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The Derivative Operator on Weighted Bergman Spaces and Quantized Number Theory

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The Derivative Operator on Weighted Bergman Spaces and Quantized Number Theory

A Dissertation submitted in partial satisfaction
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by

Timothy Logan Cobler

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ABSTRACT OF THE DISSERTATION

The Derivative Operator on Weighted Bergman Spaces and Quantized Number Theory

by

Timothy Logan Cobler

Doctor of Philosophy, Graduate Program in Mathematics
University of California, Riverside, June 2016
Dr. Michel L. Lapidus, Chairperson

Emil Artin defined a zeta function for algebraic curves over finite fields and made a conjecture about them analogous to the famous Riemann hypothesis. This and other conjectures about these zeta functions would come to be called the Weil conjectures. Much work was done in the search for a proof of these conjectures, including the development in algebraic geometry of a Weil cohomology theory for these varieties, relying on the Frobenius operator on the finite field. The zeta function is then expressed as a determinant allowing the properties of the function to relate to the properties of the operator. The search for a suitable cohomology theory and associated operator to prove the Riemann hypothesis has continued to this day. In this dissertation we study the properties of the derivative operator $D = \frac{d}{dz}$ on a particular family of weighted Bergman spaces. This operator is meant to be the replacement for the Frobenius in the general case and is first used to give a method of quantizing elliptic curves and modular forms; then to construct an operator associated to any given meromorphic function. With this construction, we show that for a wide class of functions, the function can be recovered using a regularized Berezinian determinant involv-

ing the operator constructed from the meromorphic function. This is shown in some special cases: rational functions, zeta functions of algebraic curves over finite fields, geometric zeta functions of lattice self-similar strings, the gamma function, the Riemann zeta function and culminating in a quantized version of the Hadamard factorization theorem that applies to any entire function of finite order. This shows that all of the information about the given meromorphic function is encoded into the special operator we constructed.

Contents

Chapter 1

Motivation

1.1 The Distribution of the Prime Numbers

The prime numbers have fascinated mathematicians for millenia. It has been known that there were infinitely many primes since the time of Euclid. However, the problem of precisely determining the distribution of the prime numbers among the positive integers is a much more difficult problem that has vast applications, yet remains unsolved. Finding formulas that give the n^{th} prime number for every n has continues with very little success. The primes appear to be almost randomly located among all whole numbers. For example there are arbitrarily long sequences that contain no prime numbers: $n! + 2, n! + 3, ..., n! + n$ are all composite for any $n > 2$. On the other hand, larger and larger examples of primes as close as possible, the so-called twin primes, continue to be found. Gauss and Legendre decided to approach this problem from another direction. Instead of trying to find a formula for the n^{th} prime, they looked for a formula giving the number of primes that are less than or equal to $x \in \mathbb{R}$. Despite the irregularity of the prime numbers, this counting function,

$\pi(x)$, behaves very regularly. In fact, early estimates placed $\pi(x)$ as approximately $\frac{x}{\log x}$ or more precisely, as $\mathrm{li}(x) = \int_0^x \frac{1}{\log t} dt$, the logarithmic integral, for large x. In fact, these functions give precisely the asymptotic behavior of $\pi(x)$ as stated in the Prime Number theorem, then only a conjecture.

Theorem 1.1 *(Prime Number Theorem)* $\pi(x) \sim \mathrm{li}(x)$ *as* $x \to \infty$*. (Note that* $f(x) \sim g(x)$ *as* $x \to \infty$ *means* $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$*.)*

Despite much work, including an estimate by Chebyshev, the prime number theorem proved elusive for quite some time until Riemann entered the scene.

1.2 The Riemann Zeta Function

The so-called p-series, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ had been studied for a long time and is known to converge for $p > 1$ and diverge for $p \leq 1$. Euler found a product representation and even found some special values of this series. For example, when $p = 2$ we have $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$. However, it wasn't until Bernhard Riemann that this series was studied in situations where p is to be a non-real number. Following Riemann's notation, we write $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$. This series converges absolutely in the right half plane $\{s \in \mathbb{C} : \Re(s) > 1\}$. Riemann showed that there is a meromorphic continuation of this function to the entire complex plane whose only pole at $s = 1$ is simple. It is standard to use a slight abuse of notation and denote this meromorphic continuation as $\zeta(s)$ also. In establishing this meromorphic continuation, Riemann also discovered a functional equation satisfied by $\zeta(s)$. We present his result in a different form than he gave.

Theorem 1.2 *Let* $\xi(s) = \frac{1}{2}\pi^{-\frac{s}{2}}s(s-1)\Gamma(\frac{s}{2})\zeta(s)$ *be the completed zeta function. Then* $\xi(s)$ *is entire and satisfies the functional equation* $\xi(s) = \xi(1-s)$ *for all* $s \in \mathbb{C}$.

To relate $\zeta(s)$ back to the prime number theorem, we first make the following definition.

Definition 1 Mangoldt's function, Λ, *is defined by:*

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^m \text{ for some prime } p \text{ and some } m \geq 1 \\ 0 & \text{otherwise} \end{cases},$$

and Chebyshev's function,ψ , *is given by*

$$\psi(x) = \sum_{n \leq x} \Lambda(n).$$

We will also have use for $\psi_1(x) = \int_1^x \psi(t)dt$.

It can be shown the Prime Number Theorem is equivalent to the statement $\psi_1(x) \sim \frac{1}{2}x^2$ as $x \to \infty$.

Theorem 1.3 *If* $c \geq 1$ *and* $x \geq 1$ *we have that*

$$\frac{\psi_1(x)}{x^2} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{x^{s-1}}{s(s+1)} \left(-\frac{\zeta'(s)}{\zeta(s)} \right) ds$$

The integral expression in the previous theorem gives an example of an inverse Mellin transform and it relates $\zeta(s)$ to one of the prime counting functions $\psi_1(x)$, which results can then be translated to apply to $\pi(x)$. The reader may notice that this integral being defined hinges on the fact that $\zeta(s)$ has no zeros on the half plane given by$\Re(s) > 1$. This fact can be shown using the convergence of the Euler product for $\zeta(s)$. Then in order to prove the Prime Number Theorem, the contour of this integral was shifted from

$[c - i\infty, c + i\infty]$ to $[1 - i\infty, 1 + i\infty]$. This method required the fact that $\zeta(s)$ also has no zeros on the line $\Re(s) = 1$, which was independently shown by Jacques Hadamard and Charles-Jean de la Vallée Poussin, thus completing the Prime Number Theorem.

If we could move the contour even further left of the line $\Re(s) = 1$, it would be possible to prove a more accurate version of the Prime Number Theorem. Thus knowledge of the zeros of $\zeta(s)$ is critical to knowing more about the distribution of the prime numbers. Above we mentioned that $\zeta(s)$ has no zeros on $\Re(s) \geq 1$ and the functional equation implies that there are no zeros in the left half plane $\{s \in \mathbb{C} : \Re(s) \leq 0\}$ other than the so-called trivial zeros at $s = -2, -4, -6, \ldots$. These zeros of $\zeta(s)$ cancel with poles of the $\Gamma(\frac{s}{2})$ term in the functional equation. It follows that all other zeros of $\zeta(s)$ are in the critical strip $\{s \in \mathbb{C} : 0 < \Re(s) < 1\}$. In this paper, Riemann made his famous conjecture.

Conjecture 2 *(Riemann Hypothesis) The only nontrivial zeros of $\zeta(s)$ occur when s satisfies $\Re(s) = \frac{1}{2}$.*

If the Riemann Hypothesis is true, that would imply that

$$\pi(x) = \mathrm{li}(x) + O(x^{\frac{1}{2}+\epsilon}) \text{ for any } \epsilon > 0,$$

which in a sense says that the primes are as randomly distributed as they can be.

1.3 Fractal Strings and Zeta Functions

Since Riemann introduced the complex meromorphic function $\zeta(s)$, many other 'zeta' functions have been studied in number theory, most of which share properties with $\zeta(s)$. In this section we will look at one such zeta function that serves as a motivating example for the rest of this dissertation. We will follow the treatment in [LvF13] and begin with a definition.

Definition 3 *An* ordinary fractal string $\mathcal{L}$ *is a bounded open subset* Ω *of* $\mathbb{R}$. *Any such set can be written as a countable union of disjoint open intervals. We will write these intervals in nonincreasing length order and call these lengths* $l_1 \geq l_2 \geq \cdots$. *When we refer to* $\mathcal{L}$ *we will mean this sequence of lengths.*

A classical example of a fractal string comes from the so-called middle third Cantor set. First, remove the middle third $(\frac{1}{3}, \frac{2}{3})$ of $[0,1]$, then remove the middle third of the two remaining intervals $(\frac{1}{9}, \frac{2}{9})$ and $(\frac{7}{9}, \frac{8}{9})$, then continuing this pattern indefinitely. It turns out that $\mathcal{C}$ is: a compact, perfect, totally disconnected, uncountable set whose Lebesgue measure is 0. The complement of the Cantor set in $[0,1]$ is an ordinary fractal string, which is called the Cantor String $\mathcal{C}$, with lengths $\frac{1}{3}, \frac{1}{9}, \frac{1}{9}, \ldots$. To properly study this example and others we need to talk about the 'dimension(s)' of an ordinary fractal string. However, there are many different types of dimensions that one may wish to consider, such as the Minkowski dimension of a set as follows.

Definition 4 *Let* $d(x, A)$ *denote the distance of* $x \in \mathbb{R}$ *to a subset* $A \subset \mathbb{R}$ *and let* vol_1 *denote the one-dimensional Lebesgue measure on* $\mathbb{R}$. *For an ordinary fractal string* $\mathcal{L}$ *with associated open set* Ω*: given* $\epsilon > 0$, *let* $V(\epsilon) = \text{vol}_1\{x \in \Omega : d(x, \partial\Omega) < \epsilon\}$. *Then the* dimension *of* $\mathcal{L}$ *is defined as the* inner Minkowski dimension *of* $\partial\mathcal{L}$, *given by* $D_{\mathcal{L}} = \inf\{\alpha \geq 0 : V(\epsilon) = O(\epsilon^{1-\alpha}) \text{ as } \epsilon \to 0^+\}$. *The fractal string* $\mathcal{L}$ *is said to be* Minkowski measureable, *with* Minkowski content $\mathcal{M} = \lim_{\epsilon \to 0^+} V(\epsilon)\epsilon^{-(1-D)}$ *if this limit exists in* $(0, \infty)$.

Unlike the topological dimension of a set, the Minkowski dimension isn't necessarily an integer. In fact, for the Cantor String, we have $D_{\mathcal{C}} = \log_3 2$. This dimension plays a key role in the theory of zeta functions of fractals.

Definition 5 *Given a fractal string $\mathcal{L}$, with lengths $l_1, l_2, \ldots$, the* geometric zeta function *of $\mathcal{L}$ is defined by $\zeta_{\mathcal{L}}(s) = \sum_{j=1}^{\infty} l_j^s$.*

Note that this Dirichlet series converges for $\Re(s) \geq 1$, since $\zeta_{\mathcal{L}}(1) = \sum_{j=1}^{\infty} l_j < \infty$, which equals the Lebesgue measure of the open set Ω, by our assumption that an ordinary fractal string is a bounded open subset of $\mathbb{R}$. In fact, it is possible for $\zeta_{\mathcal{L}}(s)$ to converge on a larger half-plane. The following theorem precisely characterizes the region of convergence for the Dirichlet series.

Theorem 1.4 *[LvF13] For an ordinary fractal string $\mathcal{L}$ with infinitely many lengths, the series $\zeta_{\mathcal{L}}(s) = \sum_{j=1}^{\infty} l_j^s$ converges in the half-plane $\{s \in \mathbb{C} : \Re(s) > D_{\mathcal{L}}\}$, and $D_{\mathcal{L}}$ is the smallest such real number with this property.*

This theorem shows that unlike the Riemann zeta functions, the geometric zeta function of an ordinary fractal string will not have a pole at $s = 1$. However, in [LvF13], meromorphic extensions of this zeta function are studied, and the poles of these extensions are of critical importance to our work. We will turn to another type of zeta function for an ordinary fractal string - the spectral zeta function ζ_ν. To motivate this, consider a single interval of length l. If we had a physical string of this length, the (normalized) frequencies possible are $l^{-1}, 2l^{-1}, 3l^{-1}, \ldots$. Thus the sound spectrum of the string would be the set of these possible frequencies. If you now consider vibrating an ordinary fractal string where you allow each interval l_j to vibrate independently, the sound spectrum would be consist of all $k \cdot l_j^{-1}$ for $k, j \in \mathbb{N}$. This leads to the definition of a spectral zeta function.

Definition 6 *The* spectral zeta function *of an ordinary fractal string* $\mathcal{L}$ *is defined by the equation* $\zeta_\nu(s) = \sum_{k,j=1}^{\infty} (k \cdot l_j^{-1})^{-s}$

As pointed out in [LvF13], there is a very interesting relationship between the spectral zeta function and the geometric zeta function.

Theorem 1.5 *[LvF13] For an ordinary fractal string* $\mathcal{L}$*,* $\zeta_\nu(s)$ *is holomorphic in* $\Re(s) > 1$ *and we have* $\zeta_\nu(s) = \zeta_{\mathcal{L}}(s)\zeta(s)$*, where* $\zeta(s)$ *is the Riemann zeta function.*

In the previous section, we saw that properties of the Riemann zeta function, $\zeta(s)$, led to information about the associated counting function, $\pi(x)$. In order to further this analogy with fractal zeta functions we define the associated counting functions for the geometric and spectral zeta functions of an ordinary fractal string.

Definition 7 *Given an ordinary fractal string* $\mathcal{L}$*, define the* geometric counting function *of* $\mathcal{L}$ *as* $N_{\mathcal{L}}(x) = \#\{j \geq 1 : l_j^{-1} \leq x\}$*. Similarly we define the* spectral counting function *of* $\mathcal{L}$ *as* $N_\nu(x) = \#\{f \leq x : f \text{ is a frequency of } \mathcal{L}\text{, counted with multiplicity}\}$.

Then just as there was a relationship between $\zeta_{\mathcal{L}}(s)$ and $\zeta_\nu(s)$ before, we also have the following relationship between $N_{\mathcal{L}}(x)$ and $N_\nu(x)$.

Theorem 1.6 *[LvF13]* $N_\nu(x) = N_{\mathcal{L}}(x) + N_{\mathcal{L}}\left(\frac{x}{2}\right) + \cdots = \sum_{n=1}^{\infty} N_{\mathcal{L}}\left(\frac{x}{n}\right)$ *for all* $x > 0$.

We can then think of a so-called spectral operator that takes the information about the geometry of the fractal and sends it to information about its spectrum. In

[LvF13], an heuristic idea of the spectral operator was $\mathfrak{a} : f(x) \mapsto \sum_{n=1}^{\infty} f\left(\frac{x}{n}\right)$. Thus by the previous theorem, we would have $N_\nu = \mathfrak{a} N_{\mathcal{L}}$. It was more convenient to make the change of variable $x = e^t$, and consider the spectral operator as $\mathfrak{a} : f(t) \mapsto \sum_{n=1}^{\infty} f(t - \log n)$. In [LvF13] and [HL16] this operator is initially defined heuristally and then formally and used to study various formulations of inverse spectral problems - that is, given information about the spectrum of a fractal string, can you determine geometric information about it? This question (viewed as an inverse spectral problem for fractal strings) was initially connected to the Riemann hypothesis and led to a geometric and spectral reformulation of it in [LM95].

1.4 Inverse Spectral Problems - A Formal Setting

We will now focus on the formal setting used in [HL16] for the spectral operator $\mathfrak{a}$ (See also [HL12], [HL13], and [HL14]). Let $c \in (0,1)$ and let $\mathbb{H}_c = L^2(\mathbb{R}, e^{-2ct}dt)$. Initially this c has no particular significance, but we will see later that by considering c in this range, we will focus our attention on the values of $\zeta(s)$ in the critical strip: $0 < \Re(s) < 1$ and $\Re(s) > 1$, which is crucial to the Riemann hypothesis. Then $\mathbb{H}_c$, as a weighted L^2 space, is a Hilbert space. Let $\partial_c = \frac{d}{dt}$ be the differential operator on this space with domain $D(\partial_c) = \{f \in AC(\mathbb{R}) \cap \mathbb{H}_c : f' \in \mathbb{H}_c\}$. Here $AC(\mathbb{R})$ denotes the set of functions which are absolutely continuous on $\mathbb{R}$ and thus have a derivative a.e. This operator ∂_c is also called the *infinitesimal shift* (on the real line) because $e^{-h\partial_c} f(t) = f(t-h)$. The following theorem shows another important property of ∂_c.

This operator also satisfies:

Theorem 1.7 *[HL16] Let $c \in (0,1)$ and $\mathbb{H}_c$ and ∂_c be defined as above. Then ∂_c is a* *densely defined, unbounded linear operator on $\mathbb{H}_c$ and has spectrum the vertical line $\sigma(\partial_c) = \{c + it : t \in \mathbb{R}\}$.*

Using a version of functional calculus for measurable functions applied to normal operators, the truncated infinitesimal shifts $\partial_c^{(T)}$ were also studied in [HL16], where $T > 0$. By "truncated", we mean the operator $\Psi^{(T)}(\partial_c)$ where $\Psi^{(T)}$ is a measurable, continuous, complex-valued function on the line $\Re(s) = c$ with range dense in $[c - iT, c + iT]$. Thus we have the following result on the truncated spectrum by applying a version of the spectral mapping theorem.

Theorem 1.8 *$\partial_c^{(T)}$ is a bounded linear operator on $\mathbb{H}_c$ with spectrum $\sigma(\partial_c^{(T)}) = [c - iT, c + iT]$.*

We now precisely define the spectral and truncated spectral operators, $\mathfrak{a}_\mathfrak{c}$ and $\mathfrak{a}_c^{(T)}$, respectively. In the previous section, we saw that under an appropriate change of variable, the heuristic for the spectral operator acted as $f(t) \mapsto \sum_{n=1}^{\infty} f(t - \log n)$. Recalling the reason that ∂_c was called the infinitesimal shift we note that $n^{-\partial_c} f(t) = e^{-\log n \partial_c} f(t) = f(t - \log n)$. Thus the spectral operator should be something like $\mathfrak{a} = \sum_{n=1}^{\infty} n^{-\partial_c}$. This series is precisely the standard definition of the Riemann zeta function with ∂_c replacing the complex variable s. This motivates the definitions: $\mathfrak{a}_c = \zeta(\partial_c)$ and $\mathfrak{a}_c^{(T)} = \zeta(\partial_c^{(T)})$. One of the first results about $\mathfrak{a}_c$ in [HL16] is

Theorem 1.9 *For $c > 1$, $\mathfrak{a}_c = \sum_{n=1}^{\infty} n^{-\partial_c}$, where the infinite sum converges in $B(\mathbb{H}_c)$.*

This "quantized", or operator-valued, analog, $\mathfrak{a}_c$, of the Riemann zeta function nicely formalizes the idea of the spectral operator studied in [LM95] and [LvF13]. Actually, to see this, recall that for an ordinary fractal string $\mathcal{L}$, with Minkowski dimension $D_{\mathcal{L}}$, we have that $N_{\mathcal{L}}(x) \sim x^{D_{\mathcal{L}}}$ as $x \to \infty$, with $N_{\mathcal{L}}(x) = 0$ for $x < x_0$ for some x_0. Thus under the change of variable, $x = e^t$, $N_{\mathcal{L}}(t) \sim e^{D_{\mathcal{L}} t}$ as $t \to \infty$. Hence $N_{\mathcal{L}} \in \mathbb{H}_c$ if $D_{\mathcal{L}} \leq c$. Therefore, the expression $\mathfrak{a}_c(N_{\mathcal{L}}(t))$ is defined whenever the parameter c is an upper bound for the Minkowski dimension of the fractal strings under consideration. Then the question of whether or not certain inverse spectral problems have solutions can be related to the invertibility of $\mathfrak{a}_c$. For $c > 1$ it is shown that $\mathfrak{a}_c$ is invertible with inverse given by $\mathfrak{a}_c^{-1} = \sum_{n=1}^{\infty} \mu(n) n^{-\partial_c}$, where $\mu(n)$ is the Möbius function defined to be -1 raised to the number of distinct prime factors of any square-free n and $\mu(n) = 0$ if n is not square-free. However, for values of $c < 1$, $\mathfrak{a}_c$ will also be important. Due to the Bohr-Courant Density Theorem $\mathfrak{a}_c$ is non-invertible for $\frac{1}{2} < c < 1$ because $\sigma(\mathfrak{a}_c) = \overline{\{\zeta(c+it) : t \in \mathbb{R}\}} = \mathbb{C}$ in this range of c values. Thus, the notion of quasi-invertibility of $\mathfrak{a}_c$ was defined to be whenever the truncated spectral operator $\mathfrak{a}_c^{(T)}$ is invertible for all $T > 0$. This modification of invertibility helps us here, because when we limit ourselves to $[c - iT, c + iT]$, the set of values of $\zeta(s)$ on this interval is already compact and so the closure will not add any additional points. Thus $\sigma(\mathfrak{a}_c^{(T)}) = \zeta([c - iT, c + iT])$ will only contain 0 if $\zeta(s)$ has a zero in $[c - iT, c + iT]$. We then obtain the following formulation of the Riemann Hypothesis.

Theorem 1.10 *The spectral operator $\mathfrak{a}_c$ is quasi-invertible for all $c \in (0, 1) \setminus \{\frac{1}{2}\}$ if and only if the Riemann hypothesis is true.*

Lapidus extended this result in [Lap15] and formulated the following asymmetric criterion for the Riemann hypothesis. Here, $\mathfrak{b}_c$ denotes the nonnegative self-adjoint operator $\mathfrak{b}_c = \mathfrak{a}_c \mathfrak{a}^*$

Theorem 1.11 *The following are all equivalent to the Riemann Hypothesis:*

1) For every $c \in (0, \frac{1}{2})$, $\mathfrak{a}_c$ *is invertible.*

2) For every $c \in (0, \frac{1}{2})$, $\mathfrak{b}_c$ *is invertible.*

3) For every $c \in (0, \frac{1}{2})$, $\mathfrak{b}_c$ *is bounded away from zero.*

This section summarizing the work of Herichi and Lapidus, which was motivated by earlier work with several other authors, shows a very interesting connection between fractals, functional analysis, and the Riemann zeta function. The fact that the spectra of ∂_c and $\partial_c^{(T)}$ lie on the vertical line $\Re(s) = c$ allows us to focus on the values of $\zeta(c + it)$ when considering the spectrum of $\mathfrak{a}_c$. Thus in some sense ∂_c helps to localize the information from $\zeta(s)$ to one vertical line at a time. This fact is the starting point of the research contained in this thesis, in which we find a different space and shift operator that would allow us to localize the values of $\zeta(s)$ even further - down to the value at a single point.

1.5 The Weil Conjectures

Although Riemann's Hypothesis is still an open problem today, there is an analogous result that has been shown for curves over finite fields. More precisely, let Y be a smooth, projective, geometrically connected curve over $\mathbb{F}_q$, the field with q elements. Then we can define the zeta function $\zeta(Y, s)$ of Y as: $\zeta(Y, s) = \exp\left(\sum_{n=1}^{\infty} \frac{Y_n}{n} q^{-ns}\right)$, where $Y_n =$ the number of points of Y defined over $\mathbb{F}_{q^n}$, the degree n extension of $\mathbb{F}_q$.

Weil made several conjectures about these zeta functions which were eventually proven, but I will focus on the one that is analogous to the Riemann Hypothesis:

Theorem 1.12 *The only poles of* $\zeta(Y, s)$ *lie on the lines* $\Re(s) = 0, 1$ *and the only zeros lie* *on the lines* $\Re(s) = \frac{1}{2}$.

The full proof of this result is too long to include here, but we will point out a few key parts that will lead us through the work in this thesis.

A sequence of Weil cohomology groups for the curve Y are formed, in particular H^0, H^1, H^2 are the only nontrivial groups, with $\dim H^0 = \dim H^2 = 1$ and $\dim H^1 = 2g$ where g denotes the genus of Y. Then the Frobenius map F which sends $x \to x^q$ acts on the space $\mathbb{F}_{q^n}$ for any n and in fact also induces a map on the cohomology groups $F^* : H^j \to H^j$.

Next, we examine the Lefschetz fixed point formula.

Theorem 1.13 *(Lefshetz Fixed Point Formula) Let* X *be a closed smooth manifold and let* $f : X \to X$ *be a smooth map with all fixed points nondegenerate. Then* $\sum_{j=0}^{\infty} (-1)^j \, Tr(f^* | H^j)$ *is equal to the number of fixed points of* f.

We apply this result to the n^{th} power of the Frobenius map, F^n, whose fixed points are exactly the points on the curve Y with all coordinates in $\mathbb{F}_{q^n}$. Specifically we obtain $\sum_{j=0}^{2} (-1)^j \mathrm{Tr}(F^{*^n} | H^j) = Y_n$. To proceed further, we need the next result from linear algebra.

Theorem 1.14 *If* f *is an endomorphism of a finite dimensional vector space* V *that for* $|t|$ *sufficiently small,* $\exp\left(\sum_{n=1}^{\infty} \frac{1}{n} t^n \, Tr(f^n | V) \right) = \det(I - f \cdot t | V)^{-1}$

Then, if we apply this result to the Frobenius operator F, we can proceed with the following calculation.

$$\begin{aligned}
\zeta(Y,s) &= \exp\left(\sum_{n=1}^{\infty}\frac{Y_n}{n}q^{-ns}\right)\\
&= \exp\left(\sum_{n=1}^{\infty}\frac{1}{n}\sum_{j=0}^{2}(-1)^j Tr(F^{*^n}|H^j)q^{-ns}\right)\\
&= \prod_{j=0}^{2}\left(\exp\left(\sum_{n=1}^{\infty}\frac{1}{n}(-1)^j Tr(F^{*^n}|H^j)q^{-ns}\right)\right)^{(-1)^j}\\
&= \prod_{j=0}^{2}\left(\det(I-F^*q^{-s}|H^j)\right)^{(-1)^{j+1}}\\
&= \frac{\det(I-F^*q^{-s}|H^1)}{\det(I-F^*q^{-s}|H^0)\det(I-F^*q^{-s}|H^2)}
\end{aligned}$$

Thus the zeta function of a curve Y is an alternating product of determinants of $I - q^{-s}F^*$ over the cohomology spaces. Since these spaces are finite dimensional, this equation further shows that $\zeta(Y,s)$ is a rational function of q^{-s}, which was another one of Weil's conjectures. We also see that the zeros of $\zeta(Y,s)$ are given from the eigenvalues of the operator F^* on H^1, and the poles are given from the eigenvalues on H^0 and H^2. Then it was shown that the eigenvalues of F on H^j have absolute value $q^{\frac{j}{2}}$ and thus the zeros of $\zeta(Y,s)$ satisfy $\Re(s) = \frac{1}{2}$.

1.6 A Cohomology Theory in Characteristic Zero?

Christopher Deninger has postulated that the ideas used to prove the Weil conjectures could be extended to eventually prove the Riemann hypothesis in [Den94], [Den98] and others. In particular, he envisions a cohomology theory of algebraic schemes over $Spec(\mathbb{Z})$ that would conjecturally help prove the Riemann hypothesis and other problems in analytic number theory. In his papers, he lays out some of the difficulties in doing so as well as some of the properties that such a theory would need to satisfy.

For example, the proof of the Weil conjectures outlined in the previous section relies on the Frobenius operator, but this proof has no known analogue for characteristic 0 situations such as the Riemann zeta function itself. How can we find a function in characteristic 0 that will behave in some manner similarly to the Frobenius map?

This then leads to the further question: is there some other operator one can use, where the eigenvalues on certain cohomology spaces correspond to the zeros or poles of a given zeta function (or other meromorphic function) that would work in characteristic 0? Further, can we recover the zeta function as a determinant of $I - As$? Note that the fact there are infinitely many zeros or poles for some functions under consideration will necessitate considering determinants of operators over an infinite dimensional space, unlike the situation in the Weil conjecture where all of the cohomology spaces were finite dimensional. Deninger has suggested using zeta-regularized determinants to overcome the difficulty of determinants on infinite dimensional spaces. Instead, in this thesis, the concepts of the Fredholm determinant and trace class operators as well as a different type of regularized determinants involving operators that are not trace class, but are instead in certain trace ideals.

This dissertation presents an attempt at a formalism to give a cohomology theory in characteristic 0 for certain classes of meromorphic functions, and we show that we can recover the function as an alternating product of determinants of $I - sA$ for a particular choice of operator A and cohomology spaces H^j. To show the success of this process, here are a few examples of determinant formulas to be proven later:

The determinant formula representation for a rational function.

Theorem 8 *If $f(z) = z^k g(z)$ is a rational function then $f(z) = z^k g(0) \det_{1,1}(I - zD_{g(z)})$ for any z with $f(z)$ defined.*

The determinant formula representation for zeta functions of curves over finite fields.

Theorem 9 *Let Y be a smooth, projective, geometrically connected curve over $\mathbb{F}_q$. Then,*

$$\zeta(Y, s) = \det_{1,1}\left(I - q^{-s} D_{\zeta(Y, q^{-s})}\right)$$

The determinant formula representation for the Gamma function.

Theorem 10 $\Gamma(z) = \frac{e^{-\gamma z}}{z} \det_{1,2}(I - zD_{z\Gamma(z)})$ *for any z with $\Gamma(z)$ defined.*

A quantized Euler product for $\zeta(s)$:

Theorem 11 *If $s \in \mathbb{C}$ with $\Re(s) > 1$, then $\zeta(s) = \prod_p \det(I - p^{-s} D_\phi)$ where $\phi(z) = \frac{1}{1-z}$.*

Finally we have an expression for $\zeta(s)$ as a fraction of determinants very similar to the motivating example of curves over finite fields:

Theorem 12 *If $\psi(s) = s - 1$, then we have:*

$$\zeta(s) = -\frac{e^{(\log(2\pi)-1)s}}{2} \frac{\det_{2,1}(I - sD_{\xi(s)})}{\det_{1,1}\left(I - sD_{\psi(s)}\right) \det_{1,2}\left(I - sD_{s\Gamma(s/2)}\right)}$$

This work will also be included in a paper, currently being prepared, coauthored by my advisor, Michel L. Lapidus, to be included in a volume entitled "Exploring The Riemann Zeta Function" celebrating the 190th anniversary of Riemann's birth, [CL].

Chapter 2

Some Needed Background

In our work, we will need the standard concepts of norm and inner product on a complex vector space. See [RS80] for the details. Note that we are following the convention that an inner product on a complex vector space is linear in the second argument and conjugate linear in the first. It is also standard that given an inner product, $(\cdot,\cdot)$, there is an associated norm, $\|\cdot\|$, defined by $\|x\| = \sqrt{(x,x)}$. We will also need the notions of Banach and Hilbert spaces, and will only ever deal with separable Hilbert spaces.

2.1 The Space of Bounded Operators on a Separable Hilbert Space

In this work, we will often consider linear operators on Hilbert spaces. We present a short study of possible properties for such an operator to have, beginning with the difference between bounded and unbounded operators.

Definition 13 *Given a Hilbert space H, a linear operator $A : H \to H$ is called* bounded *if*

$\sup_{x \neq 0} \frac{\|Ax\|}{\|x\|} < \infty$. *In this case, we define the norm of the operator* A *by:* $\|A\| := \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|}$. *We denote the set of all bounded linear operators on a Hilbert space* H *by* $B(H)$. *If a linear map defined on a subset of* H *is not bounded, then it is called* unbounded.

The space $B(H)$ is not just a vector space under pointwise operations, but in fact is a Banach space with norm $\|A\|$ for each $A \in B(H)$. It can be shown that any operator that is unbounded, cannot be defined on all of H. Hence in that case, the domain of A will be clearly specified. There are many additional properties an operator on H might have.

Definition 14 *An operator* $A \in B(H)$ *is called* finite rank *if the range of* A *is finite dimensional.* A *is called* compact *if it is the norm limit of a sequence of finite rank operators. We denote by* $\mathcal{J}_\infty$ *the set of all compact operators.*

We will be combining compact operators and will therefore need the following result:

Theorem 2.1 *[RS80] If* A_n *is a sequence of compact operators and* $A_n \to A$ *in the norm topology, then* A *is compact.*

We will also need the definition of the adjoint of a linear operator on a Hilbert space.

Definition 15 *Given either* $A \in B(H)$ *or a densely defined, unbounded operator* A*, the* adjoint *of the operator* A *is the operator* A^* *satisfying* $(x, Ay) = (A^*x, y)$ *for all* y *in the domain of* A *and* x *in the domain of* A^*.

Definition 16 *An operator,* A*, satisfying* $A = A^*$*, is called self-adjoint. An operator satisfying* $A^*A = AA^*$ *is called* normal.

We now recall basic theory surrounding the generalization of the notion of eigenvalues from finite dimensional vector spaces to Hilbert spaces.

Definition 17 *Given an operator A on H, bounded or unbounded, the* resolvent set $\rho(A)$ *is defined as those $\lambda \in \mathbb{C}$ such that $\lambda I - A : D(A) \to H$ is a bijection with bounded inverse. This operator, $(\lambda I - A)^{-1}$ is called the* resolvent operator of A at the point λ. *The* spectrum $\sigma(A)$ *is defined as the complement of $\rho(A)$ in $\mathbb{C}$. The* spectral radius $r(A)$ *is defined by* $r(A) := \sup_{z \in \sigma(A)} |z|$.

The first result gives a formula for the spectral radius of a bounded operator.

Theorem 2.2 *[RS80] If $A \in B(H)$, then $r(A) = \lim_{n\to\infty} \|A^n\|^{\frac{1}{n}}$. If A is normal then we have $r(A) = \|A\|$.*

We can use the above formula for a normal operator to give an upper bound on the norm of the resolvent operator as follows.

Theorem 2.3 *If T is bounded and normal and $\lambda \in \rho(T)$, then $\|(\lambda I - T)^{-1}\| \leq (d(\lambda, \sigma(T)))^{-1}$, where $d(\lambda, \sigma(T))$ is the distance from the point λ to the set $\sigma(T)$.*

Proof. First, we will show that $r((\lambda I - T)^{-1}) < (d(\lambda, \sigma(T)))^{-1}$. To do this, suppose that $\mu \in \mathbb{C}$ with $|\mu| > (d(\lambda, \sigma(T)))^{-1}$. Then $\left|\frac{1}{\mu}\right| < d(\lambda, \sigma(T))$. Thus $-\frac{1}{\mu} + \lambda \notin \sigma(T)$. Thus $((-\frac{1}{\mu} + \lambda)I - T)^{-1}$ exists and is a bounded operator. Now define the operator S by: $S = -\frac{1}{\mu}((-\frac{1}{\mu} + \lambda)I - T)^{-1}(\lambda I - T)$. Then S is a composition of bounded operators and therefore is bounded. Furthermore, we obtain:

$$\begin{aligned} S\left((\lambda I - T)^{-1} - \mu I\right) &= -\frac{1}{\mu}\left(\left(-\frac{1}{\mu}+\lambda\right)I - T\right)^{-1}(\lambda I - T)\left((\lambda I - T)^{-1} - \mu I\right) \\ &= -\frac{1}{\mu}\left(\left(-\frac{1}{\mu}+\lambda\right)I - T\right)^{-1}(I - \lambda\mu I + \mu T) \\ &= \left(\left(-\frac{1}{\mu}+\lambda\right)I - T\right)^{-1}\left(\left(-\frac{1}{\mu}+\lambda\right)I - T\right) \\ &= I. \end{aligned}$$

This shows that $\left((\lambda I - T)^{-1} - \mu I\right)$ has a bounded inverse, and thus $\mu \notin \sigma\left((\lambda I - T)^{-1}\right)$. Since this fact holds for any $|\mu| > (d(\lambda, \sigma(T)))^{-1}$, the spectral radius must satisfy the inequality: $r\left((\lambda I - T)^{-1}\right) \leq (d(\lambda, \sigma(T)))^{-1}$. By the previous theorem and the assumption that T is normal, $(\lambda I - T)^{-1}$ is also normal and thus:

$$\|(\lambda I - T)^{-1}\| = r\left((\lambda I - T)^{-1}\right) < (d(\lambda, \sigma(T)))^{-1}.$$

■

The spectrum of a compact operator on a Hilbert space has a very specific form.

Theorem 2.4 *[RS80] Let A be a compact operator on a Hilbert space H. Then:*

1) $\sigma(A)$ is a set with no non-zero limit points and

2) Every non-zero $\lambda \in \sigma(A)$ is an eigenvalue of finite (geometric) multiplicity.

Now in this work we will also need the notion of direct sums of Hilbert spaces and operators on these sums.

Definition 18 *Given a countable collection of Hilbert spaces $\{H_k : k \in \mathbb{N}\}$, we let H be the set of sequences $\{x_n\}$ satisfying $x_n \in H_n$ for each n and $\sum_n \|x_n\|^2_{H_n} < \infty$. Then H*

is a Hilbert space under the inner product $(\{x_n\},\{y_n\})_H = \sum_n (x_n, y_n)_{H_n}$ *that is denoted by* $H = \bigoplus_n H_n$. *Similarly given a sequence of bounded operators* A_n *on* H_n*, we define* $A = \bigoplus_n A_n$*, with domain* $D(A) = \{(x_n)_n \in H : (A_n x_n)_n \in H\}$ *by* $(A(x_n))_n = (A_n x_n)_n$.

The following result regarding $A = \bigoplus_n A_n$ will be used in our work.

Theorem 2.5 *Given a sequence of Hilbert spaces* H_n *and operators* $A_n \in B(H_n)$ *for each* n*, let* $A = \bigoplus_n A_n$ *on the space* $H = \bigoplus_n H_n$*. Then the following statements hold:*
1) $A \in B(H)$ *iff* $\sup_n \|A_n\|_{H_n} < \infty$
2) If each A_n *is normal, then* $\sigma(A) = \overline{\bigcup_n \sigma(A_n)}$

Proof. For 1), if we let $e = (e_n) \in H$ be such that $e_n \neq 0$ in H_n, but all other components are 0, then $\|Ae\| = \|A_n e_n\|$ so that $\|A\| \geq \sup_n \|A_n\|$. Thus if $\sup_n \|A_n\| = \infty$ then A is unbounded. On the other hand if $\sup_n \|A_n\| = M < \infty$, then for any $x = (x_n) \in H$, we have $\sum_n \|x_n\|^2 < \infty$. Thus $\|Ax\|^2 = \sum_n \|A_n x_n\|^2 \leq M\|x_n\|^2 < \infty$. Thus Ax is defined for every $x \in H$, thus implying that A is bounded.

If $\lambda \in \bigcup_n \sigma(A_n)$, then for at least one n, we have that $A_n - \lambda I$ does not have a bounded inverse. Thus $A - \lambda I = \bigoplus_n (A_n - \lambda I)$ cannot have a bounded inverse and $\lambda \in \sigma(A)$. Also, since $\sigma(A)$ is a closed set we get $\overline{\bigcup_n \sigma(A_n)} \subset \sigma(A)$. Now suppose that $\mu \notin \overline{\bigcup_n \sigma(A_n)}$. Then $\epsilon = d\left(\mu, \overline{\bigcup_n \sigma(A_n)}\right) > 0$. Furthermore $d(\mu, \sigma(A_n)) > \epsilon$ for all n. Thus $\|(A_n - \mu I)^{-1}\| \leq \frac{1}{\epsilon}$ by theorem 2.8 and so $\bigoplus (A_n - \mu I)^{-1}$ is an inverse for $A - \mu I$, with norm $\left\|\bigoplus (A_n - \mu I)^{-1}\right\| = \sup_n \|(A_n - \mu I)^{-1}\| \leq \frac{1}{\epsilon}$. Hence, $A - \mu I$ has a bounded inverse. Therefore $\mu \notin \sigma(A)$, and we conclude that $\sigma(A) = \overline{\bigcup_n \sigma(A_n)}$. ∎

2.2 Riesz Functional Calculus

When $A \in B(H)$ and $p(t)$ is a polynomial, it is straight forward to understand what we mean by $p(A)$ by using repeated compositions and additions of bounded operators. In particular, if $p(t) = \sum_{k=0}^{n} a_k t^k$, then $p(A) = \sum_{k=0}^{n} a_k A^k$, where A^k is just A composed with itself k times. However, if p is not a polynomial and is instead some more general type of function, we need a more involved process to make sense of $p(A)$. The process of associating an operator to an expression $p(A)$ is called a functional calculus and for this work, we will be using the so-called Riesz functional calculus. In this situation we will have a bounded operator A and a complex valued function $p(z)$ that is holomorphic on a neighborhood of $\sigma(A)$, the spectrum of A.

To begin, we first need to recall a basic fact from complex analysis that is the basis behind the Riesz functional calculus: a version of the Cauchy integral formula.

Theorem 2.6 *(Cauchy Integral Formula) Suppose that U is an open subset of $\mathbb{C}$, $f : U \to \mathbb{C}$ is holomorphic, and γ is a closed, rectifiable curve in U. Let $a \in \mathbb{C}$ be such that the winding number of γ around w is $n(\gamma, w) = 1$. Then*

$$f(a) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z-a} dz.$$

We then use the equation above, but instead of a being a complex number, we replace it with a bounded operator A. This shifts from the problem from that of making sense of $f(A)$, to instead, that of defining $\frac{1}{z-A} = (z-A)^{-1}$ and an integral which is operator valued. However, whenever $z \in \rho(A)$, $(z-A)^{-1}$ exists and is bounded, and so we are integrating a bounded operator $f(z)(z-A)^{-1}$, over a closed, rectifiable curve γ. We can make sense of such an integral in a normed space, of which $B(H)$ is one, using the usual ideas of Riemann summation. This leads to the following definition.

Definition 19 *Given a bounded operator A on a Hilbert space H, and a function f that is holomorphic on a neighborhood G of $\sigma(A)$, let $\gamma_1, \gamma_2, ..., \gamma_n$ be closed rectifiable curves in G, oriented positively, such that every point $a \in \sigma(A)$ has winding number $\sum_{k=1}^{n} n(\gamma_k, a) = 1$. Then we define $f(A)$ to be the bounded operator given by $\frac{1}{2\pi i}\int_\gamma f(z)(z-A)^{-1}dz$*

The reader can see [Con90] for the details that this is well-defined, but here we will just mention some of the nice properties that this functional calculus has. Fix $A \in B(H)$ and let $Hol(A)$ be the algebra of all functions that are analytic in a neighborhood of $\sigma(A)$.

Theorem 2.7 *[Con90] Fix $A \in B(H)$*

1) The map $A \mapsto f(A)$ of $Hol(A) \to B(H)$ is an algebra homomorphism

2) If $f(z) = \sum_{n=0}^{\infty} c_n z^n$ has radius of convergence greater than $r(A)$ then $f \in Hol(A)$ and $f(A) = \sum_{n=0}^{\infty} c_n A^n$.

3) If $f(z) \equiv 1$, then $f(A) = I$.

4) If $f(z) = z$, then $f(A) = A$.

5) If $f_1, f_2, ...$ are all analytic on G, $\sigma(A) \subset G$, and $f_n \to f$ uniformly on compact subsets of G, then $\|f_n(A) - f(A)\| \to 0$.

This shows that the Riesz functional calculus extends the polynomial functional calculus. In fact, it can be shown to be the unique way of defining such a functional calculus on the space $Hol(A)$. Furthermore, we have the following version of the spectral mapping theorem in this case.

Theorem 2.8 *(Spectral Mapping Theorem) [Con90] Given a Hilbert space H, an operator $A \in B(H)$, and a function f that is holomorphic on a neighborhood of $\sigma(A)$, we have that the operator $f(A)$ defined by the Riesz functional calculus satisfies $\sigma(f(A)) = f(\sigma(A))$.*

2.3 Trace Ideals and Regularized Determinants

For some of our work, we will need a stronger condition than simply being bounded. We will first look at compact operators and then further restrict to a specific family of ideals of compact operators.

Theorem 2.9 *[RS80] Let A be a compact operator on H. Then there are orthonormal sets $\{\psi_n\}$ and $\{\phi_n\}$ and positive real numbers $\mu_n(A)$, with $\mu_1(A) \geq \mu_2(A) \geq \cdots$ such that $A = \sum_n \mu_n(A)(\psi_n, \cdot)\phi_n$. Moreover the $\mu_n(A)$ are uniquely determined.*

The $\mu_n(A)$ from the previous theorem are called the *singular values* of A. We can actually describe $\{\mu_n(A)\}$ in another way. Given an operator A, the operator A^*A is a positive operator so $|A| := \sqrt{A^*A}$ makes sense. The $\mu_n(A)'s$ are exactly the eigenvalues of $|A|$. Now we can turn to Calkin's theory of operator ideals. We begin by setting up a relationship between ideals in $B(H)$ and certain sequence spaces.

Definition 20 *Fix an orthonormal set $\{\phi_n\}$ in H. Given an ideal $\mathcal{J} \neq B(H)$; we define the* sequence space associated to $\mathcal{J}$ *by $\mathcal{S}(\mathcal{J}) = \{a = (a_1, a_2, ...) | \sum a_n(\phi_n, \cdot)\phi_n \in \mathcal{J}\}$. On the other hand, given a sequence space s, let $I(s)$ be the family of compact operators A with $(\mu_1(A), \mu_2(A), ...) \in s$.*

In order for this correspondence between sequence spaces and ideals to be one-to-one, we need to restrict our sequence spaces to Calkin spaces. We then need the following operator on sequences.

Definition 21 *Given an infinite sequence, (a_n), of numbers with $a_n \to 0$ as $n \to \infty$. a_n^* is the sequence defined by $a_1^* = \max |a_i|$, $a_1^* + a_2^* = \max_{i \neq j}(|a_i| + |a_j|)$, etc. Thus $a_1^* \geq a_2^* \geq \cdots$ and the sets of a_i^* and $|a_i|$ are identical, counting multiplicities.*

This operator allows us to make the following definition.

Definition 22 *A* Calkin space *is a vector space, s, of sequences a_n with $\lim_{n\to\infty} a_n = 0$, and the so-called Calkin property: $a \in s$ and $b_n^* \leq a_n^*$ implies $b \in s$.*

With these definitions in mind we can use the following theorem to see a relation between two-sided ideals and Calkin spaces.

Theorem 2.10 *[Sim05] If s is a Calkin space, then $I(s)$ is a two-sided ideal of operators and $\mathcal{S}(I(s)) = s$. if $\mathcal{J}$ is a two-sided ideal, then $\mathcal{S}(\mathcal{J})$ is a Calkin space and $I(\mathcal{S}(\mathcal{J})) = \mathcal{J}$.*

We will now use this relation to define the ideals in the space of compact operators that we will be using.

Definition 23 *A compact operator A is said to be in the* trace ideal *$\mathcal{J}_p$, for some $p \geq 1$, if $\sum_n \mu_n(A)^p < \infty$. That is, $\mathcal{J}_p$ is the ideal that is associated to the Calkin space l^p. An element A of $\mathcal{J}_1$ is called a* trace class operator. *For $A \in \mathcal{J}_1$ we define $Tr(A) = \sum_n (\phi_n, A\phi_n)$ for any choice of orthonormal basis $\{\phi_n\}$. If $A \in \mathcal{J}_2$ then we say that A is* Hilbert-Schmidt.

Trace class operators, A, are precisely those operators for which $\mathrm{Tr}(A) = \sum_n (\phi_n, A\phi_n)$ is absolutely convergent and independent of the choice of orthonormal basis. Similarly, Hilbert Schmidt operators are those for which $\sum_n (A\phi_n, A\phi_n) = \|A\phi_n\|^2$ is convergent and independent of the choice of orthonormal basis. If A is a trace class operator, then there is a method to define a so-called Fredholm determinant, $\det(I + zA)$, which defines an entire function on $\mathbb{C}$. Operators of the form $I + zA$ for a trace class A are called *Fredholm.* This determinant can be defined in several equivalent ways. We list them here for trace class A and $z \in \mathbb{C}$:

$$\det(I + zA) := e^{\mathrm{Tr}(\log(I+zA))} \tag{2.11}$$

for small $|z|$ and analytically continued to the whole plane,

$$\det(I + zA) = \sum_{n=0}^{\infty} z^n \mathrm{Tr}(\wedge^n(A)) \tag{2.12}$$

with $\wedge^n(A)$ defined in terms of alternating algebras, and

$$\det(I + zA) = \prod_{n}^{N(A)} (1 + z\lambda_n(A)) \tag{2.13}$$

where $\lambda_n(A)$ are the nonzero eigenvalues of A and $N(A)$ is the number of such eigenvalues, which can be infinite.

A discussion concerning which of the above equations should be taken as a definition and which are to be proven appears in [Sim05]. For the work here, (2.13) will be the most convenient choice. One thing to note at this time though is that $\det(I + zA)$ does define an entire function by any of the above definitions, when A is trace class. This then shows why one can't hope to recover a meromorphic function without taking the quotient of determinants as was seen in the proof of the Weil conjectures.

Although some of the operators we will consider will not be trace class, they will at least be in one of the other trace ideals $\mathcal{J}_n$. In this case we can define a regularized determinant that will allow us to get a determinant formula for the operator. We start by considering an expression of the form $\det(I + zA)e^{-zTr(A)}$. For trace class operators A, both $\det(I + zA)$ and $e^{-zTr(A)}$ are convergent, but for Hilbert Schmidt operators neither is necessarily. And yet, when you consider the two factors together as a possibly infinite product over the eigenvalues of A, $\prod_{k=1}^{N(A)} ((1 + z\lambda_k(A)) \exp -\lambda_k(A)z)$, the combined term does converge for Hilbert Schmidt operators. This idea can in fact be extended to get a convergent infinite product expression for operators in any $\mathcal{J}_n$, which will be called the regularized determinant. First we need a lemma.

Lemma 24 *[Sim05] For $A \in B(H)$, let $\mathcal{R}_n(A) = \left[(I + A)\exp(\sum_{j=1}^{n-1}(-1)^j j^{-1} A^j)\right] - I$. Then if $A \in \mathcal{J}_n$ we have $\mathcal{R}_n(A) \in \mathcal{J}_1$.*

Proof. ([Sim05]) Let $g(z) = (1 + z)\exp(\sum_{j=1}^{n-1}(-1)^j j^{-1} z^j) - 1$. Since $\sum_{j=1}^{n-1}(-1)^j j^{-1} z^j$ is the beginning of the Taylor series for $\log(1 + z)$, we see that $\frac{g(z)}{z^n}$ is an entire function $h(z)$. Thus $g(A) = A^n h(A)$. If $A \in B(H)$, then $h(A) \in B(A)$ by the Riesz functional calculus explained in the previous section. On the other hand, $A \in \mathcal{J}_n$ implies that $A^n \in \mathcal{J}_1$ and since $\mathcal{J}_1$ is a two-sided ideal we have that $g(A) = R_n(A) \in \mathcal{J}_1$. ■

This associates a trace class operator to any given $A \in \mathcal{J}_n$ and allows us to define the regularized determinant as follows:

Definition 25 *[Sim05] For* $A \in \mathcal{J}_n$*, define* $\det_n(I+A) = \det(I + \mathcal{R}_n(A))$

With this definition, we can now conclude this section with a very similar product formula for the regularized determinant of a Hilbert Schmidt operator with each term having an exponential factor to help convergence along with some other interesting properties.

Theorem 2.14 *[Sim05] For* $A \in \mathcal{J}_n$*, we have:*
1) $\det_n(I+\mu A) = \prod_{k=1}^{N(A)} \left[(1+\mu\lambda_k(A)) \exp\left(\sum_{j=1}^{n-1} (-1)^j j^{-1} \lambda_k(A)^j \mu^j \right) \right]$
2) $|\det_n(I+A)| \leq \exp(\Gamma_n \|A\|_n^n)$ *for a suitable constant* Γ_n

3) $|\det_n(I+A) - \det_n(I+B)| \leq \|A-B\|_n \exp(\Gamma_n(\|A\|_n + \|B\|_n + 1)^n)$

4) If $A \in \mathcal{J}_{n-1}$*, then* $\det_n(I+A) = \det_{n-1}(I+A)\exp\left[(-1)^{n-1}\frac{1}{n-1} Tr(A^{n-1})\right]$ *and in particular, for* $A \in \mathcal{J}_1$*,* $\det_n(I+A) = \det(I+A)\exp\left(\sum_{j=1}^{n-1}(-1)^j j^{-1} Tr(A^j)\right)$
5) $(I+A)$ *is invertible if and only if* $\det_n(I+A) \neq 0$.

Proof. For 1), this comes immediately when using the definition of determinant for Fredholm operators as a product over the eigenvalues, since $\det_n(I+\mu A) = \det(I+R_n(\mu A))$ and the eigenvalues $\lambda_k(R_n(\mu A))$ are $(1+\mu\lambda_k)\exp(\sum_{j=1}^{n-1}(-1)^j j^{-1}\lambda_k(A)^j\mu^j - 1$ by the Spectral Mapping Theorem given earlier. For a suitable Γ_n, $|1+g(z)| \leq \exp(\Gamma_n|z|^n)$, so that 2) follows from 1) and the fact that $\sum |\lambda_j(A)|^n \leq \|A\|_n^n$. 3) follows from theorem 5.1 on page 45 of [Sim05]. For 4), use 1):

$$\begin{aligned}
\det_n(I+A) &= \prod_{k=1}^{N(A)} \left[(1+\lambda_k(A)) \exp\left(\sum_{j=1}^{n-1} (-1)^j j^{-1} \lambda_k(A)^j \right) \right] \\
&= \prod_{k=1}^{N(A)} \left[(1+\lambda_k(A)) \exp\left(\sum_{j=1}^{n-2} (-1)^j j^{-1} \lambda_k(A)^j \right) \right] \prod_{k=1}^{N(A)} \exp\left((-1)^{n-1} \frac{1}{n-1} \lambda_k(A)^{n-1} \right) \\
&= \det_{n-1}(I+A) \exp\left((-1)^{n-1} \tfrac{1}{n-1} \mathrm{Tr}\left(A^{n-1}\right) \right),
\end{aligned}$$

where each step is valid because $A \in \mathcal{J}_{n-1}$. If $A \in \mathcal{J}_1$ then we can repeat to get: $\det_n(I + A) = \det(I+A)\exp\left(\sum_{j=1}^{n-1}(-1)^j j^{-1}\mathrm{Tr}(A^j)\right)$. Finally 5) is true by 1), and, noting that since the infinite product converges, the determinant is 0 if and only one of the terms in the infinite product is 0 if and only if $0 \in \sigma(I + A)$ if and only if $I + A$ is non-invertible. ■

These "regularized determinants" will be key to our results.

2.4 Super Linear Algebra and the Berezinian Determinant

As described in [Lap08], one of the motivating ideas that initiated this direction was known relations between $\zeta(s)$ and concepts from physics, in particular, supersymmetry. In this area of physics, objects are classified as either even or odd and there is a new type of commutative law, the so-called supercommutative relation, given by the sign rule: if two elements are switched in a product, then the product has the opposite sign if both switched elements are odd. We will now look at some of the formalism used to handle situations like this in order to get a very useful notation for a supersymmetric determinant.

Definition 26 *A* super vector space V *is a vector space over the field* k *that is graded by* $\mathbb{Z}/2\mathbb{Z}$ *as* $V = V_0 \bigoplus V_1$.

Any element of V_0 is called *even* and will be said to have *parity* 0, while an element of V_1 is called *odd* and have *parity* 1. An *endomorphism* on a super vector space V is a linear transformation $A : V \to V$ that preserves the parity. We can write an endomorphism A, as $A = \begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{pmatrix}$. In this case, the supertrace of A is defined as:

$$\mathrm{str}(A) = \mathrm{Tr}(A_{00}) - \mathrm{Tr}(A_{11})$$

This definition is chosen so that the space of endormorphisms becomes a super algebra that satisfies the supercommutative relation: $\mathrm{str}(AB) = (-1)^{p(A)p(B)}\mathrm{str}(BA)$. Once we have the supertrace, we can define a superdeterminant. We wish to keep the standard relationship to be true that $\det(e^A) = e^{\mathrm{Tr}(A)}$, but in terms of supertrace, we then make the following definition:

Definition 27 *Given a diagonal endormorphism* $A = \begin{pmatrix} A_{00} & 0 \\ 0 & A_{11} \end{pmatrix}$*, with* A_{11} *invertible, the superdeterminant, or Berezinian determinant, is:* $Ber(A) = \det(A_{00}) \cdot \det(A_{11})^{-1}$.

One can define the Berezinian of an arbitrary invertible endormorphism A, but it will not be needed here. The operators that we consider will be given in diagonal block form.

Chapter 3

The Derivative Operator on Weighted Bergman Spaces

In this chapter we will study some properties of the derivative operator in a special family of weighted Bergman spaces. We begin by defining a weighted Bergman space of entire functions.

Definition 28 *Define a* weight function *to be a positive continuous function w on $\mathbb{C}$. Then we define the* weighted L^p spaces *for $1 \leq p \leq \infty$ to be $L^p_w(\mathbb{C})$ to be the space of functions on $\mathbb{C}$ such that $fw \in L^p(\mathbb{C}, d\lambda)$, where λ is the Lebesgue measure on $\mathbb{R}^2$, given the norm $\|f\|_{L^p_w} = \|fw\|_{L^p(\mathbb{R}^2)}$. Next denote by B^p_w to be the subspace of entire functions in L^p_w. B^p_w is called a* weighted Bergman space of entire functions.

Then we have a quick fact about these spaces.

Theorem 3.1 *For $p \geq 1$, B^p_w is a closed subspace of L^p_w and hence is a Banach space. Also, for $p = 2$, B^2_w is a Hilbert space.*

For the rest of this chapter we will be exploring the properties of the differential operator $D = \frac{d}{dz}$ on the space B^p_w, including its properties for particular choices of w and p.

3.1 The Operator D

In this section we focus on the results that we will be using from [AB06]. Consider the following types of weight functions: $w(z) = e^{-\phi(|z|)}$, where ϕ is a non-negative concave function on $\mathbb{R}_+ = [0, \infty)$ such that $w(0) = 0$ and $\lim\limits_{t \to +\infty} \dfrac{\phi(t)}{\log t} = +\infty$. Now we define:

$$a = \lim_{t \to +\infty} \frac{\phi(t)}{t}. \tag{3.2}$$

Then we have the following results in this situation:

Theorem 3.3 *[AB06] Let $1 \leq p \leq \infty$ and w be a weight function with constant a as above:*

1) The differentiation operator $D = \frac{d}{dz}$ is a bounded linear operator on B^p_w

2) $\|D^n\| \leq n! r^{-n} e^{\phi(r)}$, for all $r > 0$, and $n = 1, 2, \ldots$.

Proof. We will prove 2) noting that 1) follows from it. Suppose that $f \in B^w_p$ and $r > 0$. Cauchy's formula for the n^{th} derivative of f reads as:

$$D^n f(z_0) = \frac{n!}{2\pi i} \int_{|z|=r} \frac{f(z_0 + z)}{z^{n+1}} dz \tag{3.4}$$

Now we consider the case $p = \infty$. Let $z_0, z \in \mathbb{C}$ with $|z| = r$. Then since ϕ is subadditive and monotonic, we have a version of the Triangle Inequality: $\phi(|z_0 + z|) \leq \phi(|z_0| + |z|) \leq \phi(|z_0|) + \phi(|z|)$. Also, by definition of the $\infty-$norm we have $\|f\|_{\infty,w} = \sup\limits_{z \in \mathbb{C}} |f(z)| e^{-\phi(|z|)} \geq |f(z_0 +$

$z)|e^{-\phi(|z_0+z|)}$. Solving for $|f(z_0+z)|$ gives the upper bound, $|f(z_0+z)| \le \|f\|_{\infty,w}e^{\phi(|z_0+z|)} \le \|f\|_{\infty,w}e^{\phi(|z_0|)}e^{\phi(|z|)}$. Then by (3.4), we have for any $z \in \mathbb{C}$ that

$$|D^n f(z)| \le n!r^{-n} \sup_{|z|=r} |f(z_0+z)| \le n!r^{-n}\|f\|_{\infty,w}e^{\phi(|z|)}e^{\phi(r)}.$$

Thus $\|D^n f\|_{\infty,w} = \sup_{z\in\mathbb{C}} |D^n f(z)|e^{-\phi(|z|)} \le n!r^{-n}e^{\phi(r)}\|f\|_{\infty,w}$. Therefore $\|D^n\| \le n!r^{-n}e^{\phi(r)}$.

Next, we turn to the case $1 \le p < \infty$. Let $z \in \mathbb{C}$. Applying Hölder's inequality in (3.4) gives:

$$|D^n f(z)| \le \frac{n!}{(2\pi)^{\frac{1}{p}} r^n} \left(\int_0^{2\pi} |f(z+re^{i\theta})|^p d\theta \right)^{\frac{1}{p}}.$$

This leads to

$$\int_{\mathbb{C}} |D^n f(z)|^p e^{-p\phi(|z|)} d\lambda(z) \le \frac{n!^p}{2\pi r^{pn}} \int_0^{2\pi} \left(\int_{\mathbb{C}} |f(z+re^{i\theta})|^p e^{-p\phi(|z|)} d\lambda(z) \right) d\theta.$$

By a change of variable, we can rewrite this as

$$\int_{\mathbb{C}} |D^n f(z)|^p e^{-p\phi(|z|)} d\lambda(z) \le \frac{n!^p}{2\pi r^{pn}} \int_0^{2\pi} \left(\int_{\mathbb{C}} |f(z)|^p e^{-p\phi(|z-re^{i\theta}|)} d\lambda(z) \right) d\theta. \tag{3.5}$$

Using the Triangle Inequality for ϕ we have $|\phi(|z|) - \phi(|z-re^{i\theta}|)| \le \phi(|re^{i\theta}|) = \phi(r)$. Using this in the inner integral gives:

$$\begin{aligned}\int_{\mathbb{C}} |f(z)|^p e^{-p\phi(|z-re^{i\theta}|)} d\lambda(z) &= \int_{\mathbb{C}} |f(z)|^p e^{-p\phi(|z|)} e^{p(\phi(|z|)-\phi(|z-re^{i\theta}|))} d\lambda(z) \\ &\le e^{p\phi(r)} \int_{\mathbb{C}} |f(z)|^p e^{-p\phi(|z|)} d\lambda(z) \le e^{p\phi(r)} \|f\|_{p,w}^p.\end{aligned}$$

Applying this estimate to (3.5) then gives

$$\int_{\mathbb{C}} |D^n f(z)|^p e^{-p\phi(|z|)} d\lambda(z) \le \frac{n!^p}{2\pi r^{pn}} \int_0^{2\pi} e^{p\phi(r)} \|f\|_{p,w}^p d\theta = \frac{n!^p e^{p\phi(r)}}{r^p n} \|f\|_{p,w}^p$$

Thus $\|D^n f\|_{p,w} \le n!r^{-n}e^{\phi(r)}\|f\|_{p,w}$, and it follows that $\|D^n\| \le n!r^{-n}e^{\phi(r)}$. ∎

We also have the following result about the spectrum, $\sigma(D)$, of D.

Theorem 3.6 *[AB06] Under the conditions of theorem 3.3 we have the spectrum of D is $\sigma(D) = \Delta_a := \{z \in \mathbb{C} : |z| \leq a\}$.*

Proof. Let $e_\lambda(z) = e^{\lambda z}$ for $\lambda \in \mathbb{C}$. Clearly we have $De_\lambda = \lambda e_\lambda$ and so e_λ is an eigenvector of the operator D with eigenvalue λ, as long as $e_\lambda \in B_w^p$. However, if $|\lambda| < a$ and we write $z = re^{i\theta}$ and $\lambda = |\lambda|e^{i\beta}$, we have $|e_\lambda(z)e^{-\phi(|z|)}| = |e^{|\lambda|re^{i(\beta+\theta)}-\phi(r)}| = e^{|\lambda|r\cos(\beta+\theta)-\phi(r)} \leq e^{r(|\lambda|-\frac{\phi(r)}{r})}$. But by (3.2), then this function is integrable, so $e_\lambda \in B_w^p$ for $|\lambda| < a$. Then we have $\Delta_a \subset \sigma(D)$. To complete the proof we will show that the spectral radius $r(D) \leq a$. It suffices to show that $r(D) \leq a + \epsilon$ for any $\epsilon > 0$, so let $\epsilon > 0$. Then again using (3.2) we see that there is a $t_0 > 0$ such that $\phi(t) \leq (a + \epsilon)t$ for $t \geq t_0$. Thus by part 2 of Theorem 3.3 we have that $\|D^n\| \leq Cn!r^{-n}e^{(a+\epsilon)r}$ for any $r > 0$, $n = 1, 2, ...$ where the C is a constant depending only on ϵ. Minimizing this expression with respect to r gives the critical value $r = \frac{n}{a+\epsilon}$. Substituting this choice of r gives $\|D^n\| \leq C\frac{n!e^n(a+\epsilon)^n}{n^n}$. Applying Stirling's formula gives that $\|D^n\| \leq f(n)$, where $f(n)$ is asymptotic to a constant times $\sqrt{n}(a+\epsilon)^n$. Thus we have $r(D) = \lim_{n\to\infty} \|D^n\|^{\frac{1}{n}} \leq a + \epsilon$. Thus $r(D) = a$ and $\sigma(D) = \Delta_a$. ■

To further study this operator, we restrict to the special case where $p = 2$ where we actually have a Hilbert space with $(f, g) = \int_{\mathbb{R}^2} \overline{f}ge^{-2\phi(|z|)}d\lambda$. It is convenient to have a particular simple orthonormal basis to deal with, and, since we are dealing with entire functions that are guaranteed to have convergent power series, it makes sense to look at polynomials to try to find this orthonormal basis. It turns out that all we need are monomials.

Theorem 3.7 *[AB06] There exist constants c_n such that $\{u_n(z)\}$, where $u_n(z) = c_n z^n$, forms an orthonormal basis for B_w^2.*

Proof. First note that $(z^n, z^m) = 0$ if $n \neq m$. This is by a simple calculation using the fact that the weight function is radial, so that using polar coordinates, we have $(z^n, z^m) = \int_0^{2\pi} \int_0^\infty r^n e^{-in\theta} r^m e^{im\theta} e^{-2\phi(r)} r dr d\theta = \int_0^{2\pi} e^{i(m-n)\theta} d\theta \int_0^\infty r^{n+m+1} e^{-2\phi(r)} dr = 0$ if $n \neq m$. Note that the integral on r converges for any $n, m \in \mathbb{N}_0$ by the properties of our weight function. Thus the monomials form an orthogonal set. This orthogonal set is complete because every entire function has a convergent power series on $\mathbb{C}$. Thus if we choose $c_n = \frac{1}{\|z^n\|}$ we normalize our set and $\{u_n\}$ is an orthonormal basis for B_w^2. ■

Now we specialize further by choosing the weight function given by $w(z) = e^{-|z|^\alpha}$ for some $\alpha \in \mathbb{R}$ with $0 < \alpha \leq 1$. We will call the resulting Hilbert space $H_\alpha := B_w^2$. In this case we can actually find the constants c_n explicitly.

Theorem 3.8 *[AB06] If $0 < \alpha \leq 1$, then for $n \in \mathbb{N}_0$, we have that:*

$$\|z^n\|_{H_\alpha} = \frac{2\pi}{\alpha} 2^{-\frac{2}{\alpha}(n+1)} \Gamma\left[\frac{2}{\alpha}(n+1)\right]$$

Proof. Computing the norm in H_α gives:

$$\|z^n\|_{H_\alpha}^2 = \int_{\mathbb{C}} |z|^n e^{-2r^\alpha} dz = \int_0^{2\pi} 1 d\theta \int_0^\infty r^{2n+1} e^{-2r^\alpha} dr$$

For the integral over r, we make the change of variable $x = 2r^\alpha$, which changes the integral into:

$$\|z^n\|_{H_\alpha}^2 = 2\pi \int_0^\infty \left(\left(\frac{x}{2}\right)^{\frac{1}{\alpha}}\right)^{2n+1} e^{-x} \frac{1}{2\alpha(\frac{x}{2})^{\frac{\alpha-1}{\alpha}}} dx = \frac{2\pi}{\alpha} 2^{-\frac{2}{\alpha}(n+1)} \int_0^\infty x^{\frac{2(n+1)}{\alpha}-1} e^{-x} dx$$

However, the final integral is simply $\Gamma(\frac{2}{\alpha}(n+1))$. ■

Thus we can simply take the normalizing constants c_n to be the square root of the reciprocal of the formula for $\|z^n\|^2$ given above. Now examining the action of D on a typical basis element $u_n(z)$, we see that: $Du_n = D(c_n z^n) = nc_n z^{n-1} = \frac{nc_n}{c_{n-1}} u_{n-1}$. We thus obtain the following representation of D:

Theorem 3.9 *The operator D is a weighted backward shift on H_α taking a sequence of coefficients (a_n), where $f(z) = \sum_{n=0}^{\infty} a_n u_n(z)$, to $(\gamma_n a_{n+1})$, where $\gamma_n^2 = 2^{\frac{2}{\alpha}} \frac{(n+1)^2 \Gamma(\frac{2}{\alpha}(n+1))}{\Gamma(\frac{2}{\alpha}(n+2))}$.*

Proof. Using the last calculation and writing $f(z) = \sum_{n=0}^{\infty} a_n u_n$, we obtain:

$$Df(z) = \sum_{n=0}^{\infty} a_n \frac{nc_n}{c_{n-1}} u_{n-1} = \sum_{n=0}^{\infty} \frac{(n+1)c_{n+1}}{c_n} a_{n+1} u_n = \sum_{n=0}^{\infty} \gamma_n a_{n+1} u_n,$$

where $\gamma_n = \frac{(n+1)c_{n+1}}{c_n}$. It follows, using the previously calculated formula for c_n, that

$$\gamma_n^2 = \frac{(n+1)^2 c_{n+1}^2}{c_n^2} = \frac{(n+1)^2 \|z^n\|^2}{\|z^{n+1}\|^2} = 2^{\frac{2}{\alpha}} \frac{(n+1)^2 \Gamma(\frac{2}{\alpha}(n+1))}{\Gamma(\frac{2}{\alpha}(n+2))}.$$

■

The last fact we will need from [AB06] is to apply the standard asymptotic for the Gamma function to obtain $\gamma_n \sim c \cdot n^{1-\frac{1}{\alpha}}$ as $n \to \infty$, where c is a positive constant. Thus if $0 < \alpha < 1$, then $\gamma_n \to 0$ as $n \to \infty$.

3.2 Further Properties of D

Since this operator D will be the basis for the rest of the work in this thesis, we will continue to explore its properties beyond what is given in [AB06]. As a first step we will calculate the adjoint D^*.

Theorem 3.10 *Given $f \in H_\alpha$, let $f = \sum_{n=0}^{\infty} a_n u_n$ be its expansion in terms of the orthonormal basis. The adjoint of D^* is a weighted forward shift giving $D^*(a_n) = (\gamma_{n-1} a_{n-1})$.*

Proof. To calculate D^*, write $D^*f = \sum_{n=0}^{\infty} b_n u_n$. Since $\{u_n\}$ is an orthonormal basis we find the n^{th} coefficient of D^*f is equal to $(D^*f, u_n) = (f, Du_n) = (f, \gamma_{n-1}u_{n-1}) = \gamma_{n-1}a_{n-1}$. Thus we have $b_n = \gamma_{n-1}a_{n-1}$ for each $n \in \mathbb{N}$ and so D^* acts on the sequence of coefficients (a_n) as a weighted forward shift $(a_n) \mapsto (\gamma_{n-1}a_{n-1})$. ■

Now that we have the adjoint, we can immediately see that D is not self-adjoint as D is a backward shift and D^* is a forward shift. However, the following calculation with $f(z) \equiv 1$ shows that it is not even normal: $D^*Df = D^*0 = 0$, but $DD^*f = D(\gamma_0 z) = \gamma_0$. This shows that we cannot apply the functional calculus for normal operators that was used in [HL16], which is why, in Ch. 2, we setup the Riesz functional calculus, which is valid for bounded operators like D.

Next, we will use the asymptotic $\gamma_n \sim c \cdot n^{1-\frac{1}{\alpha}}$ to determine which trace ideals D will belong to, depending on α.

Theorem 3.11 *The operator D is compact on H_α for any $0 < \alpha < 1$, trace class for any $0 < \alpha < \frac{1}{2}$, Hilbert Schmidt for any $0 < \alpha < \frac{2}{3}$, and, in general, $D \in \mathcal{J}_p$ if $\alpha < \frac{p}{p+1}$ for any $p \in \mathbb{N}$.*

Proof. Let $0 < \alpha < 1$, and let $E_N : H_\alpha \to H_\alpha$ that takes a power series $\sum_{n=1}^{\infty} a_n u_n \mapsto \sum_{n=1}^{N} a_n \lambda_{n-1} u_{n-1}$, which is the composition DP_N of the derivative operator D with the projection onto the subspace of polynomials at most degree N, P_N. Each E_N is of finite rank, in fact, the range of E_N has dimension N. We claim that the norm limit of E_N is $D = \frac{d}{dz}$. Note that $\|D - E_N\| = \sup_{n>N}\{\lambda_{n-1}\}$. But $\lambda_n \sim c \cdot n^{1-\frac{1}{\alpha}} \to 0$, as $n \to \infty$, for any $0 < \alpha < 1$. Thus E_N converges to D in norm and therefore D is compact. Furthermore, we can write $D = \sum_{n=1}^{\infty} \lambda_{n-1}(u_n, \cdot)u_{n-1}$ so λ_{n-1}^* are the singular values of D. To determine when λ_{n-1}^* are in l^p, we use the Limit Comparison Test to compare $\sum_{n=1}^{\infty}(\lambda_{n-1}^*)^p$ with $\sum_{n=1}^{\infty}(n^{1-\frac{1}{\alpha}})^p$, which converges if and only if $p\left(1 - \frac{1}{\alpha}\right) < -1$. Solving this gives $\alpha < \frac{p}{p+1}$. Therefore $D \in \mathcal{J}_p$ if $\alpha < \frac{p}{p+1}$ and, in particular, is trace class if $p < \frac{1}{2}$ and Hilbert Schmidt if $p < \frac{2}{3}$.

■

From now on, we will fix an α with $0 < \alpha < \frac{1}{2}$ and simply refer to H_α as H. In this case, we have the following spectrum for D.

Theorem 3.12 *We have $\sigma(D) = \sigma_p(D) = \{0\}$.*

Proof. From the previous section we know that $\sigma(D) = \Delta_a$, where $a = \lim_{t\to\infty} \frac{\phi(t)}{t}$. Here we have $\phi(t) = t^\alpha$ for $0 < \alpha < \frac{1}{2}$. Thus we have that $\lim_{t\to\infty} \frac{t^\alpha}{t} = \lim_{t\to\infty} t^{\alpha-1} = 0$. Thus $a = 0$ and $\sigma(D) = \Delta_0 = \{0\}$. However, we also know that $f(z) \equiv 1 \in H_\alpha$, so that D has the eigenvector f corresponding to the eigenvalue 0 and the point spectrum of D is also $\sigma_p(D) = \{0\}$. ■

Finally we turn to considering the set of operators $\{e^{-sD}\}_{s\in\mathbb{C}}$. We compare this to the result for ∂_c obtained in [HL16] and mentioned in chapter 1. This theorem will show that D is the infinitesimal shift (on the complex plane).

Theorem 3.13 *The set $\{e^{-sD}\}_{s\geq 0}$ gives the group of translation operators on H.*

Proof. First note that since any $f \in H$ is an entire function, we have the convergent power series representation: $f(z-s) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z)}{n!}(-s)^n$ for any $z, s \in \mathbb{C}$. Thus

$$e^{-sD} f(z) = \sum_{n=0}^{\infty} \frac{1}{n!}(-sD)^n f(z) = \sum_{n=0}^{\infty} \frac{1}{n!}(-s)^n \frac{d^n}{dz^n} f(z) = f(z-s)$$

This shows that e^{-sD} just acts as translation by s on the space H. From this expression we also see that $\lim_{s \searrow} \|e^{-sD} f - f\| \to 0$. ∎

Chapter 4

Quantized Number Theory

Now that we have a nice differential operator D to work with on a suitable Hilbert space H we can build the basis for our "quantized number theory" by defining a new class of operators.

4.1 Local Operators

First, we give the standard definition of a quasinilpotent operator on a Hilbert space.

Definition 29 *If T is an operator on a Hilbert space H, then we say T is* quasinilpotent *if $\sigma(T) = \{0\}$, where $\sigma(T)$ denotes the spectrum of T.*

From the previous chapter, we see that D on the weighted Bergman space H_α is quasinilpotent when $0 < \alpha < 1$. For our work, we'd like to generalize the notion of quasinilpotentcy to be able to get local information at any point in the complex plane. This can be accomplished by having an operator whose spectrum is a single specified complex number.

Definition 30 *Let $z_0 \in \mathbb{C}$. A bounded linear operator S on a separable Hilbert space H is called a z_0-*local operator *if $\sigma(S) = \{z_0\}$. Thus a quasinilpotent operator is a 0-local operator. We denote by $Loc(H)$ the set of all z_0-local operators on H for any $z_0 \in \mathbb{C}$.*

Now $Loc(H)$ is not necessarily a subspace of $B(H)$, but we do have closure under some particular operations to be specified. We begin with the following result.

Lemma 31 *For a bounded linear operator S on a separable Hilbert space, the following are equivalent:*

1) S is a z_0-local operator

2) $S = T + z_0 I$ for some 0-local operator T

and if $z_0 \neq 0$

3) $S = z_0 U + z_0 I$ for some 0-local operator U.

Proof. ($1 \Rightarrow 2$) Suppose that S is a z_0-local operator. Let $T = S - z_0 I$. Then by the spectral mapping theorem applied to the function $f(z) = z - z_0$, $\sigma(T) = \{0\}$. Thus T is 0-local, which implies that $S = T + z_0 I$. ($2 \Rightarrow 3$) Suppose that $z_0 \neq 0$ and $S = T + z_0 I$ for a 0-local T. Let $U = \frac{1}{z_0} T$. Then $\sigma(U) = \{0\}$, so that U is also 0-local. Thus $S = z_0 U + z_0 I$. (2 or $3 \Rightarrow 1$) If $S = T + z_0 I$ or $S = z_0 U + z_0 I$, then applying Spectral Mapping Theorem to either $f(z) = z + z_0$ or $f(z) = z_0 z + z_0$ we obtain $\sigma(S) = \{z_0\}$ and so S is a z_0-local operator. ■

These local operators allow us to use the spectral mapping theorem to prove the following lemma which focuses on particular meromorphic functions. More precisely, we have the following lemma.

Lemma 32 *Suppose that f is a meromorphic function and that z_0 is not a pole of f. Let S be any z_0-local operator on a separable Hilbert space H. Then $f(S)$ is a $f(z_0)$-local operator. Consequently $f(z_0) \neq 0$ if and only if $f(S)$ is invertible.*

Proof. If z_0 is not a pole of f, then f is holomorphic on a neighborhood of $\sigma(S) = \{z_0\}$. Therefore, we can apply the Spectral Mapping Theorem to conclude $\sigma[f(S)] = \{f(z_0)\}$. Therefore $f(S)$ is a $f(z_0)$-local operator. Further, by definition, $f(S)$ is invertible if and only if $0 \notin \sigma(f(S))$, and we conclude that $f(z_0) \neq 0$ if and only if $f(S)$ is invertible. ■

One nice property of $Loc(H)$ is that we can define a spectrum function whose range is the complex numbers rather than subsets of the complex plane.

Definition 33 *Define a map $\hat{\sigma} : Loc(H) \to \mathbb{C}$, where if S is a z_0-local operator we have $\hat{\sigma}(S) = z_0$. Call $\hat{\sigma}$ the local spectral map.*

4.2 Quantized Complex Numbers

For the remainder of this chapter, we will use $z_0D + z_0I$ as our standard z_0-local operator where $D = \frac{d}{dz}$ is the derivative operator on the weighted Bergman space H_α with $0 < \alpha < \frac{1}{2}$ from earlier. This gives us the following result:

Theorem 4.1 *The map $\phi : \mathbb{C} \to B(H)$ given by $z \mapsto zD + zI$ gives an injective linear map between the complex numbers and the space of bounded linear operators on H. The left inverse of ϕ is the local spectral map $\hat{\sigma}$.*

Proof. First note that $\phi(z) \in B(H)$ for every $z \in C$. Then given $z_1, z_2, \alpha \in \mathbb{C}$ we have $\phi(z_1 + z_2) = (z_1 + z_2)D + (z_1 + z_2)I = (z_1 D + z_1 I) + (z_2 D + z_2 I) = \phi(z_1) + \phi(z_2)$ and $\phi(\alpha z_1) = (\alpha z_1)D + (\alpha z_1)I = \alpha(z_1 D + z_1 I) = \alpha\phi(z_1)$ so that ϕ is linear. Now if $\hat{\sigma}$ is the local spectral map then $\hat{\sigma}(\phi(z_1)) = \hat{\sigma}(z_1 D + z_1 I) = z_1$, since $z_1 D + z_1 I$ is z_1-local. Therefore, ϕ is injective. ■

We can now make the following definition.

Definition 34 *We define the* quantized complex numbers *by* $\hat{\mathbb{C}} := \phi(\mathbb{C}) = \{zD + zI : z \in \mathbb{C}\}$.

This gives us a subspace of $B(H)$ that is isomorphic to $\mathbb{C}$ as vector spaces and gives a standard setting to replace the complex variable z in meromorphic functions $f(z)$ to get $f(zD + zI)$.

4.3 Quantized Lattices and Elliptic Curves

Recall that a *lattice* in $\mathbb{C}$ is any set of the form $\Lambda = \{m\omega_1 + n\omega_2 : m, n \in \mathbb{Z}\}$ where ω_1, ω_2 are $\mathbb{R}$-linearly independent. We then define:

Definition 35 *A* quantized lattice *in $\hat{\mathbb{C}}$ is any set of the form $\hat{\Lambda} = \phi(\Lambda)$ where Λ is a lattice in $\mathbb{C}$.*

Since ϕ is injective, this gives a $1-1$ correspondence between lattices and quantized lattices. Now that we have quantized lattices we can create quantized complex tori as follows, recalling that a complex torus can be thought of as $\mathbb{C}/\Lambda$ for some lattice Λ.

Definition 36 *A* quantized complex torus *is* $\hat{\mathbb{C}}/\hat{\Lambda}$ *for some quantized lattice* $\hat{\Lambda}$.

Next we turn to concept of an elliptic curve. The standard definition is as follows.

Definition 37 *An* elliptic curve *is a smooth, projective algebraic curve of genus one.*

However, there is another characterization of elliptic curves over $\mathbb{C}$ that is more amenable to being quantized due to the following theorem.

Theorem 4.2 *Given a lattice* Λ *in* $\mathbb{C}$*, there is an elliptic curve* E *and a bijective map* $\mathbb{C}/\Lambda \to E$ *given by* $z + \Lambda \mapsto [\mathcal{P}(z), \mathcal{P}'(z), 1]$*. Conversely, given an elliptic curve,* E*, there is a lattice* Λ *that is mapped bijectively onto it in the same way. That is, there is a* $1-1$ *correspondence between elliptic curves over* $\mathbb{C}$ *and complex tori* $\mathbb{C}/\Lambda$*.*

Now that we have this we can define a quantized elliptic curve to be a quantized complex torus $\hat{\mathbb{C}}/\hat{\Lambda}$. This leads to the following relationship between elliptic curves and quantized elliptic curves.

Theorem 4.3 *The map* $\phi : \mathbb{C} \to \hat{\mathbb{C}}$ *induces a bijection between elliptic curves and quantized elliptic curves.*

Proof. Define the map ψ : {elliptic curves} $\to$ {quantized elliptic curves} as follows. Given an elliptic curve $\mathbb{C}/\Lambda$, find the associated quantized lattice $\hat{\Lambda} = \phi(\Lambda)$ and then consider the quantized elliptic curve $\hat{\mathbb{C}}/\hat{\Lambda}$. Define $\psi(\mathbb{C}/\Lambda) = \hat{\mathbb{C}}/\hat{\Lambda}$ To show that ψ is surjective, consider any quantized elliptic curve $\hat{\mathbb{C}}/\hat{\Lambda}$. Then $\hat{\Lambda}$ is a quantized lattice and so is of the form $\phi(\Lambda)$ for some lattice Λ. However every lattice Λ gives an elliptic curve $\mathbb{C}/\Lambda$, as in the previous theorem. Hence, there is an elliptic curve that ψ will take to the given quantized elliptic curve, and ψ is surjective. For injectivity, if $\hat{\mathbb{C}}/\hat{\Lambda_1} = \hat{\mathbb{C}}/\hat{\Lambda_2}$ then $\hat{\Lambda_1} = \hat{\Lambda_2}$. But then since every quantized lattice is the image of a lattice in $\mathbb{C}$ under the injective map ϕ this means ψ is injective as well. ■

The above method of quantizing elliptic curves does not give us a larger class of objects to study, but still lets us work inside $B(H)$. One interesting direction to consider in the future could then be to start with two noncommuting quasinilpotent operators in $B(H)$ and then create a quantized noncommutative analogue of $\mathbb{C}^2$ and study objects there, but this direction will not be pursued in this work.

4.4 Quantized Modular Forms

Once we have the idea of a quantized elliptic curve, we can turn our attention to the problem of constructing a rigorous definition for a suitable quantized version of modular forms. To do so, we recall the normal definition of a modular form.

Definition 38 *Let $\mathcal{H} = \{z \in \mathbb{C} : \Im(z) > 0\}$ be the upper half plane. Let $k \in \mathbb{Z}$. A function $f : \mathcal{H} \to \mathbb{C}$ is called a* modular form of weight k *if*

1) f is holomorphic on $\mathcal{H}$,

2) f is holomorphic at ∞, and

3) $f(\gamma(z)) = (cz+d)^k f(z)$ *for every* $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z})$ *and for every* $z \in \mathbb{C}$.

To quantize this definition, the only modification that we will make is in the third requirement. We will treat it as an equality of operators rather than complex numbers, where we will write $\hat{z}$ to mean $\phi(z)$, where $\phi : \mathbb{C} \to B(H)$ is the map used in the previous section. This gives the definition:

Definition 39 *Let* $\mathcal{H} = \{z \in \mathbb{C} : \Im(z) > 0\}$ *be the upper half plane. Let* $k \in \mathbb{Z}$. *A function* $f : \mathcal{H} \to \mathbb{C}$ *is a* quantized modular form of weight k *if*

1) f *is holomorphic on* $\mathcal{H}$,

2) f *is holomorphic at* ∞, *and*

3) $f(\gamma(\hat{z})) = (c\hat{z}+d)^k f(\hat{z})$ *for every* $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z})$ *and for every* $z \in \mathbb{C}$.

Note that in the above definition, since k can be negative, we need verify that $c\hat{z} + d$ is an invertible operator for a quantized elliptic curve to be well-defined. However, $\hat{\sigma}(c\hat{z} + d) = \{cz + d\}$, which never includes 0 for any $z \in \mathcal{H}$ and $c, d \in \mathbb{Z}$. Thus $c\hat{z} + d$ is always an invertible operator in the above definition. Unfortunately, our quantization once again does not yield a larger class of objects than regular modular forms as we see next.

Theorem 4.4 *A function* $f : \mathcal{H} \to \mathbb{C}$ *is a modular form of weight* k *if and only if it is a quantized modular form of weight* k.

Proof. Conditions 1) and 2) of the respective definitions are the same, so we need only focus on 3). By the functional calculus, if $f(\gamma(z)) = (cz + d)^k f(z)$ are equal as functions then applying each to the same bounded operator $\hat{z}$ must give the same operator. This is because the operator obtained by the functional calculus is unique. That is, we have $f(\gamma(\hat{z})) = (c\hat{z} + d)^k f(\hat{z})$, and, thus if f is a modular form of weight k, then it must also be a quantized modular form of weight k. On the other hand, if f is a quantized modular form and satisfies $f(\gamma(\hat{z})) = (c\hat{z} + d)^k f(\hat{z})$, then the spectrum of both operators must be equal. However, by the Spectral Mapping Theorem that means: $\{f(\gamma(z))\} = \sigma(f(\gamma(\hat{z}))) = \sigma((c\hat{z} + d)^k f(\hat{z})) = \{(cz + d)^k f(z)\}$. Thus we have equality as complex numbers $f(\gamma(z)) = (cz + d)^k f(z)$. Thus if f is a quantized modular form of weight k it must also be a modular form of weight k. ■

The method of quantizing shown here implies that quantized elliptic curves and quantized modular forms are essentially the same as their usual counterparts, but perhaps there is a more interesting way to quantize these objects that will give new information. This will not be pursued in this work, but instead a different use of the operator D will be studied.

Chapter 5

Cohomology Theory of a Meromorphic Function

5.1 The operator of a meromorphic function: A first attempt

We now consider a particular choice of the family of analytic functions $\phi_\tau(z) = z + \tau$. This gives us operators $D_\tau := \phi_\tau(D) = D + \tau I$ for which the following lemma holds.

Lemma 40 *For any $\tau \in \mathbb{C}$, $D_\tau \in B(H)$ with spectrum $\sigma(D_\tau) = \{\tau\}$. If $\tau \neq 0$, then D_τ is invertible and $D_\tau^{-1} \in B(H)$.*

Proof. Applying the functional calculus on bounded operators along with the Spectral Mapping Theorem to the operator D and the function $\phi_\tau(z)$ gives a bounded operator D_τ with spectrum $\sigma(D_\tau) = \phi_\tau(\{0\}) = \{\tau\}$. Furthermore, if $\tau \neq 0$, then $0 \notin \sigma(D_\tau)$ and it follows that D_τ has a bounded inverse. ■

This gives us a family of operators, each of whose spectra are each a single point, which can be any complex number. Recall that in the situation of the cohomology theory that helped prove the Weil conjectures, that we would like an operator whose eigenvalues on different cohomology spaces are the zeros and poles of the zeta function we are interested in. In order to obtain operators whose spectrum can represent the zero or pole set of a meromorphic function we use the following construction. If $\mathcal{Z} = \{z_1, z_2, ...\}$ is a discrete multiset of complex numbers, let H_n be a copy of the weighted Bergman space H and associate an operator D_n to be D_{z_n} on H_n. Finally, define the Hilbert space $H_{\mathcal{Z}} = \bigoplus_n H_n$ with operator $D_{\mathcal{Z}} = \bigoplus_n D_n$. This gives:

Theorem 5.1 *The operator $D_{\mathcal{Z}}$ has spectrum $\sigma(D_{\mathcal{Z}}) = \overline{\{z_1, z_2, ...\}}$ and for each $z = z_i$, the number of linearly independent eigenvectors of z_i for $D_{\mathcal{Z}}$ in $H_{\mathcal{Z}}$ is equal to the multiplicity of z in $\{z_1, z_2, ...\}$.*

Proof. For each $n \in \mathbb{N}$, let $e_n \in H_{\mathcal{Z}}$ be the element which is the constant function, with value 1 in the n^{th} component, and 0 in every other component. Then $D_{\mathcal{Z}} e_n = z_n e_n$ and so z_n is an eigenvalue with eigenvector e_n. Suppose $z_{n_1} = z_{n_2} = \cdots = z_{n_k} = z$. Then z is an eigenvalue with eigenvectors $e_{n_1}, e_{n_2}, ...e_{n_k}$ and so there are at least as many linearly independent eigenvectors of z for $D_{\mathcal{Z}}$ as the multiplicity of z in the multiset. Next, recall that the only eigenvalue of $\frac{d}{dz}$ on H is 0. Thus, the only eigenvalue of D_{z_n} is z_n. Suppose now that $D_{\mathcal{Z}} x = zx$ for some x, we must either have the n^{th} component of x being 0 or $z = z_n$ and so there cannot be any more linearly independent eigenvectors of z for $D_{\mathcal{Z}}$. Now that we know z_i is an eigenvalue of $D_{\mathcal{Z}}$ for each i, we know that $\overline{\{z_1, z_2, ...\}} \subset \sigma(D_{\mathcal{Z}})$. Now let $\lambda \in \mathbb{C} - \overline{\{z_1, z_2, ...\}}$. Then $d = \inf_{n \geq 0} |\lambda - z_n| > 0$. Since $D = \frac{d}{dz}$ is quasinilpotent, $r(D) = 0$

and so there is a N such that for every positive integer $k \geq N$, we have $\|D^k\| < \left(\frac{d}{2}\right)^k$. Then on the n^{th} component of $H_{\mathcal{Z}}$ we have $\sum_{k=0}^{\infty} \frac{D^k}{|\lambda - z_n|}$ is absolutely convergent because

$$\sum_{k=N}^{\infty} \frac{\|D^k\|}{|\lambda - z_n|} \leq \sum_{k=N}^{\infty} \frac{\left(\frac{d}{2}\right)^k}{d^k} = \frac{1}{2^{N-1}}.$$

Then we can calculate the inverse on the n^{th} component via the absolutely convergent series:

$$(\lambda I - D_{z_n})^{-1} = ((\lambda - z_n)I - D)^{-1} = \frac{1}{\lambda - z_n} \sum_{k=0}^{\infty} \frac{D^k}{\lambda - z_n}$$

Further, by the same estimate $\|(\lambda I - D_{z_n})^{-1}\| \leq C$ uniformly in n, where $C = \sum_{k=0}^{N} \frac{\|D^k\|}{d^k} + 2^{1-N}$. Therefore $\bigoplus_n (\lambda I - D_{z_n})^{-1} \in B(H_{\mathcal{Z}})$ and so $(\lambda I - D_{\mathcal{Z}})^{-1}$ exists and is bounded. That is, $\lambda \in \rho(D_{\mathcal{Z}})$. Therefore $\sigma(D_{\mathcal{Z}}) = \overline{\{z_1, z_2, ...\}}$. ■

Thus, $D_{\mathcal{Z}}$ contains all of the information from the multiset $\{z_1, z_2, ...\}$ contained in its spectrum. If we then consider the multiset to be the zeros and poles of a meromorphic function $f(z)$ then the operator $D_{\mathcal{Z}}$ contains these pieces of information of this function. However, we cannot use the determinant formulas for operators given earlier for the operator $\mathcal{D}_{\mathcal{Z}}$ to recover $f(z)$ as a whole, because with this formulation $\mathcal{D}_{\mathcal{Z}}$ is not trace class. Even looking at just a single one of the terms $D_\tau = D + \tau I$ is not compact, let alone trace class, because our Hilbert space is infinite dimensional. So while this formulation gives us some tools to study functions, we need to modify the method. If we restrict each D_τ to its eigenspace, E, the space of constant functions, thus giving us a compact operator. We also need to make a second adjustment from the original idea. Any zero or pole set of a meromorphic function will be a discrete set, and hence if there are infinitely many zeros or poles they must tend to ∞. This would then imply that the operator $D_{\mathcal{Z}}$ described here is unbounded. What allows us to repair this problem and simultaneously recover the given function $f(z)$ using determinants, is to have our set $\mathcal{Z}$ be the reciprocals of the zeros rather than the zeros themselves, and similarly for the poles. Thus the operator $D_{\mathcal{Z}}$ becomes bounded, compact, and in some cases, in a trace ideal for a wide range of functions of interest as described in the rest of this thesis.

5.2 Refining the operator of a meromorphic function

First, let $\mathcal{Z} = (z_n)$ be a sequence of complex numbers. Let $D_n = D + z_n I$ be the operator in the previous section restricted to the subspace of constant functions E. Let $\mathcal{D}_{\mathcal{Z}} = \bigoplus_n D_n$ on the space $\mathcal{E}_{\mathcal{Z}} = \bigoplus_n E$ which is a subspace of $H_{\mathcal{Z}}$ from the last section. So in actuality, this new definition of $\mathcal{D}_{\mathcal{Z}}$ is just the restriction of the operator given in the previous section to $\mathcal{E}_{\mathcal{Z}}$. First we note that this restriction still retains the main property from the last section.

Theorem 5.2 *For each $n \in \mathbb{N}$, each z_n is an eigenvalue of $\mathcal{D}_{\mathcal{Z}}$ and the number of linearly independent eigenvectors associated to z_n is equal to the number of times z_n occurs in the sequence $\mathcal{Z}$. Further, $\sigma(\mathcal{D}_{\mathcal{Z}}) = \overline{\{z_n : n = 1, 2, 3, ...\}}$*

Proof. Let e_n be the eigenfunctions from the previous proof. Then since e_n is constant in each coordinate, $e_n \in \mathcal{E}_{\mathcal{Z}}$. Thus when restricted to the space of functions constant on each coordinate, $\mathcal{D}_{\mathcal{Z}}$ retains all of its eigenvalues and eigenvectors from before. Finally, we note that $\sigma(D_{z_n}) = \{z_n\}$ from which it follows that $\sigma(\mathcal{D}_{\mathcal{Z}}) = \overline{\{z_n : n = 1, 2, 3, ...\}}$ by Theorem 2.5 2) since each D_n is normal. ∎

The next theorem shows that this restriction of the operator will truly give us what we need for our quantized number theory framework.

Theorem 5.3 *We have the following relationships between an infinite sequence* $\mathcal{Z} = (z_n)$ *and the associated operator* $\mathcal{D}_{\mathcal{Z}}$.

1) $\mathcal{D}_{\mathcal{Z}}$ *is bounded iff* (z_n) *is a bounded sequence.*

2) $\mathcal{D}_{\mathcal{Z}}$ *is self-adjoint iff* $z_n \in \mathbb{R}$ *for all* n. *3)* $\mathcal{D}_{\mathcal{Z}}$ *is compact iff* $\lim_{n\to\infty} z_n = 0$.

4) $\mathcal{D}_{\mathcal{Z}}$ *is Hilbert Schmidt iff* $\sum_{n=1}^{\infty} |z_n|^2 < \infty$.

5) $\mathcal{D}_{\mathcal{Z}}$ *is trace class iff* $\sum_{n=1}^{\infty} |z_n| < \infty$.

6) For $p \geq 1$, $\mathcal{D}_{\mathcal{Z}} \in \mathcal{J}_p$ *iff* $\sum_{n=1}^{\infty} |z_n|^p < \infty$.

If (z_n) *is a finite sequence then* $\mathcal{D}_{\mathcal{Z}}$ *is bounded, compact, and in* $\mathcal{J}_p$ *for each* $p \geq 1$.

Proof. Since $\|D_n\| = |z_n|$, for each $n \in \mathbb{N}$, we have $\|\mathcal{D}_{\mathcal{Z}}\| = \sup_n |z_n|$. Then $\mathcal{D}_{\mathcal{Z}}$ is bounded iff (z_n) is a bounded sequence. For 2), consider the sequence of operators $\overline{D}_N = \bigoplus_{n=1}^{N} D_n$, for $N \in \mathbb{N}$, as an operator on $\mathcal{E}_{\mathcal{Z}}$ by letting it act as multiplication by 0 on the remaining components. Thus $\overline{D}_N$ is a finite rank operator on $\mathcal{E}_{\mathcal{Z}}$ for each N. Then $\|\mathcal{D}_{\mathcal{Z}} - \overline{D}_N\| = \sup_{k>N} |z_k|$ and so if $\lim_{n\to\infty} z_n = 0$, we then have that $\mathcal{D}_{\mathcal{Z}}$ is the norm limit of finite rank operators and thus is compact. On the other hand, if $\lim_{n\to\infty} z_n \neq 0$ then $\{e_n\}$ is a bounded sequence of vectors such that $\{\mathcal{D}_{\mathcal{Z}} e_n\}$ has no convergent subsequence. Thus, $\mathcal{D}_{\mathcal{Z}}$ is not compact. For 3) and 4) assume that $\mathcal{D}_{\mathcal{Z}}$ is compact. Then since $\mathcal{D}_{\mathcal{Z}} e_n = z_n e_n$ and the fact that $\{e_n\}$ forms an orthonormal basis for $\mathcal{E}_{\mathcal{Z}}$ we know the singular values of $\mathcal{E}_{\mathcal{Z}}$ are $z_1^*, z_2^*, \ldots$. Thus $\mathcal{E}_{\mathcal{Z}}$ is Hilbert Schmidt iff $\sum_{n=1}^{\infty} |z_n|^2 < \infty$, and, trace class iff $\sum_{n=1}^{\infty} |z_n| < \infty$ and more generally in $\mathcal{J}_p$ iff $\sum_{n=1}^{\infty} |z_n|^p < \infty$. If (z_n) is a finite sequence then $\mathcal{D}_{\mathcal{Z}}$ is actually a finite rank operator and is trivially bounded, compact, and in $\mathcal{J}_p$ for each $p \geq 1$. ■

Now that we have a formulation that can indeed give us a trace class operator we can now state the result we will use to fully recover certain functions of interest.

Corollary 41 *If* $\{z_n\}$ *is a sequence of complex numbers satisfying* $\sum_{n=1}^{\infty} |z_n| < \infty$*, then we have* $\det(I - z\mathcal{D}_{\mathcal{Z}}) = \prod_n (1 - z_n z)$*.*

Proof. This is a direct consequence of the Equation (2.13) for trace class operators of which $\mathcal{D}_{\mathcal{Z}}$ is one when the series is absolutely summable. ■

By the previous corollary, we can now see that we will be getting an entire function out of our construction. Thus if we want to handle meromorphic functions, we will need to handle zeros and poles separately. Also we will want to choose our sequence (z_n) to be the reciprocals of the poles. With this in mind we make the following final construction for our operator of a meromorphic function.

Let $f(z)$ be a meromorphic function on $\mathbb{C}$ with $z = 0$ neither a zero nor a pole of f. Let $\{a_n\}$ be a sequence of the zeros of $f(z)$ and $\{b_n\}$ be a sequence of the poles of $f(z)$, both counting multiplicity. Define the sequences $\mathcal{Z} = (z_n)$ and $\mathcal{P} = (p_n)$ where $z_n = \frac{1}{a_n}$ and $p_n = \frac{1}{b_n}$. Define $H^0 = \mathcal{E}_{\mathcal{Z}}$ and $H^1 = \mathcal{E}_{\mathcal{P}}$ and let $H_{f(z)} = H^0 \bigoplus H^1$ where we consider this as a supersymmetric direct sum. That is, treat $H_{f(z)}$ as a super vector space with even part H^0 and odd part H^1. Define $\mathcal{D}_{\mathcal{Z}}$ and $\mathcal{D}_{\mathcal{P}}$ as before and call $\mathcal{D}_f = \mathcal{D}_{\mathcal{Z}} \bigoplus \mathcal{D}_{\mathcal{P}}$ on H_f. Then the block matrix representation of $\mathcal{D}_{f(z)}$ is $\begin{pmatrix} \mathcal{D}_{\mathcal{Z}} & 0 \\ 0 & \mathcal{D}_{\mathcal{P}} \end{pmatrix}$. Thus we can follow the example of the Berezinian determinant to define the regularized Berezinian determinant of $I - zD_{f(z)}$ as follows:

Definition 42 *With the construction as above, if $\mathcal{D}_{\mathcal{Z}} \in \mathcal{J}_m \setminus \mathcal{J}_{m-1}$ for some $m \geq 1$, and if $\mathcal{D}_{\mathcal{P}} \in \mathcal{J}_n \setminus \mathcal{J}_{n-1}$ for some $n \geq 1$, define the m,n-regularized Berezinian determinant as:*

$$\det_{m,n}(I - z\mathfrak{D}_{f(z)}) = \det_m(I - z\mathcal{D}_{\mathcal{Z}})(\det_n(I - z\mathcal{D}_{\mathcal{P}}))^{-1}$$

In the next chapter, we will examine what this construction accomplishes for several classically important functions in number theory.

Chapter 6

Applications of the Construction to Specific Functions

6.1 Rational Functions

To begin, we look at the simplest type of meromorphic functions: the rational functions. Let $f(z)$ be a rational function. Then we can write $f(z) = z^k g(z)$ for some $k \in \mathbb{Z}$ and further $g(z) = g(0)\frac{\prod_{n=1}^{s}(1 - \frac{z}{a_n})}{\prod_{n=1}^{t}(1 - \frac{z}{b_n})}$ for some finite set $\{a_1, a_2, ..., a_s, b_1, b_2, ..., b_t\}$. Construct the operator $\mathfrak{D}_g$ as given in the previous chapter. The following theorem tells us that our Berezinian determinant exactly recovers the given function f.

Theorem 6.1 *If $f(z) = z^k g(z)$ is a rational function as above then $f(z) = z^k g(0) \det_{1,1}(I - sD_{g(z)})$.*

Proof. Write out $f(z)$ as given in the preceding paragraph. Then consider the finite sequences $\mathcal{Z} = \{\frac{1}{a_1}, \frac{1}{a_2}, ..., \frac{1}{a_s}\}$ and $\mathcal{P} = \{\frac{1}{b_1}, \frac{1}{b_2}, ..., \frac{1}{b_t}\}$. The operators $D_{\mathcal{Z}}$ and $D_{\mathcal{P}}$ are both trivially trace class since both are created from finite sequences. Hence we may apply the $1,1$-regularized Berezian determinant (really just the normal Berezinian determinant since both the components are trace class) we defined to get $\det_{1,1}(I - sD_g) = g(0)z^k \det(I - D_{\mathcal{Z}}) \det(I - D_{\mathcal{P}})^{-1} = \frac{\prod_{n=1}^{s}(1-\frac{z}{a_n})}{\prod_{n=1}^{t}(1-\frac{z}{b_n})} = \frac{g(z)}{g(0)}$ and thus $f(z) = z^k g(0) \det_{1,1}(I - sD_{g(z)})$. ■

6.2 Zeta Functions of Curves Over Finite Fields

Recall that the zeta function of a curve Y over the finite field $\mathbb{F}_q$ is defined as $\zeta(Y, s) = \exp\left(\sum_{n=1}^{\infty} \frac{Y_n}{n} q^{-ns}\right)$. The proof of the Weil conjectures expressed this function as an alternating product of determinants as follows:

$$\zeta(Y, s) = \frac{\det(I - F^* q^{-s} | H^1)}{\det(I - F^* q^{-s} | H^0) \det(I - F^* q^{-s} | H^2)}.$$

One of the Weil conjectures, that $\zeta(Y, s)$ is a rational function of q^{-s}, then followed from this formula. Thus we may apply the result in the previous section about rational functions to obtain:

Theorem 6.2 *Let Y be a smooth, projective, geometrically connected curve over $\mathbb{F}_q$, the field with q elements. Then:* $\zeta(Y, s) = \det_{1,1}\left(I - q^{-s} D_{\zeta(Y, q^{-s})}\right)$.

Proof. Define $\overline{\zeta}(Y, q^{-s}) = \zeta(Y, s)$. Then $\overline{\zeta}(Y, z)$ is a rational function of z. Thus by the rational function result: $\overline{\zeta}(Y, z) = \det_{1,1}(I - zD_{\overline{\zeta}(Y,z)})$ and so replacing back gives: $\zeta(Y, s) = \det_{1,1}\left(I - q^{-s} D_{\zeta(Y, q^{-s})}\right)$

■

6.3 Geometric Zeta Function of a Self-Similar Fractal String

In this section we will consider another type of zeta function which will also become a rational function under a suitable change of variable. Following the treatment in [LvF13], we construct a self-similar fractal string as follows:

Consider a closed interval I of length L, called the *initial interval*. For some $N \geq 2$, let $\Phi_1, \Phi_2, ..., \Phi_N$ be N contraction similitudes mapping I to I, with respective scaling factors $r_1, r_2, ..., r_N$ which we assume to be ordered nonincreasingly, $1 > r_1 \geq r_2 \geq \cdots \geq r_N > 0$. Assume that $\sum_{j=1}^{N} r_j < 1$ and that the images $\Phi_j(I)$ of I, for $j = 1, 2, ..., N$, do not overlap, except possibly at the endpoints. Translate the functions Φ_j so that there is only one nonzero gap, of length g, in $I \setminus \bigcup_{j=1}^{N} \Phi_j(I)$. Further, for simplification purposes, suppose that the length of the initial interval is chosen so that $gL = 1$. Define the self-similar fractal string as the ordinary fractal string with lengths $\left(\prod_{j=1}^{N} r_j^{e_j}\right) gL = \prod_{j=1}^{N} r_j^{e_j}$, where e_j are whole numbers, which are the lengths of the gaps created by the compositions $\prod_{j=1}^{N} \Phi_j^{e_j}(I)$. Now recall that the geometric zeta function of an ordinary fractal string is defined as $\zeta_{\mathcal{L}}(s) = \sum_{n=1}^{\infty} l_n^s$ where $l_1, l_2, ...$ are the lengths associated to the fractal string. We have the following theorem:

Theorem 6.3 *[LvF13] Let $\mathcal{L}$ be a self-similar string with scaling ratios $r_1, r_2, ..., r_N$ and a single gap, which is normalized so that $gL = 1$ as above. Then the geometric zeta function of the string has a meromorphic continuation to the whole complex plane given by:*

$$\zeta_{\mathcal{L}}(s) = \frac{1}{1 - \sum_{j=1}^{N} r_j^s}$$

Self-similar strings can be separated into two types: lattice and nonlattice. A lattice self-similar fractal string is when there exists an $r > 0$, such that for each j, $r_j = r^{n_j}$ for some positive integer n_j. If no such $r > 0$ exists, then $\mathcal{L}$ is said to be nonlattice. Note that by the above theorem that for lattice self-similar strings $\mathcal{L}$, we have that $\zeta_{\mathcal{L}}$ is a rational function of r^s for any $r > 0$ satisfying the lattice property. This leads to the following:

Theorem 6.4 *Let $\mathcal{L}$ be a lattice self-similar fractal string with scaling ratios $r_1, r_2, ..., r_N$ and single gap $g = 1$ as defined above. Let $r > 0$ be such that $r_j = r^{n_j}$ for some positive integer n_j. Then $\zeta_{\mathcal{L}}(s) = \det_{1,1}(I - r^s D_{\zeta_{\mathcal{L}}(r^s)})$.*

Proof. Letting $\overline{\zeta}_{\mathcal{L}}(r^s) = \zeta_{\mathcal{L}}(s)$. Then the formula for the meromorphic extension in the previous theorem gives

$$\overline{\zeta}_{\mathcal{L}}(z) = \frac{1}{1 - \sum_{j=1}^{N} z^{n_j}}$$

which is a rational function of z with $\overline{\zeta}_{\mathcal{L}}(0) = 1$. Thus by the determinant formula for rational functions: $\overline{\zeta}_{\mathcal{L}}(z) = \det_{1,1}(I - zD_{\overline{\zeta}_{\mathcal{L}}(z)})$. Making the change of variable back gives:

$$\zeta_{\mathcal{L}}(s) = \det_{1,1}(I - r^s D_{\zeta_{\mathcal{L}}(r^s)}).$$

■

6.4 The Gamma Function

The next meromorphic function that we will turn our attention to is the Gamma function, $\Gamma(z) = \int_0^\infty x^{z-1}e^{-x}dx$. This function has numerous applications in many branches of mathematics, including our focus - number theory. One point of interest is that this function gives a meromorphic continuation of the factorial function on integers. It also appears in the functional equation for the Riemann zeta function. We have the following well known properties of the Gamma function:

Theorem 6.5 *1)* $\Gamma(z+1) = z\Gamma(z)$.

2) $\Gamma(n) = (n-1)!$ *for* $n \in \mathbb{N}$.

3) $\Gamma(z) = \dfrac{e^{-\gamma z}}{z} \displaystyle\prod_{n=1}^{\infty} \left(\left(1+\frac{z}{n}\right)^{-1} e^{\frac{z}{n}} \right)$

This infinite product representation for $\Gamma(z)$ allows us to now show that we can recover the function from the determinant of the operator construction we have laid out.

Theorem 6.6 *We have* $\Gamma(z) = \frac{e^{-\gamma z}}{z} \det_{1,2}(I - zD_{z\Gamma(z)})$.

Proof. We will apply our construction to the function $g(z) = z\Gamma(z)$. This function has simple poles at $z = -1, -2, \ldots$. Note that the residue of $\Gamma(z)$ at $z = 0$ is 1 so that $g(0) = 1$. Further $g(z)$ has no zeros so the sequence of zeros $\mathcal{Z}$ is empty, which means $\mathcal{D}_{\mathcal{Z}} = 0$. So $\det(I - z\mathcal{D}_{\mathcal{Z}}) = 1$ will be the numerator of our Berezinian determinant. Now if we consider the sequence, $\mathcal{P} = (-\frac{1}{n})$ of reciprocals of poles of $g(z)$ we see that it is not a summable series, but it is square summable. This means the associated operator $\mathcal{D}_{\mathcal{P}}$ is not trace class, but only Hilbert Schmidt. This forces us to use $\det_2$ in our definition of the regularized Berezinian determinant. In fact:

$$\det_2(I - z\mathcal{D}_{\mathcal{P}}) = \prod_{n=1}^{\infty} \left[(1+\frac{z}{n})e^{-\frac{z}{n}} \right]. \tag{6.7}$$

Thus the full Berezinian determinant involving $\mathfrak{D}_{z\Gamma(z)}$ is:

$$\begin{aligned}
\det\nolimits_{1,2}(I - zD_{z\Gamma(z)}) &= \det\nolimits_1(I - zD_{\mathcal{Z}})\det\nolimits_2(I - zD_{\mathcal{P}})^{-1} \\
&= 1 \cdot \left(\prod_{n=1}^{\infty} \left[\left(1 + \frac{z}{n}\right) e^{-\frac{z}{n}} \right] \right)^{-1} \\
&= \prod_{n=1}^{\infty} \left[\left(1 + \frac{z}{n}\right)^{-1} e^{\frac{z}{n}} \right] \\
&= z e^{\gamma z} \Gamma(z).
\end{aligned}$$

Therefore, we have that $\Gamma(z) = \frac{e^{-\gamma z}}{z} \det_{1,2}(I - zD_{\Gamma(z)})$. ■

Before we turn to the Riemann zeta function, we will need to slightly modify the determinant formula above to obtain a formula for $\Gamma\left(\frac{s}{2}\right)$ as that will appear in the completed Riemann zeta function $\xi(s)$.

Corollary 43 $\Gamma\left(\frac{s}{2}\right) = \frac{2e^{-\gamma s}}{s} \det_{1,2}(I - sD_{s\Gamma\left(\frac{s}{2}\right)})$.

Proof. As before:

$$\begin{aligned}
\det\nolimits_{1,2}(I - sD_{s\Gamma\left(\frac{s}{2}\right)}) &= \det(I - zD_{\mathcal{Z}})\det\nolimits_2(I - zD_{\mathcal{P}})^{-1} \\
&= \left(\prod_{n=1}^{\infty} \left[\left(1 + \frac{s}{2n}\right) e^{-\frac{s}{2n}} \right] \right)^{-1} \\
&= \frac{s}{2} e^{\frac{\gamma s}{2}} \Gamma\left(\frac{s}{2}\right)
\end{aligned}$$

This then gives $\Gamma\left(\frac{s}{2}\right) = \frac{2e^{-\gamma s}}{s} \det_{1,2}(I - sD_{s\Gamma\left(\frac{s}{2}\right)})$. ■

6.5 The Riemann Zeta Function

Now we turn our attention to another important example, the Riemann zeta function. First, we will consider the well known Euler product expression for $\zeta(s)$.

Theorem 6.8 *For $s \in \mathbb{C}$, with $\Re(s) > 1$, $\zeta(s) = \prod_p (1-p^{-s})^{-1}$ where the product is taken over all prime numbers p.*

To use our formulation, let $\phi(z) = \frac{1}{1-z}$. Then by the result for rational functions the first section $\phi(z) = \det(I - zD_\phi)$ which is true for every value of $z \neq 1$. Let $z = p^{-s}$ then gives $(1-p^{-s})^{-1} = \det(I - p^{-s}D_\phi)$ for $s \neq \frac{2\pi i k}{\log p}$, $k \in \mathbb{Z}$. This leads to the following operator based Euler product:

Theorem 6.9 *For $s \in \mathbb{C}$, with $\Re(s) > 1$, $\zeta(s) = \prod_p \det(I - p^{-s}D_\phi)$, where the product is taken over the primes p.*

Proof. We simply apply the determinant equality to each term in the infinite product and then use the standard Euler product convergence. Note that for $\Re(s) > 1$, we never have $s = \frac{2\pi i k}{\log p}$ for any integer k so the determinant equality does apply at each prime p. ■

Recall that $\xi(s) = \frac{1}{2}\pi^{-\frac{s}{2}} s(s-1)\Gamma\left(\frac{s}{2}\right)\zeta(s)$ is an entire function whose zeros all lie in the critical strip $\{s \in \mathbb{C} : 0 < \Re(s) < 1\}$. We have the following well known product representation for $\xi(s)$.

Theorem 6.10 $\xi(s) = \dfrac{1}{2}\pi^{-\frac{s}{2}} e^{\left(\log(2\pi)-1-\frac{\gamma}{2}\right)s} \displaystyle\prod_\rho \left[\left(1 - \frac{s}{\rho}\right) e^{\frac{s}{\rho}}\right]$, *where the product over ρ is the product over the zeros of $\xi(s)$ which are the nontrivial zeros of $\zeta(s)$.*

Now if we wish to express $\xi(s)$ using the determinant construction in this thesis, we need to consider $\mathcal{Z} = \{\frac{1}{\rho}\}$ and the convergence of $\displaystyle\sum_\rho \frac{1}{\rho^p}$. It is proven in [Edw01] that this series converges for $p = 1$, but only conditionally and so we will need $p = 2$ to get the absolute convergence needed for $D_{\mathcal{Z}} \in \mathcal{J}_2$. Since $\xi(s)$ is entire, then the set $\mathcal{P} = \{\}$

is empty and $D_{\mathcal{P}}$ is trivially trace class. Thus we will need to consider the determinant $\det_{2,1}(I - sD_{\xi(s)})$. This suggests the following theorem.

Theorem 6.11 $\xi(s) = \frac{1}{2}\pi^{-\frac{s}{2}} e^{\left(\log(2\pi)-1-\frac{\gamma}{2}\right)s} \det_{2,1}(I - sD_{\xi})$

Proof. From the preceding discussion, we begin by defining $\mathcal{Z} = \{\frac{1}{\rho}\}$, $\mathcal{P} = \{\}$, and constructing $D_{\xi(s)} = D_{\mathcal{Z}} \bigoplus D_{\mathcal{P}}$. Then we calculate:

$$\begin{aligned}
\det_{2,1}(I - sD_{\xi(s)}) &= \det_2(I - sD_{\mathcal{Z}}) \det_1(I - sD_{\mathcal{P}})^{-1} \\
&= \prod_{\rho} \left[\left(1 - \frac{s}{\rho}\right) e^{\frac{s}{\rho}} \right] \cdot (1)^{-1} \\
&= \frac{2\xi(s)}{\pi^{-\frac{s}{2}} e^{\left(\log(2\pi)-1-\frac{\gamma}{2}\right)s}}.
\end{aligned}$$

Thus, $\xi(s) = \frac{1}{2}\pi^{-\frac{s}{2}} e^{\left(\log(2\pi)-1-\frac{\gamma}{2}\right)s} \det_{2,1}(I - sD_{\xi})$ ∎

We can then combine the results for $\xi(s)$ and $\Gamma(s)$ to give an expression for $\zeta(s)$ in a similar spirit to the representation of zeta functions of curves over finite fields as follows:

Theorem 6.12 *If psi is given by $\psi(s) = s - 1$, then we have:*

$$\zeta(s) = -\frac{e^{(\log(2\pi)-1)s}}{2} \frac{\det_{2,1}(I - sD_{\xi(s)})}{\det_{1,1}\left(I - sD_{\psi(s)}\right) \det_{1,2}\left(I - sD_{s\Gamma(s/2)}\right)}$$

Proof. First note that since $\psi(s)$ is a rational function, then

$$\det_{1,1}(I - sD_{\psi(s)}) = \psi(0) \det_{1,1}(I - sD_{\psi(s)}) = -(s-1).$$

Then recalling the following three equations:

$$\Gamma\left(\frac{s}{2}\right) = \frac{2e^{-\gamma s}}{s} \det_{1,2}(I - sD_{s\Gamma\left(\frac{s}{2}\right)})$$

$$\xi(s) = \frac{1}{2}\pi^{-\frac{s}{2}} s(s-1)\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

$$\xi(s) = \frac{1}{2}\pi^{-\frac{s}{2}} e^{\left(\log(2\pi)-1-\frac{\gamma}{2}\right)s} \det_{2,1}(I - sD_\xi)$$

We solve for $\zeta(s)$ in the middle equation and substitute the other two to obtain:

$$\begin{aligned}
\zeta(s) &= \frac{2\pi^{\frac{s}{2}}\xi(s)}{s(s-1)\Gamma(\frac{s}{2})} \\
&= 2\pi^{\frac{s}{2}} \cdot \frac{\frac{1}{2}\pi^{-\frac{s}{2}} e^{\log(2\pi)-1-\frac{\gamma}{2}s} \det_{2,1}(I - sD_\xi)}{s(s-1)\frac{2e^{-\gamma s}}{s} \det_{1,2}(I - sD_{s\Gamma(\frac{s}{2})})} \\
&= \frac{e^{(\log(2\pi)-1)s}}{2} \frac{\det_{2,1}(I - sD_\xi)}{(s-1)\det_{1,2}(I - sD_{s\Gamma(\frac{s}{2})})} \\
&= -\frac{e^{(\log(2\pi)-1)s}}{2} \frac{\det_{2,1}(I - sD_\xi)}{\det_{1,1}(I - sD_{\psi(s)})\det_{1,2}(I - sD_{s\Gamma(\frac{s}{2})})}.
\end{aligned}$$

■

We will conclude this section with a different approach that gives an equivalent criterion for the Riemann Hypothesis. Let $\mathcal{Z}$ be the set of zeros of the function $\overline{\xi}(s) = \xi\left(\frac{1}{2} + is\right)$. Construct the operator $D_{\mathcal{Z}} = D_{\overline{\xi}}$. This leads to the following result.

Theorem 6.13 *The operator $D_{\overline{\xi}}$ is self-adjoint if and only if Riemann hypothesis is true.*

Proof. This follows directly from 5.3 part 2) and the fact that Riemann Hypothesis says that the zeros of $\xi\left(\frac{1}{2} + is\right)$ must all be real. ■

6.6 Entire Functions of Finite Order

In this section, we observe that the theory presented in this thesis is quite general. It will apply to all entire functions of finite order. We will begin with an overview of the concepts of rank, genus and order of an entire function.

Definition 44 *Let f be an entire function with zeros $\{a_1, a_2, ...\}$, repeated according to multiplicity and arranged such that $|a_1| \leq |a_2| \leq \cdots$. Then f is said to be of* finite rank *if there is an integer p such that $\sum_{n=1}^{\infty} |a_n|^{-(p+1)} < \infty$. If p is the smallest integer such that this occurs, then f is said to be of* rank p*; a function with only a finite number of zeros has rank 0. A function is said to be of* infinite rank *if it is not of finite rank.*

In order to define the genus of an entire function, we need to define what it means for an entire function to be written in standard form, which will require the following definition.

Definition 45 *For $n \in \mathbb{N}$, define the* elementary factor

$$E_n(z) = \begin{cases} (1-z), & \text{if } n = 0 \\ (1-z)\exp(\frac{z}{1} + \frac{z^2}{2} + \cdots \frac{z^n}{n}), & \text{if } n = 1, 2, 3, ... \end{cases}$$

To justify the definition of elementary factor, simply note that if $\sum_{n=1}^{\infty} |a_n|^{-(p+1)} < \infty$, then $\prod_{n=1}^{\infty} E_p(\frac{z}{a_n})$ converges uniformly on compact subsets of $\mathbb{C}$ and defines an entire function with zeros at $a_1, a_2, ...$ The exponential factor is what is needed to ensure convergence of the infinite product. With this definition in hand, we can, in turn, define the genus of an entire function:

Definition 46 *An entire function f has* finite genus *if the following statements hold: 1) f has finite rank p and 2) $f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} E_p\left(\frac{z}{a_n}\right)$, where $g(z)$ is a polynomial of degree q. In this case, the* genus *of f is defined by $\mu = \max(p, q)$.*

We now define the order of an entire function:

Definition 47 *An entire function f is said to be of* finite order *if there is a positive constant a and and $r_0 > 0$ such that $|f(z)| < \exp(|z|^a)$ for $|z| > r_0$. If f is not of finite order, then f is said to be of* infinite order. *If f is of finite order, then the number $\lambda = \inf\{a : |f(z)| < \exp(|z|^a)$ for $|z|$ sufficiently large$\}$ is called the* order *of f.*

Thus the order of an entire function f is a measure of the growth of $|f(z)|$ as $|z| \to \infty$ whereas the rank of f is based on the growth of the n^{th} smallest zero as $n \to \infty$. From the definitions, there is no inherent relationship between the two concepts, but with the following version of the Hadamard factorization theorem, we see that they are in fact closely related:

Theorem 6.14 *(Hadamard Factorization Theorem) [Con95] If $f(z)$ is an entire function of finite order λ, then f has finite genus $\mu \leq \lambda$ and f admits a factorization $f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} E_p(\frac{z}{a_n})$ where $g(z)$ is a polynomial of degree $q \leq \lambda$ and $p = [\lambda]$. In particular, f is of rank not exceeding p.*

Now when we apply our operator construction to a given entire function of finite order we obtain a Quantized Hadamard Factorization Theorem

Theorem 6.15 *(Quantized Hadamard Factorization Theorem) If $f(z)$ is an entire function of finite order λ, then f admits a factorization $f(z) = z^m e^{g(z)} \det_{p+1,1}(I - zD_{f(z)})$, where $g(z)$ is a polynomial of degree $q \leq \lambda$ and $p = [\lambda]$.*

Proof. By the standard Hadamard factorization theorem we can write $f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} E_p(\frac{z}{a_n})$, where $g(z)$ is a polynomial of degree $q \leq \lambda$ and $p = [\lambda]$, with the rank of f not exceeding p. That is, if $\{a_1, a_2, ...\}$ is the multiset of zeros of $f(z)$ including multiplicity then

$\sum_{n=1}^{\infty} \frac{1}{|a_n|^{p+1}} < \infty$. Thus if $\mathcal{Z} = \{\frac{1}{a_1}, \frac{1}{a_2}, ...\}$, the associated operator $D_{\mathcal{Z}} \in \mathcal{J}_{p+1}$. There are no poles of f, since it is entire and thus $\mathcal{P} = \{\}$, and it follows that $D_{\mathcal{P}}$ is trivially trace class. Then we can calculate:

$$\begin{aligned}
\det_{p+1,1}(I - zD_{f(z)}) &= \det_{p+1}(I - zD_{\mathcal{Z}}) \det_1(I - zD_{\mathcal{P}})^{-1} \\
&= \prod_{n=1}^{\infty} \left[\left(1 - \frac{z}{a_n}\right) \exp\left(\sum_{j=1}^{p} \frac{z^j}{j a_n^j} \right) \right] \cdot (1)^{-1} \\
&= \prod_{n=1}^{\infty} E_p\left(\frac{z}{a_n}\right) \\
&= \frac{f(z)}{z^m e^{g(z)}}
\end{aligned}$$

Thus we have that: $f(z) = z^m e^{g(z)} \det_{p+1,1}(I - zD_{f(z)})$. ■

In the above proof, we see that the extra convergence factor in the regularized determinants is exactly the same as the one for the elementary factor in the infinite product representation of entire functions, which validates, in some sense, the choice in this thesis for the type of regularized determinants as those on trace ideals. Further, since this works for all entire functions of finite order, this theorem will apply to every L-function in the Selberg class as that is one of the axioms.

Chapter 7

Further Directions

7.1 Examining the Cohomology Spaces

In the definition of the 'cohomology spaces' H^0 and H^1 given in chapter 5, we grouped all of the zeros into H^0 and all of the poles into H^1. However, if we wish to more closely resemble what was done in the case for zeta functions of curves over finite fields, it is natural to separate the spaces further so that each space only contained the zeros or poles on on a given real line in $\mathbb{C}$. Such a grading might look like the following:

$$H^0 = \bigoplus_{\alpha} H^0_{\alpha \in \mathbb{R}} \text{ and } H^1 = \bigoplus_{\alpha \in \mathbb{R}} H^1_{\alpha}$$

where, instead of creating $\mathcal{Z}$ and $\mathcal{P}$, as in this thesis, we create $\mathcal{Z}_\alpha$(respectively $\mathcal{P}_\alpha$) to be the multisets containing all of the zeros with real part α(respectively poles). This would necessitate a change of the regularized Berezinian determinant:

$$\det_{m,n}(I - zD_{f(z)}) = \frac{\det_m(I - zD_{\mathcal{Z}}}{\det_n(I - zD_{\mathcal{P}})},$$

to:

$$\det_{m_\alpha, n_\alpha}(I - zD_{f(z)}) = \frac{\displaystyle\prod_{\alpha\in\mathbb{R}} \det_{m_\alpha}(I - zD_{\mathcal{Z}_\alpha})}{\displaystyle\prod_{\alpha\in\mathbb{R}} \det_{n_\alpha}(I - zD_{\mathcal{P}_\alpha})}$$

where, in many situations, $\det_{m_\alpha}(I-zD_{\mathcal{Z}_\alpha})$ and $\det_{n_\alpha}(I-zD_{\mathcal{P}_\alpha})$ will be 1 except for a finite number of α and thus, the above a priori uncountable product, is really a finite product of the determinants. We did not pursue this idea any further, but it could possibly become a more natural representation for the cohomology spaces.

7.2 Geometry of a Zeta Function

In the previous section, we discussed a way to rewrite the 'cohomology' spaces H^0, H^1. However, these spaces are not truly cohomology spaces as we do not have a topology on a space of points to define cohomology in the first place. How might we create such a framework? What would a point on the curve for $\zeta(s)$ look like?

In the case of curves over finite fields, the points on the curves arose as fixed points of the associated operator, the Frobenius. More specifically, every point of the curve that was defined over $\mathbb{F}_{q^n}$ was a fixed point of the iterate F^n. These were crucial in relating the expression for the zeta function of the curve to the trace of the operator using the Lefshetz fixed point theorem.

However, in the case of the Riemann zeta function there is no obvious geometry of points associated to the Riemann zeta function. If we could discover a natural way of defining such a geometry, we might gain further insights into $\zeta(s)$.

One possible approach is to consider fixed points of iterates of the associated operator D_ζ^k. To simplify, first consider $(D + \alpha I)^k$ for a fixed $\alpha \in \mathbb{C}$. The fixed points of this map are $e^{(-\alpha+\omega_k)z}$ where ω_k runs through the k^{th} roots of unity. We could then define these functions to be the points of the curve defined on level k. Then if we consider the collection of all fixed points of all levels k, we can create an map from this space to the circle with center $-\alpha$ and radius 1 as $e^{(-\alpha+\omega_k)z} \mapsto -\alpha + \omega_k$. This shares one nice property with the case of curves over finite fields. The set of points of level d are a subset of the set of points of level k if and only if $d|k$. Exactly the same condition needed for the set of points defined over $\mathbb{F}_{q^d}$ to be a subset of $\mathbb{F}_{q^k}$.

Then, as in the case of curves over finite fields, the whole curve is some sort of closure of points defined over $\mathbb{F}_{q^k}$, so we could define the curve associated to $D + \alpha I$ to be the 'closure' of all the points on the curve of level k, which would make sense to give you the full circle of radius 1 and centered at $-\alpha$. In other words, the curve would be the set of functions $e^{(-\alpha+\omega)z}$ with $|\omega| = 1$. Then, in order to get the curve for a zeta function, such as $\xi(s)$, we would need to repeat this for every zero ρ of $\xi(s)$. Thus the curve of $\xi(s)$ could be considered as the union of all circles of radius 1 and centers at $-\rho$ where ρ is a nontrivial zero of $\zeta(s)$.

This may be promising, but there is still a lot to be done to fully realize this theory. One of the complications that arises here is that these fixed point functions do not in general live in the weighted Bergman space we have been dealing with thus far. They do live in the space of entire functions, but what is an appropriate way to turn that space into a Hilbert space? In addition, all of the results about $D = \frac{d}{dz}$ including bounded, compact, spectrum will not necessarily hold when expanding to a larger domain of functions.

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