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UNIVERSITY OF CALIFORNIA SAN DIEGO

Impact of Structure and Rotational Multi-stability in Energy Absorbing Metamaterials

A Dissertation submitted in partial satisfaction of the requirements for the degree Doctor of Philosophy

in

Engineering Sciences (Applied Mechanics)

by

Jack Pechac

Committee in charge:

Professor Michael Frazier, Chair Professor Nicholas Boechler Professor Hysun-sun Alicia Kim Professor Michael Tolley

The dissertation of Jack Pechac is approved, and it is acceptable in quality and form for publication on microfilm and electronically.

University of California San Diego

2024

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Abstract of the Dissertation

Impact of Structure and Rotational Multi-stability in Energy Absorbing Metamaterials

by

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Doctor of Philosophy in Engineering Sciences (Applied Mechanics)

University of California San Diego, 2024 Professor Michael Frazier, Chair

This dissertation outlines investigations aimed at advancing structure-based mechanical energy absorption in metamaterials. The goal of these investigations are to devise a new class of cellular metamaterials characterized by a multi-stable internal architecture and analyze the impact of those attributes on the mechanical energy absorption performance. In pursuit of this goal, three main outcomes are achieved.

The first outcome is multi-modal energy absorption in a metamaterial enabled by a rotationally multi-stable node embedded within a dual-chiral layer. Numerical simulation of several two- dimensional lattices demonstrates energy-absorbing hysteresis in their quasistatic loading curves under tensile, compressive, and shear deformations. Furthermore, this energy absorbing capacity is demonstrated in many loading directions, with directional absorption

dictated by the underlying lattice's rotational symmetry.

The second outcome is enhanced structural properties and absorption performance achieved by tuning key kinematic parameters and microstructure in order to direct the load-displacement hysteresis toward that of an ideal absorber. Theoretical analysis of the unit cell characterizes the stiffness and peak load as functions of fundamental design parameters. It is shown numerically how these may be utilized to manipulate the loading curve prior to the onset of energy absorption, allowing for implementation of the design in structural applications. The onset of energy absorption leads to a plateau in the load-displacement curve, the length of which may be tailored in a similar fashion, with the mean height of the plateau heavily influenced by the microstructure.

The third outcome is control over the directionality of the metamaterial absorption performance by mimicking the poly-crystalline microstructure of metals/alloys in metamaterials through "meta- grains" with spatially prescribed lattice orientation. Energy absorption in specific directions is optimized via the Non-dominated Sorting Genetic Algorithm II, treating target absorption values as objective functions and grain orientations as variables to be optimized. A simple bi-directional case is demonstrated experimentally to validate numerical results. The parameters dictating the polycrystalline structure are then examined by optimizing the directional stiffness of a lattice modeled with FEM beam elements In addition, we take a detour to observe other applications and phenomena in multi-stable metamaterials, namely a mechanical memory device and acoustic supratransmission.

Chapter 1

Introduction

1.1 Conventional Energy Absorbing Materials

Energy-absorbing materials⁴ are ubiquitous in nature⁵ and engineering applications demanding, e.g., impact mitigation and stress redistribution/relief.^{6,7} A loading-unloading cycle generates a characteristic hysteresis in the load-displacement diagram which, by the enclosed area, quantifies the absorbed energy. The absorption mechanism in conventional materials^{8,9} of course depends on material properties: metals with high ductility deform plastically, ceramics fracture and crack, fibrous materials tear, split, and delaminate, and materials with significant visco-elastic response (e.g. polymers, gels, foams) have additional internal friction. While conventional energy-absorbing materials have their advantages, the material-specific composition, micro/meso-scale stucture, and absorption mechanism manifest inherent limitations as well. Thin-walled metal tubes possess load-bearing capabilities and hence find application as structural elements. Their absorption of energy is however sensitive to loading direction, preferring pure axial loads to activate progressive buckling.^{10,11} Structures hence rely on internal architecture to encourage axial loading of elements, exemplifying the dependence of absorption on meso-scale structure. As another example, ceramics with high hardness and compressive strength are

commonly employed in personal and vehicular armor but lose structural integrity with fracture. Employing ceramic tiles serves to arrest crack propagation at tile boundaries, effectively toughening designs through structure alone.¹² These examples are suited to high-energy impacts such as vehicle collision and protection from projectiles and typically exhibit high peak-stresses. More delicate applications such as packaging or athletic protection prefer materials with a lower peak-stress in order to cushion protected objects but do not require structural load-bearing capability.

Polymers, gels, and cellular materials fill this niche. Compared to cellular materials – of which foams are the most widespread – polymers and gels are dense and exhibit a short deformation range for a given peak stress, limiting the energy absorption capacity. However, the pseudo-stochastic nature of the typical foam micro-structure hinders predictive modeling and is difficult to control during the manufacturing process, potentially yielding non-uniform properties.^{13–15} Moreover, in order to elicit the desired hysteretic behavior, certain cellular materials utilize destructive absorption mechanisms akin to that of metals and ceramics (e.g., plastic deformation¹⁶ and fragmentation^{17, 18}) which sacrifices re-usability. Among several desirable characteristics, the ideal energy-absorbing material is low-density, efficient with respect to mass and volume, and delivers predictable and repeatable performance under general loading, i.e., irrespective of the loading mode and direction.

1.2 Metamaterials and the Current State of the Field

Over the past decade, researchers have designed and realized mechanical metamaterials^{19–21} whose extreme and sometimes unusual mechanics^{22–26} emerge from an engineered, highly tunable micro- structure geometry rather than the composition. In the context of energy absorption, cellular metamaterials^{27–29} inherit the low density quality of foams, while enhancing predictabilility and re-usability with ordered (e.g., periodic) micro-structures amenable to analytical and computer modeling as well as fabrication via 3D-printing technologies. Although, currently, the mass-

normalized absorption capacity is not comparable to conventional materials, cellular metamaterials may serve a complementary or even principal role in a growing fraction of absorption applications stemming from their enhanced tunability and predictability. Nevertheless, the full potential of the metamaterial platform in regard to mechanical energy absorption is yet to be realized, in part, due to the limitations of current approaches. This proposal aims at overcoming these shortcomings in order to significantly advance metamaterial energy absorption capability and, thus, further promote their practical viability.

It is well-known that the mechanical properties of cellular materials vary with the relative density, $\rho_r < 1$ (i.e., the density of the cellular material to that of the solid constituent), according to the scaling law, $\Gamma_r \propto \rho_r^n$, where Γ_r is the (relative) property of interest and *n* is a positive scaling parameter.³⁰ Consequently, as n > 1 is typical, cellular materials exhibit a significant reduction in strength and stiffness as ρ_r decreases, limiting their utility in load-bearing applications. Notably, many natural^{31,32} and man-made³³ materials exhibit structure on more than one length scale which can play a significant role in shaping the macroscopic performance.^{34,35} In recent years, this structural hierarchy has been exploited for the creation of cellular metamaterials for which *n* approaches unity, yielding exceptional strength and stiffness^{36–40} as well as better damage tolerance.⁴¹ Physically, deformation via bending is minimized at each hierarchical level in order to develop a stretching-dominated system where strength and stiffness scale linearly (or nearly so) with relative density. Hierarchy inhibits crack propagation, leading to better damage tolerance. The bulk of these studies concern the metamaterial response under uni-axial compression where, among other factors (e.g., visco-elasticity), mechanical energy absorption is frequently enabled by destructive mechanisms (e.g., plastic deformation and fracture) which limit the utility of current hierarchical metamaterials for high-cycle operations. While hierarchical metamaterials have shown superior strength and stiffness in a low-density platform, an approach that enhances the mechanical energy absorption without sacrificing re-usability remains elusive, but desirable for metamaterials to serve dual structural and protective roles for extended times.

A review of the relevant metamaterial literature reveals snap-through, e.g., of

buckled/buckling beams,⁴²⁻⁵⁴ conical shells,⁵⁵ or antagonistic arrays of magnets,^{56,57} to be a well-utilized mechanism to elicit a hysteretic response. Loading causes snapping elements at the microstructural level to transition from one energetically stable equilibrium configuration to another. Typically, snapping elements are bi-stable and transitions ensue via a translation-based deformation. As the internal architecture remains elastic during the loadingunloading process, snapping metamaterials exhibit exceptional re-usability. Nevertheless, despite the myriad articles and the diversity of designs, the development of snapping metamaterials has (i) proceeded with little consideration to tailoring the absorption behavior and (ii) heavily relied on translation-based, bi-stable elements to the detriment of increased performance and utility in applications. As a consequence of (i), the literature lacks a rational strategy to affect the metamaterial absorption capacity per unit mass/volume or the efficiency thereof. These properties are of practical relevance to, e.g., aerospace applications where space and weight limitations drive development. As a consequence of (ii), as clarified below, current snapping metamaterials manifest effective energy absorption from mechanical loading of one specific type (e.g., tension) applied along a few (usually one) preferred axes. For applications, an absorption capability in response to more general loading is desirable. In particular, given a multi-axial absorption capability, command over the metamaterial directional absorption response would better enable the customization of performance to direction-dependent variations in a loading environment. Finally, no attempted has been made to structurally (rather than compositionally) improve the stiffness of snap-through-based cellular metamaterials.

While the affect of both hierarchy and snap-through on cellular metamaterial performance have been investigated separately, thus far, their combined effects within a single system has not been studied. This is especially troubling given that each of these features addresses distinct shortcomings in the performance of low-density materials: hierarchy bestows exceptional load bearing capacity³⁷ and damage tolerance,⁴¹ but poor energy absorption and repeatability;⁵⁸ snap-through enables excellent (hitherto, conditional⁵³) absorption and re-usability at the expense of stiffness and robustness against imperfections. Of a hierarchical, snap-through metamaterial,

the prevailing questions are: can the desirable attributes of each approach be adopted without inheriting the undesirable ones? do the effects of each approach support or counteract the other, and under what conditions? what new capabilities emerge from the united approach that are unavailable from the separate approaches? Understanding the combined, potentially synergistic, effects of hierarchy and multi-stable snap-through on the mechanical behavior can guide metamaterial design for specific performance and promote the utility and proliferation of cellular metamaterials in absorption applications.

1.3 Proposed Approach

The goal of this proposal is to devise a new class of cellular metamaterials characterized by hierarchical, multi-stable internal architecture and analyze the impact of those attributes on the energy absorption performance, specifically, the capacity, the efficiency, and the directionality thereof. Translation-based elements – the current dominant paradigm – place restrictions on the loading mode and direction, resulting in conditional absorption. For current metamaterial designs, this may manifest as a particular sensitivity to either horizontally-aligned tension or compression load, exclusively. This proposal utilizes a novel rotation-based snap-through element to create cellular metamaterials for general energy absorption.

As described in Chapter 2, the adoption of rotation-base snapping elements facilitates greater flexibility in metamaterial construction than previously demonstrated, benefiting the hierarchical designs considered in this proposal. The following objectives represent a combined theoretical, numerical, and experimental effort:

• **Objective 1:** Enable multi-modal and omni-directional energy absorption in cellular metamaterials. The strategy assembles a cellular bi-layer from two elastic 2D lattices of identical arrangement but opposite chirality. These layers are coupled elasto-magnetically through the aligned rotation centers which constitute rotation-based, multi-stable elements (Fig. 2.1b). Consequently, any in-plane deformation of the bi-layer in response to general mechanical loading (i.e., irrespective of loading mode or direction) is accommodated by a relative angular displacement at the rotation centers which stimulates snap-through, leading to hysteresis in the load-displacement curve indicative of energy absorption. The strategy is amenable to a number of (quasi-)periodic and amorphous internal architectures, demonstrating a flexibility in choosing the underlying lattice arrangement.

- **Objective II:** Enhance the stiffness of metamaterial architectures while, simultaneously, increasing the (mechanical) energy absorption capacity and efficiency. Success will promote metamaterials in structural settings requiring protection. We will manipulate the geometry of the multi-stable unit cell and investigate the stiffness, load bearing and absorption capacity.
- **Objective III:** Control the directionality of the metamaterial's energy absorption. This will minimize the influence of the lattice symmetry on the large-scale absorption directionality, permitting customized absorption capacity for different loading directions. We mimic the poly-crystalline microstructure of metals/alloys by creating metamaterial "grains" defined by a cellular snap-through architecture with unique lattice orientation. Rationally assemble grains of various orientation to construct a metamaterial system with the desired direction-dependent absorption profile. Sets of grain orientations are found via the Non-dominated Sorting Genetic Algorithm II, an stochastic evolutionary algorithm with elitism which excels in multi-objective optimization where many minima exist.

Chapter 2

Metamaterial design strategy for mechanical energy absorption under general loading

Abstract

This chapter addresses Objective I. It proposes a design strategy for two-dimensional, multi-stable cellular materials which leverages local rotational degrees of freedom in support of mechanical energy absorption under general loading. The approach aligns the rotation centers of two layers of identical patterning but opposite chirality such that the relative angular displacement accompanying any in-plane deformation facilitates local snap-through events which contribute to the global energy absorption performance. The rotation symmetry of the underlying (quasi-)crystalline layers governs the directionality of the energy absorption capacity. Moreover, cellular materials emerging from the proposed strategy possess an absorption capability for all loading modes – tension, compression, and shear. Simulations reveal the energy absorption capacity and the directionality thereof for several cellular bi-layers as well as the impact of key tuning parameters. The cellular materials constructed following the proposed design strategy fill the need for omni-directional, multi-modal energy absorption in a low-density, tunable, and re-usable platform. These results were published in *Extreme Mechanics Letters* in February of

 2022^{1}

2.1 Motivation

Although structurally 2D/3D, most of the cellular metamaterials described in the current literature are functionally one-dimensional, exhibiting a significant energy absorption capacity for loading along a single axis.^{42–50,56} Structures are otherwise subjected to uni-axial loading along equivalent axes of symmetry.^{51,52,55} Moreover, despite the complex loading of real-world environments, from the native (i.e., ground) configuration, current metamaterial designs accommodate simple load cases, i.e., loading in either tension,⁴² compression,^{44–48,51–53,55} or shear,^{54,57} and thus possess a specialized, rather than general energy absorption capability. The constraint on loading direction follows from the utilization of translation-based snapping elements in the metamaterial architecture, a practice which is widespread in the literature. The constraint on loading mode is a consequence of the bi-stable nature of the snapping elements, which implies that transitions from the native state may only be accomplished by loads of a single variant. While the metamaterial platform has achieved predictable, tunable, and re-usable energy absorption, a pressing challenge lies in devising a strategy for constructing cellular metamaterials with general loading energy absorption capability: effective energy absorption from loading (i) along any axis and (ii) of any variant.

Here, we propose an alternative strategy based on snap-through transitions in the rotational degrees of freedom to create cellular metamaterials capable of mechanical energy absorption under general loading. We numerically analyze the absorption capabilities of various (quasi-)crystalline metamaterials, demonstrating a flexibility of construction. We also observe a direction-dependent energy absorption reflecting the rotational symmetry of the underlying lattice arrangement. Such cellular metamaterials expand the capabilities of current uni-modal, functionally one-dimensional design strategies by permitting energy absorption from a broader spectrum of loading conditions



Figure 2.1: Construction of Multi-stable Cellular Materials. (a) Two lattices of identical arrangement but opposite chirality are aligned and corresponding rotation centers constrained to exhibit a common in-plane translation and permit a relative out-of-plane rotation, φ . Due to the opposing chirality, deforming the cellular bi-layer generates a non-zero φ . (b) Paired rotation centers, represented by discs, interact via magnetic elements along the perimeter and a torsional spring. (c) The corresponding on-site potential function, $\psi(\varphi)$, is symmetric and features multiple minima associated with (meta-)stable configurations. The barrier to transition to the first meta-stable state is labeled $\Delta \psi_S$, while the reversion barrier is $\Delta \psi_R$. Simulations utilize a (relatively) computationally inexpensive 2nd-order Fourier approximation of the magnetic component of the potential, $\psi_m(\varphi)$.

whilst maintaining predictability, tunability, and re-usability.

2.2 Structure and Methods

2.2.1 Design Strategy

In general, each cellular material design aligns a pair of two-dimensional lattices of identical (quasi-)periodic arrangement but opposite chirality such that any in-plane deformation of the resulting bi-layer is accommodated by a relative out-of-plane angular displacement, $\varphi = \varphi_t - \varphi_b$, between corresponding rotation centers in each layer (Figs. 2.1a,b). As the system deforms, a rigid rod through the common axis of rotation maintains the alignment, ensuring that paired centers of rotation undergo identical in-plane displacement while preserving the

independent rotations of the top and bottom layers, φ_t and φ_b , respectively. Similar chiral^{59–61} and anti-chiral^{62,63} cellular bi-layers have been theoretically, numerically, and experimentally investigated in the context of directional auxeticity, but lacked the critical multi-stability for exploration of energy absorption aspects. The literature also contains examples of bi-stable and anti-chiral single layers utilizing rotational freedoms for mechanical energy absorption; however, snap-through is stimulated by loading the architecture in uni-axial tension⁶⁴ or compression⁶⁵ from the native state while other loading modes and directions were not investigated. In the present study, paired rotation centers, represented by two concentric discs of mass, m, radius, r, and separated by an inter-layer spacing, h, facilitate the formation of multi-stable elements when the perimeter of each is decorated with *n* permanent magnets of opposing polarization [Figure 2.1b, Movie S4 of Supporting Information (SI)]. Among other parameters, magnetic interactions are affected by the dimensions of each magnet; however, for simplicity and without loss of generality, we idealize each magnet as a point dipole moment \mathbf{m}_i . Accordingly, the multi-stable potential function of the magnetically-coupled discs is given by $\psi_m(\varphi) = -\sum_{i=1}^n \sum_{j\neq i}^n \mathbf{m}_i \cdot \mathbf{B}_{ij}(\varphi)$, where $\mathbf{B}_{ii}(\boldsymbol{\varphi})$ is the magnetic field produced by \mathbf{m}_i at the location of \mathbf{m}_i . The resulting potential landscape is symmetric about $\varphi = 0$ with infinite ground state configurations, $\varphi = p\varphi_0$, $p \in \mathbb{Z}$, appearing at regular intervals. If the specifics of an application preclude the use of magnets, then a similar multi-stability may be achieved mechanically by substituting the magnets for sinusoidal elastic embossments along the perimeter of each disc, i.e., the angular equivalent of the translation-based multi-stable units described in Ref.⁴⁹ (Figure S2 of SI). To promote recoverability, we introduce a torsional spring of stiffness, k_t , which penalizes φ according to $\psi_t(\varphi) = k_t \varphi^2/2$ such that the total on-site potential, $\psi = \psi_m + \psi_t$, possesses a single ground state, $\varphi = 0$, and a finite number of meta-stable states of increasing energy for increasing $|\varphi|$ (Figure 2.1c).

The multi-stable elements, arranged in a (quasi-)crystalline pattern with lattice constant a and elastically connected, form a cellular bi-layer. In particular, linear springs of stiffness, k_c , couple co-planar discs in neighboring elements by attaching to their perimeter via pin joints

(Figure 2.1a). For the top (bottom) discs of coupled elements, the joints are situated at an angle α ($-\alpha$) from the line through the center of each disc forming a chiral lattice. Apparently, any loading-induced deformation of the cellular material which changes the center-to-center distance between elements involves a change in length in the interaction springs and a relative rotation between the discs comprising the multi-stable elements. Thus, we expect $|\varphi|$ and snap-through events to increase where the magnitude of the effective dilation is greatest. Following snap-through, upon settling into a meta-stable state, energy is stored in the local and non-local elastic/magnetic components of the bi-layer. Since, as demonstrated in the following, the deformation may stem from in-plane tensile and compressive loading along any axis as well as in-plane shear loading, the (quasi-)crystalline cellular metamaterials emerging from the proposed design strategy possess a general loading energy absorption capability.

The mass-spring lattices discussed in this article are a computationally efficient simplification of practical beam structures (Figure S1 of SI). However, the lack of a bending rigidity in the simplified construction causes lattices to be susceptible to collapse when loaded in compression or shear. Consequently, for stability, a bending rigidity (and an inconsequentially small axial rigidity) is re-introduced by mounting bi-layers upon non-chiral, straight-beam lattices of identical arrangement. As previously stated, physical models of similar (anti-)chiral bi-layers have been fabricated for experiments. Metamaterials following the proposed design strategy may utilize similar manufacturing approaches. Incorporating the magnets may be a challenge; however, a stated above, these may be substituted for sinusoidal elastic embossments to achieve multi-stability. In order to balance the need for bending resistance and minimizing stress, the points where beams connect to rigid discs may be more compliant than the beam interior. Since energy absorption following the proposed design strategy stems from the geometry, manufacturing across different length scales is limited by only by present technology and size effects.¹⁷

2.2.2 Simulation

In order to demonstrate the proposed strategy, we analyze *in silico* the energy absorption of various (quasi-)crystalline cellular bi-layers subject to quasi-static, in-plane loading. Simulation models include a viscous damping force, $f_d = c\phi$, in order to attenuate small-amplitude oscillations. The material response to dynamic loading will be explored in a separate article. Regardless of lattice arrangement, we consider bi-layer samples approximately $20a \times 20a$ in dimension and loaded at various orientations, θ . Specifically, as illustrated in Figure 2.3a for a square sample, the multi-stable elements are placed at the lattice points but only those within the $20a \times 20a$ region are elastically connected to form the sample; the remainder are discarded from the model. Regarding the impact of the sample dimensions on the material performance, preliminary studies showed the results from $20a \times 20a$ samples to be qualitatively similar to those obtained at greater computational cost from $60a \times 60a$ samples. The load is applied to the top boundary of the rectangular sample while the bottom remains fixed; its magnitude traces a symmetric triangle wave in time (Figure S3 of SI). While bi-axial, twisting, and concentrated loads - whether quasi-static or dynamic - are of practical relevance, for brevity, simulations concern only uni-axial tension and compression, and shear, under quasi-static conditions with various bi-layer orientations. In general, the (quasi-)crystalline samples do not possess smooth boundaries; therefore, for loading purposes, elements within 0.9a of the rectangular boundary are treated as the boundary elements. For computational economy, we approximate $\psi_m(\varphi)$ by a 2nd-order Fourier expansion (Figure 2.1c). All dimensionless material and geometric parameters are listed in the SI.

2.3 Results and Discussion

2.3.1 Energy-absorption Mechanism

Figure 2.2 depicts the response of an initially undeformed cellular bi-layer with square arrangement to separate loading-unloading cycles in tension, compression, and shear, respectively (Movies S1-S3 of SI). In particular, through snapshots, Figure 2.2 shows the lattice deformation and phase (i.e., relative rotation) distribution at different instances within a cycle. Auxetic effects due to the chirality of the underlying lattices are apparent in the deformation. In addition, snap-through phase transitions indicative of energy absorption occur irrespective of loading mode. The sign of the phase is opposite that of the local effective dilation. As $|\varphi|$, generally, increases with the magnitude of the effective dilation, snap-through initiates along the unconstrained boundaries of the structure and propagates toward the interior. Upon removal of the load, the bi-layer maintains a residual deformation as well as a smooth phase gradient as a consequence of the multiple possible states into which snapped elements may settle. Upon unloading, we find that bi-layers whose snapping elements possess an optimum number of magnets are less likely to have those elements return to the zero-energy ground configuration (see Sec. 2.3.3), an effect which increases the overall energy absorption.

Figure 2.2 also plots the force-displacement response for each loading case with labels for the corresponding configurations depicted in the snapshots. The curves illustrate the familiar hysteretic behavior associated with energy absorption as measured by the area of the enclosed region, $E = \int f du$. In particular, during the loading phase, the tension and compression cases show a well-defined plateau region – which is tunable in the metamaterial platform – over which most of the energy-absorbing snap-through events occur. Continued loading stimulates further, intermittent snapping among progressively smaller sets of elements as the system saturates. In the shear case, snapping begins early in the loading phase and proceeds at a relatively constant rate, such that a definite plateau force cannot be identified. The proposed design strategy does not exploit destructive mechanisms that hinder the re-usability of the cellular material (e.g., plastic deformation, fracture), therefore the stored energy is recoverable.

Impact of the metamaterial's geometric parameters on the absorption performance is complex; no one parameter directly controls a specific feature of the load-displacement diagram. Typically, energy absorbing materials exhibit a long plateau region at a given peak load, f_p . The appropriate distribution of geometric parameters in cellular materials may encourage simultaneous snap-through at a given f_p , maximizing the plateau length, ℓ_{plt} . For the present homogeneous system, f_p and ℓ_{plt} show a strong dependence on the attachment angle, α . Figure 2.4 shows that increasing α decreases f_p while simultaneously increasing ℓ_{plt} . Larger α more effectively converts interaction spring forces to torques acting on the multi-stable elements, permitting snap-through at lower loads. Simultaneously, higher α enable more rotation of the snapping elements which translates into greater overall displacement and, thus, increased plateau lengths.

2.3.2 Symmetry and Isotropy

The hysteretic response exhibited by the square lattice in Figure 2.2 is not limited to a single loading direction; regarding tension/compression, the energy absorption capability persists for loading along axes of arbitrary orientation, θ . Similarly, loading any two parallel planes in shear will generate a hysteresis. These results stem from the internal chiral architecture of the cellular material which guarantees that any deformation of the bi-layer is accommodated by a non-zero φ , promoting snap-through. Moreover, the chiral feature permits energy absorption irrespective of the underlying lattice arrangement: crystalline *or* quasi-crystalline. Here we examine the absorption response of crystalline square and triangular lattices, as well as quasi-crystalline P3 Penrose⁶⁶ and Ammann-Beenker (AB) tilings (Figure S4 of SI).

Figure 2.3b plots the direction-dependent energy absorption, $E(\theta)$, for cellular materials with various (quasi-)crystalline microstructures that are loaded in tension, compression, and shear. Apparently, loading bi-layers at particular orientations elicits a more efficient absorption response. These arrangement- and loading-specific orientations correspond to a lower effective stiffness, leading to larger displacements which promote absorption. Naturally, the directional response of each bi-layer is directly related to the rotational symmetry of the component lattices. While quasi-crystals lack translational symmetry, they possess a local rotational symmetry which may exceed those of traditional crystals – a feature which may elicit a more isotropic response. The square, triangular, and AB samples exhibit a directional energy absorption with four-, six-, and eight-fold rotational symmetry, respectively, which is consistent with that of the direct lattice of the internal architecture. Seemingly to the contrary, the Penrose bi-layer, characterized by an internal architecture with fifth-order rotational symmetry, produces an energy diagram with ten-fold rotational symmetry. This is due to an additional five axes of mirror symmetry that complete the D_5 dihedral group of the lattice. As an odd dihedral group, each of the ten symmetries in the group gives an inner automorphism which results in the same energy absorption.⁶⁷

In practical settings where the loading direction is unpredictable, it may be beneficial for materials to exhibit an energy absorption capacity that is isotropic. Nevertheless, energy diagrams with higher order rotational symmetry – indicating identical energy trapping for a greater number of loading directions – do not imply isotropic energy absorption for identical loading as effectiveness varies with θ . A truly isotropic material would have no variation in its energy absorption capacity with respect to orientation and, hence, produce a circle in a polar plot. Here, in order to quantify the directionality of energy absorption of various (quasi-)crystalline cellular bi-layers, we measure the coefficient of variation, $c_v = \sum^N [E(\theta_i) - E_{avg}]/[E_{avg}(N-1)]$, of the energy plots in Figure 2.3b, which approaches zero as the response becomes isotropic. Figure 2.3c summarizes the results, indicating that, of the considered arrangements, the P3 sample most closely approximates an isotropic response despite possessing a lower degree of rotational symmetry than the AB sample.

2.3.3 Exploring the Design Parameter Space

The metamaterial platform facilitates the customization of the load-displacement curve and (thus, the energy absorption) through adjustments to the material and geometric parameters defining the internal architecture, representing, potentially, a vast design space. In order to demonstrate the impact of certain design choices on the bi-layer absorption performance, we explore the parameter space of a bi-layer with square arrangement at orientation, $\theta = 0^{\circ}$. Simulations apply an identical, quasi-static loading-unloading schedule to bi-layers characterized by unique architectures and measure the corresponding absorption. Figure 2.4 and Figure 2.5a–d plot the simulation results, revealing the combination of design parameters – classified as local (e.g., k_t and n) or non-local (e.g., k_c and α) type – which maximize/minimize the absorption for a particular mode of loading.

In addition to lattice arrangement, the non-local design parameters control the receptibility of the bi-layer to energy absorption by specific modes of loading. Figure 2.5a-c, respectively, plot the absorption for bi-layers under tension, compression, and shear as a function of α and k_c . Naturally, $\alpha = \{0^\circ, 180^\circ\}$ (not shown) destroys the lattice chirality and, consequently, the ability to stimulate snap-through. As illustrated in Figure 2.5, for α away from an optimum value, the energy absorption capability of the bi-layer declines. This can be explained, in part, by the torque applied to the rotation centers by the interaction spring forces, whose lever arm is maximized at the optimal attachment angle, promoting snap-through and absorption. Away from the optimum α , snap-through becomes harder to stimulate. In addition, upon unloading, due to the non-degenerate on-site potential, ψ , and to the non-uniform phase distribution (as observed in Figure 2.2) resisted by the interaction springs, the material tends toward the undeformed state (i.e., recovery) and hence the absorption diminishes. In tension, compression, and shear, the maximum absorption is observed at $\alpha \approx 38^\circ$, $\alpha \approx 30^\circ$, and $\alpha \approx 45^\circ$, respectively. In general, as k_c increases, less energy is absorbed as the bi-layer deformation is diminished.

The local design parameters shape the on-site energy landscape responsible for snap-through, a critical effect for establishing a hysteresis in the load-displacement curve. Figure 2.5d plots the energy absorption of the cellular bi-layer (in tension) as a function of the number of magnets per disc, n, and the torsional stiffness, k_t , where an extremum at $(n = n_c, k_t) = (33, 10^{-2})$ suggests an optimum combination of these parameters which maximizes

the absorption. This can be understood by observing that as *n* moves away from the critical value, n_c , the absorption tends toward zero since either the barrier, $\Delta \psi_S$, becomes prohibitively high for the loading to stimulate snap-through ($n < n_c$) or the barrier, $\Delta \psi_R$, decreases such that (non-)local effects more effectively push elements to low energy configurations ($n > n_c$). Naturally, as the torsional spring resists the relative rotation of paired discs, increasing k_t inhibits snap-through which, in turn, diminishes the absorbed energy. Figure 2.5e summarizes for a given k_t , showing that as *n* increases, the number of meta-stable states (i.e., the opportunity for snap-through) and the depth of the corresponding energy wells simultaneously decrease.

2.4 Conclusion

In summary, this chapter proposes a strategy to construct two-dimensional, cellular metamaterials for mechanical energy absorption under general loading. The strategy assembles a cellular bi-layer from two lattices of identical arrangement but opposite chirality and coupled elasto-magnetically through the aligned rotation centers to form multi-stable elements. Subsequently, any deformation of the bi-layer in response to general mechanical loading – tension/compression along any axes and shearing of any pair of parallel planes – stimulates snap-through, leading to hysteresis in the load-displacement curve indicative of energy absorption. The strategy is amenable to a number of (quasi-)periodic arrangements, demonstrating its flexibility in lattice construction. Here, example bi-layers composed of square, triangular honeycomb, P3 Penrose, and Ammann-Beenker chiral lattices are loaded quasi-statically, revealing a direction-dependent energy absorption reflecting the rotational symmetry of the lattice arrangement. We expect anti-chiral and hybrid architectures to yield similar results.

A number of parameter studies illuminated the impact of material and geometric design choices on the absorption. In particular, for a given loading-unloading schedule, the relationship between absorption and the number of magnets is complex: too few magnets manifest large energy barriers, hindering snap-through while too many magnets risks eliminating meta-stable states altogether. Nevertheless, replacing the magnets with elastic embossments (Figure S2 of SI) would void this trade-off and, thus, in addition to being more amenable to realization via 3D-printing, may be the more attractive option.

In this chapter, we present the metamaterial energy absorption response due to quasi-static loading; however, the response to, e.g., large-amplitude, harmonic loading – especially in light of the metamaterial dispersion characteristics – has yet to be considered. In addition, while we apply the proposed strategy toward the realization of planar metamaterials, the construction and characterization of curved surfaces – a logical extension into the third dimension – represents a potential subsequent research direction.

2.5 Acknowledgements

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Figure 2.2: Loading the Cellular Material. (a) The evolution of in-plane displacement and phase distribution in a square cellular bi-layer subject to quasi-static loading in (a) tension, (b) compression, and (b) shear. The phase distribution shows the transition to progressively higher meta-stable states (i) during loading, (ii) at max loading, and (iii) following unloading. The corresponding hysteresis of the force-displacement diagram is a testament to the energy absorption capability of the bi-layer under each loading condition.



Figure 2.3: Rotational Symmetry of Cellular Bi-layers. (a) Sample of a cellular bi-layer with underlying square lattice at orientation, θ , loaded along opposing boundaries (blue). (b) The direction-dependent energy absorption of various (quasi-)crystalline lattices. The shaded regions of the polar plots highlight the unique absorption response specific to the bi-layer arrangement and loading mode, and beyond which the response repeats. (c) A comparison of the coefficient of variation amongst the different lattice arrangements, a measure for effective isotropy.



Figure 2.4: Effect of attachment angle on the peak load, f_p , and plateau length, ℓ_{plt} , for a square lattice at 0° under tensile loading.



Figure 2.5: Dependence of energy absorption on local and non-local parameters. Energy absorption, E, varies with the attachment angle α and the non-local connection stiffness k_c in (a) tension, (b) compression, and (c) shear. (d) Locally, the torsional stiffness k_t and the number of magnets on a disc *n* alter the local potential which affects absorption. (e) The number of stable states and the ratio of the energy barriers for reversion and transition vary with the number of magnets, characterizing the local potential landscape.
Chapter 3

Manipulation of structural and absorption capabilities in rotationally multi-stable metamaterials

Abstract

This chapter investigates methods for tuning the structural response of the proposed rotationally multi-stable metamaterial at the lowest structural orders, specifically at the unit cell and at the microscale. Theoretical analysis of the multi-stable element and a numerical model based on recursive coordinate reduction are employed to investigate the impact of the disc radius, attachment angle, and periodicity of the local multi-stable interaction on the structural response. Specifically, structural stiffness, peak force, plateau length, and mean plateau force are examined. It is found that while peak force and stiffness may achieve a wide range of values through variation of the disc radius and the attachment angle, this comes at the sacrifice of densification strain and hence absorption capacity. However, stiffness and peak force may be reliably tuned through variation of the multi-stable interaction's periodicity, with higher periodicities increasing load bearing capabilities. To demonstrate these effects regardless of the micro-structure employed,

these trends are observed in tensile loadings of both square and hexagonal-honeycomb lattices. These microstructures also affect the absorption plateau, with hexagonal- honeycomb structures leading to a lower mean force. Microstructure may be leveraged further with functional grading of properties in order to decrease the plateau height.

3.1 Motivation

A variety of applications for energy absorbing materials also demand load-bearing capacity, such as structures for seismic mitigation and crash boxes of automobiles. Such applications typically employ materials with high elastic moduli, with careful design of the micro-structure which aims to initiate the absorption process once a peak force has been met. Cellular metamaterials with hysteretic behavior based on snap-through, e.g., of buckled/buckling beams [24, 37–48], conical shells [49], or antagonistic arrays of magnets [50, 51] offer a means of reusable energy absorption. However the development of snapping metamaterials has proceeded with little consideration to tailoring the force vs. displacement diagram to suit the needs of practical applications. Typically the structural response is normalized, leaving the tailoring of stiffness and other load-bearing qualities in the design to selection of the composing materials elastic modulus.

While structural capability will always be limited in some capacity by material composition, we delineate structural techniques for tailoring the response in order to enhance the capabilities of any composing material. We begin at the fundamental scale of the snapping element and its connections to a neighbor. Examining kinematic variables at this level builds trends for the structural stiffness, peak force, and densification strain which are independent of microstructure. However, microstructure cannot be ignored entirely, as stress-concentration stemming from any geometric constraints or boundary conditions in the design will nucleate snap-through at elements with higher stress, and hence heterogeneously, affecting the force plateau. A myriad

of micro- structures exist which may be used to manipulate the force vs. displacement curve, but a popular method implemented in foams is that of spatial grading of density. We take a similar approach, demonstrating graded variation of the fundamental parameters in order to affect structural response.

3.2 Structure and Methods

We will adopt a slightly different numerical model than previously employed. We do so in order to eliminate as much dependence on composing material properties as possible. The connections between nodes of chapter 2 were previously modelled as linear springs, the stiffness of which served to normalize other kinematics parameters in the structure. As a result the local multi-stable interaction was defined relative to this value. We now replace the spring connections with rigid linkages, eliminating any dependence on their stiffness. Our normalizing stiffness is hence the amplitude of the multi-stable interaction, a value which may be tuned practically if electromagnets are used to produce the desired effect. The substrate beam structure is also abandoned, along with the torsional spring on the local potential, leavign a purely sinusoidal interaction. With rigid linkages, the dynamics of resulting structures are heavily constrained geometrically. For simple cases, i.e. single chains, these constraints may be used to solve for several structural properties, as done in the following sections. Beyond these extremely simple cases, analytical predictions are incredibly complex; we hence leverage numerical simulation.

3.2.1 Geometry and Restrictions

This section analyzes the unit cell of a one-dimensional chain in order to determine geometric restrictions on the disc radius r and the rotation ϕ . These limits result from impingement between connectors and discs. The unit cell consist of a single two-disc node and the two connectors attached to them. The origin of the system will be placed at the first node's center. The position of the second node, taking into account the constraints imposed by the pin joints, is given by

$$\mathbf{r}_{2} = r(\cos\phi_{t}^{1} \mathbf{E}_{x} + \sin\phi_{t}^{1} \mathbf{E}_{y}) + \ell(\cos\theta_{t} \mathbf{E}_{x} + \sin\theta_{t} \mathbf{E}_{y}) + r(\cos\phi_{t}^{1} \mathbf{E}_{x} + \sin\phi_{t}^{2} \mathbf{E}_{y}),$$

$$= r(\cos\phi_{b}^{1} \mathbf{E}_{x} + \sin\phi_{b}^{1} \mathbf{E}_{y}) + \ell(\cos\theta_{b} \mathbf{E}_{x} + \sin\theta_{b} \mathbf{E}_{y}) + r(\cos\phi_{b}^{2} \mathbf{E}_{x} + \sin\phi_{b}^{2} \mathbf{E}_{y}).$$
(3.1)

This states that the path along the top and bottom layer must both point to the center of node 2. The angle of the connectors are given by θ_t and θ_b . In practice, the angles above are independent, but assuming static equilibrium and deformation only in the *x* directions leads to equivalent states for the two nodes, eliminating the need superscripts, along with:

$$\phi_b = -\phi_t \equiv -\phi,$$

$$\theta_b = -\theta_t \equiv -\theta.$$
(3.2)

We now form geometric constraints in the system. The x and y positions of the next disc in the chain are

$$x = 2r\cos\phi + \ell\cos\theta,$$

$$y = 2r\sin\phi + \ell\sin\theta.$$
(3.3)

The assumed deformation along the x direction leads to

$$x = 2r\cos\phi + \ell\cos\theta,$$

(3.4)
$$0 = 2r\sin\phi + \ell\sin\theta.$$

The above may be used to solve for $x(\phi)$, giving

$$x = 2r\cos\phi + \sqrt{\ell^2 - 4r^2\sin^2\phi}.$$
 (3.5)

This determines the position of the second node in the chain as a function of the rotation. In the initial state, we have $\phi = \phi_0$ and the normalized position $x(\phi_0) \equiv 1$, giving ℓ as

$$\ell = \sqrt{4r^2 + 1 - 4r\cos\phi_0}.$$
(3.6)

Not all configurations are admissable. The value of x takes a natural upper bound of $2r + \ell$ at $\phi = 0$, which marks the point of tensile densification. However we must limit rotations in order to prevent impingement of bodies. A connector will impinge with the disc if it moves beyond the tangent point. This places the limit

$$\phi \le \arctan \frac{\ell}{2r}.\tag{3.7}$$

This limit marks the point of compressive densification, at which point snap-through ceases. Furthermore, we must limit *r* so as to prevent impingement of discs in the initial state, at which $\phi = \phi_0$ and x = 1. This leads to

$$r \le 0.5. \tag{3.8}$$

Limits on the initial rotation follow from the impingement condition, yielding

$$\phi_0 \le \arccos(2r). \tag{3.9}$$

These inequalities mark points of impingement and limit possible values for the system's configuration.

3.2.2 Theoretical Prediction of Peak Force and Stiffness

This section analyzes the unit cell of an infinite one-dimensional chain in order to predict the stiffness and peak force of the metamaterial as functions of the disc radius r, the initial attachment angle ϕ_0 , and the multi-stable interaction's periodicity ψ_0 . Under simplifying assumptions, the interaction between two nodes is likened to that of a nonlinear spring.

The unit cell consist of a single two-disc node and the two connectors attached to them.

Resistance to deformation is only provided by the local multi-stable moment M at the node. The external force on the system balances the moment produced from the resultant deformation, i.e.

$$\mathbf{r} \times \mathbf{F} = M \mathbf{E}_{z},$$

$$\rightarrow r(\cos \phi \mathbf{E}_{x} + \sin \phi \mathbf{E}_{y}) \times -F_{t}(\cos \theta \mathbf{E}_{x} + \sin \theta \mathbf{E}_{y}) = M \mathbf{E}_{z},$$

$$r(\cos \phi \mathbf{E}_{x} + \sin \phi \mathbf{E}_{y}) \times -F_{b}(\cos \theta \mathbf{E}_{x} + \sin \theta \mathbf{E}_{y}) = -M \mathbf{E}_{z}.$$
(3.10)

In the above, F_t and F_b are the forces carried in the top and bottom connectors. For the time being, the form of $M(\phi)$ will be left general. Once again, we hold our previous assumptions of static equilibrium and pure *x* deformation, which leads to $F_b = F_t \equiv F$. Carrying out the cross products yields

$$rF(\sin\phi\cos\theta - \cos\phi\sin\theta) = M. \tag{3.11}$$

Specifically we are interested in the force in the *x* direction, i.e. $F_x = F \cos \theta$. Eliminating explicit dependence on θ and ℓ in 3.11 gives

$$F_x = \frac{M}{r}\csc\phi - \frac{2M\cot\phi}{2r\cos\phi + \sqrt{4r^2\cos^2\phi - 4r\cos\phi_0 + 1}}.$$
 (3.12)

We see the force as a function of radius *r*, the initial attachment angle ϕ_0 , and of course the local moment *M*, which is inherently a function of the periodicity ψ_0 . The above may be used to predict the peak force for both compressive and tensile loadings by finding the extrema within the first snap through, i.e. $\phi_0 - \psi_0/2 < \phi < \phi_0 + \psi_0/2$, denoted by ϕ_c^{\min} and ϕ_c^{\max} . Prediction of the peak force is best sought numerically, as it is dependent on the local moment *M* and finding extrema through the roots of $dF_x/d\phi$ is rather complex. While presently displayed as a function of the rotation ϕ , we may invert the relation (3.5) to obtain the force as a nonlinear function of displacement, likening the interaction to that of a nonlinear spring.

We would also like to characterize the stiffness. This is given by

$$k_x = \frac{\mathrm{d}F_x}{\mathrm{d}x} = \frac{\mathrm{d}F_x}{\mathrm{d}\phi}\frac{\mathrm{d}\phi}{\mathrm{d}x}.$$
(3.13)

One may apply this to 3.12 to obtain the terms of the right hand side and evaluate at $\phi = \phi_0$ for the stiffness at the initial state, with the assumption that M(0) = 0. The initial stiffness is hence.

$$k_x(\phi_0) = \frac{M'(0)(1 - 2r\cos\phi_0)}{r\sin\phi_0} \frac{4r^2 - \ell^2 - 1}{\sqrt{-(\ell - 2r - 1)(\ell + 2r - 1)(\ell - 2r + 1)(\ell + 2r + 1)}}.$$
 (3.14)

It is important to note that the periodicity ψ_0 of the multi-stable interaction will manifest in M'. In this case, we assume $M = -\sin(4\pi \frac{\phi}{\psi_0})$. The predictions can be visualized as functions of r and ϕ_0 for various periodicities in Fig. ??.

3.2.3 Scaling for Plateau Length and Mean Force

The plateau length L_p is related simply to the maximum obtainable deformations of the unit cell, which occur at $\phi = 0$ for tension and $\phi = \arctan \frac{\ell}{2r}$ for compression. Exact values for the plateau length will vary with microstructure. The length of the unit cell at these points from (3.5) determines a proportional relation for the force plateau, giving

$$L_p \propto 2r + \sqrt{4r^2 + 1 - 4r\cos\phi_0} \qquad \text{(tension)},$$

$$L_p \propto \sqrt{8r^2 + 1 - 4r\cos\phi_0} \qquad \text{(compression)}.$$
(3.15)

This may be visualized in figure 3.4.

The mean force of this plateau is highly structurally dependent and we hence rely on parameter studies to empirically determine the effects of geometric parameters. We numerically test a 1D chain of 10 unit cells over various values of r, ϕ_0 , and ψ_0 . While these simulations do not provide quantitative predictions, the trends established for a 1D chain hold for other microstructures.

3.2.4 Simulation

Moving beyond the previous analytical predictions requires the use of numerical simulations. The structure consists of discs and connectors, both modelled as planar rigid bodies with three degrees of freedom. These bodies are connected by perfect revolute joints, each joint removing two degrees of freedom from the system. The lattice structure produces complex geometric relations between bodies, making a minimal set of coordinates for the remaining degrees of freedom nearly impossible to define for a general system. We therefore employ a method for the modelling of articulated robots known as recursive coordinate reduction in order to formulate equations of motion for the system. The resultant equations are advanced in time through a semi-explicit integration scheme m with an adaptive time step and geometric constraint-drift control.

The employed method incorporates as many constraints from the joints into the formulation of the equations of motion as possible, while enforcing the remaining joint constraints through Lagrange multipliers. Selection of the primary joints is made by first representing the system as an undirected graph, with nodes representing bodies and edges between nodes representing the revolute joints. This graph may have many circuits, or loops. A minimum spanning tree is then produced by eliminating edges such that no loops remain; in this step we make the distinction between the primary joints whose constraints are automatically satisfied and secondary joints whose constraints must be enforced through Lagrange multipliers. Minimum spanning trees carry the important property that between any two nodes, there exists only one path. We formulate the paths between a designated root node and all other nodes in the minimum spanning tree. A body's velocity may then be defined recursively as the velocity of the body before it in the path plus a contribution from rotation about the revolute joint. This is done repeatedly by travelling backwards along the path to the root node. The result is a coordinate transformation **N** from the unconstrained velocity coordinates **t** of the individual bodies to a minimal spanning coordinate set $\dot{\theta}$ representing the rotational velocities of the primary joints. This transformation is applied to the equations of motion and supplemented from constraint forces realized through Lagrange's prescription. The transformation and constraint enforcement is summarized in

$$\begin{split} \mathbf{M}\dot{\mathbf{t}} + \mathbf{C}\mathbf{t} = \mathbf{F}, & (\text{unconstrained EOM}), \\ \mathbf{t} = \mathbf{N}\dot{\boldsymbol{\theta}}, & (\text{coordinate transformation}), \\ \mathbf{N}^T \mathbf{M} \mathbf{N} \ddot{\boldsymbol{\theta}} + \mathbf{N}^T (\mathbf{C} \mathbf{N} + \mathbf{M}\dot{\mathbf{N}}) \dot{\boldsymbol{\theta}} = \mathbf{N}^T \mathbf{F}, & (\text{transformed EOM}), \\ \Psi = \mathbf{0}, & (\text{holonomic constraints}), \\ \Psi = \nabla_{\boldsymbol{\theta}} \Psi \dot{\boldsymbol{\theta}} = \mathbf{J} \dot{\boldsymbol{\theta}} = \mathbf{0}, & (\text{velocity constraints}), \\ \ddot{\Psi} = \mathbf{J} \ddot{\boldsymbol{\theta}} + \dot{\mathbf{J}} \dot{\boldsymbol{\theta}} = \mathbf{J} \ddot{\boldsymbol{\theta}} + \mathbf{b} = \mathbf{0}, & (\text{velocity constraints}), \\ \mathbf{V} = \mathbf{J} \ddot{\boldsymbol{\theta}} + \dot{\mathbf{J}} \dot{\boldsymbol{\theta}} = \mathbf{J} \ddot{\boldsymbol{\theta}} + \mathbf{b} = \mathbf{0}, & (\text{acceleration constraints}), \\ \mathbf{F}_c = \mathbf{J}^T \lambda, & (\text{constraint force}), \\ \begin{bmatrix} \mathbf{N}^T \mathbf{M} \mathbf{N} & -\mathbf{J}^T \\ \mathbf{J} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \ddot{\boldsymbol{\theta}} \\ \lambda \end{bmatrix} + \begin{bmatrix} \mathbf{N}^T (\mathbf{C} \mathbf{N} + \mathbf{M} \dot{\mathbf{N}}) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{\theta}} \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{N}^T \mathbf{F} \\ -\mathbf{b} \end{bmatrix}, & (\text{dynamical equations}). \\ & (3.16) \end{split}$$

The last equation above is a simultaneous expression of the equations of motion with the constraint force moved to the left hand side (top) and the the acceleration level constraint (bottom). It may be solved in two steps as

$$\lambda = [-\mathbf{J}(\mathbf{N}^T \mathbf{M} \mathbf{N})^{-1} \mathbf{J}^T]^{-1} (\mathbf{J}(\mathbf{N}^T \mathbf{M} \mathbf{N})^{-1} \mathbf{N}^T (\mathbf{F} - \mathbf{C} \mathbf{N} - \mathbf{M} \dot{\mathbf{N}}) + \mathbf{b}),$$

$$\ddot{\mathbf{U}} = (\mathbf{N}^T \mathbf{M} \mathbf{N})^{-1} \mathbf{J}^T \lambda + (\mathbf{N}^T \mathbf{M} \mathbf{N})^{-1} \mathbf{N}^T (\mathbf{F} - \mathbf{C} \mathbf{N} - \mathbf{M} \dot{\mathbf{N}}).$$
(3.17)

The two step process allows the same decomposition to be used for $(\mathbf{N}^T \mathbf{M} \mathbf{N})^{-1}$ in the computation of $(\mathbf{N}^T \mathbf{M} \mathbf{N})^{-1} \mathbf{J}^T$ and $(\mathbf{N}^T \mathbf{M} \mathbf{N})^{-1} \mathbf{N}^T (\mathbf{F} - \mathbf{C} \mathbf{N} - \mathbf{M} \dot{\mathbf{N}})$. It should be noted that all inversions may utilize a Cholesky decomposition.

3.3 Results and Discussion

Results are broken down into load-bearing qualities (stiffness and peak force) and characteristics of the absorption plateau (plateau length and mean force). Load bearing capacity is improved through manipulating the fundamental scale, while the absorption plateau is tuned through both the fundamental scale and the microscale structures.

3.3.1 Load-bearing Capacity

The relations defining the structural stiffness and peak force derived in 3.12 and 3.14 can be visualized for a variety of parameters in Fig. 3.1. The three panels demonstrate the effects of individually varying the radius r, attachment angle ϕ_0 , and the multi-stable periodicity ψ_0 , from left to right. Three distinct curves are shown for particular values of he varied parameter, but they are overlaid on a grey envelope which encompass the pre-snap response of a unit cell over all admissable values. The figure shows both tensile and compressive cases, which are not symmetric despite similar appearances. Panel one demonstrates that increasing the radius r decreases both the stiffness and peak force. This is simply because the moment arm increases with radius. Since the moment of the multi-stable interaction does not scale with the radius, the necessary force required to deform the unit cell decreases. The final blue curve of the panel does not actually reach a maximum with zero-derivative in a compressive loading; this is because the radius chosen for this curve results in impingement of the connector and the disc prior to snap-through. Panel two of Fig. 3.1 demonstrates the effects of the attachment angle ϕ_0 . Just like the radius, a higher attachment angle produces a larger moment arm and reduced stiffness and peak force. The final curve was once again selected to demonstrate the impingement of the disc and connector in compressive loadings. In both of these cases, stiffness and peak force are coupled.

The third and final panel of Fig. 3.1 shows the impact of the multi-stable periodicity ψ_0 . In contrast to the other parameters, the stiffness can apparently be tuned relatively independently of the peak force. Higher periodicities require the multi-stable potential to be traversed over much



Pre-snap Envelopes for Structural Stiffness and Peak Force

Figure 3.1: This figure shows the effects of the fundamental-scale parameters on the load-bearing capabilities of a system, following the predictions of 3.12 and 3.14. In the left panel, the disc radius r is varied while the attachment angle ϕ_0 and multi-stable periodicity ψ_0 are kept constant. This produces a grey envelope in which all pre-snap curves lie. Three example curves are shown, with the blue curve displaying a choice of r that leads to impingement of disc and connector prior to snap through. The middle panel displays the same, this time varying the attachment angle ϕ_0 . The right panel varies the multi-stable periodicity ψ_0 .

shorter distances, producing higher stiffnesses. However, since the amplitude of the interaction is unchanged, the peak force remains relatively constant.

The joint effects of the parameters may be observed in Figures 3.2 and 3.3, which produce $r \times \phi_0$ heat maps for the stiffness, peak force, and maximum strains for two periodicities of the multi-stable interaction. These plots provide a good visualization of the restrictions placed on the parameters by the impingement conditions. Furthermore, decrease in periodicity can be seen to further reduce the admissable parameter space in Fig. 3.3. If the critical angle at which disc-connector impingement occurs isn't sufficiently far away from the attachment angle, then no snap through will occur and no energy will be absorbed, i.e. $\phi_c - \phi_0 > \psi_0/2$.

The third columns of these plots demonstrate another important consideration: the effect on maximum strain.



Effects of Geometric Parameters for Periodicity $\psi_0 = 1^\circ$

Figure 3.2: Above the joint effects of varying the disc radius r and attachment angle ϕ_0 can be viewed for the stiffness (left column), peak force (middle column), and also the maximum strain (right column). The top row represents values for tensile loading and the bottom row for compressive loading. A portion of the map is white as it is not geometrically admissable due to the contact of discs and connectors. For the above, the multi-stable periodicity ψ_0 is 1°.

3.3.2 The Absorption Plateau

The chief characteristics of the absorption plateau are its length and its height. Ideally, the plateau is long with a constant, low value. The fundamental scale may be used to affect the plateau length for any micro-structure. the third column of Figures 3.2 and 3.3 demonstrate the maximum obtainable strain for tensile and compressive loadings as a function of the attachment angle ϕ_0 and the radius *r*. For the tensile case, the maximum strain is reached when the disc has rotated from ϕ_0 to zero. For the compressive case, the maximum is when the disc rotates from ϕ_0 to the critical angle ϕ_c where the connector impinges with the disc. Larger radii cause larger displacements upon rotation, leading to larger maximum strains for both tensile and compressive modes. However the attachment angle cannot be used to unilaterally increase strain in both modes, The further the attachment angle from the angle that determines densification (0 for tensile and ϕ_c for compressive), the greater the strain will be. These trends may be observed for a 1D chain of 10 nodes in Fig. 3.4. Each of the parameter combinations shown are extended to 97% of their maximum tensile strain. It is verified that independent increases to the radius and attachment angle



Effects of Geometric Parameters for Periodicity $\psi_0 = 18^\circ$

Figure 3.3: This aids in visualizing how the periodicity ψ_0 may affect the structural response. The largest factor is that the admissable region for compression shrinks, as snap-through may not be met prior to disc-connector contact.

increase the plateau length, while changing the periodicity has no effect. In general, the plateau reaches a value similar to that of the peak force, with a strain hardening effect due to the decreasing moment arm in tensile loadings. Oscillations due to snap through occur according to the periodicity of the multi-stable potential. While the fundamental parameters can be used to set the height of the plateau to that of the peak force, they are unable to tune the plateau independently of their effects on the pre-snap region.

The microstructure however may be chosen to manipulate the plateau height quite effectively. A myriad of options for microstructures are available, and we compare a few here. Fig. 3.6 shows the 60% maximum tensile extension loading curves of a roughly 12×12 hexagonal honeycomb lattice with several fundamental-scale variations, while Fig. 3.5 shows the same for a square lattice. The stiffness and peak force of each microstructure follow the same trends for the fundamental parameters as the 1D chain. However, comparing the two figures shows dramatically different response in the absorption plateau. The hexagonal lattice has a absorption plateau whose mean value is much lower than the peak force, in contrast to that of the square lattice. The height of the plateau is directly related to the number of nodes which are in the negative-stiffness portion of a phase transition. These nodes serve to reduce the load necessary to



Figure 3.4: Various tensile loading curves for a 10 unit long 1D chain are shown. They are taken to a 97% maximum strain. Along the first row, the radius *r* is increased, demonstrating the predictions of before. Likewise, the second column varies the attachment angle ϕ_0 and the third varies the periodicity ψ_0 .

hold a desired displacement. In a square lattice, the higher number of connections leads to a large number of nodes snapping at once, creating larger oscillations in the absorption plateau. However, in the hexagonal honeycomb lattice, transition amongst nodes is more sequential due to the lower connectivity, leading to smaller oscillations and a more constant plateau.

Beyond a heterogeneous microstructure, we take a small step here toward meso-scale structural design. Taking inspiration from density-graded foams, we demonstrate the effect of spatial grading of the fundamental parameters in a square lattice microstructure. The results can be seen in Fig. **??**. The first panel shows a square lattice with linearly increasing disc radius r from left to right, with maximum and minimum radii corresponding to the values tested in Fig. **3.5**. In this case, nodes with a higher radius have a lower radius to transition, and hence phase transition in the structure is more sequential. The absorption plateau is more constant than that of the curves displayed in the first column of Fig. **3.5**, yet the stiffness and peak force of the pre-snap through region are still higher than that of the curve corresponding to a heterogeneous structure of



Figure 3.5: Tensile loading curves for a square lattice with various fundamental-scale parameters demonstrate their effect regardless of microstructure.

the higher radius value. The next panel shows an analogous grading for the attachment angle ϕ_0 which displays a similar effect. The third panel contains the loading history for a structure whose multi-stable periodicity decreases from left to right. Here we yet again see a similar effect.

3.4 Conclusion

The fundamental scale plays a vital role in determining the macro-scale response regardless of the micro-structure employed. The disc radius *r* and initial attachment angle ϕ_0 play similar roles in affecting the structural stiffness and peak force through the same fundamental effect. Increases in these parameters increase the moment arm of any force applied to a disc, making transition to the next state easier. As a result, stiffness and peak force decrease. Increases in these parameters also lead to a higher maximum strain, and hence a longer absorption plateau. The periodicity of the multi-stable interaction ψ_0 demonstrates to tune the stiffness relatively independently of the peak force, with lower ψ_0 values increasing the stiffness. This parameter additionally has no effect on the length of the absorption plateau. However, a simple 1D chain demonstrates that this plateau



Figure 3.6: A hexagonal honeycomb lattice undergoes the same tests as the square lattice above, and displays an absorption plateau with significantly less oscillation.

oscillates between the peak force an a minimum, regardless of the tuning of the fundamental scale parameters. To manipulate the absorption plateau relative to the peak force, we turn to the microstructure. A hexagonal honeycomb lattice produces an absorption plateau which is longer and of lower mean height than a square lattice of similar size. The use of spatial gradient of the fundamental scale parameters also manipulate the height of this plateau by encouraging more sequential snap-through.

3.5 Acknowledgements

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Figure 3.7: Bridging the gap between the micro-scale and the meso-scale, this figure gives loading curves and morphologies for a square lattice with spatial variation of the fundamental-scale parameters.

Chapter 4

Mimicing polycrystalline structure in the meso-scale structure of metamaterials to control directional energy absorption

Abstract

This chapter investigates the ability of crystal grains- domains of varying shape/size in which the lattice orientation of any one domain is different from those of its neighbours-to manipulate the directional absorption in the rotationally multi-stable metamaterial. In addition, characteristics of the meso-structure, e.g. grain number and size, are examined for their impact on the directional capabilities in the context of stiffness. Artificial metamaterial grains are produced by dividing the metamaterial into a grid and rotating the sub-domains to chosen orientations. The orientations of the resultant grains are found using the Non-dominated Sorting Genetic Algorithm II (NSGA-II) to maximize absorption in the vertical direction while minimizing absorption in the horizontal direction. Experimental results validate the findings. To examine the effects of grain size and number, an FEM frame model is employed to place bounds on the maximum and minimum attainable stiffness (and first and second derivatives thereof) as a function of direction. Finally, a continuum model is employed to distinguish the effects of grain orientations from the effects of imperfect grain boundaries and voids.

4.1 Motivation

Typical metamaterial architectures possess a periodic arrangement resembling the crystal micro- structures of metals in their flawless, single-domain form. This order facilitates the development of analytical formulations linking the internal architecture to the observed performance. In particular, the architectural symmetry manifests in the directional response, producing multi-lobed "floral" patterns with rotational symmetry in polar plots of the response versus orientation, θ , of the loading axis (e.g., see Fig. 2.3). However, most metallic materials consist of several domains (i.e., grains) of varying shape/size in which the lattice orientation of any one domain is different from those of its neighbours. For a mean grain size, the random distribution of grain orientation at the meso-scale, generally, produces direction-independent (i.e., isotropic) behavior at the macro-scale. It is conceivable that the rational design of individual grains would lend control of the macro-scopic directional response. For metals/alloys, grain selection usually implies homogeneous, monolithic production for turbines and aerospace applications.⁶⁸ Selection at specific grains has application in micro-mechanical devices and is carried out through powder-bed approaches with complex thermodynamic control.⁶⁹ This proposal aims to mimic the poly-crystalline meso-structure of metals/alloys at the readily accessible scales of metamaterials. In the metamaterial platform, connections between nodes near grain boundaries present the only complication. Our particular structure benefits from the robust construction demonstrated by accommodation of quasi-crystalline layouts, wherein the number and directions of connections varies from node to node. Polycrystalline architectures of the energy absorbing metamaterial are modeled using the recursive coordinate reduction model proposed in chapter 3. We employ the Non-dominated Sorting Genetic Algorithm II (NSGA-II) in order to optimize the grain structure

to maximize the absorption in the vertical direction while minimizing the absorption in the horizontal direction. This optimized structure is fabricated using 3D printed components and tested to validate the numerical results.

To investigate the potential of polycrystallinity to influence directional properties, a finite element frame model is used to approximate bounds on attainable maximum and minimum values of the structural stiffness as a function of tested angle. In addition the first and second derivatives of the stiffness with respect to angle. These bounds are found using the NSGA-II, and are not rigorous. Nevertheless, they reveal trends and speak to the efficacy of the method. Finally, a continuum approximation to the energy absorbing metamaterial is used in order to distinguish the effects of grain orientation from those of imperfect grain boundaries and voids.

4.2 Structure and Methods

Here, we test the systems exclusively in tension for no reason other than brevity. To optimize energy we absorption, we employ the cellular metamaterial design developed in chapter 3. To evaluate the ability of a polycrystalline meso-strucuture to tune directional properties, we use a computationally efficient finite element (FE) frame model. These two models simulate discrete systems, and as such are subject to odd geometries resulting at the boundary between grains. In light of this, we employ a final continuum model which is free of odd connections at grain boundaries.

4.2.1 Grain Construction

All models study a square grid of $N_g \times N_g$ grains. The sub-grain structure is a rectangular lattice characterized by the lattice parameters a_x and a_y ; if these are equal then a square lattice is recovered. Each grain is of size $\ell_g = a_x(N_x - 1) = a_y(N_y - 1)$, where N_x and N_y are the number of nodes along a side of the grain. Each grain *i* receives an orientation γ_i . Construction of the sample involves placing all nodes in the structure and then forming connections based on positional



Figure 4.1: Directional energy absorption of a (a) homogeneous and (b) poly-crystalline metamaterial with random grain orientations. Absorption is plotted on the radial axis and the orientation of the outer isotropic continuum is plotted on the azimuth. (c) The morphology of the homogeneous system is shown, with the isotropic continuum meshed in blue and the discrete metamaterial in black. (d) Similarly, the morphology of the poly-crystalline sample is shown. Nodes which are common to both the discrete system and the continuous mesh are plotted with a black circle and have their rotational degrees of freedom constrained.

tolerances. To construct a grain, a larger lattice centered to the grain is rotated to the orientation, and nodes which lie within the boundaries of the grain are kept. This process is repeated for all grains. Finally connections are formed amongst the discrete lattice nodes. For a given node, the distance to all other nodes are calculated. If the distance to another node is less than $\sqrt{a_x^2 + a_y^2}$ but greater than 0.6 min a_x , a_y , then the two nodes are given a connection. The upper bound prevents connections along the diagonal of the sub-grain lattice, while the lower bound prevents connections between nodes which are too close.

4.2.2 Grain Selection

The goal is to orient grains to elicit a desired directional response. This presents an optimization problem: the grain orientations γ_i are taken as tuning parameters, and the absorption (or stiffness) E_j at direction θ_j are taken as objectives. To reduce the number of objectives and to allow exact value of storage to be dictated by the kinematic parameters of the system, objectives

are taken to be the ratios E_i/E_0 , i.e. normalized by the absorption (or stiffness) at $\theta = 0$. Due to the relatively small number of parameters, discontinuity in the solution space, and the existence of many local minima, we employ the Non-dominated Sorting Genetic Algorithm-II (NSGA-II).⁷⁰ A flowchart of the work-flow is given in Fig. 4.2. This stochastic algorithm proceeds by generating a population of N_p solutions each with randomly chosen grain orientations. These grain orientations are analogized to the *genome* of each population member. From here, the energy (or stiddness) of each population member is calculated at each direction of interest θ_i . Objectives are given by minimizing the function $|T_j - E_j/E_0|$ for some target value T_j . Population members are sorted into fronts according to their satisfaction of the objectives relative to other population members. This relativity amongst solutions is resolved via *domination*. A solution is dominated by another if the solution performs worse in all objectives. Solutions which are not dominated by any others are collected in the first tier of solutions, termed a *front*. Front f is the collection of solutions dominated by f - 1 other solutions. This organization into fronts comes into play when selecting parents for the production of the next generation of solutions. A binary tournament occurs for two pairs of solutions which are chosen at random. In a given pair, the population member of the higher front advances; if members are in the same front, the population member who is cumulatively further from all other solutions in terms of satisfaction of objectives advances. This is a metric for *crowding distance* and serves to promote diversity in the population. From the two advancing members, orientations for the grains of the offspring are chosen with a 50-50 chance for each orientation to come from either parent. This offspring is then mutated, with each orientation receiving the addition of a small angular deflection, the value of which is pulled from a discrete probability density function of a Gaussian profile. This reproduction process is repeated until a new population is formed. Following this, the new population is evaluated. This new population is composited with the old, and the collected population is sorted into fronts. From here the top N_p solutions, based on front and crowding distance, progress and the rest are neglected; this is termed *elitism.* The process is repeated until the objectives are satisfied to a specific tolerance or a number of prescribed generations has passed. Heuristically, it is better to start with fewer objectives and

use the resultant solutions as the initial population at the next step where more objectives are added.

4.2.3 Structure for Energy Absorption

For optimization of energy absorption, a square sample is constructed using $N_g = 2$, $a_x =$ $a_y = 4.4$ cm, and $N_x = N_y = 13$. The fundamental scale parameters are r = 1.24496 cm, $\phi_0 = \pi/4$, and $\psi_0 = 18^\circ$. The physical model is 3D printed on a Connex Objet 350 multi-material printer using VeroClear. Top and bottom discs along with connections comprise the three printed pieces of the structure. Discs are screwed together with machine screws and lock-nuts, and connections are attached between discs with machine dowels. The multi-stable potential itself is realized using 20 magnets embedded in the discs. The resultant interaction is determined experimentally and fit with a sinusoidal curve characterized by the amplitude $\Psi =$. This structure is tested along two directions, the horizontal and the vertical. Two sides of the square structure have the discs pinned to an acrylic mount, restricting their translational degrees of freedom but leaving them free to rotate. Once the recorded force reaches zero, the we leave the structure in the new stable state with residual displacement. These tests are carried out on an MTS Machine. To optimize the absorption in these two directions, a relatively low population of $N_p = 12$ is used and evolved over 150 generations. Specifically, the objective function used is $|1000 - E_{90^{\circ}}/E_{0^{\circ}}|$. To reduce computational expense, simulations of the structure are run with a much shorter final time than that of the experiment, only $t_f = 100$ in dimensionless units (the physical experiment runs for $t_f = 2857$ dimensionless time units). Furthermore, the adaptive time-step is dropped for these low-fidelity simulations. The resultant morphology is however re-evaluated with the true final time and adaptive time-stepping reincorporated.

4.2.4 Finite Element Models

To investigate the capabilities and limitations of a polycrystalline meso-structure, we employ a low-cost FE frame model which has three degrees of freedom per node (two translational and one rotational). For these studies, we will simulate and test a circular sample. This sample is identified by inscribing a circle to the node structure produced by section 4.2.1 and removing nodes which fall outside the inscription. This sample is fastened to a square isotropic continuum with a circular hole cut to accommodate the lattice, as in Fig. 4.1. The attachment sites themselves have rotational periodicity such that the frame sample can be attached at various orientations. The discrete and continuous structures are joined via a lead-follower constraint method. The outer isotropic continuum is essentially a conduit, and will have its left end fixed and right end given a prescribed displacement while the force on the right end is recorded. The isotropic continuum is modelled with a 2D plane stress approximation implemented with the FE method with Youngs modulus $E_c = 0.08$ and Poisson ratio v = 0.45. These values are chosen so that the effects of the internal frame are prevalent. The parameters for the optimization scheme are varied for these tests. This simple method results in a static problem that may be used to find the reaction force for a given displacement, producing the stiffness as opposed to absorption. In chapter 2 it was demonstrated that when displacement is prescribed, absorption is maximized along directions with higher stiffness, and the two share similar symmetry. This method is computationally efficient compared to the fully nonlinear case and is employed in subsequent optimizations for relatively small structures, yet it still captures the complex interactions at grain boundaries. For larger systems however, e.g. when the number of nodes within a grain exceeds 100, even this method loses some practicality.

For such systems, a third numerical model based on a homegenization procedure is employed following the derivations of Chen et. al.⁷¹ This method is applied to a single tetrachiral lattice and hence implies $a_x = a_y$. For us, the procedure is applied to top and bottom layers separately, and the two are constrained to share translational degrees of freedom. Derivations are supplied in Appendix B. This method assumes rigid discs, however connections are made by beams with bending stiffness at the point of connection to discs. This provides a resistive torque to rotation of the discs. Furthermore, this method does not employ a local interaction; instead the resistance to rotation is provided simply by the bending moment of the connections. The homogenization approach produces a continuum approximation of the lattice which is modeled with the FE method. Again, these are static simulations which determine the stiffness in all directions. This method loses information regarding the complex interactions that occur at grain boundaries and at the interface between the chiral metamaterial and the isotropic continuum. As a result, resultant directional dependence is rooted solely in the distinct orientations of the grains.



Figure 4.2: Flowchart outlining the Non-dominated Sorting Genetic Algorithm II.⁷⁰

4.3 **Results and Discussion**

4.3.1 Absorption in a Polycrystalline Metamaterial

First, the multi-stable interaction is characterized experimentally. The physical model used for testing consists of a 1D chain, shown in Fig. 4.3. The chain consists of 3 nodes. The bottom node is constrained in all degrees of freedom and hence cannot move or rotate. This is a chieved through attachment to a mount that connects to the MTS machine. The top node is constrained similarly, except its mount is connected to the top grip of the MTS machine, which connects to a 20N load cell. This top node is then displaced and the force recorded by the load cell. The center node has no constraints. This model differs slightly from other in that every joint contains a precision bearing, in order to minimize any play or friction, giving an accurate prediction for the multi-stable interaction. This simple geometry may be analyzed similarly to the algebra carried out in chapter 3 in 3.12. The position of the top node is

$$x = 2r\cos\pi/4 + 2\ell\cos\theta + 2r\cos\phi,$$

$$y = 0 = r\sin\pi/4 + \ell\sin\theta + r\sin\phi.$$
(4.1)

Here, θ is the angle of the arm. Rearranging gives

$$\ell \cos \theta = x/2 - \left[r \cos \pi/4 + r \cos \phi \right],$$

$$-\ell \sin \theta = r \sin \pi/4 + r \sin \phi.$$
 (4.2)

Squaring and summing gives

$$\ell^{2} = \left(x/2 - r \left[\cos \pi/4 + \cos \phi \right] \right)^{2} + r^{2} \left(\sin \pi/4 + \sin \phi \right)^{2}.$$
(4.3)

Some rearranging gives

$$\ell^2/r^2 = \left(\frac{x}{2r} - \left[\cos\pi/4 + \cos\phi\right]\right)^2 + \left(\sin\pi/4 + \sin\phi\right)^2.$$
(4.4)

A trig identity gives

$$\ell^2 / r^2 = \left(\frac{x}{2r} - \left[\cos \pi / 4 + \cos \phi\right]\right)^2 + \left(\sin \pi / 4 + \sqrt{1 - \cos^2 \phi}\right)^2.$$
(4.5)

Expanding

$$\ell^2/r^2 = x^2 - \left(\frac{x}{r} - \sqrt{2}\right)\cos\phi + \left(\frac{x}{2r} - \frac{\sqrt{2}}{2}\right)^2 - \cos\phi^2 + \sqrt{2 - 2\cos\phi^2} + \frac{3}{2}.$$
 (4.6)

This leads to

$$\frac{\ell^2}{r^2} - \frac{3}{2} - \left(\frac{x}{2r} - \frac{\sqrt{2}}{2}\right)^2 + \left(\frac{x}{r} - \sqrt{2}\right)\cos\phi = \sqrt{2 - 2\cos\phi^2}.$$
(4.7)

Squaring each side gives

$$\left[\frac{\ell^2}{r^2} - \frac{3}{2} - \left(\frac{x}{2r} - \frac{\sqrt{2}}{2}\right)^2\right]^2 + \dots$$

$$\left[\frac{\ell^2}{r^2} - \frac{3}{2} - \left(\frac{x}{2r} - \frac{\sqrt{2}}{2}\right)^2\right] \left(\frac{x}{r} - \sqrt{2}\right) \cos\phi + \left(\frac{x}{r} - \sqrt{2}\right)^2 \cos\phi^2 = 2 - 2\cos\phi^2.$$
(4.8)

This is a quadratic. This gives

$$a = 2 + \left(\frac{x}{r} - \sqrt{2}\right)^{2},$$

$$b = \left[\frac{\ell^{2}}{r^{2}} - \frac{3}{2} - \left(\frac{x}{2r} - \frac{\sqrt{2}}{2}\right)^{2}\right] \left(\frac{x}{r} - \sqrt{2}\right),$$

$$c = \left[\frac{\ell^{2}}{r^{2}} - \frac{3}{2} - \left(\frac{x}{2r} - \frac{\sqrt{2}}{2}\right)^{2}\right]^{2} - 2,$$

$$\phi = \arccos\left(\frac{-b \pm \sqrt{b^{2} - 4ac}}{2a}\right).$$

(4.9)

We hence know the rotation as a function of the displacement. The moment on the top disc is must be balanced by the forces transmitted through the two connections. The balance is

$$M\mathbf{E}_{z} = r(-\cos\phi\mathbf{E}_{x} - \sin\phi\mathbf{E}_{y}) \times -F\mathbf{e}_{c} + r(\cos\phi\mathbf{E}_{x} + \sin\phi\mathbf{E}_{y}) \times F\mathbf{e}_{c}.$$
 (4.10)

The vector above is given by

$$\mathbf{e}_c = -\cos\theta \mathbf{E}_x - \sin\theta \mathbf{E}_y. \tag{4.11}$$

The angle follows from 4.1 and is

$$\theta = \arcsin\left(r/\ell\sin\phi + r/\ell\sin\pi/4\right). \tag{4.12}$$

Returning to the moment balance, we have

$$M = 2rF(-\cos\phi\sin\phi + \sin\phi\cos\theta) \tag{4.13}$$

Thus the force magnitude is given by

$$F = \frac{M}{2r(-\cos\phi\sin\theta + \sin\theta\cos\theta)}.$$
(4.14)

The same force magnitude balances the bottom disc, so the expected force measured by the mount should be

$$F_{\text{mount}} = \frac{M\cos\theta}{r(-\cos(\phi_8 + \pi/4)\sin(\phi_9 + \alpha) + \sin(\phi_8 + \pi/4)\cos(\phi_9 + \alpha))}.$$
 (4.15)

We can use the above to find the moment as a function of displacement for the experimental setup. The force vs. displacement curve, the extracted moment, and numerical results employing the data can be visualized in Fig. 4.3. There is decent agreement, with differences attributed to dissipation in the system.

Further numerical validation of the model can be seen in Fig. 4.4. A 4×6 model was studied, where the top for nodes are displaced and the bottom 4 have their positions held. Initial numerical results shown did not have stellar agreement. In identifying discrepancies, video recording showed wobbling of the top MTS mount, allowing the top row of constrained discs to rotate slightly. This small rotation of about two degrees was applied to the numerical simulation and brought results into closer agreement. This reveals that the system possesses many configurations, which are quite sensitive to the configuration history. Small deviations arising in the physical experiment, emanating from small errors in manufacturing and assembly, have an accumulated effect over time, and cause the results to deviate from numerical predictions,



Figure 4.3: Experimental determination of the multi-stable interaction. The first panel shows the experimental measurement of the force vs. displacement of a simple 1D chain like those on the right. This data is used to determine the multi-stable interaction as a function of relative rotation, which a red dotted curve is fitted to. This fitted curve is used in a numerical simulation of the 1D chain, producing a red dotted curve in the first panel with very good agreement.

seemingly beyond about 5% strain. Still, the numerical model employed produces vastly more accurate results than models proposed for other multi-stable systems,⁵⁰ despite possessing more stable states from the rotationally multi- stable cell.

With basic validation of the numerical model, optimization and validation of the results took place. The physical model assembled can be seen in Fig. 4.6, with the experimentally recorded force vs. displacement curves. Overlaid, the numerical results from the low-fidelity test used in optimization as well as a high-fidelity simulation can be seen. The low-fidelity simulation, high-fidelity simulation, and experimental results produce mechanical energy absorption ratios of $E_{90^{\circ}}/E_{0^{\circ}} = [1.51, 1.67, 1.16]$, respectively. The curves shown are qualitatively similar, but exact snap-through events marked by the sudden drops in the force vs. displacement diagram differ in location and magnitude. The loading curve is sensitive to small variations in the system; nevertheless, the ratio of absorbed energies prevails despite these differences.



Figure 4.4: Experimental and numerical absorption in a 4×6 system. The experimental results in blue produce a stiffness in very good agreement with the numerical predictions in red and green, but the initial simulation in red predicts a very different peak force. Inspection of recordings of the experiment showed the top mount wiggling. Applying a 5° deflection to the top row of discs in the simulation produced more similar results. Still, this extra degree of freedom produces errors that accumulate through higher strain values. Nevertheless, good qualitative agreement is shown.

4.3.2 Evaluation of Polycrystalline Meso-structure

We apply the NSGA-II to the linear frame model to optimize stiffness in many directions. Three optimizations are first performed as a demonstration of efficacy. First, we target an isotropic response where each target $T_j = K_j/K_0 \equiv 1$. We examine only a 3 by 3 grid of grains with a square lattice sub-structure ($a_x = a_y = 1$) where $\ell_g = 10$. We proceed in steps where the directions are given by $\pi/6, \pi/12, \pi/18$, and $\pi/24$, and each step is given a maximum of 150 generations with 12 population members. The results of the optimization are given in Fig. 4.5. The final step produces a metamaterial with stiffness that is nearly isotropic. This result is produced in natural metals with many grains, but with optimization we are able to produce a similar result with only 9 grains.



Figure 4.5: Optimization of poly-crystalline frames. The linear frame system undergoes the NSGA-II process in steps in an attempt to produce isotropic stiffness, 6-fold rotational symmetry, and a custom shape. Each polar plot has stiffness on the radial axis and tested orientation on the azimuth. In each case, the population member with the minimum root mean squared error is displayed. A homogeneous frame system and its directional stiffness are shown on the far left. The right most polar plot has the target custom shape in red and the result of the optimization in blue. The resultant morphologies for each are displayed.

Next we turned towards producing 6-fold rotational symmetry in the overall structure, while the sub-grain structure held different rotational symmetry. In choosing objectives, we allow the first sector (i.e. $0 < \theta < \pi/3$) to become any shape, and simply try to repeat this shape in each subsequent sector. To prevent reduction to isotropy, which technically possesses 6-fold rotational symmetry, we place an addition target of $T_{\pi/6} = K_{\pi/6}/K_0 = 1.1$ in order to maintain non-constant curvature in the stiffness. The resultant 6-pointed star shown in the second column of Fig. 4.5 approximates 6-fold rotational symmetry.

The next target was chosen to be a custom shape. The desired shape and result of the optimization can be seen in the third column of Fig. 4.5. Although this shape is met reasonably, other shapes, particularly those with oscillations over short angular distances, are difficult to resolve. Further, it is difficult to know the range of relative stiffnesses that can be attained.



Figure 4.6: Optimization of poly-crstalline energy absorbing structure. The results of the optimization are shown in red, producing an relative absorption of $E_{90^{\circ}}/E_{0^{\circ}} = 1.51$. The experimental results in blue are $E_{90^{\circ}}/E_{0^{\circ}} = 1.16$. The green line depicts a higher-fidelity simulation with $E_{90^{\circ}}/E_{0^{\circ}} = 1.67$.

To this end, we aimed to maximize the relative stiffness in a multitude of directions by targeting arbitrarily large values in single-objective NSGA-II optimizations. This sets a sort of outer bound for potential custom shapes; similarly we sought minimum values for the inner bound. Beyond this, we also approximated and optimized the first and second derivatives of the directional stiffness according to

$$\frac{\mathrm{d}K(\theta)}{\mathrm{d}\theta} \approx \frac{-K(\theta+2\Delta\theta)+8K(\theta+\Delta\theta)-8K(\theta-\Delta\theta)+K(\theta-2\Delta\theta)}{12\Delta\theta}, \qquad (4.16)$$
$$\frac{\mathrm{d}^2K(\theta)}{\mathrm{d}\theta^2} \approx \frac{-K(\theta+2\Delta\theta)+16K(\theta+\Delta\theta)-30K(\theta)+16K(\theta-\Delta\theta)-K(\theta-2\Delta\theta)}{12\Delta\theta^2}.$$

To manipulate these bounds, we investigate the sub-grain structure and the grain structure. The sub-grain structure is characterized by its own anisotropy. First, a square lattice sub-grain structure is employed. The frame elements have their axial and bending stiffnesses characterized by E_f and I_f , where f indicates frame. By manipulating the ratio of these values, the structure's anisotropy can be tuned. The stiffness of homogeneous structures can be viewed in the first column



Figure 4.7: Bounds on the maximum (blue) and minimum (red) relative stiffness values and its derivatives for a system with a square sub-grain microstructure. Note that the derivatives are able to obtain negative values, and the curves plotted are absolute values.

of Fig. 4.7 for $E_f/I_f = 768000$, 768, 12]. The max/min values obtained from optimization can be viewed in the remaining columns. The top row, which has the highest degree of anisotropy in its homogeneous structure, obtains the largest bounds. The middle row is slightly less anisotropic and the bounds are somewhat reduced. Meanwhile the bottom row is nearly isotropic in its homogeneous system. Nevertheless manipulating the orientation of grains produces variations in the directional stiffness. These variations can be considered to come directly from interactions at grain boundaries, produced from non-crystalline connections and voids. Next we change the sub-grain lattice type, switching to a rectangular lattice with $a_x = 1$ and $a_y = 2$. The bounds can be seen in Fig. 4.8. This lattice presents a higher degree of anisotropy than the square lattice. While the bounds appear to be slightly larger than those of a square lattice with the same $E_f/I_f = 768$ as the middle row of Fig. 4.7, this may be attributed to the optimization process, as a grain rotation of the same magnitude covers more of the irreduceable directional response of the square lattice (a 45° region) than that of the rectangular lattice (a 90° region). This means that the rectangular lattice may be tuned more "finely", and potentially have an easier time finding minima of the objective



Figure 4.8: Bounds on the maximum (blue) and minimum (red) relative stiffness values and its derivatives for a system with a rectangular sub-grain microstructure. Note that the derivatives are able to obtain negative values, and the curves plotted are absolute values.

function.

We then investigate the parameters which dictate the grain meso-structure itself, namely the grain number and size. Square grids of constant grain size $\ell_g = 10$ are tested to produce the limiting bounds of before, varying in grain number from $N_g = 2$ to $N_g = 6$. The results in Fif. 4.9 show that bounds on the values of the stiffness do not change appreciably with increasing grain number. However, the first and second derivatives show a slight decrease in range as grain number increases. This is likely due to the change in relative size of voids compared to system size. There is some slightly anomalous behavior for a 9 grain system, in which we see a drop in the bounds which increase for a 16 grain system. This could well be due to the optimization scheme. The size of grains is then investigated, testing a $N_g = 3$ square structure for grain size varying from $l_g = 6: 12$. The bounds shown in Fig. 4.10 generally show decreasing limits as grain size increases for derivatives of the directional stiffness. As grains increase in size, the effects of non-crystalline connections and voids prevalent at grain boundaries decrease. These voids re able to manifest steeper gradients in the the directional stiffness. Nevetherless, directional variation in the bounds



Figure 4.9: Bounds on the maximum (blue) and minimum (red) relative stiffness values and its derivatives for a system with varying number of grains. Note that the derivatives are able to obtain negative values, and the curves plotted are absolute values. Random morphologies are shown to help visualize the system size.

are still observed, showing that the directional dependencies observed do not arise solely from microstructural anomalies at grain boundaries.

In the question of the existence of a "threshold" grain size, we plot the directional stiffness of a random morhpology for structures whose grain size doubles at each iteration, seen in Fig. 4.11. The demonstration suggests that the profile experiences little change beyond $\ell_g = 24$.

To further separate the role of grain orientation from grain boundary effects, we employ our continuum model of the dual-chiral layer. This model incorporates grain boundaries in the mesh, leading to no microstructural anomalies. We still observe directionally dependent bounds in Fig. 4.12, which in general are much smoother than those of the discrete structure. The continuum structure allows for efficient modelling of very large systems; the system shown is of size 300 non-dimensional length units.



Figure 4.10: Bounds on the maximum (blue) and minimum (red) relative stiffness values and its derivatives for a system with varying grain size. Note that the derivatives are able to obtain negative values, and the curves plotted are absolute values.

4.4 Conclusion

Poly-crystallinity clearly shows the ability to influence the directional absorption of the multi-stable metamaterial, validated with physical and numeric demonstrations. It also clearly affects stiffness and even the potential to control the directionality thereof. Despite some limitations, general control over the directional stiffness is leveraged to produce isotropy, 6-fold rotational symmetry, and a custom shape. These shapes are limited in relative magnitude, of which a bound is observed but not proven. More than this, these shapes are limited in their first derivative and second derivatives with respect to orientation. These bounds are increased with the relative anisotropy of a grain's comprising microstructure. The structure of the grains themselves are important as well, with more grains producing more customizeable directionalities. The size of the grains appears to take a threshold value, below which grain boundary effects have a significant effect.


Figure 4.11: Convergence of a random morphology's directional stiffness with increasing grain size, ℓ_g

4.5 Acknowledgements

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Figure 4.12: Bounds on the maximum (blue) and minimum (red) relative stiffness values and its derivatives for the continuum model (mesh shown for random morphology. Note that the derivatives are able to obtain negative values, and the curves plotted are absolute values.

Chapter 5

Mechanical Multi-level Memory from Multistable Metamaterial on Elastic Foundation

Abstract

As a detour into other applications for multi-stable metamaterials, we examine a memory device. In this letter, we couple a multi-stable metamaterial to an elastic foundation to realize a mechanical system within which the position of a transition wavefront can be precisely controlled and remotely determined. This ability is enabled, in part, by a (strain-)tunable potential energy landscape which conveys the wavefront from one stabilizing defect site to another. In separating two, acoustically distinct domains, the wavefront reflects small-amplitude waves of appropriate frequency back toward the source whereupon the time interval between excitation and echo reveals the position of the front. In a combined theoretical and numerical study, we exploit these mechanisms for mechanical multi-level memory which may find application, e.g., in soft robots as a flexible alternative to current rigid memory technologies. In general, we anticipate that the concepts presented here for a command of the transition wave position will further the development and applicability of multi-stable metamaterials. This work was published in Applied Physics Letters in 2023.²

5.1 Introduction

(Multi-)ferroics⁷² are crystalline materials whose unit cells notably exhibit multiple, energetically equivalent stable configurations (i.e., phases, states) distinguished by one or more order parameters and, possibly, by unique physical properties. Within a given sample, each stable configuration may appear simultaneously, organized into regions of uniform phase (i.e., domains) separated by an interface (i.e., domain wall). Under the influence of a conjugate field,⁷³ the domain wall may become mobile, transforming the local configuration in its path; thus, constituting a transition wave. These attributes hold considerable promise for nano-electronics applications,⁷⁴ including, e.g., high-density memory⁷⁵ and re-configurable circuits.^{76,77} The former interprets two given configurations as bits "0" and "1" while the latter leverages the enhanced conductivity of the domain wall (i.e., transition wavefront) and its ability to be re-positioned. Apparently, the ability to manipulate domains and domain walls in (multi-)ferroics is critical to developing their functionality and, therefore, remains an active area of research in condensed matter physics. A persistent challenge, however, lies in accessing the relevant length/time scales for precision control.

Recently, similar physics has been elicited at the structural level from mechanical metamaterials whose 3D-printable unit cells realize a non-convex potential function in a strain/displacement order parameter.²¹ Numerous studies have been devoted to characterizing the motion of domain walls under various conditions.^{78–83} In particular, Hwang and Arrieta⁸⁰ show monotonically decreasing elastic potentials to support uni-directional, multi-mode propagation which enables phase reversals. Jin *et al.*⁶⁴ demonstrate the utility of "hard" point defects (i.e., unit cells storing no deformation energy) in controlling the speed/direction and contour of a transition wavefront in a 2D environment. Together with an understanding of the physics of transition waves, the enhanced accessibility of the metamaterial internal architecture permits a tailorable response that has, so far, inspired proposals for, e.g., locomotion,^{84–86} signal transmission,^{79,87} tunable acoustic filtering,⁸⁸ and mechanical logic operations⁸⁹ which promise flexible, mechanical

alternatives to rigid electronics in otherwise soft robots and smart structures.

Inspired by these earlier works, in this letter, we propose a structural design leveraging a multi-stable metamaterial for mechanical multi-level (i.e., high-density) memory functionality. The concept may find utility in soft robotics, especially, the emerging paradigm of physical reservoir computing.⁹⁰ The basic operation of the proposed memory element – supported by theoretical and numerical analysis – relies on both large- and small-amplitude dynamics of the metamaterial: the former (transition waves) for the write operation and memory stability; the later (harmonic waves) for the read operation. The proposed design, method, and results are distinct from similar implementations of multi-stable architectures for mechanical (i.e., low-density) memory that do not support transition waves and store only one bit per cell.^{91,92}

Figures 5.1a,b display schematics of the proposed mechanical multi-level memory element, comprising two discrete, one-dimensional chains, i.e., the *substrate* (SS) and the *metamaterial* (MM), with periodic and uniform inter-site coupling, $s_{j+1/2}$ and k_r , respectively. The two chains are connected locally through a non-linear spring (henceforth, associated with the metamaterial) adhering to the non-convex potential, $\psi(\Delta_j)$, $\Delta_j = v_j - V_j$, providing for the formation of transition waves. The system is an adaptation of the LOC arrangement utilized by Ramakrishnan and Frazier⁸⁸ for spontaneous alterations to the effective stiffness distribution in metamaterials. The corresponding non-dimensional Lagrangian is given by [SI]

$$\mathscr{L} = \sum_{j=1}^{N} \left[\frac{1}{2} (\dot{v}_{j}^{2} + \dot{V}_{j}^{2}) - \alpha_{j} \psi(\Delta_{j}) \right] - \cdots$$
$$\frac{1}{2} \sum_{j=1}^{N-1} \left[(v_{j} - v_{j+1})^{2} + s_{j+1/2} (V_{j} - V_{j+1})^{2} \right]$$

from which the governing equations (with on-site viscous damping, η) of an arbitrary site j are

Table 5.1: Parameters

	metamaterial			substrate		Ψ			
	γ	β	J	$J_{\rm s}$	j ₀	a	b	С	d
Case I	$\begin{array}{c}-2.5\times\\10^{-4}\end{array}$	-1/5	100	200	81	0.2582	0.0172	7.4489	0.0061
Case II	$\begin{array}{c} -1.\bar{3}\times\\ 10^{-4}\end{array}$	-1/20	25	50	17	0.2582	0.0172	7.4489	0.0380

derived as

$$\ddot{v}_j + \eta \dot{v}_j + (2v_j - v_{j+1} - v_{j-1}) + \alpha_j \frac{\partial \psi(\Delta_j)}{\partial v_j} = 0, \qquad (5.1a)$$

$$\ddot{V}_j + \eta \dot{V}_j + s_{j+1/2} (V_j - V_{j+1}) + \cdots$$

$$s_{j-1/2} (V_j - V_{j-1}) + \alpha_j \frac{\partial \psi(\Delta_j)}{\partial V_j} = 0, \qquad (5.1b)$$

where $\alpha_j = (1 + \gamma j)[1 + \beta \delta_{ij}]$ is a positive scalar responsible for effecting (i) the element-wide amplification/attenuation of the on-site potential as prescribed by the parameter, $\gamma = (-\infty, \infty)$, and (ii) the nature and distribution of "soft" point defects (i.e., unit cells storing anomalous deformation energy) as controlled by the parameter, $\beta \in (-1, \infty)$, and the Kronecker delta, δ_{Ij} , $I = \{j | (j/J) \in \mathbb{N}\}$ with period $J \in \mathbb{N}$.

In order to elicit the desired functionality, the system adheres to certain design requirements. For one, $\psi(\Delta_j)$ possesses $n \ge 2$ degenerate ground states, $\Delta_j = \{\Delta_{s1}, \Delta_{s2}, \dots, \Delta_{sn}\}$, supporting the formation of a (anti-)kink wave profile, yet, is asymmetric such that the linear on-site stiffness about each ground state, $k_{si} = \psi''(\Delta_{si})$, is unique (Fig. 5.1c.i). For another, the α_j vary mono- tonically (here, linearly) across the element except at p periodically-distributed defect sites, I, where, for $\beta < 0$ ($\beta > 0$), α_j is less (greater) than the neighboring $\alpha_{j\pm 1}$ in order to create a locally attractive (repulsive) potential suitable for immobilizing transition waves. The monotonic variation otherwise supports the uni-directional propagation of both kink and anti-kink modes,⁸⁰ permitting the present state, acquired by the passage of one transition wave, to be reversed by the passage of another of the opposite topological charge. The soft defects divide the system into *p* levels of *J* unit cells, each of which may be assigned an *m*-bit binary value where $m = \lfloor \log_2(p) \rfloor$ (Fig. 5.1a). Finally, although the substrate is pliable, proper function requires $s_{j+1/2}$ to be, effectively, rigid in comparison to $k_r (\equiv 1$ when non-dimensionalized) and k_{si} , i.e., $\min(s_{j+1/2}) \gg \max(k_r, k_{si})$. This relative rigidity ensures that the inhomogeneous strain that arises in the substrate upon the application of a prescribed boundary displacement is, effectively, reproduced in the metamaterial. Moreover, the stark difference in compliance allows for the dynamics of the metamaterial subsystem to treated in isolation at low frequencies, i.e., described by Eq. (5.1a) with a foundation of fixed V_j . Of the myriad possible periodic functions, the substrate stiffness is modulated by a triangle wave

$$s_{j+1/2} = \frac{2(s_{\max} - s_{\min})}{J_s} \cdots \times \left| \left(j - j_0 - \frac{J_s}{2} \right) \operatorname{mod}(J_s) - \frac{J_s}{2} \right| + s_{\min}$$
(5.2)

with period, J_s , offset, j_0 , and minimum (maximum) value set by s_{min} (s_{max}).

Consider, for example, the case in which $\gamma, \beta < 0$ such that α_j progressively reduces the on-site potential and establishes an energy well at the defect sites. A transition wave initialized at the left boundary will propagate to the right with constant speed^{80,82} until entering and, with the aid of dissipative effects, becoming pinned (i.e., immobilized) within the potential well established by a soft defect. However, as discussed below, deforming the substrate modifies the energy landscape governing the transition wave motion, providing a mechanism for de-pinning and mobilizing the wave, even if only to become pinned again at the next defect site. The pinned position of the wavefront may be determined from the time of flight of a small-amplitude harmonic signal initiated at either boundary and reflected off the front due to the incompatible dynamics of the domains (i.e., regions of homogeneous Δ_{si} and k_{si}) that it separates. Thus determined, the current memory state is revealed as the number of consecutive levels in the configuration arbitrarily designated as, e.g., the "active", "ON", or "1" phase (here, Δ_{s2} is active); the opposing configuration fittingly labeled the "inactive", "OFF", or "0" phase. The element is the mechanical analogue of an electron CTF

memory cell, where the two mechanical phases are likened to two charge states; *J* cells and defect sites are, respectively, likened to floating gates and isolators; mechanical load likened to voltage. Altogether, controlling and reading the defect-stabilized position of the domain wall enables a multi-level memory functionality.

In this letter, we define $\psi(\Delta_j) = d\{1 - \cos[a(\Delta_j + c) + b(\Delta_j + c)^3]\}$, where *a*, *b*, *c*, and *d* are tuning parameters that maintain an asymmetric potential with degenerate ground states distinct in k_{si} . In practice, these qualities may be realized, e.g., through the proper arrangement and orientation of magnetic di-poles.⁸⁸

The nonlinearity of the metamaterial admits tailoring of its dynamic response by straining the memory element, which alters the characteristic energy landscape which dictates transition wave motion.⁹³ To illustrate this effect, we calculate the total potential energy, $\mathscr{U}(X_j)$, of the metamaterial in a finite system under various levels of prescribed strain as a function of the instantaneous position, $X_j = j$, of a (anti-)kink wavefront. In particular, we consider a system of N = 200 unit cells constructed following the set of parameters given in Case I of Table 5.1. For analytical convenience, $|\gamma|$ is chosen sufficiently small to ensure that the supported (anti-)kink exhibits an effectively permanent profile while traveling the length of the system and that the k_{si} are effectively independent of position. For this and the following studies, the extrema of the substrate stiffness variation adhere to s_{max} : $s_{min} = 3 : 1$.

Following a prescribed boundary displacement, the interiors of the metamaterial and substrate each acquire new equilibrium configurations, v_j^0 and V_j^0 . While $\mathscr{U}(X_j)$ may be computed via simulation, this approach yields an incomplete picture when the simulated transition wave becomes pinned, preventing the energy from being evaluated beyond X_j . It is more effective to utilize an analytical model with prescribed X and the continuum form of the total potential energy [SI]

$$\mathscr{U}(X) = \int_0^\ell \left[\alpha(x)\psi(v) + \frac{1}{2} \left(\frac{\partial v}{\partial x} + \frac{\partial v_0}{\partial x} \right)^2 \right] \mathrm{d}x, \tag{5.3}$$

where $\ell = Na$, and $v_0(x)$ and $V_0(x)$, respectively, model the initial strain-induced equilibrium displacements of the metamaterial and substrate. In general, exact analytical expressions for

v(x;X), $v_0(x)$, and $V_0(x)$ are difficult to obtain from the corresponding governing equations; therefore, they are approximated by fitting ansatzes to the discrete results from simulation of Eq. (5.1). As the wave profile is assumed constant across the memory element, we fit

$$v(x;X) = A \arctan\left[B_1 e^{C_1(x-X)} + B_2 e^{C_2(x-X)}\right]$$
(5.4)

to the simulated (anti-)kink at rest in an unstrained, uniform system [i.e., $(\gamma, \beta) = (0, 0)$] [SI]. For the initial equilibrium displacements, the fitting function combines a linear function and a *q*th-order Fourier expansion:

$$v_0(x) = a_0 + c_0 x + \sum_{n=1}^q a_n \cos(nx) + \sum_{n=1}^q b_n \sin(nx),$$

where the fitting function for $V_0(x)$ has the same form. For all studies, we find good agreement between fitted curves and simulation results, limiting the root-mean-square error to $e_{\rm rms} < 1.35 \times 10^{-4}$.

Figure 5.2a plots $\mathscr{U}(X)$ for various levels of initial substrate strain, $\varepsilon = \Delta \ell / \ell$, which may be physically achieved via, e.g., pneumatic actuation with flexible plumbing, electric actuation with an electro-active polymer composing the substrate, or direct mechanical loading. Here, we simply prescribe $V_1 = 0$ and $V_N \neq 0$. For $\varepsilon = 0$, there is a pronounced decrease in the potential energy centered at the location of the soft defect (x = 0) as α locally lowers the energy expenditure for switching between stable states, representing a local attractor which, aided by dissipative effects, may pin an incoming transition wave. As $|\varepsilon|$ increases, the defect energy well shallows; consequently, the attractor is less able to pin incoming waves and already pinned waves are more susceptible to mobilization by external perturbation. In addition, \mathscr{U} develops strain-dependent local extrema associated with locations where the gradient of the substrate stiffness modulation changes sign (e.g., the peak or trough of a triangle modulation) which, appropriately, either attract or repel the transition wave and, in doing so, alter the speed/direction of its propagation. For the current system, under compression (tension) a local maximum (minimum) develops in the vicinity of s_{\min} ; the opposite effect (not shown) manifests in the vicinity of s_{\max} . The observed distortion in \mathscr{U} reflects the work done by the non-uniform load applied to the metamaterial by the deformed substrate, revealing pre-strain as a mechanism for spontaneously altering the dynamics of transition waves in a host medium, post-fabrication. For the (anti-)kink initially pinned at x = 0, sufficient strain will effectuate de-pinning and mobilization; although, due to the asymmetries inherent to $\alpha(x)$, as well as the phase difference between δ_{Ij} and the substrate stiffness modulation, the critical strain at which this occurs and the direction of propagation depends on the deformation mode. These effects are illustrated in Fig. 5.2b. The first row of Fig. 5.2b tracks the simulated position the anti-kink wavefront initially pinned at x = 0 as $|\varepsilon|$ quasi-statically increases. For a sufficient strain, energy extrema *not* aligned with the defect location, ultimately, induce de-pinning and compel the newly-liberated wavefront to seek an another potential minima, which is achieved by propagating to either the system boundaries or a strain-induced minima whereupon it is pinned again (Fig. 5.2b.i,iii). For a sufficient strain, de-pinning and propagation may still occur when the maxima is aligned with defect location – though, under a significantly higher magnitude of strain – due to the asymmetry established by $\alpha(x)$ (Fig. 5.2b.ii).

Altogether, these results demonstrate the utility of soft defects and prescribed strain in modifying the energy landscape to effect the pinning, de-pinning, and propagation speed/direction control of transition waves which we exploit for mechanical multi-level memory.

We now turn our attention to the propagation of small-amplitude harmonic signals, for the hypothetical scenario of an infinite, undeformed system of homogeneous ground state configuration. The application of Bloch's theorem to the linearized and lossless form of Eq. (5.1a) yields the characteristic dispersion relations for small-amplitude waves confined to the metamaterial [SI]:

$$2 + k_{\rm si} - \frac{1}{\vartheta} - \vartheta = \omega^2, \tag{5.5}$$

where ω is the dimensionless temporal frequency and $\vartheta = e^{i\kappa a}$ is the propagation constant with dimensionless complex wavenumber, κa . In formulating Eq. (5.5), we have assumed $(\gamma, \beta) = (0,0)$, which is reasonable in the scenario where (as is the case here), in constructing the (finite) mechanical memory element, both γ and β are kept small as are the number and density of pinning

sites, and the size of the system does not result in a large discrepancy between corresponding k_{si} at each boundary. In solving Eq. (5.5) for $\vartheta(\omega)$, propagating and attenuating wave modes are extracted, respectively, as $\kappa_R a = |\text{Re}(i \ln \vartheta)|$ and $\kappa_I a = |\text{Im}(i \ln \vartheta)|$. Figure 5.3a displays the dispersion curves for a system consistent with Case II of Table 5.1, revealing the non-overlapping frequency ranges of the respective propagating modes.

For the finite system supporting a pinned transition wave which separates domains with unique k_{si} , the results in Fig. 5.3a suggests that a propagating signal stimulated in one domain cannot enter the second domain if the Fourier components are not supported there. Instead, at the domain wall, the wave energy is reflected back toward the source as an echo. To demonstrate this, we simulate the response of a finite system (N = 200) to a Gaussian-modulated harmonic pulse stimulated within the metamaterial where a pinned anti-kink separates opposing regions of k_{s2} (left) and k_{s1} (right). For an excitation at the left end, the displacement prescription has the form:

$$v_1(t) = \begin{cases} v_{s2} + A \exp\left[-\frac{1}{2}\left(\frac{t-t_0}{\sigma}\right)^2\right] \sin(\omega_c t), & t \le 2t_0 \\\\ \text{not prescribed}, & t > 2t_0 \end{cases}$$

where ω_c and σ are, respectively, the carrier frequency and the standard deviation. The velocity at the left boundary, $\dot{v}_1(t)$, is also enforced. In particular, the carrier frequency is restricted to lie within the propagating band associated with the designated active configuration. Here, we select $\omega_c = 2.4$. For the Gaussian envelope, $\sigma = 8$ and $t_0 = 6\sigma$. As only the small-amplitude response is of interest, we limit the excitation amplitude to A = 0.01. For the numerical analyses, we apply the Noh-Bathe integration scheme⁹⁴ to Eqs. (5.1) with $\eta = 1/250$. Figure 5.3b plots both the resulting displacement magnitude, $v_m(t) = |v_1(t)|$, and its smoothened, time-averaged counterpart defined by the convolution, $v_c(t) = [v_m \cdot h](t)$, where h(t) is a rectangular window function of width, $T = 16\pi/\omega_c$, centered at t. Naturally, the elapsed time, Δt , between signal generation and the returned echo [identified by $v_c(t) > v_{th}$ for $t > 2t_0$, v_{th} a threshold value] changes with distance between the boundary excitation (i.e., the source) and the domain wall. This correlation is exploited in the mechanical multi-level memory element to infer the location of the pinned domain wall and, thus, identify the current memory state.

We bring together the physics of small- and large-amplitude waves, of soft defects, as well as the ability to tune the energy landscape in order to assemble a mechanical system with multi-level memory functionality, which we demonstrate numerically. Specifically, we simulate the response of an anti-kink within a metamaterial- substrate system (N = 225) under deformation due to a prescribed, quasi-static boundary displacement. The relevant parameters are listed in Case II of Table 5.1, imbuing the metamaterial sub-system with eight pinning sites sufficient for 3-bit memory storage.

Figure 5.4a.i is an xt-contour diagram of the configuration, Δ_i in the metamaterial. It depicts the controlled advancement of a transition wave across the memory element as the system undergoes cycles of prescribed compressive, tensile, and zero strain (Figure 5.4a.ii). The transition wave is initially pinned at x = -75, representing level 0 with binary code 000. In practice, a (anti-)kink or harmonic pulse may be injected from the boundary with the aid of a solenoid actuator coupled to the boundary site. Under a non-zero strain, a change in the energy landscape releases and repels the domain wall from its current pinned position and toward another pinning site. Upon recovery of the zero strain condition, the transition wave becomes pinned at a soft defect site different from the one from which it originated (presently, the subsequent pinning site). The inset displayed Fig. 5.4 details each of these events. The process is repeated until the domain wall is pinned to the desired defect site (here, the final pinning site at x = 100, representing level 7 with binary code 111). Under the condition of zero strain and for the transition wave pinned at an arbitrary defect site, the position of the domain wall – thus, the memory state – may be determined from the time of flight of a small-amplitude signal injected into the metamaterial with appropriate carrier frequency. Figure 5.4b shows the results of such an exercise conducted in the numerical model when the transition wave is pinned at each of the eight defect sites where the more extended

times correspond to higher memory state.

Since $\gamma < 0$ and j_0 places a strain-induced local energy maximum and minimum, respectively, before and after each pinning site, the transition wave is restricted to unidirectional motion from left to right. Consequently, to return the system to the inactive phase, Δ_{s1} , a kink may be established at x = -75 and conveyed across the memory element by a similar cyclic prescription of strain (Fig. 5.5). The complete process is animated in Mov. S1.

In summary, we have devised a mechanical system within which the position of a transition wavefront can be precisely controlled and remotely determined, which we exploit for mechanical multi-level memory; yet, energy harvesting and post-fabrication tuning of mechanical properties represent additional opportunities. A realization of the proposed memory architecture may find application in soft robotics, providing a flexible (albeit, rudimentary) alternative to current, ceramic- based technologies. The essential characteristics of the system necessary for the control scheme are multi-stability, soft defects, and the proportional relationship among stiffnesses; all of these are independent of scale, and hence the proposed design, leveraging advancements in 3D-printing, is amenable to later efforts aimed at increasing memory density via miniaturization.

5.2 Supporting Information: Mechanical Multi-level Memory from Multi-stable Metamaterial on Elastic Foundation

5.2.1 Non-dimensionalization

Consider the dimensional Lagrangian of a single unit cell of the proposed system:

$$\mathscr{L} = \frac{m}{2} (\dot{v}_j^2 + \dot{V}_j^2) - \alpha_j \psi(\Delta_j) \cdots - \frac{1}{2} \left[k (v_j - v_{j+1})^2 + s_{j+1/2} (V_j - V_{j+1})^2 \right].$$
(5.6)

Non-dimensionalization allows the investigation to proceed with a reduced set of parameters, better highlighting their ultimate effect on the observed performance. Here, we express space and time in

terms of their non-dimensional counterparts (overbars), i.e., $x = a\bar{x}$ and $t = \bar{t}/\omega_0$, where $\omega_0^2 = k/m$. In addition, per the current objective as well as for aesthetic reasons, displacements are transformed as follows: $\Delta_j = (\Delta_{s2} + \Delta_{s1})/2 + (\Delta_{s2} - \Delta_{s1})\bar{\Delta}_j/2$. In substituting the dimensional variables, Eq. (5.6) becomes

$$\mathscr{L} = \frac{\mathscr{E}}{2} (\dot{\bar{v}}_j^2 + \dot{\bar{V}}_j^2) - \alpha_j \psi(\bar{\Delta}_j) \cdots \\ -\frac{1}{2} \left[\mathscr{E} (\bar{v}_j - \bar{v}_{j+1})^2 + s_{j+1/2} (\mathscr{E}/k) (\bar{V}_j - \bar{V}_{j+1})^2 \right],$$

where $\mathscr{E} = k(\Delta_{s2} - \Delta_{s1})^2/4$ is a unit of energy. Upon dividing the above by \mathscr{E} , we arrive at the following:

$$\vec{\mathscr{L}} = \frac{1}{2} (\dot{\vec{v}}_{j}^{2} + \dot{\vec{V}}_{j}^{2}) - \alpha_{j} \bar{\psi}(\bar{\Delta}_{j}) - \cdots \\
\frac{1}{2} \left[(\bar{v}_{j} - \bar{v}_{j+1})^{2} + \bar{s}_{j+1/2} (\bar{V}_{j} - \bar{V}_{j+1})^{2} \right],$$
(5.7)

which, with $\bar{\psi} = \psi/\mathscr{E}$ and $\bar{s}_{j+1/2} = s_{j+1/2}/k$, is the basis of the result given in the main report (overbars omitted for clarity).

5.2.2 The Potential under Initial Deformation

From Eq. (5.7) (omitting overbars), inserting a first-order Taylor expansion for the displacements and substrate stiffness yields the corresponding continuum approximation of the Lagrangian:

$$\mathscr{L} = \frac{1}{2} (\dot{v}^2 + \dot{V}^2) - \alpha(x) \psi(\Delta) - \cdots \\ \frac{1}{2} \left[v_{,x}^2 + \left(s + \frac{1}{2} s_{,x} \right) V_{,x}^2 \right].$$
(5.8)

In the main, the transition wave with permanent profile is conceptualized as a quasi-particle presently at x = X and moving in the non-uniform potential, $\mathscr{U}(X)$, which derives from the

point-wise potential in Eq. (5.8). As the substrate is comparatively rigid, the energy of the transition wave is effectively confined to the metamaterial; therefore, $V_{,x}^2$ is irrelevant to the problem of deducing the energy landscape influencing the motion of the transition wave. Following a prescribed boundary displacement, the metamaterial and substrate acquire initial displacements, $v_0(x)$ and $V_0(x)$, respectively. Taking these points into account and defining $v(x) = \tilde{v}(x) + v_0(x)$, the potential in Eq. (5.8) reduces to the form:

$$\phi(x) = \alpha(x)\psi(\tilde{v}) + \frac{1}{2}\left(\frac{\partial\tilde{v}}{\partial x} + \frac{\partial v_0}{\partial x}\right)^2,$$
(5.9)

where, upon defining $\tilde{c}(x) = c + v_0(x) - V_0(x)$, yields

$$\psi(\tilde{v}) = d\{1 - \cos[a(\tilde{v} + \tilde{c}) + b(\tilde{v} + \tilde{c})^3]\}.$$
(5.10)

Tilde are omitted in the main. Thus, the total potential is computed as $U(X) = \int_0^\ell \phi(x) dx$.

5.2.3 Dispersion Relations

Following Eq. 1a in the main report, with $V_j = 0$ and assuming a time-harmonic response, $\bar{\mathbf{v}} = \tilde{\mathbf{v}} e^{i\bar{\omega}\bar{t}}$, with dimensionless frequency $\bar{\omega} = \omega/\omega_0$, the matrix equation of motion for an arbitrary unit cell is given by

$$\begin{bmatrix} 1+k_{si}-\bar{\omega}^2 & -1\\ -1 & 1 \end{bmatrix} \begin{bmatrix} \tilde{v}_j\\ \tilde{v}_{j+1} \end{bmatrix} = \begin{bmatrix} f_j\\ f_{j+1} \end{bmatrix}$$
(5.11)

where and f_j and f_{j-1} are the forces applied to the current unit cell by its neighbors. According to Bloch's theorem, $v_{j+1} = \vartheta v_j$, where $\vartheta = e^{i\kappa a}$ is the phase constant. Thus, the full displacement vector, given in terms of the Bloch transformation matrix, $\mathbf{T} = \begin{bmatrix} 1 & \vartheta \end{bmatrix}^T$, is written $\bar{\mathbf{v}} = \mathbf{T}v_j$. Upon substitution into Eq. (5.11) – recognizing $\mathbf{T}^{H}\mathbf{f} = \mathbf{0}$ with \Box^{H} denoting the Hermitian transpose – the characteristic equation in ϑ of the resulting eigenvalue problem is given by Eq. 4 in the main report (overbars omitted for clarity).

Table 5.2: Transition Wave, v(x)

Α	B_1	C_1	B_2	C_2	$e_{\rm rms}$
1.2790	0.6071	-0.3609	0.7297	-0.1565	1.722×10^{-3}

5.2.4 Approximate Fit Functions

In the main article, we gain insight to the energy landscape traversed by transition waves in the metamaterial- substrate system through analytical approximations of v(x;X), $v_0(x)$, and $V_0(x)$. In particular, we fit

$$v(x;X) = A \arctan\left[B_1 e^{C_1(x-X)} + B_2 e^{C_2(x-X)}\right],$$
(5.12)

to the simulated (anti-)kink at rest in an unstrained, uniform system [i.e., $(\gamma, \beta) = (0, 0)$]. Since, for the scenarios considered in the main, both γ and β are small as are the number and density of pinning sites, the profile of the transition wave in these non-uniform conditions is, essentially, identical to that of the uniform case. Similarly, provided that strains remain small, the profile is largely unchanged. Table 5.2 gives the parameters of the function approximating the transition wave in the main report and Fig. 5.6 illustrates the quality of the fit to simulation results. For $v_0(x)$ and $V_0(x)$, the corresponding fitting functions are *q*th-order Fourier expansions paired with a linear term.

5.3 Acknowledgements

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Figure 5.1: Mechanical multi-level Memory. (a) Schematic of the mechanical (8-bit) memory element comprising p levels of J unit cells, including a defect cell. Each level is assigned a binary-encoded value for 0 to p - 1. (b) Detail of three unit cells (lattice constant a) of the substrate and metamaterial (MM) mass-spring chains comprising the element and locally coupled via non-linear springs generating a non-convex potential. (c) (i) The non-convex on-site potential (black) and corresponding local stiffness (red) supporting the formation of (ii) transition wavefronts separating domains of opposing ground state configurations, Δ_{si} , with characteristic on-site stiffness, k_{si} .



Figure 5.2: Strain-tunable Energy Landscape. (a) The energy landscape, $\mathscr{U}(X)$, encountered by a transition wave in the proposed system under compressive and tensile pre-deformation. Under deformation, local energy extrema develop, reducing the pinning ability of the soft defect energy well which, ultimately, promotes de-pinning. Embedded curve labels indicate the strain, $\varepsilon \times 10^{-3}$. (b) The simulated position of an initially-pinned anti-kink as the applied strain increases quasi-statically, revealing instances of de-pinning and re-pinning in accordance to the changing profile of $\mathscr{U}(X)$ for s_{\min} located (i) before, (ii) at, and (iii) after the soft defect. (c) Profile of the substrate stiffness.



Figure 5.3: Dynamics of Small-Amplitude Waves. (a) The propagating and attenuating frequency bands of the metamaterial homogeneous in state Δ_{s1} (red) and state Δ_{s2} (blue). (b) History of a harmonic pulse (according to v_1) injected into the metamaterial with transition wave (insets) pinned at different locations: (i) j = 51 and (ii) j = 101. The elapsed time, Δt , between signal injection and echo (threshold, $v_{th} = 7.5 \times 10^{-3}$) correlates to the distance between the injection site (j = 1) and the transition wavefront. (a) Infinite and (b) finite models consistent with Case II, Table 5.1.



Figure 5.4: Mechanical multi-level Memory Performance. (a) (i) The spatio-temporal evolution of the configuration, Δ , in the memory element subject to (ii) a periodic schedule of of compressive, tensile, and zero strain. Detail of (i) shows the domain wall (1) de-pinning and mobilization from a (permanent) soft defect at x = 0, and (2) re-pinning, respectively, within (temporary) local energy minima developed under increasing strain (i.e., $\varepsilon \uparrow$). As the strain decreases (i.e. $\varepsilon \downarrow$), (3) the gradient in ψ enforced by α drives the transition wave (4) to the soft defect pinning site at x = 25. (b) Correlation between the domain wall pinned position (syn., binary memory state) and the elapsed time, Δt , between the injection and echo of a harmonic pulse.



Figure 5.5: Memory Reset. The memory is reset by initializing and advancing a kink. Here, we prescribe an initial velocity, $\dot{v}_1 = -0.6148$, to create the kink, and then, following a settling period, advance the kink via the cyclic application of strain.



Figure 5.6: Approximate analytical fit to simulated kink. The fitting parameters and rms error are listed in Table 5.2.

Chapter 6

Non-Reciprocal Supratransmission in Mechanical Lattices with Non-Local Feedback Control Interactions

Abstract

Here we take another detour, examining a nonlinear acoustic property of multi-stable metamaterials. We numerically investigate the supratransmission phenomenon in an active non-linear system modeled by the 1D/2D discrete sine-Gordon equation with non-local feedback. While, at a given frequency, the typical passive system exhibits a single amplitude threshold marking the onset of the phenomenon, we show that the inclusion of non-local feedback manifests additional thresholds that depend upon the specific boundary from which supratransmission is stimulated, realizing asymmetric (i.e., non-reciprocal) dynamics. The results illustrate a new means of controlling non-linear wave propagation and energy transport for, e.g., signal amplification and mechanical logic. This work was published in Crystals in 2021.³

6.1 Introduction

In the context of elastodynamics, phononic crystals and metamaterials 95,96 – collectively phononic materials - are two classes of materials whose artificial microstructure provides for the management of mechanical waves. For linear, small-amplitude waves, the microstructure design regulates internal scattering and resonance phenomena such that the Fourier components of a disturbance penetrate the material within only particular frequency ranges and in all or specific directions; outside these pass bands, i.e., within the band gaps, wave propagation is prohibited, the associated wave energy decaying exponentially in space. This filtering capability has inspired proposals for a variety of phononic material applications.^{97–99} Nevertheless, non-linearities inherent to the microstructure enable a unique dynamic response for large amplitude waves. Supratransmission describes the spontaneous flow of energy within the band gap via the non-linear (non-topological) modes of a medium activated by a boundary driving of sufficient amplitude.¹⁰⁰ The effect is a generic property of non-linear systems, having been shown to emerge from both integrable and non-integrable governing equations^{101–106} even when accounting for dissipative effects¹⁰⁷ which more readily extinguish their linear counterparts. This may be exploited, e.g., for the transmission of binary, non-linear signals in lightly-damped systems,¹⁰⁸ as well as for the digital amplification of exceptionally weak signals for sensing.^{106,109,110} While these and other studies have promoted an understanding of amplitude-dependent energy transmission, investigations of systems with a directional response, beneficial in applications for greater control of energy flow, are few.

Reciprocity describes the symmetry of wave transmission between two points in space: if a source and receiver exchange positions, the corresponding frequency response function is identical even in the presence of inhomogeneities and losses.¹¹¹ In recent years, inspired by the concept of electric¹¹² and optical^{113,114} diodes and motivated by their application in communications, sensing, and the directional control of energy flow, significant effort has been directed toward the discovery and study of non-reciprocity in other domains of physics. Regarding phononic materials,^{97,115–118} inherent non-reciprocity has been demonstrated in a number of systems utilizing many strategies for microstructure design, i.e., internal architectures characterized by internal motion,¹¹⁹ time dependent^{120–122} and topological^{123–125} properties, and non-linearity.^{89,126,127} The focus of these and parallel studies is linear wave manipulation. Transition waves – non-linear (topological) modes characteristic of multi-stable systems which propagate by liberating stored elastic energy – demonstrate non-reciprocity as well;^{79,80} however, they are energetically limited to one-time operation or short propagation distances.⁸² In the context of supratransmission, Wu *et al.*¹²⁸ exploited a spatial asymmetry in the metamaterial construction to elicit non-reciprocal transmission.

While most of the previous references employ a passive material platform, a small but growing collection of work^{119,121,122,124,125,127} investigates linear wave propagation within in an active setting. Recently, feedback control integrated directly into the material architecture has opened the door to more complex interactions unavailable in traditional structures and have been shown to be capable of eliciting non-reciprocal behavior.^{125,129} Uniquely, rather than injecting energy into (or extracting from) the system by an external means, feedback involves observing the state of the system and then, following predetermined relations, generating a response that alters the present state. As a result the material behavior is inherent rather than a function of environmental conditions (e.g., temperature, external fields, pumps, actuators which behave independently of the system). At present, the supratransmission phenomenon in feedback mediated non-linear networks has yet to be investigated.

In this article, we present an approach to asymmetric (i.e., non-reciprocal) wave propagation in non-linear mechanical networks with a focus on energy transmission within the band gap. Unique in the supratransmission literature, these lattice materials incorporate active elements which impart a local, non-conservative forcing proportional to non-local degrees-of-freedom. The result of this construction is that the onset of the supratransmission phenomenon is not only a function of the driving parameters (i.e., frequency and amplitude) but the specific network boundary at which the excitation is applied and from which wave energy is



Figure 6.1: Pendulum Network with Feedback. (a) One-dimensional network of rotating pendula with elastic bands providing nearest-neighbor coupling and a feedback mechanism imposing a local influence which is proportional to the non-local displacements, $f_c = s(\varphi_{j+1} - \varphi_{j-1})$. (b) The on-site, multi-stable potential which manifests the system non-linearity. (c) The complex dispersion of small-amplitude waves of prescribed non-dimensional frequency, $\omega \tau$, with real and imaginary wavenumbers, κ_R (solid) and κ_I (dotted), respectively. The feedback effect shifts the attenuating structure about $\kappa_I = 0$, provoking non-reciprocal wave propagation.

transmitted to the medium, establishing the asymmetric dynamic behavior.

The article is organized as follows. Section 6.2 presents the non-linear governing equation of a representative, one-dimensional mechanical system with feedback. We then formulate the corresponding dispersion relation of linear, small-amplitude dynamics. In Sec. 6.3, we analyze the supratransmission characteristics of the system, demonstrating asymmetric performance. We also present results for a two-dimensional network. Section 6.4 concludes the article with a summary of the main results and proposals for future research directions.

6.2 Theory

6.2.1 Model

To demonstrate the feedback-mediated asymmetric energy transmission, we initially consider the non-linear dynamics of a one-dimensional (1D) periodic network of coupled pendula, a modified version of that analyzed by Geniet and Leon¹⁰⁰ (Fig. 6.1a). The pendulum motif is composed of a ring of radius, ℓ , and mass, m, concentrated at a single point along the circumference. The pendulum rotates (in-plane), φ , about its center which, in the presence a gravitational field of strength g, adjusts the local potential as $\psi = mg\ell(1 - \cos \varphi)$, which describes a non-convex energy landscape responsible for the network non-linearity (Fig. 6.1b). Moreover, the potential renders the system multi-stable, possessing several energetically degenerate ground states at $\varphi = 2\pi p$, $p \in \mathbb{Z}$. Utilizing elastic bands to manifest an effective (torsional) stiffness, $c = k\ell^2$, the network emerges via nearest-neighbor coupling. Uniquely, a feedback mechanism imposes an additional, non-conservative influence on each φ_i which, from myriad possible descriptions, we relate to non-local variables; specifically, the torque, f_c , applied to φ_j via feedback is proportional to the relative rotation of its nearest neighbors, i.e., $f_c = s(\varphi_{j+1} - \varphi_{j-1})$, where s denotes the proportional control gain. Physically, the feedback mechanism may emerge from active components which sense the non-local displacement and then, through a connected micro-controller and actuator, apply the calculated torque. For example, each pendulum may connect to a potentiometer which converts the angular displacement to a predetermined voltage drop measured by a programmable micro-controller. Dependent upon the voltage associated with φ_{j+1} and φ_{j-1} , the micro-controller regulates a voltage sent to drive a DC motor which applies a torque, f_c , to the pendulum at site j. For an arbitrary pendulum in the 1D network, the non-dimensional governing equation has the form

$$\ddot{\varphi}_{j} + \bar{c}(2\varphi_{j} - \varphi_{j+1} - \varphi_{j-1}) + \bar{s}(\varphi_{j+1} - \varphi_{j-1}) + \bar{r}\sin\varphi_{j} = 0,$$
(6.1)

where $\bar{c} = c\tau^2/m$, $\bar{s} = s\tau^2/m\ell^2$, and $\bar{r} = g\tau^2/m\ell$ are dimensionless parameters. The period of oscillation for an isolated pendulum, $\tau = 2\pi\sqrt{\ell/g}$, is a natural choice for the characteristic time and the normalizing parameter defining the dimensionless temporal variable, $\bar{t} = t/\tau$. Absent the feedback mechanism (i.e., $\bar{s} = 0$), Eq. (6.1) has a form that arises in several domains of science;^{130–132} however, the sine-Gordon equation with feedback appears unique to the meta-material described in this article.

For a system of *N* pendula, Eq. (6.1) forms the basis of a system of equations, $M\ddot{u} + (K + S)u + f_{NL} = 0$, where M, K, and S, respectively, are the mass, stiffness, and feedback matrices; u and f_{NL} are, accordingly, the displacement and non-linear force vectors. While the mass and stiffness matrices are symmetric, $S \neq S^{T}$, which renders the system non-Hermitian and supportive of non-reciprocal dynamics. Alternatively, K + S may emerge from a system in which the spring interactions, when deformed, manifest unequal forces at each end. Moreover, from the continuum limit of Eq. (6.1) with expansion $\varphi_{j\pm 1} \rightarrow \varphi \pm a\varphi_{,x} + a^{2}\varphi_{,xx}/2$,

$$\varphi_{,tt} - \bar{c}a^2\varphi_{,xx} + 2\bar{s}a\varphi_{,x} + \bar{r}\sin\varphi_j = 0,$$

it is apparent that the activity either injects or extracts energy (spatially) from the system depending on the sign of $\bar{s}\varphi_{,x}$. Consequently, for a positive \bar{s} and, e.g., an exponentially decreasing (increasing) displacement function, the active term tends to support (dampen) the motion of φ . Odd spatial derivatives of higher order would have a similar effect.

6.2.2 Dispersion

The proposed system supports a number of amplitude-dependent solutions owing to the non-linear on-site potential. For the particular case of small-amplitude motion, the Fourier components of an initially compact disturbance disperse as the propagation speeds depend on the corresponding wavelength. To establish the dispersion relation, we first linearize Eq. (6.1) about a (stable) equilibrium configuration and then theoretically extend the pendulum network to infinity through the application of Bloch boundary conditions on a single unit cell; thus, the dynamics of such a system are described by $\varphi_j = \Phi e^{i(\kappa j a - \omega \tau \bar{t})}$ with κ and ω , respectively, the spatial and temporal frequencies. Applied to Eq. (6.1), the Bloch wave solution entails the following dispersion relation

$$-\omega^{2}\tau^{2} + \bar{c}(2 - \gamma - 1/\gamma) + \bar{s}(\gamma - 1/\gamma) + \bar{r} = 0, \qquad (6.2)$$

where $\gamma = e^{i\kappa a}$. The complex band structure emerges from the solution, $\gamma(\omega\tau)$, which contains the real and imaginary components of the normalized wavenumber, $\kappa a = \kappa_R a + i\kappa_I a$: $\kappa_R a =$ Re[$-i\ln(\gamma)$], specifying the spatial oscillation of displacement, and $\kappa_I a = \text{Im}[-i\ln(\gamma)]$, expressing the spatial attenuation of the amplitude.

6.3 Results

6.3.1 Simulation

In the following, we numerically investigate the dynamics of a representative non-linear network with non-local feedback. To this end, we consider a finite, non-linear system of N pendula subject to a prescribed harmonic boundary displacement, $\varphi_j = \Phi \sin(\omega \tau t)$, where $\omega \tau$ is set within the band gap and Φ varies between simulations. The left (L) and right (R) boundaries are distinguished, respectively, by j = 1 and j = N. The system response is quantified by the mean energy transmitted to the network by the driven boundary over an $n \in \mathbb{Z}^+$ multiple of the excitation period, T:

$$E_{\rm in} = \frac{\bar{c}}{nT} \int_0^{nT} (\varphi_2 - \varphi_1) \dot{\varphi}_1 d\bar{t}, \qquad (6.3)$$

for the left boundary; for the right boundary, substitute the indices in Eq. (6.3) as $\{1,2\} \rightarrow \{N, N-1\}$. For sufficiently small amplitudes, the system exhibits a linear response where the energy supplied to the system, ultimately, returns to the driving due to Bragg reflection, causing E_{in} to vanish. Conversely, beyond a critical amplitude, Φ_c , the driving excites non-linear modes which penetrate the system, resulting in $E_{in} > 0$, signifying a spontaneous energy flow. While a number

of methods have been developed to predict the amplitude threshold,^{101,133–135} in this article, we intend to introduce the concept of active lattices to the supratransmission literature and initiate exploration of its effects.

Simulations evolve Eq. (6.1) for a network of N = 1000 coupled pendula with $\bar{c} = 4$ and $\bar{r} = 1$ for n = 100 periods. In order to minimize reflections from the free boundary (thus, mimicking an infinite medium), we apply a linearly increasing viscous damping, $\bar{\eta} \dot{\phi}_j$, to the final 800 sites with max $(\bar{\eta}) = 1/10$. In order to avoid the shock wave generated by vanishing initial velocities, the simulations adopt an inaugural velocity profile matching the (linear) evanescent solution at the driving frequency, i.e., $\dot{\phi}_j = -\omega\tau\Phi\cos[(j-1)\kappa_R a]e^{-(j-1)\kappa_I a}$. In simulation, each boundary is excited in turn, revealing a direction-specific energy transmission for $\bar{s} \neq 0$.

6.3.2 Supratransmission: One Dimension

Figure 6.1c graphs the dispersion diagram for instances of passive ($\bar{s} = 0$) and active ($\bar{s} = 1/3$) feedback control. Each system exhibits a single pass band separating two band gaps. Apparently, for the reference case where $\bar{s} = 0$, the diagram is symmetric about $\kappa = 0$, implying that wave propagation is independent of direction. Harmonic boundary driving in the pass band transmits wave energy unabated into the network; in the band gap, Bragg reflection confines energy to the boundary as indicated by $\kappa_I \neq 0$. Activating the feedback mechanism modifies the dispersion. While the real component of the band structure, $\kappa_R(\omega\tau)$, remains symmetric about $\kappa = 0$, the imaginary component, $\kappa_I(\omega\tau)$, shifts primarily along the wavenumber axis, breaking the diagram symmetry. This implies that wave propagation, including the supratransmission phenomenon, is direction dependent under non-local feedback control. Analyzing small-amplitude waves, Rosa and Ruzzene¹²⁵ also observed non-reciprocal behavior via a feedback effect which was attributed to a complex temporal frequency whose imaginary component resulted in a propagation direction dependent exponential growth or decay. This article describes a similar effect for non-linear waves which is attributed to a shift in the imaginary wavenumber component of the linear wave dispersion.



Figure 6.2: Asymmetric Supratransmission in One Dimension (color online). (a) Energy transmission as a function of the driving amplitude. For supratransmission, the passive system $(\bar{s} = 0)$ exhibits a common threshold amplitude, $\Phi_c = 1.77$, when excited from either the left or right boundary; conversely, the active system $(\bar{s} = 1/3)$ expresses two thresholds, $\Phi_c^L = 1.41$ and $\Phi_c^R = 2.44$, when excited from the left and right, respectively. (b) The separation of Φ_c^L and Φ_c^R over a range of \bar{s} . (c) The normalized displacement profile and energy distribution for an active system $(\bar{s} = 1/100)$ at various amplitudes. For each row, the left and right boundaries are subject to the same harmonic excitation, the amplitude of which may be above/below the critical amplitude to trigger supratransmission at the particular boundary.

Figure 6.2a plots the energy transmission efficiency of a harmonic boundary driving within the lower band gap ($\omega \tau = 0.9$) in the pendulum network as a function of the driving amplitude for two values of the control gain, $\bar{s} = 0$ and $\bar{s} = 1/3$. In general, below a critical amplitude, Φ_c , the linear, small-amplitude response of the system dominates as the total energy remains concentrated near the driven boundary. As the driving amplitude increases, however, higher harmonic modes – the multiples of the driving frequency which appear in the pass band – generated by burgeoning non-linear effects propagate into the system, contributing to a relatively small increase in E_{in} . For the reference case ($\bar{s} = 0$), beyond the critical amplitude predicted by Geniet and Leon,¹⁰⁰ $\Phi_c = 1.77$, the attenuating displacement profile of the linear solution is unstable;^{136,137} instead,



Figure 6.3: Dispersion (color online). (a) The complex dispersion surfaces for a passive, square lattice of pendula with near-neighbor interactions, including the propagating (yellow) and attenuating (red) modes. (b) The circular contour formed by the cross-section of the attenuating surface at band-gap frequency $\omega \tau = 0.9$. Active feedback, i.e., $\bar{s}_{1,2} \neq 0$, shifts the attenuation contour (red, dashed) with respect to the passive result (black), enabling direction-dependent non-reciprocal wave propagation.

the non-linear modes generated at the driven boundary subsequently propagate into the system. This is signified by the sudden increase in the transmitted energy, i.e., supratransmission. As the band diagram of the reference system is symmetric (Fig. 6.1c), the critical amplitude for supratransmission is identical for left- and right-boundary driving. However, setting $\bar{s} \neq 0$ activates the feedback mechanism and adjusts the band diagram as described previously – for $\bar{s} = 1/3$ and $\omega \tau = 0.9$, forward and backward waves are attenuated according to $\kappa_{I}a = 0.150$ and $\kappa_{I}a = 0.317$, respectively – which suggests that Φ_c differs for the same excitation on opposite boundaries. The simulation results depicted in Fig. 6.2a confirm that the amplitude thresholds for supratransmission in the feedback network do, indeed, differ: $\Phi_c^L = 1.41$ and $\Phi_c^R = 2.44$, respectively, for left- and right-boundary driving. Figure 6.2b shows how the two thresholds diverge as a function of \bar{s} .

The supratransmission demonstrated in the previous results transmits the energy of a signal with band-gap frequency through a non-linear system. To further emphasize this phenomenon, Fig. 6.2c displays snapshots of the left and right boundary regions of the active pendula network ($\bar{s} = 1/100$) subject to the same harmonic driving with various amplitudes. For visualization purposes,



Figure 6.4: Asymmetric Supratransmission in Two Dimensions (color online). (a) Finite square lattice with periodic boundary conditions (blue) and a harmonically exciting a line of nodes (red). (b) The threshold amplitude Φ_c versus orientation angle θ reveals the directional impact of activity for a different combinations of the feedback parameters (red) relative to a passive system (black).

the instantaneous displacement in the boundary region is normalized such that $\max(|\varphi_j|) = 1$. The superposed color indicates the corresponding energy distribution over the same region, $\mathscr{H}_j = \frac{1}{2}\phi_j^2 + \frac{1}{2}\bar{c}(\varphi_j - \varphi_{j-1})^2 + \psi(\varphi_j)$. As expected, for small amplitudes, the displacement profile exhibits an exponential decay and energy concentrates at the boundary. In the second row of Fig. 6.2c, the excitation amplitude exceeds the supratransmission threshold for only the left boundary driving; consequently, the evanescent response is unstable and energy propagates away from the boundary. The same excitation amplitude does not exceed the critical value for the right boundary; therefore, apart from the aforementioned higher harmonics lying within the pass band, the energy remains localized there.

6.3.3 Supratransmission: Two Dimensions

The pendulum network can be extended to obtain a two-dimensional (2D) periodic system, e.g., a square lattice with nearest-neighbor coupling. Accordingly, the non-local feedback applied to each $\varphi_{j,k}$ depends on additional terms, $f_c = \bar{s}_1(\varphi_{j+1,k} - \varphi_{j-1,k}) + \bar{s}_2(\varphi_{j,k+1} - \varphi_{j,k-1})$ where \bar{s}_1 and \bar{s}_2 are gain parameters. Considering a Cartesian frame, indices *j* and *k* designate sites along the *x* and *y* axes, respectively, with the indices increasing with the relevant coordinate. Following the same normalization scheme as before, the governing equation for a generic unit cell is

$$\ddot{\varphi}_{j,k} + \bar{c}(4\varphi_{j,k} - \varphi_{j+1,k} - \varphi_{j-1,k} - \varphi_{j,k+1} - \varphi_{j,k-1}) + \bar{s}_1(\varphi_{j+1,k} - \varphi_{j-1,k}) + \bar{s}_2(\varphi_{j,k+1} - \varphi_{j,k-1}) + \bar{r}\sin\varphi_{j,k} = 0.$$
(6.4)

Linearizing Eq. (6.4) and applying the Bloch solution, $\varphi_{j,k} = \Phi e^{i(\kappa_x j a + \kappa_y k a - \omega \tau t)}$, yields

$$-\omega^{2}\tau^{2} + \bar{c}(4 - \gamma_{x} - 1/\gamma_{x} - \gamma_{y} - 1/\gamma_{y}) + \bar{s}_{1}(\gamma_{x} - 1/\gamma_{x}) + \bar{s}_{2}(\gamma_{y} - 1/\gamma_{y}) + \bar{r} = 0,$$
(6.5)

where $\gamma_x = e^{i\kappa_x a}$ and $\gamma_y = e^{i\kappa_y a}$. Figure 6.3a plots the complex dispersion surfaces for the passive system ($\bar{s}_1 = \bar{s}_2 = 0$), revealing a single propagating surface separating two attenuating surfaces within frequency band gaps. Since, as before, our focus is in the lower band gap where the real component of the wavevector vanishes, we substitute $\gamma_x = e^{-\kappa_{I,x}a}$ and $\gamma_y = e^{-\kappa_{I,y}a}$ in Eq. (6.5) such that, for a given $\omega\tau$ in the lower band gap, the $\kappa_{I,x}$ and $\kappa_{I,y}$ which satisfy the relation trace the circular contour of the attenuating surface. Specifically, at the band-gap frequency $\omega\tau = 0.9$, the diagrams in Fig. 6.3b compare the attenuating contours of various active systems with that of the passive lattice. Apparently, similar to the earlier 1D system, in the 2D network, activity causes the attenuating contour to shift relative to the passive reference centered at (κ_x , κ_y) = (0,0), indicating the broken symmetry of the system's small-amplitude dynamics which we expect to

persist at larger amplitudes.

To investigate asymmetric supratransmission in the 2D setting, we track the energy transmitted to a square lattice of finite dimension by a set of co-linear sites oscillating in phase at a band-gap frequency (Fig. 6.4a). Beyond a critical amplitude, the driven boundary generates a wave front which propagates away from the driving and normal to it, i.e., at an angle θ with respect to the *x*-axis. Thus, the directional dependence of the supratransmission phenomena may be investigated by adjusting the slope of the driven boundary. To this end, we modify Eq. (6.3) to accommodate the additional interactions of the 2D system under consideration:

$$E_{\rm in} = \frac{\bar{c}}{nT} \sum_{m,n} \sum_{r,s} \int_0^{nT} (\varphi_{r,s} - \varphi_{m,n}) \dot{\varphi}_{m,n} \mathrm{d}\bar{t}, \qquad (6.6)$$

where $\{m, n\}$ collects the indices of driven sites and $\{r, s\}$ the indices of their nearest neighbor(s). Thus Eq. (6.6) is the time-averaged energy transmitted to the lattice by the driven boundary.

Simulations utilize a square lattice with nearest-neighbor interactions to develop a rectangular system of approximately $23a \times 800a$ in dimension with the driven sites along one of the smaller dimensions and periodic boundary conditions along the two extended dimensions. For $\bar{c} = 4$ and $\bar{r} = 1$, the excitation frequency $\omega \tau = 0.9$ lies within the band gap. For n = 25 periods, we evolve the system following Eq. (6.4) from an initial velocity, $\dot{\phi}_{j,k} = -\omega \tau \Phi \cos(\kappa_{\rm R} d) e^{\kappa_{\rm I} d}$, where *d* is the perpendicular distance from the driven boundary to the (j,k)th site and $\kappa = \kappa_{\rm R} + i\kappa_{\rm I}$ is the complex wavenumber in that direction.

For a number of orientations, θ , we determine the critical driving amplitude above which E_{in} sharply increases. The panels in Fig. 6.4b display the directional dependence of the threshold amplitude for different combinations of the feedback parameters, \bar{s}_1 and \bar{s}_2 . For the passive system, we determine a θ -independent threshold amplitude of $\Phi_c = 1.77$ which is identical to that of the 1D passive system. However, following the wavenumber contours in Fig. 6.3b, activity causes the threshold amplitude to shift relative to its passive value, the magnitude of the shift dependent upon θ . For the active scenario (\bar{s}_1, \bar{s}_2) = (1/8,0), the minimum amplitude for

supratransmission decreases for waves propagating to the right (i.e., $|\theta| < \pi/2$); increases for waves traveling to the left (i.e., $|\theta| > \pi/2$). The effect is maximized for $\theta = 0$ where the wave front aligns with the gradient, $\varphi_{,x}$, which determines the strength of f_c . Conversely, the threshold is unchanged at $\theta = \pm \pi/2$ where the effect of $\varphi_{,x}$ and, therefore, f_c vanish. Figure 6.4b displays additional orientation-amplitude results for systems with alternative feedback definitions which may be similarly understood.

6.4 Conclusion

To summarize, we numerically investigated the non-linear supratransmission phenomenon in active 1D/2D periodic networks that are characterized by a non-local feedback control. We found that such a feedback adjusts the imaginary wavenumber component across the whole of the frequency range, although asymmetrically, such that the otherwise reciprocal dynamics become direction dependent. In the context of band gap energy transmission in finite, one-dimensional networks, the critical amplitude for stimulating supratransmission via boundary excitation differs for each boundary. Similarly, for a two-dimensional system, the amplitude threshold changes with boundary orientation. These results demonstrate an alternative, extremely tunable approach toward non- reciprocal dynamics and applications.^{103, 108}

There are several research directions for subsequent studies. The influence of anisotropy and lattice type, especially those with multiple elements per lattice site (e.g. Lieb, Kagome, etc.), on the directionality of supratransmission is worth investigating. While we have considered a proportional controller with displacement inputs, a natural extension of the present work would explore controllers of derivative- and integral-type to determine the characteristics of each feedback archetype, independently followed by in combination. In addition, feedback definitions involving spatial and temporal rates have yet to be explored.

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Chapter 7

Conclusion

A new design for an energy absorbing metamaterial was proposed and analyzed. Several design scales were investigated individually and also in unison to observe effects on the capabilities of the structure.

The design enables multi-modal energy absorption in a metamaterial through a rotationally multi-stable node embedded within a dual-chiral layer. The dual chiral layer experiences opposite rotation of connection centers in order to produce the relative rotation needed to activate the rotationally multi-stable element. Numerical simulations demonstrate energy absorption in tensile, compressive, and shear loadings for a variety of lattices, all from the same native configuration. This capability previously eluded multi-stable metamaterials. The structures chare the multi-directional ability of other 2D snapping materials, although their multi-modal capacity allows even more complex loadings. These properties stem from the structural design. The simultaneous tensile and compressive capabilities are enabled by the symmetric rotational multi-stability at the fundamental scale.

Key kinematic parameters and various microstructures were then investigated in order to tune the stiffness and peak force of the pre-snap through region. With a few assumptions, the stiffness and peak load were derived as functions of the radius, attachment angle, and multistable moment periodicity. These predictions were then validated numerically in a 1D chain. After this pre-snap region, the absorption plateau's length was tailored in a similar fashion, with the peak height of the plateau heavily influenced by the fundamental parameters and the oscillations influence by the microstructrue. To manipulate the mean height, functionally graded microstructures inspired by density-graded foams were employed. These lead to more progressive snap-through in the system, leading to fewer oscillations in the absorption plateau and a relatively low mean-force.

Finally we turned our attention to the meso-scale; here we influenced the directionality of the metamaterial's energy absorption by mimicking the poly-crystalline microstructure of metals. Due to the discontinuities not only in the topology of the structure but the valeus of the solution space, we employed the Non-dominated Sorting Genetic Algorithm II to optimize material properties in several directions, treating target absorption values as objective functions and grain orientations as variables to be optimized. An experiment to maximize absorption in a single direction relative to another provided good base validation for the nuemrical results in the context of energy absorption. Next the capabilities of polycrystalline mesostructures were analyzed by optimizing the directional stiffness of a lattice modeled with FEM beam elements. The fundamental scale of these structures determines the anisotropy of the base lattice, which heavily influences the macro-scale properties of the structure. The sub-grain microstructure also plays a slight role. The grain geometries themselves also play a lsight role, with limiting values emerging for grain size and grain number.

The investigations revealed the effects of structure at the fundamental, micro-, and mes-scale on macro-scale material properties. The lowest level is the fundamental scale, comprised of the rotationally multi-stable element and the chiral connections which activate switching. The next level is the micro-scale, characterized by the arrangement of multi-stable elements in lattice structures. The highest level is the meso-scale, in this instance a novel artificial polycrystalline structure. Design across these various levels work in concert to produce impressive macro-scale performance of the structure.

The detours taken at the end illustrated an interesting application of tunable multi-stability in order to enable a mechanical multi-level memory device. In addition, we observed a tunable threshold in order to actuate the phenomenon of supratransmission.

Appendix A

Implementation of Recursive Coordinate Reduction

This appendix details the implementation of the recursive coordinate reduction method.¹³⁸ This may serve as a useful introduction to an unfamiliar practitioner. The source introduces many algorithms useful for a variety of numerical simulations, but we are interested primarily in a forward-dynamics. Moreover, as the code is implemented in MATLAB, many recursive algorithms defined in the source are slower than the powerful linear algebra routines native in MATLAB, at least in the problems encountered in these studies. It would be advisable for those familiar with C/C++ or Fortran to attempt the algorithms from the source. Nevertheless, the ideas presented will be simultaneously formulated with a MATLAB implementation with a fairly copy-and-paste-able pseudocode.

8.1 Recursive Coordinate Reduction of a Serial Chain

A "serial chain" consists of bodies linked by revolute joints. There are two ends to the chain, given by body 1 and body N_b . Each body in the chain is connected to the body before it and

the body after it, with the obvious exception of the chain ends. There are no loops or branches. The joints which connect them are similarly numbered, with joint *b* joing bodies b - 1 and *b*. Note that there is a joint for body 1 located at its center of mass which pins it buts allows it to rotate. A rigid body *b* constrained to remain in plane has 3 degrees of freedom (1 rotational and 2 translational) which respond to a moment about the normal to the plane and forces in the plane; these are collected in to the "twist" and "wrench" vectors as

$$\mathbf{t}_{b} = \begin{bmatrix} \boldsymbol{\omega}_{b} \\ \mathbf{v}_{b} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\omega}_{b} \\ \dot{x}_{b} \\ \dot{y}_{b} \end{bmatrix}, \quad \text{and} \quad \mathbf{w}_{b} = \begin{bmatrix} n_{b} \\ \mathbf{f}_{b} \end{bmatrix} = \begin{bmatrix} n_{b} \\ f_{b}^{x} \\ f_{b}^{y} \end{bmatrix}.$$
(8.1)

In the above the twist corresponds to velocities. Note that these velocities belong to the point of connection between body b and body b - 1, i.e. the joint. They are not the velocity of the center of mass of the body. This point does not apply for the angular velocity as it is the same for any point on a rigid body, but the translational velocites are affected by this distinction. These comprise the first time derivatives of the degrees of freedom of a given body in the system; all in all they may be concatenated into a total twist vector as

$$\mathbf{t} = [\mathbf{t}_1^T \quad \mathbf{t}_2^T \quad \cdots \quad \mathbf{t}_{N_b-1}^T \quad \mathbf{t}_{N_b}^T]^T.$$
(8.2)

In a code implementation, this vector is denoted twists, and to index ω_b , \dot{x}_b , and \dot{y}_b from twists, one would simply grab twists(3*b-2):3*b) for body *b*. In a serial structure (i.e. a single chain of bodies connected in sequence), the twist of body *b* is related to the twist of body b - 1 due to the pin joint between them. Let's start at the first body and move onward down the chain to illustrate the point. We will denote the position of the center of mass of a body as \mathbf{x}_b and the position of a joint as \mathbf{r}_b . Body 1 is pinned at its center of mass (i.e. $\mathbf{x}_1 = \mathbf{r}_1 = \mathbf{0}$ and has the twist

$$\mathbf{t}_1 = \begin{bmatrix} \dot{\theta}_1 \\ 0 \\ 0 \end{bmatrix}. \tag{8.3}$$

Here, θ_1 gives the orientation of body 1. Initially, the orientation will be set such that $\theta_1 = 0$. This body connects to body 2 at joint 2, whose position is given by

$$\mathbf{r}_2 = \mathbf{r}_1 + \mathbf{a}_2. \tag{8.4}$$

In the above, \mathbf{a}_2 is the vector that points from the joint associated with body 1 to the joint associated with body 2. It has the form

$$\mathbf{a}_{2} = a_{2} \cos\left(\theta_{1} + \alpha_{2}\right) \mathbf{E}_{x} + a_{2} \sin\left(\theta_{1} + \alpha_{2}\right) \mathbf{E}_{y},$$

= $a_{2} \mathbf{e}_{2}^{r}.$ (8.5)

In the above, a_2 gives the constant length of this vector, and α_2 gives the initial attachment angle. The twist of body 2 is defined as the angular velocity and the translational velocity of \mathbf{r}_2 , i.e. $\mathbf{t}_2 = [\boldsymbol{\omega}_2; \dot{\mathbf{r}}_2]$. Differentiating with respect to time gives

$$\dot{\mathbf{r}}_{2} = \dot{\mathbf{r}}_{1} + \dot{\mathbf{a}}_{2},$$

$$= \dot{\mathbf{r}}_{1} - a_{2}\dot{\theta}_{1}\sin\left(\theta_{1} + \alpha_{2}\right)\mathbf{E}_{x} + a_{2}\dot{\theta}_{1}\cos\left(\theta_{1} + \alpha_{2}\right)\mathbf{E}_{y},$$

$$= \dot{\mathbf{r}}_{1} + a_{2}\dot{\theta}_{1}\mathbf{e}_{2}^{\theta}$$
(8.6)

Let's recognize that $\omega_1 = \dot{\theta}_1$ to reach

$$\dot{\mathbf{r}}_2 = \dot{\mathbf{r}}_1 + a_2 \omega_1 \, \mathbf{e}_2^{\theta}. \tag{8.7}$$

Before we may define the twist for body 2, we must defined its angular velocity. A body confined to a plane will have its angular velocity determined via its corotational frame $\{\mathbf{e}_b^x, \mathbf{e}_b^x\}$ through

$$\boldsymbol{\omega}_b = \dot{\mathbf{e}}_b^x \cdot \mathbf{e}_b^y. \tag{8.8}$$

A lot of choices for the corotational frame could work, but in the interest of notational economy we will choose a basis that will be useful later. We will choose \mathbf{e}_b^x such that it points from joint *b* to the center of mass of body *b*, that is \mathbf{r}_b to \mathbf{x}_b . You could define this as

$$\mathbf{e}_b^x = \frac{\mathbf{x}_b - \mathbf{r}_b}{\|\mathbf{x}_b - \mathbf{r}_b\|}.$$
(8.9)

The tricky portion however is that in this scheme, **angles are defined relative to the last body**. If θ_1 is nonzero and all θ_i following it are zero, the entire serial chain will rotate as if all the joints are rigid. I hope you can visualize this, but in any case the mathematical expressions for body two would be

$$\mathbf{e}_{2}^{x} = \cos\left(\theta_{2} + \delta_{2}\right) \mathbf{e}_{1}^{r} + \sin\left(\theta_{2} + \delta_{2}\right) \mathbf{e}_{1}^{\theta},$$

$$\mathbf{e}_{2}^{y} = -\sin\left(\theta_{2} + \delta_{2}\right) \mathbf{e}_{1}^{r} + \cos\left(\theta_{2} + \delta_{2}\right) \mathbf{e}_{1}^{\theta}.$$
(8.10)

Differentiating the first with respect to time gives

$$\dot{\mathbf{e}}_{2}^{x} = -\dot{\theta}_{2}\sin\left(\theta_{2} + \delta_{2}\right)\mathbf{e}_{2}^{r} + \dot{\theta}_{2}\cos\left(\theta_{2} + \delta_{2}\right)\mathbf{e}_{2}^{\theta} + \dot{\theta}_{1}\cos\left(\theta_{2} + \delta_{2}\right)\mathbf{e}_{2}^{\theta} - \dot{\theta}_{1}\sin\left(\theta_{2} + \delta_{2}\right)\mathbf{e}_{2}^{r}.$$
 (8.11)

Let's recognize again that $\omega_1 = \dot{\theta}_1$. I am trying to differentiate that there is a difference between the joint rate $\dot{\theta}_b$ and the angular velocity of a body ω_b . We end up with

$$\dot{\mathbf{e}}_{2}^{x} = -\dot{\theta}_{2}\sin\left(\theta_{2}+\delta_{2}\right)\mathbf{e}_{2}^{r}+\dot{\theta}_{2}\cos\left(\theta_{2}+\delta_{2}\right)\mathbf{e}_{2}^{\theta}+\omega_{1}\cos\left(\theta_{2}+\delta_{2}\right)\mathbf{e}_{2}^{\theta}-\omega_{1}\sin\left(\theta_{2}+\delta_{2}\right)\mathbf{e}_{2}^{r}.$$
 (8.12)

Taking the necessary dot product gives

$$\omega_2 = (\dot{\theta}_2 + \omega_1)\sin^2(\theta_2 + \delta_2) + (\dot{\theta}_2 + \omega_1)\cos^2(\theta_b + \delta_b),$$

= $\dot{\theta}_2 + \omega_1.$ (8.13)

We can clearly see that **the angular velocity of a body** ω_b **is not simply the joint rate** $\dot{\theta}_b$. This is because the joint angle is defined relatively to the preceding joint angle. Now we may define the twist for body two as

$$\mathbf{t}_{2} = \begin{bmatrix} \boldsymbol{\omega}_{1} + \dot{\boldsymbol{\theta}}_{2} \\ \dot{r}_{1} + a_{2}\boldsymbol{\omega}_{1} \, \mathbf{e}_{2}^{\boldsymbol{\theta}} \end{bmatrix},$$

$$= \begin{bmatrix} \boldsymbol{\omega}_{1} + \dot{\boldsymbol{\theta}}_{2} \\ \dot{r}_{1} - a_{2}\boldsymbol{\omega}_{1}\sin\left(\boldsymbol{\theta}_{1} + \boldsymbol{\alpha}_{2}\right) \, \mathbf{E}_{x} + a_{2}\boldsymbol{\omega}_{1}\cos\left(\boldsymbol{\theta}_{1} + \boldsymbol{\alpha}_{2}\right) \, \mathbf{E}_{y}, \end{bmatrix}$$

$$(8.14)$$

This is actually a linear function of the twist of body 1! Let's concatenate the Cartesian components to reach

$$\mathbf{t}_{2} = \begin{bmatrix} 1 & 0 & 0 \\ -a_{2}\sin(\theta_{1} + \alpha_{2}) & 1 & 0 \\ a_{2}\cos(\theta_{1} + \alpha_{2}) & 0 & 1 \end{bmatrix} \mathbf{t}_{1} + \begin{bmatrix} \dot{\theta}_{2} \\ 0 \\ 0 \end{bmatrix}.$$
(8.15)

Now let's add a third body to the chain to really drive this home. The joint associated with this body is given by

$$\mathbf{r}_{3} = \mathbf{r}_{2} + \mathbf{a}_{3},$$

$$\mathbf{a}_{3} = a_{3} \cos\left(\theta_{2} + \alpha_{3}\right) \mathbf{e}_{2}^{r} + a_{3} \sin\left(\theta_{2} + \alpha_{3}\right) \mathbf{e}_{2}^{\theta},$$

$$= a_{3} \mathbf{e}_{3}^{r}.$$
 (8.16)

This sheds some light on the form of \mathbf{e}_b^r , as it incorporates information from all of the preceding bodies due to the relative definition of angles. Let's actually do the algebra all the way back to Cartesian coordinates to see how they change. Expanding the second line gives

$$\mathbf{e}_{3}^{r} = \cos(\theta_{2} + \alpha_{3}) \left[\cos(\theta_{1} + \alpha_{2}) \mathbf{E}_{x} + \sin(\theta_{1} + \alpha_{2}) \mathbf{E}_{y} \right] \dots + \sin(\theta_{2} + \alpha_{3}) \left[-\sin(\theta_{1} + \alpha_{2}) \mathbf{E}_{x} + \cos(\theta_{1} + \alpha_{2}) \mathbf{E}_{y} \right],$$

$$= \left[\cos(\theta_{2} + \alpha_{3}) \cos(\theta_{1} + \alpha_{2}) - \sin(\theta_{2} + \alpha_{3}) \sin(\theta_{1} + \alpha_{2}) \right] \mathbf{E}_{x} \dots + \left[\cos(\theta_{2} + \alpha_{3}) \sin(\theta_{1} + \alpha_{2}) + \sin(\theta_{2} + \alpha_{3}) \cos(\theta_{1} + \alpha_{2}) \right] \mathbf{E}_{y}.$$

$$(8.17)$$

This may be reduced using trig identities to

$$\mathbf{e}_{3}^{r} = \cos\left(\theta_{1} + \theta_{2} + \alpha_{2} + \alpha_{3}\right) \mathbf{E}_{x} + \sin\left(\theta_{1} + \theta_{2} + \alpha_{2} + \alpha_{3}\right) \mathbf{E}_{y}.$$
(8.18)

This is actually a fairly simple form. What it represents is the actual orientation of joint 3. The joint angles θ_b are relative angles, along with the offsets α_b . However simply summing all these relative angles leading up to a body or joint gives the actual absolute orientation. This makes sense even more when considering that $\{\mathbf{e}_3^r, \mathbf{e}_3^\theta\}$ is a corotational frame for body two, whose angular velocity is $\omega_2 = \omega_1 + \dot{\theta}_2 = \dot{\theta}_1 + \dot{\theta}_2$. We therefore see the integral of this in the argument of the trig functions, as $\dot{\mathbf{e}}_3^r = \omega_2 \mathbf{e}_3^\theta$. Returning to the task at hand. Let's take a time derivative:

$$\dot{\mathbf{r}}_{3} = \dot{\mathbf{r}}_{2} + a_{3}\omega_{2}\mathbf{e}_{3}^{\theta},$$

$$= \dot{\mathbf{r}}_{2} - a_{3}\omega_{2}\sin\left(\theta_{1} + \theta_{2} + \alpha_{1} + \alpha_{2}\right)\mathbf{E}_{x} + a_{3}\omega_{2}\cos\left(\theta_{1} + \theta_{2} + \alpha_{2} + \alpha_{3}\right)\mathbf{E}_{y}.$$
(8.19)

Moving on to defining the corotational basis for body 3, we have

$$\mathbf{e}_{3}^{x} = \cos\left(\theta_{3} + \delta_{3}\right) \mathbf{e}_{3}^{r} + \sin\left(\theta_{3} + \delta_{3}\right) \mathbf{e}_{3}^{\theta},$$

$$\mathbf{e}_{3}^{y} = -\sin\left(\theta_{3} + \delta_{3}\right) \mathbf{e}_{3}^{r} + \cos\left(\theta_{3} + \delta_{3}\right) \mathbf{e}_{3}^{\theta}.$$
(8.20)

Differentiating with respect to time gives

$$\dot{\mathbf{e}}_{3}^{x} = -\dot{\theta}_{3}\sin\left(\theta_{3} + \delta_{3}\right)\mathbf{e}_{3}^{r} + \dot{\theta}_{3}\cos\left(\theta_{3} + \delta_{3}\right)\mathbf{e}_{3}^{\theta} + \omega_{2}\cos\left(\theta_{3} + \delta_{3}\right)\mathbf{e}_{3}^{\theta} - \omega_{2}\sin\left(\theta_{3} + \delta_{3}\right)\mathbf{e}_{3}^{r}.$$
(8.21)

Taking the necessary dot product gives

$$\omega_3 = \omega_2 + \dot{\theta}_3. \tag{8.22}$$

The full twist is then

$$\mathbf{t}_{3} = \begin{bmatrix} 1 & 0 & 0 \\ -a_{3}\sin(\theta_{1} + \theta_{2} + \alpha_{2} + \alpha_{3}) & 1 & 0 \\ a_{3}\cos(\theta_{1} + \theta_{2} + \alpha_{2} + \alpha_{3}) & 0 & 1 \end{bmatrix} \mathbf{t}_{2} + \begin{bmatrix} \dot{\theta}_{3} \\ 0 \\ 0 \end{bmatrix}.$$
 (8.23)

I hope this illustrates the point. We will now develop the relations for a general system. For the translational components, we have

$$\mathbf{r}_{b} = \mathbf{r}_{b-1} + \mathbf{a}_{b},$$

$$= \mathbf{r}_{b-1} + a_{b}\cos\left(\theta_{b-1} + \alpha_{b}\right)\mathbf{e}_{b-1}^{r} + a_{b}\cos\left(\theta_{b-1} + \alpha_{b}\right)\mathbf{e}_{b-1}^{\theta},$$

$$= \mathbf{r}_{b-1} + a_{b}\mathbf{e}_{b}^{r},$$

$$= \mathbf{r}_{b-1} + a_{b}\cos\left(\sum_{i=1}^{b-1}\theta_{i} + \alpha_{i+1}\right)\mathbf{E}_{x} + a_{b}\sin\left(\sum_{i=1}^{b-1}\theta_{i} + \alpha_{i+1}\right)\mathbf{E}_{y}.$$
(8.24)

Differentiating with respect to time gives

$$\dot{\mathbf{r}}_{b} = \dot{\mathbf{r}}_{b-1} + \dot{\mathbf{a}}_{b},$$

$$= \dot{\mathbf{r}}_{b-1} + a_{b} \left[\omega_{b-1} \cos\left(\theta_{b-1} + \alpha_{b}\right) - \dot{\theta}_{b} \sin\left(\theta_{b-1} + \alpha_{b}\right) \right] \mathbf{e}_{b}^{r} \dots$$

$$+ a_{b} \left[\dot{\theta}_{1} \sin\left(\theta_{b-1} + \alpha_{b}\right) - \omega_{b-1} \cos\left(\theta_{b-1} + \alpha_{b}\right) \right] \mathbf{e}_{b}^{\theta},$$

$$= \dot{\mathbf{r}}_{b-1} + a_{b} \omega_{b-1} \mathbf{e}_{b}^{\theta},$$

$$= \dot{\mathbf{r}}_{b-1} - a_{b} \left(\sum_{i=1}^{b-1} \dot{\theta}_{i} \right) \sin\left(\sum_{i=1}^{b-1} \theta_{i} + \alpha_{i+1} \right) \mathbf{E}_{x} + a_{b} \left(\sum_{i=1}^{b-1} \dot{\theta}_{i} \right) \cos\left(\sum_{i=1}^{b-1} \theta_{i} + \alpha_{i+1} \right) \mathbf{E}_{y}.$$
(8.25)

The corotational frame of body b is given by

$$\begin{aligned} \mathbf{e}_{b}^{x} &= \cos\left(\theta_{b} + \delta_{b}\right) \mathbf{e}_{b}^{r} + \sin\left(\theta_{b} + \delta_{b}\right) \mathbf{e}_{b}^{\theta}, \\ \mathbf{e}_{b}^{x} &= \cos\left(\theta_{b} + \delta_{b}\right) \left[\cos\left(\sum_{i=1}^{b-1} \theta_{i} + \alpha_{i+1}\right) \mathbf{E}_{x} + a_{b} \sin\left(\sum_{i=1}^{b-1} \theta_{i} + \alpha_{i+1}\right) \mathbf{E}_{y} \right] \dots \\ &+ \sin\left(\theta_{b} + \delta_{b}\right) \left[-\sin\left(\sum_{i=1}^{b-1} \theta_{i} + \alpha_{i+1}\right) \mathbf{E}_{x} + a_{b} \cos\left(\sum_{i=1}^{b-1} \theta_{i} + \alpha_{i+1}\right) \mathbf{E}_{y} \right], \end{aligned} \tag{8.26}$$

$$\mathbf{e}_{b}^{y} &= -\sin\left(\theta_{b} + \delta_{b}\right) \mathbf{e}_{b}^{r} + \cos\left(\theta_{b} + \delta_{b}\right) \mathbf{e}_{b}^{\theta}, \\ \mathbf{e}_{b}^{y} &= -\sin\left(\theta_{b} + \delta_{b}\right) \left[\cos\left(\sum_{i=1}^{b-1} \theta_{i} + \alpha_{i+1}\right) \mathbf{E}_{x} + a_{b} \sin\left(\sum_{i=1}^{b-1} \theta_{i} + \alpha_{i+1}\right) \mathbf{E}_{y} \right] \dots \\ &+ \cos\left(\theta_{b} + \delta_{b}\right) \left[-\sin\left(\sum_{i=1}^{b-1} \theta_{i} + \alpha_{i+1}\right) \mathbf{E}_{x} + a_{b} \cos\left(\sum_{i=1}^{b-1} \theta_{i} + \alpha_{i+1}\right) \mathbf{E}_{y} \right], \end{aligned}$$

Differentiating the first gives

$$\dot{\mathbf{e}}_{b}^{x} = - = \dot{\theta}_{b} \sin\left(\theta_{b} + \delta_{b}\right) \mathbf{e}_{b}^{r} + \dot{\theta}_{b} \cos\left(\theta_{b} + \delta_{b}\right) \mathbf{e}_{b}^{\theta} + \omega_{b-1} \cos\left(\theta_{b} + \delta_{b}\right) \mathbf{e}_{b}^{\theta} - \omega_{b-1} \sin\left(\theta_{b} + \delta_{b}\right) \mathbf{e}_{b}^{r}.$$
(8.27)

Please note that although we are choosing the vectors which point from the joint associated with a body to that bodies center of mass as the corotational frame, the set $\{\mathbf{e}_b^r, \mathbf{e}_b^\theta\}$ also serves

as a corotational frame for the preceding body, and has the property that $\dot{\mathbf{e}}_b^r = \omega_{b-1} \mathbf{e}_b^{\theta}$ and $\dot{\mathbf{e}}_b^{\theta} = -\omega_{b-1} \mathbf{e}_b^r$. Taking the necessary dot product gives the angular velocity as

$$\omega_b = (\dot{\theta}_b + \omega_{b-1})\sin^2(\theta_b + \delta_b) + (\dot{\theta}_b + \omega_{b-1})\cos^2(\theta_b + \delta_b),$$

= $\dot{\theta}_b + \omega_{b-1}.$ (8.28)

The general form for the relationship of twists in the serial chain is thus

$$\mathbf{t}_{b} = \begin{bmatrix} 1 & 0 & 0 \\ -a_{b} \sin\left(\sum_{i=1}^{b-1} \theta_{i} + \alpha_{i+1}\right) & 1 & 0 \\ a_{b} \cos\left(\sum_{i=1}^{b-1} \theta_{i} + \alpha_{i+1}\right) & 0 & 1 \end{bmatrix} \mathbf{t}_{b-1} + \begin{bmatrix} \dot{\theta}_{b} \\ 0 \\ 0 \end{bmatrix}$$

$$= \mathbf{A}_{b} \mathbf{t}_{b-1} + \begin{bmatrix} \dot{\theta}_{b} \\ 0 \\ 0 \end{bmatrix}.$$
(8.29)

We will now take the time to develop a few arrays for implementation. First off, the joint angles θ_i will be the solution vector, i.e.

$$\mathbf{U} = \begin{bmatrix} \boldsymbol{\theta}_1 & \boldsymbol{\theta}_2 & \cdots & \boldsymbol{\theta}_{N_h} \end{bmatrix}^T = \boldsymbol{\theta}. \tag{8.30}$$

As seen from the arguments of the trig functions within A_b , we shall require specific sums of the joint angles. All of these can be achieved with the matrix operation

$$\begin{bmatrix} \sum_{i=1}^{1-1} \theta_i \\ \sum_{i=1}^{2-1} \theta_i \\ \sum_{i=1}^{3-1} \theta_i \\ \sum_{i=1}^{4-1} \theta_i \\ \cdots \\ \sum_{i=1}^{5-1} \theta_i \\ \sum_{i=1}^{5-1} \theta_i \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 1 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 1 & 0 & 0 \\ 1 & 1 & 1 & \cdots & 1 & 1 & 0 \end{bmatrix} \theta,$$
(8.31)
$$= AbsAng*U.$$

Any of the entries of the left hand side may be computed with AbsAng(b, :)*U. The form of AbsAng is that of a simple lower triangular matrix with zeros along the diagonal and ones for the lower diagonal entries. Note this is only true for the serial chain. and the form will change later. The matrix A_b has two other variables that may change for a given body. These will be stored as

Aij1(b) =
$$a_b$$
,
Aij2(b) = $\sum_{i=b}^{b-1} \alpha_{i+1}$. (8.32)

Since the angle offsets α_b don't change in time, we don't introduce a matrix operation for their summation. It also makes initializing them easier as you can simply use the actual orientation as opposed to the relative angle.

Now We move onto actually creating a recursive relationship between the twists and the joint rates. It is best to think of $\dot{\theta}$ as a new set of coordinates and the twists t as an old set of coordinates. We essentially want to transform between these two sets of coordinates, and hence need to define the relationship between them. To get the new coordinate which describes the motion of body *b*, one would use $\dot{\theta}$ (b). This propagation of twist throughout the chain of n+1 bodies can be recorded in the single matrix equation

$$\mathbf{A}_0 \mathbf{t}_0 + \mathbf{A} \mathbf{t} = \mathbf{N}_d \dot{\boldsymbol{\theta}}.$$
 (8.33)

In the above,

$$\mathbf{A}_{0} = \begin{bmatrix} -\mathbf{A}_{10} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix}; \quad \mathbf{A} = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ -\mathbf{A}_{21} & \mathbf{I} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & -\mathbf{A}_{n,n-1} & \mathbf{I} \end{bmatrix}; \quad \mathbf{N}_{d} = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix}$$

Solving for the twists gives

$$\mathbf{t} = \mathbf{A}^{-1} (\mathbf{N}_d \dot{\boldsymbol{\theta}} - \mathbf{A}_0 \mathbf{t}_0). \tag{8.35}$$

Define

$$\mathbf{N}_{l} \equiv \mathbf{A}^{-1} = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{A}_{21} & \mathbf{I} & \vdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{A}_{n1} & \mathbf{A}_{n2} & \cdots & \mathbf{I} \end{bmatrix}$$
(8.36)

with the note that $\mathbf{A}_{ij} = \mathbf{A}_{i,i-1} \cdots \mathbf{A}_{j+1,j}$ for i > j, e.g. $\mathbf{A}_{41} = \mathbf{A}_{43}\mathbf{A}_{32}\mathbf{A}_{21}$. We hence have

$$\mathbf{t} = \mathbf{N}_l \mathbf{N}_d \dot{\boldsymbol{\theta}} - \mathbf{N}_l \mathbf{A}_0 \mathbf{t}_0 \tag{8.37}$$

In the above, N_l and N_d are called the decoupled natural orthogonal complement (DeNOC) matrices.

8.1.1 Dynamics

For a given body *i*, the Newton-Euler equations are

$$\mathbf{M}_i \dot{\mathbf{t}}_i + \mathbf{C}_i \mathbf{t}_i = \mathbf{w}_i \tag{8.38}$$

with

$$\mathbf{M}_{i} = \begin{bmatrix} I_{i} & -m_{i}\mathbf{d}_{i}^{T}\mathbf{E} \\ m_{i}\overline{\mathbf{E}}\mathbf{d}_{i} & m_{i}\mathbf{I} \end{bmatrix},$$

$$\mathbf{C}_{i} = \begin{bmatrix} \mathbf{0} & \mathbf{0}^{T} \\ -m_{i}\omega_{i}\mathbf{d}_{i} & \mathbf{0} \end{bmatrix}$$
(8.39)

In this we have

$$\overline{\mathbf{E}} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{8.40}$$

We have yet to define the vector \mathbf{d}_i . This is the vector which points from joint *i* to the center of mass of of body #*i*. All in all, this leads to

$$\mathbf{M}_{i} = \begin{bmatrix} I_{i} & sym. sym. \\ -m_{i}d1(i)\sin(AbsAng(i,:)*U+d2(i)) & m_{i} & sym. \\ m_{i}d1(i)\cos(AbsAng(i,:)*U+d2(i)) & 0 & m_{i} \end{bmatrix},$$

$$\mathbf{C}_{i} = \begin{bmatrix} 0 & 0 & 0 \\ -m_{i}\omega_{i}d1(i)\cos(AbsAng(i,:)*U+d2(i)) & 0 & 0 \\ -m_{i}\omega_{i}d1(i)\sin(AbsAng(i,:)*U+d2(i)) & 0 & 0 \end{bmatrix}.$$
(8.41)

The above is for body *i*; defining $\mathbf{M} = \text{diag}([\mathbf{M}_1 \cdots \mathbf{M}_n)]$ and $\mathbf{C} = \text{diag}([\mathbf{C}_1 \cdots \mathbf{C}_n])$ we have

$$\mathbf{M}\dot{\mathbf{t}} + \mathbf{C}\mathbf{t} = \mathbf{w}.\tag{8.42}$$

Note we have concatenated the twists and wrenches. How does this help? Note that we have

$$\mathbf{t} = \mathbf{N}\dot{\boldsymbol{\theta}}, \quad \dot{\mathbf{t}} = \dot{\mathbf{N}}\dot{\boldsymbol{\theta}} + \mathbf{N}\ddot{\boldsymbol{\theta}}. \tag{8.43}$$

Working this into the above, we get

$$\mathbf{M}(\dot{\mathbf{N}}\dot{\boldsymbol{\theta}} + \mathbf{N}\ddot{\boldsymbol{\theta}}) + \mathbf{C}\mathbf{N}\dot{\boldsymbol{\theta}} = \mathbf{w}.$$
(8.44)

Note that $\dim(\mathbf{t}) = 3n \times 1$, while $\dim(\dot{\theta}) = n \times 1$. This gives a significant reduction in system size. We must also left-multiply by the transpose, giving

$$\mathbf{N}^T \mathbf{M} \mathbf{N} \ddot{\boldsymbol{\theta}} + \mathbf{N}^T \mathbf{M} \dot{\mathbf{N}} \dot{\boldsymbol{\theta}} + \mathbf{N}^T \mathbf{C} \mathbf{N} \dot{\boldsymbol{\theta}} = \mathbf{N}^T \mathbf{w}.$$
(8.45)

At each time step, we will need to create the **N**, **M**, and **C** matrices, then multiply and invert them accordingly.

8.2 An Example

We will examine two discs of radius *r* connected via a rod of length ℓ . The twist vector for this system is

$$\mathbf{t} = [\boldsymbol{\omega}_1 \ u_1 \ v_1 \ \boldsymbol{\omega}_2 \ u_2 \ v_2 \ \boldsymbol{\omega}_3 \ u_3 \ v_3]^T$$
(8.46)

We will label the bodies such that the first disc is body 1, the second disc is body 2, and the rod is body 3. The constraints are then expressed as

$$\mathbf{t}_{3} = \begin{bmatrix} 1 & 0 & 0 \\ -r\sin(\theta_{1} + \alpha_{3}) & 1 & 0 \\ r\cos(\theta_{1} + \alpha_{3}) & 0 & 1 \end{bmatrix} \mathbf{t}_{1} + \begin{bmatrix} \dot{\theta}_{3} \\ 0 \\ 0 \end{bmatrix},$$

$$\mathbf{t}_{2} = \begin{bmatrix} 1 & 0 & 0 \\ -\ell\sin(\theta_{3} + \theta_{1} + \alpha_{2} + \alpha_{3}) & 1 & 0 \\ \ell\cos(\theta_{3} + \theta_{1} + \alpha_{2} + \alpha_{3}) & 0 & 1 \end{bmatrix} \mathbf{t}_{3} + \begin{bmatrix} \dot{\theta}_{2} \\ 0 \\ 0 \end{bmatrix}.$$
(8.47)

The angles used are a bit strange. For the first body, $\theta_1 + \beta_{13}$ gives the absolute orientation. The translational and angular velocities of this body are

$$\dot{\mathbf{x}}_1 = u_1 \mathbf{E}_x + v_1 \mathbf{E}_y,$$

$$\boldsymbol{\omega}_1 = \dot{\boldsymbol{\theta}}_1.$$
(8.48)

This comprises gives the twist $\mathbf{t}_1 = [\boldsymbol{\omega}_1, \dot{\mathbf{x}}_1]^T$. The twist vectors correlate to the joint on the body, and this formulation implies that body 1 has a pin joint right at its mass center. This will be the origin, and is further assumed to be fixed for all time, i.e. $\mathbf{x}_1 = 0$. The joint which connects to body 3 has its position given by

$$\mathbf{r}_{13} = \mathbf{x}_1 + r\cos\left(\theta_1 + \alpha_3\right)\mathbf{E}_x + r\sin\left(\theta_1 + \alpha_3\right)\mathbf{E}_y \equiv \mathbf{x}_1 + r\mathbf{e}_x^{13},$$
with $\mathbf{e}_y^{13} = -\sin\left(\theta_1 + \alpha_3\right)\mathbf{E}_x + \cos\left(\theta_1 + \alpha_3\right)\mathbf{E}_y.$
(8.49)

This states that the corotational frame gives the position of the joint relative to the mass center of body 1, \mathbf{x}_1 . Initially, we will have $\theta_1(t = 0) = 0$ and $\beta_{13} = \pi/4$. The constant β_{ij} is included in the formulation as multiple bodies may connect to body *i*, and the position of the connections will vary despite only one degree of freedom being associated with body *i*. The velocity of this joint is given by

$$\dot{\mathbf{r}}_{13} = \dot{\mathbf{x}}_1 + r\dot{\theta}_1 \mathbf{e}_v^{13}. \tag{8.50}$$

In a matrix equation, this is

$$\begin{bmatrix} u_3 \\ v_3 \end{bmatrix} = \begin{bmatrix} -r\sin(\theta_1 + \alpha_3) & 1 & 0 \\ r\cos(\theta_1 + \alpha_3) & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ u_1 \\ v_1 \end{bmatrix}.$$
 (8.51)

Recognizing the latter vector as the twist of joint 1, this gives

$$\begin{bmatrix} u_3 \\ v_3 \end{bmatrix} = \begin{bmatrix} -r\sin(\theta_1 + \alpha_3) & 1 & 0 \\ r\cos(\theta_1 + \alpha_3) & 0 & 1 \end{bmatrix} \mathbf{t}_1.$$
(8.52)

This takes care of the translational portion of the twist for joint 3 (and body 3). We now need to determine the angular velocity of body 3 (on which joint 3 is assume to be attached). The angular velocity of a body with the corotational frame $\{\mathbf{e}_i\}$ is given by

$$\boldsymbol{\omega} = (\dot{\mathbf{e}}_x \cdot \mathbf{e}_y)\mathbf{e}_z + (\dot{\mathbf{e}}_y \cdot \mathbf{e}_z)\mathbf{e}_x + (\dot{\mathbf{e}}_z \cdot \mathbf{e}_x)\mathbf{e}_y. \tag{8.53}$$

For a body confined to a plane, this is simply

$$\boldsymbol{\omega}_{z} = \dot{\mathbf{e}}_{x} \cdot \mathbf{e}_{y}. \tag{8.54}$$

Let's verify for body one:

$$\boldsymbol{\omega}_{1} = \dot{\mathbf{e}}_{x}^{13} \cdot \mathbf{e}_{y}^{13},$$

$$= \dot{\boldsymbol{\theta}}_{1} \|\mathbf{e}_{y}^{13}\|,$$

$$= \dot{\boldsymbol{\theta}}_{1}.$$
(8.55)

Excellent. Now for body three we need to define the corotational frame. This is

$$\mathbf{e}_{x}^{32} = \cos{(\theta_{3} + \alpha_{2})} \mathbf{e}_{x}^{13} + \sin{(\theta_{3} + \alpha_{2})} \mathbf{e}_{y}^{13},$$

$$\mathbf{e}_{y}^{32} = -\sin{(\theta_{3} + \alpha_{2})} \mathbf{e}_{x}^{13} + \cos{(\theta_{3} + \alpha_{2})} \mathbf{e}_{y}^{13}.$$

(8.56)

This is to say that θ_3 defines a relative angle to the direction that \mathbf{e}_x^{13} points. Initially, we have $\theta_3 = 0$ and $\alpha_2 = -pi/4 - \arcsin(2r/\ell\sin\pi/4)$. We will require the time derivative of the first vector, and it is

$$\dot{\mathbf{e}}_{x}^{32} = -\dot{\theta}_{3}\sin(\theta_{3} + \alpha_{2})\mathbf{e}_{x}^{13} + \dot{\theta}_{1}\cos(\theta_{3} + \alpha_{2})\mathbf{e}_{y}^{13} + \dot{\theta}_{3}\cos(\theta_{3} + \alpha_{2})\mathbf{e}_{y}^{13} - \dot{\theta}_{1}\sin(\theta_{3} + \alpha_{2})\mathbf{e}_{x}^{13}.$$
(8.57)

Collecting terms to form the \mathbf{e}_x^{32} and \mathbf{e}_y^{32} , we have

$$\dot{\mathbf{e}}_x^{32} = (\dot{\theta}_3 + \dot{\theta}_1) \mathbf{e}_y^{32}.$$
 (8.58)

Since we will need it later, we also compute the derivative of the y direction, which is

$$\dot{\mathbf{e}}_{y}^{32} = -\dot{\theta}_{3}\cos{(\theta_{3} + \alpha_{2})}\mathbf{e}_{x}^{13} - \dot{\theta}_{1}\sin{(\theta_{3} + \alpha_{2})}\mathbf{e}_{y}^{13} - \dot{\theta}_{3}\sin{(\theta_{3} + \alpha_{2})}\mathbf{e}_{y}^{13} - \dot{\theta}_{1}\cos{(\theta_{3} + \alpha_{2})}\mathbf{e}_{x}^{13}.$$
(8.59)

Simplifying yields

$$\dot{\mathbf{e}}_{y}^{32} = -(\dot{\theta}_{3} + \dot{\theta}_{1})\mathbf{e}_{x}^{32}.$$
 (8.60)

The angular velocity is thus

$$\begin{split} \boldsymbol{\omega}_{1} &= \dot{\mathbf{e}}_{x}^{32} \cdot \mathbf{e}_{y}^{32}, \\ &= (\dot{\boldsymbol{\theta}}_{3} + \dot{\boldsymbol{\theta}}_{1}) \| \mathbf{e}_{y}^{32} \|, \\ &= \dot{\boldsymbol{\theta}}_{3} + \dot{\boldsymbol{\theta}}_{1}. \end{split}$$
(8.61)

Excellent. This completes our twist matrix equation from above as

$$\begin{bmatrix} \omega_{3} \\ u_{3} \\ v_{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -r\sin(\theta_{1} + \alpha_{3}) & 1 & 0 \\ r\cos(\theta_{1} + \alpha_{3}) & 0 & 1 \end{bmatrix} \mathbf{t}_{1} + \begin{bmatrix} \dot{\theta}_{1} \\ 0 \\ 0 \end{bmatrix}.$$
 (8.62)

The position of the next joint is given by

$$\mathbf{r}_{32} = \mathbf{r}_{13} + \ell \mathbf{e}_x^{32}. \tag{8.63}$$

The velocity is

$$\dot{\mathbf{r}}_{32} = \dot{\mathbf{r}}_{13} + \ell(\dot{\theta}_3 + \dot{\theta}_1) \quad \mathbf{e}_y^{32}, \\ = \dot{\mathbf{r}}_{13} + \ell(\dot{\theta}_3 + \dot{\theta}_1) \quad \left[-\sin(\theta_3 + \alpha_2)\mathbf{e}_x^{13} + \cos(\theta_3 + \alpha_2)\mathbf{e}_y^{13} \right], \\ = \dot{\mathbf{r}}_{13} + \ell(\dot{\theta}_3 + \dot{\theta}_1) \quad \left[\left(-\sin(\theta_3 + \alpha_2)\cos(\theta_1 + \alpha_3) - \cos(\theta_3 + \alpha_2)\sin(\theta_1 + \alpha_3) \right) \mathbf{E}_x + \dots \right] \\ \left(\cos(\theta_3 + \alpha_2)\cos(\theta_1 + \alpha_3) - \sin(\theta_3 + \alpha_2)\sin(\theta_1 + \alpha_3) \right) \mathbf{E}_y \right].$$
(8.64)

Using some trig identities and recognizing the angular velocity of body three, we have

$$\dot{\mathbf{r}}_{32} = \dot{\mathbf{r}}_{13} + \ell \boldsymbol{\omega}_3 \Big[-\sin\left(\theta_3 + \theta_1 + \alpha_3 + \alpha_2\right) \mathbf{E}_x + \cos\left(\theta_3 + \theta_1 + \alpha_3 + \alpha_2\right) \mathbf{E}_y \Big].$$
(8.65)

This leads to the matrix equation

$$\begin{bmatrix} u_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} -\ell \sin(\theta_3 + \theta_1 + \alpha_3 + \alpha_2) & 1 & 0 \\ \ell \cos(\theta_3 + \theta_1 + \alpha_3 + \alpha_2) & 0 & 1 \end{bmatrix} \mathbf{t}_3.$$
(8.66)

Now we require the corotational frame of body two. This is

$$\mathbf{e}_{x}^{2} = \cos{(\theta_{2} + \beta_{2})} \mathbf{e}_{x}^{32} + \sin{(\theta_{2} + \delta_{2})} \mathbf{e}_{y}^{32},$$

$$\mathbf{e}_{y}^{2} = -\sin{(\theta_{2} + \delta_{2})} \mathbf{e}_{x}^{32} + \cos{(\theta_{2} + \delta_{2})} \mathbf{e}_{y}^{32}.$$
(8.67)

We will need the derivative of the *x* vector, which is

$$\dot{\mathbf{e}}_{x}^{2} = -\dot{\theta}_{2}\sin(\theta_{2} + \delta_{2})\mathbf{e}_{x}^{32} + \cos(\theta_{2} + \delta_{2})\dot{\mathbf{e}}_{x}^{32} + \dot{\theta}_{2}\cos(\theta_{2} + \delta_{2})\mathbf{e}_{y}^{32} + \sin(\theta_{2} + \delta_{2})\dot{\mathbf{e}}_{y}^{32},$$

$$= \dot{\theta}_{2}\mathbf{e}_{y}^{2} + (\dot{\theta}_{3} + \dot{\theta}_{1})\cos(\theta_{2} + \delta_{2})\mathbf{e}_{y}^{32} - (\dot{\theta}_{3} + \dot{\theta}_{1})\sin(\theta_{2} + \delta_{2})\mathbf{e}_{x}^{32}.$$
(8.68)

This is

$$\dot{\mathbf{e}}_x^2 = \boldsymbol{\omega}_3 + \dot{\boldsymbol{\theta}}_2. \tag{8.69}$$

The angular velocity is thus

$$\omega_2 = \omega_3 + \dot{\theta}_2. \tag{8.70}$$

We recover the matrix equation for the twists, namely

$$\begin{bmatrix} \omega_2 \\ u_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\ell \sin(\theta_3 + \theta_1 + \alpha_3 + \alpha_2) & 1 & 0 \\ \ell \cos(\theta_3 + \theta_1 + \alpha_3 + \alpha_2) & 0 & 1 \end{bmatrix} \mathbf{t}_3 + \begin{bmatrix} \dot{\theta}_2 \\ 0 \\ 0 \end{bmatrix}.$$
 (8.71)

In the above, $\theta_2(t=0) = 0$ and $\delta_2 = \pi/4 + \arcsin(2r/\ell\sin\pi/4)$.

Arranging into a single matrix equation, we have

(8.72)

Okay, let's use the abbreviated forms to reach the compact equation

Now we find the inverse. This is

Let's make it detailed:

Now we can verify that our code does this. We use r = 0.2829455, $\ell = 0.7210610$, and $\alpha_0 = -0.5882820$ radians. Our code passes this test. Hooray. Note that this formulation assumes that the first body is fixed at its mass center but is allowed to rotate. This is perfectly fine for our purposes. Let's move on to the next definitions. We will require

$$\mathbf{M}_i \dot{\mathbf{t}}_i + \mathbf{C}_i \mathbf{t}_i = \mathbf{w}_i, \tag{8.76}$$

$$\mathbf{M}_{i} = \begin{bmatrix} I_{i} & -m_{i}\mathbf{d}_{i}^{T}\overline{\mathbf{E}} \\ m_{i}\overline{\mathbf{E}}\mathbf{d}_{i} & m_{i}\mathbf{I} \end{bmatrix},$$

$$\mathbf{C}_{i} = \begin{bmatrix} 0 & \mathbf{0}^{T} \\ -m_{i}\omega_{i}\mathbf{d}_{i} & \mathbf{0} \end{bmatrix},$$
(8.77)

for each body. In more detail, this is

$$\begin{bmatrix} I_i & -m_i d_i \cos \phi_i & m_i d_i \sin \phi_i \\ -m_i d_i \sin \phi_i & m_i & 0 \\ m_i d_i \cos \phi_i & 0 & m_i \end{bmatrix} \dot{\mathbf{t}}_i + \begin{bmatrix} 0 & 0 & 0 \\ -m_i \omega_i d_i \cos \phi_i & 0 & 0 \\ -m_i \omega_i d_i \sin \phi_i & 0 & 0 \end{bmatrix} \mathbf{t}_i = \mathbf{w}_i.$$
(8.78)

In the above, ϕ_i refers to the absolute angle of the body. These arguments are the same as what we derived above. Note this is only true for when the lines connecting joints in the serial chain pass through the mass centers. Explicitly, these angles are

$$\phi_2 = \theta_1 + \theta_2 + \theta_3 + \alpha_3 + \alpha_2 + \delta_2,$$

$$\phi_3 = \theta_1 + \theta_3 + \alpha_3 + \delta_2.$$
(8.79)

It doesn't make sense to define ϕ_1 as the joint for this body is at the center of mass. Similarly, $d_1 = 0$. In the above, $\delta_2 = \pi/4$ and $\delta_3 = \alpha_2$, and are constant angle offsets. The matrices above are

$$\begin{bmatrix} r^{2}/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \dot{\mathbf{t}}_{1} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{t}_{1} = \mathbf{w}_{1},$$

$$\begin{bmatrix} r^{2}/2 & -r\cos\phi_{2} & r\sin\phi_{2} \\ -r\sin\phi_{2} & 1 & 0 \\ r\cos\phi_{2} & 0 & 1 \end{bmatrix} \dot{\mathbf{t}}_{2} + \begin{bmatrix} 0 & 0 & 0 \\ -\omega_{2}r\cos\phi_{2} & 0 & 0 \\ -\omega_{2}r\sin\phi_{2} & 0 & 0 \end{bmatrix} \mathbf{t}_{2} = \mathbf{w}_{2}, \quad (8.80)$$

$$\begin{bmatrix} \ell^{2}/12 & -\ell/2\cos\phi_{3} & \ell/2\sin\phi_{3} \\ -\ell/2\sin\phi_{3} & 1 & 0 \\ \ell/2\cos\phi_{3} & 0 & 1 \end{bmatrix} \dot{\mathbf{t}}_{3} + \begin{bmatrix} 0 & 0 & 0 \\ -\omega_{3}\ell/2\cos\phi_{3} & 0 & 0 \\ -\omega_{3}\ell/2\sin\phi_{3} & 0 & 0 \end{bmatrix} \mathbf{t}_{3} = \mathbf{w}_{3}.$$

Note that we have set all masses to be 1. In the full system, this leads to



These matrices also match those output from our code.

8.3 Formation of Matrices

There are several key matrices that form the numerical equations of motion. The system is

$$\begin{bmatrix} \mathbf{N}^{T}(\mathbf{U})\mathbf{M}(\mathbf{U})\mathbf{N}(\mathbf{U}) & -\mathbf{J}^{T}(\mathbf{U}) \\ \mathbf{J}(\mathbf{U}) & \mathbf{0} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{U}} \\ \lambda \end{bmatrix} + \begin{bmatrix} \mathbf{N}^{T}(\mathbf{U})\mathbf{M}(\mathbf{U})\dot{\mathbf{N}}(\mathbf{U}) + \mathbf{N}^{T}(\mathbf{U})\mathbf{C}(\mathbf{U})\mathbf{N}(\mathbf{U}) + \mathbf{D} & \mathbf{0} \\ \mathbf{0} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{U}} \\ \mathbf{0} \end{bmatrix} \dots \\ - \begin{bmatrix} \mathbf{N}^{T}(\mathbf{U})\mathbf{F}_{\text{Nonlinear}}(\mathbf{U}) \\ \mathbf{B}(\mathbf{U}) \end{bmatrix} = 0.$$
(8.82)

I have marked in the above what is a function of the solution variable. We will need to form each of the above matrices. They can be represented a bit more concisely using

$$\mathbf{m}\ddot{\mathbf{u}} + \mathbf{c}\dot{\mathbf{u}} - \mathbf{f} = \mathbf{0}.\tag{8.83}$$

The above is a bit more standard, but is noticeably missing a stiffness term. We will eventually need to calculate the stable time step, and hence will need to linearize the above, i.e.

$$\mathbf{c}\dot{\mathbf{u}} - \mathbf{f} \approx \left(\mathbf{c}\dot{\mathbf{u}} - \mathbf{f}\right)\Big|_{\mathbf{u}_0} + \frac{\partial}{\partial \mathbf{u}}\left(\mathbf{c}\dot{\mathbf{u}} - \mathbf{f}\right)\Big|_{\mathbf{u}_0}\mathbf{u}.$$
 (8.84)

We will correspondingly need the derivatives of a few of the above matrices. In the subsequent subsections, each matrix's form is given, and derivatives if necessary.

8.3.1 The matrix N

This is the natural orthogonal complement matrix. It is decoupled into

$$\mathbf{N} = \mathbf{N}_{\ell}(\mathbf{U})\mathbf{N}_d. \tag{8.85}$$

The above can be further decomposed according to

$$\mathbf{N}_{\ell} = \mathbf{N}_0 + \mathbf{N}_t(\mathbf{U}). \tag{8.86}$$

The forms for these matrices are

$$\begin{aligned} &\text{Nd} = \text{sparse}(3*(1:\text{Nbods})'-2,(1:\text{Nbods})', \text{ones}(\text{Nbods},1), 3*\text{Nbods}, \text{Nbods}), \\ &\text{NO}(3*b-2:3*b, 3*\text{EdgeID}(e,1)-2:3*\text{EdgeID}(e,1)) = \text{eye}(3) \quad \forall e \in \mathscr{E}_b, \end{aligned} \tag{8.87}$$
$$&\text{Nt}(\text{SubDexX}, \text{SubDexY}) = [-\text{R*sin}_{\text{edgeID}}; \text{ R*cos}_{\text{edgeID}}], \end{aligned}$$

The above is stating that each index value is a sum of terms. We can make this simpler by using repeated index summation. This is achieved through

The new array is thus given by

$$Nt(Nti,Ntj) = [-sin_edgeID(N_val_indx) ; cos_edgeID(N_val_indx)].$$
 (8.89)

8.3.2 The matrix M

This is the mass matrix for the degrees of freedom of the unconstrained bodies. It gives the inertias relative to the joints on the bodies. Non-zero entries are given by

8.3.3 The matrix dN/dt

This tensor has identical indices as those of N, except the values are now the derivatives with respect to time, i.e.

8.3.4 The matrix C

This matrix is part of the evolution equation for the twists; it's essentially a stiffness for velocities. It has the form

C(Cx,Cy) = [-omega.*m*d1.*cos(AbsAng*U+d2);-omega.*m*d1.*sin(AbsAng*U+d2)] (8.92)

8.3.5 The vector F_{Nonlinear}

This force vector resulting from the local multi-stable interaction is given by

 $\mathbf{F}_{\text{Nonlinear}}(\text{NonLinDexX}) = \dots$

```
[-kt*sin(2*pi/psi0*(AbsAng(TopDiscID,:)-AbsAng(BotDiscID,:))*U) ;...
kt*sin(2*pi/psi0*(AbsAng(TopDiscID,:)-AbsAng(BotDiscID,:))*U)];.
(8.93)
```

8.3.6 The vector B

This is the vector that contains terms which are not lienar in acceleration form the constraint equations. It is given by

B= [-P-sparse(Pi,1,[cosp.*Abs_Ut_sqrd(PID) ; sinp.*Abs_Ut_sqrd(PID)])...

```
-sparse(Pi,1,[EdgePaths(PID,:)*(cos_edgeID.*Abs_Ut_sqrd(edgeID1)) ; ...
```

EdgePaths(PID,:)*(sin_edgeID.*Abs_Ut_sqrd(edgeID1))]);...

-sparse(Ci,1,[cosc1.*Abs_Ut_sqrd(chordID1) ;...

sinc1.*Abs_Ut_sqrd(chordID1)])...

-sparse(Ci,1,[EdgePaths(chordID1,:)*(cos_edgeID.*Abs_Ut_sqrd(edgeID1)) ; ...

EdgePaths(chordID1,:)*(sin_edgeID.*Abs_Ut_sqrd(edgeID1))])...

-(-sparse(Ci,1,[cosc2.*Abs_Ut_sqrd(chordID2) ;...

sinc2.*Abs_Ut_sqrd(chordID2)])...

```
-sparse(Ci,1,[EdgePaths(chordID2,:)*(cos_edgeID.*Abs_Ut_sqrd(edgeID1)) ; ...
EdgePaths(chordID2,:)*(sin_edgeID.*Abs_Ut_sqrd(edgeID1))])
```

(8.94)

For ease, let's break down the following entries:

P1 = sparse(Pi,1,[cosp.*Abs_Ut_sqrd(PID) ; sinp.*Abs_Ut_sqrd(PID)]);

```
P2 = sparse(Pi,1,[EdgePaths(PID,:)*(cos_edgeID.*Abs_Ut_sqrd(edgeID1)) ; ...
```

```
EdgePaths(PID,:)*(sin_edgeID.*Abs_Ut_sqrd(edgeID1))]);
```

C1 = sparse(Ci,1,[cosc1.*Abs_Ut_sqrd(chorID1) ;...

sinc1.*Abs_Ut_sqrd(chorID1)]);

C2 = sparse(Ci,1,...

[EdgePaths(chorID1,:)*(cos_edgeID.*Abs_Ut_sqrd(edgeID1)) ; ...

```
EdgePaths(chorID1,:)*(sin_edgeID.*Abs_Ut_sqrd(edgeID1))]);
```

```
C3 = sparse(Ci,1,...
```

```
[cosc2.*Abs_Ut_sqrd(chorID2) ; sinc2.*Abs_Ut_sqrd(chorID2)]);
```

```
C4 = sparse(Ci, 1, ...)
```

```
[EdgePaths(chorID1,:)*(cos_edgeID.*Abs_Ut_sqrd(edgeID2)) ; ...
```

```
EdgePaths(chorID2,:)*(sin_edgeID.*Abs_Ut_sqrd(edgeID1))]);
```

```
(8.95)
```

As we did before for the **N** array, we will readjust the formulation so that we use repeated indices for summation. This applies to P2, C2, and C4. We then have

The new array is thus given by

We can repeat the above for all the other vectors. The full vector is then

$$B = [-P-P1-P2; -C1-C2+C3+C4]; \qquad (8.98)$$

8.3.7 The matrix J

This array contains terms which multiply accelerations in our constraint equation. It is given by

J = [reshape([-sinp cosp]',[],1).*Jp_indices+...

reshape([-sin_edgePath(PID,:) cos_edgePath(PID,:)]',...

Nedges,[])'*AbsAng_edgeID;...

reshape([-sinc1 cosc1]',[],1).*Jc1_indices+...

reshape([-sin_edgePath(chordID1,:) cos_edgePath(chordID1,:)]',.. (8.99)

Nedges,[])'*AbsAng_edgeID...

```
-(reshape([-sinc2 cosc2]',[],1).*Jc2_indices+...
```

reshape([-sin_edgePath(chordID2,:) cos_edgePath(chordID2,:)]',...

Nedges,[])'*AbsAng_edgeID)] ;

Let's decompose this a bit, i.e.

J1 = reshape([-sinp cosp]',[],1).*Jp_indices,

J2 = reshape([-sin_edgePath(PID,:) cos_edgePath(PID,:)]',...

Nedges,[])'*AbsAng_edgeID,

J3 = reshape([-sinc1 cosc1]',[],1).*Jc1_indices,

```
J4 = reshape([-sin_edgePath(chordID1,:) cos_edgePath(chordID1,:)]',...
```

Nedges,[])'*AbsAng_edgeID,

J5 = reshape([-sinc2 cosc2]',[],1).*Jc2_indices,

J6 = reshape([-sin_edgePath(chordID2,:) cos_edgePath(chordID2,:)]',...

```
Nedges,[])'*AbsAng_edgeID)] ;
```

Then we have

$$J = [J1+J2; J3+J4-J5-J6].$$
(8.101)

(8.100)

We can get index assemblies for odd arrays as, e.g.

For the even arrays, a more explicit expression is

This is assembled via

```
J2_val = [-sin_edgeID; cos_edgeID] ;
J2i = [] ;
J2j = [];
J2_val_indx = [];
for i = 1:length(PID)
 for j = 1:length(AbsAng_edgeID(1,:))
 J2i = [J2i; i*ones(sum(AbsAng_edgeID(:,j) & EdgePaths(PID(i),:)'),1)] ;
J2j = [J2j; j*ones(sum(AbsAng_edgeID(:,j) & EdgePaths(PID(i),:)'),1)] ;
 J2_val_indx = [J2_val_indx;...
 find(AbsAng_edgeID(:,j) & EdgePaths(PID(i),:)')] ;
 end
end
J2i = [2*J2i-1;2*J2i];
J2j = [J2j; J2j] ;
J2_val_indx = [J2_val_indx; J2_val_indx+Nedges] ;
J2 = sparse(J2i,J2j,J2_val(J2_val_indx),2*length(PID),Nbods) ;
                                                                     (8.104)
```

8.4 Constraint Drift Elimination

This is where we will try to get rid of discs and connectors drifting apart over long simulations.

8.4.1 Marques, Souto, and Flores

This follows from the paper "On the constraints violation in forward dynamics of multibody systems."¹³⁹ We require the set of constraints in a column vector stated as

$$\Phi(\mathbf{U}) = \mathbf{0}.\tag{8.105}$$

These are the position constraints, defined for the bodies who have prescribed motions on the boundaries (denoted with Φ_p as well as for bodies who are joined via chords (denoted with Φ_c). This can be defined simply with

We will also require the velocity level constraint, i.e.

$$\dot{\Phi} = \frac{\partial}{\partial t} \Phi + \frac{\partial \Phi}{\partial \mathbf{U}} \dot{\mathbf{U}} = \mathbf{0}.$$
(8.107)

We will denote

$$\mathbf{J} \equiv \frac{\partial \Phi}{\partial \mathbf{U}}.\tag{8.108}$$

This is given by
- J = [reshape([-sinp cosp]',[],1).*Jp_indices+... reshape([-sin_edgePath(PID,:) cos_edgePath(PID,:)]',... Nedges,[])'*AbsAng_edgeID;...
- reshape([-sinc1 cosc1]',[],1).*Jc1_indices+...
- reshape([-sin_edgePath(chordID1,:) cos_edgePath(chordID1,:)]',... (8.109)
- Nedges,[])'*AbsAng_edgeID...
- -(reshape([-sinc2 cosc2]',[],1).*Jc2_indices+...
 reshape([-sin_edgePath(chordID2,:) cos_edgePath(chordID2,:)]',...
 - Nedges,[])'*AbsAng_edgeID)] ;

Of course we also have the acceleration level constraint:

$$\ddot{\Phi} = \frac{\partial}{\partial t} \dot{\Phi} + \frac{\partial}{\partial \mathbf{U}} \dot{\Phi} \dot{\mathbf{U}} + \frac{\partial}{\partial \dot{\mathbf{U}}} \dot{\Phi} \dot{\mathbf{U}},$$

$$= \frac{\partial^2}{\partial t^2} \Phi + \frac{\partial \mathbf{J}}{\partial \mathbf{U}} \dot{\mathbf{U}}^2 + \mathbf{J} \ddot{\mathbf{U}} = \mathbf{0}.$$
(8.110)

The above is rearranged to

$$\mathbf{J}\ddot{\mathbf{U}} + \mathbf{b} = \mathbf{0},$$
where $\mathbf{b} \equiv \frac{\partial^2}{\partial t^2} \mathbf{\Phi} + \frac{\partial \mathbf{J}}{\partial \mathbf{U}} \dot{\mathbf{U}}^2.$
(8.111)

The constraint force for such a set of constraints is given by

$$\mathbf{F}_c = \mathbf{J}^T \boldsymbol{\lambda}. \tag{8.112}$$

This leads to the combined set of equations

$$\begin{bmatrix} \mathbf{M} & -\mathbf{J}^T \\ \mathbf{J} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{U}} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_{\text{ext}} \\ -\mathbf{b} \end{bmatrix}.$$
 (8.113)

Integrating the above will ensure that $\ddot{\Phi} = 0$ throughout the simulation, but over long times the accumulation of numerical errors will violate the constraints $\dot{\Phi} = 0$ and $\Phi = 0$. This is termed "constraint drift."

We remedy this at the end of each time step. We will start by applying a correction ΔU to the uncorrected state U^u to obtain a corrected state that satisfies $\Phi = 0$, i.e.

$$\mathbf{U}^c = \mathbf{U}^u + \Delta \mathbf{U}.\tag{8.114}$$

This corrected term should satisfy

$$\mathbf{\Phi}(t, \mathbf{U}^c) = \mathbf{0}.\tag{8.115}$$

Substituting and linearizing around the uncorrected state gives

$$\Phi(t, \mathbf{U}^{u} + \Delta \mathbf{U}) = \Phi(t, \mathbf{U}^{u}) + \mathbf{J}\Delta \mathbf{U} = \mathbf{0}.$$
(8.116)

Solving for the correction gives

$$\Delta \mathbf{U} = -\mathbf{J}^{-1} \mathbf{\Phi}(t, \mathbf{U}^u). \tag{8.117}$$

However, **J** is not a square matrix and does not have an inverse. You could use MATLAB's pinv(J) to generate the Moore-Penrose pseudo-inverse, or use the expression

$$\Delta \mathbf{U} = -\mathbf{J}^T (\mathbf{J} \mathbf{J}^T)^{-1} \mathbf{\Phi}(t, \mathbf{U}^u).$$
(8.118)

After the completion of a time step, we iterate the following until the error is sufficiently small:

$$\mathbf{U}^{i+1} = \mathbf{U}^{i+1} - \mathbf{J}^T (\mathbf{J}\mathbf{J}^T)^{-1} \mathbf{\Phi}(\mathbf{U}^i).$$
(8.119)

Once the position is corrected, we correct the velocity as

$$\dot{\mathbf{U}}^{i+1} = \dot{\mathbf{U}}^{i+1} - \mathbf{J}^T (\mathbf{J}\mathbf{J}^T)^{-1} \dot{\mathbf{\Phi}} (\mathbf{U}^c, \dot{\mathbf{U}}^i).$$
(8.120)

Note that the positions are corrected first and remain the same in each iteration of the velocity correction. The velocity constraint shoud only take one iteration.

Appendix B

Homogenization of Dual-Layer Tetrachiral Lattice

This alters the derivations of the article "Micropolar Continuum modelling of Bi-dimensional Tetrachiral Lattices" by Chen et al.⁷¹ The material is 2D, and hence greek subscripts follow summation convention up to 2 but not 3. In the absence of body forces and moments, we have the momenta conservations

$$\nabla \cdot \boldsymbol{\sigma}^{*t} = \boldsymbol{\rho}^* \mathbf{u}_{,t^*t^*}^*,$$

$$\nabla \cdot \boldsymbol{\sigma}^{*b} = \boldsymbol{\rho}^* \mathbf{u}_{,t^*t^*}^*,$$

$$\nabla \cdot \mathbf{m}^{*t} + \varepsilon_{\alpha\beta} \boldsymbol{\sigma}^{*t}_{\alpha\beta},$$

$$\nabla \cdot \mathbf{m}^{*b} + \varepsilon_{\alpha\beta} \boldsymbol{\sigma}^{*b}_{\alpha\beta} = J^* \boldsymbol{\phi}_{,t^*t^*}^b.$$
(9.1)

In the above the 2D permutation symbol is defined $\varepsilon_{\alpha\beta} \equiv \varepsilon_{3\alpha\beta}$. This is accompanied by the Dirichlet and Neumann boundary conditions

$$\mathbf{u}^{*} = \mathbf{g}^{*} \quad \text{on} \quad \Gamma_{g},$$

$$\phi^{t} = \phi_{g}^{t} \quad \text{on} \quad \Gamma_{g},$$

$$\phi^{b} = \phi_{g}^{b} \quad \text{on} \quad \Gamma_{g},$$

$$\boldsymbol{\sigma}^{*t} \mathbf{n} = \mathbf{h}^{*t} \quad \text{on} \quad \Gamma_{h},$$

$$\boldsymbol{\sigma}^{*b} \mathbf{n} = \mathbf{h}^{*b} \quad \text{on} \quad \Gamma_{h},$$

$$\mathbf{m}^{*t} = \mathbf{m}^{*t}_{h} \quad \text{on} \quad \Gamma_{h},$$

$$\mathbf{m}^{*b} = \mathbf{m}^{*b}_{h} \quad \text{on} \quad \Gamma_{h}.$$
(9.2)

In the above, Γ represents the boundary of the domain and **n** represents the normal to that boundary.

We have the strain-displacement relation

$$\boldsymbol{\varepsilon} = \nabla^* \mathbf{u}^{*T} + \boldsymbol{\phi} \boldsymbol{\varepsilon}_{\alpha\beta} \mathbf{E}_{\beta} \otimes \mathbf{E}_{\alpha}. \tag{9.3}$$

Since we are in a micropolar medium, we also define the curvature

$$\boldsymbol{\kappa}^* = \nabla^* \boldsymbol{\phi}. \tag{9.4}$$

For the strains of interest in this problem, we have explicitly

where u^* and v^* are the displacements in the horizontal and vertical directions, respectively. The above relations can be summarized via matrix operation as

$$\begin{bmatrix} \boldsymbol{\varepsilon}_{11} \\ \boldsymbol{\varepsilon}_{22} \\ \boldsymbol{\varepsilon}_{12} \\ \boldsymbol{\varepsilon}_{21} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x^*} & 0 \\ 0 & \frac{\partial}{\partial y^*} \\ 0 & \frac{\partial}{\partial x^*} \\ \frac{\partial}{\partial y^*} & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{u}^* \\ \boldsymbol{v}^* \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -\boldsymbol{\phi} \\ \boldsymbol{\phi} \end{bmatrix}, \qquad (9.6)$$
$$\begin{bmatrix} \boldsymbol{\kappa}_1^* \\ \boldsymbol{\kappa}_2^* \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x^*} \\ \frac{\partial}{\partial y^*} \end{bmatrix} \boldsymbol{\phi}$$

The constitutive laws are

$$\boldsymbol{\sigma}^{*t} = \mathbf{C}^{*t} \boldsymbol{\varepsilon}^{t} + \mathbf{H}^{*t} \boldsymbol{\kappa}^{*t},$$

$$\boldsymbol{\sigma}^{*b} = \mathbf{C}^{b} \boldsymbol{\varepsilon}^{b} + \mathbf{H}^{*b} \boldsymbol{\kappa}^{*b},$$

$$\mathbf{m}^{*t} = \mathbf{H}^{*t} \boldsymbol{\varepsilon}^{t} + \mathbf{D}^{*t} \boldsymbol{\kappa}^{*t},$$

$$\mathbf{m}^{*b} = \mathbf{H}^{*b} \boldsymbol{\varepsilon}^{b} + \mathbf{D}^{*b} \boldsymbol{\kappa}^{*b}.$$

(9.7)

 C^* , H^* , and D^* are 4th, 3rd, and 2nd order tensors, respectively. Taking various symmetries into account and utilisation of a 2D Voigt notation gives the simple constitutive laws

$$\begin{bmatrix} \sigma_{11}^{*} \\ \sigma_{22}^{*} \\ \sigma_{12}^{*} \\ \sigma_{21}^{*} \end{bmatrix} = \begin{bmatrix} \lambda^{*} + 2\mu^{*} + \alpha^{*} & \lambda^{*} - \alpha^{*} & A^{*} + B^{*} & -A^{*} + B^{*} \\ \lambda^{*} - \alpha^{*} & \lambda^{*} + 2\mu^{*} + \alpha^{*} & A^{*} - B^{*} & -A^{*} - B^{*} \\ A^{*} + B^{*} & A^{*} - B^{*} & \mu^{*} + \kappa^{*} - \alpha^{*} & \mu^{*} - \kappa^{*} - \alpha^{*} \\ -A^{*} + B^{*} & -A^{*} - B^{*} & \mu^{*} - \kappa^{*} - \alpha^{*} & \mu^{*} + \kappa^{*} - \alpha^{*} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \\ \varepsilon_{21} \end{bmatrix}, \quad (9.8)$$

$$\begin{bmatrix} m_{1}^{*} \\ m_{2}^{*} \end{bmatrix} = \begin{bmatrix} \gamma^{*} & 0 \\ 0 & \gamma^{*} \end{bmatrix} \begin{bmatrix} \phi_{,1} \\ \phi_{,2} \end{bmatrix}.$$

The material properties in C^* and D^* must be populated. The discrete model has the essential geometric and kinematic parameters:

 r^* disc radius, $[L^1]$

 $a^{*} \quad \text{lattice parameter, } [L^{1}]$ $\beta = \sin^{-1} (2r^{*}/a^{*}) \quad \text{incline angle of ligament,}$ $L^{*} = a^{*} \cos \beta \quad \text{ligament length, } [L^{1}]$ $\eta = \eta_{1} = t_{a}^{*}/a^{*} = 10^{-2} \quad \text{horizontal ligament thickness,}$ $\eta_{2} = \eta_{1} \quad \text{vertical ligament thickness,}$ $E_{s}^{*} \quad \text{Young's modulus of ligament (per unit depth), } [M^{1}T^{-2}]$ $m^{*} \quad \text{mass of a unit cell, } [M^{1}]$ $J^{*} \quad \text{Micro-inertia density } (\bar{J}^{*}/a^{*2}), [M^{1}].$

This translates to the homogenized parameters

$$\begin{split} \lambda^{*} &= \frac{\eta E_{s}^{*}}{8} (1 - 2\eta^{2} + \cos 2\beta) \cos 2\beta \sec^{3} \beta, \\ \mu^{*} &= \frac{\eta E_{s}^{*}}{8} (1 + 2\eta^{2} + \cos 2\beta) \sec^{3} \beta, \\ \kappa^{*} &= \frac{\eta E_{s}^{*}}{4} (1 + 2\eta^{2} - \cos 2\beta) \sec \beta, \\ A^{*} &= -\frac{\eta E_{s}^{*}}{4} (1 - 2\eta^{2} + \cos 2\beta) \sec \beta \tan \beta, \\ \gamma^{*} &= \frac{\eta a^{*2} E_{s}^{*}}{24} (3 + 8\eta^{2} - 3\cos 2\beta), \\ \alpha^{*} &= \lambda^{*}, \\ B^{*} &= A^{*}. \end{split}$$
(9.10)

Let's take a moment to non-dimensionalize these parameters. We will select as independent dimensions the length, characterized by the lattice spacing a^* , the mass, characterized by the unit cell mass m^* , and time, characterized by the product $\sqrt{m^*/E_s^*}$. This gives the dimensionless parameters:

 $r = r^*/a^*$ disc radius,

$$a = a^*/a^* = 1 \quad \text{lattice parameter,}$$

$$\beta = \sin^{-1}(2r/a) \quad \text{incline angle of ligament,}$$

$$L = L^*/a^* = a \cos \beta \quad \text{ligament length,}$$

$$\eta = \eta_1 = t_a/a = 10^{-2} \quad \text{dimensionless horizontal ligament thickness,} \qquad (9.11)$$

$$\eta_2 = \eta_1 \quad \text{dimensionless vertical ligament thickness,}$$

$$E_s = E_s^*/E_s^* = 1 \quad \text{Young's modulus of ligament,}$$

$$m = m^*/m^* = 1 \quad \text{mass of a unit cell,}$$

$$J = J^*/(m^*) = 0.5r^2 \quad \text{Micro-inertia density.}$$

This gives the dimensionless material properties:

$$\lambda = \lambda^* / E_s^*,$$

$$\mu = \mu^* / E_s^*,$$

$$\kappa = \kappa^* / E_s^*,$$

$$A = A^* / E_s^*,$$

$$\gamma = \gamma^* / (E_s^* a^{*2}),$$

$$\alpha = \alpha^* / E_s^*,$$

$$B = B^* / E_s^*.$$

(9.12)

This gives the property tensors as $\mathbf{C} = \mathbf{C}^* / E_s^*$ and $\mathbf{D} = \mathbf{D}^* / (E_s^* a^{*2})$. Let's transform our partial derivatives now. We have $\partial_{x^*} = 1/a^* \partial_x$ and $\partial_{t^*} = \sqrt{E_s^* / m^*} \partial_t$. Now let's apply these to all of our equations thus far:

Momenta Balance :

$$1/a^*\nabla \cdot E_s^* \boldsymbol{\sigma}^t = m^*/a^{*2} \rho a^* (E_s^*/m^*) \mathbf{u}_{,tt},$$

$$1/a^*\nabla \cdot E_s^* \boldsymbol{\sigma}^b = m^*/a^{*2} \rho a^* (E_s^*/m^*) \mathbf{u}_{,tt},$$

$$1/a^*\nabla \cdot E_s^* a^* \mathbf{m}^t + E_s^* \varepsilon_{\alpha\beta} \boldsymbol{\sigma}_{\alpha\beta}^t = m^* J (E_s^*/m^*) \phi_{,tt}^t,$$

$$1/a^*\nabla \cdot E_s^* a^* \mathbf{m}^b + E_s^* \varepsilon_{\alpha\beta} \boldsymbol{\sigma}_{\alpha\beta}^b = m^* J (E_s^*/m^*) \phi_{,tt}^b,$$

Strain-Displacement :

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \\ \varepsilon_{21} \end{bmatrix} = 1/a^{*} \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ 0 & \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} & 0 \end{bmatrix} a^{*} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -\phi \\ \phi \end{bmatrix}, \qquad (9.13)$$
$$1/a^{*} \begin{bmatrix} \kappa_{1} \\ \kappa_{2} \end{bmatrix} = 1/a^{*} \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} \phi,$$

Constitutive Laws :

$$E_s^* \boldsymbol{\sigma}^t = E_s^* \mathbf{C} \boldsymbol{\varepsilon}^t,$$

$$E_s^* \boldsymbol{\sigma}^b = E_s^* \mathbf{C} \boldsymbol{\varepsilon}^b,$$

$$E_s^* a^* \mathbf{m}^t = E_s^* a^{*2} \mathbf{D}^t 1 / a^* \boldsymbol{\kappa}^t,$$

$$E_s^* a^* \mathbf{m}^b = E_s^* a^{*2} \mathbf{D}^b 1 / a^* \boldsymbol{\kappa}^b.$$

This of course gives the simplified relations

Momenta Balance :

$$\nabla \cdot \boldsymbol{\sigma}^{t} = \rho \mathbf{u}_{,tt},$$

$$\nabla \cdot E_{s}^{*} \boldsymbol{\sigma}^{b} = \rho \mathbf{u}_{,tt},$$

$$\nabla \cdot \mathbf{m}^{t} + \varepsilon_{\alpha\beta} \boldsymbol{\sigma}_{\alpha\beta}^{t} = J \phi_{,tt}^{t},$$

$$\nabla \cdot \mathbf{m}^{b} + \varepsilon_{\alpha\beta} \boldsymbol{\sigma}_{\alpha\beta}^{b} = J \phi_{,tt}^{b},$$

Strain-Displacement :

$$\begin{bmatrix} \boldsymbol{\varepsilon}_{11} \\ \boldsymbol{\varepsilon}_{22} \\ \boldsymbol{\varepsilon}_{12} \\ \boldsymbol{\varepsilon}_{21} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ 0 & \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{v} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -\phi \\ \phi \end{bmatrix}, \qquad (9.14)$$
$$\begin{bmatrix} \kappa_1 \\ \kappa_2 \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} \phi,$$

Constitutive Laws :

$$egin{aligned} & m{\sigma}^t = \mathbf{C}m{arepsilon}^t, \ & m{\sigma}^b = \mathbf{C}m{arepsilon}^b, \ & \mathbf{m}^t = \mathbf{D}^tm{\kappa}^t, \ & \mathbf{m}^b = \mathbf{D}^bm{\kappa}^b. \end{aligned}$$

This implies that the only free parameters which can be varied are *r* and η .

We begin by taking a dot product of the momenta balance of (9.14) with the weighting vector function $\boldsymbol{\omega}$ and integrating over the domain such that we obtain the weak form. Given g_i , ϕ_g , h_i , m_{h_i} , and b_i , find \mathbf{u} and $\phi \in \{\mathbf{u} \& \phi | \mathbf{u} \& \phi \in \mathbf{H}^1; \mathbf{u} = \mathbf{g} \text{ and } \phi = \phi_g \text{ on } \Gamma_g\}$ and $\boldsymbol{\omega} \in \{\boldsymbol{\omega} | \boldsymbol{\omega} \in \mathbf{H}^1; \boldsymbol{\omega} = \mathbf{0} \text{ on } \Gamma_g\}$ such that

$$\int_{\Omega} \boldsymbol{\omega} \cdot (\nabla \cdot \boldsymbol{\sigma}^{t}) d\Omega = \int_{\Omega} \boldsymbol{\rho} \boldsymbol{\omega} \cdot \mathbf{u}_{,tt} d\Omega,$$

$$\int_{\Omega} \boldsymbol{\omega} \cdot (\nabla \cdot \boldsymbol{\sigma}^{b}) d\Omega = \int_{\Omega} \boldsymbol{\rho} \boldsymbol{\omega} \cdot \mathbf{u}_{,tt} d\Omega,$$

$$\int_{\Omega} \boldsymbol{\omega}^{t} \left(\nabla \cdot \mathbf{m}^{t} + \varepsilon_{\alpha\beta} \boldsymbol{\sigma}_{\alpha\beta}^{t} \right) d\Omega = \int_{\Omega} J \boldsymbol{\omega}^{t} \boldsymbol{\phi}_{,tt}^{t} d\Omega,$$

$$\int_{\Omega} \boldsymbol{\omega}^{b} \left(\nabla \cdot \mathbf{m}^{b} + \varepsilon_{\alpha\beta} \boldsymbol{\sigma}_{\alpha\beta}^{b} \right) d\Omega = \int_{\Omega} J \boldsymbol{\omega}^{b} \boldsymbol{\phi}_{,tt}^{b} d\Omega.$$
(9.15)

We switch to indicial notation

$$\int_{\Omega} \omega_{\beta} \sigma^{t}_{\alpha\beta,\alpha} d\Omega = \int_{\Omega} \rho \,\omega_{\alpha} u_{\alpha,tt} d\Omega,$$

$$\int_{\Omega} \omega_{\beta} \sigma^{b}_{\alpha\beta,\alpha} d\Omega = \int_{\Omega} \rho \,\omega_{\alpha} u_{\alpha,tt} d\Omega,$$

$$\int_{\Omega} \omega^{t} (m^{t}_{\alpha,\alpha} + \varepsilon_{\alpha\beta} \sigma^{t}_{\alpha\beta}) d\Omega = \int_{\Omega} J \omega^{t} \phi^{t}_{,tt} d\Omega,$$

$$\int_{\Omega} \omega^{b} (m^{b}_{\alpha,\alpha} + \varepsilon_{\alpha\beta} \sigma^{b}_{\alpha\beta}) d\Omega = \int_{\Omega} J \omega^{b} \phi^{b}_{,tt} d\Omega.$$
(9.16)

and rephrase the first terms of each equation using the product rule as

$$\int_{\Omega} (\omega_{\beta} \sigma_{\alpha\beta}^{t})_{,\alpha} d\Omega - \int_{\Omega} \omega_{\beta,\alpha} \sigma_{\alpha\beta}^{t} d\Omega = \int_{\Omega} \rho \,\omega_{\alpha} u_{\alpha,tt} d\Omega,$$

$$\int_{\Omega} (\omega_{\beta} \sigma_{\alpha\beta}^{b})_{,\alpha} d\Omega - \int_{\Omega} \omega_{\beta,\alpha} \sigma_{\alpha\beta}^{b} d\Omega = \int_{\Omega} \rho \,\omega_{\alpha} u_{\alpha,tt} d\Omega,$$

$$\int_{\Omega} (\omega^{t} m_{\alpha}^{t})_{,\alpha} d\Omega - \int_{\Omega} \omega_{,\alpha}^{t} m_{\alpha}^{t} d\Omega + \int_{\Omega} \omega^{t} (\varepsilon_{\alpha\beta} \sigma_{\alpha\beta}^{t}) d\Omega = \int_{\Omega} J \omega^{t} \phi_{,tt}^{t} d\Omega,$$

$$\int_{\Omega} (\omega^{b} m_{\alpha}^{b})_{,\alpha} d\Omega - \int_{\Omega} \omega_{,\alpha}^{b} m_{\alpha}^{b} d\Omega + \int_{\Omega} \omega^{b} (\varepsilon_{\alpha\beta} \sigma_{\alpha\beta}^{b}) d\Omega = \int_{\Omega} J \omega^{b} \phi_{,tt}^{b} d\Omega.$$
(9.17)

In direct notation, the first terms appears as

$$\int_{\Omega} \nabla \cdot \left(\boldsymbol{\sigma} \boldsymbol{\omega} \right) d\Omega,
\int_{\Omega} \nabla \cdot \left(\boldsymbol{\omega} \mathbf{m} \right) d\Omega,$$
(9.18)

to which we can directly apply the divergence theorem (integration by parts in higher dimensions) to obtain

$$\int_{\Omega} \nabla \cdot (\boldsymbol{\sigma} \boldsymbol{\omega}) d\Omega = \int_{\boldsymbol{\Gamma}_h \times \boldsymbol{\Gamma}_g} \boldsymbol{\omega} \cdot (\boldsymbol{\sigma}^T \mathbf{n}) d\boldsymbol{\Gamma},$$

$$\int_{\Omega} \nabla \cdot (\boldsymbol{\omega} \mathbf{m}) d\Omega = \int_{\boldsymbol{\Gamma}_h \times \boldsymbol{\Gamma}_g} \boldsymbol{\omega} \mathbf{m} \cdot \mathbf{n} d\boldsymbol{\Gamma}$$
(9.19)

Note that $\Gamma_h \times \Gamma_g$ is the entire boundary of the domain. Noting from our weak form that $\omega = 0$ over Γ_g , the only the boundary Γ_h contributes such that

$$\int_{\Gamma_h \times \Gamma_g} \boldsymbol{\omega} \cdot (\boldsymbol{\sigma}^T \mathbf{n}) d\Gamma = \int_{\Gamma_h} \boldsymbol{\omega} \cdot (\boldsymbol{\sigma}^T \mathbf{n}) d\Gamma,$$

$$\int_{\Gamma_h \times \Gamma_g} \boldsymbol{\omega} \mathbf{m} \cdot \mathbf{n} d\Gamma = \int_{\Gamma_h} \boldsymbol{\omega} \mathbf{m} \cdot \mathbf{n} d\Gamma.$$
(9.20)

We now incorporate our natural boundary condition from (9.2) to obtain

$$\int_{\Gamma_{h}} \boldsymbol{\omega} \cdot \left(\boldsymbol{\sigma}^{T} \mathbf{n}\right) d\Gamma = \int_{\Gamma_{h}} \boldsymbol{\omega} \cdot \mathbf{h} d\Gamma,$$

$$\int_{\Gamma_{h}} \boldsymbol{\omega} \mathbf{m} \cdot \mathbf{n} \, d\Gamma = \int_{\Gamma_{h}} \boldsymbol{\omega} \mathbf{m}_{h} \cdot \mathbf{n} \, d\Gamma.$$
(9.21)

For the time being $\mathbf{m}_h = \mathbf{0}$. We switch this to indicial notation and substitute for the first term in (9.17) to find

$$\int_{\Gamma_{h}} \omega_{\alpha} h_{\alpha}^{t} d\Gamma - \int_{\Omega} \omega_{\beta,\alpha} \sigma_{\alpha\beta}^{t} d\Omega = \int_{\Omega} \rho \, \omega_{\alpha} u_{\alpha,tt} d\Omega,$$

$$\int_{\Gamma_{h}} \omega_{\alpha} h_{\alpha}^{b} d\Gamma - \int_{\Omega} \omega_{\beta,\alpha} \sigma_{\alpha\beta}^{b} d\Omega = \int_{\Omega} \rho \, \omega_{\alpha} u_{\alpha,tt} d\Omega,$$

$$- \int_{\Omega} \omega_{,\alpha}^{t} m_{\alpha}^{t} d\Omega + \int_{\Omega} \omega^{t} (\varepsilon_{\alpha\beta} \sigma_{\alpha\beta}^{t}) d\Omega = \int_{\Omega} J \omega^{t} \phi_{,tt}^{t} d\Omega,$$

$$- \int_{\Omega} \omega_{,\alpha}^{b} m_{\alpha}^{b} d\Omega + \int_{\Omega} \omega^{b} (\varepsilon_{\alpha\beta} \sigma_{\alpha\beta}^{b}) d\Omega = \int_{\Omega} J \omega^{b} \phi_{,tt}^{b} d\Omega.$$
(9.22)

Switching back to direct notation, we now utilize our constitutive relation (9.7) to replace σ_{ij} and m_{α} in the above and see that

$$\int_{\Gamma_{h}} \boldsymbol{\omega} \cdot \mathbf{h}^{t} \, \mathrm{d}\Gamma - \int_{\Omega} \nabla \boldsymbol{\omega}^{T} : (\mathbf{C}^{t} \boldsymbol{\varepsilon}^{t}) \, \mathrm{d}\Omega = \int_{\Omega} \boldsymbol{\rho} \boldsymbol{\omega} \cdot \mathbf{u}_{,tt} \, \mathrm{d}\Omega,$$

$$\int_{\Gamma_{h}} \boldsymbol{\omega} \cdot \mathbf{h}^{b} \, \mathrm{d}\Gamma - \int_{\Omega} \nabla \boldsymbol{\omega}^{T} : (\mathbf{C}^{b} \boldsymbol{\varepsilon}^{b}) \, \mathrm{d}\Omega = \int_{\Omega} \boldsymbol{\rho} \boldsymbol{\omega} \cdot \mathbf{u}_{,tt} \, \mathrm{d}\Omega,$$

$$- \int_{\Omega} \nabla \boldsymbol{\omega}^{t} \cdot (\mathbf{D}^{t} \boldsymbol{\kappa}^{t}) \, \mathrm{d}\Omega + \int_{\Omega} \boldsymbol{\omega}^{t} (\boldsymbol{\epsilon} : (\mathbf{C}^{t} \boldsymbol{\varepsilon}^{t})) \, \mathrm{d}\Omega = \int_{\Omega} J \boldsymbol{\omega}^{t} \boldsymbol{\phi}^{t}_{,tt} \, \mathrm{d}\Omega,$$

$$- \int_{\Omega} \nabla \boldsymbol{\omega}^{b} \cdot (\mathbf{D}^{b} \boldsymbol{\kappa}^{b}) \, \mathrm{d}\Omega + \int_{\Omega} \boldsymbol{\omega}^{b} (\boldsymbol{\epsilon} : (\mathbf{C}^{b} \boldsymbol{\varepsilon}^{b})) \, \mathrm{d}\Omega = \int_{\Omega} J \boldsymbol{\omega}^{b} \boldsymbol{\phi}^{b}_{,tt} \, \mathrm{d}\Omega.$$
(9.23)

Next we use our kinematic relation (9.6) to replace ε and κ with the displacement vector **u**. We find

$$\begin{split} &\int_{\Gamma_{h}}\boldsymbol{\omega}\cdot\mathbf{h}^{t}\mathrm{d}\Gamma-\int_{\Omega}\nabla\boldsymbol{\omega}^{T}:\left[\mathbf{C}^{t}\left(\begin{bmatrix}\frac{\partial}{\partial x}&0\\0&\frac{\partial}{\partial y}\\0&\frac{\partial}{\partial x}\\\frac{\partial}{\partial y}&0\end{bmatrix}\mathbf{u}+\begin{bmatrix}0\\0\\-1\\1\end{bmatrix}\boldsymbol{\phi}^{t}\right)\right]\mathrm{d}\Omega=\int_{\Omega}\boldsymbol{\rho}\boldsymbol{\omega}\cdot\mathbf{u}_{,tt}\,\mathrm{d}\Omega,\\ &\int_{\Gamma_{h}}\boldsymbol{\omega}\cdot\mathbf{h}^{b}\mathrm{d}\Gamma-\int_{\Omega}\nabla\boldsymbol{\omega}^{T}:\left[\mathbf{C}^{b}\left(\begin{bmatrix}\frac{\partial}{\partial x}&0\\0&\frac{\partial}{\partial y}\\0&\frac{\partial}{\partial x}\\\frac{\partial}{\partial y}&0\end{bmatrix}\mathbf{u}+\begin{bmatrix}0\\0\\-1\\1\end{bmatrix}\boldsymbol{\phi}^{b}\right)\right]\mathrm{d}\Omega=\int_{\Omega}\boldsymbol{\rho}\boldsymbol{\omega}\cdot\mathbf{u}_{,tt}\,\mathrm{d}\Omega,\\ &-\int_{\Omega}\nabla\boldsymbol{\omega}^{t}\cdot(\mathbf{D}^{t}\begin{bmatrix}\frac{\partial}{\partial x}\\\frac{\partial}{\partial y}\end{bmatrix}\boldsymbol{\phi}^{t})\,\mathrm{d}\Omega+\int_{\Omega}\boldsymbol{\omega}^{t}(\boldsymbol{\epsilon}:\left[\mathbf{C}^{t}\left(\begin{bmatrix}\frac{\partial}{\partial x}&0\\0&\frac{\partial}{\partial y}\\0&\frac{\partial}{\partial z}\\\frac{\partial}{\partial y}&0\end{bmatrix}\mathbf{u}+\begin{bmatrix}0\\0\\-1\\1\end{bmatrix}\boldsymbol{\phi}^{t}\right)\right])\,\mathrm{d}\Omega=\int_{\Omega}\boldsymbol{J}\boldsymbol{\omega}^{t}\boldsymbol{\phi}^{t},_{tt}\,\mathrm{d}\Omega,\\ &-\int_{\Omega}\nabla\boldsymbol{\omega}^{b}\cdot(\mathbf{D}^{b}\begin{bmatrix}\frac{\partial}{\partial x}\\\frac{\partial}{\partial y}\end{bmatrix}\boldsymbol{\phi}^{b})\,\mathrm{d}\Omega+\int_{\Omega}\boldsymbol{\omega}^{b}(\boldsymbol{\epsilon}:\left[\mathbf{C}^{b}\left(\begin{bmatrix}\frac{\partial}{\partial x}&0\\0&\frac{\partial}{\partial y}\\0&\frac{\partial}{\partial y}\\0&\frac{\partial}{\partial y}\end{bmatrix}\mathbf{u}+\begin{bmatrix}0\\0\\-1\\1\end{bmatrix}\boldsymbol{\phi}^{b}\right)\right])\,\mathrm{d}\Omega=\int_{\Omega}\boldsymbol{J}\boldsymbol{\omega}^{b}\boldsymbol{\phi}^{b},_{tt}\,\mathrm{d}\Omega. \end{split}$$

$$(9.24)$$

Now we adjust the $\nabla \omega^T$ term to obtain

$$\nabla \boldsymbol{\omega}^{T} = \begin{bmatrix} \frac{\partial}{\partial x} & 0\\ 0 & \frac{\partial}{\partial y}\\ 0 & \frac{\partial}{\partial x}\\ \frac{\partial}{\partial y} & 0 \end{bmatrix} \boldsymbol{\omega},$$

$$\nabla \boldsymbol{\omega} = \begin{bmatrix} \frac{\partial}{\partial x}\\ \frac{\partial}{\partial y} \end{bmatrix} \boldsymbol{\omega}.$$
(9.25)

We now approximate the solution with discretized functions. Let

$$\boldsymbol{\omega} \to \boldsymbol{\omega}^{h} = \sum_{i}^{NeN} \begin{bmatrix} N_{i} & 0 \\ 0 & N_{i} \end{bmatrix} \begin{bmatrix} w_{1i} \\ w_{2i} \end{bmatrix},$$

$$\boldsymbol{\omega} \to \boldsymbol{\omega}^{h} = \sum_{i}^{NeN} N_{i} \boldsymbol{\omega}_{i},$$

$$\boldsymbol{\phi} \to \boldsymbol{\phi}^{h} = \sum_{i}^{NeN} N_{i} \boldsymbol{\phi}_{i},$$

$$\mathbf{u} \to \mathbf{u}^{h} = \sum_{i}^{NeN} \begin{bmatrix} N_{i} & 0 \\ 0 & N_{i} \end{bmatrix} \begin{bmatrix} u_{i} \\ v_{i} \end{bmatrix}.$$

(9.26)

Note that *NeN* is the total number of nodes in the mesh. What we are doing in the above is approximating the vertical and horizontal displacement fields by nodal values multiplied by the shape function associated with each node. To be more explicit, the index *i* refers to a particular node, N_i , refers to the shape function which is one at that node and zero at all other nodes, and u_i and v_i refer to the horizontal and vertical displacements of that node and ϕ_i the rotation. We make this substitution into the above and find

$$\begin{split} &\int_{\Gamma_{h}}\sum_{i}^{NeN} \begin{bmatrix} N_{i} & 0\\ 0 & N_{i} \end{bmatrix} \begin{bmatrix} w_{1i}\\ w_{2i} \end{bmatrix} \cdot \mathbf{h}' \, \mathrm{d}\Gamma - \dots \\ &\int_{\Omega}\sum_{i}^{NeN} \begin{bmatrix} \frac{\partial}{\partial x} & 0\\ 0 & \frac{\partial}{\partial y}\\ 0 & \frac{\partial}{\partial x}\\ \frac{\partial}{\partial y} & 0 \end{bmatrix} \begin{bmatrix} N_{i} & 0\\ 0 & N_{i} \end{bmatrix} \begin{bmatrix} w_{1i}\\ w_{2i} \end{bmatrix} : \begin{bmatrix} \mathbf{C}' \left(\begin{bmatrix} \frac{\partial}{\partial x} & 0\\ 0 & \frac{\partial}{\partial y}\\ 0 & \frac{\partial}{\partial x}\\ \frac{\partial}{\partial y} & 0 \end{bmatrix} \begin{bmatrix} u_{i}\\ v_{i} \end{bmatrix} + \begin{bmatrix} 0\\ 0\\ -1\\ 1 \end{bmatrix} N_{i} \phi_{i}^{i} \end{pmatrix} \end{bmatrix} \, \mathrm{d}\Omega = \dots \\ &\int_{\Omega} \rho \sum_{i}^{NeN} \begin{bmatrix} N_{i} & 0\\ 0 & N_{i} \end{bmatrix} \begin{bmatrix} w_{1i}\\ w_{2i} \end{bmatrix} \cdot \sum_{i}^{NeN} \begin{bmatrix} N_{i} & 0\\ 0 & N_{i} \end{bmatrix} \begin{bmatrix} w_{1i}\\ w_{2i} \end{bmatrix} \cdot \sum_{i}^{NeN} \begin{bmatrix} N_{i} & 0\\ 0 & N_{i} \end{bmatrix} \begin{bmatrix} u_{i,nt}\\ v_{i,nt} \end{bmatrix} \, \mathrm{d}\Omega = 0, \\ &\int_{\Gamma_{h}}\sum_{i}^{NeN} \begin{bmatrix} N_{i} & 0\\ 0 & N_{i} \end{bmatrix} \begin{bmatrix} w_{1i}\\ w_{2i} \end{bmatrix} \cdot \mathbf{h}^{b} \, \mathrm{d}\Gamma - \dots \\ &\int_{\Omega}\sum_{i}^{NeN} \begin{bmatrix} \frac{\partial}{\partial x} & 0\\ 0 & \frac{\partial}{\partial y}\\ 0 & \frac{\partial}{\partial x}\\ \frac{\partial}{\partial y} & 0 \end{bmatrix} \begin{bmatrix} N_{i} & 0\\ 0 & N_{i} \end{bmatrix} \begin{bmatrix} w_{1i}\\ w_{2i} \end{bmatrix} : \begin{bmatrix} \mathbf{C}^{b} \left(\begin{bmatrix} \frac{\partial}{\partial x} & 0\\ 0 & \frac{\partial}{\partial y}\\ 0 & \frac{\partial}{\partial x}\\ \frac{\partial}{\partial y} & 0 \end{bmatrix} \begin{bmatrix} u_{i}\\ v_{i} \end{bmatrix} + \begin{bmatrix} 0\\ 0\\ -1\\ 1 \end{bmatrix} N_{i} \phi_{i}^{b} \end{bmatrix} \right] \, \mathrm{d}\Omega = \dots \\ &\int_{\Omega} \rho \sum_{i}^{NeN} \begin{bmatrix} N_{i} & 0\\ 0 & \frac{\partial}{\partial y}\\ 0 & \frac{\partial}{\partial x}\\ \frac{\partial}{\partial y} & 0 \end{bmatrix} \begin{bmatrix} w_{1i}\\ w_{2i} \end{bmatrix} \cdot \sum_{i}^{NeN} \begin{bmatrix} N_{i} & 0\\ 0 & \frac{\partial}{\partial y}\\ 0 & \frac{\partial}{\partial x}\\ \frac{\partial}{\partial y} & 0 \end{bmatrix} \begin{bmatrix} u_{i}\\ v_{i} \end{bmatrix} + \begin{bmatrix} 0\\ 0\\ -1\\ 1 \end{bmatrix} N_{i} \phi_{i}^{b} \end{bmatrix} \, \mathrm{d}\Omega = \dots \\ &\int_{\Omega} \rho \sum_{i}^{NeN} \begin{bmatrix} N_{i} & 0\\ 0 & N_{i} \end{bmatrix} \begin{bmatrix} w_{1i}\\ w_{2i} \end{bmatrix} \cdot \sum_{i}^{NeN} \begin{bmatrix} N_{i} & 0\\ 0 & N_{i} \end{bmatrix} \begin{bmatrix} u_{i,tt}\\ v_{i,tt} \end{bmatrix} \, \mathrm{d}\Omega = 0, \end{aligned}$$

and from the angular equations

$$-\int_{\Omega} \sum_{i}^{NeN} \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} N_{i} \omega_{i}^{t} \cdot (\mathbf{D}' \sum_{i}^{NeN} \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} N_{i} \phi_{i}^{t} \right) d\Omega + \dots$$

$$\int_{\Omega} \sum_{i}^{NeN} N_{i} \omega_{i}^{t} \left(\epsilon : \begin{bmatrix} \mathbf{C}' \left(\begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ 0 & \frac{\partial}{\partial x} \end{bmatrix} \sum_{i}^{NeN} \begin{bmatrix} N_{i} & 0 \\ 0 & N_{i} \end{bmatrix} \begin{bmatrix} u_{i} \\ v_{i} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} N_{i} \phi_{i}^{t} \right) \right] ((\phi_{i}^{t} - \phi_{i}^{b}) N_{i}) \right) d\Omega = \dots$$

$$\int_{\Omega} J \sum_{i}^{NeN} N_{i} \omega_{i}^{t} \sum_{i}^{NeN} N_{i} \phi_{i}^{t} d\Omega,$$

$$-\int_{\Omega} \sum_{i}^{NeN} \begin{bmatrix} \frac{\partial}{\partial x} \\ 0 \\ \frac{\partial}{\partial y} \end{bmatrix} N_{i} \omega_{i}^{b} \cdot (\mathbf{D}^{b} \sum_{i}^{NeN} \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} N_{i} \phi_{i}^{b}) d\Omega + \dots$$

$$\int_{\Omega} \sum_{i}^{NeN} N_{i} \omega_{i}^{b} \left(\epsilon : \begin{bmatrix} \mathbf{C}^{b} \left(\begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 \\ \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} N_{i} \phi_{i}^{b} \right) d\Omega + \dots$$

$$\int_{\Omega} \int_{\Omega} \sum_{i}^{NeN} N_{i} \omega_{i}^{b} \left(\epsilon : \begin{bmatrix} \mathbf{C}^{b} \left(\begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 \\ \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} N_{i} \phi_{i}^{b} \right) N_{i} \phi_{i}^{b} d\Omega + \dots$$

$$\int_{\Omega} \int_{X} \sum_{i}^{NeN} N_{i} \omega_{i}^{b} \sum_{i}^{NeN} N_{i} \phi_{i}^{b} d\Omega.$$
(9.28)

We carry out the derivative matrices and reach

$$\begin{split} &\int_{\Gamma_{h}} \sum_{i}^{NeN} \begin{bmatrix} N_{i} & 0\\ 0 & N_{i} \end{bmatrix} \begin{bmatrix} w_{1i}\\ w_{2i} \end{bmatrix} \cdot \mathbf{h}^{t} d\mathbf{\Gamma} - \dots \\ &\int_{\Omega} \sum_{i}^{NeN} \begin{bmatrix} N_{isx} & 0\\ 0 & N_{isy}\\ 0 & N_{isx}\\ N_{iy} & 0 \end{bmatrix} \begin{bmatrix} w_{1i}\\ w_{2i} \end{bmatrix} : \begin{bmatrix} \mathbf{C}^{t} \begin{pmatrix} NeN\\ \sum_{i}^{NeN} \begin{bmatrix} N_{isx} & 0\\ 0 & N_{isy}\\ N_{iy} & 0 \end{bmatrix} \begin{bmatrix} u_{i}\\ v_{i} \end{bmatrix} + \begin{bmatrix} 0\\ 0\\ -N_{i}\\ N_{i} \end{bmatrix} \phi_{i}^{t} \end{bmatrix}] d\Omega = \dots \\ &\int_{\Omega} \rho \sum_{i}^{NeN} \begin{bmatrix} N_{i} & 0\\ 0 & N_{i} \end{bmatrix} \begin{bmatrix} w_{1i}\\ w_{2i} \end{bmatrix} \cdot \sum_{i}^{NeN} \begin{bmatrix} N_{i} & 0\\ 0 & N_{i} \end{bmatrix} \begin{bmatrix} w_{1i}\\ w_{2i} \end{bmatrix} \cdot \sum_{i}^{NeN} \begin{bmatrix} N_{i} & 0\\ 0 & N_{i} \end{bmatrix} \begin{bmatrix} w_{1i}\\ w_{2i} \end{bmatrix} \cdot \mathbf{h}^{b} d\mathbf{\Gamma} - \dots \\ &\int_{\Omega} \sum_{i}^{NeN} \begin{bmatrix} N_{isx} & 0\\ 0 & N_{isx}\\ N_{iyy} & 0 \end{bmatrix} \begin{bmatrix} w_{1i}\\ w_{2i} \end{bmatrix} \cdot \mathbf{h}^{b} d\mathbf{\Gamma} - \dots \\ &\int_{\Omega} \sum_{i}^{NeN} \begin{bmatrix} N_{isx} & 0\\ 0 & N_{isx}\\ N_{iyy} & 0 \end{bmatrix} \begin{bmatrix} w_{1i}\\ w_{2i} \end{bmatrix} : \begin{bmatrix} \mathbf{C}^{b} \begin{pmatrix} NeN\\ \sum_{i}^{NeN} \begin{bmatrix} N_{isx} & 0\\ 0 & N_{isy}\\ N_{iyy} & 0 \end{bmatrix} \begin{bmatrix} u_{i}\\ w_{2i} \end{bmatrix} + \begin{bmatrix} 0\\ 0\\ -N_{i}\\ N_{i} \end{bmatrix} \phi_{i}^{b} \end{bmatrix}] d\Omega = \dots \\ &\int_{\Omega} \rho \sum_{i}^{NeN} \begin{bmatrix} N_{i} & 0\\ 0 & N_{isx}\\ N_{iyy} & 0 \end{bmatrix} \begin{bmatrix} w_{1i}\\ w_{2i} \end{bmatrix} \cdot \sum_{i}^{NeN} \begin{bmatrix} N_{i} & 0\\ 0 & N_{isx}\\ N_{iyy} & 0 \end{bmatrix} \begin{bmatrix} u_{i}\\ w_{1i}\\ w_{2i} \end{bmatrix} + \sum_{i}^{NeN} \begin{bmatrix} N_{i} & 0\\ 0 & N_{ixy}\\ N_{iyy} & 0 \end{bmatrix}] d\Omega = \dots \\ &\int_{\Omega} \rho \sum_{i}^{NeN} \begin{bmatrix} N_{i} & 0\\ 0 & N_{ixx}\\ N_{iyy} & 0 \end{bmatrix} \begin{bmatrix} w_{1i}\\ w_{2i}\\ w_{2i} \end{bmatrix} \cdot \sum_{i}^{NeN} \begin{bmatrix} N_{i} & 0\\ 0 & N_{i} \end{bmatrix} \begin{bmatrix} u_{i,m}\\ u_{i,m}\\ v_{i,m} \end{bmatrix} d\Omega, \end{split}$$

and the angular equations

$$-\int_{\Omega} \sum_{i}^{NeN} \begin{bmatrix} N_{i,x} \\ N_{i,y} \end{bmatrix} \omega_{i}^{t} \cdot \left(\mathbf{D}^{t} \sum_{i}^{NeN} \begin{bmatrix} N_{i,x} \\ N_{i,y} \end{bmatrix} \phi_{i}^{t}\right) d\Omega + \dots$$

$$\int_{\Omega} \sum_{i}^{NeN} \omega_{i}^{t} \left(\begin{bmatrix} 0 \\ 0 \\ N_{i} \\ -N_{i} \end{bmatrix} : \begin{bmatrix} \mathbf{C}^{t} \left(\sum_{i}^{NeN} \begin{bmatrix} N_{i,x} & 0 \\ 0 & N_{i,y} \\ 0 & N_{i,x} \\ N_{i,y} & 0 \end{bmatrix} \begin{bmatrix} u_{i} \\ v_{i} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -N_{i} \\ N_{i} \end{bmatrix} \phi_{i}^{t} \right) \end{bmatrix} d\Omega = \dots$$

$$\int_{\Omega} J \sum_{i}^{NeN} N_{i} \omega_{i}^{t} \sum_{i}^{NeN} N_{i} \phi_{i}^{t} d\Omega,$$

$$-\int_{\Omega} \sum_{i}^{NeN} \begin{bmatrix} N_{i,x} \\ N_{i,y} \end{bmatrix} \omega_{i}^{b} \cdot \left(\mathbf{D}^{b} \sum_{i}^{NeN} \begin{bmatrix} N_{i,x} \\ N_{i,y} \end{bmatrix} \phi_{i}^{b} \right) d\Omega + \dots$$

$$\int_{\Omega} J \sum_{i}^{NeN} \omega_{i}^{b} \left(\begin{bmatrix} 0 \\ 0 \\ N_{i} \\ -N_{i} \end{bmatrix} : \begin{bmatrix} \mathbf{C}^{b} \left(\sum_{i}^{NeN} \begin{bmatrix} N_{i,x} \\ N_{i,y} \end{bmatrix} \phi_{i}^{b} \right) d\Omega + \dots$$

$$\int_{\Omega} J \sum_{i}^{NeN} \omega_{i}^{b} \left(\begin{bmatrix} 0 \\ 0 \\ N_{i} \\ -N_{i} \end{bmatrix} : \begin{bmatrix} \mathbf{C}^{b} \left(\sum_{i}^{NeN} \begin{bmatrix} N_{i,x} & 0 \\ 0 & N_{i,y} \\ 0 & N_{i,x} \\ N_{i,y} & 0 \end{bmatrix} \end{bmatrix} \begin{bmatrix} u_{i} \\ v_{i} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -N_{i} \\ N_{i} \end{bmatrix} \phi_{i}^{b} \right) \end{bmatrix} \right) d\Omega = \dots$$

$$\int_{\Omega} J \sum_{i}^{NeN} N_{i} \omega_{i}^{b} \sum_{i}^{NeN} N_{i} \phi_{i}^{b} d\Omega.$$
(9.30)

At this point we need to combine the degrees of freedom into one vector \mathbf{d} (and also \mathbf{w}). The above becomes

$$\begin{split} &\int_{\Gamma_{k}}\sum_{i}^{NeN} \begin{bmatrix} N_{i} & 0 & 0 & 0 \\ 0 & N_{i} & 0 & 0 \end{bmatrix} \begin{bmatrix} w_{1i} \\ w_{2i} \\ \omega_{i}^{b} \\ \omega_{i}^{b} \end{bmatrix} \cdot \mathbf{h}' d\mathbf{\Gamma} - \dots \\ &\int_{\Omega}\sum_{i}^{NeN} \begin{bmatrix} N_{i,x} & 0 & 0 & 0 \\ 0 & N_{i,y} & 0 & 0 \\ 0 & N_{i,y} & 0 & 0 \end{bmatrix} \begin{bmatrix} w_{1i} \\ w_{2i} \\ \omega_{i}^{b} \\ \omega_{i}^{b} \end{bmatrix} : \begin{bmatrix} \mathbf{C}' \left(\sum_{i}^{NeN} \begin{bmatrix} N_{i,x} & 0 & 0 & 0 \\ 0 & N_{i,y} & 0 & 0 \\ 0 & N_{i,y} & 0 & N_{i} & 0 \end{bmatrix} \begin{bmatrix} u_{i} \\ v_{i} \\ \phi_{i}^{b} \\ \phi_{i}^{b} \end{bmatrix} \end{bmatrix} d\Omega = \dots \\ &\int_{\Omega} \rho \sum_{i}^{NeN} \begin{bmatrix} N_{i} & 0 & 0 & 0 \\ 0 & N_{i} & 0 & 0 \end{bmatrix} \begin{bmatrix} w_{1i} \\ w_{2i} \\ \omega_{i}^{b} \\ \omega_{i}^{b} \end{bmatrix} \cdot \sum_{i}^{NeN} \begin{bmatrix} N_{i} & 0 & 0 & 0 \\ 0 & N_{i} & 0 & 0 \end{bmatrix} \begin{bmatrix} w_{1i} \\ w_{2i} \\ \omega_{i}^{b} \\ \vdots \end{bmatrix} \cdot \sum_{i}^{NeN} \begin{bmatrix} N_{i} & 0 & 0 & 0 \\ 0 & N_{i} & 0 & 0 \end{bmatrix} \begin{bmatrix} w_{1i} \\ w_{2i} \\ \omega_{i}^{b} \\ \vdots \end{bmatrix} \cdot \mathbf{h}^{b} d\mathbf{\Gamma} - \dots \\ &\int_{\Omega} \sum_{i}^{NeN} \begin{bmatrix} N_{i,x} & 0 & 0 & 0 \\ 0 & N_{i,y} & 0 & 0 \\ 0 & N_{i,y} & 0 & 0 \end{bmatrix} \begin{bmatrix} w_{1i} \\ w_{2i} \\ \omega_{i}^{b} \\ \vdots \end{bmatrix} \cdot \begin{bmatrix} \mathbf{C}^{b} \left(\sum_{i}^{NeN} \begin{bmatrix} N_{i,x} & 0 & 0 & 0 \\ 0 & N_{i,y} & 0 & 0 \\ 0 & N_{i,y} & 0 & 0 \\ 0 & N_{i,y} & 0 & 0 \end{bmatrix} \begin{bmatrix} w_{1i} \\ w_{2i} \\ \omega_{i}^{b} \\ \vdots \end{bmatrix} : \begin{bmatrix} \mathbf{C}^{b} \left(\sum_{i}^{NeN} \begin{bmatrix} N_{i,x} & 0 & 0 & 0 \\ 0 & N_{i,y} & 0 & 0 \\ \vdots \end{bmatrix} \begin{bmatrix} w_{1i} \\ w_{2i} \\ \omega_{i}^{b} \end{bmatrix} : \begin{bmatrix} \mathbf{C}^{b} \left(\sum_{i}^{NeN} \begin{bmatrix} N_{i,x} & 0 & 0 & 0 \\ 0 & N_{i,y} & 0 & 0 \\ \vdots \end{bmatrix} \end{bmatrix} \end{bmatrix} d\Omega = \dots \\ \int_{\Omega} \rho \sum_{i}^{NeN} \begin{bmatrix} N_{i} & 0 & 0 & 0 \\ 0 & N_{i} & 0 \\ 0 & N_{i} & 0 \\ 0 \end{bmatrix} \begin{bmatrix} w_{1i} \\ w_{2i} \\ w_{2i} \\ w_{2i} \\ w_{2i} \\ w_{2i} \end{bmatrix}} \cdot \sum_{i}^{NeN} \begin{bmatrix} N_{i} & 0 & 0 & 0 \\ 0 & N_{i} & 0 \\ 0 \end{bmatrix} \begin{bmatrix} w_{1i} \\ w_{2i} \\ w_{2i} \\ w_{2i} \\ w_{2i} \\ w_{2i} \end{bmatrix} \cdot \sum_{i}^{NeN} \begin{bmatrix} N_{i} & 0 & 0 & 0 \\ N_{i} & 0 \\ 0 \end{bmatrix} \begin{bmatrix} w_{1i} \\ w_{2i} \\ w_{2i} \\ w_{2i} \\ w_{2i} \end{bmatrix} \end{bmatrix}$$

and the angular equations

$$-\int_{\Omega} \sum_{i}^{N \in \mathbb{N}} \begin{bmatrix} 0 & 0 & N_{i:x} & 0 \\ 0 & 0 & N_{i:y} & 0 \end{bmatrix} \begin{bmatrix} w_{1i} \\ w_{2i} \\ \omega_{i}^{i} \\ \omega_{i}^{i} \end{bmatrix} \cdot \left(\mathbf{D}' \sum_{i}^{N \in \mathbb{N}} \begin{bmatrix} 0 & 0 & N_{i:x} & 0 \\ 0 & 0 & N_{i:y} & 0 \end{bmatrix} \begin{bmatrix} u_{i} \\ v_{i} \\ \phi_{i}^{i} \\ \phi_{i}^{i} \end{bmatrix} \right) d\Omega + \dots \\ \int_{\Omega} \sum_{i}^{N \in \mathbb{N}} \left(\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & N_{i} & 0 \\ 0 & 0 & N_{i} & 0 \\ 0 & 0 & -N_{i} & 0 \end{bmatrix} \begin{bmatrix} w_{1i} \\ w_{2i} \\ \omega_{i}^{i} \\ \omega_{i}^{i} \\ \omega_{i}^{i} \end{bmatrix} \right) : \begin{bmatrix} \mathbf{C}' \left(\sum_{i=1}^{N \in \mathbb{N}} \begin{bmatrix} N_{i:x} & 0 & 0 & 0 \\ 0 & N_{i:y} & 0 & 0 \\ 0 & N_{i:x} & -N_{i} & 0 \\ 0 & N_{i:x} & 0 & N_{i} & 0 \end{bmatrix} \begin{bmatrix} u_{i} \\ \psi_{i} \\ \phi_{i}^{i} \\ \phi_{i}^{i} \end{bmatrix} \right) \right] d\Omega = \dots \\ \int_{\Omega} \int_{n}^{N \in \mathbb{N}} \begin{bmatrix} 0 & 0 & N_{i} & 0 \\ 0 & 0 & N_{i} & 0 \end{bmatrix} \begin{bmatrix} w_{1i} \\ w_{2i} \\ \omega_{i}^{i} \\ \omega_{i}^{i} \\ \omega_{i}^{i} \end{bmatrix} \cdot \left(\mathbf{D}^{p} \sum_{i}^{N \in \mathbb{N}} \begin{bmatrix} 0 & 0 & N_{i:x} \\ 0 & 0 & N_{i:y} \end{bmatrix} \begin{bmatrix} u_{i} \\ v_{i} \\ \phi_{i}^{i} \\ \phi_{i}^{i} \end{bmatrix} \right) d\Omega + \dots \\ \int_{\Omega} \int_{n}^{N \in \mathbb{N}} \left[\begin{bmatrix} 0 & 0 & 0 & N_{i:x} \\ 0 & 0 & 0 & N_{i} \\ \phi_{i}^{i} \\ 0 & 0 & 0 & N_{i} \end{bmatrix} \begin{bmatrix} w_{1i} \\ w_{2i} \\ \omega_{i}^{i} \\ \omega_{i}^{i} \end{bmatrix} \right) : \begin{bmatrix} \mathbf{C}' \left(\sum_{i=1}^{N \in \mathbb{N}} \begin{bmatrix} N_{i:x} & 0 & 0 & 0 \\ 0 & N_{i:y} \\ \phi_{i}^{i} \\ \phi_{i}^{j} \end{bmatrix} \right) d\Omega + \dots \\ \int_{\Omega} \int_{n}^{N \in \mathbb{N}} \left[\begin{bmatrix} 0 & 0 & 0 & N_{i:x} \\ 0 & 0 & 0 & N_{i} \\ \omega_{i}^{j} \\ \phi_{i}^{j} \end{bmatrix} \right) : \begin{bmatrix} \mathbf{C}' \left(\sum_{i=1}^{N \in \mathbb{N}} \begin{bmatrix} N_{i:x} & 0 & 0 & 0 \\ 0 & N_{i:y} \\ \phi_{i}^{i} \\ \phi_{i}^{j} \end{bmatrix} \right) d\Omega + \dots \\ \int_{\Omega} \int_{n}^{N \in \mathbb{N}} \left[\begin{bmatrix} 0 & 0 & 0 & N_{i} \\ \omega_{i} \\ \omega_{i}^{j} \\ \phi_{i}^{j} \end{bmatrix} \right) : \begin{bmatrix} \mathbf{C}' \left(\sum_{i=1}^{N \in \mathbb{N}} \begin{bmatrix} N_{i:x} & 0 & 0 & 0 \\ 0 & N_{i:y} \\ 0 & 0 & N_{i} \\ \phi_{i}^{j} \\ \phi_{i}^{j} \end{bmatrix} \right] d\Omega = \dots \\ \int_{\Omega} \int_{n}^{N \in \mathbb{N}} \left[0 & 0 & 0 & N_{i} \end{bmatrix} \begin{bmatrix} w_{1i} \\ w_{2i} \\ \omega_{i}^{j} \\ \phi_{i}^{j} \end{bmatrix} \right] \sum_{i=1}^{N \in \mathbb{N}} \left[0 & 0 & 0 & N_{i} \end{bmatrix} \begin{bmatrix} u_{i} \\ w_{1i} \\ w_{2i} \\ \phi_{i}^{j} \\ \phi_{i}^{j} \end{bmatrix} \right] d\Omega = \dots$$

$$(9.32)$$

We create the notation

We create a larger matrix by concatenating to

$$\begin{array}{l}
 B \\
 (4 \times 4NeN) = \begin{bmatrix}
 B_{1} & B_{2} & \dots & B_{NeN} \\
 4 \times 4 & 4 \times 4 & 4 \times 4
 \end{bmatrix}, \\
 P \\
 (4 \times 4NeN) = \begin{bmatrix}
 P_{1} & P_{2} & \dots & P_{NeN} \\
 4 \times 4 & 4 \times 4 & 4 \times 4
 \end{bmatrix}, \\
 Q \\
 (4 \times 4NeN) = \begin{bmatrix}
 Q_{1} & Q_{2} & \dots & Q_{NeN} \\
 4 \times 4 & 4 \times 4 & 4 \times 4
 \end{bmatrix}, \\
 L \\
 (2 \times 4NeN) = \begin{bmatrix}
 L_{1} & L_{2} & \dots & L_{NeN} \\
 2 \times 4 & 2 \times 4 & 2 \times 4
 \end{bmatrix}, \\
 M \\
 (2 \times 4NeN) = \begin{bmatrix}
 d_{1} & 4 \times 1 \\
 d_{2} & 4 \times 1 \\
 \vdots \\
 d_{NeN} & 4 \times 1
 \end{bmatrix} \quad \text{and} \quad \begin{array}{l}
 w \\
 (4NeN \times 1) =
 \begin{bmatrix}
 w_{1} & 4 \times 1 \\
 w_{2} & 4 \times 1 \\
 \vdots \\
 w_{NeN} & 4 \times 1
 \end{bmatrix}$$
 (9.34)

Matrix multiplication reveals that

$$\sum_{i}^{NeN} \mathbf{B}_{i} \mathbf{d}_{i} = \mathbf{B} \mathbf{d},$$

$$\sum_{i}^{NeN} \mathbf{P}_{i} \mathbf{w}_{i} = \mathbf{P} \mathbf{w},$$

$$\sum_{i}^{NeN} \mathbf{Q}_{i} \mathbf{w}_{i} = \mathbf{Q} \mathbf{w},$$

$$\sum_{i}^{NeN} \mathbf{L}_{i} \mathbf{d}_{i} = \mathbf{L} \mathbf{d}.$$
(9.35)

Substitution into (9.32) yields

$$\int_{\Gamma_{h}} \sum_{i}^{NeN} \begin{bmatrix} N_{i} & 0 & 0 & 0 \\ 0 & N_{i} & 0 & 0 \end{bmatrix} \mathbf{w}_{i} \cdot \mathbf{h}^{t} d\Gamma - \int_{\Omega} (\mathbf{Pw}) : (\mathbf{C}^{t} \mathbf{B}^{t} \mathbf{d}) d\Omega = \dots$$

$$\int_{\Omega} \rho \sum_{i}^{NeN} \begin{bmatrix} N_{i} & 0 & 0 & 0 \\ 0 & N_{i} & 0 & 0 \end{bmatrix} \mathbf{w}_{i} \cdot \sum_{i}^{NeN} \begin{bmatrix} N_{i} & 0 & 0 & 0 \\ 0 & N_{i} & 0 & 0 \end{bmatrix} \mathbf{w}_{i} \cdot \mathbf{h}^{b} d\Gamma - \int_{\Omega} (\mathbf{Pw}) : (\mathbf{C}^{b} \mathbf{B}^{b} \mathbf{d}) d\Omega = \dots$$

$$\int_{\Omega} \rho \sum_{i}^{NeN} \begin{bmatrix} N_{i} & 0 & 0 & 0 \\ 0 & N_{i} & 0 & 0 \end{bmatrix} \mathbf{w}_{i} \cdot \mathbf{h}^{b} d\Gamma - \int_{\Omega} (\mathbf{Pw}) : (\mathbf{C}^{b} \mathbf{B}^{b} \mathbf{d}) d\Omega = \dots$$

$$\int_{\Omega} \rho \sum_{i}^{NeN} \begin{bmatrix} N_{i} & 0 & 0 & 0 \\ 0 & N_{i} & 0 & 0 \end{bmatrix} \mathbf{w}_{i} \cdot \sum_{i}^{NeN} \begin{bmatrix} N_{i} & 0 & 0 & 0 \\ 0 & N_{i} & 0 & 0 \end{bmatrix} \mathbf{w}_{i} \cdot \sum_{i}^{NeN} \begin{bmatrix} N_{i} & 0 & 0 & 0 \\ 0 & N_{i} & 0 & 0 \end{bmatrix} \mathbf{d}_{i} d\Omega,$$

$$-\int_{\Omega} (\mathbf{L}^{t} \mathbf{w}) \cdot (\mathbf{D}^{t} \mathbf{L}^{t} \mathbf{d}) d\Omega + \int_{\Omega} (\mathbf{Q}^{t} \mathbf{w}) : (\mathbf{C}^{t} \mathbf{B}^{t} \mathbf{d}) d\Omega = \dots$$

$$\int_{\Omega} J \sum_{i}^{NeN} \begin{bmatrix} 0 & 0 & N_{i} & 0 \end{bmatrix} \mathbf{w}_{i} \sum_{i}^{NeN} \begin{bmatrix} 0 & 0 & N_{i} & 0 \end{bmatrix} \mathbf{d}_{i} d\Omega,$$

$$-\int_{\Omega} (\mathbf{L}^{b} \mathbf{w}) \cdot (\mathbf{D}^{b} \mathbf{L}^{b} \mathbf{d}) d\Omega + \int_{\Omega} (\mathbf{Q}^{b} \mathbf{w}) : (\mathbf{C}^{b} \mathbf{B}^{b} \mathbf{d}) d\Omega = \dots$$

$$\int_{\Omega} J \sum_{i}^{NeN} \begin{bmatrix} 0 & 0 & N_{i} & 0 \end{bmatrix} \mathbf{w}_{i} \sum_{i}^{NeN} \begin{bmatrix} 0 & 0 & N_{i} & 0 \end{bmatrix} \mathbf{d}_{i} d\Omega.$$

Let us now introduce

$$\mathbf{N}_{i} = \begin{bmatrix} N_{i} & 0 & 0 & 0 \\ 0 & N_{i} & 0 & 0 \end{bmatrix}, \quad \mathbf{R}_{i}^{t} = \begin{bmatrix} 0 & 0 & N_{i} & 0 \end{bmatrix}, \quad \mathbf{R}_{i}^{b} = \begin{bmatrix} 0 & 0 & 0 & N_{i} \end{bmatrix}$$
$$\mathbf{N} = \begin{bmatrix} \mathbf{N}_{1} & \mathbf{N}_{2} & \dots & \mathbf{N}_{NeN} \end{bmatrix}, \quad \mathbf{R} = \begin{bmatrix} \mathbf{R}_{1} & \mathbf{R}_{2} & \dots & \mathbf{R}_{NeN} \end{bmatrix}$$
$$\rightarrow \sum_{i}^{NeN} \mathbf{N}_{i} \mathbf{w}_{i} = \mathbf{N} \mathbf{w} \quad \text{and} \quad \sum_{i}^{NeN} \mathbf{N}_{i} \mathbf{\ddot{d}}_{i} = \mathbf{N} \mathbf{\ddot{d}}$$
$$\rightarrow \sum_{i}^{NeN} \mathbf{R}_{i} \mathbf{w}_{i} = \mathbf{R} \mathbf{w} \quad \text{and} \quad \sum_{i}^{NeN} \mathbf{R}_{i} \mathbf{\ddot{d}}_{i} = \mathbf{R} \mathbf{\ddot{d}}$$
$$(9.37)$$

We substitute to find

$$\int_{\Gamma_{h}} (\mathbf{N}\mathbf{w}) \cdot \mathbf{h}^{t} d\Gamma - \int_{\Omega} (\mathbf{P}\mathbf{w}) : (\mathbf{C}^{t} \mathbf{B}^{t} \mathbf{d}) d\Omega = \int_{\Omega} \rho(\mathbf{N}\mathbf{w}) \cdot (\mathbf{N}\mathbf{\ddot{d}}) d\Omega,$$

$$\int_{\Gamma_{h}} (\mathbf{N}\mathbf{w}) \cdot \mathbf{h}^{b} d\Gamma - \int_{\Omega} (\mathbf{P}\mathbf{w}) : (\mathbf{C}^{b} \mathbf{B}^{b} \mathbf{d}) d\Omega = \int_{\Omega} \rho(\mathbf{N}\mathbf{w}) \cdot (\mathbf{N}\mathbf{\ddot{d}}) d\Omega,$$

$$- \int_{\Omega} (\mathbf{L}^{t}\mathbf{w}) \cdot (\mathbf{D}^{t} \mathbf{L}^{t} \mathbf{d}) d\Omega + \int_{\Omega} (\mathbf{Q}^{t}\mathbf{w}) : (\mathbf{C}^{t} \mathbf{B}^{t} \mathbf{d}) d\Omega = \dots$$

$$\int_{\Omega} J \mathbf{R}^{t} \mathbf{w} \mathbf{R}^{t} \mathbf{\ddot{d}} d\Omega,$$

$$- \int_{\Omega} (\mathbf{L}^{b}\mathbf{w}) \cdot (\mathbf{D}^{b} \mathbf{L}^{b} \mathbf{d}) d\Omega + \int_{\Omega} (\mathbf{Q}^{b}\mathbf{w}) : (\mathbf{C}^{b} \mathbf{B}^{b} \mathbf{d}) d\Omega = \dots$$

$$\int_{\Omega} J \mathbf{R}^{b} \mathbf{w} \mathbf{R}^{b} \mathbf{\ddot{d}} d\Omega.$$

(9.38)

Exploiting transposes we rephrase to

$$\begin{split} &\int_{\Gamma_{h}} \mathbf{w} \cdot \mathbf{N}^{T} \mathbf{h}^{t} \, \mathrm{d}\Gamma - \int_{\Omega} \mathbf{w} \cdot \left((\mathbf{P}^{t})^{T} \mathbf{C}^{t} \mathbf{B}^{t} \mathbf{d} \right) \, \mathrm{d}\Omega = \int_{\Omega} \mathbf{w} \cdot (\rho \mathbf{N}^{T} \mathbf{N} \ddot{\mathbf{d}}) \, \mathrm{d}\Omega, \\ &\int_{\Gamma_{h}} \mathbf{w} \cdot \mathbf{N}^{T} \mathbf{h}^{b} \, \mathrm{d}\Gamma - \int_{\Omega} \mathbf{w} \cdot \left((\mathbf{P}^{b})^{T} \mathbf{C}^{t} \mathbf{B}^{b} \mathbf{d} \right) \, \mathrm{d}\Omega = \int_{\Omega} \mathbf{w} \cdot (\rho \mathbf{N}^{T} \mathbf{N} \ddot{\mathbf{d}}) \, \mathrm{d}\Omega, \\ &- \int_{\Omega} \mathbf{w} \cdot \left((\mathbf{L}^{t})^{T} \mathbf{D}^{t} \mathbf{L}^{t} \mathbf{d} \right) \, \mathrm{d}\Omega + \int_{\Omega} \mathbf{w} \cdot \left((\mathbf{Q}^{t})^{T} \mathbf{C}^{t} \mathbf{B}^{t} \mathbf{d} \right) \, \mathrm{d}\Omega - \int_{\Omega} \mathbf{w} \cdot \left[(\mathbf{R}^{t})^{T} \Psi^{\prime} ((\phi_{i}^{t} - \phi_{i}^{b}) N_{i} \right) \right] \mathrm{d}\Omega = \dots \\ &\int_{\Omega} \mathbf{w} \cdot (J(\mathbf{R}^{t})^{T} \mathbf{R}^{t} \ddot{\mathbf{d}}_{i}) \, \mathrm{d}\Omega, \\ &- \int_{\Omega} \mathbf{w} \cdot \left((\mathbf{L}^{b})^{T} \mathbf{D}^{b} \mathbf{L}^{b} \mathbf{d} \right) \, \mathrm{d}\Omega + \int_{\Omega} \mathbf{w} \cdot \left((\mathbf{Q}^{b})^{T} \mathbf{C}^{b} \mathbf{B}^{b} \mathbf{d} \right) \, \mathrm{d}\Omega - \int_{\Omega} \mathbf{w} \cdot \left[(\mathbf{R}^{b})^{T} \Psi^{\prime} ((\phi_{i}^{b} - \phi_{i}^{t}) N_{i} \right) \right] \mathrm{d}\Omega = \dots \\ &\int_{\Omega} \mathbf{w} \cdot (J(\mathbf{R}^{b})^{T} \mathbf{R}^{b} \ddot{\mathbf{d}}_{i}) \, \mathrm{d}\Omega. \end{split}$$

$$\tag{9.39}$$

The arbitrariness of the weight function allows the equivalent matrix equation

$$\int_{\Gamma_{h}} \mathbf{N}^{T} \mathbf{h}^{t} \, d\Gamma - \int_{\Omega} (\mathbf{P}^{t})^{T} \mathbf{C}^{t} \mathbf{B}^{t} \mathbf{d} \, d\Omega = \int_{\Omega} \rho \mathbf{N}^{T} \mathbf{N} \ddot{\mathbf{d}} \, d\Omega,$$

$$\int_{\Gamma_{h}} \mathbf{N}^{T} \mathbf{h}^{b} \, d\Gamma - \int_{\Omega} (\mathbf{P}^{b})^{T} \mathbf{C}^{b} \mathbf{B}^{b} \mathbf{d} \, d\Omega = \int_{\Omega} \rho \mathbf{N}^{T} \mathbf{N} \ddot{\mathbf{d}} \, d\Omega,$$

$$- \int_{\Omega} (\mathbf{L}^{t})^{T} \mathbf{D} \mathbf{L}^{t} \mathbf{d} \, d\Omega + \int_{\Omega} (\mathbf{Q}^{t})^{T} \mathbf{C}^{t} \mathbf{B}^{t} \mathbf{d} \, d\Omega = \int_{\Omega} J(\mathbf{R}^{t})^{T} \mathbf{R}^{t} \ddot{\mathbf{d}} \, d\Omega,$$

$$- \int_{\Omega} (\mathbf{L}^{b})^{T} \mathbf{D} \mathbf{L}^{b} \mathbf{d} \, d\Omega + \int_{\Omega} (\mathbf{Q}^{b})^{T} \mathbf{C}^{b} \mathbf{B}^{b} \mathbf{d} \, d\Omega = \int_{\Omega} J(\mathbf{R}^{b})^{T} \mathbf{R}^{b} \ddot{\mathbf{d}} \, d\Omega.$$

(9.40)

Let's combine the equations into the single equation

$$\int_{\Omega} \left(2\rho \mathbf{N}^{T} \mathbf{N} + J(\mathbf{R}^{t})^{T} \mathbf{R}^{t} + J(\mathbf{R}^{b})^{T} \mathbf{R}^{b} \right) \ddot{\mathbf{d}} \, \mathrm{d}\Omega + \dots$$

$$\int_{\Omega} \left[\left((\mathbf{P}^{t})^{T} - (\mathbf{Q}^{t})^{T}) \mathbf{C}^{t} \mathbf{B}^{t} + \left((\mathbf{P}^{b})^{T} - (\mathbf{Q}^{b})^{T}) \mathbf{C}^{b} \mathbf{B}^{b} + (\mathbf{L}^{t})^{T} \mathbf{D}^{t} \mathbf{L}^{t} + (\mathbf{L}^{b})^{T} \mathbf{D}^{b} \mathbf{L}^{b} \right] \mathbf{d} \, \mathrm{d}\Omega = \dots$$

$$\int_{\Gamma_{h}} \mathbf{N}^{T} \mathbf{h} \, \mathrm{d}\Gamma.$$
(9.41)

We now collocate the domains to be over each element e. Breaking up the integrals, we achieve the summation

$$\begin{split} \sum_{e}^{N_{e}} \int_{\Omega^{e}} \left(2\rho \mathbf{N}_{e}^{T} \mathbf{N}_{e} + J_{e} (\mathbf{R}_{e}^{t})^{T} \mathbf{R}_{e}^{t} + J_{e} (\mathbf{R}_{e}^{b})^{T} \mathbf{R}_{e}^{b} \right) \ddot{\mathbf{d}}_{e} \, \mathrm{d}\Omega + \dots \\ \int_{\Omega^{e}} \left[\left((\mathbf{P}_{e}^{t})^{T} - (\mathbf{Q}_{e}^{t})^{T} \right) \mathbf{C}_{e}^{t} \mathbf{B}_{e}^{t} + \left((\mathbf{P}_{e}^{b})^{T} - (\mathbf{Q}_{e}^{b})^{T} \right) \mathbf{C}_{e}^{b} \mathbf{B}_{e}^{b} + (\mathbf{L}_{e}^{t})^{T} \mathbf{D}_{e}^{t} \mathbf{L}_{e}^{t} + (\mathbf{L}_{e}^{b})^{T} \mathbf{D}_{e}^{b} \mathbf{L}_{e}^{b} \right] \mathbf{d}_{e} \, \mathrm{d}\Omega = \dots \\ \int_{\Gamma_{h}^{e}} \mathbf{N}_{e}^{T} \mathbf{h}_{e} \, \mathrm{d}\Gamma. \end{split}$$

$$(9.42)$$

In the element form, we have the following

$$\mathbf{d}_{e} = \begin{bmatrix} u_{1} & v_{1} & \phi_{1}^{t} & \phi_{1}^{b} & u_{2} & v_{2} & \phi_{2}^{t} & \phi_{2}^{b} & u_{3} & v_{3} & \phi_{3}^{t} & \phi_{3}^{b} \end{bmatrix}^{T},$$
(9.43)

$$\mathbf{B}_{e}^{t} = \begin{bmatrix} N_{1,x} & 0 & 0 & 0 & N_{2,x} & 0 & 0 & 0 & N_{3,x} & 0 & 0 & 0 \\ 0 & N_{1,y} & 0 & 0 & 0 & N_{2,y} & 0 & 0 & 0 & N_{3,y} & 0 & 0 \\ 0 & N_{1,x} & -N_{1} & 0 & 0 & N_{2,x} & -N_{2} & 0 & 0 & N_{3,x} & -N_{3} & 0 \\ N_{1,y} & 0 & N_{1} & 0 & N_{2,y} & 0 & N_{2} & 0 & N_{3,y} & 0 & N_{3} & 0 \end{bmatrix},$$
(9.44)

$$\mathbf{B}_{e}^{b} = \begin{bmatrix} N_{1,x} & 0 & 0 & 0 & N_{2,x} & 0 & 0 & 0 & N_{3,x} & 0 & 0 & 0 \\ 0 & N_{1,y} & 0 & 0 & 0 & N_{2,y} & 0 & 0 & 0 & N_{3,y} & 0 & 0 \\ 0 & N_{1,x} & 0 & -N_{1} & 0 & N_{2,x} & 0 & -N_{2} & 0 & N_{3,x} & 0 & -N_{3} \\ N_{1,y} & 0 & 0 & N_{1} & N_{2,y} & 0 & 0 & N_{2} & N_{3,y} & 0 & 0 & N_{3} \end{bmatrix},$$
(9.45)

$$\mathbf{P}_{e} = \begin{bmatrix} N_{1,x} & 0 & 0 & 0 & N_{2,x} & 0 & 0 & 0 & N_{3,x} & 0 & 0 & 0 \\ 0 & N_{1,y} & 0 & 0 & 0 & N_{2,y} & 0 & 0 & 0 & N_{3,y} & 0 & 0 \\ 0 & N_{1,x} & 0 & 0 & 0 & N_{2,x} & 0 & 0 & 0 & N_{3,x} & 0 & 0 \\ N_{1,y} & 0 & 0 & 0 & N_{2,y} & 0 & 0 & 0 & N_{3,y} & 0 & 0 & 0 \end{bmatrix},$$
(9.46)

$$\mathbf{L}_{e}^{b} = \begin{bmatrix} 0 & 0 & 0 & N_{1,x} & 0 & 0 & 0 & N_{2,x} & 0 & 0 & 0 & N_{3,x} \\ 0 & 0 & 0 & N_{1,y} & 0 & 0 & 0 & N_{2,y} & 0 & 0 & 0 & N_{3,y} \end{bmatrix},$$
(9.50)

$$\mathbf{N}_{e} = \begin{bmatrix} N_{1} & 0 & 0 & N_{2} & 0 & 0 & N_{3} & 0 & 0 & 0 \\ 0 & N_{1} & 0 & 0 & N_{2} & 0 & 0 & N_{3} & 0 & 0 \end{bmatrix},$$
(9.51)

$$\mathbf{R}_{e}^{t} = \begin{bmatrix} 0 & 0 & N_{1} & 0 & 0 & 0 & N_{2} & 0 & 0 & 0 & N_{3} & 0 \end{bmatrix},$$
(9.52)

$$\mathbf{R}_{e}^{b} = \begin{bmatrix} 0 & 0 & 0 & N_{1} & 0 & 0 & N_{2} & 0 & 0 & N_{3} \end{bmatrix},$$
(9.53)

$$\mathbf{h}_e = \begin{bmatrix} 0\\ f \end{bmatrix}. \tag{9.54}$$

Note that this is dependent on the second and third nodes being on the vertical boundary.

Now we shift to local coordinates. Our shape functions are

$$N_{1} = \frac{1}{4}(1 - \xi)(1 - \eta),$$

$$N_{2} = \frac{1}{4}(1 + \xi)(1 - \eta),$$

$$N_{3} = \frac{1}{2}(1 + \eta).$$
(9.55)

These map to our global space via

$$x = x_1 N_1 + x_2 N_2 + x_2 N_3,$$

$$y = y_1 N_1 + y_2 N_2 + y_3 N_3.$$
(9.56)

To transform our derivatives in the \mathbf{B}_e matrix, we note that

$$\frac{\partial}{\partial x} = \frac{\partial \xi}{\partial x} \frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x} \frac{\partial}{\partial \eta},$$

$$\frac{\partial}{\partial y} = \frac{\partial \xi}{\partial y} \frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial y} \frac{\partial}{\partial \eta}$$
(9.57)

We see that

$$\begin{bmatrix} N_{i,x} \\ N_{i,y} \end{bmatrix} = \begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial x} \\ \frac{\partial \xi}{\partial y} & \frac{\partial \eta}{\partial y} \end{bmatrix} \begin{bmatrix} N_{i,\xi} \\ N_{i,\eta} \end{bmatrix}.$$
(9.58)

We do not explicitly know the matrix of derivatives, but we do know

$$\begin{bmatrix} x_{,\xi} & y_{,\xi} \\ x_{,\eta} & y_{,\eta} \end{bmatrix}$$
(9.59)

by virtue of (9.56). We have the relation

$$\begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial x} \\ \frac{\partial \xi}{\partial y} & \frac{\partial \eta}{\partial y} \end{bmatrix} = \begin{bmatrix} x_{,\xi} & y_{,\xi} \\ x_{,\eta} & y_{,\eta} \end{bmatrix}^{-1} = \frac{1}{j} \begin{bmatrix} y_{,\eta} & -y_{,\xi} \\ -x_{,\eta} & x_{,\xi} \end{bmatrix}$$
(9.60)

where

$$j = \det \begin{bmatrix} x_{,\xi} & y_{,\xi} \\ x_{,\eta} & y_{,\eta} \end{bmatrix} = x_{,\xi}y_{,\eta} - x_{,\eta}y_{,\xi}, \qquad (9.61)$$

and

$$\begin{aligned} x_{,\xi} &= \sum_{i}^{3} x_{i} N_{i,\xi}, \\ x_{,\eta} &= \sum_{i}^{3} x_{i} N_{i,\eta}, \\ y_{,\xi} &= \sum_{i}^{3} y_{i} N_{i,\xi}, \\ y_{,\eta} &= \sum_{i}^{3} y_{i} N_{i,\eta}. \end{aligned}$$
(9.62)

This gives the relation

$$\begin{bmatrix} N_{i,x} \\ N_{i,y} \end{bmatrix} = \frac{1}{j} \begin{bmatrix} y, \eta & -y_{\xi} \\ -x, \eta & x_{\xi} \end{bmatrix} \begin{bmatrix} N_{i,\xi} \\ N_{i,\eta} \end{bmatrix}.$$
 (9.63)

This means our \mathbf{B}_i , \mathbf{P}_i , and \mathbf{L}_i matrices become

$$\begin{split} \mathbf{B}_{i}^{t} &= \frac{1}{j} \begin{bmatrix} y_{,\eta}N_{i,\xi} - y_{,\xi}N_{i,\eta} & 0 & 0 & 0 \\ 0 & -x_{,\eta}N_{i,\xi} + x_{,\xi}N_{i,\eta} & 0 & 0 \\ 0 & y_{,\eta}N_{i,\xi} - y_{,\xi}N_{i,\eta} & -jN_{i} & 0 \\ -x_{,\eta}N_{i,\xi} + x_{,\xi}N_{i,\eta} & 0 & 0 & 0 \\ 0 & -x_{,\eta}N_{i,\xi} + x_{,\xi}N_{i,\eta} & 0 & 0 & 0 \\ 0 & y_{,\eta}N_{i,\xi} - y_{,\xi}N_{i,\eta} & 0 & 0 & jN_{i} \end{bmatrix}, \\ \mathbf{B}_{i}^{b} &= \frac{1}{j} \begin{bmatrix} y_{,\eta}N_{i,\xi} - y_{,\xi}N_{i,\eta} & 0 & 0 & 0 \\ 0 & y_{,\eta}N_{i,\xi} - y_{,\xi}N_{i,\eta} & 0 & 0 & jN_{i} \\ -x_{,\eta}N_{i,\xi} + x_{,\xi}N_{i,\eta} & 0 & 0 & 0 \\ 0 & y_{,\eta}N_{i,\xi} - y_{,\xi}N_{i,\eta} & 0 & 0 \\ 0 & y_{,\eta}N_{i,\xi} - y_{,\xi}N_{i,\eta} & 0 & 0 \\ 0 & y_{,\eta}N_{i,\xi} - y_{,\xi}N_{i,\eta} & 0 & 0 \\ 0 & y_{,\eta}N_{i,\xi} - y_{,\xi}N_{i,\eta} & 0 & 0 \\ 0 & y_{,\eta}N_{i,\xi} - y_{,\xi}N_{i,\eta} & 0 & 0 \\ 0 & 0 & y_{,\eta}N_{i,\xi} - y_{,\xi}N_{i,\eta} & 0 \\ 0 & 0 & -x_{,\eta}N_{i,\xi} + x_{,\xi}N_{i,\eta} & 0 \\ 0 & 0 & -x_{,\eta}N_{i,\xi} + x_{,\xi}N_{i,\eta} & 0 \end{bmatrix}, \end{split}$$

$$\mathbf{L}_{i}^{b} &= \frac{1}{j} \begin{bmatrix} 0 & 0 & y_{,\eta}N_{i,\xi} - y_{,\xi}N_{i,\eta} & 0 \\ 0 & 0 & -x_{,\eta}N_{i,\xi} + x_{,\xi}N_{i,\eta} & 0 \end{bmatrix},$$

Finally we must transfer our differentials $d\Omega$ and $d\Gamma.$ This is given by

$$d\Omega = dxdy = jd\xi d\eta,$$

$$d\Gamma = \ell d\xi = \sqrt{x_{,\xi}^2 + y_{,\xi}^2} d\xi.$$
(9.65)

Now the trick is the Gaussian quadrature. The \mathbf{B}_e matrices have only linear functions, while j also has linear functions. The multiplication of matrices and division by j should lead to linear functions, but I'm not exactly sure about that so we'll do two point integration. The Gauss points and weights are

$$\xi_{1} = -\frac{1}{\sqrt{3}}, \qquad \xi_{2} = \frac{1}{\sqrt{3}}, \eta_{1} = -\frac{1}{\sqrt{3}}, \qquad \eta_{2} = \frac{1}{\sqrt{3}}, w_{1} = 1 \qquad w_{2} = 2.$$
(9.66)

We can do this in order, ending up in four terms for the mass and stiffness and two terms for the force:

$$\mathbf{m} = w_{1}w_{1}j(\xi_{1},\eta_{1})\left(\rho_{e}\mathbf{N}_{e}^{T}(\xi_{1},\eta_{1})\mathbf{N}_{e}(\xi_{1},\eta_{1}) + J_{e}\mathbf{R}_{e}^{T}(\xi_{1},\eta_{1})\mathbf{R}_{e}(\xi_{1},\eta_{1})\right) + \dots \\ w_{1}w_{2}j(\xi_{1},\eta_{2})\left(\rho_{e}\mathbf{N}_{e}^{T}(\xi_{1},\eta_{2})\mathbf{N}_{e}(\xi_{1},\eta_{2}) + J_{e}\mathbf{R}_{e}^{T}(\xi_{1},\eta_{2})\mathbf{R}_{e}(\xi_{1},\eta_{2})\right) + \dots \\ w_{2}w_{1}j(\xi_{2},\eta_{1})\left(\rho_{e}\mathbf{N}_{e}^{T}(\xi_{2},\eta_{1})\mathbf{N}_{e}(\xi_{2},\eta_{1}) + J_{e}\mathbf{R}_{e}^{T}(\xi_{2},\eta_{1})\mathbf{R}_{e}(\xi_{2},\eta_{1})\right) + \dots \\ w_{2}w_{2}j(\xi_{2},\eta_{2})\left(\rho_{e}\mathbf{N}_{e}^{T}(\xi_{2},\eta_{2})\mathbf{N}_{e}(\xi_{2},\eta_{2}) + J_{e}\mathbf{R}_{e}^{T}(\xi_{2},\eta_{2})\mathbf{R}_{e}(\xi_{2},\eta_{2})\right), \\ \mathbf{k} = \frac{w_{1}w_{1}}{j(\xi_{1},\eta_{1})}\left[\left(\mathbf{P}_{e}^{T}(\xi_{1},\eta_{1}) - \mathbf{Q}_{e}^{T}(\xi_{1},\eta_{1})\right)\mathbf{C}_{e}\mathbf{B}_{e}(\xi_{1},\eta_{1}) + \mathbf{L}_{e}^{T}(\xi_{1},\eta_{1})\mathbf{D}_{e}\mathbf{L}_{e}(\xi_{1},\eta_{1})\right] + \dots \\ \frac{w_{1}w_{2}}{j(\xi_{1},\eta_{2})}\left[\left(\mathbf{P}_{e}^{T}(\xi_{1},\eta_{2}) - \mathbf{Q}_{e}^{T}(\xi_{1},\eta_{2})\right)\mathbf{C}_{e}\mathbf{B}_{e}(\xi_{1},\eta_{2}) + \mathbf{L}_{e}^{T}(\xi_{1},\eta_{2})\mathbf{D}_{e}\mathbf{L}_{e}(\xi_{1},\eta_{2})\right] + \dots \\ \frac{w_{2}w_{1}}{j(\xi_{2},\eta_{1})}\left[\left(\mathbf{P}_{e}^{T}(\xi_{2},\eta_{1}) - \mathbf{Q}_{e}^{T}(\xi_{2},\eta_{1})\right)\mathbf{C}_{e}\mathbf{B}_{e}(\xi_{2},\eta_{2}) + \mathbf{L}_{e}^{T}(\xi_{2},\eta_{1})\mathbf{D}_{e}\mathbf{L}_{e}(\xi_{2},\eta_{1})\right] + \dots \\ \frac{w_{2}w_{2}}{j(\xi_{2},\eta_{2})}\left[\left(\mathbf{P}_{e}^{T}(\xi_{2},\eta_{2}) - \mathbf{Q}_{e}^{T}(\xi_{2},\eta_{2})\right)\mathbf{C}_{e}\mathbf{B}_{e}(\xi_{2},\eta_{2}) + \mathbf{L}_{e}^{T}(\xi_{2},\eta_{2})\mathbf{D}_{e}\mathbf{L}_{e}(\xi_{2},\eta_{1})\right] + \dots \\ \frac{w_{2}w_{2}}{j(\xi_{2},\eta_{2})}\left[\left(\mathbf{P}_{e}^{T}(\xi_{2},\eta_{2}) - \mathbf{Q}_{e}^{T}(\xi_{2},\eta_{2})\right)\mathbf{C}_{e}\mathbf{B}_{e}(\xi_{2},\eta_{2}) + \mathbf{L}_{e}^{T}(\xi_{2},\eta_{2})\mathbf{D}_{e}\mathbf{L}_{e}(\xi_{2},\eta_{2})\right], \\ \mathbf{f} = w_{1}\ell(\xi_{1},\eta = 1)\mathbf{N}_{e}^{T}(\xi_{1},\eta = 1)\mathbf{h}_{e} + w_{2}\ell(\xi_{2},\eta = 1)\mathbf{N}_{e}^{T}(\xi_{2},\eta = 1)\mathbf{h}_{e}.$$

This once constructed yields a form amenable to the FE method.

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