor $-\xi(v{\cdot}v')$ plays the rôle of the reduced matrix element in the Wigner–Eckart theorem. We will explore the consequences of Eqs. (3.28) in depth in Section 3.7.

Exercise 3.5 An even simpler case is that of the $l = 0$ multiplet. In this case the states must transform as a spinor. How would you represent matrix elements of these states? What about those of the $l = 1$ or $l = 3/2$ multiplets? This formalism can be extended[30] to deal with multiplets of arbitrary l.

3.6. Meson Decay Constants

The pseudoscalar decay constant is one of the first physical quantities studied in the context of HQET's. For a heavy-light pseudoscalar meson X of mass M_X, the decay constant f_X, we will see, scales like $1/\sqrt{M_X}$. This was known before the formal development of HQET's, although the arguments relied on models of strong interactions. The HQET will give us a systematic way of obtaining this result. Moreover, it will give us the means of studying corrections to this prediction.

The decay constant f_X is defined through

$$\langle 0|A_\mu(0)|X(p)\rangle = f_X p_\mu\ , \tag{3.29}$$

where $A_\mu = \bar{q}\gamma_\mu\gamma_5 Q$ is the heavy-light axial current, and the meson has the standard relativistic normalization

$$\langle X(p')|X(p)\rangle = 2E\delta^{(3)}(\mathbf{p}-\mathbf{p}')\ . \tag{3.30}$$

Thus, the states have mass-dimension -1. Analogous definitions can be made for other mesons. For example, for the vector meson X^* (the $l = 1/2$ partner of X), has

$$\langle 0|V_\mu(0)|X^*(p,\epsilon)\rangle = f_{X^*}\epsilon_\mu \ . \tag{3.31}$$

Note that the mass-dimensions of f_X and f_{X^*} are 1 and 2, respectively.

Consider the decay constant of the meson state in the HQET. The effective pseudoscalar decay constant $\tilde{f}_X$ is defined by

$$\langle 0|\widetilde{A}_\mu(0)|\widetilde{X}(v)\rangle = \tilde{f}_X v_\mu \tag{3.32}$$

The state in the HQET, $|\widetilde{X}\rangle$, is normalized *à la* Bjorken and Drell,[31] to $2E/M_X$ rather than to $2E$:

$$\langle\widetilde{X}(v')|\widetilde{X}(v)\rangle = 2v^0\delta^{(3)}(\mathbf{v}-\mathbf{v}') \ . \tag{3.33}$$

Actually, defining states in the HQET requires some care, but I will just assume it all works and merely refer the interested reader to the literature[21]. Obviously, since the normalization of states and the dynamics are M_Q independent, so is $\tilde{f}_X$. To relate $\tilde{f}_X$ to the physical f_X simply multiply Eq. (3.32) by $\sqrt{M_X}$, to restore the normalization of states of Eq. (3.30), and write $v_\mu = p_\mu/M_X$. Thus we arrive at

$$f_X = \tilde{f}_X/\sqrt{M_X}, \tag{3.34}$$

A useful way of quoting the result is, for the physical case of B and D mesons,

$$\frac{f_B}{f_D} = \sqrt{\frac{M_D}{M_B}} \tag{3.35}$$

As a simple application of the spin symmetry, consider the pseudoscalar decay constant f_{X^*}. Using the 4×4 notation of Section 3.5, the matrix element in Eq. (3.31) that defines the pseudoscalar constant is proportional to

$$\mathrm{Tr}\left(\gamma^\mu\gamma_5\widetilde{M}(v)\right) = \mathrm{Tr}\left(\gamma^\mu\gamma_5\left(\frac{1+\not{v}}{2}\right)\gamma_5\right) = -2v^\mu \tag{3.36}$$

The matrix element

$$\langle 0|\widetilde{V}^\mu(0)|\widetilde{X}^*(v)\epsilon\rangle = \tilde{f}_{X^*}\epsilon^\mu \tag{3.37}$$

is proportional to

$$\mathrm{Tr}\left(\gamma^\mu\widetilde{M}^*(v,\epsilon)\right) = \mathrm{Tr}\left(\gamma^\mu\left(\frac{1+\not{v}}{2}\right)\not{\epsilon}\right) = 2\epsilon^\mu \tag{3.38}$$

with the same constant of proportionality. Therefore

$$\tilde{f}_{X^*} = -\tilde{f}_X \tag{3.39}$$

The sign is unimportant, since it can be absorbed into a phase redefinition of either state. It is the magnitude that matters. Multiplying by $\sqrt{M_{X^*}} \approx \sqrt{M_X}$ to restore to the standard normalization, we have

$$f_{X^*} = -f_X M_X \tag{3.40}$$

The predictions Eq. (3.35) and Eq. (3.40) have not been tested experimentally. The difficulty is the small expected branching fraction for the decays $X \to \mu\nu$ or $X^* \to \mu\nu$, for $X = B$ and D. Alternatively, the decay constants f_X and f_{X^*} can be measured in Monte Carlo simulations of lattice QCD. There are indications from such simulations that the $1/M_Q$ corrections to the relation (3.35) are large[32].

3.7. Semileptonic decays

The semileptonic decays of a $\bar{B}$-meson to D- or D^*-mesons offer the most direct means of extracting the mixing angle $|V_{cb}|$. In order to extract this angle from experiment, theory must provide the form factors for the $\bar{B} \to D$ and $\bar{B} \to D^*$ transitions. Several means of estimating these form factors can be found in the literature. A popular method consists of estimating the form factor at one value of the momentum transfer $q^2 = q_0^2$, and then introducing the functional dependence on q^2 in some arbitrary, hopefully reasonable, way. In the pre-HQET days it was customary to estimate the form factor at q_0^2 from some model of strong interactions, like the non-relativistic constituent quark model.

The HQET gives the form factor at the maximum momentum transfer, $q^2 = q^2_{\text{max}} = (M_B - M_D)^2$ —the point at which the resulting D or D^* does not recoil in the restframe of the decaying B-meson. While the functional dependence on q^2 is a non-perturbative problem, it is already progress to have a prediction of the form factor at one point. Moreover, the HQET gives relations between the form factors. One may study these relations experimentally to test the accuracy of the HQET predictions.

The standard definition of form factors in semileptonic $\bar{B}$-meson decays is

$$\left\langle D(p')|V_\mu|\bar{B}(p)\right\rangle = f_+(q^2)(p+p')_\mu + f_-(q^2)(p-p')_\mu \tag{3.41a}$$

$$\left\langle D^*(p')\epsilon|A_\mu|\bar{B}(p)\right\rangle = f(q^2)\epsilon^*_\mu + a_+(q^2)\epsilon^{*}\!\cdot\! p(p+p')_\mu + a_-(q^2)\epsilon^{*}\!\cdot\! p(p-p')_\mu \tag{3.41b}$$

$$\left\langle D^*(p')\epsilon|V_\mu|\bar{B}(p)\right\rangle = ig(q^2)\epsilon_{\mu\nu\lambda\sigma}\epsilon^{*\nu}(p+p')^\lambda(p-p')^\sigma \tag{3.41c}$$

Here, the states have the standard normalization, Eq. (3.30), and $q^2 \equiv (p-p')^2$. The contribution to the decay rates from the form factors f_- and a_- are suppressed by m_ℓ^2/M_B^2, where m_ℓ is the mass of the charged lepton, and therefore they are often neglected.

Exercise 3.6 Compute the differential decay rates, $d\Gamma/dx\ dy$, where $x = q^2/M_B^2$ and $y = v\cdot q/M_B$, for $\bar{B} \to De\bar{\nu}$ and $\bar{B} \to D^* e\bar{\nu}$, in terms of these form factors.

In the effective theory, we would like to compute the matrix elements of the effective currents $\tilde{V}_\mu$ and $\tilde{A}_\mu$ between states of the $l = \frac{1}{2}$ multiplet. We can take advantage of the flavor and spin symmetries to write these matrix elements in terms of generalized Clebsch-Gordan coefficients and reduced matrix elements, *i.e.,* we use the Wigner-Eckart theorem. We have already introduced the relevant machinery in Section 3.5. The matrix elements of the operator $\widetilde{G} = \bar{c}_{v'}\Gamma b_v$ between B and D or D^* states, are given by (*c.f.,* Eqs. (3.28))

$$\langle D(v')|\widetilde{G}|\bar{B}(v)\rangle = -\xi(v{\cdot}v')\mathrm{Tr}\,\overline{\widetilde{D}}(v')\Gamma\widetilde{B}(v) \tag{3.42a}$$

$$\langle D^*(v')\varepsilon|\widetilde{G}|\bar{B}(v)\rangle = -\xi(v{\cdot}v')\mathrm{Tr}\,\overline{\widetilde{D}}^*(v',\varepsilon)\Gamma\widetilde{B}(v). \tag{3.42b}$$

Before expanding Eqs. (3.42), we note that the flavor symmetry implies that the B-current form factor between $\bar{B}$-meson states is given by the same reduced matrix element:

$$\langle \bar{B}(v')|\bar{b}_{v'}\Gamma b_v|\bar{B}(v)\rangle = -\xi(v{\cdot}v')\mathrm{Tr}\,\overline{\widetilde{B}}(v')\Gamma\widetilde{B}(v) \tag{3.43}$$

Using $\Gamma = \gamma^0$, and recalling that B-number is conserved, one finds that ξ is fixed at $v' = v$. With the normalization of states appropriate to the effective theory, Eq. (3.33), and expanding Eq. (3.43) at $v = v'$, one has

$$\xi(1) = 1. \tag{3.44}$$

The reduced matrix element ξ is the universal function that describes all of the matrix elements of operators $\widetilde{G}$ between $l = \frac{1}{2}$ states. It is known as the Isgur-Wise function after the discoverers of the relations (3.42) and (3.43). It is quite remarkable that the Isgur-Wise function describes both timelike form-factors (as in $\bar{B} \to De\nu$) as well as spacelike form-factors (as in $\bar{B} \to \bar{B}$). The point, of course, is that in both cases it describes transitions between infinitely heavy sources at fixed "velocity-transfer" $(v - v')^2$.

Expanding Eq. (3.42) for $\Gamma = \gamma^\mu$ or $\gamma^\mu\gamma_5$, we have

$$\langle D(v')|\widetilde{V}_\mu|\bar{B}(v)\rangle = \xi(v{\cdot}v')(v_\mu + v'_\mu) \tag{3.45a}$$

$$\langle D^*(v')\epsilon|\tilde{A}_\mu|\ \bar{B}(v)\rangle = -\xi(v{\cdot}v')[\epsilon^*_\mu(1 + v{\cdot}v') - v'_\mu\epsilon^*{\cdot}v] \tag{3.45b}$$

$$\langle D^*(v')\epsilon|\widetilde{V}_\mu|\ \bar{B}(v)\rangle = -\xi(v{\cdot}v')[-i\epsilon_{\mu\nu\lambda\sigma}\epsilon^{*\nu}v^\lambda v^\sigma] \tag{3.45c}$$

It remains to express the physical form factors in terms of the Isgur-Wise functions. We must multiply by $\sqrt{M_D M_B}$ to restore to the standard normalization of states, and express Eqs. (3.45) in terms of momenta using $v = p/M_B$ and $v' = p'/M_D$. For example, one has,

$$\langle D(p')|V_\nu|B(p)\rangle = \xi(v\cdot v')\sqrt{M_B M_D}\left(\frac{p_\nu}{M_B} + \frac{p'_\nu}{M_D}\right) \tag{3.46}$$

It follows that

$$f_\pm(q^2) = \xi(v\cdot v')\left(\frac{M_D \pm M_B}{2\sqrt{M_B M_D}}\right) \tag{3.47}$$

Similarly, f, $a_\pm$ and g can all be written in terms of $\xi(v\cdot v')$. Moreover, at $v\cdot v' = 1$, one has $q^2 = (M_B v - M_D v)^2 = (M_B - M_D)^2 \equiv q^2_{\rm max}$ so the normalization Eq. (3.44) gives

$$f_\pm(q^2_{\rm max}) = \left(\frac{M_D \pm M_B}{2\sqrt{M_B M_D}}\right) \tag{3.48}$$

This remarkable result gives the form factors, in the heavy quark limit, without uncertainties from hadronic matrix elements. Short distance corrections will be discussed below; see Sections 3.8 and 3.9.

Exercise 3.7 The same methods can be used to obtain relations among, and normalizations of, the form factors relevant to semileptonic decays of heavy baryons. The case of transitions between $l = 0$ states is simplest. The case of transitions involving higher l states can be found elsewhere.[33,30] There are three form factors, F_i, for the matrix element of the vector current between Λ_b and Λ_c states, and three more, G_i, for the matrix element of the axial current:

$$\langle \Lambda_c(v', s')|\bar{c}\gamma_\mu b|\Lambda_b(v, s)\rangle = \bar{u}^{(s')}(v')[\gamma_\mu F_1 + v'_\mu F_2 + v_\mu F_3]u^{(s)}(v) \ ,$$
$$\langle \Lambda_c(v', s')|\bar{c}\gamma_\mu\gamma_5 b|\Lambda_b(v, s)\rangle = \bar{u}^{(s')}(v')[\gamma_\mu G_1 + v'_\mu G_2 + v_\mu G_3]\gamma_5 u^{(s)}(v) \ .$$

Prove that all six are given in terms of one universal 'Isgur–Wise' function[33]. Show that the matrix element of the current is given by

$$\langle \Lambda_c(v', s')|\bar{c}_{v'}\Gamma b_v|\Lambda_b(v, s)\rangle = \zeta(v\cdot v')\bar{u}^{(s')}(v')\Gamma u^{(s)}(v) \ , \tag{3.49}$$

and that ζ is fixed at one point: $\zeta(1) = 1$. Thus (3.49) show that

$$F_1 = G_1 \qquad \text{and} \qquad F_2 = F_3 = G_2 = G_3 = 0, \tag{3.50}$$

and $G_1(1) = 1$.

3.8. Beyond Tree Level

In the previous sections we have seen how the HQET can be derived from QCD and used to obtain useful information about physical processes. The derivation of the HQET in Section 3.2 involved only tree level Feynman diagrams. Clearly one must extend this beyond tree level if the HQET is to be at all useful. After all, we want to use it to describe mesons made of a confined heavy quark and light antiquark. It is not difficult to extend the HQET to arbitrary order in perturbation theory[19,34]. We will content ourselves with an understanding of how this works at one loop, although going beyond is not much more complicated.

The alert reader may complain, justifiably, that an all orders proof is not enough. There are *bona fide* non-perturbative effects that one cannot obtain even in all orders of perturbation theory. I do not know of a non-perturbative proof of the validity of the HQET. One must not forget, though, that many important results of quantum field theory are proved perturbatively, *e.g.,* renormalizability of the S-matrix in Yang-Mills theories and the operator product expansion.

The generalization of the effective lagrangian of Eq. (3.1) beyond tree level consists of adding to it counterterms,

$$\mathcal{L}_{\text{eff}}^{(v)} = \bar{Q}_v iv{\cdot}DQ_v + \mathcal{L}_{\text{light}} + \mathcal{L}_{\text{c.t.}} \ . \tag{3.51}$$

At tree level the HQET gave us an expression for the Green functions of QCD as an expansion of powers in the residual momentum $k = p - M_Q v$. For the two point function we had, Eq. (3.11),

$$G(p;q) = \widetilde{G}_v(k;q) + \mathcal{O}\left(\Lambda/M_Q\right) \qquad \text{(tree level)} \ . \tag{3.52}$$

Beyond tree level the corrected version is still close in form to this,

$$G(p;q;\mu) = C(M_Q/\mu, g_s)\widetilde{G}_v(k;q;\mu) + \mathcal{O}\left(\Lambda/M_Q\right) \qquad \text{(beyond tree level)} \ . \tag{3.53}$$

The Green functions G and $\widetilde{G}_v$ are renormalized, so they depend on a renormalization point μ. The function C is independent of momenta or light quark masses: it is independent of the dynamics of the light degrees of freedom. It is there because the left hand side has some terms which grow logarithmicaly with the heavy mass, $\ln(M_Q/\mu)$. The beauty of Eq. (3.53) is that *all of the logarithmic dependence on the heavy mass factors out.* Better yet, since C is dimensionless, it is a function of the ratio M_Q/μ only, and not of M_Q and μ separately[3]. To find the dependence on M_Q it suffices to find the dependence on μ. This in turn is dictated by the renormalization group equation.

It is appropriate to think of the HQET as a factorization theorem, stating that, in the large M_Q limit, the QCD Green functions factorize into a universal function of M_Q, $C(M_Q/\mu, g_s)$, which depends on the short distance physics only, times a function that contains all of the information about long distance physics and is independent of M_Q, and can be computed as a Green function of the HQET lagrangian.

Exercise 3.8 There is a factorization theorem in the physics of deep inelastic scattering, expressing the cross section as a product of parton distributions times parton cross sections. Draw an analogy between those quantities in that factorization theorem and the ones in the HQET.

Of course, the theorem holds for any Green functions, and not just for the two point function.

To see how this works at one loop, we start by considering Green functions for a heavy quark with n gluons, with $n > 1$. These are convergent by power counting, and since there are no nested divergences at one loop, they are convergent. It suffices to consider one-particle irreducible (1PI) functions. In Fig. 5 the left side is calculated in the full theory and the right side in the HQET. The double line stands for the heavy propagator in the HQET.

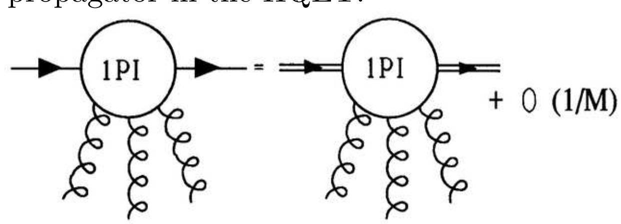

Figure 5 *Relation between Green functions in full and effective field theories.*

We can prove the validity of the equation represented in Fig. 5, diagram by diagram (there are several diagrams that contribute to each side of the equation). Consider, for definiteness, the diagrammatic equation in Fig. 6.

[3] Actually, additional μ dependence is implicit in the definition of the renormalized coupling constant g_s. This reflects itself in the explicit form of C.

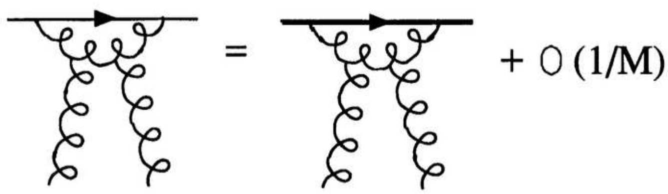

Figure 6 *Relation between Green functions in full and effective field theories in simple contribution to four point function, at one loop.*

The equation would trivially hold if we could make the propagator replacement

$$\frac{i}{\not p + \not l - M_Q} \to \left(\frac{1+\not v}{2}\right)\frac{i}{v\cdot(k+l)}$$

even inside the loop integral. Here $p = M_Q v + k$, and l is the loop momentum. In other words, in the right hand side of Fig. 6, we take the limit $M_Q \to \infty$ and then integrate, while on the left side we first integrate and then take the limit. Everyone knows that, if both integrals converge, then they agree. And that is the case for Fig. 6, and, indeed, it is also the case for any 1–loop integral with a heavy quark and $n \geq 2$ external gluons. We have established Fig. 5 for $n \geq 2$.

We are left with the 2–point ($n = 0$) and 3–point ($n = 1$) functions. These are different from the $n \geq 2$ functions in two ways. First, they receive contributions at tree level. And second, they are divergent at 1–loop. Choose some method of regularization. Dimensional regularization is particularly useful as it preserves gauge invariance (or, more precisely, BRST invariance). The comparison between full and effective theories is simplest if the same gauge and regularization choices are made. For concreteness, consider Fig. 7.

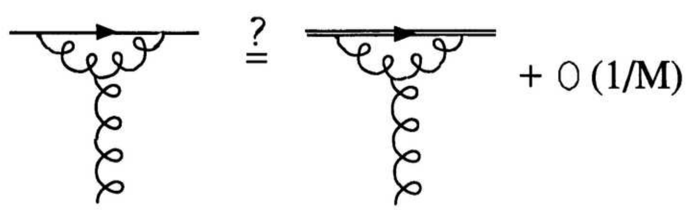

Figure 7 *Relation between infinite Green functions in full and effective field theories at one loop: three point function.*

Since both sides are finite, we can argue as before. But we run into trouble when we try to remove the regulator. One must renormalize the Green functions by adding counter terms, but there is no guarantee that the counterterms satisfy the same relation as the regulated Green functions of Fig. 7. To elucidate the relation between counterterms, take a derivative from both sides of Fig. 7 with respect to either the residual momentum k_μ or the gluon external momentum of q_μ. This makes the diagrams finite and the regulator can be removed. Thus, at 1–loop, the relations

$$\frac{\partial}{\partial k_\mu} G^{(2,1)} = \frac{\partial}{\partial k_\mu} \tilde{G}_v^{(2,1)} + \mathcal{O}\left(\Lambda/M_Q\right) \tag{3.54}$$

and

$$\frac{\partial}{\partial q_\mu} G^{(2,1)} = \frac{\partial}{\partial q_\mu} \tilde{G}_v^{(2,1)} + \mathcal{O}(\Lambda/M_Q) \tag{3.55}$$

hold. The counterterms, or at least the difference between them, are k_μ and q_μ independent. It is a simple algebraic exercise to show, then, that the difference between counterterms is of the form

$$aG^{(2,1)0} + b\tilde{G}_v^{(2,1)0} \tag{3.56}$$

where the superscript '0' stands for tree level, and a and b are infinite constants, *i.e.*, independent of k_μ and q_μ. Thus, one can subtract the 1–loop Green functions by standard counterterms, and establish the equality of Fig. 7.

A similar argument can be constructed for the 2–point function. One must take two derivatives with respect to k_μ, but that is as it should, since the counterterms are linear in momentum.

We have therefore established that, to 1–loop, the renormalized Green functions in the full and effective theories agree. The alert reader must be puzzled as to the fate of the function $C(M_Q/\mu, g_s)$ of Eq. (3.53). What has happened is that the constant b in the counterterm in Eq. (3.56) is, in general, M_Q dependent. Indeed, if we take derivatives with respect to M_Q, as in (3.54) or (3.55), the degree of divergence is not changed, and one cannot argue that a or b are M_Q independent. The relation between renormalized Green functions that we have derived contains hidden M_Q–dependence in the renormalization prescription for the Green functions in the HQET.

Given two different renormalization schemes, the corresponding renormalized Green functions $\tilde{G}$ and $\tilde{G}'$ are related by a finite renormalization

$$\tilde{G} = z(\mu, g_s)\tilde{G}'$$

Choosing $\tilde{G}$ to be the mass–independent subtracted Green function, and $\tilde{G}'$ the one in our peculiar subtraction scheme, we have that the relation between full and effective theories becomes

$$G^{(2,1)}(p, q; \mu) = C\left(M_Q/\mu, g_s\right) \tilde{G}_v^{(2,1)}(k, q; \mu) + \mathcal{O}\left(\Lambda/M_Q\right)$$

as advertised in Section 3.2. Here, C is nothing but this finite renormalization $z(\mu, g_s)$. That we can use the same function C for all Green functions can be established by using the same wave functions renormalization prescription for gluons in the full and effective theories. Otherwise, an additional factor of $z_A^{n/2}$ would have to be included in the relation between $G^{(2,n)}$ and $\tilde{G}^{(2,n)}$. This completes the argument.

It is worth mentioning that the discussion above *assumes* the renormalizability, preserving BRST invariance, of the effective theory. Although, to my knowledge, this has not been established, there is no obvious reason to doubt that the standard techniques apply in this case.

3.9. External Currents

We will often be interested in computing Green functions with an insertion of a current. Consider, the current

$$J_\Gamma = \bar{q}\,\Gamma Q \tag{3.57}$$

in the full theory, where Γ is some Dirac matrix, and q a light quark. In the effective theory, this is replaced according to

$$J_\Gamma(x) \to e^{-iM_Q v\cdot x}\tilde{J}_\Gamma(x)\ , \tag{3.58}$$

where

$$\tilde{J}_\Gamma = \bar{q}\,\Gamma Q_v\ , \tag{3.59}$$

and it is understood that in $\tilde{J}_\Gamma$ the heavy quark is that of the HQET, satisfying, in particular, $\not{v} Q_v = Q_v$. The exponential factor in Eq. (3.58) reminds us to take the large momentum out through the current, allowing us to keep the external momentum of light quarks and gluons small. The relation between full and effective theories takes the form of an approximate equation between Green functions —and eventually amplitudes— of insertions of these currents:

$$G_{J_\Gamma}(p,p';q;\mu) = C(M_Q/\mu, g_s)^{1/2} C_\Gamma(M_Q/\mu, g_s)\widetilde{G}_{v,\tilde{J}_\Gamma}(k,k';q;\mu) + \mathcal{O}(\Lambda/M_Q)\ , \tag{3.60}$$

where p and p' are the momenta of the heavy quark and the external current, k and k' the corresponding residual momenta, $p = M_Q v + k$, $p' = M_Q v + k'$, and q stands for the momenta of the light degrees of freedom. The factor $C^{1/2}\,C_\Gamma$ accounts for the logarithmic mass dependence, as explained earlier. We see that an additional factor, namely, C_Γ, is needed in this case to account for the different scaling behavior of the currents in the full and effective theories. It is convenient to think of the replacement of currents, not as given by Eq. (3.58), but rather by

$$J_\Gamma(x) \to e^{-iM_Q v x} C_\Gamma(M_Q/\mu, g_s) \tilde{J}_\Gamma(x) \ . \tag{3.61}$$

In fact, Eq. (3.60), and therefore the replacement in Eq. (3.61), are not quite correct. To reproduce the matrix elements of the current J_Γ of Eq. (3.57), it is necessary to sum over matrix elements of several different 'currents' in the effective theory. The operator $\tilde{J}_\Gamma$ of Eq. (3.59) is just one of them. In addition, one may have to introduce such operators as $\bar{q}\not{v}\Gamma Q_v$. The correct replacement is therefore

$$J_\Gamma(x) \to e^{-iM_Q v x} \sum_i C_\Gamma^{(i)}(M_Q/\mu, g_s) \widetilde{\mathcal{O}}^{(i)}(x) \ . \tag{3.62}$$

Here $\widetilde{\mathcal{O}}^{(i)}(x)$ is the collection of the operators of dimension 3 with appropriate quantum numbers. The first operator in the sum, call it $\widetilde{\mathcal{O}}^{(0)}$, is there even at tree level, and corresponds to the operator $\tilde{J}_\Gamma$ of Eq. (3.59).

Another case of interest is that of the insertion of a current of two heavy quarks

$$J_\Gamma = \bar{Q}'\Gamma Q \ . \tag{3.63}$$

The replacement now is

$$J_\Gamma(x) \to e^{-iM_Q v x + iM_{Q'} v' \cdot x} \sum_i \widehat{C}_\Gamma^{(i)}(\frac{M_Q}{\mu}, \frac{M_{Q'}}{M_Q}, v{\cdot}v', g_s) \widehat{\mathcal{O}}_\Gamma^{(i)}(x) \ . \tag{3.64}$$

Again, $\widehat{\mathcal{O}}^{(i)}(x)$ stands for the complete list of operators of dimension 3 in the effective theory with the right quantum numbers. Also, the operator $\widehat{\mathcal{O}}^{(0)} = \bar{Q}'_{v'}\Gamma Q_v$ appears in the sum at tree level.

This deserves some explanation. The Green functions now include two heavy quarks. The functions $\widehat{C}$ connecting these full and effective Green functions will now, in general, depend on both M_Q and $M_{Q'}$. Moreover, we can not argue that $\widehat{C}$ are independent of the velocities v and v'. In fact, this was true of the simpler case considered in Section 3.8; but there, C could only depend on v_μ through $v^2 = 1$. In the case at hand there is an additional invariant on which $\widehat{C}$ can depend, namely $v{\cdot}v'$.

The explicit functional dependence on M_Q in the functions C_Γ and $\widehat{C}_\Gamma$ can be obtained from a study of their dependence on the renormalization point μ. For clarity of presentation we neglect operator mixing for now. When necessary, this can be incorporated without much difficulty. Taking a derivative $d/d\mu$ on both sides of Eqs. (3.53) and (3.61), we find

$$\mu \frac{d}{d\mu} C_\Gamma = (\gamma_\Gamma - \tilde{\gamma}_\Gamma) C_\Gamma \tag{3.65}$$

where γ_Γ and $\tilde{\gamma}_\Gamma$ are the anomalous dimensions of the currents J_Γ and $\tilde{J}_\Gamma$ in the full and effective theories, respectively. Of particular interest are the cases $\Gamma = \gamma^\mu$ and $\Gamma = \gamma^\mu\gamma_5$. These correspond, in the full theory, to conserved and partially conserved currents, and therefore the corresponding anomalous dimensions vanish, giving

$$\mu \frac{dC_\Gamma}{d\mu} = -\tilde{\gamma}_\Gamma \, \tilde{C}_\Gamma \qquad (\Gamma = \gamma^\mu, \, \gamma^\mu\gamma_5) \ . \tag{3.66}$$

Before we solve this equation, we recall that

$$\mu \frac{d}{d\mu} = \mu \frac{\partial}{\partial \mu} + \beta(g_s) \frac{\partial}{\partial g_s} \ . \tag{3.67}$$

Here β is the QCD β–function, with perturbative expansion

$$\frac{\beta(g)}{g} = -b_0 \frac{g^2}{16\pi^2} + b_1 \left(\frac{g^2}{16\pi^2} \right)^2 + \ldots \quad , \tag{3.68}$$

and

$$b_0 = 11 - \frac{2}{3} n_f \ , \tag{3.69}$$

where n_f is the number of quarks in the theory. For our purposes, n_f should *not* include the heavy quark. This is explained in the famous paper by Appelquist and Carrazone[35]; it simply reflects the fact that the logarithmic scaling of g_s is not affected by heavy quark loops, since these are suppressed by powers of M_Q. Now, the solution to (3.66) is standard:

$$C_\Gamma(\mu, g_s) = \exp\left(- \int_{\bar{g}_s(\mu_0)}^{g_s} dg' \, \frac{\tilde{\gamma}_\Gamma(g')}{\beta(g')} \right) C_\Gamma(\mu_0, \bar{g}_s(\mu_0)) \tag{3.70}$$

where $\bar{g}_s$ is the running coupling constant defined by

$$\mu' \frac{d\bar{g}_s(\mu')}{d\mu'} = \beta(\bar{g}_s(\mu')) \qquad , \qquad \bar{g}_s(\mu) = g_s \ . \tag{3.71}$$

Choosing $\mu_0 = M_Q$, and restoring the dependence on M_Q, we have then

$$C_\Gamma(M_Q/\mu, g_s) = \exp\left(-\int_{\bar{g}_s(M_Q)}^{\bar{g}_s(\mu)} dg' \, \frac{\tilde{\gamma}_\Gamma(g')}{\beta(g')} \right) C_\Gamma(1, \bar{g}_s(M_Q)) \ . \tag{3.72}$$

Therefore, the problem of determining $C_\Gamma(M_Q/\mu, g_s)$ breaks down into two parts. One is the determination of the anomalous dimensions $\tilde{\gamma}_\Gamma$. The other is the calculation of $C_\Gamma(1, \bar{g}_s(M_Q))$. Both can be done perturbatively, and $C_\Gamma(M_Q/\mu, g_s)$ can thus be computed, provided μ and M_Q are large enough so that $\bar{g}_s(\mu)$ and $\bar{g}_s(M_Q)$ are small. One finds, for example, that in leading order C_Γ is Γ independent and there is no mixing:

$$C_\Gamma(M_Q/\mu, g_s) = \left(\frac{\bar{\alpha}_s(M_Q)}{\bar{\alpha}_s(\mu)} \right)^{a_I} \ , \tag{3.73}$$

where $\bar{\alpha}_s \equiv \bar{g}_s^2/4\pi$, and[36–37] $a_I \equiv -c_1/2b_0 = -6/(33 - 2n_f)$.

Exercise 3.9 Obtain an expression for C_Γ analogous to that of Eq. (3.72) but without assuming that the anomalous dimension γ_Γ of Eq. (3.65) vanishes.

We now turn to the computation of the coefficient $\widehat{C}_\Gamma$ for the current of two heavy quarks in Eq. (3.64). A new difficulty arises. Because $\widehat{C}_\Gamma$ depends on three dimensionful quantities, namely the masses M_Q and $M_{Q'}$, and the renormalization point μ, its functional dependence is not determined from the renormalization group equation (even if we neglect the implicit dependence of g_s on μ). Two different approximations have been developed to deal with this problem:

I) Treat the ratio $M_{Q'}/M_Q$ as a dimensionless parameter, and study the dependence of $\widehat{C}_\Gamma$ on $M_{Q'}/\mu$ through the renormalization group[38]. This is just like what was done for the heavy-light case, so we can transcribe the result:

$$\widehat{C}_\Gamma\left(\frac{M_{Q'}}{\mu}, \frac{M_{Q'}}{M_Q}, v{\cdot}v', g_s\right) \approx \exp\left(-\int_{\bar{g}_s(M_{Q'})}^{\bar{g}_s(\mu)} dg' \, \frac{\hat{\gamma}_\Gamma(g')}{\beta(g')} \right) \widehat{C}_\Gamma\left(1, \frac{M_{Q'}}{M_Q}, v{\cdot}v', \bar{g}_s(M_{Q'})\right) . \tag{3.74}$$

Again

$$\widehat{C}_\Gamma\left(1, \frac{M_{Q'}}{M_Q}, v{\cdot}v', \bar{g}_s(M_{Q'})\right) = 1 + \mathcal{O}(\bar{\alpha}_s(M_{Q'})) \ . \tag{3.75}$$

But, now, the correction of order $\bar{\alpha}_s(M_{Q'})$ is a function of $M_{Q'}/M_Q$. This method has the advantage that the complete functional dependence on $M_{Q'}/M_Q$ is retained, order by order in $\bar{\alpha}_s(M_{Q'})$. Nevertheless, it fails to re–sum the leading–logs between the scales $M_{Q'}$ and M_Q, *i.e.*, it does not include the effects of running of the QCD coupling constant between M_Q and $M_{Q'}$. Therefore, this method is useful when $M_{Q'}/M_Q \sim 1$, or, equivalently, when $(\bar\alpha_s(M_{Q'}) - \bar\alpha_s(M_Q))/\bar\alpha_s(M_Q) \ll 1$.

II) Treat the ratio $M_{Q'}/M_Q$ as small. Expand first in a HQET treating Q as heavy and Q' as light. The corrections are not just of order Λ/M_Q but also $M_{Q'}/M_Q$, but this is assumed to be small (even if much larger than Λ/M_Q). Then expand from this HQET, in powers of $\Lambda/M_{Q'}$, by constructing a new HQET where both Q and Q' are heavy[29]. The calculation of $\widehat{C}_\Gamma$ then proceeds in two steps. The first gives a factor just like that of the heavy-light current, in (3.72)

$$\exp\left(-\int_{\bar g_s(M_Q)}^{\bar g_s(\mu)} dg' \frac{\tilde\gamma_\Gamma(g')}{\beta(g')}\right) C_\Gamma(1, \bar g_s(M_Q)) \ . \tag{3.76}$$

The second factor is as in method I, above, but neglecting $M_{Q'}/M_Q$. Moreover, the current $\tilde J_\Gamma$ is not conserved, so the anomalous dimension to be used is not $-\hat\gamma_\Gamma$ but $\tilde\gamma_\Gamma - \hat\gamma_\Gamma$. Finally, we must make explicit the fact that in the first and second steps the appropriate β-functions differ in the number of active quarks. We therefore label the one in the second step β' and the corresponding running coupling constant $\bar g'_s$. The second factor is

$$\exp\left(\int_{\bar g_s(M_{Q'})}^{\bar g_s(\mu)} dg' \frac{\tilde\gamma_\Gamma(g')}{\beta(g')} - \int_{\bar g'_s(M_{Q'})}^{\bar g'_s(\mu)} dg'' \frac{\hat\gamma_\Gamma(g'')}{\beta'(g'')}\right) \widehat{C}_\Gamma(1, 0, v{\cdot}v', \bar g_s(M_{Q'})) \ . \tag{3.77}$$

Combining factors gives

$$\widehat{C}_\Gamma\left(\frac{M_{Q'}}{\mu}, \frac{M_{Q'}}{M_Q}, v{\cdot}v', g_s\right) \approx \exp\left(-\int_{\bar g_s(M_Q)}^{\bar g_s(M_{Q'})} dg' \frac{\tilde\gamma_\Gamma(g')}{\beta(g')} - \int_{\bar g'_s(M_{Q'})}^{\bar g'_s(\mu)} dg'' \frac{\hat\gamma_\Gamma(g'')}{\beta'(g'')}\right) \times C_\Gamma(1, \bar g_s(M_Q)) \widehat{C}_\Gamma(1, 0, v{\cdot}v', \bar g_s(M_{Q'})) \ . \tag{3.78}$$

The advantage of method II over method I is that it does include the effects of running between M_Q and $M_{Q'}$. The disadvantage is that it neglects powers of $M_{Q'}/M_Q$. (Actually, the result can be improved by reincorporating the $M_{Q'}/M_Q$ dependence, as a power series expansion in this ratio).

For example, in method II eqn. (3.64) becomes, in leading order,[29]

$$\bar c\gamma^\mu b \to \left(\frac{\bar\alpha_s(m_b)}{\bar\alpha_s(m_c)}\right)^{a_I} \left(\frac{\bar\alpha'_s(m_c)}{\bar\alpha'_s(\mu)}\right)^{a_L} \bar c_{v'}[(1+\kappa)\gamma^\mu + (\lambda_b - \lambda_c(v{\cdot}v'))\not v\gamma^\mu]b_v \tag{3.79}$$

for $\Gamma = \gamma^\mu$, and

$$\bar{c}\gamma^\mu\gamma_5 b \to \left(\frac{\bar{\alpha}_s(m_b)}{\bar{\alpha}_s(m_c)}\right)^{a_I} \left(\frac{\bar{\alpha}'_s(m_c)}{\bar{\alpha}'_s(\mu)}\right)^{a_L} \bar{c}_{v'}[(1+\kappa)\gamma^\mu\gamma_5 - (\lambda_b + \lambda_c(v\cdot v'))\not{v}\gamma^\mu\gamma_5]b_v \tag{3.80}$$

for $\Gamma = \gamma^\mu\gamma_5$, where

$$\begin{aligned} \lambda_b = \frac{\alpha_s(m_b)}{3\pi}, \qquad \lambda_c(v\cdot v') &= \frac{2\alpha_s(m_c)}{3\pi} r(v\cdot v') \ , \\ a_L(v\cdot v') &= \frac{8}{33-2n_f}[v\cdot v' r(v\cdot v') - 1] \ , \\ r(x) &\equiv \frac{1}{\sqrt{x^2-1}} \ln\left(x + \sqrt{x^2-1}\right) \ , \end{aligned} \tag{3.81}$$

and κ is of order α_s but a subleading log.

Exercise 3.10 Why is the term involving κ in Eqs. (3.79) and (3.80) a subleading log, while those involving λ_b and λ_c are leading logs?

3.10. Form factors in order α_s

The predicted relations between form factors, and normalizations at $q^2_{\rm max}$, are only approximate. Indeed, several approximations were made in obtaining those results. Corrections that arise from subleading order in the $1/M$ expansion will be considered in Chapter 4. Here we will discuss corrections of order α_s.

As observed in Section 3.9, the vector and axial–vector currents of the full theory, $\bar{c}\Gamma b$, match onto a linear combination of 'currents', *i.e.*, dimension 3 operators, in the effective theory. At one loop, the correspondence between vector and axial currents in the full and effective theories is given by Eqs. (3.79) and (3.80). The constant λ_b and the function λ_c arise only from 1-loop matching, and are scheme independent. The constant κ receives contributions both from matching at 1-loop, and from 2-loops anomalous dimensions. Leaving out the latter would give a meaningless, scheme dependent, result. Although κ has been computed, it is interesting to note that predictions can be made solely form the 1-loop matching computation.

Indeed, comparing Eqs. (3.45) with Eqs. (3.41), we see that at zeroth order in $\bar{\alpha}_s(m_b)$ or $\bar{\alpha}_s(m_c)$ we have

$$a_+ + a_- = 0. \tag{3.82}$$

Plugging Eq. (3.80) into Eq. (3.42) we see that, to order $\bar{\alpha}_s(m_c)$ and $\bar{\alpha}_s(m_b)$ there is a computable correction to this combination of form factors, namely

$$\frac{a_+ + a_-}{a_+} = -4\frac{m_c}{m_b}\left[\frac{\bar{\alpha}_s(m_b)}{3\pi} + \frac{2\bar{\alpha}_s(m_c)}{3\pi} r(v\cdot v')\right] \tag{3.83}$$

The constant κ, although difficult to compute, does not change the relations between form factors since it simply rescales the leading order predictions in Eq. (3.45) by the common factor of $(1+\kappa)$. It does, however, affect the predicted normalization of form factors at $q^2_{\rm max}$. Since at $v'=v$ the effective vector current is again $\bar c_{v'}\gamma_\mu b_v$, but rescaled by $(1+\kappa+\lambda_b-\lambda_c(1))$, the correction to Eq. (3.48) is

$$f_\pm(q^2_{\rm max}) = (1+\kappa+\lambda_b-\lambda_c(1))\left(\frac{\alpha_s(m_b)}{\alpha_s(m_c)}\right)^{a_I}\left(\frac{M_D\pm M_B}{2\sqrt{M_BM_D}}\right). \qquad (3.84)$$

We emphasize that retaining the constant κ in Eq. (3.84) is inconsistent, because not all next to leading logs are included. For a calculation of the sub-leading logs, including the constant κ, see Ref. [39].

4. $1/M_Q$

4.1. The Correcting Lagrangian

One of the main virtues of the HQET is that, in contrast to *models* of the strongly bound hadrons, it lets us study systematically the corrections arising from the approximations we have made. To be sure, we've made several approximations already, even within the zeroth order expansion in Λ/M_Q. For example, we have computed the logarithmic dependence on M_Q, *i.e.,* the functions $C^{(i)}_\Gamma$ and $\widehat{C}^{(i)}_\Gamma$ of Eqs. (3.62) and (3.64), using perturbation theory. In this Section we turn to the corrections of order Λ/M_Q.

The HQET lagrangian was derived, in Section 3.2, by putting the heavy quark almost on-shell and expanding in powers of the residual momentum, k_μ, or light quark or gluon momentum, q_μ, over M_Q, which we generally wrote as Λ/M_Q. Let us again derive the effective lagrangian, keeping track, this time, of the terms of order Λ/M_Q.

We will rederive $\mathcal{L}^{(v)}_{\rm eff}$, including $1/M_Q$ corrections, working directly in configuration space[40]. The heavy quark equation of motion is

$$(i\not{D}-M_Q)Q=0 \qquad (4.1)$$

We can put the quark almost on shell by introducing the redefinition

$$Q=e^{-iM_Qv\cdot x}\tilde Q_v \qquad (4.2)$$

In terms of $\tilde{Q}_v$, the equation of motion is

$$[i\not{D} + M_Q(\not{v} - 1)]\tilde{Q}_v = 0 \tag{4.3}$$

If we separate the $(1+\not{v})$ and $(1-\not{v})$ components of $\tilde{Q}_v$, we see that, as expected, the latter is very heavy and decouples in the infinite mass limit. To project out the components,

$$\tilde{Q}_v = \tilde{Q}_v^{(+)} + \tilde{Q}_v^{(-)} \tag{4.4}$$

where

$$\tilde{Q}_v^{(\pm)} = \left(\frac{1 \pm \not{v}}{2}\right)\tilde{Q}_v \ , \tag{4.5}$$

we multiply Eq. (4.3) by $(\frac{1\pm\not{v}}{2})$. Thus we have the equations

$$iv{\cdot}D\tilde{Q}_v^{(+)} = -\left(\frac{1+\not{v}}{2}\right)i\not{D}\tilde{Q}_v^{(-)} \tag{4.6}$$

and

$$iv{\cdot}D\tilde{Q}_v^{(-)} + 2M_Q\tilde{Q}_v^{(-)} = \left(\frac{1-\not{v}}{2}\right)i\not{D}\tilde{Q}_v^{(+)} \tag{4.7}$$

These equations can be solved self-consistently by assuming that $\tilde{Q}_v^{(+)}$ is order $(M_Q)^0$ while $\tilde{Q}_v^{(-)}$ is order M_Q^{-1}. A recursive solution follows. From Eq. (4.7)

$$\tilde{Q}_v^{(-)} = \frac{1}{2M_Q}\left(\frac{1-\not{v}}{2}\right)i\not{D}\tilde{Q}_v^{(+)} - i\frac{v{\cdot}D}{2M_Q}\tilde{Q}_v^{(-)} \tag{4.8}$$

Substituting into Eq. (4.6) and dropping terms of order $1/M_Q^2$, we have

$$iv{\cdot}D\tilde{Q}_v^{(+)} = -\left(\frac{1+\not{v}}{2}\right)i\not{D}\frac{1}{2M_Q}\left(\frac{1-\not{v}}{2}\right)i\not{D}\tilde{Q}_v^{(+)} \tag{4.9}$$

The right hand side involves

$$\left(\frac{1+\not{v}}{2}\right)\not{D}\left(\frac{1-\not{v}}{2}\right)\not{D}\left(\frac{1+\not{v}}{2}\right) = \left(\frac{1+\not{v}}{2}\right)\left[D^2 - (v{\cdot}D)^2 + \tfrac{1}{2}g_s\sigma^{\mu\nu}G_{\mu\nu}\right]\left(\frac{1+\not{v}}{2}\right) \tag{4.10}$$

where $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$ and $G_{\mu\nu} = \frac{1}{ig_s}[D_\mu, D_\nu]$ is the QCD field strength tensor. This equation of motion is obtained from the lagrangian

$$\mathcal{L}_{\rm eff}^{(v)} = \bar{Q}_v iv{\cdot}DQ_v + \frac{1}{2M_Q}\bar{Q}_v\left[D^2 - (v{\cdot}D)^2 + \frac{g_s}{2}\sigma^{\mu\nu}G_{\mu\nu}\right]Q_v \tag{4.11}$$

Here I have reverted to the notation Q_v for $\tilde{Q}_v^{(+)}$. How to include higher order terms in the $1/M_Q$ expansion into $\mathcal{L}_{\rm eff}^{(v)}$ should be clear.

Exercise 4.1 Find the effective lagrangian describing a heavy scalar field to first order in $1/M$.

The $1/M_Q$ term in $\mathcal{L}_{\text{eff}}^{(v)}$ is treated as small. If it is not, it does not make sense to talk about a HQET in the first place. It is therefore appropriate to use perturbation theory to compute its effects. In this perturbative expansion, the corrections of order $1/M_Q$ to Green functions, and therefore to physical observables, are computed by making a single insertion of the perturbation

$$\Delta\mathcal{L} = \frac{1}{2M_Q}\bar{Q}_v\left[D^2 - (v\cdot D)^2 + \frac{g_s}{2}\sigma^{\mu\nu}G_{\mu\nu}\right]Q_v \ . \tag{4.12}$$

The symmetries of the HQET, discussed at length in Sections 3.1 and 3.3, are broken by $\Delta\mathcal{L}$. Under the $SU(N_f)$–flavor symmetry, $\Delta\mathcal{L}$ transforms as a combination of the Adjoint and Singlet representations, while only the chromo-magnetic moment operator

$$\bar{Q}_v\sigma^{\mu\nu}G_{\mu\nu}Q_v \tag{4.13}$$

breaks the spin-$SU(2)$ symmetry: it transforms as a 3 of spin–$SU(2)$.

A single insertion of $\Delta\mathcal{L}$ does include all orders in QCD, and it will often prove difficult to make precise calculations of $1/M_Q$ effects. Since $\Delta\mathcal{L}$ is treated as a simple insertion in Green functions, its treatment in the HQET is entirely analogous to that of current operators of Section 3.9. There are coefficient functions that connect the HQET results with the full theory. It is convenient to include them directly into the effective lagrangian as[40,41,42]

$$\Delta\mathcal{L} = \frac{1}{2M_Q}\bar{Q}_v\left[c_1 D^2 + c_2(v\cdot D)^2 + \tfrac{1}{2}c_3 g_s\sigma^{\mu\nu}G_{\mu\nu}\right]Q_v. \tag{4.14}$$

Here

$$c_i = c_i(M_Q/\mu, g_s) \tag{4.15}$$

can be determined through the methods discussed extensively in Section 3.9. In leading-log, one finds[40]

$$\begin{aligned} c_1 &= -1 \\ c_2 &= 3\left(\frac{\bar{\alpha}_s(\mu)}{\bar{\alpha}_s(M_Q)}\right)^{-8/(33-2n_f)} - 2 \\ c_3 &= -\left(\frac{\bar{\alpha}_s(\mu)}{\bar{\alpha}_s(M_Q)}\right)^{-9/(33-2n_f)} . \end{aligned} \tag{4.16}$$

4.2. The Corrected Currents

Just as the lagrangian is corrected in order $1/M_Q$, any other operator is too. In particular, the current operators studied in Section 3.9, are modified in this order. At tree level, these corrections are given by the change of variables of last Section:

$$J_\Gamma = \bar{q}\Gamma Q \to \bar{q}\Gamma e^{-iM_Q v\cdot x}\left[Q_v + \frac{1}{2M_Q}\left(\frac{1-\not{v}}{2}\right) i\not{D} Q_v\right] . \tag{4.17}$$

Beyond tree level, this sum of two terms has to be replaced by a more general sum over operators of the right dimensions and quantum numbers. The replacement is

$$J_\Gamma \to e^{-iM_Q v\cdot x}\left(\sum_i \widetilde{C}^{(i)}_\Gamma \bar{q}\Gamma_i Q_v + \frac{1}{2M_Q}\sum_j \widetilde{D}^{(j)}_\Gamma \mathcal{O}_j\right) \tag{4.18}$$

where $\mathcal{O}_j$ are operators of dimension 4 that include, for example, the operators

$$\bar{q}\Gamma i\not{D} Q_v \qquad , \qquad \bar{q}\Gamma i(v\cdot D)Q_v \qquad , \qquad \bar{q}\not{v}\Gamma i\not{D} Q_v \; . \tag{4.19}$$

A complete set of operators, and the corresponding coefficients, $\widetilde{D}^{(i)}_\Gamma$, for the cases $\Gamma = \gamma^\mu$ and $\Gamma = \gamma^\mu\gamma_5$, can be found in Refs. [41,43] in the leading-log approximation.

The case of two heavy currents is similar. A straightforward calculation gives

$$\begin{aligned} J_\Gamma = \bar{Q}'\Gamma Q \to & e^{-iM_Q v\cdot x + iM_{Q'} v'\cdot x}\Big[\bar{Q}'_{v'}\Gamma Q_v \\ & + \frac{1}{2M_Q}\bar{Q}'_{v'}\Gamma\left(\frac{1-\not{v}}{2}\right) i\not{D} Q_v + \frac{1}{2M_{Q'}}\bar{Q}'_{v'} i\overleftarrow{\not{D}}\left(\frac{1-\not{v}'}{2}\right)\Gamma Q_v\Big] \end{aligned} \tag{4.20}$$

Again, beyond tree level we must replace this expression by a more general sum over operators of dimension four,

$$\begin{aligned} J_\Gamma \to e^{-iM_Q v\cdot x + iM_{Q'} v'\cdot x}\Bigg[& \sum_i \widehat{C}^{(i)}_\Gamma \bar{Q}'_{v'}\Gamma_i Q_v + \frac{1}{2M_Q}\sum_j \widehat{D}^{(j)}_\Gamma \mathcal{O}_j \\ & + \frac{1}{2M_{Q'}}\sum_j \widehat{D}'^{(j)}_\Gamma \mathcal{O}_j \Bigg] \end{aligned} \tag{4.21}$$

It is worth pointing out that, in the computation of the coefficient functions $\widetilde{D}^{(j)}_\Gamma$, $\widehat{D}^{(j)}_\Gamma$ and $\widehat{D}'^{(j)}_\Gamma$, there is a contribution from the term of order $(1/M_Q)^0$. In computing the coefficient functions to order $1/M_Q$ one must not forget graphs with one insertion of the zeroth order term in the current and one insertion of the first order term in the HQET lagrangian.

4.3. Corrections of order m_c/m_b

In the case of semileptonic decays of a beauty hadron to charmed hadron, we introduced earlier an approximation method ("Method II" in Section 3.8) in which m_c/m_b was treated as a small parameter. Now, $m_c/m_b \sim 1/3$ and you may justifiably worry that this is not a good expansion parameter. We will see in this Section that the corrections are actually of the order of $\alpha_s/\pi(m_c/m_b)$ and therefore small. Moreover, they are explicitly calculable.

The strategy is[41] to look at those corrections of order $1/m_b$ which may be accompanied by a factor of m_c. In the first step of the approximation scheme we construct a HQET for the b-quark, treating the c-quark as light. We must, of course, keep terms of order $1/m_b$ in this first step. The second step is to go over to a HQET in which the m_c-quark is also heavy. For now, we care only about terms in this HQET that have positive powers of m_c.

In the first step, the hadronic current $\bar{c}\Gamma b$, with $\Gamma = \gamma^\mu$ or $\Gamma = \gamma^\mu\gamma_5$, is replaced according to Eq. (4.18). The question is, which terms in Eq. (4.18) can give factors of m_c when we replace the c-quark by a HQET quark, $c_{v'}$. Recall that, once we complete the second step, all of the m_c dependence is explicit. The answer is that any operators in Eq. (4.18) which have a derivative acting on the c-quark will give a factor of m_c. From Eq. (4.2) we see that a derivative $i\partial_\mu$ acting on the charm quark becomes, in the effective theory, the operation $m_c v'_\mu + i\partial_\mu$. So the prescription is simple: take J_Γ in Eq. (4.18) and replace

$$i\partial_\mu \to m_c v'_\mu \tag{4.22}$$

in those terms where $i\partial_\mu$ is acting on the charm quark.

For example, if the operator

$$\frac{1}{m_b}\bar{c}\, i\overleftarrow{\not{D}}\,\Gamma b_v \tag{4.23}$$

is generated at some order in the loop expansion, it gives an operator

$$-\frac{m_c}{m_b}\bar{c}_{v'}\not{v}'\Gamma b_v = -\frac{m_c}{m_b}\bar{c}_{v'}\Gamma b_v \tag{4.24}$$

after step two is completed.

It is really interesting to note that the resulting correction does not introduce any new unknown form factors. For example, the matrix element of (4.24) between a $\bar{B}$ and a D is given by Eq. (3.42) only with an additional factor of $-m_c/m_b$ in front.

The calculation described here has been performed in the leading-log approximation in Ref. [41]. The correction to the vector current is

$$\Delta V_\mu = \frac{m_c}{m_b}\bar{c}_{v'}(a_1\gamma_\mu + a_2 v_\mu + a_3 v'_\mu)b_v \tag{4.25}$$

where the coefficients $a_i = a_i(\mu)$, written in terms of

$$z = \frac{\bar{\alpha}_s(m_c)}{\bar{\alpha}_s(m_b)} \tag{4.26}$$

are

$$\begin{aligned}
a_1 &= \frac{5}{9}(v{\cdot}v' - 1) - \frac{1}{18}z^{-\frac{6}{25}} + \frac{2v{\cdot}v' + 12}{27}z^{-\frac{3}{25}} - \frac{34v{\cdot}v' - 9}{54}z^{\frac{6}{25}} - \frac{8}{25}v{\cdot}v' z^{\frac{6}{25}}\ln z \\
a_2 &= \frac{5}{9}(1 - 2v{\cdot}v') - \frac{13}{9}z^{-\frac{6}{25}} - \frac{44v{\cdot}v' - 6}{27}z^{-\frac{3}{25}} - \frac{14v{\cdot}v' - 18}{27}z^{\frac{6}{25}} \\
a_3 &= \frac{15}{9} - \frac{2}{3}z^{-\frac{3}{25}} - z^{\frac{6}{25}}
\end{aligned} \tag{4.27}$$

In particular, this gives a contribution to the form factor, at $v = v'$, of

$$\frac{m_c}{m_b}(a_1 + a_2 + a_3)|_{v{\cdot}v'=1} \simeq .07 \tag{4.28}$$

This is not negligible! It is reassuring that this type of corrections can be extracted explicitly. On the other hand, it should be remembered that both corrections of order $(m_c/m_b)^2$ and of subleading-log order can still be considerable and should be, but have not been, computed.

4.4. Corrections of order $\bar{\Lambda}/m_c$ and $\bar{\Lambda}/m_b$.

Corrections to the form factors for semileptonic decays of B's and Λ_b's that arise from the terms of order $1/m_c$ in the effective lagrangian Eq. (4.11) and the currents Eqs. (4.18) and (4.21) are, in principle, as large or larger than those considered in the previous Section. It is a welcome surprise that the corrections to the combination of form factors that contribute to the semileptonic decay vanish at the endpoint $v{\cdot}v' = 1$. Thus, the predicted normalization of form factors persists, although, as we will see, not so the relations between form factors.

The decay[44] $\Lambda_b \to \Lambda_c e\nu$ is simpler to analyze than the decays[45] $\bar{B} \to De\nu$ and $\bar{B} \to D^* e\nu$. Moreover, it turns out that for the baryonic decay some relations between form factors survive at this order. For these reasons, we will present here the baryonic case. We will briefly return to the decay of the meson at the end of this section, where we will describe the result.

There are two types of corrections to consider[44], coming from either the modified lagrangian of from the modified current. We start by considering the former. The c_1 and c_2 terms in the effective lagrangian (4.14) transform trivially under the spin symmetry, contributing to the form factors in the same proportion as the leading term in Eq. (3.49). This effectively renormalizes the function ζ but does not affect relations between form factors.

Moreover, the normalization at the symmetry point $v{\cdot}v' = 1$ is not affected. This is a straightforward application of the Ademollo-Gatto theorem. If j_μ is a symmetry generating current of a hamiltonian H_0, then corrections to the matrix element of the current, at zero momentum, from a symmetry-breaking perturbation to the hamiltonian, ϵH_1, are of order ϵ^2. In the case at hand the Ademollo-Gatto theorem implies that corrections to the normalization of ζ at the symmetry point are of order $(1/m_c)^2$.

The chromomagnetic moment operator in the lagrangian (4.14) does not give a contribution at all. The spin symmetries imply

$$\begin{aligned} &\langle\Lambda_c(v',s')|\mathrm{T}\int d^4x\,(\bar{c}_{v'}\sigma^{\mu\nu}G_{\mu\nu}c_{v'})(x)(\bar{c}_{v'}\Gamma b_v)(0)|\Lambda_b(v,s)\rangle \\ &\qquad = \zeta_{\mu\nu}(v,v')\bar{u}^{(s')}(v')\sigma^{\mu\nu}\left(\frac{1+\not{v}}{2}\right)\Gamma u^{(s)}(v). \end{aligned} \tag{4.29}$$

The function $\zeta_{\mu\nu}$ must be an antisymmetric tensor and must therefore be proportional to $v'_\mu v_\nu - v'_\nu v_\mu$. But

$$\left(\frac{1+\not{v}'}{2}\right)\sigma^{\mu\nu}\left(\frac{1+\not{v}'}{2}\right)v'_\mu = 0. \tag{4.30}$$

This, we see, is an enormous simplification. There is no analogous simple reason for the matrix element of the chromomagnetic moment operator to vanish in the case of a meson transition. The chromomagnetic matrix element gives, in that case, uncalculable corrections to the relations between form factors.

We turn next to the contribution from the modification to the current. We need the matrix element of the local operators of order $1/m_c$ in Eq. (4.21). These operators are all constructed of one derivative acting on either heavy quark in the quark bilinear. Consider the matrix element

$$\langle\Lambda_c(v',s')|\bar{c}_{v'}i\overleftarrow{D}_\mu\Gamma b_v|\Lambda_b(v,s)\rangle = \bar{u}^{(s')}(v')\Gamma u^{(s)}(v)[Av_\mu + Bv'_\mu], \tag{4.31}$$

where the form of the right hand side follows again from the spin symmetries. The form factors A and B are not independent. Rather, they are given in terms of ζ. To see this, note that, contracting with v'_μ and using the equations of motion,

$$B = -v{\cdot}v' A. \tag{4.32}$$

Also, if the mass of the $l = 1/2$ state in the effective theory is $\bar\Lambda$, then

$$\langle \Lambda_c(v', s')| i\partial_\mu(\bar c_{v'} \Gamma b_v)|\Lambda_b(v, s)\rangle = \bar\Lambda(v_\mu - v'_\mu)\zeta(v{\cdot}v')\bar u^{(s')}(v')\Gamma u^{(s)}(v). \tag{4.33}$$

Contracting with v_μ, using the equations of motion and Eq. (4.32) we have

$$A(1 - (v{\cdot}v')^2) = \bar\Lambda(1 - v{\cdot}v')\zeta \tag{4.34}$$

Therefore, the matrix element of interest is

$$\langle \Lambda_c(v', s')|\bar c_{v'} i\overleftarrow{\not{D}}\,\Gamma b_v|\Lambda_b(r, s)\rangle = \bar\Lambda\zeta(v{\cdot}v')\frac{v_\nu - (v{\cdot}v')v'_\nu}{1 + v{\cdot}v'}\bar u^{(s')}(v')\gamma^\nu \Gamma u^{(s)}(v) \tag{4.35}$$

where $\Gamma = \gamma^\mu$ or $\gamma^\mu\gamma_5$. Putting it all together, using the leading log expression for the coefficients $\widehat{D}^{(j)}_\Gamma$ in the current of Eq. (4.21), one finds

$$\begin{aligned} F_1 &= G_1\left[1 + \frac{\bar\Lambda}{m_c}\left(\frac{1}{1 + v{\cdot}v'}\right)\right] \\ F_2 &= G_2 = -G_1\frac{\bar\Lambda}{m_c}\left(\frac{1}{1 + v{\cdot}v'}\right) \\ F_3 &= G_3 = 0 \end{aligned} \tag{4.36}$$

Moreover,

$$G_1(1) = \left(\frac{\bar\alpha_s(m_b)}{\bar\alpha_s(m_c)}\right)^{a_I} \tag{4.37}$$

as before. Up to an unknown constant, $\bar\Lambda$, there are still five relations among six form factors. We can estimate $\bar\Lambda$ by writing $\bar\Lambda = M_{\Lambda_c} - m_c = (M_{\Lambda_c} - M_D) + (M_D - m_c)$. If the 'constituent' quark mass in the D meson is $\simeq 300MeV$, then $\bar\Lambda \simeq 700MeV$. With this, we can estimate the next order corrections to be of the order of $\left(\bar\Lambda/2m_c\right)^2 \sim 5\%$. There are, of course, additional computable corrections, of order $\bar\Lambda/2m_b$ and $\alpha_s(m_c)/\pi\left(\bar\Lambda/2m_c\right)$.[46]

The result of $1/m_c$ corrections to the mesonic transitions is quite different. There both the matrix elements of the correction to the current and of the time order product with the chromo-magnetic moment operator lead to new form factors. The result is that there are incalculable corrections, of order $\bar\Lambda/2m_c$, to all the leading order relations between form factors. Even if $\bar\Lambda$ is smaller in this case, presumably $\bar\Lambda \sim 300$MeV, these corrections may be large, say 10%–20%. Remarkably, at the symmetry point, $v'{\cdot}v = 1$, there are no corrections of order $\bar\Lambda/2m_c$ to the leading order predictions. Thus, one may still extract the mixing angle $|V_{cb}|$ with high precision from measurements at the end of the spectrum of the semileptonic decay rates for $B \to De\nu$ and $B \to D^*e\nu$.

5. Inclusive Semileptonic B-Meson Decays.

Well before the development of HQETs, it was commonly held that the inclusive rate of decay of a heavy meson should be well approximated by the decay rate of the heavy quark. Intuitively, the heavy quark just sits at rest (in the rest frame of the meson) surrounded by 'brown muck' for which it acts as a static source of color. When the heavy quark decays, the complications of strong interactions come in, dictating how the rate is divided up between exclusive channels, but the sum total must be the quark's rate of decay.

With the advent of HQET and HQ symmetries one may prove the validity of this assertion. Moreover, and more importantly, this is done by setting up an expansion for the decay rate in inverse powers of the heavy mass. With a systematic expansion one may investigate the accuracy of the result for the not-really-so-heavy D and B mesons.

5.1. Kinematics.

The decay rate is

$$d\Gamma = \frac{\kappa}{2M} \sum_X |\langle X|j_\mu|\bar{B}\rangle \ell^\mu|^2 d\Phi_X \tag{5.1}$$

where j_μ is the hadronic $\Delta B = -1$ current,

$$d\Phi_X \propto \delta^{(4)}(P - p_X - p_e - p_{\bar\nu})\, \Pi_i \frac{d^3p_i}{2E_i} \tag{5.2}$$

is the phase space for final states and

$$\ell_\mu = \bar{v}(p_\nu)\gamma_\mu(1-\gamma_5)u(p_e) \tag{5.3}$$

is the lepton current. The factor κ includes the coupling constants $G_F^2|V_{qb}|^2$.

All of the interesting dynamics is in the hadronic tensor

$$h_{\mu\nu} = \sum_X (2\pi)^3\delta^{(4)}(P - p_x - q)\langle\bar{B}|j_\nu^\dagger(0)|X\rangle\langle X|j_\mu(0)|\bar{B}\rangle \tag{5.4}$$

where the sum symbol includes the integral over phase space for the final state X.

We will investigate the properties of the hadronic tensor $h_{\mu\nu}$ indirectly, by studying the related tensor

$$T_{\mu\nu} = i\int d^4x\, e^{-iq\cdot x}\langle\bar{B}(p)|T(j_\nu^\dagger(x)j_\mu(0))|\bar{B}\rangle \tag{5.5}$$

Both tensors can be written in terms of Lorentz invariant form factors, thus:

$$h_{\mu\nu} = g_{\mu\nu} h_1 + p_\mu p_\nu h_2 + \cdots \qquad (5.6a)$$

$$T_{\mu\nu} = g_{\mu\nu} T_1 + p_\mu p_\nu T_2 + \cdots \qquad (5.6b)$$

Exercise 5.1 I have intentionally left the list of form factors incomplete. The ellipsis in Eqs. (5.6) stand for a finite number of terms. Ennumerate them. There is one term that can be eliminated by the requirement of time reversal invariance of the strong interactions. Which one is it?

The form factors T_i are more amenable to theoretical investigations because $T_{\mu\nu}$ is a Green function. The h_i form factors can be obtained from these by cutting:

$$h_i = \frac{1}{\pi}\mathrm{Im}T_i \qquad \text{for} \qquad p\cdot q < m_B^2 \qquad (5.7)$$

Exercise 5.2 To see this,

(i) Separate the integral in Eq. (5.5) into $x^0 > 0$ and $x^0 < 0$ parts. Insert a complete set of states between the two currents, and make the dependence on x explicit.

(ii) Carry out the integrations, using, eg,

$$\int d^4x\, \theta(x^0) e^{i(P-p_X-q)\cdot x} = (2\pi)^3 \delta(\vec{P} - \vec{p}_X - \vec{q}) \frac{i}{P^0 - p_X^0 - q^0 + i\epsilon}$$

and use the standard identity

$$\frac{1}{z \pm i\epsilon} = \mathrm{PP}\frac{1}{z} \mp i\pi\delta(z) \ ,$$

where PP stands for the principal part, to separate "imaginary" from "real" parts (these are in quotation marks because by them I just mean the parts that come from the PP terms and from the δ term, respectively).

(iii) In the "imaginary" part you will recognize one term is just $h_{\mu\nu}$. Write out explicitly the other term in the "imaginary" part and show that it vanishes if $P\cdot q < m_B$. (Hint: What is the lowest mass of the intermediate state X?).

(iv) Complete the proof by relating form factors and taking, where appropriate, the real or imaginary parts.

5.2. The Analytic Structure of The Hadronic Green Function

The hadronic form factors T_i and h_i are Lorentz invariant functions of the momenta P and q. There are only two invariant variables, namely $\hat{Q} \equiv \sqrt{q^2/m_B^2}$ and $z \equiv P\cdot q/m_B^2$. Here and throughout a hat on a dimensionful quantity means taking out the dimensions by apropriate powers of m_B. We will study the behaviour of $T_{\mu\nu}$ for fixed $\hat{Q}^2$ as a function of z. This variable is of interest because $1-z$ measures how far one is from the resonant region. To see this, consider any final state X contributing to the inclusive decay, so $q = P - p_X$. Then $z = 1 - P\cdot p_X/m_B^2$, or, in the rest frame of the decaying meson, $1 - z = E_X/m_B$. Only states with small energy contribute to the region of $z \approx 1$, and only single resonances contribute when z is large enough.

Exercise 5.3 Show that the physical region extends from $z = 0$ (neglect the lepton mass) to

$$z = \frac{1}{2}(1 + \hat{Q}^2 - \hat{m}_\pi^2) \equiv z_0$$

Since $T_{\mu\nu}$ is a Green function we can study its behaviour as a function of complex momenta. In particular we can hold $\hat{Q}$ fixed and study $T_i(z)$ as functions of complex z. What do we know about the complex z plane? You established above that the segment of the real line $(0, z_0)$ corresponds to the physical region for the inlcusive semileptonic $\bar{B}$ decay. And we know that the right end of the segment corresponds to the resonance region. See Fig. 8.

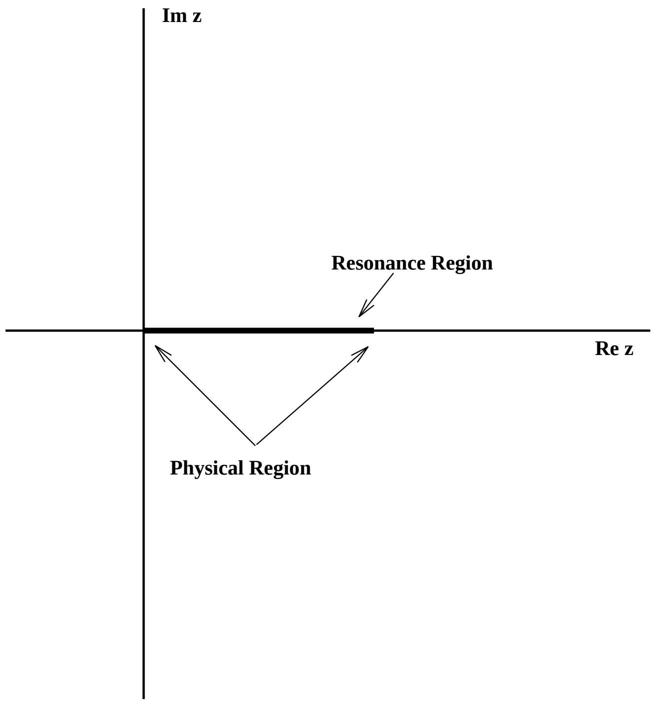

Figure 8 *The location of the physical and resonant regions in the complex z-plane. The variable z is the energy of the final hadronic system in the semileptonic B decay, in the B restframe.*

Associated with the physical process there is a cut in the z plane. The cut extends from $z_0 = \frac{1}{2}(1 + \hat{Q}^2 - \hat{m}_\pi^2)$ to $-\infty$ on the real axis. There is, in addition, a cut that extends from $z_1 = \frac{1}{2}(3 - \hat{Q}^2)$ out to $+\infty$ on thereal axis. See Fig. 9.

Exercise 5.4 What are the physical processes that correspond to the cuts in the z plane?

If the presence and location of these cuts does not seem obvious, the reader may use Landau conditions[4] to determine the analytic properties of a Green function in

[4] See, for example, Ref. [31]

perturbation theory. She will then find the same cuts, save for the fact that the masses will turn out to be those of quarks rather than of the physical bound states.

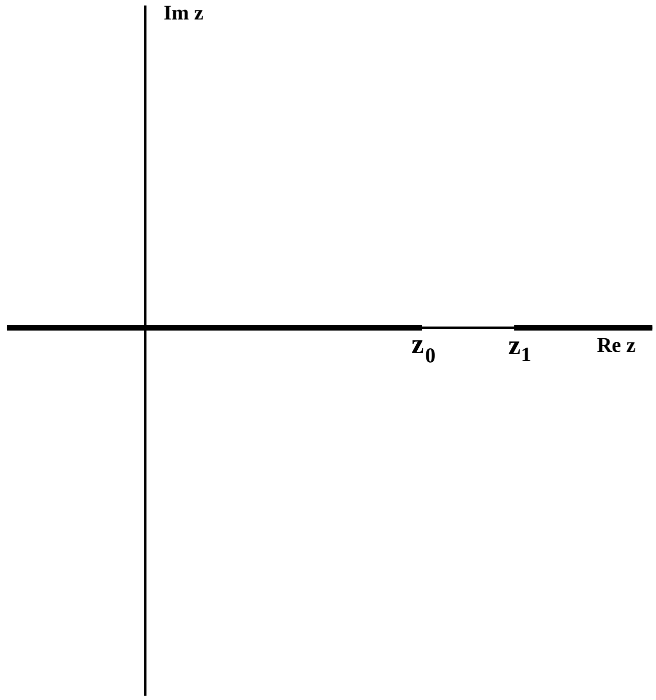

Figure 9 *The location of the cuts in the complex z-plane.*

As will become evident later, a perturbative computation will be trustworthy provided z is nowhere near the resonance region. We should stay away from the resonance region by at least a typical hadronic scale, say the rho mass, m_ρ, or the chiral symmerty breaking scale, Λ_χ. This is why the problem at hand is non-trivial: to compute the form factors h_i from the imaginary part of a Green function one needs to deal with the resonant region where perturbation theory is not valid.

The situation is much better if we are willing to settle for slightly less, namely for averages over the variable z of the decay rate (at fixed $\hat{Q}$). Integrating both sides of Eq. (5.7) one has

$$\begin{aligned}\int_0^{z_0} dz\ h_i(z) &= \int_0^{z_0} dz\ \frac{1}{\pi}\mathrm{Im}T_i(z) \\ &= \frac{1}{\pi}\int_C dz\ T_i(z)\end{aligned} \tag{5.8}$$

where the contour C runs above the cut from 0 to z_0, around the branch point and back to 0 under the cut; see Fig. 10.

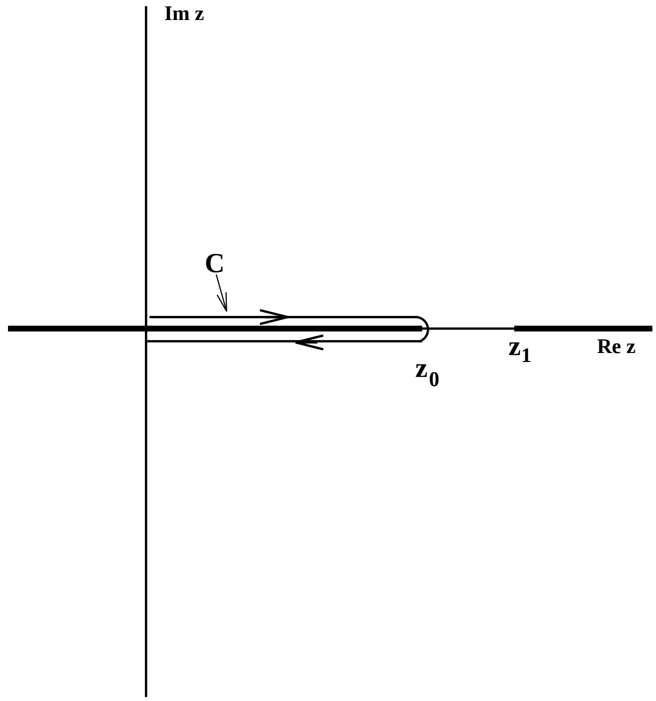

Figure 10 *Contour of integration, C, in the dispersion relation (5.8).*

We can trade the contour C for a new one C' that stays away from the resonance region. C' is restircted to start just above the origin of the complex z plane and to finish just under it. This is justified because there are no singularities other than the cuts. See Fig 11. Provided z_0 and z_1 are well separated there is no problem getting the contour C' to lie well away from the cut, exept in the vicinity of $z = 0$. But in particular one can have the cut away from the resonance region in its entirety, and one may apply perturbation theory to the computation of the Green function.

Exercise 5.5 Under what circumstances can the cuts meet? In other words, determine the kinematic regeme for which $z_1 < z_0$. What does this imply for inclusive $b \to ce\bar{\nu}$ and $b \to ue\bar{\nu}$?

The only thing missing is a means of computing the Green function perturbatively. The problem is that we need to compute a matrix element between meson states, and this usually involves complicated dynamics. As we will see in the next section it is possible to use the HQET and the HQ symmetries to compute the matrix element. Before turning to that problem, we conclude with a simple observation that extends the above result. The average in Eq. (5.8) uses one of many possible measures, namely simply dz. One could just as well consider

$$\int_0^{z_0} dz\, f(z) h_i(z) \tag{5.9}$$

Clearly we want the function $f(z)$ to be regular on the region of integration. More generally we would like its singularities in the complex z plane to be as far away from the contour C' as posible. This suggest having them at infinity, ie, using an entire function. Particularly interesting is the set of functions $f_n(z) = (z_0 - z)^n$, which measure the relative importance of the non-resonant and resonant regions.

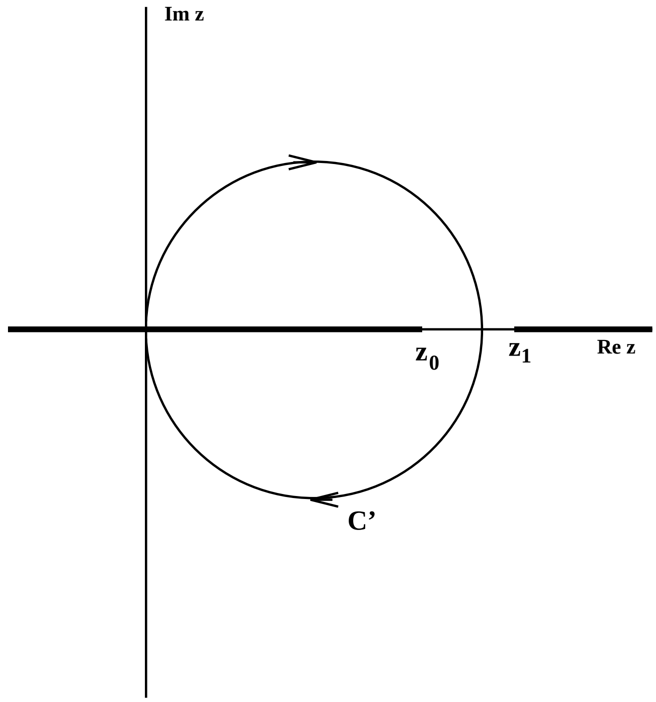

Figure 11 *Deformed contour of integration, C', in the dispersion relation (5.8). Note that it stays away from the physical region everywhere except near z=0, provided Z_0 and z_1 are well separated.*

5.3. An HQET based OPE

The Operator Product Expansion (OPE) has been used with great succes to separate perturbative from non-perturbative effects in deep inelastic nucleon scattering. The coefficient functions may be computed perturbatively, while the non-perturbative information is encoded in the matrix elements of the operators.

The common OPE cannot be applied to $T_{\mu\nu}$. The problem is that it is an expansion in inverse powers of Q. The matrix elements of operators of heavy mesons can in fact grow with powers of the heavy mass m_B. Since in the physical region $Q < m_B$, the ratio m_B/Q is not a good expansion parameter.

But there is a way one can produce a useful OPE for $T_{\mu\nu}$. The idea is simple: expand in inverse powers of the largest scale around, namely m_B. Clearely we should be using the techniques of the HQET to produce such an expansion. Let $P = m_B v$.

Consider any matrix element (physical or unphysical) of the time ordered product $T(j_\nu^\dagger(x)j_\mu(0))$. The idea is to take the momentum of the b quark to be $p = m_b v + k$ and to expand in inverse powers of m_b.

Consider, in particular, the matrix element between b quarks of momentum p. The tree level Feynman graph is in Fig. 12. We insist that the momentum in the intermediate light quark, $q = u$ or c, be $m_b v + \cdots$ and expand in inverse powers of the mass. Arbitrary quantities, like Q^2, may scale with m_b (so we hold, eg, $\hat{Q}^2$ fixed as $m_b \to \infty$).

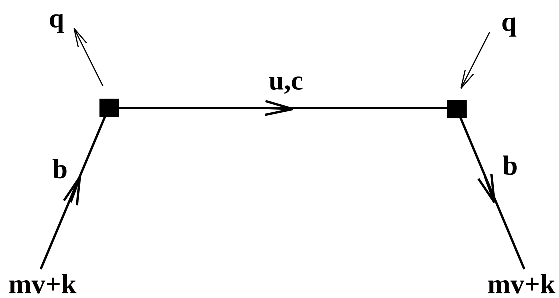

Figure 12 *Tree level Green function for the product of currents.*

To see how the expansion works, let us compute the Feynamn diagram of Fig. 12. To be concise we consider the current $j_\Gamma = \bar{q}\Gamma b$, where Γ stands for any Dirac gamma matrix. Then we have the tree level Green function and its expansion:

$$\begin{aligned}\Gamma\frac{i}{m_b\not{v} + \not{k} - \not{q} - m_q}\Gamma &= \frac{1}{m_b}\Gamma\frac{i}{\not{v} - \hat{\not{q}} - \hat{m}_q + \not{k}/m_b}\Gamma \\ &= \frac{1}{m_b}\Gamma\frac{i}{\not{v} - \hat{\not{q}} - \hat{m}_q}\left[1 - \frac{\not{k}}{m_b}\frac{1}{\not{v} - \hat{\not{q}} - \hat{m}_q} + \cdots\right]\Gamma\end{aligned} \tag{5.10}$$

This matrix element can then be used to determine the coefficients of operators in an OPE-like relation between operators:

$$\begin{aligned}&\int d^4x\, e^{-iq\cdot x}T(\bar{b}\Gamma q(x)\; \bar{q}\Gamma b(0)) \\ &\quad = \frac{1}{m_b}\left[\bar{b}_v\Gamma\frac{i}{\not{v} - \hat{\not{q}} - \hat{m}_q}\Gamma b_v - \frac{1}{m_b}\bar{b}_v\Gamma\frac{i}{\not{v} - \hat{\not{q}} - \hat{m}_q}i\not{D}\frac{1}{\not{v} - \hat{\not{q}} - \hat{m}_q}\Gamma b_v + \cdots\right]\end{aligned} \tag{5.11}$$

where the ellipsis stand for an expansion in powers of $1/m_b$ which has operators of ever increasing dimensions. Going beyond tree level will modify the specific coefficient functions but the form of the expansion will remain practically the same (some new operators may be introduced beyond tree level).

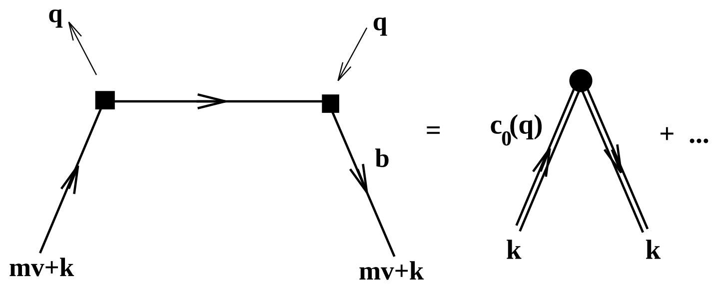

Figure 13 *Representation of the combined OPE-HQET at tree level.*

Note that the right hand side of Eq. (5.11) is written in terms of the quark operators of the HQET, while the left hand side involves the original quark fields. This is what makes this OPE different from the common one (used, say, in deep inelastic scattering). And there is no way out of it: since we are expanding the full theory time ordered product in powers of $1/m_b$, we must use HQET fields for the result. A useful pictorial representation of this is presented in Fig. 13, which shows the HQET-OPE at tree level. The diagrams on the right hand side of the equation have double lines, reminding us they represent quarks in the HQET.

The coefficients in the OPE of Eq. (5.11) have poles in $z - z_0$ of increasingly higher order. Symbolically we may write for the expansion of the operator product

$$\frac{1}{m_b}\sum_n \frac{1}{(2m_b)^n} c_n(\hat{Q}^2, z)\mathcal{O}^{(n)} \tag{5.12}$$

with coeficient functions that behave as

$$c_n \sim \frac{1}{(z - z_0)^{n+1}} \tag{5.13}$$

Beyond tree level there will be cuts in additions to these poles. Although the nature of the singularities change, the discussion that follows remains essentially the same.

The first two terms in the OPE expansion of $T_{\mu\nu}$ can be computed! The first one involves the matrix element between $\bar{B}$-mesons of a dimension-3 operator. But this is fixed by spin/flavor symmetries:

$$\langle\overline{B}|\bar{b}_v\Gamma(\not{v} + \not{\hat{q}} + \hat{m}_q)\Gamma b_v|\overline{B}\rangle = -\xi(1)\mathrm{tr}\overline{\widetilde{B}}(v)\Gamma(\not{v} + \not{\hat{q}} + \hat{m}_q)\Gamma\widetilde{B}(v) \tag{5.14}$$

with $\xi(v\cdot v')$ the Isgur-Wise function, $\xi(1) = 1$. The second one involves a dimension-4 operator. The matrix element of this vanishes:

$$\langle\overline{B}|\bar{b}_v\Gamma D_\mu b_v|\overline{B}\rangle = 0 \tag{5.15}$$

There are therefore no corrections of order $1/m_b$. More on this below.

Exercise 5.6 Prove Eq. (5.15). *Hint:* Consult section 4.4.

We see that one can compute the rate for inclusive semiletonic B decay. With the tree level coefficient functions just computed, the matrix elements in Eqs. (5.14) and (5.15), and neglecting higher order terms in $1/m_b$, one obtains for the decay rate the same result as if computing the decay rate of a free b quark!

Exercise 5.7 Prove this assertion. In fact, we can do even better because we need not integrate over Q^2. Prove that the moments

$$\int_0^{z_0} dz\ (z_0 - z)^n \frac{d\Gamma(B \to Xe\nu)}{\mathrm{d}z\ \mathrm{d}Q^2}$$

are the same as for the free quark decay. **This is an important exercise.** You will have to put together all the pieces of the argument discussed above, and then more.

The deviations from the free decay rate are parameterized by the matrix elements of the operators of dimension 5 and higher. In this sense we have managed to separate long from short distance effects.

Note that for z on the real axis, the expansion breaks down as $z \to z_0$. More precisely, in this limit the coefficients c_n of Eq. (5.12) grow arbitrarily large. This is in accord with the statement we had made previously that the perturbative calculation of the Green function breaks down in the resonant region, $z \sim z_0$.

A physically interesting choice of the measure f in Eq. (5.9) is $f_1 = z - z_0$, because it measures the deviation from the free quark result. The quantity $m_b^2 f_1(v \cdot q/m_b)$ is the invariant mass-squared of the hadronic final state minus m_q^2. If the final state quark produced by the b decay is on its mass shell, then $f_1 = 0$.

Exercise 5.8 Prove that with $f_1 = z - z_0$ the contribution from the parton model to the average in Eq. (5.9) vanishes. Discuss corrections to this result arising from the nonperturbative terms and from the perturbative corrections to the coefficients in the expansion of Eq. (5.12).

Exercise 5.9 $|V_{ub}|$ is determined from a measurement of inclusive charmles semileptonic B decay. Since $V_{ub}| \ll |V_{cb}|$, the rate for $B \to Xe\bar{\nu}$ is dominated by the decay into charm. Short of reconstructing the hadronic state X, one is forced to consider the spectrum of electron energies, E_e. Find the maximum electron energies, $E_e^{c,\max}$ and $E_e^{u,\max}$, in the decay into charm and charmless final states, respectively. What does the region $E_e^{c,\max} < E_e < E_e^{u,\max}$, correspond to in terms of our variables z and Q?

6. Chiral Lagrangian with Heavy Mesons

6.1. Generalities

Chiral symmetry and soft pion theorems have been used in particle physics for several decades now with great success. The most efficient way of extracting information from chiral symmetry is by writing a phenomenological lagrangian for pions that incorporates both the explicitly realized vector symmetry and the non-linearly realized spontaneously broken axial symmetry.[47] Theorems that simultaneously use heavy quark symmetries and chiral symmetries are most expediently written by means of a phenomenological lagrangian for pions and heavy mesons that incorporates these symmetries.[48,49]

In the limit $m_b \to \infty$, the $\overline{B}$ and the $\overline{B}^*$ mesons are degenerate, and to implement the heavy quark symmetries it is convenient to assemble them into a "superfield" $H_a(v)$:

$$H_a(v) = \frac{1+\not{v}}{2}\left[\overline{B}_a^{*\mu}\gamma_\mu - \overline{B}_a\gamma^5\right] . \tag{6.1}$$

Here v^μ is the fixed four-velocity of the heavy meson, and a is a flavor $SU(3)$ index corresponding to the light antiquark. Because we have absorbed mass factors $\sqrt{2m_B}$ into the fields, they have dimension 3/2; to recover the correct relativistic normalization, we will multiply amplitudes by $\sqrt{2m_B}$ for each external $\overline{B}$ or $\overline{B}^*$ meson.

The chiral lagrangian contains both heavy meson superfields and pseudogoldstone bosons, coupled together in an $SU(3)_L \times SU(3)_R$ invariant way. The matrix of pseudogoldstone bosons appears in the usual exponentiated form $\xi = \exp(\mathrm{i}\mathcal{M}/f)$, where

$$\mathcal{M} = \begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi_0 + \frac{1}{\sqrt{6}}\eta & K^0 \\ K^- & \overline{K}^0 & -\sqrt{\frac{2}{3}}\,\eta \end{pmatrix} , \tag{6.2}$$

and f is the meson decay constant. The bosons couple to the heavy fields through the covariant derivative and axial vector field,

$$D^\mu_{ab} = \delta_{ab}\partial^\mu + V^\mu_{ab} = \delta_{ab}\partial^\mu + \tfrac{1}{2}\left(\xi^\dagger\partial^\mu\xi + \xi\partial^\mu\xi^\dagger\right)_{ab} ,$$
$$A^\mu_{ab} = \frac{\mathrm{i}}{2}\left(\xi^\dagger\partial^\mu\xi - \xi\partial^\mu\xi^\dagger\right)_{ab} = -\frac{1}{f}\partial_\mu\mathcal{M}_{ab} + \mathcal{O}(\mathcal{M}^3) .$$

Lower case roman indices correspond to flavor $SU(3)$. Under chiral $SU(3)_L \times SU(3)_R$, the pseudogoldstone bosons and heavy meson fields transform as

$$\begin{aligned} \xi &\to L\xi U^\dagger = U\xi R^\dagger \\ A^\mu &\to U A^\mu U^\dagger \\ H &\to H U^\dagger \\ (D^\mu H) &\to (D^\mu H) U^\dagger \end{aligned}$$

where the matrix U is a nonlinear function of the pseudogoldstone boson matrix $\mathcal{M}$, implicitly defined by the transformation law for ξ. Then $\Sigma = \xi^2$transforms as follows:

$$\Sigma \to L \Sigma R^\dagger \ .$$

The effective lagrangian is an expansion in derivatives and in inverse powers of the heavy quark mass. The kinetic energy terms take the form

$$\mathcal{L}_{\rm kin} = \frac{1}{8} f^2\, \partial^\mu \Sigma_{ab}\, \partial_\mu \Sigma^\dagger_{ba} - {\rm Tr}\left[\overline{H}_a(v) iv \cdot D_{ba} H_b(v)\right] \ .$$

Here the trace is in the space of 4×4 Dirac matrices that define the "superfields" $H_a(v)$ in Eq. (6.1). The leading interaction term is of dimension four,

$$\mathcal{L}_{\rm int} = g\, {\rm Tr}\left[\overline{H}_a(v) H_b(v)\, \not{A}_{ba} \gamma_5\right] \ , \tag{6.3}$$

where g is an unknown parameter, of order one in the constituent quark model. The analogous term in the charm system is responsible for the decay $D^* \to D\pi$. Expanding the term in the lagrangian in (6.3) to linear order in the Goldstone Boson fields, $\mathcal{M}$, we find the explicit forms for the $D^* D \mathcal{M}$ and $D^* D^* \mathcal{M}$ couplings

$$\left[\left(\frac{-2g}{f}\right) D^{*\nu} \partial_\mu \mathcal{M} D^\dagger + {\rm h.c.}\right] + \left(\frac{2gi}{f}\right) \epsilon_{\mu\nu\lambda\kappa} D^{*\mu} \partial^\nu \mathcal{M} D^{*\lambda} v^\kappa \ . \tag{6.4}$$

Using this one can compute the partial width

$$\begin{aligned} \Gamma(D^{*+} \to D^0 \pi^+) &= \frac{g^2}{6\pi f^2} |\vec{p}_\pi|^3 \\ \Gamma(D^{*+} \to D^+ \pi^0) &= \frac{g^2}{12\pi f^2} |\vec{p}_\pi|^3 \end{aligned}$$

The ACCMOR collaboration has reported an upper limit of 131 KeV on the D^* width.[50] The branching fractions for $D^{*+} \to D^0 \pi^+$ and $D^{*+} \to D^+ \pi^0$ are

$(68.1 \pm 1.0 \pm 1.3)\%$ and $(30.8 \pm 0.4 \pm 0.8)\%$, respectively, as measured by the CLEO collaboration.[51] Using $f = 130$ MeV, one obtains the limit $g^2 < 0.5$. Even if the D^* decay width is too small to measure, radiative D^* decays provide an indirect means for determining the coupling g, and provide a lower bound $g^2 \gtrsim 0.1$.[52]

Since charmed and beauty baryons are long lived, one can write down phenomenological lagrangians for their interactions with pions. These are as well justified and should be as good an approximation as the lagrangian for heavy mesons discussed above. The treatment is rather similar, and we refer the interested reader to the literature.[53]

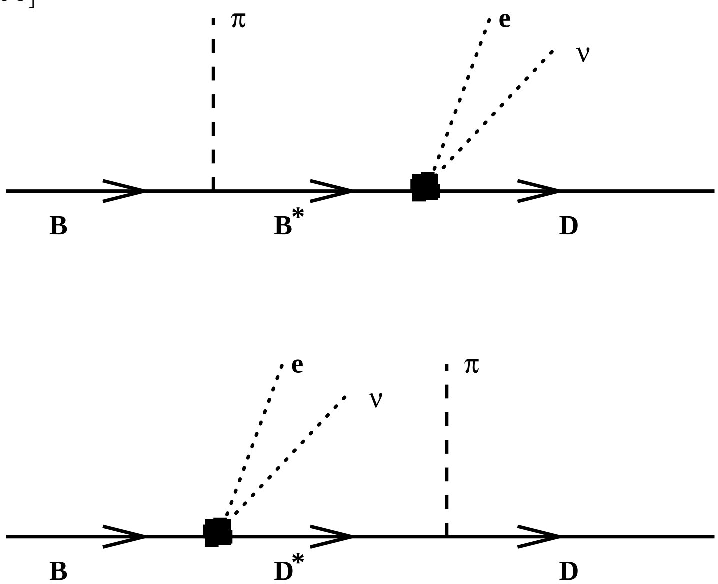

Figure 14 *Feynman diagrams for* $B \to De\nu$

6.2. $B \to De\nu$ *And* $B \to D^*\pi E\nu$

As a first example of an application consider a soft pion theorem that relates the amplitudes for $B \to D^* e\nu$ and $B \to D^*\pi e\nu$, for small pion momentum.[49] The heavy quark current is represented in the phenomenological lagrangian approach by

$$J^{\bar{c}b}_\mu = \bar{h}^{(c)}_{v'}\gamma_\mu(1-\gamma_5)h^{(b)}_v \to -\xi(v{\cdot}v')\mathrm{Tr}\overline{H}^{(c)}_a(v')\gamma_\mu(1-\gamma_5)H^{(b)}_a(v) + \cdots \qquad (6.5)$$

where the ellipsis denote terms with derivatives, factors of light quark masses m_q, or factors of $1/M_Q$, and $\xi(v{\cdot}v')$ is the Isgur-Wise function.[5] The leading term in

[5] The symbol ξ is traditionally used both for the Isgur-Wise function and for the exponential of the meson fields. To distinguish between them, whenever context may not be sufficient, we denote the former as a function of velocities, $\xi(v{\cdot}v')$.

Eq. (6.5) is independent of the pion field. Therefore, it is pole diagrams that dominate the amplitude for semileptonic $B \to D\pi$ and $B \to D^*\pi$ transitions; see Fig. 14. These pole diagrams are calculable in this approach, and are determined by the Isgur-Wise function and the coupling g.

A straightforward calculation gives[49]

$$\begin{aligned}\langle D(v')\pi^a(q)|J_\mu^{\bar{c}b}|B(v)\rangle = iu(B)^* \tfrac{1}{2}\tau^a u(D)\sqrt{M_B M_D}\frac{g}{f}\xi(v{\cdot}v') \\ \times \left\{ \frac{1}{v{\cdot}q}[i\epsilon_{\mu\nu\lambda\kappa}q^\nu v'^\lambda v^\kappa + q\cdot(v+v')v_\mu - (1+v{\cdot}v')q_\mu \right. \\ \left. - \frac{1}{v'{\cdot}q}[i\epsilon_{\mu\nu\lambda\kappa}q^\nu v'^\lambda v^\kappa + q\cdot(v+v')v'_\mu - (1+v{\cdot}v')q_\mu \right\}\end{aligned}$$

where $u(M)$ stands for the isospin wavefunction of meson M. A similar but lengthier expression is found for $B \to D^*\pi e\nu$. Even if the coupling g is close to its upper limit, this expression makes a small contribution to the inclusive semileptonic rate.

6.3. Violations To Chiral Symmetry

Phenomenological lagrangians are particularly well suited to explore deviations from symmetry predictions. We begin by introducing symmetry breaking terms into the phenomenological lagrangian. The light quark mass matrix $m_q = \mathrm{diag}(m_u, m_d, m_s)$ parametrizes the violations to flavor $SU(3)_V$. To linear order in m_q and lowest order in the derivative expansion, the correction to the phenomenological lagrangian is

$$\begin{aligned}\Delta\mathcal{L} = \lambda_0 \ \left[m_q\Sigma + m_q\Sigma^\dagger\right]^a{}_a \\ + \lambda_1 \mathrm{Tr}\overline{H}^{(Q)a} H_b^{(Q)} \left[\xi m_q \xi + \xi^\dagger m_q \xi^\dagger\right]^b{}_a \\ + \lambda_1' \mathrm{Tr}\overline{H}^{(Q)a} H_a^{(Q)} \ \left[m_q\Sigma + m_q\Sigma^\dagger\right]^b{}_b\end{aligned} \tag{6.6}$$

The coefficients λ_0, λ_1 and λ_1' are fixed by non-perturbative strong interaction effects, but may be determined phenomenologically. We postpone consideration of mass relations obtained from this lagrangian until we have introduced heavy quark spin symmetry breaking terms into the lagrangian too.

Exercise 6.1 The operator $\left[\xi m_q \xi - \xi^\dagger m_q \xi^\dagger\right]$ is parity odd. One could add a term $\mathrm{Tr}\overline{H}^{(Q)a} H_b^{(Q)}\gamma_5 \left[\xi m_q \xi - \xi^\dagger m_q \xi^\dagger\right]^b{}_a$. Why did I leave it out?

The decay constants for the D and D_s mesons, defined by

$$\langle 0|\bar{d}\gamma_\mu\gamma_5 c|D^+(p)\rangle = i f_D p_\mu \tag{6.7}$$

and

$$\langle 0|\bar{s}\gamma_\mu\gamma_5 c|D_s(p)\rangle = i f_{D_s} p_\mu \ , \tag{6.8}$$

determine the rate for the purely leptonic decays $D^+ \to \mu^+\nu_\mu$ and $D_s \to \mu^+\nu_\mu$. These are likely to be measured in the future.[54] In the chiral limit, where the up, down and strange quark masses go to zero, flavor $SU(3)_V$ is an exact symmetry and so $f_{D_S}/f_{D^+} = 1$. However $m_s \neq 0$, so this ratio will deviate from unity.

In the chiral lagrangian approach the current operators in Eqs. (6.7) and (6.8) are represented by

$$J^\lambda_{a(M)} = \frac{i\alpha}{2}\mathrm{Tr}[\gamma^\lambda\gamma_5 H_b(v)\xi^\dagger_{ba}] \tag{6.9}$$

This is the lowest order only in an expansion in derivatives. Chiral symmetry violation in the relation between f_{D_S} and f_{D^+} arises directly from symmetry breaking terms that can be added to the current in Eq. (6.9) and indirectly in loop diagrams through the symmetry breaking in the masses of the virtual particles. The latter involves, at one loop, the Feynman diagrams in Fig. 15, where a dashed line stands for a light pseudoscalar propagator. Neglecting the up and down quark masses in comparison with the strange quark mass, the loop graphs give[55,56]

$$f_{D_S}/f_{D^+} = 1 - \frac{5}{6}\left(1+3g^2\right)\frac{M_K^2}{16\pi^2 f^2}\ln\left(M_K^2/\mu^2\right) + \cdots \tag{6.10}$$

Here μ is the renormalization point. The contribution from η loops has been written in terms of M_K using the Gell-Mann–Okubo formula $M_\eta^2 = 4M_K^2/3$, and the contribution from pion loops, proportional to $M_\pi^2 \ln M_\pi^2$, has been neglected. The ellipsis denote terms proportional to the strange quark mass (recall $M_K^2 \sim m_s$) without a log. Theoretically, as $m_s \to 0$ the log term, which has non-analytic dependence on m_s, dominates. The neglected terms, as well as the contribution from the symmetry breaking terms in the current are analytic in m_s.

Exercise 6.2 Display the symmetry breaking additions to the current in Eq. (6.10) to lowest order in m_q. This will introduce new unknown parameters. Compute the modification to $f_{D_S}/f_{D^+} - 1$ from these terms.

The dependence of the new parameters in the current on the subtraction point μ cancels that of the logarithm. Without additional information on these new parameters we have no real predictive power. But one can venture a guess by arguing, as above, that the log term dominates. What value should we assign the renormalization point μ? If μ is of order the chiral symmetry breaking scale then the new parameters have no implicit large logarithms. Numerically, using $\mu = 1$ GeV, the result is that

$$f_{D_s}/f_{D^+} = 1 + 0.064\ (1 + 3g^2), \tag{6.11}$$

or $f_{D_s}/f_{D^+} = 1.16$ for $g^2 = 0.5$.

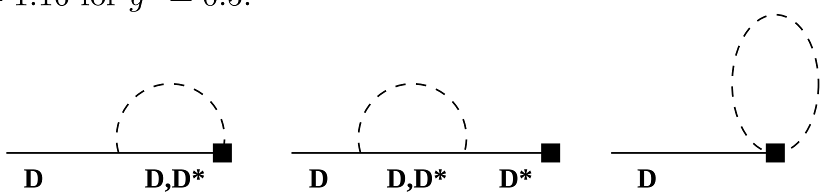

Figure 15 *Feynman Diagrams in the calculation of* f_{D_s}/f_D.

The same formula also holds for f_{B_s}/f_B. In fact, to leading order in $1/M_Q$ the ratio is independent of the the flavor of the heavy quark. Consequently,

$$\frac{f_{B_s}/f_B}{f_{D_s}/f_D} = 1 \tag{6.12}$$

to leading order in $1/M_Q$ and all orders in the light quark masses. Now, Eq. (6.12) also holds as a result of chiral symmetry, for any m_c and m_b. That is f_{B_s}/f_B and f_{D_s}/f_D are separately unity in the limit in which the light quark masses are equal. This means that deviations from unity in Eq. (6.12) must be small, $O(m_s) \times O(1/m_c - 1/m_b)$.[57] This ratio of ratios is observed to be very close to unity in a variety of calculations.[58] This may be very useful, since it suggests obtaining the ratio f_{B_s}/f_B of interest in the analysis of $B - \overline{B}$ mixing (see below) from the ratio f_{D_s}/f_D, measurable from leptonic D and D_s decays.

Exercise 6.3 The hadronic matrix elements needed for the analysis of $B - \overline{B}$ mixing are

$$\langle \overline{B}(v)|\bar{b}\gamma^\mu(1-\gamma_5)d\ \bar{b}\gamma^\mu(1-\gamma_5)d|B(v)\rangle = \frac{8}{3} f_B^2 B_B\ ,$$

$$\langle \overline{B}_s(v)|\bar{b}\gamma^\mu(1-\gamma_5)s\ \bar{b}\gamma^\mu(1-\gamma_5)s|B_s(v)\rangle = \frac{8}{3} f_{B_s}^2 B_{B_s}\ ,$$

where the right hand side of these equations define the parameters B_{B_s} and B_B. In the $SU(3)_V$ symmetry limit $B_{B_s}/B_B = 1$. For non-zero strange quark mass, the ratio is no longer unity. Show that the chiral log is

$$\frac{B_{B_s}}{B_B} = 1 - \frac{2}{3}\left(1 - 3g^2\right)\frac{M_K^2}{16\pi^2 f^2}\ln\left(M_K^2/\mu^2\right) + \cdots$$

Again, $M_\eta^2 = 4M_K^2/3$ has been used. Using $\mu = 1$ GeV, $f = f_K$, and $g^2 = 0.5$, the correction is $B_{B_s}/B_B \approx 0.95$. *Hint:* The hard part is to represent the four-fermion operators in the effective theory. If stuck, consult Ref. [55]

Violations to chiral symmetry in $B \to D$ semileptonic decays have also been studied. One obtains that a different Isgur-Wise function must be used for each flavor of light spectator quark[56]

$$\frac{\xi_s(v\cdot v')}{\xi_{u,d}(v\cdot v')} = 1 + \frac{5}{3} g^2 \Omega(v\cdot v') \frac{M_K^2}{16\pi^2 f^2} \ln\left(M_K^2/\mu^2\right) + \lambda'(\mu, v\cdot v') M_K^2 + \cdots \tag{6.13}$$

where $\lambda'(\mu, v\cdot v')$ is the analytic counter-term, and

$$\Omega(x) = -1 + \frac{2+x}{2\sqrt{x^2-1}} \ln\left(\frac{x+1+\sqrt{x^2-1}}{x+1-\sqrt{x^2-1}}\right) + \frac{x}{4\sqrt{x^2-1}} \ln\left(\frac{x-\sqrt{x^2-1}}{x+\sqrt{x^2-1}}\right)$$

or, expanding about $x = 1$,

$$\Omega(x) = -\frac{1}{3}(x-1) + \frac{2}{15}(x-1)^2 - \frac{2}{35}(x-1)^3 + \cdots$$

Using $g^2 = 0.5$ and $\mu = 1$ GeV, and neglecting the counterterm one obtains

$$\frac{\xi_s(v\cdot v')}{\xi_{u,d}(v\cdot v')} = 1 - 0.21\,\Omega(v\cdot v') + \cdots$$

or a 5% correction at $v\cdot v' = 2$.

Exercise 6.4 Display the Feynman graphs that contribute to the ratio in Eq. (6.13).

6.4. Violations To Heavy Quark Symmetry

In a similar spirit one can consider the corrections in chiral perturbation theory to predictions that follow from heavy quark spin and flavor symmetries. These are effects that enter at order $1/M_Q$, so the first step towards this end is to supplement the phenomenological lagrangian with such terms. In particular, there are $SU(3)_V$ preserving terms of order $1/M_Q$ that violate spin symmetry in the lagrangian, such as[55,59]

$$\Delta\mathcal{L}_{\rm int} = \frac{\lambda_2}{M_Q} {\rm Tr} \overline{H}^{(Q)a} \sigma^{\mu\nu} H_a^{(Q)} \sigma_{\mu\nu} + \frac{g_2}{M_Q} {\rm Tr}\left[\overline{H}_a(v) \not{A}_{ba} \gamma_5 H_b(v)\right] . \tag{6.14}$$

In addition there are contributions to the lagrangian in order $1/M_Q$ that violate flavor but not spin symmetries. These can be characterized as introducing M_Q dependence in the couplings g, λ_1 and λ_1' of Eqs. (6.3) and (6.6). At the same order as these corrections, there is a term that violates both spin and $SU(3)_V$ symmetries

$$\Delta\mathcal{L}_{\rm int} = \frac{\lambda_3}{M_Q} {\rm Tr}\left[\overline{H}^{(Q)a} \sigma^{\mu\nu} H_b^{(Q)} \sigma_{\mu\nu}\right] \left[\xi m_q \xi + \xi^\dagger m_q \xi^\dagger\right]^b{}_a \tag{6.15}$$

Exercise 6.5 Make a complete analysis of terms that violate $SU(3)_V$ and heavy-quark symmetries simultaneously, to leading order in both m_q and $1/M_Q$. Note that our list is shorter since we are going to concentrate shortly on the spectrum only.

Spin symmetry violation is responsible for "hyperfine" splittings in spin multiplets. To leading order these mass splittings are computed in terms of the spin symmetry violating coupling of Eq. (6.14)

$$\Delta_B \equiv M_{B^*} - M_B = -\frac{8\lambda_2}{m_b} \tag{6.16}$$

That the mass splittings scale like $1/M_Q$ seems to be well verified in nature:

$$\frac{M_{D^*} - M_D}{M_{B^*} - M_B} \approx \frac{M_B}{M_D}$$

Measured Mass Splittings

$X - Y$	$M_X - M_Y$(MeV)
$D_s - D^+$	99.5 ± 0.6[60]
$D^+ - D^0$	$4.80 \pm 0.10 \pm 0.06$[61]
$D^{*+} - D^{*0}$	$3.32 \pm 0.08 \pm 0.05$[61]
$D^{*0} - D^0$	$142.12 \pm 0.05 \pm 0.05$[61]
$D^{*+} - D^+$	$140.64 \pm 0.08 \pm 0.06$[61]
$D_s^* - D_s$	141.5 ± 1.9[60]
$B_s - B$	82.5 ± 2.5[61] or 121 ± 9[62]
$B^0 - B^+$	0.01 ± 0.08[60]
$B^* - B$	$46.2 \pm 0.3 \pm 0.8$[63] or45.4 ± 1.0[62]
$B_s^* - B_s$	47.0 ± 2.6[62]
$(D^{*0} - D^0)$ $-(D^{*+} - D^+)$	$1.48 \pm 0.09 \pm 0.05$[61]

Armed with the machinery of chiral lagrangians that include both spin and chiral symmetry violating terms, one can compare hyperfine splitting for different flavored mesons. There is a wealth of experimental information to draw from; see Table 3. Breaking of flavor $SU(3)_V$ and heavy quark flavor symmetries by electromagnetic effects is not negligible. It is readily incorporated into the lagrangian in terms of the charge matrices $Q_Q = \text{diag}(2/3, -1/3)$ and $Q_q = \text{diag}(2/3, -1/3, -1/3)$,[64] which must come in bilinearly. For example, terms involving Q_q^2 correspond to replacing $m_q \to Q_q$ in Eqs. (6.6) and (6.15). The electromagnetic effects of the light quarks can be neglected if one considers only mesons with d and s light quarks. The electromagnetic shifts in the hyperfine splittings Δ_{X_q} and Δ_{X_q} ($X = D, B$, $q = d, s$) differ on account of different b and c charges, but they cancel in the difference of splittings

$$\Delta_{X_s} - \Delta_{X_d} = (M_{X_s^*} - M_{X_s}) - (M_{X_d^*} - M_{X_d})$$

The only term in the phenomenological lagrangian that enters this difference is Eq. (6.15). This immediately leads to

$$(M_{B_s^*} - M_{B_s}) - (M_{B_d^*} - M_{B_d}) = \frac{m_c}{m_b}\left(\frac{\bar{\alpha}_s(m_c)}{\bar{\alpha}_s(m_b)}\right)^{\frac{9}{25}}\left[(M_{D_s^*} - M_{D_s}) - (M_{D_d^*} - M_{D_d})\right] \tag{6.17}$$

We have included here the short distance QCD effect.[40]

One expects Eq. (6.17) holds to much better accuracy than the separate relations for each hyperfine splitting in Eq. (6.16). Recall that $SU(3)_V$ breaking by light quark masses and electromagnetic interactions have been accounted for in leading order. Moreover, the result is trivially generalized by replacing the quark mass matrix in Eqs. (6.6) and (6.15), by an arbitrary function of the light quark mass matrix. It is seen from Table 3 that this relation works well. The left side is 1.2 ± 2.7 MeV while the right side is 3.0 ± 6.3 MeV.

Since both sides of Eq. (6.17) are consistent with zero and both are proportional to the interaction term in Eq. (6.15), it must be that the coupling λ_3 is very small.[64] From the difference of hyperfine splittings in the charm sector

$$-\frac{8\lambda_3}{m_c}(m_s - m_d) = 0.9 \pm 1.9 \text{ MeV}$$

while

$$M_{D_s} - M_{D_d} = 4\lambda_1(m_s - m_d) - \frac{12\lambda_3}{m_c}(m_s - m_d) = 99.5 \pm 0.6 \text{ MeV}$$

leading to $|\lambda_3/\lambda_1|$ less than ~ 20 MeV. This is smaller than expected by about an order of magnitude. With such a small coefficient it is clear that the next-to-leading terms and the loop corrections may play an important role. In particular they may invalidate the simple $1/M_Q$ scaling of Eq. (6.17).[65] There is no obvious breakdown of chiral perturbation theory, even though the leading coupling (λ_3) is anomalously small.[66]

At one loop, the expressions for the mass shifts involve large $O(m_s \ln m_s)$ and $O(m_s^{3/2})$ (non-analytic) terms.[56,66] The coupling λ_3 is not anomalously small at one loop. Instead, the smallness of the difference of hyperfine splittings in Eq. (6.17) is the result of a precise cancellation between one loop and tree level graphs. Explicitly,[66]

$$\left(M_{X_s} - M_{X_s^*}\right) - \left(M_{X_d} - M_{X_d^*}\right) = \frac{5}{3} g^2 \left(\frac{8\lambda_2}{M_Q}\right) \frac{M_K^2}{16\pi^2 f^2} \ln\left(M_K^2/\mu^2\right) - \frac{8\lambda_3}{M_Q} m_s$$

With $g^2 = 0.5$ and $\mu = 1$ GeV, the chiral log is 30 MeV, so the λ_3 counterterm must cancel this to a precision of better than 10%.

The $1/M_Q$ corrections to the masses M_X and M_{X^*} drop out of the combination $M_X + 3M_{X^*}$. The combination $(M_{X_s} + 3M_{X_s^*}) - (M_{X_d} + 3M_{X_d^*})$ is a measure of $SU(3)_V$ breaking by a non-vanishing m_s (or $m_s - m_d$ if the d quark mass is not neglected). It can be computed in the phenomenological lagrangian. To one loop[66]

$$\begin{aligned} \frac{1}{4}\left(M_{X_s} + 3M_{X_s^*}\right) - \frac{1}{4}\left(M_{X_d} + 3M_{X_d^*}\right) &= 4\lambda_1 m_s - g^2 \left(1 + \frac{8}{3\sqrt{3}} \frac{1}{2}\right) \frac{M_K^3}{16\pi f^2} \\ &\quad - 4\lambda_1 m_s \left(\frac{25}{18} + \frac{9}{2} g^2\right) \frac{M_K^2}{16\pi^2 f^2} \ln\left(M_K^2/\mu^2\right) \end{aligned} \tag{6.18}$$

The pseudoscalar splittings $(M_{D_s} - M_{D_d})$ and $(M_{B_s} - M_{B_d})$ have been measured; see Table 3. Also, $\frac{1}{4}(M_{X_s} + 3M_{X_s^*}) - \frac{1}{4}(M_{X_d} + 3M_{X_d^*}) = \frac{3}{4}[(M_{X_s^*} - M_{X_s}) - (M_{X_d^*} - M_{X_d})] + (M_{X_s} - M_{X_d})$, and the term in square brackets is less than a few MeV, as we saw above. The combination $(M_{X_s} + 3M_{X_s^*}) - (M_{X_d} + 3M_{X_d^*})$ in Eq. (6.18) is first order in m_s but has no corrections at order $1/M_Q$. Thus, one expects a similar numerical result for B and D systems. Experimentally, $(M_{B_s} - M_{B_d})/(M_{D_s} - M_{D_d})$ is consistent with unity; see Table 3. The formula in Eq. (6.18) has a significant contribution from the M_K^3 term which is independent of the splitting parameter λ_1. The M_K^3 term gives a negative contribution to the splitting of ~ -250 MeV for $g^2 = 0.5$. The chiral logarithmic correction effectively corrects the tree level value of the parameter λ_1; for $\mu = 1$ GeV and $g^2 = 0.5$, the term $4\lambda_1 m_s$ gets a correction ≈ 0.9 times its tree level value. Thus, the one-loop value of $4\lambda_1 m_s$ can be significantly greater than the value determined at tree-level of approximately 100 MeV.

Chiral perturbation theory can be used to estimate the leading corrections to the form factors for semileptonic $B \to D$ or D^* decays which are generated at low momentum, below the chiral symmetry breaking scale. Of particular interest are corrections to the predicted normalization of form factors at zero recoil, $v{\cdot}v' = 1$. According to Luke's theorem (see Section 4.4), long distance corrections enter first at order $1/M_Q^2$. Deviations from the predicted normalization of form factors that arise from terms of order $1/M_Q^2$ in either the lagrangian or the current are dictated by non-perturbative physics. But there are computable corrections that arise from the terms of order $1/M_Q$ in the lagrangian. These must enter at one-loop, since Luke's theorem prevents them at tree level, and result from the spin and flavor symmetry breaking in the hyperfine splittings Δ_D and Δ_B. Retaining only the dependence on the larger Δ_D, the correction to the matrix elements at zero recoil are[67]

$$\langle D(v)|J_\mu^{\bar{c}b}|B(v)\rangle = 2v_\mu \left(1 - \frac{3g^2}{2}\left(\frac{\Delta_D}{4\pi f}\right)^2 \left[F(\Delta_D/M_\pi) + \ln(\mu^2/M_\pi^2)\right] + C(\mu)/m_c^2\right) \tag{6.19a}$$

$$\langle D^*(v,\epsilon)|J_\mu^{\bar{c}b}|B(v)\rangle = 2\epsilon_\mu^* \left(1 - \frac{g^2}{2}\left(\frac{\Delta_D}{4\pi f}\right)^2 \left[F(-\Delta_D/M_\pi) + \ln(\mu^2/M_\pi^2)\right] + C'(\mu)/m_c^2\right) \tag{6.19b}$$

where C and C' stand for tree level counter-terms and

$$F(x) \equiv \int_0^\infty dz \frac{z^4}{(z^2+1)^{3/2}} \left(\frac{1}{[(z^2+1)^{1/2}+x]^2} - \frac{1}{z^2+1}\right) \tag{6.20}$$

As before, no large logarithms will appear in the functions C and C' if one takes $\mu \approx 4\pi f \sim 1$ GeV. With this choice, formally, their contributions are dwarfed by the term that is enhanced by a logarithm of the pion mass. Numerically, with $g^2 = 0.5$ the logarithmically enhanced term is -2.1% and -0.7% for D and D^*, respectively.

The function F accounts for effects of order $(1/m_c)^{2+n}$, $n = 1, 2, \ldots$ It is enhanced by powers of $1/M_\pi$ over terms that have been neglected. Consequently it is expected to be a good estimate of higher order $1/m_c$ corrections. With $\Delta_D/M_\pi \approx 1$, one needs $F(1) = 14/3 - 2\pi$ and $F(-1) = 14/3 + 2\pi$ for a numerical estimate; with μ and g^2 as above, this term is 0.9% and -2.0% for D and D^*, respectively.

To put it differently, if the function F is expanded in powers of $1/m_c$, then since the resulting terms have inverse powers of $1/M_\pi$ they cannot be confused with local counterterms. Thiese terms play the same role as the non-analytic logs of the previous sections.

Frequently the non-analytic corrections to relations that follow from the symmetries are uncomfortably large. A case of much interest is the relation between the form factors $f_\pm$ and h for $B \to K$ transitions, relevant to the short distance process $b \to se^+e^-$,

$$\langle \overline{K}(p_K) \,|\, \overline{s}\gamma^\mu b \,|\, \overline{B}(p_B)\rangle = f_+ \,(p_B + p_K)^\mu + f_- \,(p_B - p_K)^\mu \,,$$
$$\langle \overline{K}(p_K) \,|\, \overline{s}\sigma^{\mu\nu} b \,|\, \overline{B}(p_B)\rangle = \mathrm{i}h \left[(p_B + p_K)^\mu (p_B - p_K)^\nu - (p_B + p_K)^\nu (p_B - p_K)^\mu\right],$$

and the form factors for $B \to \pi e\nu$,

$$\langle \overline{\pi}(p_\pi) \,|\, \overline{u}\gamma^\mu b \,|\, \overline{B}(p_B)\rangle = \hat{f}_+ \,(p_B + p_\pi)^\mu + \hat{f}_- \,(p_B - p_\pi)^\mu \,.$$

In the combined large mass and chiral limits only one of these form factors is independent:

$$m_b h = f_+ = -f_- = \hat{f}_+ = -\hat{f}_- \tag{6.21}$$

In this limit, the ratio of rates for $B \to K e^+e^-$ and $B \to \pi e\nu$ is simply given, in the standard model of electroweak interactions, by $|V_{ts}/V_{ub}|^2$, times a perturbatively computable function of the top quark mass. If the relation (6.21) held to good accuracy one could thus measure a ratio of fundamental standard model parameters.[6]

The non-analytic, one-loop corrections to the relations in Eq. (6.21) have been computed.[69] The results are too lengthy to display here. Numerically, the violation to $SU(3)_V$ symmetry is found to be at the 40% level.[7]

The phenomenological lagrangian that we have been considering extensively neglects the effects of states with heavy-light quantum numbers other than the pseudoscalar – vector-meson multiplet. The splitting between multiplets is of the order of 400 MeV and is hardly negligible when one considers $SU(3)_V$ relations involving both π and K mesons. For example, consider the effect of the scalar –

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[6] Another application of this relation has been discussed by I. Dunietz[68]. Assuming factorization in $B \to \psi X$, ratios of CKM elements can be extracted from these two body hadronic decays.

[7] The large violation of $SU(3)_V$ symmetry affects as well the results of Dunietz (see previous footnote).

pseudovector-meson multiplet. One can incorporate its effects into the phenomenological lagrangian. To this end, assemble its components into a "superfield", akin to that in Eq. (6.1) for the pseudoscalar – vector multiplet:[70]

$$S_a(v) = \frac{1+\not{v}}{2}\left[B'^{\mu}_{1a}\gamma_\mu\gamma^5 - B^*_{0a}\right] . \tag{6.22}$$

The phenomenological lagrangian has to be supplemented with a kinetic energy and mass for S,

$$\mathrm{Tr}\left[\overline{S}_a(v)(iv\cdot D_{ba} - \Delta\delta_{ba})S_b(v)\right] ,$$

where Δ is the mass splitting for the excited S from the ground state H, and with coupling terms

$$g'\,\mathrm{Tr}\left[\overline{S}_a(v)S_b(v)\,\not{A}_{ba}\gamma^5\right] + (h\,\mathrm{Tr}\left[\overline{H}_a(v)S_b(v)\,\not{A}_{ba}\gamma^5\right] + \mathrm{h.c.})\,.$$

In terms of these one can now compute additional corrections to quantities such as f_{D_s}/f_D in Eq. (6.10). Numerically the corrections are not small,[71] $f_{D_s}/f_D = 1 + 0.13h^2$ for $M_{D_0^*} = 2300$ MeV (or $f_{D_s}/f_D = 1 + 0.08h^2$ for $M_{D_0^*} = 2400$ MeV), assuming the strange mesons to be 100 MeV heavier. Similarly, corrections to the Isgur-Wise function can be computed, and are not negligible.[71]

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