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Classical and Quantum Aspects of Black Holes and Spacetime

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by

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Abstract

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Understanding the microscopic structure of black holes and, more generally, of arbitrary spacetime regions, is one of the fundamental problems of quantum gravity. The holographic principle suggests that the information content of a spacetime region is encoded in degrees of freedom on the boundary of that region. To this aim, progress in AdS/CFT suggests that the emergence of spacetime from boundary degrees of freedom entails a deep connection between gravity and entanglement. In this thesis, we attempt to gain insight into this problem by following two different approaches.

A particularly important step towards understanding the emergence of spacetime is explaining the origin of black hole entropy. Given that black holes are subregions of spacetime, a "bottom-up" approach to black hole entropy would first require understanding the gravitational degrees of freedom on the boundaries of subregions, at both the classical and quantum levels. A particularly powerful way to shed light on these degrees of freedom is by characterizing the symmetries and charges of gravitational theories with internal boundaries. In the first part of this thesis, we primarily focus on this question at the classical level. We start by considering subregions behind causal horizons, which we treat as null boundaries of the spacetime. We then extend the analysis to causal diamonds. We apply this formalism to event horizons, using the algebra of charges to derive the entropy of a black hole. Lastly, we study the measurability of gravitational charges at asymptotic boundaries when quantum corrections are included.

While the first part of this thesis focuses on the purely gravitational aspects of black holes and spacetime, the second part aims to uncover the profound relationship between these concepts and quantum field theory (QFT). The classic example of this is the quantum null energy condition (QNEC), a novel inequality in quantum field theory relating energy and entanglement which was discovered through the classical focusing theorem in general relativity. We first study the relationship between the QNEC and quantum focusing using AdS/CFT. We then study the QNEC purely using QFT, and show that it is always saturated, which displays a deep connection between energy and entanglement. We also use black hole entropy to derive energy-minimizing states in QFT which are naturally understood in terms of modular flow. Finally, we derive the holographic dual of this modular flow.

To my parents,

Alamelu and Sekar,

for their unconditional sacrifices, dedication, and love.

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Chapter 1

Introduction

Progress over the last few decades has provided a plethora of evidence that spacetime ultimately emerges from more fundamental building blocks. The road that led to this conventional wisdom can arguably be traced back to Bekenstein's argument that black holes must have an entropy proportional to its area [1], shown to hold true by Hawking [2]. Explicitly, the Bekenstein-Hawking entropy is

$$S_{\mathrm{BH}} = \frac{\mathcal{A}_{\mathrm{BH}}}{4G\hbar} \tag{1.0.1}$$

The simplest black holes are vacuum solutions to Einstein's equations that are entirely characterized by just three numbers: mass, electric charge, and spin. Bekenstein's claim was quite profound, as it suggested that spacetime itself can have microstates, since it has an entropy. What's more, this entropy would have to be proportional to the area of the black hole, unlike the volume law entropies that most conventional matter systems obey. A reasonable leap, based on this property, is that the degrees of freedom which account for the black hole entropy actually live on the horizon of the black hole.

The Bekenstein-Hawking formula served as a precursor to the holographic principle, realized most concretely by the AdS/CFT correspondence. Originally discovered by Maldacena [3], it states that quantum gravity in $d+1$ dimensional anti-de Sitter (AdS) spacetime is dual to conformal field theory (CFT) on the d dimensional boundary of the spacetime. However, it was the Ryu-Takayanagi (RT) formula [4] that actually shed light on the emergence of spacetime. The RT formula states that

$$S_{\mathrm{CFT}} = \frac{\mathcal{A}_{\mathrm{RT}}}{4G\hbar} + S_{\mathrm{out}} \tag{1.0.2}$$

where S_{CFT} is the von-Neumann entropy of a state reduced to some subregion of the boundary CFT, A is the codimension-two minimal area surface anchored to the boundary subregion, also known as the RT surface, and S_{out} is the entropy of bulk quantum fields across the RT surface. At leading order, the RT formula provides a deep connection between a purely geometric quantity in the bulk and an information theoretic quantity on the boundary. It suggests that spacetime geometry, in some sense, emerges from boundary entanglement [5].

By abstracting away from AdS/CFT, one can actually assign a *generalized entropy* to any codimension-two gravitational subregion in any spacetime:

$$S_{\text{gen}} = \frac{\mathcal{A}}{4G\hbar} + S_{\text{out}} \tag{1.0.3}$$

where A is the area of the subregion and S_{out} is the entropy across the boundary of the subregion, also known as the entangling surface. The generalized entropy is of paramount significance; it beautifully unifies geometry and entanglement for arbitrary spacetime regions, and encompasses all of the special cases mentioned thus far.

Better understanding the microscopic origin of the generalized entropy entails two particularly important questions:

- What can we learn about the gravitational degrees of freedom of arbitrary subregions?
- How does the generalized entropy serve as a bridge from gravity to QFT?

We now expand upon these two questions in turn.

1.1 Gravitational Subregions

In order to shed light on the entropy of black holes, and on the generalized entropy more broadly, we must first understand how subsystems are characterized in gravity. Black holes, for example, are specific subsystems of a given gravitational theory, which we can think of as the spacetime region behind the event horizon. We can treat the event horizon as a boundary of the spacetime, as would be the case from the perspective of an external observer, and ask what information about the black hole is encoded, by gravity, on the horizon. We can ask the same question for arbitrary spacetime regions by replacing the event horizon with the boundary of the causal diamond of some codimension-two subregion. Perhaps the most familiar is the example of null infinity, an asymptotic boundary which encodes the dynamics of radiating spacetimes.

All these examples require studying gravity on manifolds with codimension-one boundaries. The non-trivial physics follows from the fact that the diffeomorphism invariance of gravity is broken by the presence of a boundary. This means that a subset of the diffeomorphisms get promoted to physical symmetries of the theory when the boundary is present. This results in non-trivial charges on the boundary that capture important aspects of the boundary degrees of freedom associated to the subregion. In the first part of the thesis, we derive these symmetries and charges in general relativity on manifolds with null boundaries, and use them to gain insight into black hole entropy.

Our main tool throughout this part of the thesis will be the covariant phase space formalism, which is an elegant framework for dealing with Hamiltonian dynamics in gravity without giving up manifest covariance. The basic idea is to treat the internal boundary as an auxiliary background structure in the field configuration space of the gravitational theory, i.e. it does not vary from one field configuration to another, *as a smooth manifold*. Furthermore, there may be geometric structures associated to the boundary (e.g. the normal to the boundary) which we also take to be background structures. All remaining geometric structures on the boundary, which will typically be those induced by the metric, will comprise the boundary degrees of freedom that are acted on non-trivially by the broken diffeomorphisms. The subset of these diffeomorphisms which preserve the background structures will comprise the symmetries of the theory. The covariant phase space formalism allows one to readily derive these symmetries, and calculate their charges.

In Chapter 2, we systematically apply the covariant phase space formalism to general relativity in the presence of null boundaries. We consider completely general null surfaces, meaning they can be non-stationary. Since this allows for degrees of freedom to enter or leave the subregion, the Hamiltonian charges will not be integrable, as is the case for any open subsystem. However, having integrable charges is important, as they are needed in order to capture the interaction of the boundary degrees of freedom with sources, such as radiating black holes. They are also necessary for constructing a global algebra of charges. Wald and Zoupas augmented the standard covariant phase space formalism to handle exactly this [6], although they only applied their formalism to null infinity. We adapt this formalism to finite null boundaries, fix a particular boundary structure, and show that general relativity has an infinite number of symmetries and charges at null boundaries. The boundary structure that we fix consists of the null normal and the surface gravity on the null surface. The symmetry group at finite null boundaries ends up being quite similar to the well-known BMS group at null infinity, except it has two kinds of supertranslations and involves all diffeomorphisms of the base manifold of the null surface. We then compute the charge algebra and show that there is no central extension of the symmetry algebra, and discuss applications of our results to conservation laws in black hole spacetimes and the black hole memory effect.

The analysis in Chapter 2 is restricted to complete null surfaces, i.e. null surfaces whose null geodesic generators extend infinitely in either direction. However this leaves out a very important class of null surfaces: causal diamonds. If there exists a finite subregion version of holography, one might expect that the relevant boundary degrees of freedom live on the causal diamond associated to the subregion. In Chapter 3, we thus extend the previous results to the case where the internal boundary is a causal diamond, with the main subtlety being the asymptotics of bulk and boundary fields at the corners of the diamond, as well as matching conditions at the bifurcation surface. We then show that there exists an infinite number of conservation laws between the past and future components of the causal diamond, analogous to the ones between past and future null infinity. We also derive a Wald-like entropy formula [7] for causal diamonds.

In Chapter 4 we finally come to the question of black hole entropy. We first revisit the analysis in Chapter 2, generalizing the field space to include perturbations which don't fix the surface gravity, i.e. we remove it from the set of background structures. We study this field space in detail, focusing in particular on boundary anomalies generated by the difference between the action of the field space transformation and the spacetime transformation of boundary degrees of freedom. The generalized field space allows us to consider diffeomorphisms in stationary black hole spacetimes which form a Virasoro algebra in spacetime. The naive charges associated to these Virasoro vector fields turn out to be non-integrable, so we use the Wald-Zoupas formalism to derive integrable versions of them and construct an algebra of charges by making use of a modified Dirac bracket for dynamical subregions. We show that this charge algebra contains a central extension which, after using the Cardy formula [8], yields *twice* the Bekenstein-Hawking entropy of the black hole. We provide an interpretation for this result in terms of gravitational edge modes living on the bifurcation surface.

Lastly, in order for boundary symmetries and charges in gravity to be useful for understanding the generalized entropy, we must understand their properties when quantum corrections are included. In Chapter 5 we attempt to make progress on this by taking the boundary to be at null infinity, which allows us the most control. We study quantum fluctuations in the asymptotic charges on cross-sections of null infinity and show that the fluctuations grow without bound as we approach null infinity, even if the charges are smeared out across some finite interval of null infinity. We argue that this must happen in order to be consistent with the covariant entropy bound [9, 10]. We then show that the only way to obtain well-defined charges is if they are smeared across at least semi-infinite intervals on null infinity, which has important implications for the algebra of charges at null infinity.

1.2 Gravity and Entanglement

We have emphasized the profundity of the potential connections between spacetime and entanglement. Most of the evidence for this claim comes from AdS/CFT. However, we have at our disposal the generalized entropy, which is defined for any background, and links geometry and entanglement in a simple way. Therefore, we should be able to use it cleverly to gain insights into the relationship between these two concepts. Absent a concrete duality, we will provide evidence for this connection by using classical gravity to learn new things about QFT, which can then be *independently* proven in QFT. Such a paradigm would suggest that classical gravity somehow knows about the entanglement properties of quantum fields.

The generalized entropy has its own storied past, once again going back to Bekenstein [11]. He argued that when both black holes and matter are involved, it is the generalized entropy which must always increase, amounting to a generalized second law (GSL). The GSL combines the area law of classical black holes and the ordinary second law of thermodynamics. The GSL was proven relatively recently by Aron Wall [12], using the modern form of the generalized entropy. In [13], the authors considered another property of gravity: the classical focusing theorem. This theorem essentially captures the attractive nature of gravity, and says

$$\theta' \leq 0 \tag{1.2.1}$$

where θ is the expansion of a congruence of light rays emanating from a codimension-two subregion, and the derivative is along the light rays. One can promote this to a semi-classical quantum gravity statement by replacing θ with a quantum expansion $\Theta = S'_{\text{gen}}$. In [13], a quantum focusing conjecture (QFC) was formulated, which simply states

$$\Theta' \leq 0 \tag{1.2.2}$$

By taking the non-gravitational limit of the QFC, one obtains the quantum null energy condition (QNEC):

$$S'' \leq \frac{2\pi}{\hbar}\langle T_{vv}\rangle \tag{1.2.3}$$

where v is an affine parameter along the light rays. This is a pure QFT statement, and can therefore be proven using standard QFT techniques, as was done in [14, 15, 16]. Thus, a highly non-trivial propery of QFT, one which relates energy to entanglement, was arrived at starting from a very simple property of classical gravity.

In the second part of this thesis, we use this archetype to learn more about the entanglement properties of QFT, as well as further connections between gravity and entanglement. In Chapter 6 we study the QNEC and QFC using holography, following [17]. We use a property called entanglement wedge nesting (EWN) to show that the QNEC can be established holographically for curved spacetimes on the boundary. We also show that the QFC yields the QNEC whenever EWN yields the QNEC, on curved spacetimes.

In Chapter 7 we study the QNEC in generic QFTs with conformal UV fixed points, using the languge of defect CFTs. We study the operator product expansion (OPE) of displacement operators in the contact limit, which corresponds to taking two derivatives of the entanglement entropy along the same null generator. We argue that in this limit, the only operator which contributes to the OPE is the stress tensor. This shows that the QNEC is always saturated, meaning that the second null derivative of the entropy is always *equal* to the associated energy. This implies a profound connection between energy and entanglement in QFT. We go further and compute the second derivative of the entropy along different light rays, in states close to the vacuum, and show that the result can be written as the expectation value of spin-2 light ray operators.

In Chapter 8 we revisit the question of black hole entropy, but this time from the perspective of the coarse-grained entropy. In [18], a prescription was given for coarse-graining the interior of classical black holes. The black holes were defined using marginally trapped surfaces, as opposed to the event horizon. They showed that the entropy resulting from this coarse-graining coincided with the Bekenstein-Hawking entropy. We generalized this result to include semi-classical black holes, such as those which emit Hawking radiation, by giving a coarse-graining procedure for the generalized entropy of quantum marginally trapped surfaces. We then take the non-gravitational limit, and show that the coarse-graining procedure coincides with a particular manifestation of modular flow called Connes cocycle (CC) flow [16]. This flow generates energy-minimizing states, thus linking coarse-graining of black holes to energy minimization in QFT.

Lastly, in Chapter 9, we study the CC flow in the context of AdS/CFT, in order to better understand the role of modular flow in bulk emergence. In the simplest case of the

Rindler vacuum, the CC flow corresponds to an ordinary boost. We formulate a gravitational analogue of this for general spacetimes called the kink transform, which takes a set of initial data on a smooth Cauchy slice and maps it to initial data on a “kinked” slice, with the kink across some codimension-two subregion of the slice. The extrinsic curvature shock due to the kink has the effect of introducing a relative boost to the initial data. We show that this kink transform only satisfies the gravitational constraint equations if it is performed across an extremal surface. We show that the kink transform is the bulk dual of the CC flow when it is done across the RT surface.

Part I

Subregions in Gravitational Theories

Chapter 2

Symmetries and Charges of General Relativity at Null Boundaries

2.1 Introduction

It is well known that gauge transformations of a diffeomorphism invariant theory can become genuine symmetries of the theory at boundaries of the spacetime. In general relativity, diffeomorphisms of asymptotically flat spacetimes that preserve the fall-off conditions for the metric near null infinity yield the standard BMS group [19, 20, 21]. Similarly, in QED there exists an infinite set of symmetries at null infinity comprised of large gauge transformations [22, 23]. Associated to the various symmetries are global conserved charges which act as generators of the symmetries [24, 25]. There are in addition localized charges such as Bondi mass which quantify the amount of charge in subregions of the spacetime boundary, which can be calculated using a variety of formalisms [24, 26, 27].

More recently, it has been found that stationary black holes also possess an infinite number of symmetries beyond the usual horizon Killing symmetries [28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38] (see [39] for older work on this topic, and [40] for the electromagnetic case). The new symmetries are diffeomorphisms which preserve the near horizon geometry under specific gauge conditions, and a subclass of them are similar to the supertranslations at null infinity. These horizon supertranslations give rise to contributions to the global charges associated with supertranslations, in addition to the contribution from null infinity. In [33, 34, 32] it was suggested that this enlarged group of horizon symmetries and its associated charges and conservation laws play a role in how information is released as a black hole evaporates, and may lead to a resolution of the information loss paradox (see also [41, 42]). At the least, a complete analysis of supertranslation conservation laws in black hole spacetimes cannot be undertaken without first knowing what the supertranslation charges and fluxes are on general, non-stationary event horizons. It is therefore of considerable interest to gain a deeper, more unified understanding of such symmetries and charges.

A natural question is whether supertranslations are symmetries of general relativity at *any* null surface, with stationary horizons and null infinity being special cases. This would give null boundaries in general relativity quite a rich structure from the phase space point of view, and put supertranslations on far more general footing. As one of the main results of this paper, we systematically calculate the group and algebra of symmetries of general relativity at a null boundary at a finite location in spacetime, and show that this is indeed the case. We do so using covariant phase space methods, which clarifies the geometric meaning of the symmetries. The symmetry group is the semidirect product of the group of diffeomorphisms of the base space (typically the two-sphere) with a nonabelian group of supertranslations, which contains angle-dependent displacements of affine parameter as well as angle-dependent rescalings of affine parameter[1]. The results apply to nonstationary black hole horizons as well as cosmological horizons.

We next turn to the charges and conservation laws associated with these symmetries. We distinguish between *global charges* and associated global conservation laws – the independence of integrals over Cauchy surfaces Σ of the choice of Cauchy surface – and *localized charges* and localized conservation laws, which involve integrals over hypersurfaces Σ that are not Cauchy surfaces. For the global charges, we compute explicitly the contribution to the charges from integrals over event horizons. The complete charges and complete formulation of the conservation laws requires an understanding of how the symmetries of the event horizon mesh with asymptotic symmetries at null infinity. This has been worked out in some special cases [33, 34], but the general case is a subject for future investigations.

Localized charges, for example the Bondi mass at cross sections of future null infinity, are associated with localized conservation laws that express the difference between the charges at two successive cross sections with the integral of a flux over the intervening region of the boundary. These charges are not generators of symmetries on phase space. Wald and Zoupas [24] give a general prescription for computing such charges, by starting with the integral of a symplectic current that defines the variation of the global charge, and restricting the domain of integration to a hypersurface which is not a Cauchy surface, in order to attempt to obtain the charge contained within some of the degrees of freedom of the theory. This quantity is not in general a total variation and so cannot be integrated up in phase space to obtain the charge. Wald and Zoupas give a prescription for adding a correction term that overcomes this obstacle, thus allowing the definition of finite charges. Their prescription gives the conventional answers for localized charges and fluxes at null infinity [24].

In this paper we describe how to adapt the prescription to a finite null surface, and calculate the charges and fluxes of the symmetry algebra at the surface. In particular, we obtain simple expressions for the supertranslation charges and fluxes. The result applies to a very general class of null surfaces including, most importantly, non-stationary event horizons. The fluxes manifestly satisfy the property that they vanish on stationary solutions at the null surface, as one would desire if the charges are to be physically meaningful.

[1] Our symmetry group does not coincide exactly with any of the several different groups in Refs. [37, 39, 31, 30, 28, 29], since we preserve a particular geometric structure on the null surface which defines our field configuration space, and other authors preserve other quantities such as the near horizon geometry.

An interesting question is the physical interpretation of the localized charges at the null surface. At null infinity, such an interpretation of supertranslation charges is provided by the memory effect. The supertranslation that relates two different stationary regions (or vacua) can be measured as a gravitational wave memory [42, 43]. Outgoing radiation can be though of as causing a transition from one vacuum to another. A similar situation likely occurs at a black hole horizon, when accretion of radiation causes a transition from one state to a supertranslated state, with the supertranslation being measurable by near-horizon observers as a memory effect. While some aspects of this memory have been uncovered [34] there are still open questions.

Aside from the above motivations, which are centered around black holes, an understanding of the gravitational symmetry algebra at a null surface is important in and of itself: null surfaces play a crucial role in information theoretic constraints and dynamics within field theory and semi-classical gravity [44, 12, 45], in holographic settings and action formulations [46, 47, 48], in derivations of the generalized second law [12], and even in quantum gravity [49, 50]. The covariant phase space formalism for spacetimes with boundary is also important in studying the contribution of edge modes to entanglement entropy in gauge theories and gravity [51, 52]. As such, a complete description of the symmetries and charges of general non-stationary solutions at null surfaces could provide further insight into gravity, just as it did at null infinity.

Our work is complementary to the recent derivation of Hopfmuller and Friedel of boundary currents for arbitrary null surfaces and associated local conservation laws, for arbitrary vector fields tangent to the null surface [53]. Earlier treatments of the symplectic structure of general relativity on null surfaces and in 2+2 formulations can be found in Refs. [54, 55, 56, 57].

The paper is organized as follows. Section 2.2 reviews the covariant phase space formulation of boundary symmetries and conserved charges of diffeomorphism covariant theories, and Sec. 2.3 establishes our conventions for describing the local geometry of null surfaces. In Sec. 2.4 we define a universal intrinsic structure for null hypersurfaces, and derive its invariance group and algebra. Section 2.5 defines a covariant phase space for general relativity with a null boundary, and shows the associated symmetry algebra of linearized diffeomorphisms is the same as that of the universal intrinsic structure. The global and localized charges associated with these symmetries are discussed in Sec. 2.6, and global conservation laws in Sec. 2.7. Section 2.8 shows that for event horizons, the algebra of global charges under Dirac brackets coincides with the algebra of linearized diffeomorphisms under Lie brackets. Section 4.6 discusses other applications to black holes and concludes.

Notation and conventions

We use the sign convention $(-,+,+,+)$ throughout. We use the following conventions for tensor indices:

- Tensors on the spacetime M will be denoted by lowercase Roman abstract indices a,

b, c etc. from the first half of the alphabet.

- Tensors on the null surface $\mathcal{N}$ will be denoted by lowercase Roman abstract indices i, j, k etc. from the second half of the alphabet.

- Tensors built on the vector space of covectors w_i orthogonal to the normal ℓ^i at a point on $\mathcal{N}$ will be denoted by uppercase Roman abstract indices A, B, C etc.

Boldface quantities like $\boldsymbol{\omega}$ will denote differential forms. In Sec. 2.2 we will work in d spacetime dimensions, but in the remainder of the paper we will specialize to 4 spacetime dimensions.

2.2 Review of the covariant phase space formalism

In this section we review the generally covariant phase space framework for describing symmetries in a diffeomorphism covariant theory on a manifold M with boundary ∂M [58, 59, 24, 60, 61, 62, 63]. We mostly follow the notations and terminology of Wald and Zoupas [24], with one or two exceptions noted below. The framework is very general and can be applied to arbitrary theories and boundary conditions. It was applied to vacuum general relativity at null infinity in Ref. [24], and will be applied to vacuum general relativity at finite null boundaries in later sections of this paper.

A summary of the properties of the various charges and conservation laws reviewed in this section is given in Table 2.1.

Definitions of field configuration space and covariant phase space

We consider a d-dimensional manifold M with boundary ∂M, on which we want to define a theory of some dynamical fields ϕ, tensors[3] on M (we suppress tensor indices on ϕ). In the following sections of the paper we will specialize to vacuum general relativity for which $\phi = g_{ab}$. The boundary of M can consist of a number of different components $\mathcal{B}_j$,

$$\partial M = \cup_j \; \mathcal{B}_j. \tag{2.2.1}$$

The boundary components can either be at a finite location, as for a black hole horizon, or can be asymptotic boundaries. In the latter case the manifold M will be the unphysical spacetime of the conformal completion framework.

Two prototypical examples of setups we will want to consider are shown in Figure 2.1. In the first, the manifold M is the domain of outer communications of a black hole formed in a gravitational collapse, and the boundary elements are future null infinity $\mathscr{I}^+$, past null infinity $\mathscr{I}^-$, and the future event horizon $\mathcal{H}^+$. In the second, the manifold is the domain

[3] One can also include dynamical fields that are gauge-covariant fields defined on a principal bundle over M [64].

Property	Noether charge	Boundary symmetry "charge variation"	Localized (Wald-Zoupas) boundary symmetry charge	Global symmetry generator charge
Symbol	Q_ξ	$\not\delta\mathcal{Q}_{\xi,j}$ [2]	$\mathcal{Q}_\xi^{\rm loc}$	$\mathcal{Q}_\xi$
Defining equations	(2.2.6), (2.2.7), (2.6.7)	(2.2.14)	(2.2.25), (2.2.24), (2.2.26), (2.2.27)	(2.2.13) with Σ a Cauchy surface
Always well defined?	Yes	Yes	Requires the existence of presymplectic potential $\boldsymbol{\Theta}$ satisfying certain properties	Yes (assuming validity of conjecture of Sec. 2.2)
Interpretation as generator of symmetry?	No	No	No	Yes
Depends on?	Field configuration ϕ, $(d-2)$-surface $\mathcal{S}$, boundary symmetry ξ^a at $\mathcal{S}$	Field configuration ϕ, field variation $\delta\phi$, $(d-2)$-surface $\mathcal{S}$, boundary symmetry ξ^a at $\mathcal{S}$	Field configuration ϕ, $(d-2)$-surface $\mathcal{S}$, boundary symmetry ξ^a at $\mathcal{S}$	Field configuration ϕ, global boundary symmetry ξ^a (assuming global conservation laws valid)
Nature of associated conservation law	Conserved Noether current (2.2.6) on spacetime	Conserved presymplectic current (A.9.3) on spacetime	Exact $(d-1)$-form (2.2.25), (2.2.29) on component of boundary	Conjectured law is that integral of symplectic current (A.9.3) over Cauchy surface Σ and then in phase space independent of Σ (Sec. 2.7). Established in some special cases

Table 2.1: A summary of the properties of the various charges and conservation laws reviewed in this section.

of outer communications of an eternal black hole, and the boundary elements contain in addition the past event horizon $\mathcal{H}^-$. We will also be concerned with the boundaries $\mathcal{H}^+_\pm$, $\mathscr{I}^+_\pm$ etc of these boundary elements, where $\mathcal{H}^+_+$ ($\mathscr{I}^+_+$) is to be interpreted as the limit of cuts $\mathcal{S}$ of $\mathcal{H}^+$ ($\mathscr{I}^+$) in the limit as $\mathcal{S}$ approaches future timelike infinity i^+, $\mathcal{H}^+_-$ is the bifurcation two-sphere in the second case, and $\mathscr{I}^+_-$ is the limit of cuts tending to spatial infinity i^0.

A crucial role in the formalism is the definition of a field configuration space $\mathscr{F}$ of fields ϕ on M. The fields are required to be smooth on $\mathcal{M}$ and to obey suitable boundary conditions at each boundary component $\mathcal{B}_j$ and at their intersections. A key goal of this paper is to determine appropriate boundary conditions for vacuum general relativity, for a boundary component which is a general null surface $\mathcal{N}$ at a finite location in spacetime. These boundary conditions should allow the computation of symmetries and charges. Boundary conditions that achieve this are specified in Sec. 2.5 below.

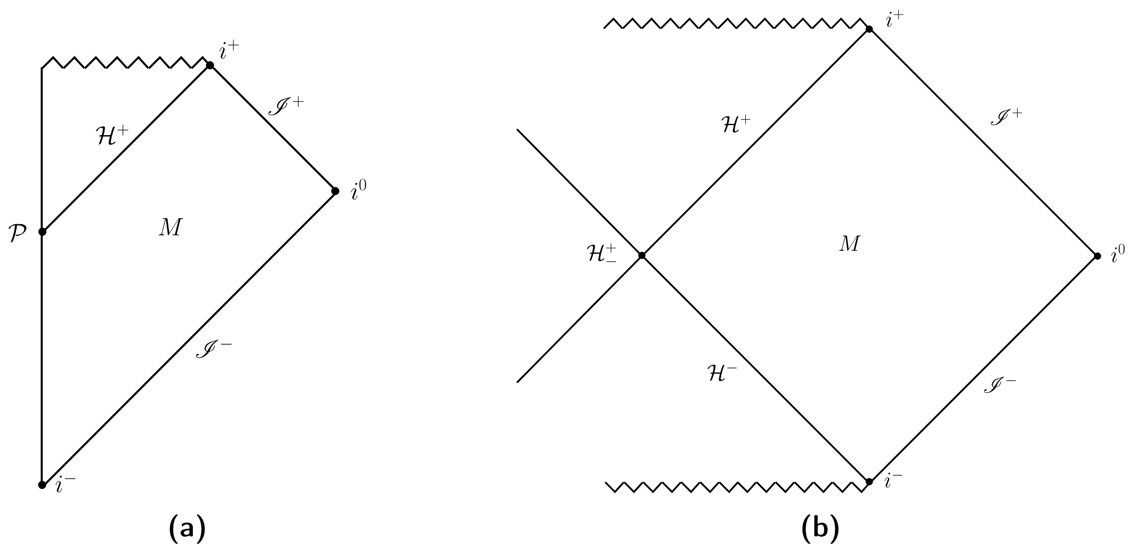

Figure 2.1: An illustration of two situations we will consider for the spacetime M. (a) M is taken to be the domain of outer communications of a black hole formed in a gravitational collapse, with boundary elements $\mathscr{I}^-$, $\mathscr{I}^+$ and $\mathcal{H}^+$. (b) M is taken to be the domain of outer communications of an eternal black hole, with boundary elements $\mathscr{I}^-$, $\mathscr{I}^+$, $\mathcal{H}^-$ and $\mathcal{H}^+$.

Definitions of currents

We next review how conserved currents associated with spacetime symmetries are obtained from the Lagrangian [24]. We assume that the dynamics of the theory is obtained from a d-form Lagrangian

$$\boldsymbol{L} = \boldsymbol{L}(\phi) \tag{2.2.2}$$

which depends locally and covariantly on the fields ϕ. Such a Lagrangian is independent of any "background fields". Under a field variation $\phi \to \phi + \delta\phi$ the variation of the Lagrangian can always be written as

$$\delta \boldsymbol{L} = \boldsymbol{E}(\phi) \cdot \delta\phi + d\boldsymbol{\theta}(\phi, \delta\phi), \tag{2.2.3}$$

where the tensor-valued d-form $\boldsymbol{E}(\phi)$ represents the equations of motion and $\cdot$ represents contraction over any suppressed tensor indices. The $(d-1)$-form $\boldsymbol{\theta}(\phi, \delta\phi)$ is the presymplectic potential, which is locally and covariantly constructed out of ϕ and $\delta\phi$ and finitely many of their derivatives. The subspace of $\mathscr{F}$ satisfying the equations of motion $\boldsymbol{E} = 0$ forms the covariant phase space $\overline{\mathscr{F}}$ of the theory.

Given two independent field variations $\delta_1\phi$ and $\delta_2\phi$ we define the presymplectic current

$$\boldsymbol{\omega}(\phi, \delta_1\phi, \delta_2\phi) = \delta_1\boldsymbol{\theta}(\phi, \delta_2\phi) - \delta_2\boldsymbol{\theta}(\phi, \delta_1\phi). \tag{2.2.4}$$

If ϕ satisfies the equations of motion and $\delta_1\phi$ and $\delta_2\phi$ satisfy the linearized equations of motion, then the presymplectic current is conserved,

$$d\boldsymbol{\omega} = 0. \tag{2.2.5}$$

We also define, for any vector field ξ^a on spacetime, the Noether current $(d-1)$-form $\boldsymbol{j}_\xi$ by

$$\boldsymbol{j}_\xi = \boldsymbol{\theta}(\phi, \pounds_\xi \phi) - i_\xi \boldsymbol{L}, \tag{2.2.6}$$

where i_ξ denotes contraction of the vector field with the differential form on the first index. It follows from Eqs. (2.2.3) and (2.2.6) that $d\boldsymbol{j}_\xi = 0$ on shell. For any local and covariant theory it can be shown that the Noether current can always be written in the form (see [65, 66])

$$\boldsymbol{j}_\xi = d\boldsymbol{Q}_\xi + \xi^a \boldsymbol{C}_a, \tag{2.2.7}$$

where $\boldsymbol{Q}_\xi(\phi)$ is the Noether charge $(d-2)$-form and $\boldsymbol{C}_a(\phi)$ are the constraints which vanish when the equations of motion hold. Taking a variation of the Noether current (2.2.6) and using Eqs. (2.2.3), (2.2.4) and (2.2.7) we get for on-shell perturbations

$$\boldsymbol{\omega}(\phi, \delta\phi, \pounds_\xi \phi) = d[\delta \boldsymbol{Q}_\xi - i_\xi \boldsymbol{\theta}(\phi, \delta\phi)]. \tag{2.2.8}$$

Definition of presymplectic form on covariant phase space

We next define the quantity

$$\Omega_\Sigma(\phi, \delta_1\phi, \delta_2\phi) = \int_\Sigma \boldsymbol{\omega}(\phi, \delta_1\phi, \delta_2\phi), \tag{2.2.9}$$

where Σ is any hypersurface embedded in M. We would like to use the definition (2.2.9) specialized to a Cauchy surface Σ to define the presymplectic form[4] of the theory, a two-form on the covariant phase space $\overline{\mathscr{F}}$. There are a number of properties that we would like Ω_Σ to satisfy, some of which inform and restrict the definition of field configuration space $\mathscr{F}$. These properties are:

- *Invariance under gauge transformations:* One might expect that $\Omega_\Sigma(\phi, \delta_1\phi, \delta_2\phi)$ should be invariant under independent linearized diffeomorphisms acting on $\delta_1\phi$ and $\delta_2\phi$. This would require that $\Omega_\Sigma(\phi, \delta\phi, \pounds_\xi\phi) = 0$ for any $\phi \in \overline{\mathscr{F}}$ and for any vector fields ξ^a and variations $\delta\phi$ for which $\delta\phi$ and $\pounds_\xi\phi$ are tangent to $\overline{\mathscr{F}}$. However, this is not true in general. Instead, from Eqs. (A.9.3) and (2.2.9) we have that, on shell, for a Cauchy surface Σ,
$$\Omega_\Sigma(\phi, \delta\phi, \pounds_\xi\phi) = \int_{\partial\Sigma} \delta\boldsymbol{Q}_\xi(\phi) - i_\xi \boldsymbol{\theta}(\phi, \delta\phi), \tag{2.2.10}$$
where $\partial\Sigma$ is the boundary of Σ, a $d-2$-surface in ∂M. This quantity vanishes for vector fields whose support lies in the interior of M, but not in general for vector fields which are nonzero on the boundary ∂M. As is well known, the fact that these

[4]The presymplectic form Ω_Σ is usually degenerate. One can factor the configuration space $\mathscr{F}$ by the orbits of the degeneracy subspaces of Ω_Σ to obtain a phase space Γ on which there exists a nondegenerate symplectic form [60]. However this will not be needed in what follows.

diffeomorphisms do not correspond to degeneracy directions of the presymplectic form reflects the fact that the the corresponding degrees of freedom are physical and not gauge[5].

- *Finiteness at asymptotic boundaries:* The definition (2.2.9) is invariant under local deformations of the hypersurface Σ when on-shell, from Eq. (2.2.5). We would like the presymplectic form (2.2.9) to have a well defined limit as Σ approaches $\mathscr{I}^+$ or $\mathscr{I}^-$, which will be true if the presymplectic current $\boldsymbol{\omega}$ has a well defined limit on those boundaries. Boundary conditions at $\mathscr{I}^+$ and $\mathscr{I}^-$ that are sufficient to ensure this are given by Wald and Zoupas [24] (see their footnote 16). These boundary conditions supplement the standard definition of asymptotic flatness at null infinity [67] by specializing the gauge[6], and are necessary for $\boldsymbol{\omega}$ to have a finite limit. In the context of null boundaries at finite locations discussed in this paper, we will also for convenience specialize the gauge at the boundary (see Sec. 2.5 below). There is a tension between gauge specializations at the boundary and the fact that some of the diffeomorphism degrees of freedom on the boundary become physical: one does not want to restrict physical degrees of freedom in the definition of the field configuration space $\mathscr{F}$. A general strategy for dealing with this tension is discussed in Sec. 2.5 below.

- *Independence of choice of Cauchy surface:* In order for Ω_Σ to define a presymplectic form on the covariant phase space $\overline{\mathscr{F}}$, one would like it to be independent of the choice of Cauchy surface Σ. While the integral (2.2.9) is invariant under local deformations of the hypersurface Σ, when one takes a limit to the boundary of spacetime there can nonzero contributions to the limiting integral from "corners" of the spacetime where boundary elements intersect, such as spatial infinity i^0. One would like to specialize the definition of the field configuration space $\mathscr{F}$ to eliminate such contributions. This issue is closely related to the question of the validity of the global conservation laws discussed in Sec. 2.7 below.

Global charges that generate boundary symmetries

We now turn to a discussion of spacetime symmetries, which we will also call boundary symmetries since only the action of the symmetry near the boundary ∂M of spacetime will be important [24]. Infinitesimal diffeomorphisms are parametrized by vector fields ξ^a on M, under which fields transform as $\phi \to \phi + \delta\phi$, where

$$\delta\phi = \pounds_\xi \phi. \tag{2.2.11}$$

Fix attention on one component $\mathcal{B}_j$ of the boundary ∂M. We denote by G_j the set of smooth vector fields ξ^a on M such that the diffeomorphism generated by ξ^a preserves the boundary

[5] One can choose to restore full diffeomorphism invariance by performing the Stueckelberg trick and introducing new physical degrees of freedom on the boundary, so-called edge modes [51, 52, 63].

[6] Here by gauge we mean both diffeomorphism freedom and choice of conformal factor.

∂M, and such that for any solution $\phi \in \overline{\mathscr{F}}$, the transformed solution $\phi + \pounds_\xi \phi$ satisfies any boundary conditions at $\mathcal{B}_j$ imposed on fields in $\mathscr{F}$, to linear order in ξ^a. We will call such a vector field a *representative of an infinitesimal boundary symmetry at* $\mathcal{B}_j$. We also define G to be the set of smooth vector fields whose diffeomorphisms preserve ∂M and map $\overline{\mathscr{F}}$ to $\overline{\mathscr{F}}$ under pullback, which we call *representatives of infinitesimal boundary symmetries* [7].

Consider now a representative of an infinitesimal boundary symmetry ξ^a. We would like to construct a charge $\mathcal{Q}_\xi$, a function on $\mathscr{F}$, which generates the boundary symmetry (2.2.11). This means that $\mathcal{Q}_\xi$ should satisfy [24]

$$\delta \mathcal{Q}_\xi = \Omega_\Sigma(\phi, \delta\phi, \pounds_\xi \phi) = \int_\Sigma \boldsymbol{\omega}(\phi, \delta\phi, \pounds_\xi \phi) \tag{2.2.12}$$

for all $\phi \in \overline{\mathscr{F}}$ and for all $\delta\phi$, $\pounds_\xi\phi$ tangent to $\mathscr{F}$, where Σ is a Cauchy surface. The charge $\mathcal{Q}_\xi$ can be interpreted as a Hamiltonian[8] in the special case when ξ is a timelike vector field. We call the charges (2.2.12) *global charges* since they are obtained by an integral over a complete Cauchy surface and so involve all the degrees of freedom in the theory, in contrast to the localized charges discussed in Sec. 2.2 below.

We next discuss the conditions under which the boundary symmetry generator charge $\mathcal{Q}_\xi$ will exist. Since Eq. (2.2.12) is attempting to define an exact one-form on field configuration space, the right hand side should be a closed one-form. It follows from Eq. (A.9.3) that the variation of the charge is a surface term on-shell:

$$\delta \mathcal{Q}_\xi = \int_{\partial\Sigma} \delta \boldsymbol{Q}_\xi - i_\xi \boldsymbol{\theta}(\phi, \delta\phi). \tag{2.2.13}$$

If the boundary $\partial\Sigma$ consists of a number of disconnected components $\mathcal{S}_j$, then $\delta\mathcal{Q}_\xi = \sum_j \delta\mathcal{Q}_{\xi,j}$ where

$$\delta \mathcal{Q}_{\xi,j} = \int_{\mathcal{S}_j} \delta \boldsymbol{Q}_\xi - i_\xi \boldsymbol{\theta}(\phi, \delta\phi). \tag{2.2.14}$$

Taking a second variation and using the definition (2.2.4) of the presymplectic current gives [24]

$$0 = (\delta_1\delta_2 - \delta_2\delta_1)\mathcal{Q}_\xi = -\int_{\partial\Sigma} i_\xi \boldsymbol{\omega}(\phi, \delta_1\phi, \delta_2\phi). \tag{2.2.15}$$

[7]The set G will generally be a proper subset of $\cap_j G_j$, because of boundary conditions imposed at intersections of boundary elements in the definition of $\mathscr{F}$ (for example continuity at a bifurcation two-sphere in an eternal black hole spacetime). See Sec. 2.7 below for further discussion.

[8]Here we depart slightly from the terminology used by Wald and Zoupas [24], who call all such charges Hamiltonians and denote them by H_ξ. The definition of Wald and Zoupas – their Eq. (8) – is also more general since they do not impose that Σ be a Cauchy surface. We will return to this generalization in Sec. 2.2 below.

The quantity (2.2.15) must vanish for all $\delta_1\phi$ and $\delta_2\phi$ tangent to $\overline{\mathscr{F}}$ in order for the charge $\mathcal{Q}_\xi$ to exist. When it does vanish[9], the definition (2.2.12) determines the charge on $\overline{\mathscr{F}}$ up to constants of integration on phase space, which can be specified by demanding that the charge vanish on a reference solution on each connected component of $\overline{\mathscr{F}}$ [24]. This prescription is discussed in more detail in the more general context of localized charges in Sec. 2.2 below.

In all cases that we are aware of, the condition (2.2.15) is satisfied whenever Σ is taken to be a Cauchy surface, as here. While we are not aware of a general proof, there is a physical argument indicating that the condition should be satisfied: a non-vanishing pullback of the symplectic current to $\partial\Sigma$ in (2.2.15) reflects an interaction between degrees of freedom that have been included in the integral (2.2.12) and those that have been excluded, and Cauchy surfaces include all of the degrees of freedom. Some examples of cases where the condition (2.2.15) is satisfied include:

- Spacetimes in general relativity that are asymptotically flat at spatial infinity i^0 and vacuum in a neighborhood of i^0, and spacelike Cauchy surfaces Σ that extend to i^0. In this case the presymplectic current extends continuously to the boundary but has vanishing pullback there [24].

- Asymptotically flat spacetimes in vacuum general relativity with no horizons, with Σ taken to be future null infinity $\mathscr{I}^+$, with certain fall off conditions on the News tensor. Consider the integrand in the obstruction (2.2.15), in the limit where the cut $\mathcal{S}$ of $\mathscr{I}^+$ approaches $\mathscr{I}^+_+$ or $\mathscr{I}^+_-$, i.e., i^+ or i^0. Denoting affine parameter by u, the integrand is given by Eq. (72) of [24] and scales like a symmetry generator $\sim u$, times a shear tensor $\sim u^0$, times a News tensor. Hence if the News tensor decays faster than $1/|u|$ as $|u| \to \infty$ the result vanishes:
$$\int_{\mathscr{I}^+_\pm} i_\xi \boldsymbol{\omega} = 0. \tag{2.2.16}$$
In the Christodoulou-Klainerman class of spacetimes [69] the News decays like $|u|^{-3/2}$.

- In the previous example, if the spacetime contains in addition a future event horizon $\mathcal{H}^+$, then the Cauchy surface can be taken to be $\mathcal{H}^+ \cup \mathscr{I}^+$ and the integral (2.2.12) will contain contributions from both $\mathcal{H}^+$ and $\mathscr{I}^+$:
$$\delta \mathcal{Q}_\xi = \int_{\mathcal{H}^+} \boldsymbol{\omega}(\phi, \delta\phi, \pounds_\xi\phi) + \int_{\mathscr{I}^+} \boldsymbol{\omega}(\phi, \delta\phi, \pounds_\xi\phi). \tag{2.2.17}$$
Here the first term will depend only on the limiting form of the symmetry ξ^a near $\mathcal{H}^+$, and the second term only on the limiting form near $\mathscr{I}^+$. The integrability analysis described above can be applied to each of these terms separately. In Appendix A.7 we show that the condition (2.2.15) is satisfied for the integral over $\mathcal{H}^+$ under certain conditions (as well as for the integral over $\mathscr{I}^+$).

[9] Note that in general the second term in Eq. (2.2.13) can give a nonvanishing contribution, so that the charge differs from the Noether charge, even when the obstruction (2.2.15) vanishes. This occurs for example for ADM charges at spatial infinity [68].

To summarize this discussion, the definition (2.2.12) should be sufficient to compute global charges $\mathcal{Q}_\xi$ that generate boundary symmetries when Σ is a Cauchy surface. See the review article by Strominger [25] for several specific calculations of charges of this type. In Sec. 2.6 below we will compute explicitly the contribution to such charges from boundary elements that are null surfaces at a finite location in spacetime, and in Sec. 2.7 we will discuss global conservation laws that are satisfied by global charges $\mathcal{Q}_\xi$.

Boundary symmetry algebras of linearized diffeomorphisms

We next discuss the symmetry algebras associated with each component $\mathcal{B}_j$ of the boundary ∂M of spacetime. These are obtained from the set G_j of representatives of infinitesimal boundary symmetries at $\mathcal{B}_j$ by modding out the trivial representatives whose charges (2.2.13) vanish [24]. Specifically, we define an equivalence relation on representatives ξ^a by

$$\xi^a \sim \xi'^a \quad \text{if} \quad \xi^a \,\hat{=}\, \xi'^a \text{ and } \int_{\mathcal{S}} (\delta \boldsymbol{Q}_\xi - i_\xi \boldsymbol{\theta}) = \int_{\mathcal{S}} (\delta \boldsymbol{Q}_{\xi'} - i_{\xi'} \boldsymbol{\theta}). \tag{2.2.18}$$

Here the notation $\hat{=}$ means equal when evaluated on $\mathcal{B}_j$, and the integrals must coincide for all $\phi \in \overline{\mathscr{F}}$ and $\delta\phi$ tangent to $\overline{\mathscr{F}}$ and for all cross sections $\mathcal{S}$ of $\mathcal{B}_j$. We define the symmetry algebra

$$\mathfrak{g}_j = G_j / \sim, \tag{2.2.19}$$

which for example gives the BMS algebra at null infinity [24]. In Sec. 2.5 below we will derive the corresponding symmetry algebra for a null surface at a finite location.

We similarly define the global symmetry algebra $\mathfrak{g} = G/\sim$, where now the equivalence relation is defined by imposing Eq. (2.2.18) at all cross sections $\mathcal{S}$ of all boundary components $\mathcal{B}_j$. In general $\mathfrak{g}$ will be a proper subalgebra of the direct sum algebra

$$\bigoplus_j \mathfrak{g}_j, \tag{2.2.20}$$

because of boundary conditions imposed at the intersections of boundary components in the definition of $\mathscr{F}$, cf. the discussions in Sec. 2.2 above and 2.7 below.

Localized (Wald-Zoupas) charges, fluxes and conservation laws

We now turn to a discussion of a different type of charge which we call *localized charges*, whose physical interpretation is roughly the amount of charge in a subset of the degrees of freedom of the theory. Studies of this type of charge have a long history in general relativity. For example, there have been many attempts made to define the total mass in a finite region of space, using various notions of quasilocal mass [70], but no natural and generally accepted definition has emerged. On the other hand, as is well known, the total amount of 4-momentum[10] radiated through any finite region of future null infinity is uniquely defined

[10] Or more generally any BMS charge.

[26, 27]. Wald and Zoupas [24] give a very general prescription for defining localized charges of this type at a boundary of spacetime, for any diffeomorphism invariant theory and for a large class of boundary conditions. They show that their general prescription gives the conventional results [26, 27] for BMS charges at null infinity. In this subsection we review and specialize slightly their general construction, and in Sec. 2.6 below we apply it to compute localized charges at a spacetime boundary consisting of a null surface at a finite location.

One trivial kind of localization was already encountered in Sec. 2.2 above. In the example (2.2.17), the charge variation $\delta\mathcal{Q}_\xi$ was expressed as a sum of an integral over the future event horizon $\mathcal{H}^+$ and an integral over future null infinity $\mathscr{I}^+$, each of which individually satisfies the integrability condition (2.2.15). Here we want to go further and consider charges localized to subregions of boundary components.

Consider a region $\Delta\mathcal{B}_j$ of a boundary $\mathcal{B}_j$ whose boundary consists of two crosssections $\mathcal{S}$ and $\mathcal{S}'$, and a representative ξ^a of an infinitesimal boundary symmetry at $\mathcal{B}_j$. Given a solution $\phi \in \overline{\mathscr{F}}$, we would like to define an exact 3-form $d\boldsymbol{\mathcal{Q}}^{\rm loc}_\xi$ on $\mathcal{B}_j$ for which the charge in the region $\Delta\mathcal{B}_j$ is

$$\int_{\Delta\mathcal{B}_j} d\boldsymbol{\mathcal{Q}}^{\rm loc}_\xi = \mathcal{Q}^{\rm loc}_\xi(\mathcal{S}') - \mathcal{Q}^{\rm loc}_\xi(\mathcal{S}), \tag{2.2.21}$$

where

$$\mathcal{Q}^{\rm loc}_\xi(\mathcal{S}) = \int_{\mathcal{S}} \boldsymbol{\mathcal{Q}}^{\rm loc}_\xi \tag{2.2.22}$$

is the charge at crosssection $\mathcal{S}$. We will call the quantity (2.2.22) a localized or Wald-Zoupas charge. The prototypical example of a quantity like this is the Bondi mass at a cross section $\mathcal{S}$ of $\mathscr{I}^+$, which is the total mass of the spacetime minus the mass radiated up to $\mathcal{S}$. In Sec. 2.6 we will define a similar quantity at cuts of a null boundary, which for a future event horizon will be the total charge at the bifurcation twosphere of the black hole (if any) plus the total charge accreted by the black hole up to the cut $\mathcal{S}$[11].

In the limit $\Delta\mathcal{B}_j \to \mathcal{B}_j$, the quantity (2.2.21) should reduce to the contribution from $\mathcal{B}_j$ to the global charge $\mathcal{Q}_\xi$. A natural candidate prescription for defining a $d-2$-form $\boldsymbol{\mathcal{Q}}^{\rm loc}_\xi$ that would achieve this is given by taking $\Sigma = \Delta\mathcal{B}_j$ in the definition (2.2.12), or, from Eqs. (2.2.13) and (2.2.21),

$$\delta\boldsymbol{\mathcal{Q}}^{\rm loc}_\xi = \delta\boldsymbol{Q}_\xi - i_\xi\boldsymbol{\theta}. \tag{2.2.23}$$

However, the corresponding charge (2.2.22) will generally not exist because of the obstruction (2.2.15). One would like to modify the right hand side of Eq. (2.2.23) in such a way as to remove this obstruction, without changing the integral on the left hand side of (2.2.21) in the limit $\Delta\mathcal{B}_j \to \mathcal{B}_j$. One would also like to find a natural prescription for this modification that yields unique charges. One could then interpret Eq. (2.2.21) as a localized conservation law, which equates a flux through a region of $\mathcal{B}_j$ with the difference between the charges at the two crosssections. (A distinct kind of global conservation law involving global charges $\mathcal{Q}_\xi$ is discussed in Sec. 2.7 below.)

[11] Our orientation convention is such that (2.2.21) is valid at $\mathscr{I}^+$ when $\mathcal{S}$ is to the future of $\mathcal{S}'$, while at a future event horizon $\mathcal{H}^+$ it is valid when $\mathcal{S}'$ is to the future of $\mathcal{S}$.

Wald and Zoupas [24] suggested a prescription of this kind that gives unique answers under certain conditions, which can be summarized as follows (we omit some subtleties related to taking the limit to asymptotic boundaries that will not be relevant for our application):

1. Compute the pullback $\boldsymbol{\omega}(\overline{\phi}, \overline{\delta_1\phi}, \overline{\delta_2\phi})$ to the boundary component $\mathcal{B}_j$ of the presymplectic current $\boldsymbol{\omega}(\phi, \delta_1\phi, \delta_2\phi)$. Here the barred fields are the dynamical fields on the boundary induced by the solution $\phi \in \overline{\mathscr{F}}$ and linearized solutions $\delta_1\phi, \delta_2\phi$ tangent to $\overline{\mathscr{F}}$, obtained by taking pullbacks of these fields (and possibly their derivatives) to the boundary.
2. Choose a presymplectic potential $\boldsymbol{\Theta}(\overline{\phi}, \overline{\delta\phi})$ on $\mathcal{B}_j$ for the pullback $\boldsymbol{\omega}$, that is, a $d-1$-form which satisfies
$$\boldsymbol{\omega}(\overline{\phi}, \overline{\delta_1\phi}, \overline{\delta_2\phi}) = \delta_1\boldsymbol{\Theta}(\overline{\phi}, \overline{\delta_2\phi}) - \delta_2\boldsymbol{\Theta}(\overline{\phi}, \overline{\delta_1\phi}). \tag{2.2.24}$$
We require that the dependence of $\boldsymbol{\Theta}$ on the dynamical fields on the boundary, as well as the dependence on fields in any universal background structure on $\mathcal{B}_j$ inherent in the definition of the field configuration space $\mathscr{F}$, be local and covariant[12]. (See Secs. 2.4 and 2.5 for more details on universal background structures.)
3. Add the term $i_\xi\boldsymbol{\Theta}$ to the right hand side of Eq. (2.2.23), thus giving from Eq. (2.2.22) the following formula for the variation of the localized charge:
$$\delta\mathcal{Q}^{\rm loc}_\xi(\mathcal{S}) = \int_{\mathcal{S}} \delta\boldsymbol{\mathcal{Q}}^{\rm loc}_\xi = \int_{\mathcal{S}} \delta\boldsymbol{Q}_\xi - i_\xi\boldsymbol{\theta} + i_\xi\boldsymbol{\Theta}. \tag{2.2.25}$$
4. Now repeating the computation that led to Eq. (2.2.15) shows that the obstruction now vanishes. The definition (2.2.25) therefore determines the charge $\mathcal{Q}^{\rm loc}_\xi(\mathcal{S})$ on $\overline{\mathscr{F}}$ up to constants of integration on phase space, which can be specified by demanding that the charges vanish on a reference solution[13] ϕ_0 on each connected component of $\overline{\mathscr{F}}$,
$$\mathcal{Q}^{\rm loc}_\xi(\mathcal{S})\Big|_{\phi=\phi_0} = 0, \tag{2.2.26}$$
for all symmetry representatives ξ^a and cuts $\mathcal{S}$ [24].
5. In order to reduce the non-uniqueness in the boundary presymplectic potential $\boldsymbol{\Theta}$, we impose the requirement that
$$\boldsymbol{\Theta}(\overline{\phi}, \overline{\delta\phi}) = 0 \tag{2.2.27}$$

[12] What this means is as follows. The presympletic potential $\boldsymbol{\Theta}$ depends on a field configuration ϕ, its variation $\delta\phi$, a universal background structure on $\mathcal{B}_j$ which we denote by $\mathfrak{p}$, and on the boundary $\mathcal{B}_j$: $\boldsymbol{\Theta} = \boldsymbol{\Theta}(\phi, \delta\phi, \mathfrak{p}, \mathcal{B}_j)$. Locality and covariance requires that for any diffeomorphism $\psi : M \to M$,
$$\psi_*\boldsymbol{\Theta}(\phi, \delta\phi, \mathfrak{p}, \mathcal{B}_j) = \boldsymbol{\Theta}(\psi_*\phi, \psi_*\delta\phi, \psi_*\mathfrak{p}, \psi^{-1}(\mathcal{B}_j)),$$
where ψ_* is the pullback. If we specialize to diffeomorphisms which preserve the boundary, $\psi^{-1}(\mathcal{B}_j) = \mathcal{B}_j$, and the universal background structure on the boundary, $\psi_*\mathfrak{p} = \mathfrak{p}$, then $\psi_*\boldsymbol{\Theta}(\phi, \delta\phi, \mathfrak{p}, \mathcal{B}_j) = \boldsymbol{\Theta}(\psi_*\phi, \psi_*\delta\phi, \mathfrak{p}, \mathcal{B}_j)$.

[13] And on all solutions related to ϕ_0 by linearized diffeomorphisms. See Appendix A.5 for further discussion of this point.

for all $\overline{\delta\phi}$ whenever $\overline{\phi}$ is stationary[14] at $\mathcal{B}_j$. We also impose that the reference solution ϕ_0 be stationary at $\mathcal{B}_j$.

The motivation for the fifth requirement is as follows [24]. It is natural on physical grounds to demand that the flux $d\boldsymbol{\mathcal{Q}}_\xi^{\rm loc}$ vanish for solutions which are stationary at the boundary $\mathcal{B}_j$. Taking the exterior derivative of the integrand in Eq. (2.2.25) and using Eq. (A.9.3) and the fact that d and δ commute we get

$$\begin{aligned} \delta d\boldsymbol{\mathcal{Q}}_\xi^{\rm loc} &= \boldsymbol{\omega}(\phi;\delta\phi,\pounds_\xi\phi) + d\left[i_\xi\boldsymbol{\Theta}(\phi;\delta\phi)\right] = \boldsymbol{\omega}(\phi;\delta\phi,\pounds_\xi\phi) + \pounds_\xi\boldsymbol{\Theta}(\phi;\delta\phi) \\ &= \delta\boldsymbol{\Theta}(\phi;\pounds_\xi\phi). \end{aligned} \tag{2.2.28}$$

To integrate this on $\overline{\mathscr{F}}$, note that $\boldsymbol{\mathcal{Q}}_\xi^{\rm loc}$ must vanish identically on ϕ_0 by Eq. (2.2.26), while $\boldsymbol{\Theta}(\phi_0,\delta\phi)$ vanishes by Eq. (2.2.27). Thus we obtain

$$d\boldsymbol{\mathcal{Q}}_\xi^{\rm loc} = \boldsymbol{\Theta}(\phi;\pounds_\xi\phi), \tag{2.2.29}$$

and so the flux vanishes identically on stationary solutions as desired, by Eq. (2.2.27).

A useful method of parameterizing choices of $\boldsymbol{\Theta}$ that automatically satisfy all the requirements apart from the stationary requirement (2.2.27) is

$$\boldsymbol{\Theta} = \boldsymbol{\theta} - \delta\boldsymbol{\alpha}, \tag{2.2.30}$$

where the first term on the right hand side is the pullback of the presymplectic potential $\boldsymbol{\theta}$, and $\boldsymbol{\alpha}$ is some $d-1$-form on $\mathcal{B}_j$ constructed from $\overline{\phi}$. Inserting this into Eq. (2.2.25), integrating in the covariant phase space $\overline{\mathscr{F}}$ and using Eq. (2.2.26) now gives

$$\mathcal{Q}_\xi^{\rm loc}(\mathcal{S}) = \int_{\mathcal{S}} \boldsymbol{Q}_\xi - i_\xi\boldsymbol{\alpha}, \tag{2.2.31}$$

if the right hand side vanishes on the reference solution $\phi = \phi_0$. In section 2.6 we will show that at a null boundary for vacuum general relativity one can choose $\boldsymbol{\alpha}$ so that $\boldsymbol{\Theta}$ satisfies the criteria outlined above, with the definition of stationary of footnote 14 replaced by the weaker notion of shear free and expansion free.

Finally, the global charges $\mathcal{Q}_\xi$ discussed in Sec. 2.2 above can often be written in terms of the localized charges $\mathcal{Q}_\xi^{\rm loc}(\mathcal{S})$ discussed here, specialized to specific cross sections $\mathcal{S}$:

$$\mathcal{Q}_\xi = \sum_j \mathcal{Q}_\xi^{\rm loc}(\mathcal{S}_j) = \sum_j \int_{\mathcal{S}_j} \boldsymbol{\mathcal{Q}}_\xi^{\rm loc}, \tag{2.2.32}$$

where the boundary $\partial\Sigma$ of a Cauchy surface Σ is a union $\partial\Sigma = \cup_j \mathcal{S}_j$ of disconnected components $\mathcal{S}_j$. The relation (2.2.32) will hold when the correction term $i_\xi\boldsymbol{\Theta}$ in the definition

[14] By "stationary at $\mathcal{B}_j$" we mean that there exists a representative τ^a of an infinitesimal boundary symmetry at $\mathcal{B}_j$ which is timelike and satisfies the Killing equation on $\mathcal{B}_j$ and to first order in deviations off $\mathcal{B}_j$. This is a weaker notion than used in [24].

(2.2.25) of the localized charge vanishes on $\partial\Sigma$, from the definition (2.2.13), if the same reference solution is used for the localized and global charges. We expect the correction term $i_\xi \boldsymbol{\Theta}$ to generically vanish on $\partial\Sigma$ when Σ is a Cauchy surface. Some examples where this occurs are:

- At future null infinity $\mathscr{I}^+$, the correction term $i_\xi \boldsymbol{\Theta}$ is proportional to the generator ξ^a times the News tensor (Eq. (73) of [24]). Letting u denote an affine parameter along $\mathscr{I}^+$, the generator scales as $\sim |u|$ as $u \to \pm\infty$, and so if the News tensor decays faster than $1/|u|$, the contributions from the boundaries $\mathscr{I}^+_\pm$ of $\mathscr{I}^+$ will vanish [cf. the discussion before Eq. (2.2.17) above].

- For a future event horizon $\mathcal{H}^+$, we show in Appendix A.7 that the contribution to the correction term from the future boundary $\mathcal{H}^+_+$ (the limit to i^+) of the horizon vanishes, if the shear obeys a suitable decay condition near $\mathcal{H}^+_+$. We also show that the contribution from a bifurcation two-sphere $\mathcal{H}^+_-$ vanishes.

Explicit expressions for $\mathcal{Q}^{\text{loc}}_\xi(\mathcal{S})$ for cross sections $\mathcal{S}$ of future null infinity $\mathscr{I}^+$ are given in Eqs. (92) and (98) of Wald and Zoupas [24], and specialized to Bondi coordinates in Eq. (3.5) of Ref. [71]. For cross sections of an arbitrary null surface, our result for $\mathcal{Q}^{\text{loc}}_\xi(\mathcal{S})$ is given in Eq. (2.6.27) below.

Potential ambiguities in global and localized charges

We next discuss some ambiguities that can arise in the definitions and constructions outlined above of global and localized charges [24, 72, 63]. Wald and Zoupas show that these ambiguities can be resolved in vacuum general relativity at future null infinity. We will similarly argue that they can be resolved at null boundaries at finite locations. However, they may be significant for other theories or at other types of boundary.

First, the definition (2.2.3) of the presymplectic potential $\boldsymbol{\theta}$ determines it up to a closed form. Since we require that $\boldsymbol{\theta}$ be local and covariant this closed form is also exact [73]. The corresponding ambiguities are

$$\boldsymbol{\theta}(\phi, \delta\phi) \rightarrow \boldsymbol{\theta}(\phi, \delta\phi) + d\boldsymbol{Y}(\phi, \delta\phi), \tag{2.2.33a}$$

$$\boldsymbol{\omega}(\phi, \delta_1\phi, \delta_2\phi) \rightarrow \boldsymbol{\omega}(\phi, \delta_1\phi, \delta_2\phi) + d\left[\delta_1 \boldsymbol{Y}(\phi, \delta_2\phi) - \delta_2 \boldsymbol{Y}(\phi, \delta_1\phi)\right] \tag{2.2.33b}$$

for some $(d-2)$-form $\boldsymbol{Y}$. These give rise to the following transformations of the presymplectic potential $\boldsymbol{\Theta}$ and of the localized charge $\mathcal{Q}^{\text{loc}}_\xi(\mathcal{S})$:

$$\boldsymbol{\Theta}(\phi, \delta\phi) \rightarrow \boldsymbol{\Theta}(\phi, \delta\phi) + d\boldsymbol{Y}(\phi, \delta\phi), \tag{2.2.34a}$$

$$\mathcal{Q}^{\text{loc}}_\xi(\mathcal{S}) \rightarrow \mathcal{Q}^{\text{loc}}_\xi(\mathcal{S}) + \int_{\mathcal{S}} \boldsymbol{Y}(\phi, \pounds_\xi \phi). \tag{2.2.34b}$$

One can demand that the maximum number of derivatives of the fields ϕ or their variations $\delta\phi$ in the $(d-2)$-form $\boldsymbol{Y}$ be two less then the number of derivatives appearing in the Lagrangian.

This requirement is in some sense natural, since otherwise the number of derivatives in $\boldsymbol{\theta}$ from Eq. (2.2.33a) exceeds what one would naively expect from Eq. (2.2.3). In Sec. 2.6 below we argue that this requirement eliminates the ambiguity (2.2.33) for vacuum general relativity.

Second, the definition (2.2.24) of the presymplectic potential $\boldsymbol{\Theta}$ determines it only up a transformation of the form

$$\boldsymbol{\Theta}(\phi, \delta\phi) \to \boldsymbol{\Theta}(\phi, \delta\phi) + \delta \boldsymbol{W}(\phi), \tag{2.2.35}$$

where $\boldsymbol{W}$ is constructed locally and covariantly from the field ϕ and from any universal background structure on $\mathcal{B}_j$. The localized charge transforms under this ambiguity as

$$\mathcal{Q}_\xi^{\rm loc}(\mathcal{S}) \to \mathcal{Q}_\xi^{\rm loc}(\mathcal{S}) + \int_{\mathcal{S}} i_\xi \boldsymbol{W}. \tag{2.2.36}$$

From the requirement (2.2.27) it follows that $\delta\boldsymbol{W}(\phi)$ must vanish for all solutions ϕ that are stationary at $\mathcal{B}_j$, and for all linearized solutions $\delta\phi$. If one additionally assumes that $\boldsymbol{W}$ depends analytically on the fields, it follows that $\boldsymbol{W} = 0$ at future null infinity $\mathscr{I}^+$ in vacuum general relativity [24]. We give a similar argument in Sec. 2.6 below to show that the ambiguity $\boldsymbol{W}$ vanishes at finite null surfaces, if we assume that the maximum number of derivatives appearing in $\boldsymbol{W}$ is one less than the number of derivatives appearing in the Lagrangian.

Third, one can redefine the Lagrangian by an exact form, $\boldsymbol{L} \to \boldsymbol{L} + d\boldsymbol{K}$, without changing the equations of motion of the theory. The corresponding transformations of the presymplectic potential $\boldsymbol{\theta}$, presymplectic current $\boldsymbol{\omega}$, Noether charge $d-2$-form $\boldsymbol{Q}_\xi$ and the integrands $\delta\boldsymbol{Q}_\xi - i_\xi\boldsymbol{\theta}$ and $d\boldsymbol{\mathcal{Q}}_\xi^{\rm loc}$ of the symmetry generator charge (2.2.13) and localized charge (2.2.25) are given by

$$\begin{aligned}
\boldsymbol{\theta}(\phi, \delta\phi) &\to \boldsymbol{\theta}(\phi, \delta\phi) + \delta\boldsymbol{K}(\phi), && (2.2.37a)\\
\boldsymbol{\omega}(\phi, \delta_1\phi, \delta_2\phi) &\to \boldsymbol{\omega}(\phi, \delta_1\phi, \delta_2\phi), && (2.2.37b)\\
\boldsymbol{Q}_\xi(\phi) &\to \boldsymbol{Q}_\xi(\phi) + i_\xi\boldsymbol{K}(\phi), && (2.2.37c)\\
\delta\boldsymbol{Q}_\xi - i_\xi\boldsymbol{\theta} &\to \delta\boldsymbol{Q}_\xi - i_\xi\boldsymbol{\theta}, && (2.2.37d)\\
\boldsymbol{\mathcal{Q}}_\xi^{\rm loc}(\phi) &\to \boldsymbol{\mathcal{Q}}_\xi^{\rm loc}(\phi). && (2.2.37e)
\end{aligned}$$

While this transformation does affect the Noether charge, it does not affect the symmetry generator charge $\mathcal{Q}_\xi$ and localized charge $\mathcal{Q}_\xi^{\rm loc}$ that are of the most interest for this paper.

2.3 Review of the local geometry of null hypersurfaces

Foundations

In this section we review the local geometry of null hypersurfaces [74, 75], in order to fix our notations and conventions. For the remainder of the paper we specialize to $3+1$ spacetime dimensions. Suppose we are given a spacetime (M, g_{ab}), and a null hypersurface $\mathcal{N}$ in M whose topology is $\mathcal{Z} \times \mathbb{R}$ for some base space $\mathcal{Z}$. We denote by ℓ_a a choice of future directed, null normal to the surface $\mathcal{N}$. This normal is not unique but can be rescaled according to

$$\ell_a \to e^{\sigma} \ell_a, \tag{2.3.1}$$

where σ is any smooth function on $\mathcal{N}$. We define the non-affinity κ, a function on $\mathcal{N}$, by

$$\ell^a \nabla_a \ell^b \mathrel{\hat{=}} \kappa l^b. \tag{2.3.2}$$

As a reminder here we are using $\mathrel{\hat{=}}$ to mean equality when restricted to $\mathcal{N}$. The non-affinity transforms under the rescaling (4.3.1) as

$$\kappa \to e^{\sigma}(\kappa + \pounds_\ell \sigma). \tag{2.3.3}$$

We will adopt the terminology that any quantity f which transforms under the transformation (4.3.1) as

$$f \to e^{-n\sigma} f \tag{2.3.4}$$

has *scaling weight* n.

We can identify the tangent space $T_p(\mathcal{N})$ to $\mathcal{N}$ at a point p with the subspace of the tangent space $T_p(M)$ consisting of vectors v^a with $v^a \ell_a = 0$. Since $\ell^a \equiv g^{ab} \ell_b$ lies in this subspace we can identify it with a vector field ℓ^i on $\mathcal{N}$, the integral curves of which are the null generators of the null surface. (Recall that we use lowercase Roman indices $i, j, \ldots$ to denote tensors intrinsic to $\mathcal{N}$.) Next, the pullback map takes covectors w_a on M evaluated on $\mathcal{N}$ to covectors w_i on $\mathcal{N}$. We denote this pullback map by

$$w_a \to \Pi^a_i w_a, \tag{2.3.5}$$

thereby defining the quantity Π^a_i. The pullback of the null normal covector ℓ_a vanishes identically by definition, since all vectors on $\mathcal{N}$ are orthogonal to ℓ_a:

$$\Pi^a_i \ell_a = 0. \tag{2.3.6}$$

A question that often arises in computations is when can a contraction $w_a v^a$ of spacetime tensors be replaced by a corresponding contraction $w_i v^i$ of tensors intrinsic to $\mathcal{N}$. First, given w_a and v^a, while w_i can be defined using the pullback, the quantity v^i is not necessarily well defined; it is defined only when $\ell_a v^a = 0$. When this condition is satisfied, the contractions coincide:

$$\ell_a v^a = 0 \qquad \Longrightarrow \qquad w_a v^a = w_i v^i. \tag{2.3.7}$$

A similar issue arises in going from three dimensions down to two dimensions. We denote by W_p the two dimensional subspace of the dual space $T_p(\mathcal{N})^*$ consisting of covectors w_i that satisfy $w_i \ell^i = 0$. We will denote by abstract indices A, B etc. tensors built on W_p. When can a contraction $w_i v^i$ of tensors on $\mathcal{N}$ be replaced by a corresponding contraction $w_A v^A$

of tensors in W_p and W_p^*? The answer in this case is the opposite of that for going from four to three dimensions. First, given w_i and v^i, the quantity v^A is always well defined by considering v^i as a linear map on $T_p(\mathcal{N})^*$ and restricting its action to W_p (we shall call this operation a pullback). On the other hand, it is necessary that $w_i\ell^i = 0$ in order that w_A be defined. When this condition is satisfied, the contractions coincide:

$$w_i\ell^i = 0 \qquad \Longrightarrow \qquad w_iv^i = w_Av^A. \tag{2.3.8}$$

Geometric fields defined on a null hypersurface

We denote by q_{ij} the induced metric on $\mathcal{N}$

$$q_{ij} = \Pi_i^a\Pi_j^b g_{ab}, \tag{2.3.9}$$

which has signature $(0,+,+)$. Taking the pullback of the relation $\ell_a = g_{ab}\ell^b$ and using Eq. (2.3.7) gives

$$q_{ij}\ell^j = 0, \tag{2.3.10}$$

i.e., ℓ^i is a eigenvector of the induced metric with eigenvalue zero. It follows that we can regard q_{ij} as a tensor in $W_p \otimes W_p$, which we write as q_{AB}. This has a unique inverse in $W_p^* \otimes W_p^*$ which we write as q^{AB}. We will use q_{AB} and q^{AB} to freely raise and lower capital Roman indices.

The second fundamental form of the surface $\mathcal{N}$ is given by

$$K_{ij} = \Pi_i^a\Pi_j^b\nabla_a\ell_b. \tag{2.3.11}$$

Since ℓ_a is normal to a hypersurface we have $\ell_{[a}\nabla_b\ell_{c]} \mathrel{\hat{=}} 0$ or $\nabla_{[a}\ell_{b]} \mathrel{\hat{=}} \ell_{[a}w_{b]}$ for some w_b, and taking the pullback and using (2.3.6) gives

$$K_{[ij]} = 0. \tag{2.3.12}$$

Similarly, lowering the index in Eq. (2.3.2), taking the pullback and using Eqs. (2.3.6) and (2.3.7) gives

$$\ell^iK_{ij} = 0. \tag{2.3.13}$$

It follows that K_{ij} lies in $W_p\otimes W_p$ and so can be written as K_{AB}. We can uniquely decompose the second fundamental form as

$$K_{AB} = \frac{1}{2}\theta q_{AB} + \sigma_{AB}, \tag{2.3.14}$$

where θ is the expansion and the shear σ_{AB} is traceless, $q^{AB}\sigma_{AB} = 0$. This equation can also be written as $K_{ij} = \theta q_{ij}/2 + \sigma_{ij}$.

The second fundamental form is related to the Lie derivative of the induced metric. Taking the pullback of the identity $\pounds_\ell g_{ab} = 2\nabla_{(a}\ell_{b)}$ and using the fact that the pullback commutes with the Lie derivative gives

$$K_{ij} = \frac{1}{2}\pounds_\ell q_{ij}. \tag{2.3.15}$$

Consider next the object

$$\Pi_i^a \nabla_a \ell^b. \tag{2.3.16}$$

This tensor is orthogonal to the normal on the b index, since $\ell_b \Pi_i^a \nabla_a \ell^b = \Pi_i^a \nabla_a (\ell_b \ell^b)/2 = 0$, since $\ell_b \ell^b = 0$ on $\mathcal{N}$ and the derivative is along the surface. Therefore this quantity is an intrinsic tensor which we write as

$$\mathcal{K}_i{}^j, \tag{2.3.17}$$

called the Weingarten map [75]. From Eqs. (2.3.2) and (2.3.7) it follows that

$$\mathcal{K}_i{}^j \ell^i = \kappa \ell^j. \tag{2.3.18}$$

Similarly taking the pullback of the relation $\nabla_a \ell^b g_{bc} = \nabla_a \ell_c$ and using (2.3.7) and (2.3.9) gives that

$$\mathcal{K}_i{}^j q_{jk} = K_{ik}. \tag{2.3.19}$$

It follows from Eqs. (2.3.12), (2.3.13), (2.3.18) and (2.3.19) that the Weingarten map $\mathcal{K}_i{}^j$ has six independent nonzero components in general in four spacetime dimensions, three of which are determined by the second fundamental form K_{ij}, and one of which is determined by the non-affinity κ, leaving two additional independent components [see Appendix A.1 for more details, especially Eqs. (A.1.13c) and (A.1.13d)].

Next, a choice of volume form ε_{abcd} on spacetime determines a volume form ε_{ijk} on $\mathcal{N}$ as follows. We consider three-forms $\bar{\varepsilon}_{abc}$ on $\mathcal{N}$ which satisfy

$$4\bar{\varepsilon}_{[abc}\ell_{d]} \mathrel{\widehat{=}} \varepsilon_{abcd}, \tag{2.3.20}$$

and then take the pullback of these three-forms:

$$\varepsilon_{ijk} = \Pi_i^a \Pi_j^b \Pi_k^c \; \bar{\varepsilon}_{abc}. \tag{2.3.21}$$

Although $\bar{\varepsilon}_{abc}$ is not unique, its pullback ε_{ijk} is. We define the antisymmetric tensor ε^{ijk} by

$$\varepsilon^{ijk} \varepsilon_{ijk} = 3!, \tag{2.3.22}$$

and the two-form ε_{ij} by

$$\varepsilon_{ij} = -\varepsilon_{ijk} \ell^k. \tag{2.3.23}$$

Under the scaling transformation (4.3.1) the various quantities defined in this subsection transform as

$$\begin{aligned}
q_{ij} &\to q_{ij}, && \text{(2.3.24a)}\\
K_{ij} &\to e^{\sigma} K_{ij}, && \text{(2.3.24b)}\\
\mathcal{K}_i{}^j &\to e^{\sigma} \left(\mathcal{K}_i{}^j + D_i \sigma \, \ell^j \right), && \text{(2.3.24c)}\\
\theta &\to e^{\sigma} \theta, && \text{(2.3.24d)}\\
\varepsilon_{ijk} &\to e^{-\sigma} \varepsilon_{ijk}, && \text{(2.3.24e)}\\
\varepsilon^{ijk} &\to e^{\sigma} \varepsilon^{ijk}, && \text{(2.3.24f)}\\
\varepsilon_{ij} &\to \varepsilon_{ij}, && \text{(2.3.24g)}
\end{aligned}$$

where D_i is any derivative operator on $\mathcal{N}$.

Divergence operator

Although there is no preferred derivative operator on $\mathcal{N}$, one can define a divergence operation $v^i \to \hat{D}_i v^i$ on vector fields via

$$\hat{D}_i v^i = \frac{1}{2}\varepsilon^{ijk} D_k(\varepsilon_{ijm} v^m), \tag{2.3.25}$$

where D_i is again any derivative operator on $\mathcal{N}$. The right hand side is independent of the choice of D_i since it enters as an exterior derivative.

We can relate this divergence operator to the four dimensional divergence operator as follows. A vector field v^i on $\mathcal{N}$ corresponds to a unique vector field v^a on $\mathcal{N}$ with $v^a \ell_a \mathrel{\hat{=}} 0$. Now choose an extension of v^a to a neighborhood of $\mathcal{N}$ in M. The linearized diffeomorphism associated with v^a maps $\mathcal{N}$ into itself, and therefore preserves the normal ℓ_a up to a rescaling. Therefore there exists a function ϖ on $\mathcal{N}$ which depends on v^a such that

$$\pounds_v \ell_a \mathrel{\hat{=}} \varpi \ell_a. \tag{2.3.26}$$

The relation between the two divergence operators is[15]

$$\nabla_a v^a \mathrel{\hat{=}} \hat{D}_i v^i + \varpi. \tag{2.3.27}$$

The divergence of the normal is

$$\hat{D}_i \ell^i = \theta. \tag{2.3.28}$$

This follows from the relation (2.3.27), the definition (2.3.26) of ϖ, and the trace of Eq. (A.1.12).

Stationary regions of null hypersurfaces

As discussed in Sec. 2.2 above, we shall call a region of a null surface *stationary* if there is a choice of normal covector τ_a in that region which satisfies Killings equation on the surface and to first order in deviations off the surface,

$$\pounds_\tau g_{ab} \mathrel{\hat{=}} 0, \tag{2.3.29a}$$
$$\nabla_c \pounds_\tau g_{ab} \mathrel{\hat{=}} 0. \tag{2.3.29b}$$

[15]This relation can be derived by specializing to a coordinate system $(r, y^1, y^2, y^3) = (r, y^\Gamma)$ for which the hypersurface $\mathcal{N}$ is given by $r = 0$ and with $\ell_a \mathrel{\hat{=}} (dr)_a$. Writing the volume form as $\boldsymbol{\varepsilon} = e^\Upsilon dr \wedge dy^1 \wedge dy^2 \wedge dy^3$ for some function Υ, the left hand side of Eq. (2.3.27) can be written as

$$e^{-\Upsilon}\partial_r(e^\Upsilon v^r) + e^{-\Upsilon}\partial_\Gamma(e^\Upsilon v^\Gamma) = \partial_r v^r + e^{-\Upsilon}\partial_\Gamma(e^\Upsilon v^\Gamma).$$

The first term on the right hand side here is ϖ, while the second term is the intrinsic divergence $\hat{D}_i v^i$, by Eqs. (2.3.20), (2.3.21) and (2.3.25).

We will denote the corresponding value of κ by κ_τ, the surface gravity. Taking the pullback of Eq. (2.3.29a) and using the fact that the pullback commutes with the Lie derivative gives

$$\pounds_\tau q_{ij} = 0, \tag{2.3.30}$$

and it follows from Eq. (2.3.15) that

$$K_{ij} = 0, \tag{2.3.31}$$

i.e. that the surface is shear free and expansion free.

It follows from the condition (2.3.31) together with Eqs. (A.1.13) that the rotation one-form defined by

$$\omega_i = -\mathcal{K}_i{}^j n_j, \tag{2.3.32}$$

where n_i is any covector with $n_i \ell^i = -1$, is independent of the choice of n_i. This is true only for null surfaces that satisfy (2.3.31). Under the transformation (4.3.1) ω_i transforms as $\omega_i \to \omega_i + D_i\sigma$, from Eqs. (2.3.24c) and (A.1.13).

We define

$$\omega_{\tau\, i} = \omega_i|_{\vec{\ell}=\vec{\tau}} \tag{2.3.33}$$

to be the rotation one-form ω_i specialized to the choice of representative $\ell^i = \tau^i$ [16]. Now Eq. (2.3.29) together with Eq. (C.3.6) of Wald [67] imply that $\pounds_\tau \nabla_a \tau^b \mathrel{\hat{=}} 0$, and taking a pullback yields $\pounds_\tau \mathcal{K}_i{}^j = 0$. Combining this with Eqs. (A.1.10) and (A.1.13) now shows that the nonaffinity and rotation one-form are Lie transported along the null surface:

$$\pounds_\tau \kappa_\tau = 0, \qquad \pounds_\tau \omega_{\tau\, i} = 0. \tag{2.3.34}$$

More generally, the Bardeen-Carter-Hawking derivation [76] of the zeroth law of black hole thermodynamics,

$$D_i \kappa_\tau = 0, \tag{2.3.35}$$

applies in this context, assuming the Einstein equations and the dominant energy condition. In the remainder of the paper we will be working in the context of vacuum general relativity, for which (2.3.35) will be satisfied in stationary regions.

Orthonormal basis formalism

Finally, it is sometimes useful for computational purposes to choose an auxiliary null vector field n^a on $\mathcal{N}$ which together with ℓ^a forms part of an orthonormal basis. Some aspects of the formalism described above simplify when described in the language of an orthonormal basis, although that language does carry the baggage of an arbitrary choice. While the main results of this paper will not require a choice of auxiliary null vector, we will translate our results into the language of the orthonormal basis formalism since it is widely used. Details of the relation between the covariant and orthonormal basis formalisms for null surfaces are given in Appendix A.1.

[16] See Ashtekar [74] for an alternative method of defining $\omega_{\tau\, i}$.

2.4 Universal intrinsic structure of a null hypersurface

In this section we will describe an intrinsic geometric structure on null hypersurfaces $\mathcal{N}$ that is determined by the spacetime geometry. It is universal in the sense that for a given $\mathcal{N}$ any two such structures are diffeomorphic. We will define the structure in Sec. 2.4, and in Sec. 2.4 we will describe the symmetry group of diffeomorphisms from $\mathcal{N}$ to $\mathcal{N}$ that preserve the structure. The corresponding Lie algebra is described in Sec. 2.4; we will show in Sec. 2.5 that this symmetry algebra coincides with that obtained from a particular definition of covariant phase space for general relativity with a null boundary in the Wald-Zoupas approach. Section 2.4 discusses preferred subalgebras associated with stationary regions of the null hypersurface. Finally in Sec. 2.4 we discuss how the group and algebra are modified in the case where the null hypersurface has a boundary $\partial\mathcal{N}$ in M.

Definition of intrinsic structure

Consider a manifold $\mathcal{N}$ which is equipped with a smooth, nowhere vanishing vector field ℓ^i and a smooth function κ. Letting $\mathcal{Z}$ denote the manifold of integral curves, we assume that $\mathcal{N}$ is diffeomorphic to the product $\mathcal{Z} \times \mathbb{R}$. We define an equivalence relation on such pairs (ℓ^i, κ) by saying that two pairs are equivalent if they are related by a rescaling of the form [cf. Eqs. (4.3.1) and (2.3.3) above]

$$\ell^i \rightarrow e^\sigma \ell^i, \tag{2.4.1a}$$

$$\kappa \rightarrow e^\sigma(\kappa + \pounds_\ell \sigma) \tag{2.4.1b}$$

where σ is a smooth function on $\mathcal{N}$. We denote by

$$\mathfrak{u} = [\ell^i, \kappa] \tag{2.4.2}$$

the equivalence class associated with (ℓ^i, κ). A choice of equivalence class is the desired intrinsic geometric structure on $\mathcal{N}$.

Suppose now we are given a spacetime (M, g_{ab}) with null boundary $\mathcal{N}$. The spacetime geometry then determines a structure $[\ell^i, \kappa]$ in the manner described in Sec. 2.3 above: the vector ℓ^i is obtained by raising the index on a choice of normal covector, and κ is the non-affinity of that vector. The resulting equivalence class $[\ell^i, \kappa]$ is independent of the choice of normalization of the covector, by the equivalence relation (2.4.1).

The intrinsic structure determines a class of foliations of $\mathcal{N}$ as follows. Choose a cross section $\mathcal{S}$ of $\mathcal{N}$, a surface which each integral curve intersects exactly once, which will be diffeomorphic to the base space $\mathcal{Z}$. Out of the equivalence class $[\ell^i, \kappa]$, pick a member $(\ell_0^i, 0)$ for which the non-affinity vanishes, by starting with a general member (ℓ^i, κ) and solving the differential equation $\kappa + \pounds_\ell \sigma = 0$ for the scaling function σ. Now Lie drag the cross section

$\mathcal{S}$ along integral curves of ℓ_0^i. The resulting foliation[17] will be level sets of a coordinate u which is determined by the properties that $u = 0$ on $\mathcal{S}$ and $\ell_0^i D_i u = 1$. In addition, if θ^A is any coordinate system on $\mathcal{S}$, one can extend the definition of these coordinates to $\mathcal{N}$ by demanding that they be constant along the integral curves, thereby generating a coordinate system (u, θ^A) on $\mathcal{N}$ for which $\vec{\ell}_0 = \partial_u$.

We will say that an intrinsic structure is *complete* if all of the generators of $\mathcal{N}$ can be extended to arbitrary values of affine parameter in both directions (where u is an affine parameter if $\vec{\ell} = \partial_u$ with $\kappa = 0$). For example, the future light cone of a point $\mathcal{P}$ in Minkowski spacetime (with $\mathcal{P}$ itself removed) is not complete when the intrinsic structure induced by the flat Minkowski metric is used, since all of the generators start at $\mathcal{P}$. By contrast, the event horizon in maximally extended Schwarzschild is complete (see Appendix A.5). We will study both types of intrinsic structure later in this paper.

Given two different complete intrinsic structures $\mathfrak{u} = [\ell^i, \kappa]$ and $\mathfrak{u}' = [\ell'^i, \kappa']$ on $\mathcal{N}$, there exists a diffeomorphism $\varphi : \mathcal{N} \to \mathcal{N}$ which maps $\mathfrak{u}$ onto $\mathfrak{u}'$. In this sense the complete intrinsic structure is *universal*, in the same way that an intrinsic structure of a different kind on future null infinity is universal in the BMS construction [77]. The existence of the diffeomorphism φ can be shown as follows. Choose a cross section $\mathcal{S}$ of $\mathcal{N}$, and using $\mathfrak{u}$ construct a coordinate u on $\mathcal{N}$ in the manner discussed above. Define the diffeomorphism

$$\Phi = (u, \pi) : \mathcal{N} \to \mathbb{R} \times \mathcal{Z}, \tag{2.4.3}$$

where $\pi : \mathcal{N} \to \mathcal{Z}$ is the natural projection obtained by taking each point to the corresponding integral curve. Starting from the intrinsic structure $\mathfrak{u}'$ one can similarly define a diffeomorphism Φ', and then $\varphi = \Phi^{-1} \circ \Phi'$ maps $\mathfrak{u}$ onto $\mathfrak{u}'$.

Symmetry group of a complete intrinsic structure

We now turn to a discussion of the symmetry group $G_\mathfrak{u}$ of diffeomorphisms $\varphi : \mathcal{N} \to \mathcal{N}$ which preserve a universal structure $\mathfrak{u}$. Remarkably, the structure of this group is very similar to that of the BMS group at null infinity, but with two important differences. First, the Lorentz group at null infinity is replaced by the group Diff$(\mathcal{Z})$ of diffeomorphisms of the base space $\mathcal{Z}$, typically the two-sphere S^2. This replacement is not surprising, since the conformal freedom that is used at null infinity to map the induced metric onto a metric of constant curvature is not present for general null surfaces. Second, the abelian subgroup of supertranslations at null infinity is replaced by a nonabelian subgroup, which contains angle-dependent displacements of affine parameters and rescalings of affine parameters.

From the definition (2.4.1) of the equivalence class, it follows that a diffeomorphism $\varphi : \mathcal{N} \to \mathcal{N}$ is a symmetry in $G_\mathfrak{u}$ if, for a given representative (ℓ^i, κ) in $\mathfrak{u}$, the pullback φ_*

[17] The class of foliations generated in this way has considerable freedom. One can pick the initial cross section $\mathcal{S}$ arbitrarily, and in addition one can pick a second arbitrary cross section $\mathcal{S}'$ disjoint from $\mathcal{S}$ and arrange for it to belong to the foliation, by exploiting the rescaling freedom $\ell_0^i \to e^\sigma \ell_0^i$ with $\pounds_{\ell_0} \sigma = 0$. However, once $\mathcal{S}$ and $\mathcal{S}'$ are specified, the foliation is uniquely determined.

acts as a scaling transformation for some smooth scaling function $\beta = \beta(\varphi)$ on $\mathcal{N}$ [cf. Eqs. (2.4.1) above]:

$$\varphi_* \ell^i = e^{\beta} \ell^i, \tag{2.4.4a}$$
$$\varphi_* \kappa = e^{\beta}(\kappa + \pounds_\ell \beta). \tag{2.4.4b}$$

If we choose a different representative (ℓ'^i, κ') with $\ell'^i = e^{\sigma} \ell^i$, then we find from (2.4.4) that

$$\varphi_* \ell'^i = e^{\beta'} \ell'^i, \tag{2.4.5a}$$
$$\varphi_* \kappa' = e^{\beta'}(\kappa' + \pounds_{\ell'} \beta'), \tag{2.4.5b}$$

where

$$\beta' = \beta + \varphi_* \sigma - \sigma. \tag{2.4.6}$$

Hence φ will be a symmetry if (2.4.4) is satisfied for any choice of representative.

Specialize now to a choice of coordinate system (u, θ^A) and representative of the kind discussed in Sec. 2.4 above, where $\kappa = 0$ and $\vec{\ell} = \partial_u$. Then the general solution for a diffeomorphism that satisfies (2.4.4) is $(u, \theta^A) \to (\overline{u}, \overline{\theta}^A)$, where

$$\overline{u}(u, \theta^A) = \alpha(\theta^A) + e^{-\beta(\theta^A)} u \tag{2.4.7a}$$
$$\overline{\theta}^A(u, \theta^B) = \overline{\theta}^A(\theta^B). \tag{2.4.7b}$$

This group of transformations contains a number different subgroups:

- The subgroup with $\alpha = 0$, $\beta = 0$, which consists of arbitrary diffeomorphisms on the base space $\mathcal{Z}$, Diff$(\mathcal{Z})$. In many applications this will be Diff(S^2), the diffeomorphisms of the two-sphere. These transformations have also been called *superrotations* [33].

- The subgroup with $\overline{\theta}^A = \theta^A$, parameterized by $\alpha(\theta^A)$ and $\beta(\theta^A)$. These transformations consist of reparameterizations of the generators of the null surface[18]. We will call these transformations *supertranslations*, following common use [34, 33, 39, 78, 31, 79, 80, 30, 28, 36], and because of the analogy with the supertranslations of the BMS group.

- The subgroup of the supertranslation group with $\beta = 0$, $\overline{\theta}^A = \theta^A$, which is parameterized by $\alpha(\theta^A)$. We will call these transformations *affine supertranslations* since they consist of angle-dependent displacements in affine parameter (as opposed to angle-dependent displacements in Killing parameter or Killing supertranslations [33, 39, 78, 31, 79, 80, 30, 28], to be discussed in Sec. 2.4 below.)

[18]This supertranslation subgroup of symmetries played an important role in Wall's proof of the generalized second law [12].

- The subgroup of the supertranslation group with $\alpha = 0$, $\overline{\theta}^A = \theta^A$, which is parameterized by $\beta(\theta^A)$. These transformations consist of constant *rescalings* of affine parameter on each generator. (Note however that if $\tau = \ln(u)/\kappa$ is a Killing parameter, the transformations consist of angle-dependent displacements in τ; see Sec. 2.4.)

The first subgroup preserves the foliation associated with the coordinate system (u, θ^A), while the last three preserve the integral curves. The affine supertranslation and supertranslation subgroups do not depend on the choice of coordinate system or representative, and can be invariantly defined. The rescaling and Diff$(\mathcal{Z})$ subgroups, by contrast, do depend on these choices. Their status is analogous to that of Lorentz subgroups of the BMS group: there are many such subgroups, but no natural or unique choice.

The symmetry algebra associated with the group of transformations is given by the linearization of Eq. (2.4.7), which yields the vector field

$$\vec{\chi} = \left[\alpha(\theta^A) - \beta(\theta^A)u\right]\partial_u + X^A(\theta^B)\partial_A, \tag{2.4.8}$$

where X^A is arbitrary. The algebra of these generators under Lie brackets is

$$\left[(\alpha_1 - \beta_1 u)\partial_u + X_1^A\partial_A, (\alpha_2 - \beta_2 u)\partial_u + X_2^A\partial_A\right] = (\alpha_3 - \beta_3 u)\partial_u + X_3^A\partial_A \tag{2.4.9}$$

with

$$\begin{aligned}
\alpha_3 &= -\alpha_1\beta_2 + X_1^A\partial_A\alpha_2 + \alpha_2\beta_1 - X_2^A\partial_A\alpha_1, &\text{(2.4.10a)}\\
\beta_3 &= -X_1^A\partial_A\beta_2 + X_2^A\partial_A\beta_1, &\text{(2.4.10b)}\\
X_3^A &= X_1^B\partial_B X_2^A - X_2^B\partial_B X_1^A. &\text{(2.4.10c)}
\end{aligned}$$

While these explicit coordinate expressions are convenient, it can be difficult to discern which aspects of the structures are specific to the choice of coordinate system. We now turn to an analysis of the symmetry algebra which is covariant and does not depend on a choice of coordinates.

Symmetry algebra of a complete intrinsic structure

The Lie algebra $\mathfrak{g}_\mathfrak{u}$ of infinitesimal symmetries in $G_\mathfrak{u}$ consists of vector fields χ^i on $\mathcal{N}$ which obey the linearized versions of Eqs. (2.4.4):

$$\begin{aligned}
\pounds_\chi \ell^i &= \beta\ell^i, &\text{(2.4.11a)}\\
\pounds_\chi \kappa &= \beta\kappa + \pounds_\ell\beta. &\text{(2.4.11b)}
\end{aligned}$$

As before, if these equations are satisfied for one representative (ℓ^i, κ) of the equivalence class, they will be satisfied for all representatives. The function β depends on both the symmetry χ^i and on the representative ℓ^i, $\beta = \beta(\chi^i, \ell^i)$, and the dependence on the normalization is given by the linearized version of Eq. (2.4.6):

$$\beta(\chi^i, e^\sigma\ell^i) = \beta(\chi^i, \ell^i) + \pounds_\chi\sigma. \tag{2.4.12}$$

The general solution of Eqs. (2.4.11) for χ^i, with a choice of representative and coordinate system (u, θ^A) for which $\kappa = 0$ and $\vec{\ell} = \partial_u$, is given by Eq. (2.4.8) above.

The algebra $\mathfrak{g}_\mathfrak{u}$ inherits the Lie bracket structure of the space of vector fields on $\mathcal{N}$. From the definition of the symmetry group $G_\mathfrak{u}$ as a subgroup of Diff$(\mathcal{N})$, it follows that $\mathfrak{g}_\mathfrak{u}$ is closed under this Lie bracket. This closure was also shown in Eq. (2.4.10) above, and can also be checked directly in the covariant context: if $\vec{\chi}_1$ and $\vec{\chi}_2$ are two vector fields which satisfy Eqs. (2.4.11), then $\vec{\chi}_3 = [\vec{\chi}_1, \vec{\chi}_2]$ also satisfies Eqs. (2.4.11) with

$$\beta(\vec{\chi}_3) = \pounds_{\chi_1}\beta_2 - \pounds_{\chi_2}\beta_1, \tag{2.4.13}$$

where $\beta_1 = \beta(\vec{\chi}_1)$ and $\beta_2 = \beta(\vec{\chi}_2)$.

We now argue that the symmetry algebra has the structure

$$\mathfrak{g}_\mathfrak{u} \cong \mathrm{diff}(\mathcal{Z}) \ltimes (\mathfrak{b} \ltimes \mathfrak{s}_0), \tag{2.4.14}$$

where $\ltimes$ denotes semidirect sum and the various algebras are as follows:

- diff$(\mathcal{Z})$ is the algebra of linearized diffeomorphisms of the base space $\mathcal{Z}$, *i.e.*, vector fields on $\mathcal{Z}$.
- $\mathfrak{s}_0$ is the abelian algebra of linearized affine supertranslations, consisting of vector fields of the form
$$\chi^i = f\ell^i \tag{2.4.15}$$
where the function f on $\mathcal{N}$ satisfies
$$\pounds_\ell f + \kappa f = 0. \tag{2.4.16}$$
- $\mathfrak{b}$ is an abelian algebra of linearized rescalings such that $\mathfrak{b} \ltimes \mathfrak{s}_0 \cong \mathfrak{s}$, where $\mathfrak{s}$ is the algebra of linearized supertranslations. This is the algebra consisting of vector fields of the form (2.4.15) where the function f satisfies
$$\pounds_\ell(\pounds_\ell f + \kappa f) = 0. \tag{2.4.17}$$

We now turn to the derivation of the structure (2.4.14). We define the subspace

$$\mathfrak{s} = \left\{\chi^i \in \mathfrak{g}_\mathfrak{u} | \chi^i = f\ell^i \text{ for some } f\right\}. \tag{2.4.18}$$

By comparison with Eqs. (2.4.7) and (2.4.8), we see that this subspace consists of the linearized supertranslations. Inserting the definition (2.4.18) into Eqs. (2.4.11) yields that the function f satisfies the condition (2.4.17) with

$$\beta(f\vec{\ell}) = -\pounds_\ell f. \tag{2.4.19}$$

The condition (2.4.17) is invariant under the scaling transformations

$$\vec{\ell} \to e^{\sigma}\vec{\ell}, \qquad f \to e^{-\sigma} f, \tag{2.4.20}$$

so the subspace $\mathfrak{s}$ is parameterized by functions of scaling weight 1 on $\mathcal{N}$ [cf. Eq. (2.3.4)]. The subspace $\mathfrak{s}$ is closed under the Lie bracket and so is a subalgebra; we have

$$[f_1\vec{\ell}, f_2\vec{\ell}] = (f_1 \pounds_\ell f_2 - f_2 \pounds_\ell f_2)\vec{\ell}. \tag{2.4.21}$$

Since the right hand side is nonvanishing in general, the subalgebra is nonabelian. Finally, for any $f\vec{\ell} \in \mathfrak{s}$ and any $\vec{\chi} \in \mathfrak{g}_\mathfrak{u}$, we have from Eqs. (2.4.11) that

$$[f\vec{\ell}, \vec{\chi}] = -\left[\pounds_\chi f + \beta(\vec{\chi}) f\right] \vec{\ell}. \tag{2.4.22}$$

Hence $[\mathfrak{s}, \mathfrak{g}_\mathfrak{u}] \subseteq \mathfrak{s}$, so $\mathfrak{s}$ is a Lie ideal of $\mathfrak{g}_\mathfrak{u}$.

Next, we define the subalgebra $\mathfrak{s}_0$ of $\mathfrak{s}$ by

$$\mathfrak{s}_0 = \left\{ f\ell^i | \pounds_\ell f + \kappa f = 0 \right\}. \tag{2.4.23}$$

By comparison with Eqs. (2.4.7) and (2.4.8), we see that this subalgebra consists of the linearized affine supertranslations, and it follows from Eq. (2.4.21) that it is abelian. The definition (2.4.23) is invariant under the rescalings (2.4.20). If we choose a representative (ℓ^i, κ) of the equivalence class with $\kappa = 0$, it follows that f is constant along generators and so can be regarded as a function on $\mathcal{Z}$. There is a residual rescaling freedom of the form (2.4.20) with $\pounds_\ell \sigma = 0$ that preserves $\kappa = 0$. Hence, the algebra $\mathfrak{s}_0$ can be identified with functions on the base space $\mathcal{Z}$ of scaling weight 1, from Eq. (2.3.4), just like supertranslations on $\mathscr{I}$[19].

Next, if $f_1\vec{\ell}$ and $f_2\vec{\ell}$ are elements of $\mathfrak{s}$, it follows from Eq. (2.4.17) that

$$\left(\pounds_\ell + \kappa\right)\left(f_1 \pounds_\ell f_2 - f_2 \pounds_\ell f_2\right) = 0. \tag{2.4.24}$$

Combining this with Eq. (2.4.21) shows that

$$[\mathfrak{s}, \mathfrak{s}] \subseteq \mathfrak{s}_0, \tag{2.4.25}$$

so $\mathfrak{s}_0$ is a Lie ideal of $\mathfrak{s}$. We define the quotient algebra $\mathfrak{b} \cong \mathfrak{s}/\mathfrak{s}_0$. This consists of equivalence classes of elements of $\mathfrak{s}$, where $f_1\vec{\ell} \sim f_2\vec{\ell}$ if $\pounds_\ell f_1 + \kappa f_1 = \pounds_\ell f_2 + \kappa f_2$. Elements of $\mathfrak{b}$ can be parameterized in terms of functions[20] on $\mathcal{Z}$ of scaling weight zero, and they correspond to

[19]Unlike the case with the BMS algebra, there is no preferred translation subalgebra of the affine supertranslation algebra $\mathfrak{s}_0$. Even if $\mathcal{Z}$ is topologically S^2, there is no universal metric on $\mathcal{Z}$, so it is not possible to single out a 4-dimensional subalgebra of translations by the first four spherical harmonics. Also, there is no scaling-invariant notion of constant functions on $\mathcal{Z}$, so there is not even a natural way to single out "time-translations".

[20]Essentially the functions $\pounds_\ell f + \kappa f$ projected to $\mathcal{Z}$, where $f\vec{\ell} \in \mathfrak{s}$.

linearized rescalings, cf. Eq. (2.4.8) above. It follows from Eq. (2.4.25) that $\mathfrak{b}$ is abelian, and so we obtain

$$\mathfrak{s} \cong \mathfrak{b} \ltimes \mathfrak{s}_0 \tag{2.4.26}$$

where both $\mathfrak{b}$ and $\mathfrak{s}_0$ are abelian.

We next argue that the quotient algebra $\mathfrak{g}_\mathfrak{u}/\mathfrak{s}$ is isomorphic to the algebra of linearized diffeomorphisms on the base space $\mathcal{Z}$,

$$\mathfrak{g}_\mathfrak{u}/\mathfrak{s} \cong \text{diff}(\mathcal{Z}), \tag{2.4.27}$$

which when combined with Eq. (2.4.26) gives the algebra structure[21] (2.4.14). The algebra $\mathfrak{g}_\mathfrak{u}/\mathfrak{s}$ consists of equivalence classes $[\chi^i]$ of vector fields χ^i in $g_\mathfrak{u}$, where two vector fields are equivalent if they differ by an element $f\ell^i$ in $\mathfrak{s}$. Pick a cross section $\mathcal{S}$ of $\mathcal{N}$, and denote by n_i the unique normal covector to $\mathcal{S}$ whose normalization is fixed by $n_i\ell^i = -1$. Given an equivalence class $[\chi^i]$, one can find a member χ^i with $\chi^i n_i = 0$ on $\mathcal{S}$, by using the freedom to add terms of the form $f\ell^i$ and using the fact that solutions to Eq. (2.4.17) can be freely specified on an initial cross section $\mathcal{S}$. This member χ^i can then be regarded as a vector field χ^A on $\mathcal{S}$, and by using the natural identification of $\mathcal{S}$ and $\mathcal{Z}$, as a vector field on $\mathcal{Z}$. We have thus defined a mapping from $\mathfrak{g}_\mathfrak{u}/\mathfrak{s}$ to diff$(\mathcal{Z})$. One can check that this mapping is onto, and it follows from Eqs. (2.4.11) that the identification of $\mathfrak{g}_\mathfrak{u}/\mathfrak{s}$ and diff$(\mathcal{Z})$ is independent of the choice of cross section $\mathcal{S}$. Thus we have derived the decomposition (2.4.14) of the algebra $\mathfrak{g}_\mathfrak{u}$.

For the computations of charges in Sec. 2.6 below, it will be useful to use an explicit decomposition of symmetry generators $\vec{\chi}$ into different pieces. However, because of the semi-direct structure $\mathfrak{g} \cong \text{diff}(\mathcal{Z}) \ltimes \mathfrak{s}$, there is no natural way to decompose a generator χ^i into a $\mathfrak{s}$-part and a diff$(\mathcal{Z})$-part. Such a decomposition requires an arbitrary choice of origin in $\mathfrak{s}$. We make such a choice by choosing a smooth covector n_i on $\mathcal{N}$, normalized so that

$$n_i\ell^i = -1. \tag{2.4.28}$$

The generator χ^i can then be uniquely decomposed as

$$\chi^i = f\ell^i + X^i, \tag{2.4.29}$$

where

$$X^i n_i = 0. \tag{2.4.30}$$

Here the first term $f\ell^i$ parameterizes the supertranslations, and the second term X^i parameterizes the diffeomorphisms on the base space.

In order for both terms on the right hand side of Eq. (2.4.29) to belong to $\mathfrak{g}_\mathfrak{u}$, from Eq. (2.4.17) it is necessary that

$$\pounds_\ell(\pounds_\ell + \kappa)(\chi^i n_i) = 0. \tag{2.4.31}$$

[21] The subalgebra $\mathfrak{s}_0$ is also a Lie ideal of $\mathfrak{g}_\mathfrak{u}$, but $\mathfrak{g}/\mathfrak{s}_0 \ncong \text{diff}(\mathcal{Z}) \ltimes \mathfrak{b}$.

Using Eq. (2.4.11), this will be automatically satisfied if n_i obeys the equation

$$\pounds_\ell(\pounds_\ell + \kappa)n_i + D_i\kappa = 0, \tag{2.4.32}$$

where D_i is any derivative operator on $\mathcal{N}$. This equation is invariant under the rescalings (2.4.1), since n_i transforms as $n_i \to e^{-\sigma}n_i$ from Eq. (2.4.28). If we choose n_i to be the normal covector to a foliation of surfaces in the natural class of foliations discussed in Sec. 2.4 above, normalized according to (2.4.28), then the condition (2.4.32) is satisfied.

Preferred subalgebra for stationary regions of a null hypersurface: Killing supertranslations

Stationary regions of the hypersurface $\mathcal{N}$ that intersect all the generators determine a preferred subalgebra $\mathfrak{t}$ of the supertranslation algebra $\mathfrak{s}$. This algebra is the set of vector fields χ^i in $\mathfrak{s}$ for which

$$\pounds_\tau\chi^i = 0 \tag{2.4.33}$$

in the stationary region, where τ^a is the Killing vector field which is normal to $\mathcal{N}$. Since (τ^i, κ_τ) is a representative of the equivalence class, we have from Eq. (2.4.17) that all elements χ^i of $\mathfrak{s}$ satisfy

$$\pounds_\tau(\pounds_\tau + \kappa_\tau)\chi^i = 0. \tag{2.4.34}$$

Hence it follows from Eqs. (2.3.34) and (2.4.33) that all solutions of Eq. (2.4.33) in the stationary region can be extended to vector fields on all of $\mathcal{N}$ which lie in $\mathfrak{s}$.

To get some insight into the nature of this subalgebra[22], specialize to a representative (ℓ^i, κ) and a coordinate system (u, θ^A) where $\vec{\ell} = \partial_u$ and $\kappa = 0$, where the general solution for χ^i is given by Eq. (2.4.8). Then the Killing field τ^i will be of the form $\tau^i = \kappa_\tau(u - u_0)\ell^i$, by Eqs. (2.3.3), (2.3.34) and (2.3.35), where κ_τ is a constant and u_0 is a function of θ^A but independent of u. The subalgebra $\mathfrak{t}$ is then given by the condition

$$\alpha - \beta u_0 = 0, \tag{2.4.35}$$

and consists of vector fields of the form $\chi^i = -(\beta/\kappa_\tau)\,\tau^i$. The corresponding transformation (2.4.7) can be expressed as

$$\overline{\tau} = \tau - \frac{\beta}{\kappa_\tau}, \tag{2.4.36}$$

where we have defined a Killing parameter τ by $\vec{\tau} = d/d\tau$. We will call these angle-dependent displacements of Killing parameter *Killing supertranslations*. They have been studied in Refs. [33, 39, 78, 31, 79, 80, 30, 28] (although they are often called just supertranslations). The intersection of the Killing supertranslation subalgebra $\mathfrak{t}$ with the affine supertranslation subalgebra $\mathfrak{s}_0$ will generically have dimension 0.

[22] The pullback τ^i of the Killing field is itself a member of the subspace $\mathfrak{t}$, giving a preferred one-dimensional subspace of “translations”.

We note that the Killing supertranslation subalgebra $\mathfrak{t}$ can be defined under the slightly weaker hypothesis that the region of $\mathcal{N}$ is *weakly isolated* in the sense of Ashtekar [74], which implies that it is shear and expansion free, satisfies Eqs. (2.3.34), and possesses a preferred choice of normal up to constant rescalings.

Symmetry groups of null hypersurfaces with boundaries

Our analysis so far has been restricted by the assumptions that the null hypersurface $\mathcal{N}$ has topology $\mathcal{Z} \times \mathbb{R}$, and that the intrinsic structure is complete, that is, that the generators of the null surface extend to infinite affine parameters in both directions. We now discuss how the symmetry group is modified when these assumptions are relaxed. Specifically, we will consider incomplete intrinsic structures. These generally occur when the null hypersurface $\mathcal{N}$ has a nontrivial topological boundary $\partial\mathcal{N}$ in M[23]. Rather than give a general analysis of the different possibilities, we will discuss two specific examples.

The first example is the future light cone of a point $\mathcal{P}$ in a spacetime which is spherically symmetric about $\mathcal{P}$. This could be the future event horizon of a black hole in a spherically symmetric gravitational collapse spacetime. Or, it could be the future light cone of a point in Minkowski spacetime. The null hypersurface still has topology $\mathcal{Z} \times \mathbb{R} \simeq S^2 \times \mathbb{R}$ (if the point $\mathcal{P}$ is excluded), but has the nontrivial boundary $\partial\mathcal{N} = \{\mathcal{P}\}$. The induced intrinsic structure is incomplete if the metric is smooth in a neighborhood of $\mathcal{P}$, as all the generators start at $\mathcal{P}$.

The second example is the future event horizon in the maximally extended Schwarzschild spacetime, on one of the two branches. In this case the boundary of $\mathcal{N}$ is the bifurcation twosphere, and the induced intrinsic structure is again incomplete, as all the generators start on the bifurcation twosphere.

In these cases, the definition of the symmetry group is modified to include the requirement that it preserve the boundary:

$$G_{\mathfrak{u}} = \{\varphi : \mathcal{N} \to \mathcal{N} \mid \varphi_* \mathfrak{u} = \mathfrak{u}, \quad \varphi(\partial\mathcal{N}) = \partial\mathcal{N}\}\,. \tag{2.4.37}$$

In the first case of a single point, $\partial\mathcal{N} = \{\mathcal{P}\}$, the corresponding Lie algebra consists of the vector fields χ^i which satisfy Eqs. (2.4.11) and in addition the condition

$$\chi^i\big|_{\partial\mathcal{N}} = 0. \tag{2.4.38}$$

This removes the affine supertranslations and but not the rescalings or diff(S^2) diffeomorphisms. If one chooses an affine coordinate system (u, θ^A) of the type described in Sec. 2.4, specialized so that $u = 0$ on $\partial\mathcal{N}$, then the transformation group (2.4.7) is modified by the condition.

$$\alpha = 0. \tag{2.4.39}$$

[23]The hypersurface $\mathcal{N}$ can have a nontrivial boundary only when $\mathcal{N}$ is a proper subset of the boundary ∂M of M, as it will be in typical applications, since $\partial\partial M = \{\}$.

In the second case of the bifurcation twosphere, the condition (2.4.38) is replaced by the requirement that the vector field be tangent to $\partial\mathcal{N}$ on $\partial\mathcal{N}$,

$$\chi^i n_i\big|_{\partial\mathcal{N}} = 0, \tag{2.4.40}$$

where n_i is the normal to $\partial\mathcal{N}$. The modification to the algebra is the same as in the first case, given by the condition (2.4.39).

We note that in this context the Killing supertranslation subalgebra $\mathfrak{t}$ associated with stationary regions of the null surface will generically have dimension 0, by Eqs. (2.4.35) and (2.4.39). This is discussed further in Sec. 2.7 below (footnote 29).

2.5 General relativity with a null boundary: covariant phase space

As discussed in Sec. 2.2, the starting point of the Wald-Zoupas framework is the definition of a field configuration space $\mathscr{F}$ of kinematically allowed field configurations, and the corresponding covariant phase space $\overline{\mathscr{F}} \subset \mathscr{F}$ obtained by restricting attention to on-shell field configurations. In this section we give a particular version of these definitions for general relativity in the presence of a null boundary in $3+1$ dimensions. The definition is given in Sec. 2.5, and in Sec. 2.5 we show that the symmetry group and algebra associated with this field configuration space coincide with those of the universal intrinsic structure of the null surface discussed in Sec. 2.4.

Definition of field configuration space

Consider a manifold M with boundary, for which a manifold $\mathcal{N}$ is a portion of the boundary. We would like to consider the space $\mathscr{F}_0$ consisting of smooth metrics g_{ab} on M for which the boundary $\mathcal{N}$ is null and for which the induced boundary structure on $\mathcal{N}$ is complete. This space $\mathscr{F}_0$ is not the field configuration space $\mathscr{F}$ we seek, since it contains a considerable amount of diffeomorphism redundancy. We will obtain our definition of $\mathscr{F}$ by fixing some of this freedom.

The kinds of fixing of diffeomorphism freedom that we will allow will be restricted by three general considerations:

- They must be global on the field configuration space, not restricted to on-shell configurations.
- They must be local to the boundary in the sense that the diffeomorphisms needed to enforce the gauge condition can be computed from degrees of freedom on the boundary.
- Field configurations (metrics in this case) and their derivatives evaluated on the boundary induce on the boundary certain geometric structures, which can be divided into

universal and non-universal structures. The universal structures are the same for all field configurations (up to boundary diffeomorphisms), while the non-universal ones depend on the field configuration. We restrict attention to fixings of the diffeomorphism freedom that involve only the universal structures.

The diffeomorphism (and conformal freedom) fixings used at future null infinity by Wald and Zoupas [24] are also of this type.

As a side note, as discussed in Sec. 2.2 above, gauge in this context is not synonymous with diffeomorphism freedom, since there are some diffeomorphisms that act on the boundary which do not correspond to degeneracies of the symplectic form on phase space (a more fundamental notion of gauge). Some of the diffeomorphism freedom we fix in going from $\mathscr{F}_0$ to $\mathscr{F}$ is not gauge in this sense. For this reason, it would be desirable to consider a larger field configuration space that includes all metric variations that are not degeneracy directions of the symplectic form. In Appendix A.8 we explore a modification of our definition of the field configuration space which yields a modified and larger algebra of symmetries and a modified set of charges. The main drawback of this modification is that it is no longer possible to obtain uniqueness of the prescription for defining localized charges by demanding that fluxes vanishes for stationary solutions, as discussed in Sec. 2.2 above. It is possible that a unique prescription may be obtained from some other criterion.

Our definition of the field configuration space $\mathscr{F}$ proceeds as follows. We start by defining a particular geometric structure on $\mathcal{N}$ which we will call a *boundary structure*. We consider triples $(\ell^a, \kappa, \hat{\ell}_a)$ of fields on $\mathcal{N}$, where ℓ^a is a smooth, nowhere vanishing vector field, κ is a smooth function, $\hat{\ell}_a$ is a choice of normal covector[24] to $\mathcal{N}$, and

$$\ell^a \hat{\ell}_a \mathrel{\widehat{=}} 0. \tag{2.5.1}$$

Recall that we are using $\mathrel{\widehat{=}}$ to mean equality when restricted to $\mathcal{N}$. We define two such triples $(\ell^a, \kappa, \hat{\ell}_a)$ and $(\ell'^a, \kappa', \hat{\ell}'_a)$ to be equivalent if they are related by the rescaling

$$\begin{aligned} \ell'^a &\mathrel{\widehat{=}} e^\sigma \ell^a, && (2.5.2a)\\ \kappa' &\mathrel{\widehat{=}} e^\sigma(\kappa + \pounds_\ell \sigma), && (2.5.2b)\\ \hat{\ell}'_a &\mathrel{\widehat{=}} e^\sigma \hat{\ell}_a, && (2.5.2c) \end{aligned}$$

where σ is a smooth function on $\mathcal{N}$. We denote by

$$\mathfrak{p} = [\ell^a, \kappa, \hat{\ell}_a] \tag{2.5.3}$$

the equivalence class associated with $(\ell^a, \kappa, \hat{\ell}_a)$. A choice of equivalence class is the desired boundary structure on $\mathcal{N}$.

[24]This normal covector was denoted ℓ_a earlier in the paper. We introduce the separate notation $\hat{\ell}_a$ because the context here of the definition of a boundary structure does not involve a metric, and to clarify that there are two independent tensor fields in the definition.

It is clear that a choice of boundary structure $\mathfrak{p} = [\ell^a, \kappa, \hat{\ell}_a]$ determines a unique universal intrinsic structure $\mathfrak{u}$: choose a representative $(\ell^a, \kappa, \hat{\ell}_a)$, discard $\hat{\ell}_a$, and note that from Eq. (2.5.1) that ℓ^a can be regarded as an intrinsic vector field ℓ^i. Then from (ℓ^i, κ) form the equivalence class $\mathfrak{u} = [\ell^i, \kappa]$ under the equivalence relation (2.4.1). From Eqs. (2.4.1) and (2.5.2) the result is independent of the representative $(\ell^a, \kappa, \hat{\ell}_a)$ initially chosen. We will denote this induced intrinsic structure by $\mathfrak{u}(\mathfrak{p})$. Our boundary structures contain more information than the intrinsic structures, which will be necessary for the definition of the field configuration space. We will say that a boundary structure $\mathfrak{p}$ is complete if the corresponding intrinsic structure $\mathfrak{u}$ is complete.

In addition, a metric g_{ab} on M for which the boundary $\mathcal{N}$ is null determines a unique boundary structure $\mathfrak{p}$, just as for intrinsic structures discussed in Sec. 2.4. Pick a normal covector $\hat{\ell}_a$ to $\mathcal{N}$, raise the index to obtain $\ell^a = g^{ab}\hat{\ell}_b$, and compute the non-affinity κ using the metric via Eq. (2.3.2). Then from the triple $(\ell^a, \kappa, \hat{\ell}_a)$ form the equivalence class $\mathfrak{p} = [\ell^a, \kappa, \hat{\ell}_a]$. The result is independent of the choice of initial normal covector, by the equivalence relation (2.5.2).

Given a boundary structure $\mathfrak{p}$, we now define the field configuration space $\mathscr{F}_{\mathfrak{p}}$ to be the set of smooth metrics g_{ab} on M which satisfy on $\mathcal{N}$ the relations

$$\ell^a \quad \hat{=} \quad g^{ab}\hat{\ell}_b, \tag{2.5.4a}$$

$$\ell^a \nabla_a \ell^b \quad \hat{=} \quad \kappa \ell^b. \tag{2.5.4b}$$

From Eqs. (2.5.1) and (2.5.4a) it follows that the boundary $\mathcal{N}$ is null with respect to g_{ab}, so that $\mathscr{F}_{\mathfrak{p}} \subset \mathscr{F}_0$. Also if the conditions (2.5.4) are satisfied by one representative $(\ell^a, \kappa, \hat{\ell}_a)$, they will be satisfied by all representatives, from Eqs. (2.3.3) and (2.5.2). Hence $\mathscr{F}_{\mathfrak{p}}$ is well defined and depends only on $\mathfrak{p}$. (An equivalent definition of $\mathscr{F}_{\mathfrak{p}}$ is the set of smooth metrics on M for which $\mathcal{N}$ is null and whose associated boundary structures agree with $\mathfrak{p}$). We define the corresponding covariant phase space $\overline{\mathscr{F}}_{\mathfrak{p}}$ to be the set of metrics in $\mathscr{F}_{\mathfrak{p}}$ which satisfy the equations of motion.

Note that the order of definitions being used in this construction is the opposite of that which is normally used. Normally, one first picks the spacetime metric, then defines the covariant version of the null normal by raising the index as in Eq. (2.5.4a), and defines the non-affinity function κ via Eq. (2.5.4b). Here, instead, we first choose the quantities ℓ^a, $\hat{\ell}_a$, and κ, and then specialize the spacetime metric g_{ab} to enforce Eqs. (2.5.4).

It may appear that the conditions (2.5.4) we are imposing on the metric are overly restrictive. In fact, they do not restrict the physical degrees of freedom in the sense that $\mathscr{F}_{\mathfrak{p}}$ is obtained from $\mathscr{F}_0$ by a fixing of the diffeomorphism freedom. More precisely, given a complete boundary structure $\mathfrak{p}$, and given any metric g_{ab} on M for which $\mathcal{N}$ is null and for which the boundary structure induced by g_{ab} is complete, one can find a diffeomorphism $\psi : M \to M$ which takes $\mathcal{N}$ into $\mathcal{N}$ for which $\psi_* g_{ab}$ satisfies the conditions (2.5.4). This is proved in Appendix A.2.

We next show that the mapping $\mathfrak{p} \to \mathscr{F}_{\mathfrak{p}}$ is injective, so that if $\mathscr{F}_{\mathfrak{p}} = \mathscr{F}_{\mathfrak{p}'}$ then $\mathfrak{p} = \mathfrak{p}'$. This property will be used in Sec. 2.5 below. Let $(\ell^a, \kappa, \hat{\ell}_a)$ be a representative of $\mathfrak{p}$, and

$(\ell'^a, \kappa', \hat{\ell}'_a)$ be a representative of $\mathfrak{p}'$. Since $\hat{\ell}_a$ and $\hat{\ell}'_a$ are both normals to $\mathcal{N}$, they are related by a rescaling, and hence by adjusting our choice of representative we can without loss of generality take $\hat{\ell}_a = \hat{\ell}'_a$. Now pick a metric g_{ab} which belongs to both $\mathscr{F}_\mathfrak{p}$ and $\mathscr{F}_{\mathfrak{p}'}$. Applying Eq. (2.5.4a) to both $\mathfrak{p}$ and $\mathfrak{p}'$ we find that $\ell^a = \ell'^a$, and it follows from Eq. (2.5.4b) that $\kappa = \kappa'$. Hence we have $\mathfrak{p} = \mathfrak{p}'$.

Symmetry algebra of the field configuration space

We now show that for a complete boundary structure $\mathfrak{p}$, the symmetry algebra associated with the field configuration space $\mathscr{F}_\mathfrak{p}$ coincides with the algebra $\mathfrak{g}_\mathfrak{u}$ of the universal intrinsic structure $\mathfrak{u}$ of the null surface discussed in Sec. 2.4, where $\mathfrak{u} = \mathfrak{u}(\mathfrak{p})$ is the intrinsic structure obtained from $\mathfrak{p}$ discussed in Sec. 2.5.

We start by defining the group of diffeomorphisms on M whose pullbacks preserve the boundary and the field configuration space:

$$H_\mathfrak{p} = \{\psi : M \to M \mid \psi(\mathcal{N}) = \mathcal{N},\ \psi_* \mathscr{F}_\mathfrak{p} = \mathscr{F}_\mathfrak{p}\ \}. \tag{2.5.5}$$

These diffeomorphisms induce diffeomorphisms of the boundary: for any ψ in $H_\mathfrak{p}$ we define

$$\varphi = \psi|_\mathcal{N}, \tag{2.5.6}$$

and since ψ preserves the boundary, φ is a diffeomorphism from $\mathcal{N}$ to $\mathcal{N}$. Next, since ψ preserves the boundary, the pullback of the normal must be a rescaling of the normal, so we have

$$\psi_* \hat{\ell}_a = e^\gamma \hat{\ell}_a, \tag{2.5.7}$$

where $\gamma = \gamma(\psi, \hat{\ell}_a)$ is a smooth function on $\mathcal{N}$ which depends on the diffeomorphism and on the normalization of the normal covector $\hat{\ell}_a$. From Eq. (2.5.7) we find for the dependence on the normalization [*cf.* Eq. (2.4.6) above]

$$\gamma(\psi, e^\sigma \hat{\ell}_a) = \gamma(\psi, \hat{\ell}_a) + \psi_* \sigma - \sigma, \tag{2.5.8}$$

for any smooth function σ on $\mathcal{N}$.

Next from the definition (2.5.5) we have

$$\mathscr{F}_\mathfrak{p} = \psi_* \mathscr{F}_\mathfrak{p} = \mathscr{F}_{\psi_* \mathfrak{p}}, \tag{2.5.9}$$

where the action of the pullback ψ_* on the boundary structure $\mathfrak{p}$ is defined by its action on a representative $(\ell^a, \kappa, \hat{\ell}_a)$. Now using the injectivity property of the mapping $\mathfrak{p} \to \mathscr{F}_\mathfrak{p}$ proved in Sec. 2.5, we obtain

$$\psi_* \mathfrak{p} = \mathfrak{p}. \tag{2.5.10}$$

From the definition (2.5.2) of the equivalence class, it follows that for a given representative $(\ell^a, \kappa, \hat{\ell}_a)$ in $\mathfrak{p}$, the pullback ψ_* acts as a scaling transformation for some smooth scaling function $\beta = \beta(\psi)$ on $\mathcal{N}$ [cf. Eq. (2.4.4) above]:

$$\psi_* \ell^a \ \hat{=} \ e^\beta \ell^a, \tag{2.5.11a}$$
$$\psi_* \kappa \ \hat{=} \ e^\beta (\kappa + \pounds_\ell \beta), \tag{2.5.11b}$$
$$\psi_* \hat{\ell}_a \ \hat{=} \ e^\beta \hat{\ell}_a. \tag{2.5.11c}$$

In the first two of these equations we can replace ℓ^a with ℓ^i, by Eq. (2.5.1), and we can replace ψ_* with φ_*. These two equations then coincide with the defining equations (2.4.4) for the group $G_\mathfrak{u}$ of boundary symmetries $\varphi : \mathcal{N} \to \mathcal{N}$ that preserve the intrinsic structure $\mathfrak{u}$ associated with $\mathfrak{p}$. Combining Eqs. (2.5.7) and (2.5.11c) yields that

$$\gamma(\psi) = \beta(\varphi). \tag{2.5.12}$$

Hence we have shown that

$$H_\mathfrak{p} = \{\psi : M \to M \mid \psi(\mathcal{N}) = \mathcal{N}, \ \varphi \in G_\mathfrak{u}, \ \beta(\varphi) = \gamma(\psi) \ \}, \tag{2.5.13}$$

where φ is the diffeomorphism (2.5.6) induced on the boundary. A bulk diffeomorphism ψ is a symmetry if it preserves the boundary, if the induced boundary diffeomorphism is a symmetry of the intrinsic structure on the boundary, and if the scaling function $\gamma(\psi)$ defined by Eq. (2.5.7) satisfies Eq. (2.5.12).

We next specialize these results to infinitesimal diffeomorphisms. Linearized diffeomorphisms on M are parameterized in terms of vector fields ξ^a on M, and the boundary is preserved if these vector fields are tangent to the boundary,

$$\xi^a \hat{\ell}_a \ \hat{=} \ 0. \tag{2.5.14}$$

We define

$$\vec{\chi} = \vec{\xi}|_\mathcal{N}, \tag{2.5.15}$$

and it follows from the condition (2.5.14) that we can regard $\vec{\chi}$ as an intrinsic vector field χ^i on $\mathcal{N}$, as in Sec. 2.4 above. The definition (2.5.7) of the scaling function γ becomes

$$\pounds_\xi \hat{\ell}_a \ \hat{=} \ \gamma(\xi^a, \hat{\ell}_a) \hat{\ell}_a, \tag{2.5.16}$$

while the dependence (2.5.8) on the normalization of the normal becomes

$$\gamma(\xi^a, e^\sigma \hat{\ell}_a) = \gamma(\xi^a, \hat{\ell}_a) + \pounds_\xi \sigma. \tag{2.5.17}$$

The linearized version of the constraint (2.5.12) is

$$\gamma(\xi^a) = \beta(\chi^i). \tag{2.5.18}$$

Defining $\mathfrak{h}_\mathfrak{p}$ to be the Lie algebra corresponding to the group $H_\mathfrak{p}$, we find by linearizing the result (2.5.13) that

$$\mathfrak{h}_\mathfrak{p} = \left\{ \xi^a \text{ on } M | \ \xi^a \hat{\ell}_a \ \hat{=} \ 0, \ \chi^i \in \mathfrak{g}_\mathfrak{u}, \ \beta(\chi^i) = \gamma(\xi^a) \ \right\}, \tag{2.5.19}$$

where χ^i is given by the restriction (2.5.15) to the boundary and $\beta(\chi^i)$ is defined by Eq. (2.4.11a).

Finally, to obtain the physical symmetry algebra, we need to factor out the trivial diffeomorphisms for which the symmetry generator charge variation (2.2.13) vanishes, using the equivalence relation $\sim$ defined in Eq. (2.2.18). In Sec. 2.6 below we compute the charge variation (2.2.13) explicitly, and in Appendix A.3 we show that it vanishes for all metric perturbations if and only if χ^i and $\gamma(\xi^a)$ both vanish. Hence the quotient set $\mathfrak{h}_\mathfrak{p}/\sim$ is parameterized by χ^i and γ, but from Eq. (2.5.18) γ is determined by χ^i. We conclude from Eq. (2.5.13) that

$$\mathfrak{h}_\mathfrak{p}/\sim \;\;\cong\;\; \mathfrak{g}_\mathfrak{u}, \tag{2.5.20}$$

as claimed.

To summarize, infinitesimal symmetries are in one-to-one correspondence with symmetries $\chi^i \in \mathfrak{g}_\mathfrak{u}$ of the intrinsic structure $\mathfrak{u}$. However, all representatives ξ^a whose restriction to the boundary is χ^i must also obey the constraint (2.5.18).

Boundary conditions on the variation of the metric

In our application of the Wald-Zoupas formalism we will need to consider variations of the metric of the form

$$g_{ab} \to g_{ab} + \delta g_{ab} = g_{ab} + h_{ab}. \tag{2.5.21}$$

We assume that both the original and varied metric lie in the configuration space $\mathscr{F}_\mathfrak{p}$, so that they both satisfy conditions (2.5.4). In this subsection, we will derive some resulting boundary conditions on the metric variation h_{ab} that will be useful in later sections of the paper. Specifically these conditions are

$$h_{ab}\ell^b \;\;\hat{=}\;\; 0, \tag{2.5.22a}$$
$$\nabla_c(h_{ab}\ell^a\ell^b) \;\;\hat{=}\;\; 0. \tag{2.5.22b}$$

Note that the condition (2.5.22b) is independent of the definition of ℓ^a off the surface, because of Eq. (2.5.22a). As a consistency check of our computations, we show in Appendix A.4 that the conditions (2.5.22) are automatically satisfied for a metric perturbation of the form

$$h_{ab} = \pounds_\xi g_{ab} \tag{2.5.23}$$

generated by a representative ξ^a of a symmetry in the algebra discussed in the previous section.

We now turn to the derivation of Eqs. (2.5.22). Equation (2.5.22a) follows from taking the variation of Eq. (2.5.4a) and noting that ℓ^a and $\hat{\ell}_a$ are fixed under the variation. By varying the definition (2.5.4b) of non-affinity and noting that κ is fixed under the variation we find

$$\nabla_a h_{bc}\ell^a\ell^c - \nabla_b h_{ac}\ell^a\ell^c/2 = 0. \tag{2.5.24}$$

We can rewrite the first term as $\ell^a \nabla_a(\ell^c h_{bc}) - (\ell^a \nabla_a \ell^c) h_{bc}$. The first term here vanishes by Eq. (2.5.22a) since the derivative is along the surface $\mathcal{N}$, while the second vanishes by Eqs. (2.5.4b) and (2.5.22a). Thus we obtain $\nabla_b h_{ac} \ell^a \ell^c = 0$, which is equivalent to Eq. (2.5.22b) by Eq. (2.5.22a).

It follows from Eq. (2.5.22a) that we can regard h^{ab} restricted to $\mathcal{N}$ as an intrinsic tensor h^{ij} on $\mathcal{N}$. We can also construct the down index versions

$$h_i{}^j = q_{ik} h^{kj} \tag{2.5.25a}$$
$$h_{ij} = q_{ik} q_{jl} h^{kl} = \Pi_i^a \Pi_j^b h_{ab}. \tag{2.5.25b}$$

These quantities satisfy

$$h_{ij} \ell^i \mathrel{\hat{=}} h_i{}^j \ell^i \mathrel{\hat{=}} 0, \tag{2.5.26}$$

from Eqs. (2.3.7) and (2.5.22a). In four spacetime dimensions, h^{ij} contains six independent components, $h_i{}^j$ five, and h_{ij} three. We will express charge variations in Sec. 2.6 below in terms of $h_i{}^j$.

A useful quantity involving the metric perturbation that will appear in the charge variations can be defined as follows. Defining $h_a = h_a{}^b \hat{\ell}_b$, we have from Eqs. (2.5.4a) and (2.5.22a) that h_a vanishes on $\mathcal{N}$. Hence there exists a one-form Γ_a on $\mathcal{N}$ so that

$$\nabla_{[a} h_{b]} \mathrel{\hat{=}} \hat{\ell}_{[a} \Gamma_{b]}. \tag{2.5.27}$$

The quantity Γ_a depends linearly on $h_a{}^b$ and its first derivatives, including in directions off the surface $\mathcal{N}$, but is independent of the background metric and connection. It does depend on how one extends the definition of $\hat{\ell}_a$ off the surface $\mathcal{N}$. However if we impose on this extension the condition

$$\nabla_{[a} \hat{\ell}_{b]} \mathrel{\hat{=}} 0, \tag{2.5.28}$$

then Γ_a is uniquely determined. From Eq. (2.5.22b) it satisfies

$$\Gamma_a \ell^a = 0. \tag{2.5.29}$$

It is invariant under a rescaling of the normal $\ell^a \to e^\sigma \ell^a$. We also define the pullback $\Gamma_i = \Pi_i^a \Gamma_a$.

2.6 Global and localized charges for a null boundary component

From the perspective of the covariant phase space, we have seen in the last two sections that general relativity in the presence of null boundaries has quite a rich structure as encapsulated in the the infinite dimensional symmetry algebra $\mathfrak{g}_{\mathfrak{u}}$. With these symmetries at hand, in this section we move on to the calculation of the corresponding charges and fluxes. We compute the Noether charge Q_ξ in Sec. 2.6, its variation δQ_ξ in Sec. 2.6, the boundary symmetry generator $\mathcal{Q}_\xi$ and its variation in Sec. 2.6, and the localized charge or Wald-Zoupas charge $\mathcal{Q}_\xi^{\rm loc}$ and its flux in Sec. 2.6.

Appendix A.8 computes the corresponding charges for a modified definition of field configuration space. As mentioned in the previous section the drawback of the modification is that one looses uniqueness in the prescription for defining localized charges.

Noether charge

For general relativity in vacuum, the Lagrangian, presymplectic potential 3-form and Noether charge 2-form are given by [24]

$$
\begin{aligned}
L_{abcd} &= \tfrac{1}{16\pi}\varepsilon_{abcd}\ R, && (2.6.1a)\\
\theta_{abc} &= \tfrac{1}{16\pi}\varepsilon_{abc}{}^{d}\ (g^{ef}\nabla_d h_{ef} - \nabla^e h_{de}), && (2.6.1b)\\
Q_{\xi\ ab} &= -\tfrac{1}{16\pi}\varepsilon_{abcd}\ \nabla^c\xi^d. && (2.6.1c)
\end{aligned}
$$

The ambiguity (2.2.33) in the presymplectic potential can be resolved in the case of general relativity by demanding that the total number of derivatives of the metric g_{ab} or metric perturbation h_{ab} in the 2-form Y_{ab} be two less than the number of derivatives appearing in the Lagrangian. One can readily convince oneself that there is no 2-form Y_{ab} that depends on g_{ab}, ε_{abcd} and h_{ab} that depends linearly on h_{ab} and has no derivatives.

We now evaluate the pullback of the Noether charge 2-form to $\mathcal{N}$. From Eqs. (2.3.20) and (2.3.6) we find

$$Q_{\xi\ ij} = -\frac{1}{16\pi}\Pi^a_i\Pi^b_j(\overline{\varepsilon}_{abc}\hat{\ell}_d - \overline{\varepsilon}_{dab}\hat{\ell}_c + \overline{\varepsilon}_{cda}\hat{\ell}_b - \overline{\varepsilon}_{bcd}\hat{\ell}_a)\nabla^c\xi^d = -\frac{1}{16\pi}\Pi^a_i\Pi^b_j\overline{\varepsilon}_{abc}q^c \tag{2.6.2}$$

where $q^c = 2\hat{\ell}_d\nabla^{[c}\xi^{d]}$. Since $\hat{\ell}_a q^a = 0$ it follows from Eqs. (2.3.7) and (2.3.21) that we can rewrite this expression in terms of tensors intrinsic to $\mathcal{N}$.

$$Q_{\xi\ ij} = -\frac{1}{16\pi}\varepsilon_{ijk}q^k. \tag{2.6.3}$$

We can rewrite q^c as

$$q^c = g^{cd}\pounds_\xi\hat{\ell}_d + \pounds_\xi\ell^c - 2\xi^b\nabla_b\ell^c, \tag{2.6.4}$$

where we have used the validity of Eq. (2.5.4a) on $\mathcal{N}$ and the fact that $\xi^b\nabla_b$ differentiates along the surface, by Eq. (2.5.14). Next, using the definitions (2.4.11a) and (2.5.7) of β and γ and the condition (2.5.18) we obtain

$$q^c = (\beta+\gamma)\ell^c - 2\xi^b\nabla_b\ell^c = 2\beta\ell^c - 2\xi^b\nabla_b\ell^c. \tag{2.6.5}$$

From Eqs. (2.3.7) and (2.5.14) the contracted b index in the second term can be replaced by an intrinsic index k, and we can then use the definition (2.3.16) of the Weingarten map. Inserting the result into (2.6.3) and using (2.5.15) gives

$$Q_{\xi\ ij} = \frac{1}{8\pi}\varepsilon_{ijk}\left[\chi^l\mathcal{K}_l{}^k - \beta(\chi^i)\ell^k\right]. \tag{2.6.6}$$

This expression is invariant under the scaling transformation (2.4.1), from the transformation properties (2.3.24c), (2.4.12) and (2.3.24e).

Suppose now that we are given a cross section $\mathcal{S}$ of $\mathcal{N}$. The Noether charge associated with that cross section is given by integrating the two form (2.6.6). Letting n_i denote the unique normal covector to $\mathcal{S}$ in $\mathcal{N}$ with the normalization (2.4.28), we obtain from the definition (2.3.23)

$$Q_\xi(\mathcal{S}) = \int_{\mathcal{S}} \boldsymbol{Q}_\xi = \frac{1}{8\pi}\int_{\mathcal{S}} \varepsilon_{ij}\left[\chi^l n_k \mathcal{K}_l{}^k + \beta(\chi^i)\right]. \tag{2.6.7}$$

Variation of Noether charge

We next turn to computing the variation of the Noether charge under a variation of the metric of the form (2.5.21). Of all the quantities which appear in the expression (2.6.6) for the pullback of the Noether charge two-form, only the volume form ε_{ijk} and the Weingarten map $\mathcal{K}_l{}^k$ vary as the metric is varied. Using $\delta\varepsilon_{abcd} = h\varepsilon_{abcd}/2$ with $h = g^{ab}h_{ab} = q^{AB}\delta q_{AB}$ we obtain

$$16\pi\delta Q_{\xi\ ij} = \varepsilon_{ijk}\left[h\chi^l\mathcal{K}_l{}^k - h\beta(\chi^i)\ell^k + 2\chi^l\delta\mathcal{K}_l{}^k\right]. \tag{2.6.8}$$

To compute the variation of the Weingarten map we define $K_a{}^b = \nabla_a\ell^b$, which when we pullback the a index is orthogonal to $\hat{\ell}_b$ on the b index and reduces to $\mathcal{K}_l{}^k$. Taking a variation we find

$$2\Pi^a_i\delta K_a{}^b = \Pi^a_i\left[-\nabla^b h_{ac} + \nabla_a h^b{}_c + \nabla_c h_a{}^b\right]\ell^c. \tag{2.6.9}$$

We now use the definition (2.5.27) of Γ_a to rewrite the first two terms in (2.6.9), and rewrite the last term in terms of a Lie derivative. This yields

$$2\Pi^a_i\delta K_a{}^b = \Pi^a_i\left[\hat{\ell}_a\Gamma^b - \Gamma_a\hat{\ell}^b - h^{bc}\nabla_a\hat{\ell}_c + h_a{}^c\nabla^b\hat{\ell}_c + \pounds_\ell h_a{}^b + h^c{}_a\nabla_c\ell^b - h_c{}^b\nabla_a\ell^c\right]. \tag{2.6.10}$$

The first term here vanishes by Eq. (2.3.6). We can replace the $\hat{\ell}^a$ with ℓ^a in the second and third terms, using the condition (2.5.4a) and the fact that the derivative is along the surface. We rewrite the fourth term using the condition (2.5.28) as $h_a{}^c\nabla_c\hat{\ell}^b = h_a{}^c\nabla_c\ell^b$, where we have used the fact that the derivative is along the surface by Eq. (2.5.22a). We thus obtain

$$2\Pi^a_i\delta K_a{}^b = \Pi^a_i\left[-\Gamma_a\ell^b + \pounds_\ell h_a{}^b + 2h^c{}_a\nabla_c\ell^b - 2h^b{}_c\nabla_a\ell^c\right]. \tag{2.6.11}$$

Now the individual terms on the right hand side are all orthogonal[25] to $\hat{\ell}_b$, by Eqs. (2.5.1) and (2.5.22a). Hence they all give rise to tensors intrinsic to $\mathcal{N}$. Also the contractions on the c index in the last two terms can be replaced by intrinsic contractions, by Eqs. (2.3.7) and (2.5.22a). Using the definition (2.3.16) finally gives

$$\delta\mathcal{K}_i{}^j = -\frac{1}{2}\Gamma_i\ell^j + \frac{1}{2}\pounds_\ell h_i{}^j + h^k{}_i\mathcal{K}_k{}^j - h^j{}_k\mathcal{K}_i{}^k. \tag{2.6.12}$$

[25] For the second term this is because $\hat{\ell}_b\pounds_\ell h_a{}^b = \pounds_\ell(h_a{}^b\hat{\ell}_b) - h_a{}^b\pounds_\ell\hat{\ell}_b = -h_a{}^b\ell^c\pounds_\ell g_{bc} = -h_a{}^b\nabla_b(\ell_c\ell^c)/2 - \kappa h_a{}^b\ell_b = 0$.

From Eq. (2.6.8) we obtain for the variation of the pullback of the Noether charge two-form

$$\delta Q_{\xi\ ij} = \frac{1}{16\pi}\varepsilon_{ijk}\left[h\chi^l \mathcal{K}_l{}^k - h\beta(\chi^i)\ell^k - \chi^l\Gamma_l\ell^k + \chi^l\pounds_\ell h_l{}^k + 2\chi^l h^m{}_l\mathcal{K}_m{}^k - 2\chi^l h^k{}_m\mathcal{K}_l{}^m\right]. \tag{2.6.13}$$

Global charges that generate boundary symmetries

As described in Sec. 2.2 above, the charge $\mathcal{Q}_\xi$ that generates a boundary symmetry ξ^a has a variation $\delta\mathcal{Q}_\xi$ which is an integral of the form (2.2.12) over a Cauchy surface Σ, and can be expressed as the surface integral (2.2.13) over the boundary $\partial\Sigma$. We now assume that a cross section $\mathcal{S}$ of the null surface $\mathcal{N}$ is a component of the boundary $\partial\Sigma$. This gives

$$\delta\mathcal{Q}_\xi = \int_{\mathcal{S}} \delta\mathcal{Q}_{\xi\ ab} + \ldots \tag{2.6.14}$$

where the ellipses represent integrals over the remaining components of $\partial\Sigma$ (for example spatial infinity). Here the integrand is the two-form

$$\delta\mathcal{Q}_{\xi\ ab} = \delta Q_{\xi\ ab} - \xi^c\theta_{cab} \tag{2.6.15}$$

and θ_{abc} is the presymplectic potential three-form (2.6.1b). Pulling back this expression to $\mathcal{N}$ using Eqs. (2.3.20) and (2.3.21) gives

$$\theta_{ijk} = \frac{1}{16\pi}\varepsilon_{ijk}\ell^f\left(\nabla_f h - \nabla_e h_f{}^e\right). \tag{2.6.16}$$

The second term can be rewritten using the boundary conditions (2.5.22a) and (2.5.22b) as $-l^f\nabla_e h_f{}^e = h_f{}^e\nabla_e\ell^f$, which then allows using the definition (2.3.16). This yields

$$\theta_{ijk} = \frac{1}{16\pi}\varepsilon_{ijk}\left[\pounds_\ell h + h_i{}^j\mathcal{K}_j{}^i\right]. \tag{2.6.17}$$

In this expression the trace h of the metric perturbation can be written as $q^{AB}h_{AB}$, from the condition (2.5.22a), while the second term in the brackets can be written as[26]

$$h_{AB}\mathcal{K}^{AB} = h_{AB}\left(\frac{1}{2}\theta q^{AB} + \sigma^{AB}\right), \tag{2.6.18}$$

from Eqs. (2.3.8), (2.3.13), (2.3.14), (2.3.19) and (2.5.26). Finally using Eqs. (2.6.13) and (2.6.17) yields for the pullback of the perturbed symmetry generator two-form (2.6.15)

$$\begin{aligned}\delta\mathcal{Q}_{\xi\ ij} \;=\; & \frac{1}{16\pi}\varepsilon_{ijk}\Big[h\chi^l\mathcal{K}_l{}^k - h\beta(\chi^i)\ell^k - \chi^l\Gamma_l\ell^k + \chi^l\pounds_\ell h_l{}^k + 2\chi^l h^m{}_l\mathcal{K}_m{}^k \\ & -2\chi^l h^k{}_m\mathcal{K}_l{}^m - \chi^k\pounds_\ell h - \chi^k h_i{}^j\mathcal{K}_j{}^i\Big].\end{aligned} \tag{2.6.19}$$

[26] An alternative form is $h_{ij}K_{kl}q^{ik}q^{jl}$ where q^{ij} is any tensor that satisfies $q^{ij}q_{ik}q_{jl} = q_{kl}$, from Eq. (2.3.8).

In general this expression is not a total variation, and so cannot be integrated up to compute a finite charge corresponding to the first term in the symmetry generator (2.6.14). To see this, we compute the pullback (2.2.15) to $\mathcal{N}$ of the presymplectic current, contracted with χ^i. As shown in Sec. 2.2 above, when this quantity is nonzero the variation (2.6.19) is not a total variation. Taking a variation of the expression (2.6.17) for the pullback of the presymplectic potential, and using the formula (2.6.12) for the variation of the Weingarten map, we obtain

$$\begin{aligned}\chi^i \omega_{ijk}(h^{lm}, h'^{pq}) &= \frac{1}{16\pi}\chi^i \varepsilon_{ijk}\left[\frac{1}{2}h\pounds_\ell h' + \frac{1}{2}\pounds_\ell h_m{}^l h_l'{}^m + \frac{1}{2}hh_l'{}^m \mathcal{K}_m{}^l + 2h_m{}^p h_l'{}^m \mathcal{K}_p{}^l\right] \\ &\quad -(h \leftrightarrow h'). \end{aligned} \tag{2.6.20}$$

The integral of this quantity over a cross section $\mathcal{S}$ is nonvanishing in general. It does vanish in the case when χ^i is tangent to $\mathcal{S}$, that is, when it is a generator of the diffeomorphism symmetries. It also vanishes if we demand that both the background and perturbed configurations are shear and expansion free on $\mathcal{S}$, and in particular if they are stationary on $\mathcal{S}$.

We now specialize to the case where the null surface $\mathcal{N}$ is the future event horizon $\mathcal{H}^+$ of a black hole. The boundary of the horizon consists of the asymptotic boundary $\mathcal{H}^+_+$ at future timelike infinity, together with a bifurcation twosphere $\mathcal{H}^+_-$ in the case of an eternal black hole. In Appendix A.7 we show that the obstruction (2.6.20) vanishes on the bifurcation twosphere $\mathcal{H}^+_-$. We also show that the obstruction vanishes on the future boundary $\mathcal{H}^+_+$, assuming certain fall off conditions on the shear along the horizon towards future timelike infinity, and we argue that these fall off conditions are physically reasonable. Hence for horizons, Eq. (2.6.14) can be used directly to compute the contribution from the horizon to global charges.

We also show in Appendix A.7 that the correction term $i_\xi \boldsymbol{\Theta}$ in the definition (2.2.25) of the localized charge vanishes on the boundaries $\mathcal{H}^+_\pm$, under the same assumptions as above. Hence from Eq. (2.2.32) the contribution from the horizon to the global charge can be written as

$$\mathcal{Q}^{\rm loc}_\xi(\mathcal{H}^+_+) - \mathcal{Q}^{\rm loc}_\xi(\mathcal{H}^+_-). \tag{2.6.21}$$

Here $\mathcal{Q}^{\rm loc}_\xi$ is the localized charge which is computed explicitly in the next subsection, cf. Eq. (2.6.27).

Global conservation laws involving these global charges are discussed further in Sec. 2.7 below.

Localized (Wald-Zoupas) charges and fluxes

We now turn to a computation of localized charges $\mathcal{Q}^{\rm loc}_\xi(\mathcal{S})$ for cross sections $\mathcal{S}$ of a null surface $\mathcal{N}$. As explained in Sec. 2.2 above, the integrand in the expression (2.2.25) for this charge is given by adding to the pullback of the right hand side of Eq. (2.6.15) a term $\chi^k\Theta_{ijk}$, where the presymplectic potential Θ_{ijk} is of the form [cf. Eq. (2.2.30)]

$$\Theta_{ijk} = \theta_{ijk} - \delta\alpha_{ijk} \tag{2.6.22}$$

that is necessary for the right hand side to be a total variation. In addition, Θ_{ijk} is required to have the property that it vanish on backgrounds for which the null surface is shear free and expansion free. Then the charge is given by the expression (2.2.22) where the integrand is

$$\mathcal{Q}^{\rm loc}_{\xi\ ij} = Q_{\xi\ ij} - \chi^k \alpha_{ijk}, \tag{2.6.23}$$

up to an overall constant of integration on phase space. We verify that this constant of integration vanishes by showing that the right hand side of Eq. (2.6.23) vanishes on our reference solution, and by assuming that the left hand side vanishes on this solution. This computation is carried out in Appendix A.5.

We choose the 3-form $\boldsymbol{\alpha}$ on $\mathcal{N}$ given by

$$\alpha_{ijk} = \frac{1}{8\pi}\theta\varepsilon_{ijk}, \tag{2.6.24}$$

where θ is the expansion (2.3.28). Computing its variation yields

$$\delta\alpha_{ijk} = \frac{1}{16\pi}\varepsilon_{ijk}(h\theta + \pounds_\ell h). \tag{2.6.25}$$

Combining this with Eqs. (2.6.17), (2.6.22) and (2.6.18) gives for the presymplectic potential on $\mathcal{N}$

$$\Theta_{ijk} = \frac{1}{16\pi}\varepsilon_{ijk}\left[h_i{}^j\mathcal{K}_j{}^i - h\theta\right] = \frac{1}{16\pi}\varepsilon_{ijk}h^{AB}\left(\sigma_{AB} - \frac{1}{2}q_{AB}\theta\right). \tag{2.6.26}$$

This choice of presymplectic potential on a null surface was independently previously suggested in Eq. (8.2.20) of a thesis by Morales [81]. For backgrounds for which the null surface is shear free and expansion free, it follows from Eq. (2.6.26) that $\boldsymbol{\Theta}$ vanishes, as required. The two-form (2.6.23) is now obtained by combining Eqs. (2.6.6) and (2.6.24), which gives

$$\mathcal{Q}^{\rm loc}_{\xi\ ij} = \frac{1}{8\pi}\varepsilon_{ijk}\left[\chi^l\mathcal{K}_l{}^k - \theta\chi^k - \beta(\chi^i)\ell^k\right]. \tag{2.6.27}$$

It follows from the transformation properties (2.3.24) and (2.4.12) that this two-form is invariant under the rescaling (4.3.1).

We next argue that the expression (2.6.27) we have derived for the localized charge is unique. As discussed in Sec. 2.2 above, the presymplectic potential $\boldsymbol{\Theta}$ will be unique if there does not exist a 3-form $\boldsymbol{W}(\phi)$ on the boundary $\mathcal{N}$ that is locally and covariantly constructed out of the fields and of the universal structure, with the property that its variation $\delta\boldsymbol{W}$ vanishes identically on solutions for which the null boundary is shear free and expansion free. We assume that $\boldsymbol{W}$ depends analytically on the fields, and that the maximum number of derivatives in the expression for $\boldsymbol{W}$ is one less than the number of derivatives appearing in the Lagrangian, or one in this case.

The various geometrical quantities on which $\boldsymbol{W}$ can depend are reviewed in Sec. 2.3 above. The restriction on the number of derivatives in $\boldsymbol{W}$ eliminates other quantities, not reviewed in Sec. 2.3, that can be used to construct candidate expressions for $\boldsymbol{W}$, such as $\varepsilon_{ijk}\pounds_\ell R$ where R is the Ricci scalar. Using the finite number of quantities in Sec. 2.3 one can show by inspection that there are no expressions with the right properties. For example the expressions $\kappa\varepsilon_{ijk}$ and $\varepsilon_{[ij}D_{k]}\theta$ not invariant under the transformation (2.3.24), while the expression $(\sigma_{AB}\sigma^{AB}/\theta)\varepsilon_{ijk}$ is invariant but does not depend analytically on the fields. We conclude that $\boldsymbol{W}=0$ and so $\boldsymbol{\Theta}$ and $\mathcal{Q}_\xi^{\rm loc}(\mathcal{S})$ are uniquely determined by our assumptions and by the Lagrangian $\boldsymbol{L}$.

Finally, the on-shell flux $d\boldsymbol{\mathcal{Q}}_\xi^{\rm loc}$ associated with the localized charge is given by the symplectic potential $\boldsymbol{\Theta}$ evaluated at $h_{ab}=\pounds_\xi g_{ab}$, from Eq. (2.2.29). From the expression (2.6.26) for $\boldsymbol{\Theta}$, combined with Eqs. (2.3.7) and (2.3.16) to transform from three dimensional notation to four dimensional notation, we obtain

$$(d\boldsymbol{\mathcal{Q}}_\xi^{\rm loc})_{ijk}=\frac{1}{8\pi}\varepsilon_{ijk}\,\nabla_a\xi_b\left(\nabla^{(a}\ell^{b)}-g^{ab}\theta\right). \tag{2.6.28}$$

Alternatively, the flux can be obtained by taking an exterior derivative of the two-form (2.6.27)

$$(d\boldsymbol{\mathcal{Q}}_\xi^{\rm loc})_{ijk}=\frac{1}{8\pi}\varepsilon_{ijk}\hat{D}_p\left[\chi^m\mathcal{K}_m{}^p-\theta\chi^p-\beta\ell^p\right], \tag{2.6.29}$$

where $\hat{D}_p$ is the divergence operator (2.3.25). It follows from the transformation properties (2.3.24) and (2.4.12) that this flux is invariant under the rescaling (4.3.1). We show in Appendix A.6 that the two expressions (2.6.28) and (2.6.29) for the flux coincide. This serves as a consistency check of the formalism.

Charges and fluxes for specific symmetry generators

We now specialize as before to a cross section $\mathcal{S}$ with normal n_i. For the special case of a supertranslation with $\chi^k=f\ell^k$, integrating the 2-form (2.6.27) over $\mathcal{S}$ and using Eqs. (2.3.18), (2.4.19), (2.3.23) and (2.4.28) gives the charge

$$\mathcal{Q}_f^{\rm loc}(\mathcal{S})=\frac{1}{8\pi}\int_{\mathcal{S}}\varepsilon_{ij}\left[\theta f-\pounds_\ell f-\kappa f\right]. \tag{2.6.30}$$

For stationary null surfaces, these charges vanish identically for affine supertranslations, for which $\pounds_\ell f+\kappa f=0$. The corresponding flux through a region $\Delta\mathcal{N}$ of $\mathcal{N}$ is given by integrating the expression (2.6.29):

$$\Delta\mathcal{Q}_f^{\rm loc}=\frac{1}{8\pi}\int_{\Delta\mathcal{N}}\varepsilon_{ijk}\hat{D}_p\left[(f\kappa-\theta f+\pounds_\ell f)\ell^p\right]. \tag{2.6.31}$$

This can be simplified using the formula (2.3.28), the symmetry condition (2.4.17) and Raychaudhuri's equation in vacuum to give

$$\Delta\mathcal{Q}_f^{\rm loc}=\frac{1}{8\pi}\int_{\Delta\mathcal{N}}\varepsilon_{ijk}f(\theta\kappa-\theta^2-\pounds_\ell\theta)=\frac{1}{8\pi}\int_{\Delta\mathcal{N}}\varepsilon_{ijk}f\left(\sigma_{AB}\sigma^{AB}-\frac{1}{2}\theta^2\right). \tag{2.6.32}$$

We next consider diff$(\mathcal{Z})$ generators of the form $\chi^i = X^i$ where $X^i n_i = 0$, making use of the decomposition (2.4.29). Here n_i is the normal to the cross section $\mathcal{S}$, and we also demand that it obey the differential equation (2.4.32) on $\mathcal{N}$, in order that X^i be an element of the symmetry algebra $\mathfrak{g}_{\mathfrak{u}}$. For such generators the pullback to $\mathcal{S}$ of $\varepsilon_{ijk}X^k$ vanishes, and so from Eqs. (2.6.23) and (2.6.24) the localized charge and Noether charge coincide. From Eq. (2.6.27) the localized charge is

$$\mathcal{Q}^{\text{loc}}_X(\mathcal{S}) = \frac{1}{8\pi}\int_{\mathcal{S}} \varepsilon_{ij}\left[X^l \mathcal{K}_l{}^k n_k + \beta\right], \tag{2.6.33}$$

and the corresponding flux from Eq. (2.6.29) is

$$\Delta\mathcal{Q}^{\text{loc}}_X = \frac{1}{8\pi}\int_{\Delta\mathcal{N}} \varepsilon_{ijk}\hat{D}_p\left[X^m \mathcal{K}_m{}^p - \theta X^p - \beta\ell^p\right]. \tag{2.6.34}$$

Stationary regions of the null surface

We now specialize to stationary regions of the null surface to obtain explicit forms for the various charges. In stationary regions the general charge (2.6.27) reduces to

$$\mathcal{Q}^{\text{loc}}_\xi(\mathcal{S}) = -\frac{1}{8\pi}\int_{\mathcal{S}} \varepsilon_{ij}(\chi^l\omega_l - \beta), \tag{2.6.35}$$

by Eqs. (2.3.31), (A.1.13) and (2.4.28). The integrand here has a vanishing Lie derivative with respect to the Killing field τ^i, so the result is independent of $\mathcal{S}$ as one would expect. To see this, take the Lie derivative of the integrand with respect to ℓ^i, and simplify using Eqs. (2.3.30), (2.4.11) and (A.1.10) to obtain $-\varepsilon_{ij}[\chi^k \pounds_\ell \omega_k - \pounds_\chi \kappa]/(8\pi)$. Now specializing without loss of generality to the choice of representative $\ell^i = \tau^i$ and using Eqs. (2.3.34) and (2.3.35) shows that the expression vanishes.

We next specialize to the choice of normal $\ell^i = \tau^i$, and to a coordinate system (τ, θ^A) for which the Killing field is $\vec{\tau} = \partial/\partial\tau$, and we write the rotation one-form (2.3.33) and symmetry generator as

$$\boldsymbol{\omega}_\tau = \omega_{\tau\,A}(\theta^B)d\theta^A + \kappa_\tau d\tau \tag{2.6.36}$$

and

$$\vec{\chi} = \left[\hat{\alpha}(\theta^A)e^{-\kappa_\tau\tau} + \hat{\beta}(\theta^A)\right]\frac{\partial}{\partial\tau} + X^A(\theta^B)\frac{\partial}{\partial\theta^A}. \tag{2.6.37}$$

Here $\hat{\alpha}$ parameterizes the affine supertranslations, $\hat{\beta}$ the Killing supertranslations[27], and X^A the diff(S^2) transformations or superrotations. From Eq. (2.4.11a) we obtain $\beta = \kappa_\tau\hat{\alpha}\exp[-\kappa_\tau\tau]$ and substituting into Eq. (2.6.35) gives

$$\mathcal{Q}^{\text{loc}}_\xi(\mathcal{S}) = -\frac{1}{8\pi}\int_{\mathcal{S}} \varepsilon_{ij}(X^A\omega_{\tau\,A} + \hat{\beta}\kappa_\tau). \tag{2.6.38}$$

[27]Note that these parameters $\hat{\alpha}$ and $\hat{\beta}$ do not coincide with the parameters α and β of Eq. (2.4.8).

2.7 Global conservation laws involving black holes

Although our analysis has considered arbitrary null surfaces in the preceding sections, our main interest lies with black holes. Accordingly, in the next few sections, we take $\mathcal{N}$ to be the future event horizon $\mathcal{H}^+$ of a black hole. This can either be a black hole formed in gravitational collapse, or an eternal black hole, as in Fig. 2.1 above. We note that our analysis is limited to smooth horizons, and that generic horizons are not smooth because of generators that join the horizon. We leave the analysis of charges and symmetries associated with nonsmooth horizons for future work.

In this section we consider global conservation laws involving black hole horizons. As discussed in the introduction, we distinguish between *localized* conservation laws that involve only one component $\mathcal{B}_j$ of the spacetime boundary, and *global* conservation laws that involve entire Cauchy surfaces. The foundation for both types of laws is the fact that the expression (2.2.12) for the variation of the global charge $\mathcal{Q}_\xi$ is invariant under local deformations of the hypersurface Σ when on shell, from Eq. (2.2.5). In situations where the charge variation is a total variation and the condition (2.2.15) is satisfied (which happens for internal symmetries), one does not need to distinguish between these two types of conservation laws. More generally, the localized conservation laws require the application of the Wald-Zoupas procedure, while the global laws do not, as argued in Sec. 2.2 above.

As discussed in the introduction, in the past few years an infinite set of new global conservation laws in gauge theories have been discovered, associated with "large" gauge transformations which are not trivial at infinity [22, 25]. Similar conservation laws have been argued for in the gravitational case [82, 25], although completely rigorous derivations have yet to be given. One of the key motivations for studying horizon symmetries and charges is the realization that the associated global conservation laws place constraints on black hole evaporation, and that the (electric parity superrotation) charges constitute "soft hair" that may play a role in how information is released as a black hole evaporates [82, 23, 42, 33, 34, 83] (see also [25] for a complete review). In this section we review the status of these conservation laws and the implications of our results for their formulation.

Consider for example a spacetime with no horizons for which the only components of the boundary are $\mathscr{I}^+$, $\mathscr{I}^-$ and the points at infinity i^-, i^0 and i^+, and specialize to vacuum general relativity. Since the charge variation (2.2.12) is invariant under local deformations of the Cauchy surface Σ, one can deform Σ into the distant past and also into the distant future. Then with appropriate sign conventions one obtains a conservation law of the form

$$\delta\mathcal{Q}_\xi(\mathscr{I}^-) + \delta\mathcal{Q}_\xi(i^-) + \delta\mathcal{Q}_\xi(i^0) = \delta\mathcal{Q}_\xi(\mathscr{I}^+) + \delta\mathcal{Q}_\xi(i^+), \tag{2.7.1}$$

where each term is an integral of the form (2.2.12) over the corresponding hypersurface or an appropriate limit of such integrals converging to one of the points at infinity, assuming such limits exist. If we specialize to spacetimes for which the Bondi mass vanishes at $\mathscr{I}^+_+$, the future limit of $\mathscr{I}^+$, and at $\mathscr{I}^-_-$, the past limit of $\mathscr{I}^-$, then the terms at future and past timelike infinity should vanish [82] giving

$$\delta\mathcal{Q}_\xi(\mathscr{I}^-) + \delta\mathcal{Q}_\xi(i^0) = \delta\mathcal{Q}_\xi(\mathscr{I}^+). \tag{2.7.2}$$

The contribution from spatial infinity in this equation need not vanish in general[28].

To derive a global conservation law one needs to show that the various limiting integrals exist, and that the contribution $\delta\mathcal{Q}_\xi(i^0)$ vanishes. Then integrating in phase space would yield a relation of the form

$$\mathcal{Q}_\xi(\mathscr{I}^-) = \mathcal{Q}_\xi(\mathscr{I}^+), \tag{2.7.3}$$

which constrains gravitational scattering [82]. We expect that imposing suitable boundary conditions at i^0 in the definition of $\mathscr{F}$ should eliminate the term $\delta\mathcal{Q}_\xi(i^0)$ (this is closely related to the matching conditions proposed in Ref. [25]). In addition these boundary conditions should reduce the global symmetry algebra to a diagonal subalgebra of $\mathrm{BMS}^- \oplus \mathrm{BMS}^+$, with an appropriate identification of BMS^- and BMS^+, as argued by Strominger [82]. See Refs. [84, 85, 86, 87, 88] for more detailed analyses of spatial infinity and of the validity of conservation laws of the form (2.7.3).

Consider now the generalization of this discussion to include horizons [34, 33]. For the black hole formed from gravitational collapse shown in Fig. 2.1, and for a representative ξ^a of the global symmetry algebra, following the argument that led to Eq. (2.7.1) we obtain

$$\delta\mathcal{Q}_\xi(\mathscr{I}^-) + \delta\mathcal{Q}_\xi(i^-) + \delta\mathcal{Q}_\xi(i^0) = \delta\mathcal{Q}_\xi(\mathscr{I}^+) + \delta\mathcal{Q}_\xi(i^+) + \delta\mathcal{Q}_\xi(\mathcal{H}^+). \tag{2.7.4}$$

Here each term is an integral of the form (2.2.12) over the corresponding hypersurface or an appropriate limit of such integrals converging to one of the points at infinity. The term $\delta\mathcal{Q}_\xi(i^-)$ can be eliminated as described above. *A priori*, the symmetry generator ξ^a appearing in this equation can have independent limits at the horizon $\mathcal{H}^+$ and at null infinity $\mathscr{I}^+/\mathscr{I}^-$. However, just as for i^0, we expect that imposing appropriate boundary conditions at future timelike infinity i^+ should eliminate the term $\delta\mathcal{Q}_\xi(i^+)$, and impose the appropriate relation between the limits of ξ^a at $\mathcal{H}^+$ and at $\mathscr{I}^+/\mathscr{I}^-$. Note that this viewpoint differs from that of Ref. [33], which used the specific prescription of maintaining global Bondi coordinates to link generators at $\mathscr{I}^-$ to those at $\mathcal{H}^+$. However the specific identification for supertranslations obtained there seems inevitable for the case of spherical symmetry[29]. For more general generators and situations, the appropriate identification of the generators is an interesting question for future study, and will need to be resolved in order to obtain the general form of the global conservation law.

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[28] For example, suppose that in Eq. (2.2.12) ϕ is the Minkowski metric, $\delta\phi$ is the linearized Schwarzschild solution, and ξ^a is a vector field which asymptotes to one timelike Killing vector field τ_-^a of the Minkowski background at $\mathscr{I}^-$ and to another τ_+^a at $\mathscr{I}^+$. Then $\delta\mathcal{Q}_\xi(i^0)$ is proportional to $P_a(\tau_+^a - \tau_-^a)$, where P_a is the ADM 4-momentum.

[29] With the following minor adjustment: assuming that a supertranslation on $\mathscr{I}^-$ corresponds to some specific element of the symmetry algebra on $\mathcal{H}^+$, as found in Ref. [33], the restriction (2.4.39) implies that the algebra element is a linear combination of a Killing supertranslation and an affine supertranslation, instead of a pure Killing supertranslation. In the notation of Eq. (2.6.37) near $\mathcal{H}_+^+$, this linear combination will be of the form

$$\vec{\chi} = \hat{\beta}(1 + u_0 e^{-\kappa_\tau \tau})\partial_\tau,$$

where $u_0 = u_0(\theta^A)$ is determined by the conditions (2.4.38) or (2.4.40) at early times. The affine supertranslation correction term does not contribute to localized charges in stationary regions or to global charges.

Finally, assuming such an identification has been derived, our explicit expressions for localized charges can be used to obtain an explicit and nonperturbative form of the resulting global conservation law. Integrating Eq. (2.7.4) in phase space and making use of Eq. (2.2.32), this form is

$$\mathcal{Q}_\xi^{\rm loc}(\mathscr{I}_-^+) - \mathcal{Q}_\xi^{\rm loc}(\mathscr{I}_+^+) - \mathcal{Q}_\xi^{\rm loc}(\mathcal{H}_+^+) = \mathcal{Q}_\xi^{\rm loc}(\mathscr{I}_+^-) - \mathcal{Q}_\xi^{\rm loc}(\mathscr{I}_-^-). \tag{2.7.5}$$

Note that the third term on the left hand side is the operator that creates soft graviton hair on a black hole horizon in the quantum theory, when ξ^a is a Killing supertranslation associated with the asymptotic Killing field near future timelike infinity, as explained by Hawking, Perry and Strominger [34, 33]. Our result (2.6.27) for this operator improves on existing treatments [34, 33] in that it is nonperturbative and not a variation[30].

2.8 Algebra of symmetry generator charges and central charges

As is well known, the algebra of the global symmetry generator charges $\mathcal{Q}_\xi$ under Dirac brackets need not coincide with the symmetry algebra $\mathfrak{g}$ of the vector fields ξ^a under Lie brackets, and can instead be a central extension of that algebra [89, 60, 90, 91]. This phenomena already arises in classical mechanics [92]. For example, there is a nontrivial central extension for 2+1 dimensional gravity with a negative cosmological constant with a certain choice of AdS boundary conditions, as shown by Brown and Henneaux [89]. There is no central extension for BMS generators in 3+1 dimensional general relativity [93], and we show in this section that the same is true for the symmetry algebra of charges at event horizons in general relativity, assuming certain fall off conditions on the shear near future timelike infinity. Thus, there is no central extension of the algebra for the symmetry algebra of global charges derived in this paper.

Algebra of symmetry generator charges in general contexts

We first review in this subsection the theory of central extensions [89, 60, 90, 91] in general contexts, and in the following subsection we will apply it to black hole event horizons.

The first step in the computation of the algebra of global charges $\mathcal{Q}_\xi$ is the computation of the Dirac bracket. For the specific case of vacuum general relativity at a future event horizon, a careful derivation of the Dirac bracket including the effects of zero modes has been given by Hawking, Perry and Strominger [33]. Here, for the discussion in a general context, we will assume that a Dirac bracket can be found for which the global charges implement the symmetries in the sense[31]

$$\{F[\phi], \mathcal{Q}_\xi\} = \delta_\xi F[\phi], \tag{2.8.1}$$

[30] The explicit form of this operator is given by Eq. (2.6.38) above, since the horizon is asymptotically stationary, assuming the fall-off conditions on the shear of Appendix A.7.

[31] Our sign convention for Eq. (2.8.1) is the opposite of that of Ref. [33] and agrees with that of Ref. [94].

where the variation δ_ξ is defined by

$$\delta_\xi F[\phi] = F[\phi + \pounds_\xi \phi] - F[\phi]. \tag{2.8.2}$$

Here F is any function on the covariant phase space $\overline{\mathscr{F}}$ (i.e. functional of field configurations ϕ), and the right hand side is understood to be linearized in ξ^a. Combining this with the definition (2.2.12) yields

$$\left\{\mathcal{Q}_\xi, \mathcal{Q}_{\tilde{\xi}}\right\} = -\Omega_\Sigma(\phi, \pounds_\xi\phi, \pounds_{\tilde{\xi}}\phi). \tag{2.8.3}$$

An alternative formal derivation of Eq. (2.8.3) is as follows. We write the presymplectic form (2.2.9) as $\Omega_{\mathscr{A}\mathscr{B}}$, where the indices $\mathscr{A}, \mathscr{B}, \ldots$ represent tensors on $\overline{\mathscr{F}}$. The definition (2.2.12) of global charges can be written in this notation as

$$\nabla_{\mathscr{A}} \mathcal{Q}_\xi = \Omega_{\mathscr{A}\mathscr{B}} v_\xi^{\mathscr{B}}, \tag{2.8.4}$$

where $v_\xi^{\mathscr{B}}$ is the vector field on the covariant phase space that assigns to each solution ϕ the linearized solution $\pounds_\xi\phi$. We assume the existence of a Dirac bracket on functions $F, G : \overline{\mathscr{F}} \to \mathbf{R}$ of the form

$$\{F, G\} = \Omega^{\mathscr{A}\mathscr{B}} \nabla_{\mathscr{A}} F \nabla_{\mathscr{B}} G \tag{2.8.5}$$

where $\Omega^{\mathscr{A}\mathscr{B}}$ satisfies

$$\Omega_{\mathscr{A}\mathscr{B}} \Omega^{\mathscr{B}\mathscr{C}} \Omega_{\mathscr{C}\mathscr{D}} = \Omega_{\mathscr{A}\mathscr{D}}. \tag{2.8.6}$$

Now inserting the charge definition (2.8.4) into the bracket (2.8.5) and using Eq. (2.8.6) gives $\left\{\mathcal{Q}_\xi, \mathcal{Q}_{\tilde{\xi}}\right\} = -\Omega_{\mathscr{A}\mathscr{B}} v_\xi^{\mathscr{A}} v_{\tilde{\xi}}^{\mathscr{B}}$, which is equivalent to Eq. (2.8.3).

Next, the relation (2.8.3) can be rewritten using the formulae (2.2.10) and (2.8.2) as

$$\left\{\mathcal{Q}_\xi, \mathcal{Q}_{\tilde{\xi}}\right\} = -\int_{\partial\Sigma} \left[\delta_\xi \boldsymbol{Q}_{\tilde{\xi}}(\phi) - i_{\tilde{\xi}} \boldsymbol{\theta}(\phi, \pounds_\xi \phi)\right]. \tag{2.8.7}$$

We now specialize to situations where the presymplectic potential $\boldsymbol{\Theta}$ exists, and where the correction term $i_\xi \boldsymbol{\Theta}$ in the definition (2.2.25) of the localized charge vanishes on $\partial\Sigma$ for all ξ^a. As discussed in Sec. 2.2 above, we expect this to be generically valid when Σ is a Cauchy surface. Taking a variation of Eq. (2.2.32) and combining with Eqs. (2.2.13) and (2.8.7) gives

$$\left\{\mathcal{Q}_\xi, \mathcal{Q}_{\tilde{\xi}}\right\} = -\int_{\partial\Sigma} \delta_\xi \boldsymbol{\mathcal{Q}}_{\tilde{\xi}}^{\text{loc}}(\phi). \tag{2.8.8}$$

Now let $\psi_\varepsilon : M \to M$ be the one parameter family of diffeomorphisms that move points along integral curves of ξ^a. Since these diffeomorphisms preserve the boundaries and the universal structures on the boundaries, and since by construction $\boldsymbol{\mathcal{Q}}_{\tilde{\xi}}^{\text{loc}}$ is local and covariant in the sense of footnote 12, it follows from the argument in that footnote that[32]

$$\psi_{\varepsilon *} \boldsymbol{\mathcal{Q}}_{\tilde{\xi}}^{\text{loc}}(\phi) = \boldsymbol{\mathcal{Q}}_{\psi_{\varepsilon *}\tilde{\xi}}^{\text{loc}}(\psi_{\varepsilon *}\phi). \tag{2.8.9}$$

[32]Note that it is important for this argument that $\boldsymbol{\mathcal{Q}}_{\tilde{\xi}}^{\text{loc}}$ does not depend on arbitrary choices such as a choice of representative of an equivalence class in the universal structure.

Now differentiating with respect to ε and setting $\varepsilon = 0$ gives the identity

$$\pounds_\xi \boldsymbol{\mathcal{Q}}^{\rm loc}_{\tilde{\xi}}(\phi) = \delta_\xi \boldsymbol{\mathcal{Q}}^{\rm loc}_{\tilde{\xi}}(\phi) + \boldsymbol{\mathcal{Q}}^{\rm loc}_{\pounds_\xi \tilde{\xi}}(\phi). \tag{2.8.10}$$

Inserting this into Eq. (2.8.8) finally gives

$$\left\{ \mathcal{Q}_\xi, \mathcal{Q}_{\tilde{\xi}} \right\} = \mathcal{Q}_{[\xi,\tilde{\xi}]} + K_{\xi,\tilde{\xi}}, \tag{2.8.11}$$

where $[\xi, \tilde{\xi}]^a = \pounds_\xi \tilde{\xi}^a$ is the Lie bracket and

$$K_{\xi,\tilde{\xi}} = -\int_{\partial\Sigma} \pounds_\xi \boldsymbol{\mathcal{Q}}^{\rm loc}_{\tilde{\xi}}. \tag{2.8.12}$$

Equation (2.8.11) shows that when the quantity $K_{\xi,\tilde{\xi}}$ is non vanishing, the algebra of charges will differ from the algebra of vector fields.

A priori the quantity $K_{\xi,\tilde{\xi}}$ could depend on the background solution ϕ. However, a theorem due to Brown and Henneaux [95] shows that there is no such dependence, and so the algebra of charges consists at most of a central extension of the algebra of vector fields. A formal version of the argument is as follows [60, 96]:

$$\nabla_{\mathscr{A}} \mathcal{Q}_{[\xi,\tilde{\xi}]} = \Omega_{\mathscr{A}\mathscr{B}} v^{\mathscr{B}}_{[\xi,\tilde{\xi}]} = -\Omega_{\mathscr{A}\mathscr{B}} \pounds_{v_\xi} v^{\mathscr{B}}_{\tilde{\xi}} = -\pounds_{v_\xi} \left(\Omega_{\mathscr{A}\mathscr{B}} v^{\mathscr{B}}_{\tilde{\xi}} \right) = -\pounds_{v_\xi} \nabla_{\mathscr{A}} \mathcal{Q}_{\tilde{\xi}}. \tag{2.8.13}$$

Here we have used the charge definition (2.8.4), then the fact that the mapping $\xi^a \to -v^{\mathscr{A}}_\xi$ is a Lie algebra homomorphism[33], then the fact that $\Omega_{\mathscr{A}\mathscr{B}}$ is a closed two form on $\overline{\mathscr{F}}$, and finally the definition (2.8.4) again. Continuing we obtain

$$\nabla_{\mathscr{A}} \mathcal{Q}_{[\xi,\tilde{\xi}]} = -\nabla_{\mathscr{A}} \left(v^{\mathscr{B}}_\xi \nabla_{\mathscr{B}} \mathcal{Q}_{\tilde{\xi}} \right) = -\nabla_{\mathscr{A}} \left(v^{\mathscr{B}}_\xi \Omega_{\mathscr{B}\mathscr{C}} v^{\mathscr{C}}_{\tilde{\xi}} \right) = \nabla_{\mathscr{A}} \left\{ \mathcal{Q}_\xi, \mathcal{Q}_{\tilde{\xi}} \right\}, \tag{2.8.14}$$

where we have used Eqs. (2.8.4), (2.8.5) and (2.8.6). It follows from Eq. (2.8.11) that $\nabla_{\mathscr{A}} K_{\xi,\tilde{\xi}} = 0$, as claimed.

Symmetry algebra of global charges at event horizons

We now show that the contribution[34] to the central charges (2.8.12) from a future event horizon vanishes, assuming certain fall off conditions on the shear near future timelike infinity. This generalizes a result of Guo, Hwang and Wu who show that the central charges vanish on the horizon of a stationary, axisymmetric black hole for a large class of generators [97].

[33] This follows from the fact that v_ξ maps any functional $F[\phi]$ to $F[\phi + \pounds_\xi \phi] - F[\phi]$ to linear order, so $\left[v_\xi, v_{\tilde{\xi}} \right] F[\phi] = F[\phi + (\pounds_{\tilde{\xi}} \pounds_\xi - \pounds_\xi \pounds_{\tilde{\xi}})\phi] - F[\phi]$.

[34] As discussed in Secs. 2.2 and 2.2 above, a symmetry ξ^a can have different limiting forms on different boundaries $\mathcal{B}_j$, and more than one can contribute to the central charge (2.8.12), depending on the Cauchy surface Σ.

Consider a connected component $\mathcal{S}$ of $\partial\Sigma$ which lies in the event horizon $\mathcal{H}^+$. Using Cartan's formula together with Eq. (2.6.29) we find that the contribution from $\mathcal{S}$ to the central charges (2.8.12) can be written as

$$-\int_{\mathcal{S}} i_\xi d\boldsymbol{\mathcal{Q}}^{\rm loc}_{\tilde\xi} = -\frac{1}{8\pi}\int_{\mathcal{S}} \chi^i \varepsilon_{ijk}\hat{D}_p\left[\tilde\chi^m \mathcal{K}_m^{\ p} - \theta\tilde\chi^p - \tilde\beta\ell^p\right]. \tag{2.8.15}$$

Now $\mathcal{S}$ cannot lie in the interior of $\mathcal{H}^+$, otherwise Σ would not be a Cauchy surface. We consider two different cases:

- The cross section $\mathcal{S}$ coincides with a component of the boundary $\mathcal{H}^+$, for example the bifurcation two-sphere $\mathcal{H}^+_-$ in an eternal black hole spacetime. Since χ^i must be tangent to $\mathcal{S}$ in this case, as argued in Sec. 2.4 above, it follows that the quantity (2.8.15) vanishes.

- The cross section $\mathcal{S}$ represents the future asymptotic boundary $\mathcal{H}^+_+$ of $\mathcal{H}^+$. Now as discussed in Appendix A.7, event horizons are asymptotically stationary. Assuming exact stationarity and using the condition (2.3.31), the quantity (2.8.15) reduces to

$$-\frac{1}{8\pi}\int_{\mathcal{S}} \chi^i \varepsilon_{ijk}\hat{D}_p\left[\tilde\chi^m\omega_m\ell^p - \tilde\beta\ell^p\right] = -\frac{1}{8\pi}\int_{\mathcal{S}} \chi^i\varepsilon_{ijk}\pounds_\ell\left[\tilde\chi^m\omega_m - \tilde\beta\right]. \tag{2.8.16}$$

  Here ω_m is the rotation one-form (2.3.32) and we have used Eqs. (2.3.28), (2.3.14) and (2.3.31). The Lie derivative in the integrand on the right hand side of Eq. (2.8.16) vanishes, as shown after Eq. (2.6.35), and so the result vanishes. In this analysis we have set the shear σ_{ij} and expansion θ to zero. Assuming instead the falloff conditions $\sigma_{ij}, \theta \sim v^{-p}$ with $p > 1$ of Appendix A.7, where v is affine parameter, one can show by an analysis similar to that of Appendix A.7 that the contribution of the shear and expansion to the expression (2.8.15) vanishes in the limit $v \to \infty$. Hence the contribution from $\mathcal{H}^+_+$ to the central charge (2.8.12) vanishes.

Symmetry algebras of localized charges

One can also consider the algebra of localized charges $\mathcal{Q}^{\rm loc}_\xi(\mathcal{S})$. The Poisson bracket of two such charges $\mathcal{Q}^{\rm loc}_\xi(\mathcal{S})$ and $\mathcal{Q}^{\rm loc}_{\tilde\xi}(\tilde{\mathcal{S}})$ will in general depend on the two-surfaces $\mathcal{S}$ and $\tilde{\mathcal{S}}$, but if one specializes to a stationary region of the null surface $\mathcal{N}$ the bracket becomes independent of the two-surfaces. It will be of the form[35]

$$\left\{\mathcal{Q}^{\rm loc}_\xi(\phi), \mathcal{Q}^{\rm loc}_{\tilde\xi}(\phi)\right\} = \mathcal{Q}^{\rm loc}_{[\xi,\tilde\xi]}(\phi) + K^{\rm loc}_{\xi,\tilde\xi}(\phi), \tag{2.8.17}$$

where the anomalous term $K^{\rm loc}_{\xi,\tilde\xi}(\phi)$ will in general depend on the background solution ϕ [98], in contrast to the situation (2.8.11) for the global charges. While we do not consider the

[35]Barnich and Troessaert have shown that the anomalous term $K^{\rm loc}_{\xi,\tilde\xi}(\phi)$ vanishes for the case of BMS generators at null infinity in 3+1 general relativity [98].

algebra (2.8.17) in this paper, we note that a recent paper by Haco, Hawking, Perry and Strominger has computed such an algebra for Kerr black holes, and used it to derive the Bekenstein-Hawking entropy [99, 100]. The symmetry algebra $\mathfrak{g}$ of vector fields used there is not the same as the algebra (2.4.11) used in this paper, and may be related to the extended algebra we discuss in Appendix A.8.

2.9 Discussion, applications and future directions

In this final section we recap our main results, discuss some implications and applications, and discuss some open questions and future directions.

Recap

In this paper, we have applied the covariant phase space formalism to general relativity with a null boundary. By an appropriate gauge-fixing at the boundary we defined a field configuration space, and derived the conditions for linearized diffeomorphisms to preserve this configuration space. Factoring out by the degeneracies left us with the infinite dimensional symmetry algebra $\mathfrak{g} = \mathrm{diff}(S^2) \ltimes \mathfrak{s}$, where $\mathfrak{s}$ is the set of supertranslations at $\mathcal{N}$ i.e. vector fields $\chi^i = f\ell^i$ satisfying $\pounds_\ell(\pounds_\ell f + \kappa f) = 0$. Supertranslations were therefore found to be symmetries of general relativity at general null boundaries. We then calculated the general form of the global conserved charges, and the localized charges and fluxes associated to $\mathfrak{g}$ by way of the Wald-Zoupas prescription. In particular, we found explicit expressions for the supertranslation localized charges and fluxes. These expressions are unique when we impose the condition that the potential $\boldsymbol{\Theta}$ for the presymplectic current on the null surface vanish when the surface is shear free and expansion free.

Black holes: localized conservation laws and horizon memory

We next discuss the implications and interpretation in the event horizon context of the localized conservation laws that we have derived.

As discussed in Sec. 2.2 above, given any two cross sections $\mathcal{S}$ and $\mathcal{S}'$ of the event horizon, we have for each symmetry generator a localized conservation law of the form

$$\int_{\Delta\mathcal{N}} d\boldsymbol{\mathcal{Q}}^{\mathrm{loc}}_\xi = \int_{\mathcal{S}'} \boldsymbol{\mathcal{Q}}^{\mathrm{loc}}_\xi - \int_{\mathcal{S}'} \boldsymbol{\mathcal{Q}}^{\mathrm{loc}}_\xi, \tag{2.9.1}$$

where $\Delta\mathcal{N}$ is the region of $\mathcal{N}$ between $\mathcal{S}$ and $\mathcal{S}'$ and explicit expressions for the charge and flux are given in Eqs. (2.6.27) and (2.6.29). Now since the event horizon has a boundary (either an initial event $\mathcal{P}$ or a bifurcation two-sphere), some of the symmetry generators χ^i of the algebra discussed in Sec. 2.4 do not preserve the boundary. As discussed in Sec. 2.4, those generators must be excluded from the global algebra $\mathfrak{g}$ that is relevant for global conservations laws. Nevertheless, the conservation law (2.9.1) is valid for all generators. This is because the derivation of the law (2.9.1) is local, and is not invalidated if the vector field violates the required boundary conditions at $\partial\mathcal{N}$ if $\partial\mathcal{N}$ is disjoint from $\Delta\mathcal{N}$.

In order to get some insight into the physical interpretation of the charges in (2.9.1), we specialize to stationary regions. The three different types of generators are:

- *Affine supertranslations:* The associated charges vanish identically in stationary regions, as noted in Sec. 2.6 above.
- *Superrotations or* diff(S^2) *generators:* The corresponding charges in stationary regions are given by the first term in Eq. (2.6.38) above. The curl (magnetic parity) piece of X^A yields the horizon angular momentum multipoles of Ashtekar [101], while the gradient (electric parity) piece gives additional charges.
- *Killing supertranslations:* The charge in this case is given by the second term in Eq. (2.6.38).

These charges all vanish for a Schwarzschild black hole, except for the $l = m = 0$ component of the Killing supertranslation charge in (2.6.38)[36]. However, as explained in Ref. [33], one can turn on an infinite number of non trivial charges by acting on the metric with symmetry transformations. If we write the charges as $\mathcal{Q}^{\rm loc}_\xi(\mathcal{S}, g_{ab})$, including the dependence on the metric g_{ab}, then it follows from covariance and the fact that the charges are independent of $\mathcal{S}$ in stationary regions that

$$\mathcal{Q}^{\rm loc}_\xi(\mathcal{S}, g_{ab} + \pounds_{\tilde\xi} g_{ab}) = \mathcal{Q}^{\rm loc}_\xi(\mathcal{S}, g_{ab}) - \mathcal{Q}^{\rm loc}_{\pounds_{\tilde\xi}\xi}(\mathcal{S}, g_{ab}) \tag{2.9.2}$$

to linear order in $\tilde\xi^a$. Hence one can compute the charges on a transformed background in terms of the charges on the original background by making use of the algebra (2.4.10) of symmetry generators. It follows from this algebra that acting on the Schwarzschild metric with a superrotation turns on an infinite number of Killing supertranslation charges, and similarly acting with a Killing supertranslation turns on an infinite number of superrotation charges.

We next turn to a consideration of *stationary to stationary transitions*, which helps to clarify the nature of the charges and conservation laws just as at future null infinity. Suppose that there are two different stationary regions of the horizon separated by a region which is non-stationary[37]. Then the stationary regions are associated with two different Killing supertranslation algebras $\mathfrak{t}_1$ and $\mathfrak{t}_2$. This is analogous to the status of Poincaré subalgebras of the BMS algebra at null infinity. Just as there, one can find a finite supertranslation σ for which

$$\mathfrak{t}_2 = \sigma \mathfrak{t}_1 \sigma^{-1}, \tag{2.9.3}$$

[36] Here we define the splitting of a general generator into supertranslation and superrotation pieces by identifying the coordinates in (2.6.37) with ingoing Eddington-Finkelstein coordinates.

[37] Actually it is not possible to have the first region be exactly stationary, since by Raychaudhuri's equation in vacuum the expansion θ must monotonically decrease to zero in affine parameterization; it can only be approximately stationary.

so that the two subalgebras are related by a supertranslation. Specifically, in the notation of Eq. (2.4.35), if the two subalgebras are given by $\alpha - \beta u_1 = 0$ and $\alpha - \beta u_2 = 0$, where u_1 and u_2 are functions just of θ^A, then one can take σ to be the affine supertranslation $u \to u + (u_2 - u_1)$. This supertranslation is presumably is related to an analog of gravitational wave memory on the horizon [34]. The details of how such memory can be defined and measured is an interesting topic for future study. The transition is also associated with net changes in (electric parity) superrotation charges, as at $\mathscr{I}^+$.

Finally, our formalism does not furnish an analog of the Bondi mass on black hole horizons, that is, a prescription for computing the mass of the black hole at an arbitrary cross section $\mathcal{S}$ of the horizon. This is so for two reasons. First, it would be necessary to specify a preferred symmetry generator (or preferred four-dimensional subgroup of translations for a 4-momentum) from the algebra in order to obtain such a definition. While there is a preferred generator for each stationary region (the Killing vector), in general horizons are non-stationary, and there is no preferred generator or preferred four-dimensional subgroup of translations. Second, even when given a generator associated with a stationary region, the corresponding charge is proportional to the area of the black hole (as used in derivations of the first law), not the mass. In this sense horizons are not similar to future null infinity.

The limit to future null infinity

The symmetry algebra for a general null surface that we have derived is larger than the BMS algebra which applies to the asymptotic boundary of future null infinity $\mathscr{I}^+$. An interesting question is how the symmetries and charges of the two algebras are related, for a family of null surfaces that limit to $\mathscr{I}^+$ in an asymptotically flat spacetime. One might expect that the localized charges $\mathcal{Q}_\xi^{\text{loc}}$ have finite limits for a subalgebra of the symmetry algebra isomorphic to the BMS algebra, and that the limits of those charges coincide with the BMS charges. In fact, this does not occur, and none of the localized charges $\mathcal{Q}_\xi^{\text{loc}}$ have finite limits. This occurs because of our choice of reference solution, in effect a different choice of reference solution is necessary in order for finite limiting charges to be obtained at $\mathscr{I}^+$. Details of this comparison will be discussed elsewhere [**CentralExt**].

Generalizations

While our results are specific to $d = 4$ spacetime dimensions, they generalize straightforwardly to all spacetime dimensions $d \geq 4$, with appropriate changes in numerical coefficients. Our analysis does not depend on details of Greens functions or on asymptotic fall off conditions which can be dimension dependent. This is in contrast to the situation at future null infinity, where the generalization of the symmetry group, charges and memory to higher dimensions is much more involved [102, 43, 103, 104]. Thus supertranslation and base-space diffeomorphism (superrotation) symmetries are universal symmetries of all null surfaces in vacuum general relativity.

It would also be useful to generalize our analysis to allow for the presence of matter. We expect that the symmetry algebra and expressions for charges will not be modified, but that the flux expressions will acquire corrections involving the stress-energy tensor, as in the BMS context.

Generalizations to other theories of gravity will be more involved. In particular, the symmetry algebra obtained from the Wald-Zoupas procedure can depend on the Lagrangian through the explicit expression for the charge variation in Eq. (2.2.18), and may no longer coincide with the specific intrinsic symmetry algebra of Sec. 2.4 (although it may still posess an intrinsic characterization).

Our symmetry algebra is analogous to the BMS symmetry algebra at future null infinity. In that context it has been suggested that the BMS algebra can be usefully extended to include additional symmetries, which do correspond to soft theorems and to new types of gravitational wave memory [105, 25, 106, 107]. However these generators are not obtained from the Wald-Zoupas construction and their status as symmetries on phase space is still unclear. Perhaps the algebra computed here of symmetries on finite null surfaces could be similarly extended.

Finally, as discussed in Sec. 2.7, a key open question in the black hole context is the restriction on the global algebra of symmetry generators imposed by boundary conditions near future timelike infinity, that should determine the identification of symmetry generators on the horizon and at future null infinity. This identification is necessary in order to formulate the general form of the global conservation law associated with the global charges on the horizon.

Chapter 3

Symmetries, Charges, and Conservation Laws at Causal Diamonds in General Relativity

3.1 Introduction

Understanding the description of finite subsystems in diffeomorphism invariant theories is an important problem in both classical and quantum gravity. Over the years there has been a lot of progress in uncovering various aspects of gravitational subsystems by studying the covariant phase space formalism in the presence of boundaries [60, 68, 24, 108, 109, 51, 52, 110, 25, 111]. The presence of a boundary promotes a subset of the boundary preserving diffeomorphisms to symmetries of the covariant phase space. These boundary symmetries then result in non-trivial boundary charges which can be thought of as capturing aspects of the degrees of freedom contained within the subregion.

Null boundaries are particularly important as they play a fundamental role in gravitational thermodynamics [12, 112, 53, 113], as well as in holography and quantum gravity [49, 44, 114]. Moreover, it was recently conjectured that the symmetries and charges at stationary event horizons are relevant to the black hole information problem [34, 33, 100]. A particularly important class of null surfaces are the boundaries of causal diamonds, which are fundamental to the description of gravitational subregions. Black hole thermodynamics [68, 76, 115, 12] and entanglement entropy in AdS/CFT [4] have demonstrated that geometric properties of causal horizons are deeply related to the thermodynamic and statistical properties of spacetime. This strongly suggests that these deep connections generalize to any causal diamond in any spacetime. While there have been important insights in this direction [116, 117, 118], a complete understanding for arbitrary gravitational subregions remains elusive. A potential avenue of progress lies in the covariant phase space formalism applied to gravity at the boundaries of causal diamonds.

In [119] the covariant phase space of general relativity at null boundaries was studied in detail. There it was shown that there exists an infinite-dimensional symmetry algebra for general relativity at *all* null boundaries, including non-stationary event horizons, in any spacetime. For null boundaries of the form $N = \mathcal{Z} \times \mathbb{R}$, where $\mathcal{Z}$ is the space of null generators, the algebra takes the form $\mathrm{diff}(\mathcal{Z}) \ltimes \mathfrak{s}$ with the "supertranslation" subalgebra $\mathfrak{s}$ consisting of angle-dependent translations and rescalings of affine parameter along the null surface. The charges (at cross-sections of the null surface) and fluxes associated to the symmetries were computed from the covariant phase space formalism using the prescription given by Wald and Zoupas [24].

In this paper we use the results of [119] at the boundaries of causal diamonds. We consider causal diamonds obtained from the intersection of the chronological past and future of timelike separated points in a convex normal neighborhood. The boundary of such a causal diamond is a null surface N with a 2-sphere bifurcation edge B. We use Gaussian null coordinates adapted to the causal diamond boundary to show that both N and B for any causal diamond in any spacetime can be identified. The resulting reduced symmetry algebra which preserves B takes the form $\mathrm{diff}(\mathbb{S}^2) \ltimes \mathfrak{b}$ with $\mathfrak{b}$ consisting of the angle-dependent rescalings along the null generators. The angle-dependent translations along the null generators are eliminated by requiring the bifurcation edge B to be preserved.

By considering the behavior of geometric fields on N near its corners, we show that the boundary charges and the boundary presymplectic potential vanish in the limit to the corners of the causal diamond. From this we show that the Wald-Zoupas fluxes are Hamiltonian generators on the covariant phase space, which in particular provides an infinite family of boost generators for any smooth causal diamond in general relativity. This is similar in spirit to the boost generator at Killing horizons, which is also the vacuum modular Hamiltonian, where in the present context the boost generators act on the gravitational data associated to the causal diamond. Furthermore, we show that the reduced symmetry algebra at N has a non-trivial center. The charges associated to the elements of the center are precisely the boost generators, whose values are proportional to the area of B. Thus there exists a Wald entropy [68], and a quasi-local first law, for any smooth causal diamond in general relativity.

Using the smoothness of the spacetime metric and the vector fields representing the null boundary symmetries we then show that the Wald-Zoupas fluxes associated to the symmetries are conserved between the past and future components of N. This gives an infinite set of conservation laws for finite subregions in general relativity on any spacetime. This is analogous to the conservation laws between past and future null infinity [82, 120, 121, 122] except, in this case, the smoothness of fields at B are much simpler to analyze. Just as the asymptotic conservation laws between past and future null infinity place an infinite number of constraints on gravitational scattering (conjectured to hold even in the quantum theory [82]), the conservation laws we derive for finite causal diamonds likely place important constraints on the properties of scattering in local gravitational subsystems.

The rest of the paper is organized as follows. In section 3.2 we review the formulation of symmetries and the associated charges and fluxes in general relativity from [119], and reformulate them in terms of Gaussian null coordinates. In section 3.3 we adapt this formalism to the boundaries of causal diamonds and detail the reduction of the symmetry algebra of a general null surface to a subalgebra which preserves the structure on a causal diamond. We also investigate the behavior of the fields and charges near the corners of the causal diamond and show that the fluxes associated to the symmetries at a causal diamond are also Hamiltonian generators on the corresponding phase space. In section 3.4 we show that the smoothness of the spacetime metric at the causal diamond implies an infinite number of conservation laws for the fluxes through the null boundaries. In section 3.5 we compute the charges associated to the central elements of the symmetry algebra and show that these take the form of a "first law". We end with section 3.6 summarizing and discussing the potential applications of our results. In appendix A.9 we collect the essential ingredients of the covariant phase space formalism and the Wald-Zoupas prescription for calculating charges and fluxes. In appendix A.10 we analyze the structure of the symmetry algebra at a causal diamond and show how it arises as a non-trivial central extension.

Notation and conventions

We follow the conventions of [67]. We use abstract indices $a, b, \dots$ to denote tensor fields, e.g. g_{ab} is the spacetime metric, and indices $A, B, \dots$ to denote components of tensor fields in some coordinate system on $\mathbb{S}^2$, e.g. q_{AB} is a metric on $\mathbb{S}^2$. Boldface quantities like $\boldsymbol{\omega}$ will denote differential forms.

We also use the following terminology for the charges associated to the symmetry algebra at a null boundary N. Quantities associated to null boundary symmetries evaluated as integrals over cross-sections of the null boundaries will be called "charges", while the difference of these charges on two cros-sections evaluated as an integral over a portion of the null boundary will be called "fluxes". When certain conditions are satisfied the fluxes can also be considered as Hamiltonian generators on the null boundary phase space (see eqs. (A.9.10) and (A.9.11)).

3.2 Null boundary symmetries and charges

In this section we briefly review the basic formalism and results of [119], namely the symmetries and charges at a null boundary in general relativity. We then recast the null boundary phase space, and the resulting symmetries and charges, in terms of Gaussian null coordinates. This will prove to be useful when considering causal diamonds.

The relationship between the covariant approach of [119] which is intrinsic to the null boundary and the coordinate-based approach in section 3.2 is the same as that between the intrinsic universal structure approach [123, 26, 77] and one based on Bondi coordinates [19, 20, 21] or the conformal Gaussian null coordinates [124, 43] at null infinity in asymptotically-flat spacetimes.

Universal structure and symmetries on null boundaries

Consider a spacetime (M, g_{ab}) with null boundary N. For now we will assume the null generators of N are complete, i.e. $N \cong \mathcal{Z} \times \mathbb{R}$, where $\mathcal{Z}$ is the space of null generators. Later we will consider null surfaces with boundary, which is the relevant setting for causal diamonds. The null boundary N is naturally equipped with the equivalence class $[\ell^a, \kappa]$ where ℓ^a is the null generator of N, κ is the non-affinity defined by[1]

$$\ell^b \nabla_b \ell^a \mathrel{\widehat{=}} \kappa \ell^a, \tag{3.2.1}$$

and the equivalence class $[\ell^a, \kappa]$ is defined by the rescaling freedom

$$\ell^a \mapsto e^\beta \ell^a \quad ; \quad \kappa \mapsto e^\beta (\kappa + \pounds_\ell \beta) \tag{3.2.2}$$

where β is a smooth function on N.

In [119] it was shown that the structure $[\ell^a, \kappa]$ is *universal* in the sense that different such structures on N, as induced by different background metrics, are all related by diffeomorphisms (we shall show this explicitly in section 3.2 in a Gaussian null coordinate system). We can then define the field configuration space $\mathscr{F}$ to be the set of smooth metrics g_{ab} on a manifold M with null boundary N which is equipped with the universal structure $[\ell^a, \kappa]$. The covariant phase space is the subset $\overline{\mathscr{F}} \subset \mathscr{F}$ consisting of on-shell metrics satisfying the vacuum Einstein equation.

The group of symmetries on a null boundary N is the subgroup of diffeomorphisms on M which preserves the null boundary N and the universal structure on it. It will be easier to work with the symmetry algebra instead of the group. The symmetry algebra on the null boundary consists of vector fields ξ^a on M which are tangent to N and preserve the linearized version of the equivalence relation eq. (3.2.2). This results in the conditions

$$\begin{aligned} \pounds_\xi \ell^a &\mathrel{\widehat{=}} \beta \ell^a \\ \pounds_\xi \kappa &\mathrel{\widehat{=}} \beta\kappa + \pounds_\ell \beta, \end{aligned} \tag{3.2.3}$$

where β is some smooth function on N which depends on the vector field ξ^a. The detailed structure of the resulting symmetry algebra $\mathfrak{g}$ was derived in [119] and can be summarized as follows. The vector fields of the form $\xi^a \mathrel{\widehat{=}} f\ell^a$ with $\pounds_\ell(\pounds_\ell + \kappa) f \mathrel{\widehat{=}} 0$ form an infinite-dimensional abelian Lie ideal $\mathfrak{s} \subset \mathfrak{g}$ of *supertranslations*. The quotient algebra $\mathfrak{g}/\mathfrak{s}$ is isomorphic to $\mathrm{diff}(\mathcal{Z})$, the algebra of smooth diffeomorphisms of the space of null generators $\mathcal{Z}$. There is an additional Lie ideal $\mathfrak{s}_0 \subset \mathfrak{s}$ of *affine supertranslations* given by $\xi^a \mathrel{\widehat{=}} f\ell^a$ with $(\pounds_\ell + \kappa) f \mathrel{\widehat{=}} 0$. Hence the symmetry algebra $\mathfrak{g}$ can be written as

$$\mathfrak{g} \cong \mathrm{diff}(\mathcal{Z}) \ltimes (\mathfrak{b} \ltimes \mathfrak{s}_0) \tag{3.2.4}$$

where $\mathfrak{b} \cong \mathfrak{s}/\mathfrak{s}_0$.

[1]We use the notation $\mathrel{\widehat{=}}$ to mean 'equality on N' throughout the paper.

The charges and fluxes associated to this symmetry algebra were also derived in [119] using the Wald-Zoupas procedure. Writing the covariant expression for these charges would require introducing a significant amount of formalism and notation. Instead we will derive the symmetry algebra, and express the charges and fluxes, using Gaussian null coordinates in section 3.2.

Gaussian null coordinates

It will be convenient to introduce coordinates adapted to the null surface N called *Gaussian null coordinates* (GNC) [**Penrose**], which have been used in a variety of contexts [29, 43, 81, 57, 125]. We briefly review the construction of GNC below. Since our main interest is in causal diamonds, we will now restrict (for convenience) to the case of 4-dimensional spacetimes where the space of null generators is a 2-sphere $\mathscr{Z} = \mathbb{S}^2$, but our results can be readily generalized.

Let ℓ^a be an affinely-parameterised ($\kappa = 0$) null normal to N and let v be an affine parameter along these null generators i.e., v is some function on N such that $\ell^a \nabla_a v \mathrel{\widehat{=}} 1$. Now let $S \cong \mathbb{S}^2$ be a cross-section of N such that $v|_S = 0$, and let x^A be a coordinate system on S. We extend the coordinate functions x^A to all of N by parallel-transport along the null generators, $\ell^a \nabla_a x^A \mathrel{\widehat{=}} 0$. This defines a coordinate system (v, x^A) on N.

To define a coordinate system in a neighborhood of N, let u be a function in such a neighborhood so that $u|_N = 0$ on N. Then, $\ell_a \equiv -du$ is the normal to N and the vector field $n^a \equiv \partial/\partial u$ is transverse (i.e. not tangent) to N. To fix coordinates away from N we choose u such that n^a is an affinely-parameterised null vector field i.e. $n^a n_a = 0$ and $n^b \nabla_b n^a = 0$. Then we extend the coordinates (v, x^A) away from N by parallel transport along n^a. The coordinate functions (u, v, x^A) define a GNC in a neighborhood of the null surface N. It follows from the above definition of the GNC that in these coordinates the spacetime metric satisfies [125]

$$\begin{aligned} g_{uu} = g_{uA} = 0 \quad ; \quad g_{uv} = -1 \\ g_{vv} \mathrel{\widehat{=}} g_{vA} \mathrel{\widehat{=}} \partial_u g_{vv} \mathrel{\widehat{=}} 0 \end{aligned} \tag{3.2.5}$$

and thus we can write the line element in the form (this is equivalent to the form in [**Penrose**])

$$\begin{aligned} &ds^2 = -W dv^2 - 2du dv + q_{AB}(dx^A - W^A dv)(dx^B - W^B dv) \\ \text{where} \quad &W|_{u=0} = \partial_u W|_{u=0} = W^A|_{u=0} = 0 \end{aligned} \tag{3.2.6}$$

and W, W^A, q_{AB} are functions of (u, v, x^A), and can be considered as tensors on $\mathbb{S}^2$ which depend on (u, v). The tensor q_{AB} defines a Riemannian metric on the 2-spheres of constant u and v. The extensions of the null generator ℓ^a and the auxilliary null vector n^a in the neighborhood of N are given by

$$\ell^a \equiv \partial_v - \tfrac{1}{2} W \partial_u + W^A \partial_A \quad ; \quad n^a \equiv \partial_u \tag{3.2.7}$$

The shear σ_{AB} and expansion θ of N are given by the relation

$$\tfrac{1}{2}\partial_v q_{AB} \mathrel{\hat{=}} \sigma_{AB} + \tfrac{1}{2}q_{AB}\theta \tag{3.2.8}$$

where $\sigma_{AB}q^{AB} = 0$, while the Hájíček rotation 1-form of the $u =$ constant cross-sections is given by

$$\omega_A \mathrel{\hat{=}} -q_A{}^a n_b \nabla_a \ell^b = -\tfrac{1}{2}\partial_u(q_{AB}W^B) \tag{3.2.9}$$

We emphasize that the above construction of the GNC can be carried out in any spacetime in a neighborhood of any null surface. Now let (M_1, g_1) and (M_2, g_2) be two spacetimes with null surfaces N_1 and N_2 along with the GNCs (u_1, v_1, x_1^A) and (u_2, v_2, x_2^A), as constructed above, respectively. Without any loss of generality we can identify a neighborhood of N_1 in (M_1, g_1) with that of N_2 in (M_2, g_2) by identifying the corresponding GNCs $(u_1, v_1, x_1^A) = (u_2, v_2, x_2^A)$. Thus, we can identify all the spacetimes under consideration, and work on a single manifold M with boundary N such that the configuration space $\mathscr{F}$ consists of all the metrics in M for which N is a null surface and the metrics take the form eq. (3.2.6) in a GNC in a neighborhood of N. From the above construction we see that while the induced metric q_{AB} depends on the particular choice of the spacetime metric in $\mathscr{F}$, the null generator ℓ^a is common to all metrics in $\mathscr{F}$. Thus, the null surface N along with the affine null generator ℓ^a are *universal*. Note that we can construct the GNC even with a non-affinely parametrized ℓ^a, which leads to the universal structure used in [119] as described in section 3.2.

Null boundary symmetries, charges and fluxes in GNC

The symmetry algebra at the null boundary N consists of the vector fields ξ^a generating infinitesimal coordinate transformations which preserve the GNC form of the metric in eq. (3.2.6). We expand the vector field ξ^a in the GNC to first order in u as,

$$\xi^a \equiv (f_0 + uf_1)\partial_v + (\beta_0 + u\beta_1)\partial_u + (X_0^A + uX_1^A)\partial_A + O(u^2) \tag{3.2.10}$$

To preserve the location $u = 0$ of the null surface N, ξ^a must be tangent to N and hence $\beta_0 = 0$. To preserve the form of the metric eq. (3.2.6) we have

$$\pounds_\xi g_{uu} = \pounds_\xi g_{uv} = \pounds_\xi g_{uA} = 0 \tag{3.2.11a}$$
$$\pounds_\xi g_{vA} = O(u) \tag{3.2.11b}$$
$$\pounds_\xi g_{vv} = O(u^2) \tag{3.2.11c}$$

Evaluating eq. (3.2.11a) at $u = 0$ we have

$$\pounds_\xi g_{uu} = 0 \implies f_1 = 0 \tag{3.2.12a}$$
$$\pounds_\xi g_{uv} = 0 \implies \beta_1 = -\partial_v f_0 \tag{3.2.12b}$$
$$\pounds_\xi g_{uA} = 0 \implies q_{AB}X_1^A = \partial_A f_0 \tag{3.2.12c}$$

The conditions eq. (3.2.11b) imply

$$\partial_v X_0^A = 0 \tag{3.2.13}$$

while eq. (3.2.11c), evaluated to $O(u)$ using eq. (3.2.13), gives

$$\partial_v \beta_1 = 0 \implies \partial_v^2 f_0 = 0 \tag{3.2.14}$$

where the second condition follows from eq. (3.2.12b). Similar conditions were derived independently in [125].

From eqs. (3.2.13) and (3.2.14) we conclude that the symmetries on the null boundary N are characterized by (α, β, X^A) where α and β are functions and X^A is a vector field on $\mathbb{S}^2$. In a neighborhood of N this symmetry is represented by a vector field ξ^a, which in GNC takes the form

$$\xi^a \equiv (\alpha - v\beta)\partial_v + u\beta\partial_u + \left[X^A + uq^{AB}\partial_B(\alpha - v\beta)\right]\partial_A + O(u^2) \tag{3.2.15}$$

Note that the vector field ξ^a at N, i.e. $u = 0$, is parametrized entirely by (α, β, X^A) and is independent of the choice of metric in the configuration space $\mathscr{F}$.

We now analyze the structure of the symmetry algebra $\mathfrak{g}$ generated by such vector fields. Consider two symmetries $\xi_1 = (\alpha_1, \beta_1, X_1^A)$ and $\xi_2 = (\alpha_2, \beta_2, X_2^A)$ in $\mathfrak{g}$. Their Lie bracket can be computed using their representations in terms of vector fields as in eq. (3.2.15). A straightforward computation gives

$$\begin{aligned} \left[(\alpha_1, \beta_1, X_1^A), (\alpha_2, \beta_2, X_2^A)\right] &= (\alpha, \beta, X^A) && (3.2.16a)\\ \text{where} \quad \alpha &= -\alpha_1\beta_2 + \alpha_2\beta_1 + X_1^A\partial_A\alpha_2 - X_2^A\partial_A\alpha_1 && (3.2.16b)\\ \beta &= X_1^A\partial_A\beta_2 - X_2^A\partial_A\beta_1 && (3.2.16c)\\ X^A &= [X_1, X_2]^A = X_1^B\partial_B X_2^A - X_2^B\partial_B X_1^A && (3.2.16d) \end{aligned}$$

where the last line is the Lie bracket of vector fields on $\mathbb{S}^2$. Note that the sign of β in Eq. 4.10b [119] is incorrect and has been corrected in eq. (3.2.16c) above.

From eq. (3.2.16) it is easy to deduce the following structure of $\mathfrak{g}$. If $X_1^A = 0$ then $X^A = 0$, i.e. symmetries of the form $(\alpha, \beta, 0)$ form an abelian Lie ideal $\mathfrak{s} \subset \mathfrak{g}$ of *supertranslations*. The quotient $\mathfrak{g}/\mathfrak{s}$ is then isomorphic to the Lie algebra $\mathrm{diff}(\mathbb{S}^2)$ represented by symmetries of the form $(0, 0, X^A)$. There is an additional Lie ideal in $\mathfrak{g}$ which is given as follows. In eq. (3.2.16), taking $\beta_1 = X_1^A = 0$ we get $\beta = X^A = 0$, that is, symmetries of the form $(\alpha, 0, 0)$ are also an abelian Lie ideal $\mathfrak{s}_0 \subset \mathfrak{g}$ called *affine supertranslations*. The quotient $\mathfrak{b} \cong \mathfrak{s}/\mathfrak{s}_0$ of all the supertranslations by $\mathfrak{s}_0$ is represented by symmetries of the form $(0, \beta, 0)$. Thus, the symmetry algebra $\mathfrak{g}$ on any null boundary has the structure (same as in eq. (3.2.4))

$$\mathfrak{g} \cong \mathrm{diff}(\mathbb{S}^2) \ltimes (\mathfrak{b} \ltimes \mathfrak{s}_0) \tag{3.2.17}$$

It was shown in [119] that this symmetry algebra coincides with the definition given by Wald and Zoupas [24], reviewed below eq. (A.9.4).

Remark 3.2.1 (Symmetry group at N). The symmetry group can also be obtained by considering finite coordinate transformations of the GNC which preserve the metric form eq. (3.2.6). In particular at N, i.e. $u = 0$, we have the coordinate transformations $(v, x^A) \mapsto (\overline{v}, \overline{x}^A)$ with

$$\overline{v} = \alpha(x^A) + e^{-\beta(x^A)}v \quad ; \quad \overline{x}^A = \overline{x}^A(x^B) \tag{3.2.18}$$

Thus, the symmetry group consists of arbitrary diffeomorphisms of $\mathbb{S}^2$ along with angle-dependent translations (given by $\alpha(x^A)$) and angle-dependent rescalings (given by $\beta(x^A)$) along the null generators.

When the null surface N has additional structure which is also universal — i.e., common to *all* the spacetimes under consideration — the symmetry algebra can be reduced further. For instance, when all the spacetimes have a Killing vector field in a neighborhood of N which becomes tangent to N, the symmetries proportional to this Killing field provide a preferred 1-dimensional subalgebra of $\mathfrak{g}$ (see Sec. 4.4 [119]). Similarly, if the null surface is stationary for all spacetimes, so that the shear and expansion of N vanish, then the symmetry algebra can be reduced so that $\beta =$ constant and X^A is a conformal Killing field on $\mathbb{S}^2$ i.e., an element of the Lorentz algebra (see Sec. IV.B [126]).[2] We show in section 3.3 that when N is the null boundary of causal diamonds a similar reduction of the symmetry algebra occurs due to the presence of a preferred cross-section corresponding to the bifurcation edge. Specifically, since the bifurcation edge is a preferred cross-section of the null boundary of a causal diamond, only those symmetry vector fields which preserve its location i.e. are tangent to the bifurcation surface are permitted in the symmetry algebra. This has the effect of eliminating the affine supertranslations $\xi^a \equiv \alpha\partial_v \in \mathfrak{s}_0$ from the symmetry algebra (see also Sec. 4.5 [119]).

The charges and fluxes associated to the null boundary algebra $\mathfrak{g}$ were computed in [119] using the covariant phase space formalism along with the Wald-Zoupas prescription. It was also shown that the ambiguities in the symplectic current and the Wald-Zoupas prescription do not affect the resulting charges and fluxes. We do not repeat the full analysis of [119], but below we write down the relevant expressions for the boundary presymplectic potential $\mathbf{\Theta}(g; \delta g)$, the Wald-Zoupas (WZ) charges $\mathcal{Q}_\xi$ and fluxes $\mathcal{F}_\xi$ for vacuum general relativity, derived in [119], in terms of GNC.

The boundary presymplectic potential on N is given by

$$\mathbf{\Theta}(g; \delta g) = \frac{1}{16\pi}\boldsymbol{\varepsilon}_3 \left(\sigma^{AB} - \tfrac{1}{2}q^{AB}\theta\right) q_A{}^a q_B{}^b \delta g_{ab} \tag{3.2.19}$$

where $\boldsymbol{\varepsilon}_3 \equiv \varepsilon_{abc}$ is the 3-volume element on N.

Let S be any cross-section of N with area-element $\boldsymbol{\varepsilon}_2 \equiv \varepsilon_{ab}$ and ΔN be a region of N bounded by two cross-sections. The charges (on S) and fluxes (through ΔN) associated to

[2] Note that the reduction of diff($\mathbb{S}^2$) to the Lorentz algebra in the stationary case relies crucially on the $\mathbb{S}^2$ topology of the cross-sections.

a supertranslation $\xi^a \mathrel{\widehat{=}} f\ell^a = (\alpha - v\beta)\ell^a$ are

$$\begin{aligned} \mathcal{Q}_f[S] &= \frac{1}{8\pi}\int_S \boldsymbol{\varepsilon}_2 \left[(\alpha - v\beta)\theta + \beta\right] \\ \mathcal{F}_f[\Delta N] &= \frac{1}{8\pi}\int_{\Delta N} \boldsymbol{\varepsilon}_3 \, (\alpha - v\beta)\left(\sigma_{AB}\sigma^{AB} - \tfrac{1}{2}\theta^2\right) \end{aligned} \tag{3.2.20}$$

while those associated to a diff($\mathbb{S}^2$) generator X^A (taken to be tangent everywhere to the $v =$ constant cross-sections of N) are given by

$$\begin{aligned} \mathcal{Q}_X[S] &= \frac{1}{8\pi}\int_S \boldsymbol{\varepsilon}_2 \left(-\omega_A X^A\right) \\ \mathcal{F}_X[\Delta N] &= \frac{1}{8\pi}\int_{\Delta N} \boldsymbol{\varepsilon}_3 \left(\sigma_{AB} - \tfrac{1}{2}\theta q_{AB}\right) D^A X^B \end{aligned} \tag{3.2.21}$$

where the Hájíček rotation 1-form ω_A (in the GNC foliation) is as defined in eq. (3.2.9), and D_A is the derivative operator with respect to q_{AB} in the foliation given by the GNC.

Note that the supertranslation charge expression eq. (3.2.20) can be evaluated on any choice of cross-section, while the charge expression eq. (3.2.21) only holds on the cross-sections of the $v =$ constant foliation.[3] If the diff($\mathbb{S}^2$) charge is to be evaluated on some arbitrary foliation of N with normal $\hat{n}_a$ such that $\ell^a \hat{n}_a \mathrel{\widehat{=}} -1$, then we have instead

$$\mathcal{Q}_{\hat{X}}[\hat{S}] = \frac{1}{8\pi}\int_{\hat{S}} \hat{\boldsymbol{\varepsilon}}_2 \left(\beta_{\hat{X}} - \hat{\omega}_A \hat{X}^A\right) \tag{3.2.22}$$

where now $\hat{X}^A$ is taken to be tangent to the cross-sections of the chosen foliation, $\hat{\omega}_A$ is the corresponding Hájíček rotation 1-form, while $\beta_{\hat{X}}$ is given by

$$\beta_{\hat{X}} = -\hat{X}^A \hat{q}_A{}^a \pounds_\ell \hat{n}_a \tag{3.2.23}$$

Remark 3.2.2 (Fluxes from the charges)*.* Given the charge expressions eqs. (3.2.20) and (3.2.21), the corresponding fluxes can be obtained using the vacuum Einstein equations $R_{ab} = 0$ on N. Specifically we have

$$\begin{aligned} R_{ab}\ell^a\ell^b = 0 \implies \partial_v\theta &= -\tfrac{1}{2}\theta^2 - \sigma_{AB}\sigma^{AB} \\ R_{ab}q_A{}^a\ell^b = 0 \implies \partial_v\omega_A &= -\theta\omega_A - D^B\sigma_{AB} + \tfrac{1}{2}D_A\theta \end{aligned} \tag{3.2.24}$$

where D_A is the derivative operator with respect to q_{AB} in the foliation given by the GNC. These are the *Raychaudhuri equation* and the *Damour-Navier-Stokes equation* respectively [75, 127].

[3]The dependence of the diff($\mathbb{S}^2$) charge expression on the foliation is a result of the semidirect structure of $\mathfrak{g}$ in eq. (3.2.6), that is, there does not exist any unique choice of a diff($\mathbb{S}^2$) subalgebra of $\mathfrak{g}$. This is similar to the status of the Lorentz algebra within the BMS algebra at null infinity.

3.3 Causal Diamonds

In this section we apply the above results to the boundaries of causal diamonds, with the appropriate modifications necessary for null surfaces with boundary. We begin by recalling the definition of causal diamonds and the structure on their null boundaries.

In a given spacetime (M, g), consider two points $p^+, p^- \in M$ such that p^+ is in a convex normal neighborhood of p^- and is in its chronological future, i.e., p^+ is "inside the future light cone" of p^-. The intersection of the chronological past of p^+ with the chronological future of p^- defines a *causal diamond* or a *double cone* (see for instance [128, 129, 117, 118]). We assume that the causal diamond is "small enough" so that the null generators emanating from $p^\pm$ form smooth null surfaces $N^\pm$ respectively, which intersect at a smooth 2-surface B, the *bifurcation edge*, which is topologically $\mathbb{S}^2$. We denote the null boundary of the causal diamond by $N = N^+ \cup N^-$ (see fig. 3.1).

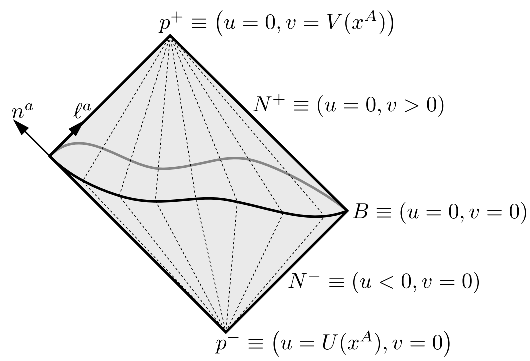


Figure 3.1: Diagram of a causal diamond in a spacetime (M, g). The points $p^\pm$ denote the corners of the causal diamond and B is the bifurcation edge while $N^\pm$ denote the future/past null surfaces joining B to $p^\pm$, respectively. The functions v and u are affine coordinates with affine null normals ℓ^a and n^a on $N^\pm$.

We now investigate the null boundary symmetries and charges defined in [119] for the null boundary N of a causal diamond. It will be convenient to use the formulation in terms of GNC as detailed in section 3.2, which we adapt to a causal diamond as follows.

Unlike a general null surface, the boundary of a causal diamond has a preferred cross-section determined by the bifurcation edge B. At B there exist *unique* null vector fields ℓ^a and n^a both future-directed such that ℓ^a is tangent to N^+, n^a is tangent to N^-, and $\ell^a n_a|_B = -1$. We can extend these vector fields to $N^\pm$ so that ℓ^a is the affine null generator of N^+, n^a is the affine null generator of N^-, and $\ell^a n_a|_N = -1$. Let x^A be some coordinates on B; we pick the affine parameter v of $\ell^a \equiv \partial/\partial v$ on N^+, and similarly u of $n^a \equiv \partial/\partial u$ on N^-, such that $B \equiv (u = 0, v = 0)$. Since, ℓ^a is future-directed, the coordinate v increases moving towards p^+ from $v = 0$ at B. Note that in a general spacetime N^+ will have both a shear and an expansion, which depend on the space of generators, hence the value of the affine-parameter v at the corner p^+ will depend on the null generator along which we approach p^+ i.e. $v|_{p^+} = V(x^A)$. Similarly, on N^- the affine parameter u along n^a decreases moving towards p^- from $u = 0$ at B and p^- lies at $u|_{p^-} = U(x^A)$. These are depicted in fig. 3.1.

As described in section 3.2, we can extend (u, v, x^A) to form a GNC in a neighborhood of the causal diamond.[4] Since we have two null surfaces we obtain two different GNCs, one based on N^+ which we denote by (u_+, v_+, x^A_+) and another based on N^- denoted by (u_-, v_-, x^A_-). In general, these two coordinate systems will not agree in a neighborhood of B and will be related by a coordinate transformation that preserves neither GNC. We will not need the explicit form of the transformations between these coordinates but we note that (by construction)

$$(u_\pm, v_\pm)|_B = (u = 0, v = 0) \quad ; \quad x^A_+\big|_B = x^A_-\big|_B \tag{3.3.1}$$

and

$$\ell^a \equiv \partial_v = \partial_{v_+}\big|_B = \partial_{v_-}\big|_B \quad ; \quad n^a \equiv \partial_u = \partial_{u_+}\big|_B = \partial_{u_-}\big|_B \tag{3.3.2}$$

The spacetime metric g_{ab}, which we assume is smooth written in either coordinate system, coincides at B.

We define the 3-volume elements $\varepsilon^\pm_{abc}$ on $N^\pm$ and the 2-area elements $\varepsilon^\pm_{ab}$ on the cross-sections of $N^\pm$ as follows:

$$\begin{aligned} &\text{on } N^+: \quad \varepsilon^+_{abc} = n^d \varepsilon_{dabc} \quad ; \quad \varepsilon^+_{ab} = -\ell^c \varepsilon^+_{cab} = -\ell^c n^d \varepsilon_{dcab} \\ &\text{on } N^-: \quad \varepsilon^-_{abc} = -\ell^d \varepsilon_{dabc} \quad ; \quad \varepsilon^-_{ab} = -n^c \varepsilon^-_{cab} = n^c \ell^d \varepsilon_{dcab} \end{aligned} \tag{3.3.3}$$

Note that on N^+ these conventions are the same as those of [119] while on N^- the sign of ε^-_{abc} is the opposite of that in [119]. We have chosen these conventions so that the area elements on the bifurcation edge B induced from $N^\pm$ coincide, that is,

$$\varepsilon^+_{ab}\big|_B = \varepsilon^-_{ab}\big|_B \tag{3.3.4}$$

Similar to the case of a general null surface, we can now identify the boundaries of any two causal diamonds in any two spacetimes by identifying the GNCs $(u_\pm, v_\pm, x^A_\pm)$. Note that with this identification the bifurcation edge $B \equiv (u = 0, v = 0)$ is common to all causal diamonds and is *universal*, but the corners $u = U(x^A)$ and $v = V(x^A)$ depend on the specific choice of causal diamond and spacetime metric, and are thus not universal. Henceforth we will work with the covariant phase space $\overline{\mathscr{F}}$ of general relativity at the boundary N of a causal diamond where the bifurcation edge B is a common universal surface for all spacetimes in $\overline{\mathscr{F}}$.

[4]To define the GNC in a neighborhood of B, we need to extend the null surface N^+ smoothly "a little" to the past of B, and similarly extend N^- to the future of B. We assume, henceforth, that this has been done.

Reduced symmetry algebra $\mathfrak{g}_{\rm CD}$ at causal diamonds

Since the bifurcation edge B is universal the symmetry algebra for a causal diamond must preserve B. Consider the null boundary symmetry algebra on the future null surface N^+. From the form of the vector fields ξ^a in eq. (3.2.15) we see that the symmetries on N^+ which preserve the surface $B \equiv (v = 0)$ are the ones which satisfy $\alpha(x^A)|_{N^+} = 0$. In other words, the bifurcation edge B breaks the affine supertranslation symmetry. Similarly, the affine supertranslations of the symmetry algebra at the past surface N^- are also broken.

A priori it seems we have two independent symmetries for the causal diamond: one induced from N^+ and the other from N^-, with the respective affine supertranslations set to vanish. However, there is a natural isomorphism between the future and past symmetries which follows from the smoothness of the vector field ξ^a in spacetime. To see this let ξ^a be a smooth vector field in the spacetime M which is a representative of a symmetry on $N^\pm$ respectively, preserving the bifurcation edge B. In the GNCs $(u_\pm, v_\pm, x^A_\pm)$ based on the null surfaces $N^\pm$ we have (see eqs. (3.2.15) and (3.3.2))

$$\begin{aligned} \xi^a &\equiv \beta_+(-v_+\partial_{v_+} + u_+\partial_{u_+}) + X^A_+\partial_{A_+} + \dots \\ &\equiv \beta_-(-u_-\partial_{u_-} + v_-\partial_{v_-}) + X^A_-\partial_{A_-} + \dots \end{aligned} \tag{3.3.5}$$

where as before $\dots$ denotes the subleading terms in the respective GNCs. Note that while the GNCs do not match in a neighborhood of B, from eqs. (3.3.1) and (3.3.2) and the smoothness of ξ^a at B we can conclude that

$$\beta_+|_B = -\,\beta_-|_B \quad ; \quad X^A_+\big|_B = X^A_-\big|_B \tag{3.3.6}$$

This implies a natural isomorphism between the symmetry algebras on N^+ and N^- given by the matching conditions eq. (3.3.6) at B. Thus the elements of the symmetry algebra $\mathfrak{g}_{\rm CD}$ on the boundary of a causal diamond are given by (β, X^A); for definiteness we choose $(\beta, X^A) = (\beta_+, X^A_+) = (-\beta_-, X^A_-)$ to represent an element in $\mathfrak{g}_{\rm CD}$.

The Lie brackets of the algebra $\mathfrak{g}_{\rm CD}$ can be derived from eq. (3.2.16) by setting $\alpha_1 = \alpha_2 = 0$. We have

$$\begin{aligned} \big[(\beta_1, X^A_1), (\beta_2, X^A_2)\big] &= (\beta, X^A) \\ \text{where} \quad \beta &= X^A_1\partial_A\beta_2 - X^A_2\partial_A\beta_1 \\ X^A = [X_1, X_2]^A &= X^B_1\partial_B X^A_2 - X^B_2\partial_B X^A_1 \end{aligned} \tag{3.3.7}$$

If $X^A_1 = 0$ then $X^A = 0$ hence symmetries of the form $(\beta, 0)$ form an infinite-dimensional abelian Lie ideal $\mathfrak{b}$ of *boost supertranslations*. Thus,

$$\mathfrak{g}_{\rm CD} \cong \mathrm{diff}(\mathbb{S}^2) \ltimes \mathfrak{b} \tag{3.3.8}$$

Further, if $\beta_1 = $ constant and $X^A_1 = 0$ then $\beta = X^A = 0$, that is, the symmetries of the form $(\beta = \text{constant}, 0)$ commute with any element of $\mathfrak{g}_{\rm CD}$ and thus form a 1-dimensional

Lie subalgebra $\mathfrak{b}_0$ of central elements which we call *boosts*.[5] Consider the quotient $\mathfrak{g}_{\rm CD}/\mathfrak{b}_0 \cong \text{diff}(\mathbb{S}^2) \ltimes (\mathfrak{b}/\mathfrak{b}_0)$. Then $\mathfrak{g}_{\rm CD}$ has the structure of a *central extension* of $\mathfrak{g}_{\rm CD}/\mathfrak{b}_0$ by the abelian Lie algebra $\mathfrak{b}_0$; the fact that this is a *non-trivial* central extension is shown in appendix A.10.[6]

We show in section 3.5 that the charges associated to the central elements in $\mathfrak{b}_0$ can be interpreted as providing a "first law" for causal diamonds.

Behavior of the fields and charges near the corners

In this section we analyze the behavior of the relevant fields and charges on N^+ near the corner p^+; similar results hold also for N^- at p^-. Essentially, for causal diamonds in smooth spacetimes the behavior of the fields of interest near p^+ is the same as that of a light cone in Minkowski spacetime with some subleading corrections away from p^+. We invoke the results of [130, 131] below. The main consequence of interest for our purposes is that in the limit to the corner p^+ along N^+ we have

$$\mathcal{Q}_\xi \to 0 \quad ; \quad i_\xi \boldsymbol{\Theta}(\delta g) \to 0 \tag{3.3.9}$$

for all symmetries $\xi \in \mathfrak{g}_{\rm CD}$, and all metric perturbations δg_{ab} which are smooth at p^+ in any spacetime. The limit $\mathcal{Q}_\xi \to 0$ near the corner ensures that the total flux $\mathcal{F}_\xi$ associated to the symmetries through all of N^+ is finite and is, in fact, equal to the charge at B. As we will show in section 3.4, along with smoothness of the spacetime at B, this gives infinitely many conservation laws between the incoming and outgoing fluxes through any causal diamond. The implication of the limit $i_\xi \boldsymbol{\Theta}(\delta g) \to 0$ is as follows: a perturbation of the total flux on N^+ can be written as (see eq. (A.9.10))

$$\delta \mathcal{F}_\xi = \int_{N^+} \boldsymbol{\omega}(g; \delta g, \pounds_\xi g) + \int_B i_\xi \boldsymbol{\Theta}(\delta g) - \int_{p^+} i_\xi \boldsymbol{\Theta}(\delta g) \tag{3.3.10}$$

where the integral over the corner p^+ should be interpreted as a limit of integrals on cross-sections of N^+ which suitably limit to p^+ as described below. Since $i_\xi \boldsymbol{\Theta}(\delta g) \to 0$ in the limit to p^+ and $i_\xi \boldsymbol{\Theta}(\delta g)|_B = 0$ (as ξ^a is tangent to B for any symmetry in $\mathfrak{g}_{\rm CD}$), we have

$$\delta \mathcal{F}_\xi = \int_{N^+} \boldsymbol{\omega}(g; \delta g, \pounds_\xi g) \tag{3.3.11}$$

for all symmetries $\xi \in \mathfrak{g}_{\rm CD}$ and all perturbations δg_{ab} which are smooth at p^+ and B. Thus the flux $\mathcal{F}_\xi$, viewed as a function on the covariant phase space on N^+, generates a Hamiltonian flow associated to the symmetry ξ^a (see eq. (A.9.11)).

[5]The terminology "boosts" for elements of $\mathfrak{b}_0$ is motivated by the fact that if one considers a bifurcate Rindler horizon in Minkowski spacetime, instead of a causal diamond, then Lorentz boosts which preserve the Rindler horizon are precisely elements in $\mathfrak{b}_0$.

[6]Note that if one eliminates the non-constant boost supertranslations, for instance by imposing a weakly isolated horizon structure when N is stationary, then the central extension also becomes trivial (see for instance Sec. IV.B [126]).

We now describe the arguments leading to the above described result, referring to [130, 131] for the details. Since the value of the GNC coordinate v at the corner is direction-dependent, v not a regular coordinate at p^+. Similarly, the cross-sections $v =$ constant do not limit to p^+. Thus, to analyze the behavior of the symmetries and charges we need a coordinate system on N^+ which is more suited to the structure near p^+. In a neighborhood of p^+ such a coordinate system can be constructed as follows. Consider the tangent space Tp^+ at p^+, and let $y^i = (y^0, y^1, y^2, y^3)$ be coordinates in Tp^+ so that $y^i|_p^+ = 0$ and the coordinate vector fields ∂_i are orthonormal, with ∂_0 being timelike and future-directed. Defining

$$r^2 = (y^1)^2 + (y^2)^2 + (y^3)^2 \quad ; \quad u = y^0 - r \tag{3.3.12}$$

the past-directed light cone in Tp^+ from p^+ is then given by $u = 0$, and coordinatized by (r, x^A) where x^A are coordinates on the space of past-directed null directions at p^+ isomorphic to $\mathbb{S}^2$.

There exists an exponential map from Tp^+ to a sufficiently small neighbourhood of p^+ so that y^i are coordinates in this neighborhood, called *Riemann normal coordinates*. In such a neighborhood, using (u, r, x^A) as coordinates, the metric g_{ab} takes the form (see [130, 131], note we have changed some signs to conform to our orientation conventions)

$$ds^2 = \mu du^2 - 2\nu du dr - 2\nu_A du dx^A + q_{AB} dx^A dx^B \tag{3.3.13}$$

The analysis of [130, 131] then shows that near the corner p^+ the metric components in eq. (3.3.13) behave as

$$\mu = 1 + O(r^2) \quad ; \quad \nu = 1 + O(r^4) \quad ; \quad \nu_A = O(r^3) \quad ; \quad q_{AB} = r^2 q^0_{AB} + O(r^4) \tag{3.3.14}$$

where q^0_{AB} is the unit-metric on $\mathbb{S}^2$. Here, for any tensor $T_{AB\ldots}$ we use $O(r^k)$ to denote that $T_{AB\ldots} = r^k t_{AB\ldots}$ for some $t_{AB\ldots}$ which, in general, has a non-vanishing limit as a tensor field on $\mathbb{S}^2$ as $r \to 0$. Roughly speaking, to leading order the metric g_{ab} near p^+ behaves as the Minkowski metric at the corner of a light cone.

The expansion and shear of N^+ have the behavior

$$\theta = -\frac{2}{r} + O(r^3) \quad ; \quad \sigma_{AB} = O(r^3) \tag{3.3.15}$$

The normal to the foliation by $r =$ constant surfaces is $\hat{n}_a = dr$. The Hájíček rotation 1-form on N^+ relative to the foliation by r is essentially the quantity denoted by ξ_A in [130, 131], which satisfies

$$\hat{\omega}_A = O(r^2) \tag{3.3.16}$$

We have put a "hat" on the rotation 1-form to emphasize its dependence on the foliation.

To consider the limit of the charges associated to the symmetries on N^+, we now relate these coordinates to the GNC used in the main arguments above. The non-affinity $\hat{\kappa}$ of the null generator $\hat{\ell}^a \equiv -\partial_r$ is given by

$$\hat{\kappa} = -\partial_r \ln \nu = O(r^3) \tag{3.3.17}$$

and thus $\hat{\ell}^a$ is an affine null generator of N^+ up to $O(r^3)$. Thus near p^+, we can identify $\hat{\ell}^a$ with the GNC null generator $\ell^a \equiv \partial_v$ and the coordinate r with GNC coordinate v as

$$\ell^a = \hat{\ell}^a + O(r^3) \quad ; \quad v - V(x^A) = -r + O(r^4) \tag{3.3.18}$$

Note that, as is to be expected, the cross-sections of N^+ given by $r =$ constant and those given by $v =$ constant do not coincide. In particular, their normals $\hat{n}_a \equiv dr$ and $n_a \equiv -dv$ are related by

$$n_a = \hat{n}_a + \partial_A V dx^A + O(r^3) \tag{3.3.19}$$

Now consider a symmetry $\xi^a = -v\beta\partial_v + X^A\partial_A$ in GNC where, as before, X^A is tangent to the $v =$ constant cross-sections. We rewrite this vector field as $\xi^a = \hat{f}\hat{\ell}^a + \hat{X}^A\partial_A$ so that $\hat{X}^A$ is tangent to the $r =$ constant cross-sections. From eqs. (3.3.18) and (3.3.19) we have

$$\hat{f} = -(V - r)\beta - X^A\partial_A V + O(r^3) \quad ; \quad \hat{X}^A = X^A + O(r^3) \tag{3.3.20}$$

and also

$$\beta_{\hat{X}} = -\hat{X}^A \hat{q}_A{}^b \pounds_{\hat{\ell}} \hat{n}_b = O(r^2) \tag{3.3.21}$$

The WZ charge $\mathcal{Q}_\xi$ (see eqs. (3.2.20) and (3.2.22)) evaluated on some cross-section S_r with $r =$ constant is then

$$\mathcal{Q}_\xi[S_r] = \frac{1}{8\pi}\int_{S_r} \hat{\boldsymbol{\varepsilon}}_2 \, \left[\hat{f}\theta + \beta + \hat{\beta}_{\hat{X}} - \hat{\omega}_A\hat{X}^A\right] \tag{3.3.22}$$

From eqs. (3.3.20) and (3.3.21) we see that

$$\mathcal{Q}_\xi[p^+] = \lim_{r\to 0} \mathcal{Q}_\xi[S_r] = 0 \tag{3.3.23}$$

where we have used that $\hat{\boldsymbol{\varepsilon}}_2 = r^2\boldsymbol{\varepsilon}_2^0 + O(r^4)$ with $\boldsymbol{\varepsilon}_2^0$ the area-element of the unit-sphere.

Next consider the integral of $i_\xi\boldsymbol{\Theta}(\delta g)$ on the cross-sections S_r

$$\int_{S_r} i_\xi\boldsymbol{\Theta}(\delta g) = -\frac{1}{16\pi}\int_{S_r} \hat{\boldsymbol{\varepsilon}}_2 \; \hat{f}\left(\sigma^{AB} - \tfrac{1}{2}q^{AB}\theta\right)\hat{q}_A{}^a \hat{q}_B{}^b \delta g_{ab} \tag{3.3.24}$$

Any metric perturbation δg_{ab} which is smooth at p^+ has smooth components in the Riemann normal coordinates y^i described above, and its spherical components behave as $\hat{q}_A{}^a\hat{q}_B{}^b\delta g_{ab} = O(r^2)$. Thus, we have

$$\lim_{r\to 0}\int_{S_r} i_\xi\boldsymbol{\Theta}(\delta g) = 0 \tag{3.3.25}$$

3.4 Conservation laws at causal diamonds

We now show that there exist an infinite-number of conservation laws associated to the symmetry algebra $\mathfrak{g}_{\text{CD}}$ between fluxes through N^- and N^+ for any causal diamond. These conservation laws follow directly from the smoothness of the relevant fields at the bifurcation edge B.

First we show that the smoothness of the spacetime at B implies that the charges corresponding to the symmetries in $\mathfrak{g}_{\rm CD}$ evaluated at B are equal. From eqs. (3.2.20), (3.2.21) and (3.3.6), the charges at B induced from $N^\pm$ are

$$\begin{aligned}\mathcal{Q}_\xi[B] &= +\frac{1}{8\pi}\int_B \boldsymbol{\varepsilon}_2^+ \ \left(+\beta - \omega_A^+ X^A\right)\\ \mathcal{Q}_\xi[B] &= -\frac{1}{8\pi}\int_B \boldsymbol{\varepsilon}_2^- \ \left(-\beta - \omega_A^- X^A\right)\end{aligned} \tag{3.4.1}$$

where the difference in the sign of these expressions is due to our conventions for the area elements on $N^\pm$ given in eq. (3.3.3) and the matching conditions on the symmetries eq. (3.3.6). To show that these charges are equal we need to consider the relation between the Hájíček rotation 1-forms ω_A^+ and ω_A^- which can be obtained as follows. Let $\ell^a_\pm$ and $n^a_\pm$ be the extensions in the respective GNCs of the null vector fields ℓ^a and n^a on B. Then we can compute

$$\begin{aligned}\omega_A^+\big|_B &= -(q^+)_A{}^c n_b^+ \nabla_c^+ \ell_+^b = (q^+)_A{}^c \ell_b^+ \nabla_c^+ n_+^b\\ &= (q^+)_A{}^c \ell_b^- \nabla_c^+ n_-^b\\ &= (q^-)_A{}^c \ell_b^- \nabla_c^- n_-^b\\ &= -\ \omega_A^-\big|_B\end{aligned} \tag{3.4.2}$$

where in the first line we have used $n_a^+ \ell_+^a|_B = n_a \ell^a = -1$, in the second line we have used the fact that ℓ^a and n^a are continuous at B (see eq. (3.3.2)), in the third line the continuity of the induced metric q_{ab} and the spacetime derivative operator ∇ (which follows from the smoothness of the metric g_{ab}) and in the last line the definition of ω_A^- at B. Thus, from the smoothness of the spacetime metric and the continuity of the GNCs at B we have[7]

$$\omega_A^+\big|_B = -\ \omega_A^-\big|_B \tag{3.4.3}$$

Combining eq. (3.4.3) with eqs. (3.3.4), (3.3.6) and (3.4.1) we have

$$\mathcal{Q}_\xi[B] \text{ from } N^+ = \mathcal{Q}_\xi[B] \text{ from } N^- \tag{3.4.4}$$

Next, we consider the fluxes through $N^\pm$ given by

$$\begin{aligned}\mathcal{F}_\xi[N^+] &= \mathcal{Q}_\xi[B] - \mathcal{Q}_\xi[p^+]\\ \mathcal{F}_\xi[N^-] &= \mathcal{Q}_\xi[B] - \mathcal{Q}_\xi[p^-]\end{aligned} \tag{3.4.5}$$

[7]In the Newman-Penrose notation [132] eq. (3.4.3) is simply the identity $\beta + \overline{\alpha} = -(-\beta - \overline{\alpha})$, while in the Geroch-Held-Penrose notation [133] it is $\beta - \overline{\beta}' = -(-\beta + \overline{\beta}')$, which follow from the continuity of the spin-coefficients of the spacetime derivative operator ∇ at B.

Note that the flux on N^+ is *outgoing* while that on N^- is *incoming* relative to the causal diamond (in accordance with our conventions eq. (3.3.3)). As shown in section 3.3 the charges at the corners $p^\pm$ vanish and thus from eq. (3.4.4) we have

$$\mathcal{F}_\xi[N^+] = \mathcal{F}_\xi[N^-] \tag{3.4.6}$$

That is, the *incoming* flux through N^- is equal to the *outgoing* flux through N^+ for any symmetry in $\mathfrak{g}_{\text{CD}}$. Thus, there are infinitely-many conservation laws associated to the symmetry algebra on any causal diamond in any spacetime in general relativity.

Remark 3.4.1 (Affine supertranslations). Note that in section 3.3 we eliminated the affine supertranslations $\alpha(x^A) \neq 0$ from the symmetry algebra of the causal diamond by demanding that the bifurcation surface B be preserved under the symmetries. If we had kept α then $i_\xi \boldsymbol{\Theta}(\delta g)|_B \neq 0$ — since such vector fields are not tangent to B — and thus, the flux of the affine supertranslations is not a Hamiltonian generator on the phase space of the null boundary. Furthermore, the affine supertranslations $\alpha_+ \ell^a$ defined on N^+ and $\alpha_- n^a$ defined on N^- cannot be matched at B, as the corresponding vector fields are not continuous. Even if one imposes the condition $\alpha_+(x^A) = \alpha_-(x^A)$ by hand, the charges corresponding to the affine supertranslations at B (see eq. (3.2.20)) do not match since the expansions $\theta_\pm$ along $N^\pm$ need not be equal at B in general. Thus, there do not exist any conservation laws at a causal diamond in general spacetimes analogous to eq. (3.4.6) for the affine supertranslations.

Remark 3.4.2 (Non-affine parametrization of the null generators). For convenience we chose the null generators of $N^\pm$ to be affinely-parametrized, but our result is invariant under this choice. One can construct a GNC on a null surface relative to an arbitrarily parametrized null generator (with $\kappa \neq 0$). The resulting symmetry algebra is then as described in [119] and section 3.2. The affine supertranslations are eliminated by the condition $f|_B = 0$, in which case the boost supertranslations in $\mathfrak{b}$ are parametrized by the function $-(\pounds_\ell + \kappa) f$ which is invariant under arbitrary rescalings of the null generators. For the boosts in $\mathfrak{b}_0$ we have $-(\pounds_\ell + \kappa) f = \text{constant}$. The remainder of our analysis can also be generalized in a similar fashion; we only note that since $f|_B = 0$, the non-affinities $\kappa_\pm$ of the generators of $N^\pm$ do not enter into the matching of the symmetries and charges at B and the resulting conservation laws.

3.5 Central charges and area of the bifurcation edge

As discussed in section 3.3, the symmetry algebra $\mathfrak{g}_{\text{CD}}$ at the boundary of the causal diamond can be viewed as a non-trivial central extension of $\mathfrak{g}_{\text{CD}}/\mathfrak{b}_0$ by the 1-dimensional abelian subalgebra $\mathfrak{b}_0$ of boosts. One expects the charges of such central elements to be of special significance. We show below that these charges are directly related to the area of B, in analogy with the Wald entropy formula for black holes [68]. A similar result was found in [51] through different considerations.

From the results of section 3.4 we have that[8]

$$\mathcal{F}_\beta[N^+] = \mathcal{Q}_\beta[B] = \frac{\beta}{8\pi}\text{Area}(B) \tag{3.5.1}$$

for any $(\beta = \text{constant}, X^A = 0) \in \mathfrak{b}_0$. This can be written in a more illuminating form as follows: consider the vector field $\xi^a|_{N^+} = -v\beta\ell^a$ corresponding to an element in $\mathfrak{b}_0$, so that

$$\xi^b\nabla_b\xi^a = \kappa_{(\beta)}\xi^a \quad ; \quad \kappa_{(\beta)} = -\beta = \text{constant} \tag{3.5.2}$$

and thus

$$\mathcal{F}_\beta[N^+] = \mathcal{Q}_\beta[B] = -\frac{\kappa_{(\beta)}}{2\pi} \times \frac{1}{4}\text{Area}(B) \tag{3.5.3}$$

If we interpret the charge $\mathcal{Q}_\beta[B]$ as an "energy", the $\frac{1}{4}$Area(B) as an "entropy" and $-\frac{\kappa_{(\beta)}}{2\pi}$ as a "temperature", relative to the vector field ξ^a, then eq. (3.5.3) takes the form of a "first law" [67, 68, 75]. Note that if ξ^a is future-directed on N^+ we have $\beta < 0$ and so $\kappa_{(\beta)} > 0$ and the temperature is negative. This difference in sign compared to the temperature of bifurcate Killing horizons essentially arises due to the fact that the future-directed null generator ℓ^a points "inwards" on N^+. Such a negative temperature was also found for causal diamonds in maximally symmetric spacetimes in [117]. Also note that for asymptotically-flat stationary black holes the scaling of the horizon Killing vector field K^a is fixed by the requirement that at spatial infinity K^a asymptotes to a future-directed unit-normalized timelike Killing vector field (plus a rotational vector field). This completely fixes the scaling of the surface gravity, and hence the temperature, of the black hole. In contrast, there is no natural normalization for the boost vector fields at a causal diamond so we get an entire 1-dimensional family of surface gravities $\kappa_{(\beta)}$ and temperatures corresponding to the symmetries $\mathfrak{b}_0$.

The charges $\mathcal{Q}_\beta[B]$ associated to elements of $\mathfrak{b}_0$ are central even in the sense of a Poisson bracket on the phase space of N^+, i.e. the boost charges Poisson-commute with all the other charges. This can be seen as follows: since the fluxes $\mathcal{F}_\xi$ are Hamiltonian on the phase space of N^+ (see eq. (3.3.11)), we can define their Poisson bracket as (see also Sec. 8 [119])

$$\{\mathcal{F}_{\xi_1}, \mathcal{F}_{\xi_2}\} = -\int_{N^+} \boldsymbol{\omega}(g; \pounds_{\xi_1}g, \pounds_{\xi_2}g) \tag{3.5.4}$$

for any two symmetries $\xi_1, \xi_2 \in \mathfrak{g}_{\text{CD}}$. Since $i_\xi\boldsymbol{\Theta}|_B = 0$ and $\mathcal{F}_\xi[N^+] = \mathcal{Q}_\xi[B]$ for any such symmetry it follows from the analysis of Sec. 8 [119] that

$$\{\mathcal{Q}_{\xi_1}[B], \mathcal{Q}_{\xi_2}[B]\} = \mathcal{Q}_{[\xi_1,\xi_2]}[B] - \int_B \pounds_{\xi_1}\boldsymbol{\mathcal{Q}}_{\xi_2} \tag{3.5.5}$$

where in the final term $\boldsymbol{\mathcal{Q}}_\xi$ is the 2-form whose integral on B gives the charges eq. (3.4.1). If this term is vanishing then the Poisson algebra of charges is isomorphic to the Lie algebra

[8] It can be verified that for a causal diamond in Minkowski spacetime where B is a sphere of radius R, our conventions eq. (3.3.3) give $\int_B \boldsymbol{\varepsilon}_2^+ = \text{Area}(B) = +4\pi R^2$.

of spacetime symmetries in $\mathfrak{g}_{\rm CD}$ (see Sec. 8 [119] for details). In the present case this term does indeed vanish; we have

$$\int_B \pounds_{\xi_1} \boldsymbol{\mathcal{Q}}_{\xi_2} = \int_B \left[i_{\xi_1} d\boldsymbol{\mathcal{Q}}_{\xi_2} + d\left(i_{\xi_1} \boldsymbol{\mathcal{Q}}_{\xi_2} \right) \right] = 0 \tag{3.5.6}$$

where the first term vanishes since any $\xi_1 \in \mathfrak{g}_{\rm CD}$ is tangent to B, while the second term vanishes upon integration over B. Thus the Poisson algebra of the charges on B is isomorphic to the Lie algebra $\mathfrak{g}_{\rm CD}$. The boost charges eq. (3.5.3) associated to the central elements in $\mathfrak{b}_0$ are then also central charges on the phase space on N^+ in the sense of the Poisson algebra.

We emphasize that the appearance of central charges in the above analysis is quite different from that of previous approaches. In particular, we work in full nonlinear general relativity at any causal diamond in any spacetime satisfying the vacuum Einstein equations without any restriction to "near horizon" geometries of stationary black holes (as done in [108, 35, 134]). Since the Einstein equations are not conformally invariant there is no conformal symmetry or Virasoro algebra at the causal diamond in the general case we have considered. In fact, since we have allowed for non-vanishing shear, the induced 2-metrics on the cross-sections of the causal diamond are not conformally related (see eq. (3.2.8)). We always work with smooth vector fields as representatives of the symmetries (as opposed to the singular vector fields considered in [100]). Furthermore, as discussed above, the Poisson algebra of the charges in our case is isomorphic to the Lie algebra of symmetries with no additional central extension (in contrast with [52, 100, 108]). The central extension we obtain already exists in the structure of the spacetime symmetry algebra $\mathfrak{g}_{\rm CD}$.

3.6 Discussion

We studied the covariant phase space formalism at the boundaries of causal diamonds in vacuum general relativity. In suitable Gaussian null coordinates, we showed that one can identify all causal diamonds and their bifurcation edges across all spacetimes, and that the symmetry algebra at the null boundaries of the casual diamond takes the form $\mathfrak{g}_{\rm CD} \cong \text{diff}(\mathbb{S}^2) \ltimes \mathfrak{b}$ where $\text{diff}(\mathbb{S}^2)$ maps different null generators of the causal diamond boundary into each other and $\mathfrak{b}$ consists of angle-dependent rescalings of the affine parameter along the null generators. Suitable smoothness conditions at the corners of the causal diamond imply that the Wald-Zoupas charges vanish at the corners — so that the total flux across the null boundary is equal to the charge at the bifurcation edge — and that the Wald-Zoupas fluxes define Hamiltonian generators of the symmetries on the null boundary phase space. The smoothness of the symmetry vector fields and the fields at the bifurcation edge then give rise to an infinite-number of conservation laws for the Wald-Zoupas fluxes between the past and future components of the causal diamond boundary. We also showed that the charge associated to the central elements of the symmetry algebra — i.e. the elements of the subalgebra $\mathfrak{b}_0$ consisting of the angle-independent supertranslations — is related to the area of the bifurcation edge through a "first law" similar to the Wald entropy formula for stationary black holes.

While our analysis focused on causal diamonds in classical vacuum general relativity we expect that it can be generalized to include matter fields described by a suitable QFT on curved spacetimes. For instance, it was shown in [129] that a comparison of suitable states defined on causal diamonds in different spacetimes can be used to extract properties of the local curvature of the spacetimes. Similarly, in [135] it was shown that the relative entropy of quantum states in linearized general relativity in an asymptotically-flat black hole spacetime is related to the area of the black hole and Bondi flux at null infinity. The infinite-dimensional symmetry algebra at the causal diamond could be useful to analyze other properties of a QFT on curved spacetimes.

It has been conjectured that the conservation laws at null infinity strongly constrain the scattering matrix of quantum gravity in asymptotically-flat spacetimes [82]. Similarly, we expect the conservation laws derived in section 3.4 can be used to constrain the transition amplitudes in quantum gravity on local causal diamonds. To do this one needs to suitably quantize the gravitational degrees of freedom on the null boundary (see for instance [114, 49, 50]) and promote the charges and fluxes to operators with the bracket structure eq. (A.2.3) in the corresponding quantum theory. We leave further investigation of this problem to future work.

We also expect that our analysis can be extended to causal diamonds at an asymptotic boundary in a spacetime, an interesting example of which arises in the AdS/CFT duality. In this context, for an asymptotically-AdS spacetime, the entanglement entropy of a CFT state defined on a causal diamond lying on the asymptotic boundary (conformal to Minkowski spacetime with one fewer dimensions) is dual to the area of the Ryu-Takayanagi surface in the bulk spacetime [4, 136]. The Ryu-Takayanagi surface can itself be considered as the bifurcation edge of an "entanglement wedge". Presumably, our analysis can be suitably generalized to this case, taking into account the asymptotic AdS boundary conditions. We expect that the resulting symmetries are related to the boundary modular Hamiltonian, and that the associated charges and fluxes could provide further insight into the bulk dual of boundary modular flow following [137, 138, 139].

Chapter 4

Anomalies in Gravitational Charge Algebras of Null Boundaries and Black Hole Entropy

4.1 Introduction and summary

Observables in general relativity tend to be global in nature, owing to the fact that diffeomorphisms are gauge symmetries of the theory. This large gauge redundancy causes the Hamiltonian of the theory to be localized to the asymptotic boundary, and diffeomorphism-invariant observables must be constructed relationally, using the fixed structures at the asymptotic boundary as points of reference [140, 141, 142]. Nonetheless, there exist notions of quasilocal observables that describe degrees of freedom inside of spatial subregions. In particular, several approaches to understanding the origin of black hole entropy deal with quasilocal charges on the event horizon [143, 144, 145, 146, 147, 148, 149, 150]. Moreover, charges associated with $\mathscr{I}$ in asymptotically flat space [6, 151, 152, 153, 154] and more general null surfaces [119, 155, 156, 157, 158, 159, 160] have received recent attention, due to their potential relevance to quantum gravity and flat space holography.

The appearance of quasilocal observables when considering subregions can be understood in terms of symmetry breaking. The introduction of a fixed boundary partially violates the diffeomorphism symmetry present in the theory, causing some transformations that were formerly considered gauge to become physical [143, 161]. The charges associated with the broken diffeomorphisms localize on the boundary of the subregion, and hence are referred to as edge modes [146, 162, 163]. The connection to black hole entropy comes from the proposal that the edge modes represent the degrees of freedom counted by the Bekenstein-Hawking entropy of a surface, given by $S_{\mathrm{BH}} = \frac{A}{4G}$, with A the area of the surface. The fact that the edge modes are localized on the boundary qualitatively explains the scaling with area, but in some examples the numerical coefficient can be computed in a precise manner. As first shown by Strominger for BTZ black holes in AdS_3 [164] using the Brown-Henneaux central charge [165], and subsequently generalized by Carlip to generic Killing horizons [143, 166], if the quasilocal charge algebra includes a Virasoro algebra, the entropy can be derived by applying the Cardy formula for the entropy of a 2D conformal field theory [167]. The rationale behind this procedure is that the Virasoro algebra is the symmetry algebra of 2D CFTs, so it is natural to conjecture that the quantization of the edge modes is given by a CFT, with the central charge determined by the classical brackets of the quasilocal charges. The precise agreement between the Cardy entropy and the Bekenstein-Hawking entropy then provides *a posteriori* justification for associating the entropy with edge mode degrees of freedom.

In most constructions in which the entropy arises from the Cardy formula applied to a boundary charge algebra, boundary conditions are needed to ensure the charges are integrable. The need for boundary conditions arises because the vector fields generating the symmetry have a transverse component to the codimension-2 surface on which the charge is being evaluated. This means they are generating a transformation that moves the bounding surface, and hence without boundary conditions, symplectic flux can leak out of the subregion as the system evolves. Imposing the boundary conditions ensures that the subregion behaves as a closed system, but gives the boundary the status of a physical barrier, preventing exchange of information between the subregion and its complement. When viewing the boundary as an arbitrary partition used to define a subregion, one would like a definition of quasilocal charges that does not employ such restrictive boundary conditions, and need not require conservation under time evolution. In the place of conservation, one seeks an independent definition of the flux of the quasilocal charge through the subregion boundary, so that the charge instead obeys a continuity equation. For general relativity and other diffeomorphism-invariant theories, Wald and Zoupas provided such a construction of quasilocal charges using covariant phase space techniques [6], and its application to null boundaries at a finite location was considered in [119].

Another reason for utilizing the Wald-Zoupas prescription is that in some cases, there is no obvious boundary condition that ensures integrability of the quasilocal charges. Such was the situation encountered by Haco, Hawking, Perry, and Strominger (HHPS) [149], who identified a set of near-horizon Virasoro symmetries for Kerr black holes, inspired by the hidden conformal symmetry of the near horizon wave equation identified in [168]. These symmetries suggest a possible extension of the results of the Kerr/CFT correspondence [169, 170], which deals with extremal Kerr black holes, to a holographic description of more general horizons. There does not exist a local boundary condition one can impose on the dynamical fields that is preserved by the HHPS vector fields, while simultaneously ensuring integrability of the corresponding charges.[1] Hence, the Wald-Zoupas procedure is needed to define the quasilocal charges.

A specific form of the flux in the Wald-Zoupas prescription was conjectured in [149], and was also used in various subsequent works generalizing the construction [150, 172, 171,

[1]There can be weaker, integrated boundary conditions that ensure integrability for special choices of the parameters defining the transformation, as described in [171].

173]. The goal of the present work is to derive the necessary Wald-Zoupas prescription for these constructions from first principles. In order to do so, there are three main technical challenges that need to be resolved.

First, there are a number of ambiguities that arise when carrying out the Wald-Zoupas construction, some of which affect the final result for the entropy. The most important ambiguity is in the ability to shift the symplectic potential on the bounding hypersurface by total variations, which subsequently affects the definitions of the charges and fluxes. To resolve this issue, we first reformulate the Wald-Zoupas procedure in section 4.2 using Harlow and Wu's presentation of the covariant phase space formalism with boundaries [174]. Doing so allows for an efficient parameterization of the ambiguities that can appear in terms of boundary and corner terms in the variational principle. Rather than imposing boundary conditions to eliminate some terms that appear in the variations, as was done in [174], we interpret the nonzero boundary terms as representing a symplectic flux through the boundary. Explicitly, we decompose the pullback $\boldsymbol{\theta}$ of the symplectic potential current into boundary ℓ, corner β, and flux $\mathcal{E}$ terms:

$$\boldsymbol{\theta} + \delta\ell = d\beta + \mathcal{E}. \tag{4.1.1}$$

Resolving the ambiguities in the Wald-Zoupas prescription then amounts to finding a preferred choice for the flux term $\mathcal{E}$.

We propose a principle for fixing this ambiguity in section 4.2, namely that $\mathcal{E}$ should be of Dirichlet form, meaning it involves variations only of intrinsic quantities on the surface. It therefore is expressible as

$$\mathcal{E} = \pi^{ij}\delta g_{ij}, \tag{4.1.2}$$

where δg_{ij} is the variation of the induced metric on the bounding hypersurface, and π^{ij} are the conjugate momenta constructed from extrinsic quantities. For null hypersurfaces, the variation of the null generator δl^i is also considered an intrinsic quantity, so the Dirichlet form of the flux in this case reads

$$\mathcal{E} = \pi^{ij}\delta g_{ij} + \pi_i \delta l^i. \tag{4.1.3}$$

The terminology "Dirichlet" refers to the fact that vanishing flux is equivalent to Dirichlet boundary conditions for this choice. The Dirichlet flux condition is a novel proposal in the context of the Wald-Zoupas construction, in contrast with previous proposals which employed properties of the flux in stationary solutions to partially fix its form [7, 119]. However, it is familiar from the Brown-York procedure for quasilocal energy [175], and has a natural interpretation in the context of holography. We also argue that this form of the flux is preferred from the perspective of gluing subregions together in the gravitational path integral [176]. As a byproduct of fixing this form of the flux, we can also employ Harlow and Wu's [174] resolution of the standard Jacobson-Kang-Myers ambiguities in the covariant phase space formalism [177, 178], leading to unambiguous definitions of the quasilocal charges.

The second issue to address is the problem of constructing a bracket for the quasilocal charges that defines their algebra. Poisson brackets are not available when employing the Wald-Zoupas procedure, since we are dealing with an open system with respect to the symplectic flux. Therefore, in section 4.2, we instead utilize the bracket defined by Barnich and Troessaert in [98] for nonintegrable charges. It has the advantage of representing the algebra satisfied by the vector fields generating the symmetry transformations, up to abelian extensions. We further show that the algebra extension has a simple expression

$$K_{\xi,\zeta} = \int_{\partial\Sigma} \left(i_\xi \Delta_{\hat{\zeta}} \ell - i_\zeta \Delta_{\hat{\xi}} \ell \right) \tag{4.1.4}$$

in terms of $\Delta_{\hat{\xi}}\ell$, the anomalous transformation with respect to the symmetry generator ξ^a of the boundary term ℓ in (4.1.1). The anomaly operator $\Delta_{\hat{\xi}}$, defined in (4.2.1), directly measures the failure of an object to transform covariantly under the diffeomorphism generated by ξ^a, and hence we immediately see that algebra extensions only appear when the boundary term ℓ is not covariant with respect to the transformation. Because the Barnich-Troessaert bracket coincides with the Poisson bracket when the charges are integrable, this formula for the extension applies in the case of integrable charges as well. This shows quite generally that central charges and abelian extensions appear as a type of classical anomaly associated with the boundary term in the variational principle. This statement is directly analogous to the appearance of holographic Weyl anomalies in AdS/CFT [179, 180, 181, 182].

The third issue to address is finding a decomposition of the symplectic potential for general relativity when restricted to a null boundary $\mathcal{N}$. This question has been treated in previous analyses [119, 183, 155, 156, 184, 185]; however, most of these employ boundary conditions that are too strong to allow for the symmetries generated by the HHPS vector fields. In our analysis in section 4.3, we employ the weakest possible boundary conditions that ensure the presence of a null surface, and in which the variations of all quantities are entirely determined in terms of δg_{ab}. This is done by fixing the normal covector, $\delta l_a = 0$, and imposing nullness by requiring that $l^a l^b \delta g_{ab} = 0$ on $\mathcal{N}$. The covector l_a is thus viewed as a background structure introduced into the theory in order to define the boundary. Because it is a background structure, no issues arise if the symmetry generators do not preserve it; in fact, the failure of l_a to be preserved by the symmetry generators is the sole source of noncovariance in the construction, and hence is responsible for the appearance of a nonzero central charge. By contrast, it is crucial that the vector fields satisfy $l^a l^b \pounds_\xi g_{ab} = 0$ on $\mathcal{N}$, since this arises from a boundary condition imposed on the dynamical metric; violating it would cause the symmetry transformations to be ill-defined. The HHPS vector fields satisfy this condition, as do any vectors which preserve the null surface.

The result of the decomposition of the symplectic potential is given in equations (4.3.26)–(4.3.30), in which the Dirichlet form of $\mathcal{E}$ is decomposed into $\frac{d(d-1)}{2}$ canonical pairs on the null surface. The decomposition that we find has appeared before in [183], and related decompositions can be found in [155, 156]. The boundary term ℓ that arises in the decomposition is constructed from the inaffinity k of the null generator l^a, and has appeared in previous analyses on null boundary terms in the action for general relativity [183, 155, 185]. In particular, we find additional flux terms beyond those employed in [149, 171], whose presence is necessary to ensure that the flux is independent of the choice of auxiliary null vector n_a.

With all this in place, we give a systematic analysis in section 4.4 of the quasilocal charges in the HHPS construction, as well as the generalization to arbitrary bifurcate, axisymetric Killing horizons [149, 171]. The symmetry algebra consists of two copies of the Virasoro algebra, and the central charges are computed to be

$$c = \overline{c} = \frac{3A}{\pi G(\alpha + \overline{\alpha})}, \tag{4.1.5}$$

where α and $\overline{\alpha}$ are two parameters characterizing the symmetry generators, and are related to the choice of left and right temperatures. These values of c, $\overline{c}$ are twice the value given in [149, 171], and consequently, when applying the Cardy formula in section 4.5, we find that the entropy is twice the Bekenstein-Hawking entropy of the horizon. We take this as an indication that the quasilocal charge algebra is sensitive to degrees of freedom associated with the complementary region. In particular, we note that the factor of 2 could be explained if the central charge appearing in the Barnich-Troessaert bracket was associated with a pair of quasilocal charge algebras, one on each side of the dividing surface. This interpretation is further motivated by the conjectured edge mode contribution to entanglement entropy in gravitational theories, which employ such a pair of quasilocal charges at an entangling surface [146]. The doubling of c, $\overline{c}$ would then be intimately related to the fact that we are considering an open system that is interacting with its complement. Conversely, if the charges were instead integrable so that they lived in a closed system, we would expect the standard entropy to arise via the Cardy formula. We demonstrate that this is the case in sections 4.5 and 4.5 by showing that a different boundary term is needed in order to find integrable generators. The new boundary term halves the value of the central charges and the entropy, and also leads to agreement between the microcanonical and canonical Cardy formulas.

In section 4.6, we further discuss the interpretation of these results, and describe some directions for future work.

Note added: This work is being released in coordination with [186], which explores some related topics.

Notation

We work in arbitrary spacetime dimension d with metric signature $(-,+,+,\ldots)$. Spacetime tensors will be written with abstract indices $a, b, \ldots$, such as the metric g_{ab}. We denote null hypersurfaces by $\mathcal{N}$, and indices $i, j, \ldots$ will denote tensors defined on $\mathcal{N}$, such as q_{ij} and l^k. An equality that only holds at the location of $\mathcal{N}$ in spacetime will be written as $\widehat{=}$. Differential forms will often be written without indices, and, when necessary, we distinguish a form θ defined on spacetime from its pullback $\boldsymbol{\theta}$ to $\mathcal{N}$ using boldface. The null normal to $\mathcal{N}$ will be denoted l_a, and the auxiliary null vector will be denoted n^a. The volume form on spacetime is denoted ϵ, and occasionally it will be written as ϵ_a or ϵ_{ab} when the displayed indices are being contracted; the undisplayed indices are left implicit. The volume form on $\mathcal{N}$ induced from l_a will be denoted η, and the horizontal spatial volume form on $\mathcal{N}$ will be denoted μ. The notation for the contraction of a vector v^a into a differential form m is $i_v m$. The notation for operations defined on $\mathcal{S}$, the space of solutions to the field equations, is described in section 4.2 below, including definitions of $I_{\hat{\xi}}$, $L_{\hat{\xi}}$, δ, and $\Delta_{\hat{\xi}}$.

4.2 Quasilocal charge algebra

We begin by reviewing the covariant phase space construction in section 4.2, before turning to the construction of quasilocal charges in section 4.2, and their algebra in section 4.2. Section 4.2 explains the relation between the Wald-Zoupas construction [6] and the recent work by Harlow and Wu on the covariant phase space with boundaries [174]. This yields an unambiguous definition of the quasilocal charges by the arguments of [174], once the form of the flux $\mathcal{E}$ has been specified. To fix this final ambiguity, we require that the flux be of Dirichlet form, and we discuss the motivation for this choice coming from the combined variational principle for the subregion and its complement. The algebra of charges is then defined in section 4.2, where we give a general expression for the extension of the algebra in terms of the anomaly of the boundary term appearing in the symplectic potential decomposition.

Covariant phase space

The main tool we employ in constructing the quasilocal charge algebra is the covariant phase space [187, 188, 189, 190, 191].[2] It provides a canonical description of field theories without singling out a preferred time foliation, and therefore is well-suited for handling diffeomorphism-invariant theories, such as general relativity. Covariance is achieved by working with the space $\mathcal{S}$ of solutions to the field equations, as opposed to the space of initial data on a time slice.

$\mathcal{S}$ can be viewed as an infinite-dimensional manifold, on which many standard differential-geometric techniques apply. Fields such as the metric g_{ab} can be viewed as functions on $\mathcal{S}$, and their variations, such as δg_{ab}, are one-forms. The operation δ of taking variations can be viewed as the exterior derivative on $\mathcal{S}$, and forms of higher degree can be built by taking exterior derivatives and wedge products in the usual way. The product of two differential forms α and β on $\mathcal{S}$ will always implicitly be a wedge product, so that $\alpha\beta = (-1)^{\deg(\alpha)\deg(\beta)}\beta\alpha$, which allows the symbol $\wedge$ to exclusively denote the wedge product between differential forms on the spacetime manifold $\mathcal{M}$. We denote by I_V the operation of contracting a vector field V on $\mathcal{S}$ with a differential form. Functions of the form $h_{ab} = I_V \delta g_{ab}$ are simply solutions to the linearized field equations, and so the vector fields on $\mathcal{S}$ are seen to coincide with the space of linearized solutions.

[2]We largely follow the notation of [163] when working with the covariant phase space.

Since diffeomorphisms of $\mathcal{M}$ are gauge symmetries of general relativity, they define an important subclass of linearized solutions $h_{ab} = \pounds_\xi g_{ab}$, where ξ^a is a spacetime vector field. The corresponding vector field on $\mathcal{S}$ generating this transformation will be called $\hat{\xi}$, which satisfies $I_{\hat{\xi}} \delta g_{ab} = \pounds_\xi g_{ab}$. Note also that $I_{\hat{\xi}} \delta g_{ab} = L_{\hat{\xi}} g_{ab}$, where $L_{\hat{\xi}}$ is the Lie derivative along the vector $\hat{\xi}$ in $\mathcal{S}$, and hence $L_{\hat{\xi}}$ and $\pounds_\xi$ agree when acting on the metric g_{ab}. The action of $L_{\hat{\xi}}$ on higher order differential forms on $\mathcal{S}$ can be computed via the Cartan formula $L_{\hat{\xi}} = I_{\hat{\xi}} \delta + \delta I_{\hat{\xi}}$. Any differential form α that is locally constructed from dynamical fields and for which $L_{\hat{\xi}} \alpha = \pounds_\xi \alpha$ will be called *covariant* with respect to $\hat{\xi}$. Since we later work with noncovariant objects as well, it is useful to define the anomaly operator

$$\Delta_{\hat{\xi}} = L_{\hat{\xi}} - \pounds_\xi, \tag{4.2.1}$$

as in [156], which measures the failure of a local object to be covariant. We therefore also refer to $\Delta_{\hat{\xi}} \alpha$ as the *noncovariance* or *anomaly* of α with respect to $\hat{\xi}$. As we will see, $\Delta_{\hat{\xi}}$ plays a prominent role in characterizing the extensions that appear in quasilocal charge algebras, and the anomalies it computes are, in many ways, classical analogs of the anomalies that appear in quantum field theories. In particular, as we show in appendix A.11, $\Delta_{\hat{\xi}}$ satisfies

$$[\Delta_{\hat{\xi}}, \Delta_{\hat{\zeta}}] = -\Delta_{\widehat{[\xi,\zeta]}}, \tag{4.2.2}$$

which, when imposed on the functionals of the theory, is the direct analog of the Wess-Zumino consistency condition for quantum anomalies [192].[3]

The covariant phase space arises from $\mathcal{S}$ by imbuing it with a presymplectic form. To construct it, one begins with the Lagrangian of the theory, L, which is a spacetime top form whose variation satisfies

$$\delta L = E^{ab} \delta g_{ab} + d\theta, \tag{4.2.3}$$

where $E^{ab} = 0$ are the classical field equations, and θ is a one-form on $\mathcal{S}$ and a $(d-1)$-form on spacetime called the symplectic potential current. For general relativity, the various quantities are

$$L = \frac{1}{16\pi G}(R - 2\Lambda)\epsilon \tag{4.2.4}$$

$$E^{ab} = \frac{-\epsilon}{16\pi G}\left(R^{ab} - \frac{1}{2} R g^{ab} + \Lambda g^{ab}\right) \tag{4.2.5}$$

$$\theta = \frac{1}{16\pi G}\epsilon_a \left(g^{bc}\delta\Gamma^a_{bc} - g^{ac}\delta\Gamma^b_{bc}\right), \tag{4.2.6}$$

where the variation of the Christoffel symbol is

$$\delta\Gamma^a_{bc} = \frac{1}{2} g^{ad}\left(\nabla_b \delta g_{dc} + \nabla_c \delta g_{bc} - \nabla_d \delta g_{bc}\right), \tag{4.2.7}$$

[3]See [193] for a discussion of the Wess-Zumino consistency condition in the context of holographic Weyl anomalies.

and we recall that ϵ_a still denotes the spacetime volume form, with uncontracted indices not displayed.

The $\mathcal{S}$-exterior derivative of θ defines the symplectic current $\omega = \delta\theta$, and its integral over a Cauchy surface Σ for the region of spacetime under consideration yields the presymplectic form,

$$\Omega = \int_\Sigma \omega. \tag{4.2.8}$$

Ω is called "presymplectic" because it contains degenerate directions corresponding to diffeomorphisms of $\mathcal{M}$. Since diffeomorphisms are symmetries of the Lagrangian, they lead to Noether currents that are conserved on shell, given by

$$J_\xi = I_{\hat{\xi}}\theta - i_\xi L. \tag{4.2.9}$$

Because $dJ_\xi = 0$ identically for all vectors ξ^a, the Noether current can be written as the exterior derivative of a potential, $J_\xi = dQ_\xi$, which is locally constructed from the metric; for general relativity, this potential is [194, 7],

$$Q_\xi = \frac{-1}{16\pi G}\epsilon^a{}_b \nabla_a \xi^b. \tag{4.2.10}$$

The degeneracy of Ω follows straightforwardly from computing the contraction with $I_{\hat{\xi}}$,

$$-I_{\hat{\xi}}\Omega = \int_{\partial\Sigma}\Big(\delta Q_\xi - i_\xi\theta\Big), \tag{4.2.11}$$

using the fact that θ is covariant, $I_{\hat{\xi}}\delta\theta + \delta I_{\hat{\xi}}\theta = \pounds_\xi\theta$ [178]. Since this contraction localizes to a boundary integral, any diffeomorphism that acts purely in the interior is a degenerate direction of Ω. The phase space $\mathcal{P}$ is a quotient of $\mathcal{S}$ by the degenerate directions, onto which Ω descends to a nondegenerate symplectic form [191].

Quasilocal charges

According to (4.2.11), diffeomorphisms with support near the Cauchy surface boundary $\partial\Sigma$ are not degenerate directions; rather, they lead to a notion of quasilocal charges associated with the subregion defined by Σ. In the case that ξ^a at $\partial\Sigma$ is vanishing or tangential, the term $i_\xi\theta$ in (4.2.11) drops out when pulled back to $\partial\Sigma$, and a Hamiltonian for the transformation can be defined by

$$H_\xi = \int_{\partial\Sigma} Q_\xi, \tag{4.2.12}$$

which generates the symmetry transformation on phase space via Hamilton's equations,

$$\delta H_\xi = -I_{\hat{\xi}}\Omega. \tag{4.2.13}$$

When ξ^a is not tangential to $\partial\Sigma$, $-I_{\hat{\xi}}\Omega$ generally cannot be written as a total variation, unless boundary conditions are imposed so that $\int_{\partial\Sigma} i_\xi\theta = \delta B_\xi$ for some quantity B_ξ.

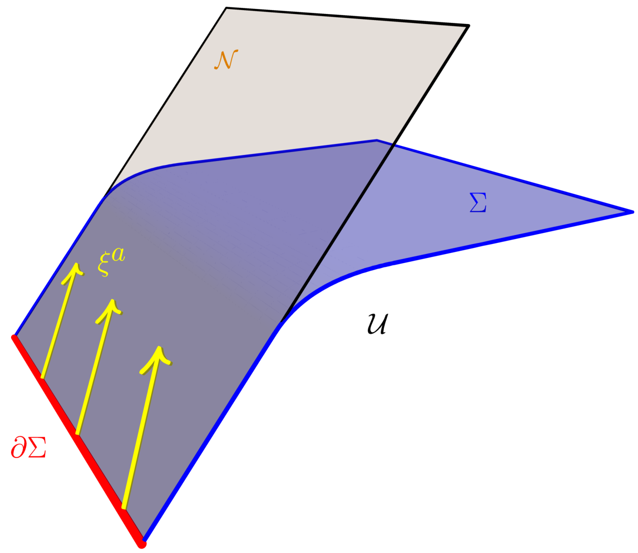


Figure 4.1: In the Wald-Zoupas construction, one seeks to construct quasilocal charges for a transformation generated by ξ^a, which is tangent to a hypersurface $\mathcal{N}$ bounding an open subregion $\mathcal{U}$ to the right of $\mathcal{N}$. The charges are constructed as integrals over a codimension-2 surface $\partial\Sigma$, bounding a Cauchy surface Σ for the subregion. The vector field ξ^a can have both tangential and normal components to $\partial\Sigma$. In this figure, $\mathcal{N}$ is a null hypersurface, and the Cauchy surface has been chosen to include a segment of $\mathcal{N}$.

Such boundary conditions are natural when $\partial\Sigma$ sits at an asymptotic boundary, but not at boundaries associated with subregions of a larger system, where the boundary conditions are generically inconsistent with the global dynamics. Instead, one can define a quasilocal charge associated with the transformation following the Wald-Zoupas prescription [6]. The quasilocal charge is not conserved since it fails to satisfy Hamilton's equation (4.2.13), but it satisfies a modified equation that relates the nonconservation to a well-defined flux through the boundary of the subregion.

Here, we give a presentation of the Wald-Zoupas construction, using the formalism developed by Harlow and Wu [174] for dealing with boundaries in the covariant phase space.[4] The Wald-Zoupas construction begins with a subregion of spacetime $\mathcal{U}$, bounded by a hypersurface $\mathcal{N} = \partial\mathcal{U}$ (see figure 4.1). Later $\mathcal{N}$ will be taken to be a null hypersurface, but the present discussion applies more generally for any signature of $\mathcal{N}$. On $\mathcal{N}$, one looks for a decomposition of the pullback $\boldsymbol{\theta}$ of the symplectic potential of the following form

$$\boldsymbol{\theta} = -\delta\ell + d\beta + \mathcal{E} \tag{4.2.14}$$

where ℓ is referred to as the *boundary term*, β is the *corner term*, and $\mathcal{E}$ is the *flux term*. The reason for this terminology becomes apparent from the variational principle for the theory

[4]See also [195] for a similar recent application of Harlow and Wu's formalism to the Wald-Zoupas construction.

defined in the subregion $\mathcal{U}$ [174, 196]. The action for the subregion is

$$S = \int_{\mathcal{U}} L + \int_{\mathcal{N}} \ell, \tag{4.2.15}$$

and by the decomposition (4.2.14) the variation satisfies

$$\delta S = \int_{\mathcal{U}} E^{ab}\delta g_{ab} + \int_{\mathcal{N}} (\mathcal{E} + d\beta)\,, \tag{4.2.16}$$

and so the action is stationary when the bulk field equations $E^{ab} = 0$ hold and boundary conditions are chosen to make $\mathcal{E}$ vanish, with the $d\beta$ term localizing to the boundary of $\mathcal{N}$, i.e. the corner. In the Wald-Zoupas setup, boundary conditions to make $\mathcal{E}$ vanish are not imposed; instead, $\mathcal{E}$ is used to construct the fluxes of the quasilocal charges. In [6], the combination $\mathcal{E}+d\beta$ is referred to as a potential for the pullback of ω to $\mathcal{N}$, since by equation (4.2.14) we see that[5]

$$\delta(\mathcal{E} + d\beta) = \delta\boldsymbol{\theta} = \boldsymbol{\omega}. \tag{4.2.17}$$

The corner term β is used to modify the symplectic form for the subregion.[6] This is done by extending $d\beta$ to an exact form on all of $\mathcal{U}$, and then treating $\theta - d\beta$ as the symplectic potential current. The symplectic form then becomes

$$\Omega = \int_{\Sigma} \omega - \int_{\partial\Sigma} \delta\beta. \tag{4.2.18}$$

We can then evaluate the contraction of Ω with a diffeomorphism generator ξ^a that is parallel to $\mathcal{N}$, but not necessarily to $\partial\Sigma$,

$$\begin{aligned} -I_{\hat{\xi}}\Omega &= \int_{\partial\Sigma} \Big(\delta Q_\xi - i_\xi\theta + I_{\hat{\xi}}\delta\beta\Big) \\ &= \int_{\partial\Sigma} \Big(\delta Q_\xi + i_\xi\delta\ell - \delta I_{\hat{\xi}}\beta\Big) - \int_{\partial\Sigma} \Big(i_\xi\mathcal{E} - \Delta_{\hat{\xi}}\beta\Big). \end{aligned} \tag{4.2.19}$$

The first term is the total variation of a quantity

$$H_\xi = \int_{\partial\Sigma} \Big(Q_\xi + i_\xi\ell - I_{\hat{\xi}}\beta\Big), \tag{4.2.20}$$

which we call the quasilocal charge for the transformation. The second term in (4.2.19) represents the failure of the quasilocal charge to be an integrable generator of the symmetry.

[5] In [6] the combination $\mathcal{E} + d\beta$ was denoted Θ.

[6] This type of modification, for example, gives the difference between the covariant Iyer-Wald symplectic form and the standard ADM symplectic form, see [197], and also recent discussions of this point in [174, 198].

Assuming that β is covariant, so that $\Delta_{\hat{\xi}}\beta = 0$, the obstruction to integrability of the charge is simply given by the integral of the flux density $i_\xi\mathcal{E}$. With slight modifications, the case where $\Delta_{\hat{\xi}}\beta \neq 0$ can be handled, and is described in appendix A.13. Equation (4.2.19) can be rearranged slightly to take the form of a modified Hamilton's equation,

$$\delta H_\xi = -I_{\hat{\xi}}\Omega + \int_{\partial\Sigma} i_\xi\mathcal{E} \tag{4.2.21}$$

To further the interpretation of $\mathcal{E}$ as a flux of H_ξ, we note first that the integrand of (4.2.20) is defined on all of $\mathcal{N}$, and its exterior derivative can be computed as

$$\begin{aligned} d\left(Q_\xi + i_\xi\ell - I_{\hat{\xi}}\beta\right) &= I_{\hat{\xi}}\boldsymbol{\theta} - i_\xi L - i_\xi d\ell + \pounds_\xi\ell - I_{\hat{\xi}}d\beta \\ &= I_{\hat{\xi}}\mathcal{E} - \Delta_{\hat{\xi}}\ell - i_\xi(L + d\ell) \end{aligned} \tag{4.2.22}$$

Integrating this relation on a segment $\mathcal{N}_1^2$ of $\mathcal{N}$ between two cuts S_2 and S_1, and using that ξ^a is parallel to $\mathcal{N}$ yields

$$H_\xi(S_2) - H_\xi(S_1) = \int_{\mathcal{N}_1^2}\left(I_{\hat{\xi}}\mathcal{E} - \Delta_{\hat{\xi}}\ell\right). \tag{4.2.23}$$

This can be interpreted as an anomalous continuity equation for the quasilocal charge H_ξ: the difference in the charge between two cuts is simply given by the flux $F_\xi = \int_{\mathcal{N}_1^2} I_{\hat{\xi}}\mathcal{E}$, up to an anomalous contribution from $\Delta_{\hat{\xi}}\ell$. This anomalous term in the flux vanishes if ℓ is covariant with respect to ξ^a; however, we will find that on null surfaces, the most natural choice for the flux term $\mathcal{E}$ requires a boundary term that is not covariant. Note that this equation differs from the standard continuity equation derived in the Wald-Zoupas and related constructions [119, 6, 195, 158], which assume a covariant boundary term, so that $\Delta_{\hat{\xi}}\ell$ drops out. This is the first indication that the noncovariance of the boundary term can be interpreted as an anomaly, since it behaves as an explicit violation of a contintuity equation for the quasilocal charges. In quantum field theory, anomalies play a similar role to that of $\Delta_{\hat{\xi}}\ell$, where they lead to explicit violations of the Ward identities.

Up to this point, we have placed no restrictions on the precise form of the flux $\mathcal{E}$. Equation (4.2.14) does not uniquely specify $\mathcal{E}$, since it can always be shifted by terms of the form $\mathcal{E} \to \mathcal{E} - \delta b - d\lambda$ by making compensating changes $\ell \to \ell - b$, $\beta \to \beta + \lambda$. These ambiguities in $\mathcal{E}$ are similar in appearance to the standard Jacobson-Kang-Myers ambiguities [177, 178] in the definition of the symplectic potential current, in which $\theta \to \theta + \delta b' + d\lambda'$. Although the (b, λ) and (b', λ') ambiguities are in principle distinct, they can be used in tandem to leave $\mathcal{E}$ invariant, by setting $(b, \lambda) = (b', \lambda')$. Additionally, the charge densities $h_\xi = Q_\xi + i_\xi\ell - I_{\hat{\xi}}\beta$ are also unchanged, provided one shifts the Noether potential by $Q_\xi + i_\xi b' + I_{\hat{\xi}}\lambda'$, as was recently emphasized by [174]. These transformations of Q_ξ simply follow from its definition as a potential for the Noether current J_ξ (4.2.9) as long as one assumes that b' is covariant (no assumption on the covariance properties of γ' is needed).

Thus, in order to avoid the ambiguities just described, we need to fix the form of the flux $\mathcal{E}$. As discussed in [196, 199], different choices for $\mathcal{E}$ are related to different boundary conditions one would impose to make the flux vanish. The principle we will advocate for in this work is that the flux take a Dirichlet form, which,[7] for $\mathcal{N}$ timelike or spacelike, means it is written as

$$\mathcal{E} = \pi^{ij}\delta g_{ij}, \tag{4.2.24}$$

where δg_{ij} is the metric variation pulled back to $\mathcal{N}$, constituting the intrinsic data on the surface, and π^{ij} is a symmetric-tensor-valued top form on $\mathcal{N}$ constructed from the extrinsic data, and interpreted as the conjugate momenta to δg_{ij}. The intrinsic data on a null surface is slightly different since the induced metric is degenerate, and so it is taken to also include variations of the null generator δl^i, leading to the null Dirichlet flux condition

$$\mathcal{E} = \pi^{ij}\delta g_{ij} + \pi_i \delta l^i. \tag{4.2.25}$$

Dependence on non-intrinsic components of the metric, such as the lapse and shift, is removed by the choice of corner term, which further fixes the ambiguities in specifying the flux. Imposing the Dirichlet form on $\mathcal{E}$ greatly reduces the freedom in its definition, since most of the ambiguities will involve variations of quantities constructed from the extrinsic geometry of $\mathcal{N}$. We will find that for general relativity, the Dirichlet requirement fixes $\mathcal{E}$ essentially uniquely.[8]

One reason for favoring the Dirichlet form of the flux comes from considering the variational principle for a subregion $\mathcal{U}$ and its complement $\overline{\mathcal{U}}$. When gluing the subregions across the boundaries $\mathcal{N}$ and $\overline{\mathcal{N}}$, the Dirichlet form of $\mathcal{E}$ is used when kinematically matching the intrinsic quantities on $\mathcal{N}$. Viewed from one side, this takes the form of a Dirichlet condition, with the value of g_{ij} on one side fixed by the value on the other side. Upon identifying $\mathcal{N}$ with $\overline{\mathcal{N}}$, matching g_{ij}, and imposing the bulk field equations, the variation of the action is given by

$$\delta\left(\int_{\mathcal{U}} L + \int_{\mathcal{N}} \ell + \int_{\overline{\mathcal{N}}} \overline{\ell} + \int_{\overline{\mathcal{U}}} L\right) = \int_{\mathcal{N}} (\pi^{ij} - \overline{\pi}^{ij})\delta g_{ij} + \text{corner term}. \tag{4.2.26}$$

Stationarity of the action then dynamically sets $\pi^{ij} - \overline{\pi}^{ij} = 0$, or more generally equal to the distributional stress energy on $\mathcal{N}$ if present, according to the junction conditions [204, 205]. If instead a Neumann form for the flux $\mathcal{E}^N = -g_{ij}\delta\pi^{ij}$ were employed, the matching condition would kinematically set $\pi^{ij} = \overline{\pi}^{ij}$, and then $g_{ij} - \overline{g}_{ij}$ would dynamically be set to zero. In this case, there does not appear to be a straightforward way to allow for distributional stress-energy on $\mathcal{N}$. In vacuum, the end result is classically the same, with

[7] This coincides with the "canonical boundary conditions" discussed in [199].

[8] For asymptotic symmetries, it can be important to include objects constructed from the intrinsic curvature of the metric, in order to have finite symplectic fluxes at infinity, which then modifies π^{ij} when imposing the Dirichlet form [179, 180, 181, 182, 200, 201, 202, 203]. Such terms will not be important for our analysis of a null boundary at a finite location.

both g_{ij} and π^{ij} matching at $\mathcal{N}$, although already the Dirichlet form has the advantage of allowing for the presence of distributional stress-energy. In a quantum description, these two options differ even more. Since the path integral receives contributions from off-shell configurations, the Dirichlet matching appears to be preferred, since the Neumann matching allows for discontinuities in the intrinsic metric, which produce distributionally ill-defined curvatures [205].[9] We further discuss the Dirichlet matching condition in section 4.6.

Barnich-Troessaert bracket

Having defined the quasilocal charges H_ξ given by (4.2.20) for the diffeomorphisms generated by ξ^a, we now consider the problem of computing their algebra. In standard Hamiltonian mechanics, this is given by the Poisson bracket constructed from the symplectic form of the system. When the charges are integrable, so that they satisfy Hamilton's equation (4.2.13), the Poisson bracket can be evaluated by contracting the vector fields generating the symmetry into the symplectic form,

$$\{H_\xi, H_\zeta\} = -I_{\hat{\xi}} I_{\hat{\zeta}} \Omega = -\left(H_{[\xi,\zeta]} + K_{\xi,\zeta}\right). \tag{4.2.27}$$

The second equality in this equation is a statement of the fact that Poisson brackets must reproduce the Lie bracket of the vector fields ξ^a, ζ^a, up to a central extension, denoted $K_{\xi,\zeta}$.[10]

For quasilocal charges, their failure to satisfy Hamilton's equations due to the flux term in (4.2.21) prevents a naive application of (4.2.27) to their brackets. Instead, Barnich and Troessaert [98] proposed a modification to the bracket that accounts for the nonconservation of the charges due to the loss of flux from the subregion. When the corner term β is covariant, their bracket is given by

$$\{H_\xi, H_\zeta\} = -I_{\hat{\xi}} I_{\hat{\zeta}} \Omega + \int_{\partial\Sigma} \Big(i_\zeta I_{\hat{\xi}} \mathcal{E} - i_\xi I_{\hat{\zeta}} \mathcal{E}\Big), \tag{4.2.28}$$

where we see that the bracket is modified by the fluxes $F_\xi = \int_{\partial\Sigma} I_{\hat{\xi}} \mathcal{E}$ identified in the Wald-Zoupas construction. A heuristic way to understand this equation is as follows: imagine adding an auxiliary system which collects the flux lost through $\mathcal{N}$ when evolving along ξ^a (for example, this could just be the phase space associated with the complementary region $\overline{\mathcal{U}}$). The total system consisting of the subregion and the auxiliary system is assumed to have a Poisson bracket defined on it, such that $\hat{\xi}$ is a symmetry of the bracket in the usual sense. The Hamiltonian for $\hat{\xi}$ should be a sum of the quasilocal Hamiltonian H_ξ and a term

[9]These singularities are unlike conical defects, whose curvature is well-defined as a distribution and are therefore valid configurations in the path integral.

[10]There are two related reasons for the minus sign appearing in (4.2.27). The first is that the Poisson bracket reproduces the Lie bracket $[\hat{\xi}, \hat{\zeta}]_{\mathcal{S}}$ of vector fields on $\mathcal{S}$, which, as shown in (A.11.3), is minus the spacetime Lie bracket for field-independent vector fields. It arises because diffeomorphisms give a left action on spacetime, but a right action on $\mathcal{S}$. The second reason is that the Hamiltonians are representing the Lie algebra of the diffeomorphism group, whose Lie bracket is minus the vector field Lie bracket [206].

H_ξ^{aux} associated with the auxiliary system. Hamilton's equation for the total system then reads

$$I_{\hat{\xi}}\delta H_\zeta = \{H_\xi + H_\xi^{\text{aux}}, H_\zeta\}. \tag{4.2.29}$$

The contribution from $\{H_\xi^{\text{aux}}, H_\zeta\}$ should compute the flux of H_ζ into the auxiliary system due to an infinitesimal change of $\partial\Sigma$ along ξ^a, which is just the integral of $i_\xi I_{\hat{\zeta}}\mathcal{E}$, given our identification of $I_{\hat{\zeta}}\mathcal{E}$ with the flux density. Equation (4.2.29) then becomes

$$I_{\hat{\xi}}\delta H_\zeta = \{H_\xi, H_\zeta\} + \int_{\partial\Sigma} i_\xi I_{\hat{\zeta}}\mathcal{E}, \tag{4.2.30}$$

which reduces to (4.2.28) after using the expression (4.2.21) for δH_ζ. Going forward, we will take (4.2.28) as the definition of the bracket for the quasilocal charges, and delay further discussion of its interpretation to section 4.6.

An important property of the Barnich-Troessaert bracket is that it reproduces the Lie bracket algebra of the vector fields, up to abelian extensions [98, 207]. This can be explicitly verified using the expression (4.2.20) for the quasilocal charges, and an exact expression for the extension $K_{\xi,\zeta}$ can be given. After a short calculation (see appendix A.12), one finds

$$\{H_\xi, H_\zeta\} = -\left(H_{[\xi,\zeta]} + K_{\xi,\zeta}\right) \tag{4.2.31}$$

$$K_{\xi,\zeta} = \int_{\partial\Sigma}\Big(i_\xi\Delta_{\hat{\zeta}}\ell - i_\zeta\Delta_{\hat{\xi}}\ell\Big). \tag{4.2.32}$$

Hence, we arrive at one of the main results of this work, namely, that the extension $K_{\xi,\zeta}$ is determined entirely by the noncovariance of the boundary term, $\Delta_{\hat{\xi}}\ell$. As an immediate corollary, we see that the extension $K_{\xi,\zeta}$ always vanishes if the boundary term ℓ is covariant with respect to the generators ξ^a. Equation (4.2.32) remains valid even when boundary conditions are imposed to ensure the transformation has integrable generators. In this case, the fluxes in (4.2.28) vanish, and we see that the Barnich-Troessaert bracket reduces to a Dirac bracket on the subspace of field configurations that satisfy the boundary conditions. This therefore gives a universal formula for the central extension in these cases, in addition to the more general cases involving nonintegrable generators.

It is worth emphasizing that the central charge appears in this formula because we have chosen to fix a background structure in defining the boundary, which gives rise to nonzero anomalies $\Delta_{\hat{\xi}}\ell$. However, the value of $K_{\xi,\zeta}$ does not depend on the choice of constant added to the Hamiltonians, which, for example, could be chosen to ensure that the Hamiltonians vanish in a given background solution. More precisely, different choices for these constant shifts can only change the extension by trivial constant terms of the form $C_{[\xi,\zeta]}$, which will not change the 2-cocycle that $K_{\xi,\zeta}$ represents for the Lie algebra of the vector fields ξ^a, ζ^a. In particular, $C_{[\xi,\zeta]}$ cannot be chosen to cancel $K_{\xi,\zeta}$ if the extension comes from a nontrivial 2-cocycle, as occurs in the Virasoro example we consider in section 4.4.

In general, the new generators $K_{\xi,\zeta}$ are not central, since they are allowed to transform nontrivially under the action of another generator H_χ. Instead, they give an abelian extension of the algebra by defining their brackets to be

$$\{H_\chi, K_{\xi,\zeta}\} = I_{\hat{\chi}}\delta K_{\xi,\zeta} \tag{4.2.33}$$

$$\{K_{\xi,\zeta}, K_{\chi,\psi}\} = 0. \tag{4.2.34}$$

This algebra closes provided $I_{\hat{\chi}}\delta K_{\xi,\zeta}$ is expressible as a sum of other generators $K_{\xi',\zeta'}$, and the Jacobi identity holds as long as $K_{\xi,\zeta}$ satisfies a generalized cocycle condition [98],

$$I_{\hat{\chi}}\delta K_{\xi,\zeta} + K_{[\chi,\xi],\zeta} + (\text{cyclic } \chi \to \xi \to \zeta) = 0. \tag{4.2.35}$$

Of course, when the right hand side of (4.2.33) vanishes, $K_{\xi,\zeta}$ represents a central extension of the algebra.

We verify the above cocycle condition for (4.2.32) in appendix A.12. We should expect this to be the case because $K_{\xi,\zeta}$ in (4.2.32) is of the form of a trivial field-dependent 2-cocycle, in the terminology of [98].[11] That is, it can be expressed as

$$K_{\xi,\zeta} = I_{\hat{\zeta}}\delta B_\xi - I_{\hat{\xi}}\delta B_\zeta - B_{[\xi,\zeta]}, \qquad B_\xi \equiv \int_{\partial\Sigma} i_\xi \ell \tag{4.2.36}$$

Despite this terminology, $K_{\xi,\zeta}$ is certainly not required to be trivial as a cocycle for the Lie algebra generated by the vector fields. This will be explicitly demonstrated for the algebra considered in section 4.4, in which case $K_{\xi,\zeta}$ becomes the nontrivial central extension of the Witt algebra to Virasoro.

Finally, it is worth noting that the corner term β, although important in arriving at the Dirichlet form (4.2.24) or (4.2.25) for the flux, is not important for obtaining the correct algebra for the quasilocal charges, including the extension $K_{\xi,\zeta}$. Algebraically, the β term in the quasilocal charge is functioning as a trivial extension of the algebra, since the β terms do not mix with other terms when deriving the identity (4.2.31), as discussed in appendix A.12. This is the reason that the central charges computed in [149, 171] were correctly identified, even without taking corner terms into account.

4.3 Symplectic potential on a null boundary

In this section, we apply the covariant phase space formalism to null boundaries. We decompose the symplectic potential into boundary, corner, and flux terms, and describe the resulting canonical pairs on the null surface. This generalizes the calculation in [119] (see also [156, 184]) by weakening the boundary conditions imposed on the field configurations. The expression for the anomalous transformation of the boundary term under diffeomorphisms is derived, and shown to arise from fixing a choice of scaling frame on the null boundary.

[11] For an interpretation of this field-dependent extension in terms of a Lie algebroid in the example of BMS_4 asymptotic symmetries, see [208].

Geometry of null hypersurfaces

We start by briefly reviewing the geometric fields on a null hypersurface and their salient properties, following [119]. For a detailed review see [75]. Consider a spacetime $(\mathcal{M}, g_{ab})$ and a null hypersurface $\mathcal{N}$ in $\mathcal{M}$. To begin with, we have the null normal l_a to $\mathcal{N}$. An important property of null surfaces is that l_a has no preferred normalization, unlike for spacelike or timelike surfaces. Consequently, we can rescale it according to

$$l_a \to e^f l_a. \tag{4.3.1}$$

We refer to a choice of f as a *scaling frame*. From l_a we can construct the null generator tangent to $\mathcal{N}$ by raising the index, $l^a = g^{ab} l_b$. Associated to the null generator is the inaffinity k,[12] defined by

$$l^a \nabla_a l^b \mathrel{\widehat{=}} k l^b, \tag{4.3.3}$$

where we have introduced the notation $\mathrel{\widehat{=}}$ to denote equality at $\mathcal{N}$. The inaffinity will play a central role in this paper.

We denote by Π^a_i the pullback to $\mathcal{N}$. Recall that indices $i, j, \ldots$ are intrinsic to $\mathcal{N}$. Using the pullback, we can now enumerate the various objects needed for our analysis. The (degenerate) induced metric q_{ij} on $\mathcal{N}$ is simply the pullback of g_{ab},

$$q_{ij} = \Pi^a_i \Pi^b_j g_{ab}. \tag{4.3.4}$$

Next, note that $l_b \Pi^a_i \nabla_a l^b \mathrel{\widehat{=}} 0$ hence the tensor

$$\Pi^a_i \nabla_a l^b \tag{4.3.5}$$

is actually intrinsic to $\mathcal{N}$. Therefore, we denote it by

$$S^i{}_j, \tag{4.3.6}$$

and refer to it as the shape tensor, or Weingarten map [75]. We can extract the inaffinity from the shape tensor through $S^i{}_j l^j = k l^i$. From $S^i{}_j$, we can obtain the extrinsic curvature of $\mathcal{N}$,

$$K_{ij} = q_{jk} S^k{}_i, \tag{4.3.7}$$

[12] The inaffinity is often denoted κ, but we use k to distinguish it from the surface gravity κ, which is defined on $\mathcal{N}$ by the relation

$$\nabla_a (l^2) \mathrel{\widehat{=}} -2\kappa l_a. \tag{4.3.2}$$

For Killing horizons, $k = \kappa$, but for general null surfaces, these two quantities differ; see, e.g., [209] for a discussion of the difference in the case of conformal Killing horizons. The definition (4.3.2) of the surface gravity is most directly related to its appearance in the Hawking temperature $T_H = \frac{\kappa}{2\pi}$ [2, 210], which is why we continue to use κ to denote it, and instead use k for the inaffinity.

which can be decomposed into its familiar form

$$K_{ij} = \sigma_{ij} + \frac{1}{d-2}\Theta q_{ij}, \tag{4.3.8}$$

where σ_{ij} is the shear and Θ is the expansion.

Lastly, we can define induced $(d-1)$ and $(d-2)$ volume forms on $\mathcal{N}$ as follows. Given a spacetime volume form ϵ, we can define a $(d-1)$ volume form $\tilde{\eta}$ by

$$\epsilon \mathrel{\widehat{=}} -l \wedge \tilde{\eta}. \tag{4.3.9}$$

Note that $\tilde{\eta}$ is fully determined by a choice of l_a up to the addition of terms of the form $l \wedge \sigma$ for some $(d-2)$ form σ. However, given a choice of l_a, the pullback of $\tilde{\eta}$ to $\mathcal{N}$ is unique. We simply denote this pullback by η, as we will only be using the pullback henceforth. Given the pullback η, we can define a $(d-2)$ volume form μ by

$$\mu = i_l \eta \tag{4.3.10}$$

which is uniquely determined by η.

We now list the transformation properties of the geometric fields defined above under the rescaling (4.3.1):

$$q_{ij} \to q_{ij}, \tag{4.3.11a}$$

$$\mu \to \mu, \tag{4.3.11b}$$

$$\eta \to e^f \eta, \tag{4.3.11c}$$

$$K_{ij} \to e^f K_{ij}, \tag{4.3.11d}$$

$$S^i{}_j \to e^f (S^i{}_j + \partial_j f\, l^i). \tag{4.3.11e}$$

We emphasize that this corresponds to a rescaling in a given background geometry. In the next section we will discuss the scale factor f on field space.

We end this section by introducing an auxiliary null vector n^a on $\mathcal{N}$, as it will prove convenient in later computations. We fix the freedom in the relative normalization of n^a by imposing $l_a n^a = -1$. We can use n^a to write the pullback and induced metric as spacetime tensors,

$$\Pi_a^b = \delta_a^b + l_a n^b, \tag{4.3.12a}$$

$$q_{ab} = g_{ab} + 2 l_{(a} n_{b)}. \tag{4.3.12b}$$

Raising the indices yields a tensor q^{ab} that is tangent to $\mathcal{N}$ since $q^{ab} l_b = 0$. It therefore defines a tensor q^{ij} intrinsic to $\mathcal{N}$, which defines a partial inverse of q_{jk} on the subspace of vectors that annihilate $n_i = \Pi_i^a n_a$. The mixed index tensor $q^i{}_j = q^{ik} q_{kj}$ is then a projector onto this subspace.

We can also use n^a to define the Hájíčekone-form,

$$\varpi_a = -q_a^c n^b \nabla_c l_b. \tag{4.3.13}$$

This pulls back to a one-form ϖ_i on $\mathcal{N}$, and under rescaling (4.3.1), it transforms by

$$\varpi_i \to \varpi_i + q^j{}_i \partial_j f \tag{4.3.14}$$

Using q^{ij} to raise the index of K_{ij}, we can give a complete decomposition of the shape tensor,

$$S^i{}_j = l^i(\varpi_j - k n_j) + K^i{}_j. \tag{4.3.15}$$

This equation emphasizes the difference between the shape tensor $S^i{}_j$ and the extrinsic curvature K_{ij} on a null hypersurface, unlike the case of a spacelike or timelike hypersurface where the two quantities have essentially the same content. An important point to keep in mind is that the quantities on $\mathcal{N}$ that depend on n_a are q^{ij}, $q^i{}_j$, n_i, $K^i{}_j$, and ϖ_i, while the quantities appearing in (4.3.11) are independent of n_a.

Boundary conditions

We now describe the field configuration space for gravitational theories with a null boundary $\mathcal{N}$ in terms of the boundary conditions imposed at $\mathcal{N}$. An important part of this specification is the choice of a background structure derived from structures defined by the boundary. A background structure is a set of fields which are constant across the field space. Fixing these fields is the source of noncovariance in the gravitational charge algebra, and ultimately is responsible for the appearance of central charges.

To this aim, we start by letting $\mathcal{N}$ be a hypersurface embedded in $\mathcal{M}$, specified by a normal covector field l_a. We do not yet impose that $\mathcal{N}$ is a null surface. Consequently, since this specification is independent of the metric, it follows that[13]

$$\delta l_a \mathrel{\widehat{=}} 0. \tag{4.3.16}$$

We take the background structure to solely consist of l_a, since all other quantities relevant for the symplectic form decomposition are constructed from l_a using the metric.[14] Now, in order to impose that $\mathcal{N}$ is a null surface for all points in the field space, we must constrain the metric perturbation δg_{ab}. This amounts to the boundary condition

$$l^a l^b \delta g_{ab} \mathrel{\widehat{=}} 0. \tag{4.3.17}$$

We do not impose any further boundary conditions, so our field configuration space is simply the set of all metrics g_{ab} on a manifold $\mathcal{M}$ with boundary $\mathcal{N} \subset \partial\mathcal{M}$ such that (4.3.16) and (4.3.17) are satisfied. This background structure is natural, if not necessary, from the point of view of the gravitational path integral: when we integrate over bulk metrics, we want a

[13] In principle we can allow l_a to rescale under variations according to $\delta l_a \mathrel{\widehat{=}} \delta a \; l_a$, but this would unnecessarily introduce an arbitrary non-metric degree of freedom that has no relation to the dynamical degrees of freedom of the theory.

[14] In particular, we do not impose any constraints on the auxiliary null vector n^a, apart from the trivial constraint resulting from fixing the relative normalization $n^a l_a \mathrel{\widehat{=}} -1$.

null surface as a boundary condition, which must be imposed as a delta function constraint on the dynamical metric, leaving the normal to the surface a non-dynamical variable.

This is a larger field space than that of [119], where the boundary conditions $\delta k = 0$ and $l^b \delta g_{ab} \mathrel{\hat{=}} 0$ were additionally imposed. Although both sets of boundary conditions lead to the same solution space globally, they differ from the point of view of the subregion $\mathcal{U}$, where they represent different choices of boundary degrees of freedom. Any additional boundary conditions, beyond the condition (4.3.17) to ensure $\mathcal{N}$ is null, eliminate physical degrees of freedom from the subregion, since these boundary conditions do not correspond to fixing a degenerate direction of the subregion symplectic form. Imposing the stronger boundary conditions is equivalent to gauge fixing the global field space using Gaussian null coordinates in the neighborhood of $\mathcal{N}$, as was done in various works [28, 211]. As we will see in section 4.4, the diffeomorphisms of interest to us satisfy neither $\delta k = 0$ nor $l^b \delta g_{ab} \mathrel{\hat{=}} 0$, so we cannot impose these conditions. In [119], these additional boundary conditions comprised the minimal set necessary for satisfying the Wald-Zoupas stationarity condition $\mathcal{E}(g_0, \delta g) = 0$ for all δg, where g_0 is a solution in which $\mathcal{N}$ is stationary. This stationarity condition has been argued to be a way of fixing the standard ambiguity in defining quasiloal charges [6, 119]; however, we do not see it as being necessary for the construction to make sense. In its place, we have instead the Dirichlet flux condition (4.2.24). Thus, we have imposed the minimal set of boundary conditions needed to specify gravitational kinematics on a manifold with a null boundary.

We now derive expressions for the variations of k and Θ, which will be needed in the next section when decomposing the symplectic potential. To begin with, we note that[15]

$$\delta l^a \mathrel{\hat{=}} (l^b n^c \delta g_{bc}) l^a - q^{ab} \delta g_{bc} l^c. \tag{4.3.18}$$

Using the definition $\Theta = q^{ab} \nabla_a l_b$ of the expansion, and the decomposition (4.3.12b), we find

$$\delta\Theta = -\left(\sigma^{ab} + \frac{\Theta}{d-2} q^{ab}\right) \delta g_{ab} - 2 l_c \delta \Gamma^c_{ab} l^a n^b - l_c \delta \Gamma^c_{ab} g^{ab}. \tag{4.3.19}$$

Separately, using $k = -n^b l^a \nabla_a l_b$, we have

$$\delta k = (k n^b - \varpi^b) l^a \delta g_{ab} + l_c \delta \Gamma^c_{ab} l^a n^b. \tag{4.3.20}$$

In arriving at these expressions we have used that $l_a \delta n^a \mathrel{\hat{=}} -n^a \delta l_a \mathrel{\hat{=}} 0$, which is simply a result of fixing the relative normalization $n^a l_a \mathrel{\hat{=}} -1$ across phase space, combined with $\delta l_a \mathrel{\hat{=}} 0$. In this sense, the expressions for $\delta\Theta$ and δk are independent of δn^a. Thus, combining these two expression, we find

$$\delta(\Theta + 2k) = 2(k n^b - \varpi^b) l^a \delta g_{ab} - \left(\sigma^{ab} + \frac{\Theta}{d-2} q^{ab}\right) \delta g_{ab} - l_c \delta \Gamma^c_{ab} g^{ab}. \tag{4.3.21}$$

[15]In [156] the l^a component of δl^a was made to vanish by relaxing the condition $\delta l_a = 0$, instead setting it to $\delta l_a = -n^b l^c \delta g_{bc} l_a$. Doing this requires a different fixed background structure, which amounts to fixing n_c on the horizon. Since they impose no additional constraints on the metric variation, the field space in [156] is the same as ours, but their analysis differs in the choice of background structure.

Lastly, the variation of η is given by

$$\delta\eta = \frac{1}{2} g^{ab}\delta g_{ab}\,\eta \tag{4.3.22}$$

Symplectic potential

So far we have only discussed the kinematics, which is valid for any theory of gravity. We now take our theory of gravity to be general relativity. By restricting the field space to on-shell configurations, i.e. metrics which solve Einstein's equations, we can obtain the associated covariant phase space $\mathcal{P}$ as outlined in section 4.2. The symplectic potential current in general relativity pulled back to $\mathcal{N}$ can be written (momentarily setting $16\pi G = 1$)

$$\boldsymbol{\theta} = \eta\left(\frac{1}{2} l^c \nabla_c\left(g^{bc}\delta g_{bc}\right) - l_a g^{bc}\delta\Gamma^a_{bc}\right), \tag{4.3.23}$$

where the bolded tensor $\boldsymbol{\theta}$ indicates that it has been pulled back to $\mathcal{N}$. We wish to decompose the above expression into boundary, corner, and flux terms, according to the general construction described in section 4.2.

We start by noting that $d\mu = \Theta\eta$. Using this relation, we have

$$d\left(\frac{1}{2} g^{ab}\delta g_{ab}\,\mu\right) \mathrel{\hat{=}} \frac{1}{2} l^c\nabla_c(g^{ab}\delta g_{ab})\,\eta + \frac{1}{2}\Theta g^{ab}\delta g_{ab}\,\eta. \tag{4.3.24}$$

The second and first terms in (4.3.23) appear explicitly in (4.3.21) and (4.3.24) respectively, so we can simply solve for them using these relations. Combining this with (4.3.22), we can write the symplectic potential as

$$\begin{aligned}\boldsymbol{\theta} &= \delta\Big[(\Theta + 2k)\eta\Big] + d\left[\frac{1}{2} g^{ab}\delta g_{ab}\mu\right] \\ &\quad + \eta\left[\sigma^{ab}\delta g_{ab} + 2\varpi^a l^b\delta g_{ab} - \left(k - \frac{\Theta}{d-2}\right) q^{ab}\delta g_{ab} - \Theta g^{bc}\delta g_{bc}\right].\end{aligned} \tag{4.3.25}$$

We can shift the Θ contribution in the boundary term into the corner term by noting that $\delta(\Theta\eta) = d\delta\mu$. Note that this shift is an example of an additional ambiguity in the decomposition (4.2.14) of $\boldsymbol{\theta}$ in separating the corner and boundary terms. In the present context, this shift will not affect any central charges since $\Theta\eta$ is covariant, but in principle this ambiguity can be resolved using the corner improvements discussed in appendix A.13.

Finally, by making use of (4.3.18) we arrive at our desired decomposition of the symplectic potential:

$$\boldsymbol{\theta} = -\delta\ell + d\beta + \pi^{ij}\delta q_{ij} + \pi_i\delta l^i, \tag{4.3.26}$$

where, restoring the factors of $16\pi G$, the various terms in the decomposition are

$$\ell = -\frac{k\eta}{8\pi G}, \tag{4.3.27}$$

$$\beta = \frac{1}{16\pi G}(\eta_a \delta l^a + g^{ab}\delta g_{ab}\mu), \tag{4.3.28}$$

$$\pi^{ij} = \frac{\eta}{16\pi G}\left[\sigma^{ij} - \left(k + \frac{d-3}{d-2}\Theta\right) q^{ij}\right], \tag{4.3.29}$$

$$\pi_i = -\frac{\eta}{8\pi G}(\varpi_i + \Theta n_i). \tag{4.3.30}$$

This decomposition of the symplectic potential on a null boundary is essentially equivalent to the one found in [183], while it differs slightly from the expressions in [155, 156, 119] due to differences in choices of boundary conditions.

The flux terms in (4.3.26) are in Dirichlet form, as required by our general prescription. The quantity π^{ij} defines the conjugate momenta to δq_{ij}, the horizontal components of the variation of the induced degenerate metric on $\mathcal{N}$. The $\frac{d(d-3)}{2}$ components of the shear make up the momenta associated with gravitons, while the scalar $k + \frac{d-3}{d-2}\Theta$ is a scalar momentum identified in [156] as a gravitational pressure. The other momenta π_i are conjugate to δl^i. It can further be decomposed into a vector piece constructed from the Hájíček form ϖ_i conjugate to spatial variations of l^i, and a scalar energy density constructed from Θ, conjugate to variations that stretch l^i. Together, π^{ij} and π_i comprise the null analog of the Brown-York stress tensor, which is usually defined for timelike hypersurfaces [175].[16]

We now discuss the dependence of the terms in the decomposition on arbitrary choices of background quantities. In writing (4.3.26) we introduced a choice of auxiliary null normal n^a. Fixing the relative normalization of n^a still leaves the freedom $n^a \to n^a + V^a + \frac{1}{2}V^2 l^a$, where V^a is any vector such that $n_a V^a = l_a V^a = 0$. However, both the boundary term (4.3.27) and corner term (4.3.28) are manifestly independent of n^a hence it follows that the flux term is independent of n^a, since $\boldsymbol{\theta}$ must be. While the total flux term is independent of n^a, π^{ij} and π_i will in general transform into one another under a change of n^a.

While we have fixed the fluctuation of the scale factor f when defining our phase space, we still would like to characterize how various quantities depend on its background value. From (4.3.11), we have the following transformation properties of the various terms in the decomposition (4.3.26) under a background rescaling:

$$\ell \to \ell - \frac{\eta}{8\pi G} l^i \partial_i f, \tag{4.3.31a}$$

$$\pi^{ij} \to \pi^{ij} - \frac{\eta q^{ij}}{16\pi G} l^k \partial_k f, \tag{4.3.31b}$$

[16] A slightly different construction in [212, 213] found a null Brown-York stress tensor without the scalar component of π_i, but with an additional component conjugate to deformations that violate the nullness condition $l^a l^b \delta g_{ab} = 0$. Another approach by [214] obtained a null boundary stress tensor as a limit of the Brown-York stress tensor on the stretched horizon. Their expression differs somewhat from the one presented here.

$$\pi_i \to e^{-f}\left(\pi_i - \frac{\eta}{8\pi G}\partial_i f\right). \tag{4.3.31c}$$

Anomalous transformation of boundary term

Having fixed the boundary term, we now derive its noncovariance under diffeomorphisms. We will find that it transforms anomalously, with the anomaly arising directly from fixing a choice of scaling frame (4.3.16). To see this, we first compute $\pounds_\xi l_a$ when ξ^a is tangent to $\mathcal{N}$, i.e. $\xi^b l_b \mathrel{\widehat{=}} 0$. We have

$$\pounds_\xi l_a \mathrel{\widehat{=}} 2\xi^b \nabla_{[b} l_{a]} + \nabla_a(\xi^b l_b). \tag{4.3.32}$$

Hypersurface orthogonality implies that $\nabla_{[b} l_{a]} \mathrel{\widehat{=}} v_{[b} l_{a]}$ for some v_a. Moreover, $\nabla_a(\xi^b l_b) \propto l_a$ on $\mathcal{N}$. Therefore,

$$\pounds_\xi l_a \mathrel{\widehat{=}} w_\xi l_a. \tag{4.3.33}$$

Recall that the anomaly operator is defined as $\Delta_{\hat{\xi}} = L_{\hat{\xi}} - \pounds_\xi$. Therefore, since $\delta l_a = 0$, we find $\Delta_{\hat{\xi}} l_a \mathrel{\widehat{=}} -w_\xi l_a$.

We also need the noncovariance of the induced volume element. Since ϵ depends only on the metric, $\Delta_{\hat{\xi}}\epsilon = 0$. Therefore, using (4.3.22), we just have

$$\Delta_{\hat{\xi}}\eta = w_\xi \eta. \tag{4.3.34}$$

Moreover, applying the anomaly operator to $l^b \nabla_b l_a = k l_a$, we find

$$\Delta_{\hat{\xi}} k = -w_\xi k - l^a \nabla_a w_\xi \tag{4.3.35}$$

Putting things together, we have the anomalous transformation of the boundary term:

$$\Delta_{\hat{\xi}}(k\eta) = -(l^c \nabla_c w_\xi)\eta \tag{4.3.36}$$

This is one of the main results of this paper. From (4.2.32), we see that the non-vanishing of the central charge is a consequence of choosing l_a to be the background structure. We discuss the significance of this in section 4.6. In section 4.4, we evaluate this anomaly explicitly for the Virasoro generators on a Killing horizon.

The expression (4.2.28) for the Barnich-Troessaert bracket that we employ in the next section applies when β is covariant, without needing the corner improvements discussed in appendix A.13. It is easy to see that our choice of corner term (4.3.28) does in fact satisfy this. First note that $\Delta_{\hat{\xi}}\mu = 0$, which handles the second term in (4.3.28). For the first term, we have $\Delta_{\hat{\xi}}(\eta_a \delta l^a) = (\Delta_{\hat{\xi}}\eta_a)\delta l^a + \eta_a \delta \Delta_{\hat{\xi}} l^a = w_\xi \eta_a \delta l^a - \eta_a \delta(w_\xi l^a) = 0$, since $\delta w_\xi = 0$. It follows that the corner term is covariant, $\Delta_{\hat{\xi}}\beta = 0$, as desired.

As a final note, the fact that the central charge can be expressed as a trivial field-dependent cocycle [98] according to (4.2.36) means that there always exists a choice of the flux and boundary terms that makes any extensions in the quasilocal charge algebra

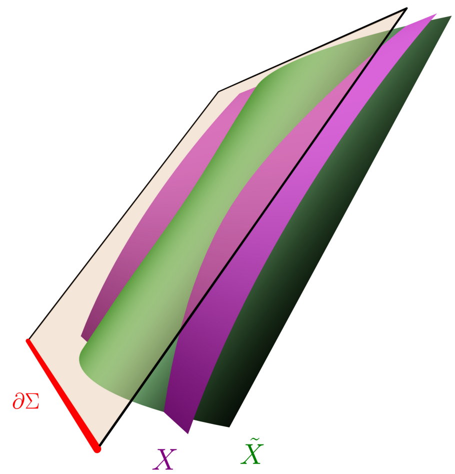


Figure 4.2: Two different choices of stretched horizons are shown, as the level sets of the functions X and $\tilde{X}$, which lead to different scaling frames for l_a on the null surface.

vanish. Moreover, this choice of flux term would be covariant and rescaling invariant, and was the choice used in [156, 158]. However, consider what would happen if a similar choice were made for asymptotic symmetries: for example, for AdS_3 asymptotics, one can choose a boundary term other than the Gibbons-Hawking-York term, in which case the Brown-Henneaux analysis would produce a central charge with $c \neq \frac{3R}{2G}$, with R the AdS radius [165]. The flux term in these cases no longer corresponds to Dirichlet boundary conditions. In holographic setups, these modified boundary conditions lead to CFTs coupled to dynamical metrics [215], producing complications that are usually avoided in standard AdS/CFT with Dirichlet boundary conditions. We therefore draw inspiration from AdS/CFT in imposing that the flux term take Dirichlet form, complementary to the path integral argument in section 4.2.

Stretched horizon

We mentioned in section 4.3 that fixing l_a corresponds to a type of frame choice. Here, we will relate this choice to the arbitrariness in choosing a sequence of stretched horizons that approach the null surface. A stretched horizon for a null surface plays a similar role to an asymptotic cutoff surface when discussing asymptotic infinity. These are especially relevant in AdS/CFT, where different choices of the radial cutoff correspond to different conformal frames in the dual theory. This then strengthens the relation between the scaling frame for l_a and the choice of conformal frame for the degrees of freedom associated with the quasilocal charges.

To see the relation, we let X denote a function whose level sets define the sequence of stretched horizons approaching $\mathcal{N}$ at $X = 0$. We let l_a be the (unnormalized) normal form to the X foliation,

$$l_a = \nabla_a X, \tag{4.3.37}$$

which is spacelike for $X > 0$ and null at $X = 0$. Any reparameterization of the form $X \to \tilde{X}(X)$ defines the same foliation, and its effect on the normal is simply to rescale l_a by $\tilde{F}'(X)$. Hence l_a at $\mathcal{N}$ only rescales by a constant $\tilde{X}'(0)$. We therefore see that the scaling frame of l_a is determined by the choice of stretched horizon foliation, up to overall constant rescalings.

A different foliation of stretched horizons can be obtained by reparameterizing by an arbitrary function of the coordinates $X \to \tilde{X}(X, x^i)$, subject to the constraint $\tilde{X}(0, x^i) = 0$, so that the foliation still approaches $\mathcal{N}$ (see figure 4.2). The null normal is now rescaled by the position dependent function $\partial_X \tilde{X}(0, x^i)$, corresponding to a change of scaling frame.

4.4 Virasoro symmetry

As an application of the null boundary covariant phase space we have just constructed, we now specialize to the case of bifurcate, axisymmetric Killing horizons. These have been the subject of many previous analyses, in which quasilocal charge algebras have been used to derive expressions for the entropy of the Killing horizon [143, 149, 164, 166, 169, 216]. The standard procedure is to find a set of vector fields in the near-horizon region whose Lie brackets yield one or two copies of the Witt algebra. Upon computing the quasilocal charge algebra, one generally finds a central extension. The resulting Virasoro algebra is the symmetry algebra of a 2D CFT, suggesting that the quantization of the near horizon charge algebra should have a CFT description. The asymptotic density of states in such a theory is controlled by the Cardy formula, and by applying it in conjunction with the central charge computed from the quasilocal charge algebra, one arrives at the Bekenstein-Hawking entropy.

This procedure for arriving at the horizon entropy has been applied in a variety of different situations, often differing in the precise details of which symmetry algebra is used and what boundary conditions are imposed [144, 147, 148, 217, 150]. Here, by means of example, we provide evidence for the claim that the central charge occurring in these setups is always computed by the general formula (4.2.32) in terms of the noncovariance of the boundary Lagrangian for the null surface. The example we will analyze is the set of symmetry generators found for axisymmetric Killing horizons in [171], which generalize the near horizon conformal symmetries of the Kerr black hole proposed by Haco, Hawking, Perry, and Strominger (HHPS) [149]. We show that the null surface Wald-Zoupas construction described above produces a formula for the central charge which, via the Cardy formula, leads to an entropy that is twice the Bekenstein-Hawking entropy of the horizon. We argue that this factor of 2 could arise if the central charge was sensitive to both sets of edge modes, one on either side of the bifurcation surface, coupled together by the Dirichlet flux matching condition. To make a contradistinction, we compare to the case where boundary conditions are found to make the quasilocal charges integrable, and show that a different central charge results, and no factor of 2 appears. This thereby gives a derivation of the appropriate "counterterms" (i.e. fluxes) that had previously been conjectured to be necessary for the construction in [149, 171].

Near-horizon expansion

We begin by reviewing the expansion of the metric near a bifurcate Killing horizon, following a construction of Carlip [166, 171]. Let l^a be the horizon-generating Killing vector, which is timelike in the exterior region, and becomes the null normal on the bifurcate Killing horizon $\mathcal{H}$. A canonical choice of radial vector can be made using the gradient of the norm of l^a,

$$\rho^a = -\frac{1}{2\kappa}\nabla^a\left(l\cdot l\right), \tag{4.4.1}$$

where κ is the surface gravity, which is constant on account of the zeroth law of black hole mechanics [218]. The normalization of ρ^a is chosen so that it coincides with l^a on $\mathcal{H}$, and as a consequence of Killing's equation, one finds that $l\cdot\rho = 0$ and $[l,\rho]=0$ everywhere. If in addition the horizon is axisymmetric, meaning it possesses a rotational Killing vector ψ^a that commutes with l^a, it follows that $\psi\cdot\rho = 0$ and $[\psi,\rho]=0$. This allows us to choose coordinates (t, r_*, ϕ) such that (l^a, ρ^a, ψ^a) are the corresponding coordinate basis vectors, and in this coordinate system, $g_{tr_*} = g_{\phi r_*} = 0$. The radial coordinate r_* is analogous to the tortoise coordinate in the Schwarzschild solution, with the horizon positioned at $r_*\to-\infty$. The remaining coordinates will be denoted θ^A.

One can demonstrate that the norm of the radial vector near the horizon satisfies [166]

$$\rho\cdot\rho = -\left(l\cdot l\right) + \mathcal{O}\left[\left(l\cdot l\right)^2\right], \tag{4.4.2}$$

and hence as a function of r_*, the Killing vector norm satisfies the differential equation

$$\partial_{r_*}\left(l\cdot l\right) = \rho^a\nabla_a\left(l\cdot l\right) = 2\kappa\left(l\cdot l\right) + \mathcal{O}\left[\left(l\cdot l\right)^2\right] \tag{4.4.3}$$

whose solution is

$$\left(l\cdot l\right) = -e^{2\kappa r_*} + \mathcal{O}\left[e^{4\kappa r_*}\right], \tag{4.4.4}$$

where the integration constant has been absorbed by the shift freedom in the definition of the tortoise coordinate, $r_*\to r_* + f(\theta^A)$. This behavior suggests a reparameterization of the radial coordinate,

$$x = \frac{1}{\kappa}e^{\kappa r_*}, \quad\implies\quad \partial_x^a = \frac{1}{\kappa x}\rho^a \tag{4.4.5}$$

in terms of which the Killing vector norm has the expansion

$$\left(l\cdot l\right) = -\kappa^2 x^2 + \mathcal{O}\left[x^4\right]. \tag{4.4.6}$$

This also implies that ∂_x^a is unit normalized to leading order in the near-horizon expansion, which means x coincides with the radial geodesic distance to the bifurcation surface at this order. This fully determines the x coordinate, and in terms of it, the near-horizon metric exhibits a Rindler-like expansion,

$$ds^2 = -\kappa^2 x^2 dt^2 + dx^2 + \psi^2 d\phi^2 + q_{AB} d\theta^A d\theta^B - 2x^2 \kappa dt \left(N_\phi d\phi + N_A d\theta^A\right) + \dots \tag{4.4.7}$$

where the $\dots$ denotes higher order terms which do not play a role in the remainder of the analysis of the near horizon symmetries. Here, we have used the shift freedom $\phi \to \phi + G(\theta^A)$ to eliminate any $d\phi d\theta^A$ terms that generically appear.

The Rindler coordinates degenerate on the future and past horizons, so it is useful to define Kruskal coordinates which are regular on the horizon,

$$U = -xe^{-\kappa t} \tag{4.4.8a}$$
$$V = xe^{\kappa t}, \tag{4.4.8b}$$

in terms of which the metric becomes

$$ds^2 = -dUdV + \psi^2 d\phi^2 + q_{AB} d\theta^A d\theta^B + (UdV - VdU)(N_\phi d\phi + N_A d\theta^A) + \dots \tag{4.4.9}$$

The Killing vector and radial vector have simple expressions in terms of Kruskal coordinates,

$$l^a = \kappa(V\partial_V^a - U\partial_U^a) \tag{4.4.10}$$
$$\rho^a = \kappa(V\partial_V^a + U\partial_U^a), \tag{4.4.11}$$

which demonstrates that near the bifurcation surface at $U = V = 0$, l^a acts like a boost while ρ^a acts like a dilatation.

The future horizon $\mathcal{H}^+$ in Kruskal coordinates is located at $U = 0$, and on the horizon the generator is $l^a = \kappa V \partial_V^a$. The natural choice of auxiliary null covector there is then $n_a = -\frac{1}{\kappa V}\nabla_a V + \frac{1}{2}\left|\frac{dV}{\kappa V}\right|^2 l_a$, where the term proportional to l_a just ensures that n_a is null on all of $\mathcal{H}^+$. The spacetime volume form is given by

$$\epsilon = \frac{1}{2} dU \wedge dV \wedge \mu = -l \wedge \eta, \tag{4.4.12}$$

where the induced volume form on the horizon is

$$\eta = \frac{1}{\kappa V} dV \wedge \mu. \tag{4.4.13}$$

The past horizon $\mathcal{H}^-$ is at $V = 0$, where the generator is $l^a = -\kappa U \partial_U^a$ and the auxiliary null covector is $n_a = \frac{1}{\kappa U}\nabla_a U + \frac{1}{2}\left|\frac{dU}{\kappa U}\right|^2 l_a$. The conventions we use to define the volume forms are slightly different than on the future horizon. We choose the volume form on the past horizon to be

$$\eta = -\frac{1}{\kappa U} dU \wedge \mu, \tag{4.4.14}$$

to maintain the relationship $\mu = i_l \eta$. This means that the spacetime volume is related to η on the past horizon by

$$\epsilon = l \wedge \eta, \tag{4.4.15}$$

and these conventions ensure that μ limits to the same volume form on the bifurcation surface when approached on $\mathcal{H}^+$ or on $\mathcal{H}^-$. Because of (4.4.15), the decomposition of $\boldsymbol{\theta}$ picks up an overall minus sign relative to the expression (4.3.23). This means that on $\mathcal{H}^-$, the boundary term has a relative minus sign compared to (4.3.27)

$$\ell = \frac{k\eta}{8\pi G} \qquad (\text{on } \mathcal{H}^-). \tag{4.4.16}$$

Expression for the noncovariance

The results of section 4.2 show that any extension of the quasilocal symmetry algebra is determined by the noncovariance of the boundary term, $\Delta_{\hat{\xi}}\ell$. The noncovariance of this quantity and the various other structures defined on a generic null surface were determined in section 4.3 in terms of the scalar w_ξ which shows up in the noncovariance of the normal form to the horizon, l_a. To apply these formulas in computations of the algebra extensions, we therefore need an expression for w_ξ on a Killing horizon.

This can be derived on $\mathcal{H}^+$ by first noting that if ξ^a is tangent to the null surface $\mathcal{N} = \mathcal{H}^+$, the value of $\pounds_\xi l_a$ does not depend on how l_a is chosen away from $\mathcal{N}$. Since l_a and ρ_a coincide on $\mathcal{N}$, we can compute $w_\xi l_a = \pounds_\xi l_a \mathrel{\hat{=}} \pounds_\xi \rho_a = \nabla_a(\xi\cdot\rho)$, since $(d\rho)_{ab} = 0$ due to its definition as a gradient in equation (4.4.1). To continue the calculation, we express ξ^a in terms of the basis $(l^a, \rho^a, \psi^a, \partial_A^a)$ as $\xi^a = \xi^\rho \rho^a + V^a$, where V^a is some combination of l^a, ψ^a, and ∂^a_A. Since $l\cdot\rho = \psi\cdot\rho = 0$ everywhere, and $\partial_A\cdot\rho = \mathcal{O}[x^3]$, when evaluated on the horizon, only the ξ^ρ component survives in the gradient. Hence we find, using (4.4.2),

$$\nabla_a(\xi\cdot\rho) \mathrel{\hat{=}} \xi^\rho\nabla_a(\rho\cdot\rho) \mathrel{\hat{=}} -\xi^\rho\nabla_a\,(l\cdot l) \mathrel{\hat{=}} 2\kappa\xi^\rho\, l_a. \tag{4.4.17}$$

This leads to the simple expression,

$$w_\xi = 2\kappa\xi^\rho \qquad (\text{on } \mathcal{H}^+), \tag{4.4.18}$$

so we see that the noncovariance comes entirely from the dilatation component of ξ^a, i.e. the component parallel to ρ^a. Note that although w_ξ does not depend on how l_a is extended off of $\mathcal{N}$, it does depend on the extension of ξ^a in the vicinity of $\mathcal{N}$. To demonstrate this point, we note that because l^a and ρ^a coincide on $\mathcal{N}$, one cannot separate ξ^a into its l^a and ρ^a components using its value on $\mathcal{N}$ alone. Only after looking at its behavior as you move away from $\mathcal{N}$ can its l^a and ρ^a components be distinguished, and then only the ρ^a component contributes to the noncovariance.

The analysis on the past horizon $\mathcal{H}^-$ is similar and leads to

$$w_\xi = 2\kappa\xi^\rho \qquad (\text{on } \mathcal{H}^-). \tag{4.4.19}$$

Virasoro vector fields

Having introduced the near-horizon expansion of the metric, we now turn to the choice of vector fields generating the near-horizon symmetries. Motivated by the hidden conformal symmetry of scattering amplitudes in Kerr [168], HHPS proposed a set of vector fields for Kerr black holes whose algebra consisted of two commuting copies of the Witt algebra. This algebra was identified by foliating the near-horizon region by approximately AdS_3 slices, and writing down the corresponding asymptotic symmetry generators. The construction of these symmetry generators was extended to Schwarzschild black holes in [219], which also proposed a two-parameter generalization in the choice of vector fields, with the two parameters coinciding with notions of left and right temperatures. The construction was further extended to arbitrary axisymmetric Killing horizons in [171], which similarly identified an algebra $\mathrm{Diff}(S^1)_\alpha \times \mathrm{Diff}(S^1)_{\overline{\alpha}}$, consisting of two commuting copies of the Witt algebra, and labeled by two parameters $(\alpha, \overline{\alpha})$ which coincide with choices of temperatures. In this section, we will analyze this latter algebra for general choices of $(\alpha, \overline{\alpha})$, and show in section 4.4 that the quasilocal charge algebra leads to an expression for the central charges.

One way to describe the symmetry algebra is to present it in terms of a geometric structure that it preserves. To this end, we define the following "conformal coordinates" depending on the two parameters $(\alpha, \overline{\alpha})$ [171]:

$$W^+ = V e^{\alpha\phi} \tag{4.4.20a}$$

$$W^- = -U e^{\overline{\alpha}\phi} \tag{4.4.20b}$$

$$y = e^{\frac{\alpha+\overline{\alpha}}{2}\phi}. \tag{4.4.20c}$$

The 2π periodicity of ϕ requires that these coordinates be identified according to $(W^+, W^-, y) \sim (e^{2\pi\alpha}W^+, e^{2\pi\overline{\alpha}}W^-, e^{\pi(\alpha+\overline{\alpha})}y)$. We then form the following tensor

$$C_{ab} = -\frac{1}{y^2}\nabla_a W^+ \nabla_b W^- = \Big(\nabla_a V + \alpha V \nabla_a \phi\Big)\Big(\nabla_b U + \overline{\alpha} U \nabla_b \phi\Big) \tag{4.4.21}$$

where the second equality demonstrates that C_{ab} is well-defined in light of the periodicity of the conformal coordinates. The near-horizon symmetries are defined to simply be the transformations that preserve C_{ab}. A trivial set of such transformations are simply those parallel to the transverse directions, $V^A\partial_A$. They preserve the bifurcation surface of the horizon, and hence do not require the Wald-Zoupas prescription, nor do they lead to algebra extensions when represented in terms of quasilocal charges. We therefore focus on the nontrivial transformations that act in the (t, r_*, ϕ) plane.

Using the first expression for C_{ab} in (4.4.21), it is straightforward to see that the vector fields that satisfy $\pounds_\xi C_{ab} = 0$ are of the form

$$\xi_n^a = F_n(W^+)\partial_+^a + \frac{1}{2}F_n'(W^+) y \partial_y^a \tag{4.4.22}$$

$$\overline{\xi}_n^a = \overline{F}_n(W^-)\partial_-^a + \frac{1}{2}\overline{F}_n'(W^-) y \partial_y^a. \tag{4.4.23}$$

In order to be single-valued, the functions $F_n, \overline{F}_n$ must satisfy $F_n(W^+e^{2\pi\alpha}) = F_n(W^+)e^{2\pi\alpha}$, $\overline{F}_n(W^-e^{2\pi\overline{\alpha}}) = \overline{F}_n(W^-)e^{2\pi\overline{\alpha}}$, and hence they can be expanded in modes,

$$F_n = \alpha W^+ \left(W^+\right)^{\frac{in}{\alpha}} \tag{4.4.24}$$

$$\overline{F}_n = -\overline{\alpha} W^- \left(W^-\right)^{-\frac{in}{\overline{\alpha}}} . \tag{4.4.25}$$

We can then compute the Lie brackets of these vector fields, and find that their algebra is given by two commuting copies of the Witt algebra,

$$[\xi_m, \xi_n] = i(n-m)\xi_{m+n} \tag{4.4.26}$$

$$[\overline{\xi}_m, \overline{\xi}_n] = i(n-m)\overline{\xi}_{m+n} \tag{4.4.27}$$

$$[\xi_m, \overline{\xi}_n] = 0 \tag{4.4.28}$$

Although preservation of the tensor C_{ab} uniquely specifies the near-horizon symmetry generators, there is still a question as to why this is a useful criterion to impose. While we do not have a completely satisfactory answer, we can point out some interesting features of C_{ab} that may inform future investigations into its significance. First we note that the vector fields also preserve the following contravariant tensor,

$$D^{ab} = -y^2 \partial_+^a \partial_-^b = \partial_V^a \partial_U^b = \frac{1}{2\kappa^2 x^2}(l^a + \rho^a)(l^b - \rho^b), \tag{4.4.29}$$

for any choice of $(\alpha, \overline{\alpha})$. From this, one can also construct the projectors

$$(P_+)^b{}_a = C_{ac}D^{bc} = \nabla_a W^+ \partial_+^b = \left(\frac{\nabla_a V}{\kappa V} + \frac{\alpha}{\kappa}\nabla_a\phi\right)\kappa V \partial_V^b \tag{4.4.30}$$

$$(P_-)^b{}_a = C_{ca}D^{cb} = \nabla_a W^- \partial_-^b = \left(\frac{\nabla_a U}{\kappa U} + \frac{\overline{\alpha}}{\kappa}\nabla_a\phi\right)\kappa U \partial_U^b \tag{4.4.31}$$

which are also preserved. On $\mathcal{H}^+$, the upper index of $(P_+)^a{}_b$ is parallel to the horizon generator, and so by pulling back the lower index to $\mathcal{H}^+$, one arrives at a vertical projector for vectors on $\mathcal{H}^+$ onto l^a. Such a projector is an example of an Ehresmann connection for the horizon, viewed as a fiber bundle with fibers consisting of the null flow lines of l^a. It is, in fact, a flat connection, with horizontal directions given by the surfaces of constant W^+. However, this connection produces a nontrivial holonomy upon completing a 2π rotation in ϕ, which results in $V \to Ve^{-2\pi\alpha}$ (see [171] for a depiction of this spiraling behavior of the conformal coordinates). $(P_-)^a{}_b$ similarly defines a flat Ehresmann connection on the past horizon, with 2π holonomy $U \to Ue^{-2\pi\overline{\alpha}}$.

The relevance of such Ehresmann connections in the study of Carroll geometries on null surfaces [220] was recently emphasized in [221], so investigating the relationship between Carroll geometries and the near-horizon Virasoro symmetries may lead to a deeper understanding as to their fundamental origin. Note, however, it is important that the generators are defined to preserve C_{ab} in a neighborhood of the bifurcation surface; it is not enough to simply find vector fields that preserve P_+ and P_- on each of the respective horizons. This is because the behavior of ξ_n^a off of the horizon determines the noncovariances, which in turn determine extensions of the quasilocal charge algebra. Since C_{ab} contains the information about both projectors, the geometric interpretation of the symmetry generators seems to involve not only the Ehresmann connections on each individual horizon, but also how they relate to each other in forming a bifurcate horizon.

As discussed in section 4.4, the noncovariances depend on the ρ^a component of the symmetry generators. This can be computed by transforming the vector fields (4.4.22) and (4.4.23) back to the (t, r_*, ϕ) coordinate system, in which they are expressed in terms of l^a, ρ^a, and ψ^a. Using (4.4.5), (4.4.8), and (4.4.20), this leads to

$$\xi_n^a = \frac{(W^+)^{\frac{in}{\alpha}}}{\alpha+\overline{\alpha}}\left[\frac{\alpha\overline{\alpha}}{\kappa}l^a + \alpha\psi^a + in\left(\frac{\overline{\alpha}-\alpha}{2\kappa}l^a + \psi^a\right)\right] - \frac{in}{2\kappa}\left(W^+\right)^{\frac{in}{\alpha}}\rho^a \tag{4.4.32}$$

$$\overline{\xi}_n^a = \frac{(W^-)^{-\frac{in}{\overline{\alpha}}}}{\alpha+\overline{\alpha}}\left[\frac{\alpha\overline{\alpha}}{\kappa}l^a - \overline{\alpha}\psi^a + in\left(\frac{\overline{\alpha}-\alpha}{2\kappa}l^a + \psi^a\right)\right] - \frac{in}{2\kappa}\left(W^-\right)^{-\frac{in}{\overline{\alpha}}}\rho^a, \tag{4.4.33}$$

Note that the prefactor $(W^+)^{\frac{in}{\alpha}} = V^{\frac{in}{\alpha}}e^{in\phi}$ in ξ_n^a has an oscillating singularity as the past horizon at $V \to 0$ is approached. This means that the ξ_n^a vector fields have no well-defined limit to the past horizon, and so their quasilocal charges will be constructed on the future horizon. Similarly, the prefactor $(W^-)^{-\frac{in}{\overline{\alpha}}} = (-U)^{-\frac{in}{\overline{\alpha}}}e^{-in\phi}$ in $\overline{\xi}_n^a$ has no limit to the future horizon $U \to 0$, and so the corresponding quasilocal charges will be evaluated on $\mathcal{H}^-$. With this in mind, we can read off the expression for the noncovariances associated with these vector fields using (4.4.18) and (4.4.19), which gives

$$w_{\xi_n} = -in\left(W^+\right)^{\frac{in}{\alpha}} \qquad (\text{on } \mathcal{H}^+) \tag{4.4.34}$$

$$w_{\overline{\xi}_n} = -in\left(W^-\right)^{-\frac{in}{\overline{\alpha}}} \qquad (\text{on } \mathcal{H}^-). \tag{4.4.35}$$

We now demonstrate that these vector fields do not preserve the boundary conditions $\delta k = 0$, $\delta l^a \mathrel{\widehat{=}} 0$, or $n_a \delta l^a \mathrel{\widehat{=}} 0$ that have been employed in previous works [119, 156, 28, 211]. On $\mathcal{H}^+$,

$$I_{\hat{\xi}_n}\delta k = -n(n-i\alpha)\frac{\kappa}{\alpha}\left(W^+\right)^{\frac{in}{\alpha}}, \tag{4.4.36}$$

$$I_{\hat{\xi}_n}\delta l^a = \frac{n(n-i\alpha)}{\alpha+\overline{\alpha}}\left(W^+\right)^{\frac{in}{\alpha}}\left[-l^a + \frac{\kappa}{\alpha}\psi^a\right], \tag{4.4.37}$$

which clearly violates all three conditions pointwise. These conditions are also violated pointwise by the $\overline{\xi}_n^a$ generators on $\mathcal{H}^-$,

$$I_{\hat{\overline{\xi}}_n}\delta k = -n(n+i\overline{\alpha})\frac{\kappa}{\overline{\alpha}}\left(W^-\right)^{-\frac{in}{\overline{\alpha}}} \tag{4.4.38}$$

$$I_{\hat{\overline{\xi}}_n}\delta l^a = \frac{n(n+i\overline{\alpha})}{\alpha+\overline{\alpha}}\left(W^-\right)^{-\frac{in}{\overline{\alpha}}}\left[l^a + \frac{\kappa}{\overline{\alpha}}\psi^a\right]. \tag{4.4.39}$$

This therefore necessitates the use of the weaker boundary conditions described in section 4.3.

Central charges

With all this in place, we can proceed to the calculation of the central extension of the quasilocal charge algebra. We denote the quasilocal charges for ξ_n^a by L_n, and the charges for $\overline{\xi}_n^a$ by $\overline{L}_n$. Their values are given by the general expression (4.2.20), evaluated on $\mathcal{H}^+$ for the L_n generators and on $\mathcal{H}^-$ for the $\overline{L}_n$ generators. Note that because the background is rotationally symmetric, all of the charges L_n, $\overline{L}_n$ except for L_0, $\overline{L}_0$ vanish, since the generators (4.4.32), (4.4.33) come with angular dependence $e^{in\phi}$, which integrates to zero on $\partial\Sigma$. Of course, their variations, which enter the calculation of the brackets, need not vanish. Since the vector fields ξ_0^a and $\overline{\xi}_0^a$ are linear combinations of the horizon-generating and rotational Killing vectors, l^a and ψ^a, the L_0, $\overline{L}_0$ charges will be linear combinations of the Noether charges for the Killing vectors, namely, the horizon area A and angular momentum J_H. The zero mode generators evaluate to

$$L_0 = \frac{\alpha}{\alpha+\overline{\alpha}} J_H \tag{4.4.40}$$

$$\overline{L}_0 = -\frac{\overline{\alpha}}{\alpha+\overline{\alpha}} J_H, \tag{4.4.41}$$

where the horizon angular momentum J_H is given by the Noether charge for the rotational Killing vector ψ^a,

$$J_H = \int_{\partial\Sigma} Q_\psi = \frac{1}{4G}\int d\theta^A \sqrt{q}|\psi| N_\phi(\theta^A). \tag{4.4.42}$$

The area contribution has dropped from these expressions because the quasilocal charge H_l for l^a, which normally is proportional to the area, vanishes upon including the Dirichlet boundary term $i_l\ell$ from (4.2.20). This is somewhat unintuitive because l^a vanishes as the bifurcation surface is approached; however, the contraction with ℓ has a nonzero value in the limit. The vanishing of this boost Noether charge was similarly observed in the analysis of a phase space bounded by a timelike hypersurface with Dirichlet boundary conditions [174, 198].

The discussion of section 4.2 showed that the Barnich-Troessaert bracket of the charges must reproduce the algebra of the vector fields, up to abelian extensions. Hence, for the ξ_n^a vector fields, the bracket of the charges can be written

$$\{L_m, L_n\} = -i\Big[(n-m)L_{m+n} + K_{m,n}\Big], \tag{4.4.43}$$

where $K_{m,n}$ is determined by the explicit formula (4.2.32),

$$K_{m,n} = -i\int_{\partial\Sigma}\Big(i_{\xi_m}\Delta_{\hat{\xi}_n}\ell - i_{\xi_n}\Delta_{\hat{\xi}_m}\ell\Big). \tag{4.4.44}$$

To evaluate this, we first note that the expression (4.3.36) for the noncovariance of $k\eta$ and the expression (4.4.34) for w_{ξ_n} gives

$$\Delta_{\hat{\xi}_n}\ell = \frac{\eta}{8\pi G} l^a \nabla_a w_{\xi_n} = \frac{\eta}{8\pi G}\frac{n^2\kappa}{\alpha}\left(W^+\right)^{\frac{in}{\alpha}}. \tag{4.4.45}$$

For the quantity $i_{\xi_m}\eta$, note that the ψ^a component will not contribute to this expression when evaluated on a surface of constant V. Recalling that $\rho^a = l^a$ on $\mathcal{H}^+$, we have

$$i_{\xi_m}\eta = \frac{(W^+)^{\frac{in}{\alpha}}}{\alpha+\overline{\alpha}}\left(\frac{\alpha\overline{\alpha}}{\kappa} - im\frac{\alpha}{\kappa}\right) i_l\eta = \frac{(W^+)^{\frac{in}{\alpha}}}{\alpha+\overline{\alpha}}\frac{\alpha}{\kappa}\left(\overline{\alpha} - im\right)\mu. \tag{4.4.46}$$

Then we find that

$$i_{\xi_m}\left(\Delta_{\hat{\xi}_{-m}}\ell\right) = -im^2\frac{(m+i\overline{\alpha})}{(\alpha+\overline{\alpha})}\frac{\mu}{8\pi G}, \tag{4.4.47}$$

and subtracting the term with $m \leftrightarrow -m$ and integrating over the surface gives a result proportional to the horizon area A,

$$K_{m,-m} = \frac{A}{4\pi G(\alpha+\overline{\alpha})}m^3. \tag{4.4.48}$$

Any other extension term $K_{m,n}$ with $m \neq -n$ vanishes, again due to rotational invariance and the overall $e^{-i(m-n)\phi}$ dependence of the integrand. We verify in appendix A.14 that the variations of the quantities $K_{m,n}$ with $m \neq -n$ are consistent with having identically zero quasilocal charges associated with them, which means that the only nontrivial extension terms are $K_{m,-m}$. Hence, the extension is in fact central, and the algebra obtained is the Virasoro algebra,

$$\{L_m, L_n\} = -i\Big[(n-m)L_{m+n} + \frac{c}{12}m^3\delta_{m,-n}\Big] \tag{4.4.49}$$

with central charge

$$c = \frac{3A}{\pi G(\alpha+\overline{\alpha})}. \tag{4.4.50}$$

The analysis for the $\overline{\xi}^a_n$ generators is similar. The calculations need to be done on the past horizon due to the singularity in $\overline{\xi}^a_n$ on the future horizon. As explained in section 4.4, this flips the sign of the boundary term ℓ in the decomposition of the symplectic form. This then gives

$$\Delta_{\hat{\overline{\xi}}_n}\ell = -\frac{\eta}{8\pi G} l^a \nabla_a w_{\overline{\xi}_n} = -\frac{\eta}{8\pi G}\frac{n^2\kappa}{\overline{\alpha}}\left(W^-\right)^{-\frac{in}{\overline{\alpha}}} \tag{4.4.51}$$

$$i_{\overline{\xi}_n}\eta = \frac{(W^-)^{-\frac{in}{\overline{\alpha}}}}{\alpha+\overline{\alpha}}\frac{\overline{\alpha}}{\kappa}(\alpha+in)\mu \tag{4.4.52}$$

$$i_{\overline{\xi}_m}\left(\Delta_{\hat{\overline{\xi}}_{-m}}\ell\right) = -im^2\frac{(m-i\alpha)}{(\alpha+\overline{\alpha})}\frac{\mu}{8\pi G} \tag{4.4.53}$$

From this last expression, we can compute the extension

$$\overline{K}_{m,-m} = -i \int_{\partial\Sigma} \left(i_{\overline{\xi}_m} \Delta_{\hat{\overline{\xi}}_{-m}} \ell - i_{\overline{\xi}_{-m}} \Delta_{\hat{\overline{\xi}}_m} \ell \right) \tag{4.4.54}$$

$$= \frac{A}{4\pi G(\alpha + \overline{\alpha})} m^3. \tag{4.4.55}$$

As before, the $\overline{L}_n$ generators are then seen to satisfy a Virasoro algebra with central charge

$$\overline{c} = \frac{3A}{\pi G(\alpha + \overline{\alpha})}, \tag{4.4.56}$$

which is the same value as c given in (4.4.50). Note that $c, \overline{c}$ given in (4.4.50), (4.4.56) are twice the values computed in [171, 149]. This factor of 2 will have an effect on the entropy computed in section 4.5.

Frame dependence

Although the null normal is fixed to coincide with the Killing horizon generator in the definition of the near-horizon phase space, we would like to understand how the central charges depend on the choice of background scaling frame. This is relevant because the choice of frame was related to the choice of stretched horizon in section 4.3, and since this frame has parallels to a choice of Weyl frame in a CFT, we would like the central charge to be insensitve to this choice. Under the rescaling transformation (4.3.1), the parameter w_ξ characterizing the noncovariance of l_a transforms according to

$$w_\xi \to w_\xi + \pounds_\xi f. \tag{4.4.57}$$

Using (4.3.36), this then leads to a change in the anomaly of the boundary term by

$$\Delta_{\hat{\xi}} \ell \to \Delta_{\hat{\xi}} \ell - \frac{\eta}{8\pi G} \pounds_l \pounds_\xi f. \tag{4.4.58}$$

For the ξ_n^a generators on $\mathcal{H}^+$, this results in an extra contribution to $K_{m,-m}$ given by the integral over the bifurcation surface of the following quantity:

$$\frac{\mu}{2\pi G} \frac{m}{(\alpha + \overline{\alpha})^2} \left[\alpha(m^2 + \overline{\alpha}^2) V \frac{\partial f}{\partial V} + \frac{\partial}{\partial \phi} \left((\alpha\overline{\alpha} - m^2) f + (\alpha + \overline{\alpha}) \frac{\partial f}{\partial V} \right) \right]. \tag{4.4.59}$$

The term involving a total ϕ derivative integrates to zero, and hence does not affect the central charge. The term that can affect the result is the one proportional to $V \frac{\partial f}{\partial V}$ in the limit $V \to 0$. If f is a regular function of V at $V = 0$, this term drops out and the central charge is unaffected. To get a nonzero contribution from it, we would need $f \sim \lambda \log V$, corresponding to a rescaling of l^a by V^λ. This then affects the rate at which l^a vanishes (or blows up) as the bifurcation surface is approached. For example, given the form of l^a in

(4.4.10), we see that $\lambda = -1$ rescales l^a to an affine parameterization, since V is an affine parameter.

In order to arrive at an unambiguous value of the central charge, we must disallow transformations that affect the rate at which l^a vanishes as $V \to 0$. This means choosing a normalization so that it vanishes linearly with respect to an affine parameter as bifurcation surface is approached, just as the horizon-generating Killing vector does. Note that this still allows for rescalings of the generator in a ϕ or θ^A-dependent manner, or, relatedly, making a different choice of the affine parameter with respect to which l^a vanishes linearly. However, it rules out using an affinely parameterized generator when analyzing bifurcate null horizons. Using the Killing parameterization of the null generator is natural for Killing horizons, but it may be that other choices are preferred for different setups. Note that in [149, 171], it seems that a nonstandard choice of this normalization was used, which happened to set any contribution to the central charge from the flux to zero except the Hájíčekterm. It would be interesting to explore these other normalizations in more detail in the future.

4.5 Entropy from the Cardy formula

The relevance of equations (4.4.50) and (4.4.56) for the central charges is that they contain information about the entropy of the horizon. To see how this comes about, we need to associate a quantum system with the near-horizon degrees of freedom. It is well known that in a theory with gauge symmetry such as general relativity, the introduction of a spatial boundary breaks some of the gauge invariance, thereby producing additional degrees of freedom on the boundary that would otherwise not have been present [162, 161, 146]. The edge modes that arise in this fashion are acted on by the quasilocal charges identified in the previous sections, and thus represent a classical system with Virasoro symmetry. The quantization of this system should respect the symmetry, and since two dimensional conformal field theories share this symmetry algebra, we are led to the postulate that the quantum system should be a 2D CFT. In such a theory, the asymptotic density of states depends in a universal way on the central charge according to the Cardy formula [167]. We will find that applying this formula in the context of a Killing horizon shows that the entropy of the CFT is directly related to the entropy of the horizon.

Canonical Cardy formula

The Cardy formula comes in two flavors: microcanonical and canonical. The canonical formula applies to a CFT in a thermal state at high temperatures, and states that the entropy is given by

$$S_{\text{Cardy}} = \frac{\pi^2}{3}(c\,T + \overline{c}\,\overline{T}), \tag{4.5.1}$$

where T and $\overline{T}$ are known as the left and right temperatures; they are the thermodynamic potentials conjugate to the L_0 and $\overline{L}_0$ charges.

To apply this formula in the context of a Killing horizon, we need to identify the temperatures. This can be done in a manner similar to the determination of the Hawking temperature in terms of the horizon surface gravity. We would expect the density matrix for quantum fields just outside of the horizon to be in the Frolov-Thorne vacuum [169, 222, 170], which is thermal with respect to the horizon-generating Killing vector l^a. This means the density matrix should be of the form

$$\rho \sim e^{-\frac{2\pi}{\kappa}\omega_l}, \tag{4.5.2}$$

where $\omega_l = -k_a l^a$ is the frequency of a mode with wavevector k_a, relative to l^a, and the coefficient $\frac{2\pi}{\kappa}$ is the inverse Hawking temperature. Since l^a can be expressed in terms of the left and right Virasoro vector fields via $\frac{1}{\kappa}l^a = \frac{1}{\alpha}\xi_0^a + \frac{1}{\overline{\alpha}}\overline{\xi}_0^a$, the density matrix can equivalently be written

$$\rho \sim e^{-\frac{2\pi}{\alpha}\omega_0 - \frac{2\pi}{\overline{\alpha}}\overline{\omega}_0} \tag{4.5.3}$$

where now $\omega_0 = -k_a\xi_0^a$, $\overline{\omega}_0 = k_a\overline{\xi}_0^a$ are the frequencies with respect to the Virasoro zero mode generators. This then leads us to identify the left and right temperatures

$$T = \frac{\alpha}{2\pi}, \qquad \overline{T} = \frac{\overline{\alpha}}{2\pi}. \tag{4.5.4}$$

With these temperatures in hand, the Cardy formula (4.5.1) applied using the computed values (4.4.50), (4.4.56) for $c, \overline{c}$ yields

$$S_{\text{Cardy}} = 2\left(\frac{A}{4G}\right). \tag{4.5.5}$$

Somewhat unexpectedly, we arrive at *twice* the entropy of the horizon. To interpret this result, recall that the central charges were computed using the Barnich-Troessaert bracket of quasilocal charges. This bracket was employed because the quasilocal charges are not integrable, since they are associated with evolution up the horizon, during which symplectic flux leaks out. In order to justify such a calculation, one should introduce an auxiliary system that collects the lost symplectic flux, allowing integrable generators and Poisson brackets to be defined on the total system. Since we postulated that the edge modes on one side of the horizon are described by a 2D CFT, it is equally natural to assume that the auxiliary system is another copy of the same CFT, associated with edge modes on the other side of the horizon. This is the picture that would appear when cutting a global Cauchy surface for the full spacetime across the bifurcation surface, in which case the left wedge and its edge modes are the only additional degrees of freedom in the space, and hence must comprise the auxiliary system that collects the fluxes from the right wedge. If we assume that the Barnich-Troessaert bracket computes the central charge of the total system, we would arrive at twice the value of the central charge for one of the CFTs. This would explain the appearance of the factor of 2 in (4.5.5), since it is counting the entropy associated with edge modes on both sides of the horizon. If we then traced out the auxiliary system, we would expect the entropy to be exactly half the value computed above, and hence would arrive at the correct horizon entropy,

$$S = \frac{A}{4G}. \tag{4.5.6}$$

This conjectural resolution will be expanded upon in section 4.6. In order to support this interpretation by way of contrast, we turn now to a case where the quasilocal charges are in fact integrable, so that no fluxes or auxiliary systems are needed.

Integrable charges

The other possibility that would produce the correct entropy is if the boundary term ℓ were half the value given in equation (4.3.27). This would correspond to different boundary conditions than Dirichlet, since the flux would now contain an additional contribution proportional to δk. Although this appears unnatural from the perspective of gluing subregions discussed in section 4.2, if we were only interested in integrable charges so that the subregion could be treated as a closed system, any boundary condition that results in integrability is valid. In this section, we will show that such modified boundary conditions are necessary if demanding that the HHPS charges be integrable.

A useful property of the Barnich-Troessaert bracket is that if boundary conditions are imposed to make the charges integrable, it reduces to the Dirac bracket of these charges on the submanifold of phase space defined by imposing the boundary conditions as constraints. The integrable charges therefore need not be considered quasilocal, but rather are legitimate Hamiltonians generating the symmetry on the constrained phase space. Note, however, that the vector fields generating the symmetry must preserve the boundary condition imposed, i.e. they must be tangent to the constraint submanifold, since otherwise they do not produce well-defined transformations of the constrained fields.

Finding a boundary condition that ensures vanishing symplectic flux but is also preserved by the vector fields (4.4.32) and (4.4.33) is somewhat nontrivial, since the vector fields tend to violate any local condition fixing the intrinsic or extrinsic quantities on the horizon, see equations (4.4.36), (4.4.37), (4.4.38), and (4.4.39). However, as discussed in [171], one can consider more general conditions that are preserved by the symmetry generators, involving integrals of variations of quantities over portions of the horizon. Assuming such a condition is found, the fact that the fluxes then vanish consequently implies that the bracket $\{L_n, L_{-n}\}$ can be computed simply from contracting the vector fields $\hat{\xi}_n$, $\hat{\xi}_{-n}$ into the symplectic form Ω.[17] This computation was already performed in [171], and the resulting central charges are

$$c = \frac{24}{(\alpha+\overline{\alpha})^2}\left(\frac{\overline{\alpha}A}{8\pi G} + J_H\right) \tag{4.5.7}$$

$$\bar{c} = \frac{24}{(\alpha+\overline{\alpha})^2}\left(\frac{\alpha A}{8\pi G} - J_H\right). \tag{4.5.8}$$

[17] As discussed in section 4.2, the central charge is independent of the choice of corner term β.

On the other hand, the general formula (4.2.32) for the extension in terms of $\Delta_{\hat{\xi}}\ell$ still remains valid, albeit with a possibly different choice of boundary term than $\ell = \frac{-k}{8\pi G}\eta$. The simplest generalization is to take

$$\ell = \frac{-ak}{8\pi G}\eta, \tag{4.5.9}$$

with a some constant. In order to ensure that the values of L_0 and $\overline{L}_0$ are the same when computed on either the future or past horizon, we must then choose the boundary term on the past horizon to be $\frac{ak}{8\pi G}\eta$. Doing this produces the central charges

$$c = \bar{c} = \frac{3aA}{\pi G(\alpha + \overline{\alpha})}. \tag{4.5.10}$$

Equating the above two expressions for c and $\bar{c}$ yields the conditions

$$\alpha - \overline{\alpha} = \frac{16\pi G J_H}{A}, \qquad a = \frac{1}{2}. \tag{4.5.11}$$

The first condition restricts the parameters $\alpha, \overline{\alpha}$ defining the symmetry generators, and was identified in [171] as a necessary condition for integrability of the charges. The second condition $a = \frac{1}{2}$ shows that the boundary term ℓ is half of the value used when imposing a Dirichlet flux condition. It implies that the central charges are now half of the value computed in section 4.4,

$$c = \bar{c} = \frac{3A}{2\pi G(\alpha + \overline{\alpha})}, \tag{4.5.12}$$

and consequently the entropy coming from the canonical Cardy formula (4.5.1) now agrees with the horizon entropy,

$$S_{\text{Cardy}} = \frac{A}{4G}. \tag{4.5.13}$$

Microcanonical Cardy formula

The canonical Cardy formula requires the left and right temperatures as inputs, which were identified for the horizon using properties of the Frolov-Thorne vacuum for quantum fields outside of the horizon. A more microscopic derivation of the entropy would utilize the microcanonical Cardy formula, which expresses the entropy in terms of the density of states at fixed, large values of L_0, $\overline{L}_0$. The microcanonical expression for the entropy is

$$S_{\mu\text{Cardy}} = 2\pi\left(\sqrt{\frac{cL_0}{6}} + \sqrt{\frac{c\overline{L}_0}{6}}\right). \tag{4.5.14}$$

To apply this formula, we need the values of the charges L_0 and $\overline{L}_0$. Note that we should expect the microcanoncial formula to work only in the case that the charges are integrable, since only then do L_0, $\overline{L}_0$ represent global charges for a closed system. This is consistent with standard thermodynamics, in which the microcanonical ensemble counts the number of states within a fixed energy band of a closed system, while the canonical ensemble is used for an open system interacting with a bath at fixed temperature.

According to the discussion in section 4.5, integrability of the charges requires that the boundary term ℓ be on future horizon

$$\ell = -\frac{k\eta}{16\pi G}, \tag{4.5.15}$$

and the past horizon expression is just $\ell = \frac{k\eta}{16\pi G}$, which are half the values they take under Dirichlet flux matching. This boundary term enters explicitly into the expression for the charges via equation (4.2.20), and making the choice (4.5.15) is important for finding the right entropy from the microcanonical Cardy formula.

Including the contribution from the boundary term (4.5.15), we now find that the zero mode charges are

$$L_0 = \frac{\alpha}{\alpha+\overline{\alpha}}\left(\frac{\overline{\alpha}A}{16\pi G} + J_H\right) = \frac{\alpha^2}{(\alpha+\overline{\alpha})}\frac{A}{16\pi G} \tag{4.5.16}$$

$$\overline{L}_0 = \frac{\overline{\alpha}}{\alpha+\overline{\alpha}}\left(\frac{\alpha A}{16\pi G} - J_H\right) = \frac{\overline{\alpha}^2}{(\alpha+\overline{\alpha})}\frac{A}{16\pi G}, \tag{4.5.17}$$

where the latter equalities in these equations employ the integrability condition (4.5.11) determining $\alpha - \overline{\alpha}$. Using these values in the microcanonical Cardy formula (4.5.14) with the central charges (4.5.12) gives

$$S_{\mu\text{Cardy}} = \frac{A}{4G}, \tag{4.5.18}$$

in agreement with the canonical result (4.5.13) and coinciding with the horizon entropy.

4.6 Discussion

In this work, we revisited the Wald-Zoupas construction of quasilocal charges and fluxes for subregions with null boundaries, with the goal of systematically deriving the central charges that have appeared in several recent works on symmetries near Killing horizons [149, 172, 150, 219, 171, 173]. This required generalizing the treatment in [119] of the Wald-Zoupas procedure for null boundaries by allowing for the most general boundary conditions consistent with the presence of a null hypersurface. In the process, we arrived at a general formula (4.2.32) for the algebra extension that appears in the quasilocal charge algebra, which would be applicable in other investigations of near horizon symmetries. We showed that the central charge arises from fixing l_a as the background structure, which we related to a choice of stretched horizon. In this context, the central charge arises as an anomaly, in a manner quite analogous to the holographic Weyl anomaly appearing in AdS/CFT due to noncovariance of the gravitational action under changes in the radial cutoff. Applying the Cardy formula to the central charges of a bifurcate, axisymmetric Killing horizon obtained using the Dirichlet flux condition yielded twice the entropy of the horizon, and we argued that the factor of 2 could be indicative of a complementary set of edge modes on the other side of the horizon. We now expand upon the possible significance of these results, and end with some future directions.

Algebra extension as a scaling anomaly

The formula (4.2.32) for the algebra extension $K_{\xi,\zeta}$ shows that extensions only arise when the boundary term ℓ is not covariant with respect to the transformations generated by ξ^a, ζ^a. In several other treatments of symmetries at null boundaries, the boundary term was chosen to be covariant, and equation (4.2.32) therefore explains the vanishing of the central extensions in those cases [119, 156, 158]. The fact that the extension is always of the form of a trivial field-dependent cocycle [98] given by equation (4.2.36), means that the boundary term can always be chosen to be covariant so as to eliminate the extension $K_{\xi,\zeta}$. However, such a choice is in conflict with the Dirichlet form of the flux, and hence describes a physically different setup. Put another way, there is nontrivial physics in the choice of boundary term, and we should not view different choices of this term as a type of gauge freedom.

By imposing the Dirichlet flux condition, we were inevitably led to fluxes and boundary terms that were not covariant under the boundary symmetries. This noncovariance seems to be a feature, rather than a bug, as it gives rise to the central charge which ultimately accounts for the horizon entropy. The source of noncovariance came from fixing a choice of the null normal l_a. This can be viewed as a choice of frame, since there is generally no preferred normalization of l_a when the surface is null. The choice of l_a bears resemblance to the choice of radial cutoff when describing asymptotic symmetries, or, equivalently, the choice of conformal factor when dealing with the conformal compactification. In holographic renormalization, the appearance of conformal anomalies in the dual CFT is known to be related to anomalous transformations of boundary terms in the gravitational action with respect to the radial cutoff [179, 180, 181, 182, 223]. Changing the radial cutoff then affects the induced metric in the limit that the conformal boundary is approached, and hence coincides with a choice of Weyl frame in the CFT.

To strengthen the analogy between this notion of conformal frame and the scaling frame of l_a, we showed in section 4.3 that a preferred normalization of l_a is determined if one specifies a sequence of stretched horizons that asymptote to the null surface. As has been remarked before, there are multiple ways to stretch the horizon [224], and here we see that this ambiguity has a precise analog in terms of the scaling frame of l_a. Furthermore, the ambiguity in stretching the horizon, or equivalently, choosing the scaling frame of l_a, is actually responsible for the appearance of the central charges in the horizon symmetry algebra. The radial vector ρ^a introduced in equation (4.4.1) generates transformations that change the stretched horizon foliation pointwise, acting like a dilatation about the bifurcation surface. Intriguingly, we showed in section 4.4 that the ρ^a component of the symmetry generators is solely responsible for producing anomalous transformations of objects on the horizon. This suggests that ρ^a should be thought of as generating changes in the scaling frame of the horizon CFT, just as the radial vector in AdS generates Weyl transformations for the holographic CFT. The central charge in the horizon quasilocal charge algebra appears as a classical diffeomorphism anomaly coming from $\Delta_{\hat{\xi}}\ell$, and experience with holographic anomalies tells us that it should be interpreted as a quantum anomaly in a dual quantum description [179, 180, 181, 182]. The Virasoro central charge indeed has this interpretation in 2D CFTs, where it appears as an anomaly in the CFT stress tensor [225].

The interpretation of the central charge as an anomaly may help explain why computations involving the Cardy formula do such a good job of capturing the black hole entropy. It is somewhat surprising that a set of Virasoro symmetry generators appear for Killing horizons of arbitrary dimension, when standard holographic reasoning would suggest that a higher dimensional CFT should appear for higher dimensional black holes. It is also surprising that seemingly disparate symmetry algebras, including BMS_3 [144, 148, 217], Virasoro-Kač-Moody [150], Heisenberg [147], or just a single copy of Virasoro [143, 166], all seem to reproduce the black hole entropy when a Cardy-like formula is available, even though each of these symmetries would coincide with physically different quantum theories. Some insight into this situation comes from recalling that the Cardy formula is derived using the anomalous tranformation of the stress tensor when performing a change in conformal frame from the plane to the cylinder [8, 167]. The conformal anomaly determines the vacuum expectation value of the stress tensor, which is attributed to a Casimir energy associated with putting the theory on a cylinder. Modular invariance then relates this vacuum energy to the high temperature density of states, from which one arrives at the Cardy formula for a CFT. The central charge appears in this formula in its capacity as an anomaly coefficient, and it may be that this conformal anomaly controls the density of states in more general contexts when an exact 2D CFT description is not valid.[18] In such a scenario, the extension in the quasilocal algebra would continue to characterize the rescaling anomaly, and one might hope that a suitable generalization of the Cardy formula would still reproduce the black hole entropy. Note, however, that modular invariance is a crucial input in the derivation of the Cardy formula, and hence it should play an important role in arriving at the correct entropy.

Barnich-Troessaert bracket and Dirichlet matching

The Barnich-Troessaert bracket given in (4.2.28) played an important role in defining the algebra satisfied by the quasilocal charges. As of yet, however, there is no derivation of this bracket from first principles. The main technical problem is in coming up with an object which replaces the Poisson bracket when dealing with an open subsystem, which can lose symplectic flux through a boundary. There has been some work addressing this problem for general phase spaces with boundaries [227, 228, 229, 230], but it remains to be seen exactly the connection between these works and the present context of quasilocal charges in

[18] For example, a version of the Cardy formula for higher-dimensional CFTs was derived in [226].

gravity. The heuristic derivation of the bracket in section 4.2 describes how it might arise by including an auxiliary system which collects the lost symplectic flux, but it would clearly be interesting to carry out such a construction in full detail.

A step toward deriving the Barnich-Troessaert bracket was taken by Troessaert in [207], who interpreted the quasilocal symmetry transformations in terms of a family of phase spaces parameterized by a set of boundary sources. These boundary sources are simply the values taken by the fields appearing in the flux. For the Dirichlet form of the flux the, intrinsic metric q_{ij} and null generator l^i constitute the sources. This interpretation is inspired by holography, where the holographic dictionary relates boundary values of the fields to sources in the dual CFT, and their conjugate momenta to expectation values of the sourced operators [231, 232]. In this case, the momenta π^{ij} and π_i from equations (4.3.29) and (4.3.30) should have the interpretation of the holographic stress tensor for the null boundary, similar to the Brown-York stress tensor on the timelike boundary in standard examples of AdS/CFT [180]. Dirichlet conditions also play an important role in holography, since other boundary conditions can lead to conformal field theories with fluctuating sources or metrics, whose interpretation as a well-defined theory is less clear [215]. Troessaert describes the quasilocal symmetries as "external symplectic symmetries," which are transformations that act on the boundary sources as well as the dynamical fields, and demonstrates that the Barnich-Troessaert bracket arises in a natural way on this enlarged phase space. External symplectic symmetries have also appeared in the context of asymptotically flat spaces, where superrotations have been shown to be of this character [233].

The interpretation of the Barnich-Troessaert bracket in terms of an enlarged phase space decomposed into smaller phase spaces of fixed Dirichlet field values is similar to the description of fixed area states in holography [234, 235]. Specifically, in the latter construction, a bulk Cauchy slice is split across the Ryu-Takayanagi (RT) surface [4, 236], and a general state in the gravitational Hilbert space is decomposed into superselection sectors corresponding to area eigenstates of the RT surface, each of which classically corresponds to a fixed Dirichlet boundary condition (albeit for a codimension-two boundary as opposed to a codimension-one boundary). This description in terms of fixed area states was important for reproducing the correct Renyi spectrum of holographic states. The analogue of the external symplectic transformations are operators that belong to neither the algebra of the entanglement wedge nor its complement. In other words, such transformations would not preserve the center. Fixed area states appeared earlier in a slightly different context in [176], where it was argued that the Bekenstein-Hawking entropy arises from summing over all fixed area configurations of a black hole in Euclidean gravity. Therefore, it might not be all that coincidental that we needed to fix the Dirichlet form in the symplectic potential in order to reproduce the Bekenstein-Hawking entropy from the Cardy formula; investigating the connection between the present work and these other works would be an interesting next step.

Ultimately, the Barnich-Troessaert bracket should arise from a Poisson bracket on a larger phase space, consisting of a subregion and its complement. When gluing together the two subregion phase spaces to construct the global phase space, each choice for the form of the flux $\mathcal{E}$ corresponds to a specific matching of the boundary variables. As discussed in section

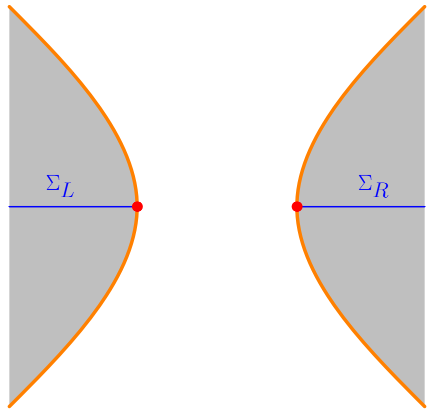
4.2, the Dirichlet flux is used to kinematically match the metric on the dividing surface, while the discontinuity in momenta π^{ij} and π_i are dynamically set equal to the boundary stress energy by the combined variational principle for the subregion and its complement, yielding a version of the junction conditions for general relativity [204, 205, 237]. Matching the intrinsic data is preferred over matching the momenta, since jumps in intrinsic data lead to distributionally ill-defined curvatures, which we expect to be excluded from the gravitational path integral. In a complete derivation of the Barnich-Troessaert bracket, we therefore expect the Dirichlet flux condition to play an important role.

Edge modes and the factor of 2

A surprising result of this work is the appearance of the additional factor of 2 in the central charges (4.4.50), (4.4.56) and entropy (4.5.5) when using the Dirichlet flux condition to define the quasilocal charges. This hints at the existence of a pair of CFTs describing the degrees of freedom near the horizon. The gluing picture described in section 4.6 supports this interpretation, since in such a description, one would naturally construct a pair of quasilocal charge algebras before combining them into a global phase space. Once this procedure is carried out, it may be that the Barnich-Troessaert bracket computes the algebra associated with the global Virasoro charges of the two CFTs combined, which would lead to a central charge that is twice the value associated with the single CFT on one side. The canonical Cardy formula then returns the total entropy assuming the CFT is in a global thermal state, but if we are interested in the entropy associated only with degrees of freedom outside of the horizon, we would first have to trace out the additional interior degrees of freedom. This would have the effect of halving the value of the entropy obtained, which leads to the correct entropy formula, $S = \frac{A}{4G}$.

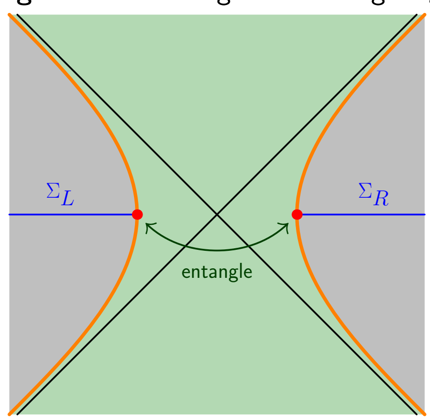
A contrasting setup was analyzed in sections 4.5 and 4.5, in which the quasilocal charges were specialized to integrable ones. This required a different boundary term that resulted in central charges and an entropy that were both half the values obtained using the Dirichlet flux, and hence correctly gave the horizon entropy. Integrability of the charges allows the subregion to be viewed as a closed system, in which case the central charge we compute would have to be associated with only a single CFT. A further consistency check in this case was agreement with the microcanonical Cardy formula, which holds since the system is isolated. The interpretation of the Dirichlet matching condition then seems to be that it necessarily entails a description in terms of an open system, and the Barnich-Troessaert bracket computes the total central charge associated with both sets of quasilocal charges. On the other hand, the boundary term necessary for integrable charges seems to be associated with one-sided generators, which, at least for the special choice of parameters given in equation (4.5.11), do not require a gluing construction. Of course, it may be that there is some other justification for using the alternative boundary term over the Dirichlet one in a gluing construction, and it would be interesting to explore this possibility further.

This picture in terms of a pair of CFTs arises naturally when interpreting the horizon entropy as an entanglement entropy. In a theory with gauge symmetry such as general



(a)

Figure 4.3: Subregions before gluing



(a)

Figure 4.4: Connected geometry after gluing

Figure 4.5: Depiction of the gluing procedure. In (4.3) we show two disconnected subregions, bounded by timelike stretched horizons in orange. The boundaries of the respective Cauchy surfaces Σ_L and Σ_R are given by the red dots. In (4.4), we imagine gluing the subregions by entangling the edge modes on $\partial\Sigma_L$ with those on $\partial\Sigma_R$. This entanglement should build up the geometry of the intervening space. For the nonextremal horizons considered in this paper, the stretched horizons can approach the bifurcate null horizon, and the gluing occurs accross the bifurcation surface, with the entanglement building up the geometry of the interior.

relativity, the quantum mechanical Hilbert space does not factorize into a tensor product associated with a subregion and its complement. However, one can form an extended Hilbert space [238] that does factorize by introducing additional edge mode degrees of freedom on the boundary which are acted on by a quasilocal charge algebra closely related to the ones considering in the present work [163, 146]. The physical Hilbert space is then identified with a particular subspace of the extended Hilbert space, which is constructed in a way analogous to the gluing construction described above. This gluing procedure produces entanglement between the edge modes, which ultimately contributes to the entropy of the state [238], and in some cases can be the dominant contribution.

In the context of this work, since the quasilocal symmetries contain a Virasoro algebra, we expect each set of edge modes to be described in terms of a CFT. In order to apply the Cardy formula, this CFT must be modular-invariant, which is an additional assumption beyond requiring that the edge mode theory furnish a representation of the Virasoro algebra. In fact, if one worked with an irreducible representation of Virasoro, the density of states would grow like a CFT with central charge $c = 1$, which is clearly insufficient to reproduce the horizon entropy.[19] A possible way to view the effect of modular invariance on the edge mode description is to think of it as an additional symmetry that acts on the edge modes, which then implies additional degeneracy of the states when the edge mode theory is quantized. This additional dengeneracy coming from modular invariance appears to be important for arriving at the correct value of the Bekenstein-Hawking entropy.

The gluing procedure for the edge modes should entangle the pair of CFTs at the boundaries into something like a thermofield double state. This creates a picture that is quite familiar from holography, where entanglement between a pair of CFTs builds up a connected black hole geometry in the bulk [239, 5, 240] (see figure 4.5). The difference when working on the horizon is that when gluing at the bifurcation surface, the two sets of edge modes are coincident, as opposed to being spatially separated by the AdS interior. Nevertheless, one might attribute the smooth region to the future of the bifurcation surface as arising from the edge mode entanglement, similar to how smooth bulk geometries arise from entanglement in holography. If one instead worked on the stretched horizons, there would be a small spatial region between the gluing surfaces which could be thought of as built up from edge mode entanglement.

In a limit where the horizon approaches extremality with $\kappa \to 0$, the stretched horizon picture begins to look like standard derivations of holographic dualities [241, 242]. The additional ingredient in AdS/CFT is the appearance of a long AdS throat, separating the stretched horizon from what would have been a bifurcation surface, were it not infinitely far away. Associated with this throat is the existence of a decoupling limit between modes deep within the throat and excitations in the distant asymptotically flat region, which allows the CFT dual to the AdS throat to be treated as a closed system. This decoupling limit is not available for the nondegenerate horizons considered in this paper, and the CFT associated with the quasilocal charges must be thought of as interacting with degrees of freedom in the exterior. The need to employ the Wald-Zoupas procedure due to the presence of fluxes can be viewed as an indication of this lack of decoupling.[20] Although nonstandard in traditional treatments of AdS/CFT, recent works on black hole evaporation in holography have employed

[19] We thank Alex Maloney for discussions on this point.

[20] Note also that since we are considering a CFT coupled to an auxiliary system, it is not immediately clear that the standard Cardy formula still applies. It may turn out that this formula is corrected due to the interactions, and this could yield an alternative resolution of the factor of 2 issue. We thank Tom Hartman for this suggestion .

a similar setup, where the standard Dirichlet boundary conditions in AdS are relaxed to allow fluxes of Hawking radiation to escape into an auxiliary asymptotically flat region [243, 244]. Time translation in such a setup should then be viewed as an external symplectic symmetry of the AdS subregion, and the definitions of energy and the boundary symmetry algebra would require the Wald-Zoupas procedure and the Barnich-Troessaert bracket. Understanding the quasilocal symmetry algebras of horizons may therefore provide additional insights into the black hole evaporation process and information paradox.

Future work

This work raises a number of questions that motivate further investigation. Foremost amongst these is the interpretation of the Barnich-Troessaert bracket and its relation to the gluing of subregions. Deriving the bracket from a gluing construction would make progress towards confirming the conjectured origin of the factor of 2 appearing in the central charge with Dirichlet flux matching. Beyond that, the gluing construction would demonstrate a way to describe a localized subregion in gravity, from which one could ask additional questions about local gravitational observables. On the quantum side, this gluing procedure gives a way to embed the global gauge-invariant Hilbert space of the theory into an extended Hilbert space, and allows notions of entanglement entropy for a subregion to be defined. It should also have a description in terms of the sewing of path integrals [176, 245, 246], which may also lead to further justifications of the Dirichlet matching condition.

Although the main application of this work was an analysis of the Virasoro vector fields for Killing horizons, the general formalism we developed is much more broadly applicable. In particular, the expression (4.2.32) for the central extension in terms of the anomalous transformation of the boundary term in the action applies quite generally, and hence can be utilized for a variety of symmetry algebras and types of hypersurfaces. One interesting application would be to investigate the various extended symmetry algebras that have been proposed for asymptotically flat space with these methods [151, 152, 247, 154, 248]. In particular, there may be some connection between the null boundary stress tensor we found in this paper and the celestial stress tensor found for 4D asymptotically flat spaces in [249], although we expect that suitable counterterms to regulate this expression will be needed [200, 202]. It would also be interesting to explore the relation between these boundary terms and fluxes and the recent work on effective actions for superrotation modes [250].

More generally, one could look at symmetry algebras associated with arbitrary null surfaces [119, 158], and analyze the extensions that appear using the Dirichlet flux condition. One intriguing aspect of some of these symmetry algebras is that they include factors of Diff(S^2), which is known to have no nontrivial central extensions. However, the Barnich-Troessaert bracket generically produces abelian extensions, which do exist for Diff(S^2). It would be interesting to see if these extensions have any connection to anomalies in a putative quantum description, and whether one can find a Cardy-like formula related to the abelian extensions.

In [119] a BMS-like algebra was found on arbitrary null surfaces, which can be written as a semidirect sum $\mathrm{diff}(S^2) \ltimes \mathfrak{s}$, where $\mathfrak{s}$ consists of the generators of the form $\xi^a = f l^a$. As discussed in section 4.3, [119] employed the boundary condition $\delta k = 0$, which constrains the function f to satisfy $\pounds_l(\pounds_l + k)f = 0$, so these generators form a pointwise $\mathbb{R} \ltimes \mathbb{R}$ subalgebra corresponding to position-dependent translations and boosts along the null surface, the former of which correspond to supertranslations. We can readily see from our general expression (4.2.32) along with the choice of boundary term (4.3.27) on a null surface that the $\delta k = 0$ boundary condition makes the central charge trivially vanish. As explained in [119], if we lift the $\delta k = 0$ condition, then the only modification to the algebra is that now f can be any function on the null surface; such vector fields were considered for example in [158]. In particular, if we consider two generators $\xi^a = f l^a$ and $\tilde{\xi}^a = \tilde{f} l^a$, the extension $K_{\xi,\tilde{\xi}}$ computed from (4.2.32) will be nonzero for an arbitary null surface. A step towards understanding the universality of the Bekenstein-Hawking entropy from the Cardy formula would therefore entail a better understanding of this enlargement of the $\mathbb{R} \ltimes \mathbb{R}$ subalgebra and the resulting abelian extension.

The Wald-Zoupas construction we described in this work required the symmetry generators to be tangent to a hypersurface that bounds the subregion of interest. However, diffeomorphisms which move the bounding hypersurface should also possess quasilocal charges. Treating such transformations would require additional analysis of the decomposition of the symplectic potential at the null surface, and a characterization of the noncovariances that can arise from such transformations, but in principle a similar set of techniques should allow quasilocal charges to be defined for these surface deformations. Carrying this out in detail would be a useful next step.

Another generalization would be to investigate higher curvature theories using the Wald-Zoupas procedure. We anticipate this being more challenging due to the presence of higher time derivatives in the action. In particular, we should not expect the Dirichlet flux condition to be available in general, with the exception of Lovelock theories, for which the null boundary terms corresponding to Dirichlet conditions are known [251]. Determining a suitable generalization of that condition would be the main obstacle one would need to overcome. The analysis of [252] on near horizon symmetries of extremal black holes in higher curvature theories may give some insights into this problem.

Finally, an open question related to the Virasoro symmetry generators considered in [149] is with regards to their geometrical significance. In the extremal limit, the generators become symmetries of a warped AdS_3 throat [169, 170], but away from extremality their interpretation is less clear. In [168], the parameters α and $\overline{\alpha}$ were determined by a hidden conformal symmetry of the scalar wave equation in the near-horizon region. Determining how this symmetry relates to preservation of the tensor C_{ab} defined in (4.4.21) would lead to further insights on the relation between the near-horizon Virasoro generators and null boundary data.

Chapter 5

Asymptotic Charges Cannot be Measured in Finite Time

5.1 Communication Without Energy?

Alice would like to send Bob a message. Alice lives on a small, massive planet. Bob occupies a Dyson sphere of large radius r_B and negligible mass, which surrounds Alice in an otherwise empty, asymptotically flat spacetime (see Fig. 5.1). It would be simplest for Alice to send Bob a radio signal, or some gravitational waves. Unfortunately, their sleep schedules are out of sync, so that Bob would not be awake when Alice's signal arrives. Instead, they come up with an ingenious protocol, which makes it unnecessary for Bob to intercept any signal from Alice.

Their protocol is as follows. Long ago, before Bob traveled to the Dyson sphere, Alice told Bob the mass M_0 of her planet. She promised not to radiate any of it away until the agreed time when the message is to be sent. That fateful night, she radiates away a certain portion of the mass of her planet. The radiation passes through Bob's sphere while he sleeps, without interacting, and is lost forever.

But when Bob wakes up, he measures the new Bondi mass M of Alice's planet. This can be done at arbitrary distance, by measuring the surface integral that defines the Bondi mass (see Eqs. (5.1.1) and (5.1.2) below).

Alice and Bob have agreed on a code, whereby the possible values of M are binned into discrete intervals, and each interval means a particular message. For example, suppose that Alice's planet has initial mass $M_0 = 10^{24}$ kg, and Bob is able to measure the final Bondi mass M to a resolution of 1 kg. Then Alice can choose from among 10^{24} messages. Upon measuring M, Bob gains an amount $\log 10^{24}$ of information, or about 80 bits.

Alice and Bob believe that their scheme will work, given a sufficiently long but fixed, finite retarded time δu for Bob to perform measurements after he wakes up, no matter how big the Dyson sphere is. That is, it should succeed in the limit as $r_B \to \infty$ at fixed retarded time $u \equiv t - r$ and fixed δu (see Fig. 5.1).

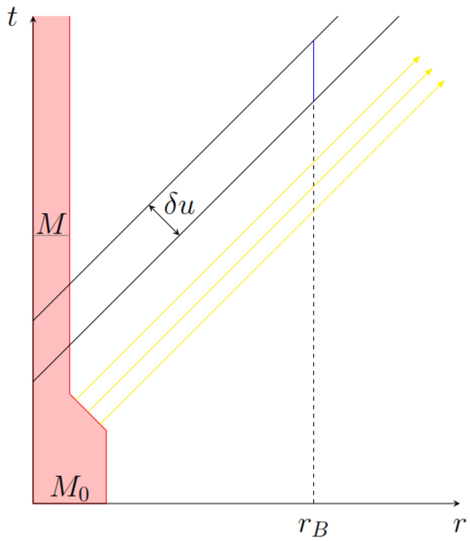


Figure 5.1: If distant observer Bob could measure the Bondi mass of Alice's planet, then Bob could receive information from Alice, without receiving energy. This would contradict recently proven bounds on distant communication channel capacities. In our example, Alice has radiated away some portion of her planet, but Bob does not intercept this radiation (yellow arrows). Instead, Bob later tries to measure how much mass is still left, in some fixed amount of time δu, at arbitrarily large radius r_B. We resolve the contradiction by showing that quantum fluctuations ruin Bob's measurement. The Bondi mass cannot be observed in finite time.

The restriction to fixed u and δu at arbitrarily large r_B is very important to Bob, because he likes to finish all his work before his mid-afternoon nap. It is also important to many theorists, who wish to associate a Bondi mass (and other charges) to a "cut," or cross-section, of future null infinity $\mathscr{I}^+$, which lies at infinite r and is parametrized by u. Of course, no measurement can be performed truly instantaneously, so Bob instead pursues the more modest goal of measuring the Bondi mass in some finite retarded time interval of length δu.

The formal definition of the Bondi mass is associated with a constant-u cut of future null infinity, $\mathscr{I}^+$ (see Fig. 5.2). To make contact with this definition, we consider the limit of a very large Dyson sphere, $r_B \to \infty$, at fixed retarded time u_0 in the metric

$$ds^2 = -\left(1 - \frac{2m_B}{r}\right) du^2 - 2du\, dr + r^2 d\Omega^2 + \dots \tag{5.1.1}$$

The ellipsis indicates terms subleading in $1/r$ that we will not need. Here m_B is the Bondi mass aspect. Its integral over a 2-sphere cut of $\mathcal{I}^+$ yields the Bondi mass:

$$M = \frac{1}{4\pi} \int_{S^2} d^2\Omega\; m_B \tag{5.1.2}$$

To claim that an asymptotic observer can measure the Bondi mass in finite time, is to claim that M can be determined by measurements in a distant region $\mathcal{R}$ in Fig. 5.2. Here $\mathcal{R}$ is bounded on the inside by an arbitrarily large radius r_B, and in the past and future by the lightsheets $u = u_0 \pm \frac{\delta u}{2}$.

However, if this protocol succeeded, we would have a paradox. Building on universal entropy bounds [253, 254, 255, 14, 256, 257], it was recently shown that communication from Alice to Bob is constrained by a universal limit on the mutual information that can be achieved [258].

In the limit as $r_B \to \infty$, the amount of information that can be gained by Bob is of order $E\delta u$, where E is the average energy of the signal that is actually received by his detectors. More precisely, the entropy in the detection region is bounded by the modular energy K in the interval δu:

$$K = \int d^2\Omega \int_{u_1(\Omega)}^{u_2(\Omega)} du\, g(u)\, \mathcal{T}(u,\Omega) \ . \tag{5.1.3}$$

Here Ω is the angle on the sphere at $\mathscr{I}^+$; $\mathcal{T} = \lim r^2 T_{uu}$ is the energy flux arriving on $\mathscr{I}^+$ per unit angle and unit retarded time; and $g(u)$ is a positive definite function. (For a free field, $g(u) = \frac{(u_2-u)(u-u_1)}{u_2-u_1}$.) But K vanishes because $\mathcal{T}$ vanishes: Bob receives no energy at all. He missed the radiation Alice sent earlier, and by the time he measures the mass or charge, there is no radiative flux at all. The entropy is closely related to the Holevo quantity [258], which bounds the mutual information between Alice and Bob. Hence, Bob cannot learn anything from Alice in this protocol.

In light of this contradiction, it is natural to go back and ask where the troublesome bound on communication [258] came from. It was obtained [256, 257] as a limit of the "Quantum Bousso bound," which was proven for free field theories in [255] and for interacting theories in [14]. Ultimately, this entropy bound arose from the conjecture [259, 260] that the entropy in a region is bounded by the cross-sectional area loss along a *lightsheet* traversing the region, measured in Planck units. Here, the lightsheet is a family of parallel light-rays that pass through the asymptotic region. Radiation will focus such light-rays, and the area they span will contract by an amount that remains fixed in Planck units, as the location of the family is taken to infinite distance. The curvature due to the Schwarzschild metric of Alice's planet will also focus the light-rays (through a shear term), but it is easy to check that the resulting area loss goes to zero as the lightsheet is taken off to null infinity.

Thus, Alice and Bob's protocol must fail: it cannot be possible to extract information by measuring a conserved charge in fixed finite time at arbitrarily large distance. In this paper, we will show how it fails. We find that, in the limit as $r_B \to \infty$ at fixed δu, quantum fluctuations dominate and prevent Bob from measuring the conserved charge.[1]

This does not mean, of course, that it is impossible to measure a conserved charge at great distances. It just cannot be done in fixed finite time. As long as the duration of the measurement scales as an appropriate positive power of r, it is possible to determine the charge. But then the measurement cannot be associated with a finite neighborhood of a cut

[1]Astronomical determinations of mass are performed in the opposite limit, $\delta u \gg r_B$, and so are unconstrained by our analysis. For example, the mass of the Sun can be found by measuring the period of Earth and applying Kepler's Third Law. In such an experiment one has $r_B = 1$ A.U. ≈ 8 min $\ll \delta u \sim 1$ year.

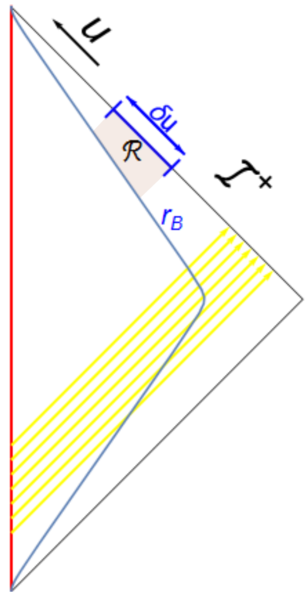


Figure 5.2: Penrose diagram of the process we consider. The red line represents Alice's worldline. The yellow arrows are the radiation emitted by Alice and reaching $\mathscr{I}^+$ without interacting with Bob (blue worldline) whose detectors are only on for a retarded time interval δu.

at future null infinity. Rather, the support of any successful measurement must approach (at least) a semi-infinite region of $\mathscr{I}^+$ in the large r limit. Similar comments apply to charges defined at spatial infinity, such as the ADM mass. They are defined by taking $r \to \infty$ at fixed t rather than fixed u. Again the duration of the measurement must scale as a positive power of r to control fluctuations.

Outline In Sec. 5.2 we begin with warm-up problem: we consider charge fluctuations near future null infinity in massless QED. We turn to the gravitational case in Sec. 5.3. An appendix contains details of our calculations.

5.2 Bondi electric charge

In standard QED, the charged particles are massive. Here we consider massless QED, as a closer analogue to the above thought-experiment where Alice uses a massless field (gravitons) to radiate away part of her planet's mass. Translated to the setting of massless QED, the paradox outlined above persists: Alice's planet now starts out with some nonzero charge Q_0, and Alice reduces this charge to Q by emitting massless charged particles. The charged radiation crosses Bob's sphere while he sleeps, so when he later attempts to determine Q, he does so by measuring the radial electric field E_r integrated over his Dyson sphere, and applying Gauss's law:

$$Q = r_B^2 \oint E_r(\Omega) d^2\Omega \ , \tag{5.2.1}$$

where Ω is the solid angle on the sphere.

The fluctuation of the electric charge in some region, $\langle Q^2 \rangle$, can be computed by integrating the two-point function of the timelike component of the current density, $\langle j^0(x) j^0(y) \rangle$. Note that Bob does not attempt to measure Q by integration of a charge density over a volume. Bob has access only to an asymptotic region, so naturally he would try to measure Q by integrating the radial electric field over the boundary of the volume. But by Gauss's law, this is the same operator. Here we find it easier to evaluate its fluctuations using the volume form of the operator.

In any CFT, the two-point function is fixed by conformal invariance. In flat space the $U(1)$ current two-point function just takes the form [261],

$$\langle j^0(x) j^0(y) \rangle = \kappa \frac{|\vec{\Delta}|^2 + (\Delta^0)^2}{\Delta^8} \ , \tag{5.2.2}$$

where $\Delta = x - y$, and the constant κ is theory dependent. For massless Dirac fermions, the current and the propagator are given by [262]

$$j^\mu = \overline{\psi} \gamma^\mu \psi \ , \tag{5.2.3}$$

$$\langle \overline{\psi}(x) \psi(y) \rangle = -\frac{i}{2\pi^2} \frac{\gamma_\mu (x^\mu - y^\mu)}{(x-y)^4} \ , \tag{5.2.4}$$

which leads to $\kappa_{(\frac{1}{2})} = -\frac{1}{\pi^4}$. For comparison, in massless scalar QED one has[2]

$$j^\mu = i \left(\phi \partial^\mu \phi^* - \phi^* \partial^\mu \phi \right) \ , \tag{5.2.5}$$

$$\langle \phi^*(x) \phi(y) \rangle = \frac{1}{4\pi^2 (x-y)^2}, \tag{5.2.6}$$

which gives $\kappa_{(0)} = -\frac{1}{4\pi^4}$.

In the 2-point functions (5.2.4) and (5.2.6), an $i\epsilon$ prescription must be specified. The choice

$$\Delta^0 \to \Delta^0 - i\epsilon \tag{5.2.7}$$

allows for only non-negative energy states in the spectrum. In the complex Δ^0-plane this corresponds to a contour prescription that cuts above both poles in Eq. (5.2.2). In what follows, this prescription will be implicit.

[2]This is the leading order result. Scalar QED is not really scale-invariant, due to the nontrivial renormalization group flow of the couplings. Unlike a massless fermion field, ϕ can gain a mass by renormalization. Even if one tunes the field to be massless, there will still be a logarithmic screening of the QED coupling constant as we flow to the IR. However, since we find a power law divergence for $\langle Q^2 \rangle$ at leading order, it does not seem possible that this divergence can be removed by a logarithmic effect. Thus we expect our qualitative conclusions to be the same for massless scalar QED, as for the fermion.

The total charge inside a spatial region V at the time t_B of Bob's measurement is

$$Q[V] = \int_V d^3x\, j^0(x) \ ; \tag{5.2.8}$$

but as an operator this would have divergent fluctuations. To obtain a well-defined operator, we smear over a finite time,

$$Q = \int dt\, Q[V(t)]\, w(t) \ . \tag{5.2.9}$$

The weight function $w(t)$ is normalized so that $\int_{-\infty}^{\infty} w(t)dt = 1$. It should peak in a finite time interval of characteristic size δt, centered on t_B; and it should fall off rapidly outside this interval. Our choice

$$w(t) = \frac{\delta t}{\pi} \frac{1}{(t-t_B)^2 + \delta t^2} \ , \tag{5.2.10}$$

facilitates the application of contour integration methods. Any other choice with a fast enough fall off should lead to the same qualitative behavior.

For $V(t)$, we must choose the volume enclosed by Bob's Dyson sphere, which is a round ball centered at the origin. Because its radius is much greater than the expected support of the charge (Alice's planet), $\langle Q \rangle$ will not depend on its precise choice. Thus we can allow for a time-dependent radius, for example as

$$r(t) = r_B + \alpha(t - t_B) \ . \tag{5.2.11}$$

Physically, this corresponds to the freedom to let Bob's Dyson sphere expand or contract during the measurement.[3] This turns out to give Bob more freedom to suppress fluctuations, but nevertheless we will find that they diverge.

We are interested in the limit as Bob's radius goes to infinity along a lightcone, $r_B = t_B + u_B \to \infty$, so that Q becomes the Bondi charge. By an overall time shift, we may set the fixed retarded time of Bob's measurement to zero, $u_B = 0$. We can then fix the retarded time duration of Bob's measurement, as the interval $-\frac{\delta u}{2} < u < \frac{\delta u}{2}$. That is, the weight function (5.2.10) should have support when Bob's world tube (5.2.11) lies in this interval, but not outside it. To this end we choose

$$\delta t = \frac{\delta u}{1-\alpha}. \tag{5.2.12}$$

Note that the proper time duration of Bob's measurement is then given by

$$\delta\tau = \delta u \sqrt{\frac{1+\alpha}{1-\alpha}}. \tag{5.2.13}$$

[3]One might worry that $r(t)$ is negative for $t < t_B - \frac{r_B}{\alpha}$. However, since this happens only at the tail of the weight function $w(t)$ (Eq. (5.2.10)), it does not affect our results. For example, the choice $r(t) = r_B(1 - \alpha \tanh(r_B)) + \alpha t \tanh(t)$, which has the same behavior as Eq. (5.2.11) at large t and is nowhere negative, leads to the same asymptotic behavior.

Intuitively, we might expect that fluctuations will be more suppressed for greater $\delta\tau$, i.e., for Bob's sphere expanding at great velocity, $\alpha \to 1$. However, as we shall see this is not sufficient to control the fluctuations as $r_B \to \infty$.

To evaluate $\langle Q^2 \rangle$, we now write it as

$$\langle Q^2 \rangle = \int d^d x \int d^d \Delta \, w(x^0) w(y^0) \theta(\vec{x}) \theta(\vec{x} - \vec{\Delta}) \langle j^0(0) j^0(\Delta) \rangle, \tag{5.2.14}$$

where $\theta = 1$ inside the volume V and $\theta = 0$ outside.

Here we summarize how this calculation goes. More details can be found in the Appendix. The integral over $d^3\vec{x}$ yields the volume of the intersection of two balls separated by $|\vec{\Delta}|$. By spherical symmetry, the integral over $d^3\vec{\Delta}$ reduces to a one-dimensional integral which we evaluate. We subsequently perform the dx^0 and $d\Delta^0$ integrations using contour methods. Here one has to be careful to choose a contour that properly avoids branch cuts. This yields an expression for $\langle Q^2 \rangle$ as a function of $r_B, \delta t$, and thus via Eq. (5.2.12), of $r_B, \delta u, \alpha$.

$$\begin{aligned} \langle Q^2 \rangle &= -\kappa \left(\pi^2 \frac{(1-\alpha)^3 r_B^2}{3(\alpha+1)\delta u^2} + \frac{\pi^2}{6} \log \left(\frac{4(1-\alpha)^3 r_B^2}{(\alpha+1)\delta u^2} \right) \right) \\ &- \frac{\kappa \pi^2}{12(\alpha^2 - 1)} + O\left(r_B^{-1}\right) \end{aligned} \tag{5.2.15}$$

We can now take the limit $r_B \to \infty$. For $\alpha = 0$, we find an expected area law divergence. For other choices of α, it is possible to have $\langle Q^2 \rangle$ diverge slower than that. To accomplish the goal of making $\langle Q^2 \rangle$ grow as slow as possible with r_B, the optimal choice of α satisfies

$$1 - \alpha^{\text{opt}} \propto \sqrt{\frac{\delta u}{r_B}}, \tag{5.2.16}$$

No choice of α can make $\langle Q^2 \rangle$ diverge slower than that, and in particular, no choice of α can make the charge fluctuations finite when $r_B \to \infty$. For the optimal choice above, the divergence goes as the fourth-root of the area,

$$\langle Q^2 \rangle^{\text{opt}} \sim \sqrt{\frac{r_B}{\delta u}} \, . \tag{5.2.17}$$

The results above are for four dimensional Minkowski space, but the same analysis can be performed in any dimension (though we have only been able to get analytic results in even dimensions). Here we quote the results in two[4] and six dimensions:

$$\langle Q^2 \rangle_{D=2} \propto \log \left(\frac{\left(\delta u^2 + (1-\alpha)^4 {r_B}^2\right)^2}{\left(1-\alpha^2\right)^2 \delta u^4} \right) \tag{5.2.18}$$

$$\langle Q^2 \rangle_{D=6} \propto \frac{(1-\alpha)^6 {r_B}^4}{(\alpha+1)^2 \delta u^4} + O\left({r_B}^2\right) \tag{5.2.19}$$

[4] Since QED is confining in 2D, one cannot give the 2D result the same interpretation as in higher dimensions.

We see that for constant α, we always get an area law $\langle Q^2\rangle_D \sim \left(\frac{r_B}{\delta u}\right)^{D-2}$. The optimal choice of α is always given by Eq. (5.2.16) for any D; this yields

$$\langle Q^2\rangle_D^{\text{opt}} \sim r_B^{(D-2)/4} \sim \delta\tau^{D-2} \ . \tag{5.2.20}$$

This divergence thwarts Bob's plans of measuring the charge and thus prevents him from receiving Alice's message. Since no information is transmitted, the apparent paradox described in the previous section is resolved.

5.3 Bondi mass

In the previous section we showed that, due to quantum fluctuations, the Bondi electric charge cannot be measured in a finite interval of $\mathscr{I}^+$. Here we repeat this analysis, but for the Bondi mass. For concreteness, we consider a massless scalar field non-minimally coupled to gravity. However, since the two point function of T_{00} is completely fixed (up to a multiplicative factor) in any scale-invariant theory with a stress-tensor, our conclusions apply equally well to spinors, gauge fields, and interacting fixed points.

The action and stress-energy tensor for a non-minimally coupled scalar are given by

$$S = -\frac{1}{2}\int d^4\sqrt{-g}\left(D_\mu\phi D^\mu\phi + \xi R\phi^2\right), \tag{5.3.1}$$

and

$$\begin{aligned} T_{\alpha\beta} &= (1-2\xi)D_\alpha\phi D_\beta\phi + \left(2\xi - \frac{1}{2}\right) D_\mu\phi D^\mu\phi g_{\alpha\beta} \\ &+ 2\xi g_{\alpha\beta}\phi D^2\phi - 2\xi\phi D_\alpha D_\beta\phi. \end{aligned} \tag{5.3.2}$$

Using this stress-energy tensor and $\langle\phi(0)\phi(\Delta)\rangle = \frac{1}{\Delta^2}$, we get

$$\langle T_{00}(x)T_{00}(y)\rangle = 8\left(30\xi^2 - 10\xi + 1\right)\frac{3\vec{\Delta}^4 + 10\Delta_0^2\vec{\Delta}^2 + 3\Delta_0^4}{\left(\Delta_0^2 - \vec{\Delta}^2\right)^6}. \tag{5.3.3}$$

Using the same smearing as in the previous section, we can now calculate the fluctuations of the energy,

$$\langle M^2\rangle = \int d^4x\int d^4\Delta\, w(x^0)w(y^0)\theta(\vec{x})\theta(\vec{x}-\vec{\Delta})\langle T_{00}(x)T_{00}(y)\rangle, \tag{5.3.4}$$

by performing the same integrals as in the QED case, the details of which are relegated to the Appendix.

As in the $U(1)$ case, we choose to evaluate the operator and its fluctuations as a volume integral, not a surface integral. This is now more subtle, because strictly the Bondi mass is defined *only* as a surface integral over a family of topological 2-spheres $\{S_\alpha\}$ that approach a cut S of null infinity [263]:

$$M = -\lim_{S_\alpha \to S} \frac{1}{8\pi} \int_{S_\alpha} \varepsilon_{abcd} \nabla^c \zeta^d \tag{5.3.5}$$

where ζ^a is an asymptotic time translation Killing vector field. Here we work in a perturbative limit, where backreaction in the bulk is small. Then an approximate Gauss law still holds, and the Bondi mass can also computed as a volume integral

$$M = \int_{\tilde{\Sigma}} d^3x \; T_{00} \tag{5.3.6}$$

over the portion $\tilde{\Sigma}$ of a Cauchy surface Σ enclosed by S. Moreover, we can reach arbitrarily large M even in the perturbative regime, by considering matter of low density spread over a large region. Hence we expect that our result for the fluctuations of M will be general.

We find

$$\begin{aligned} \langle M^2 \rangle \;\; &= \;\; 8\left(30\xi^2 - 10\xi + 1\right)\pi^2 \left(\alpha^2 \delta u^2 + 4(1-\alpha)^4 {r_B}^2\right)^3 \\ &\times \;\; \left((1-\alpha)^4\left(3\alpha^2+1\right){r_B}^2 - \left(\alpha^2 - 5\right)\frac{\delta u^2}{4}\right) \\ &\times \;\; \left(15(1-\alpha)(\alpha+1)^3 \delta u^4 \left(\delta u^2 + 4(1-\alpha)^4 {r_B}^2\right)^3\right)^{-1} \end{aligned} \tag{5.3.7}$$

For $\alpha = 0$ this gives

$$\langle M^2 \rangle = 8\left(30\xi^2 - 10\xi + 1\right) \frac{16\pi^2 r_B^6 \left(5\delta u^2 + 4r_B^2\right)}{15\delta u^4 \left(\delta u^2 + 4r_B^2\right)^3}. \tag{5.3.8}$$

Once again, it is possible to tame this divergence by a better choice of α. The optimal value remains $\alpha^{\text{opt}} \propto 1 - \left(\frac{r_B}{\delta u}\right)^{-1/2}$, which gives

$$\langle M^2 \rangle^{\text{opt}} = \frac{\left(30\xi^2 - 10\xi + 1\right) 2^{5/2} \pi^2}{30 \delta u^{5/2}} \sqrt{r_B} + O\left(\frac{1}{r_B^{1/2}}\right). \tag{5.3.9}$$

We therefore see that the Bondi energy also has unbounded fluctuations as we approach finite intervals of null infinity.

5.4 Discussion

We argued that entropy bounds preclude gauge charges from being well-defined quantum observables on cuts or finite intervals of $\mathscr{I}^+$. We confirmed this by showing that unbounded fluctuations preclude a measurement of the electric charge or the Bondi mass, in finite time at arbitrarily large radius.[5]

It is important to emphasize the quantum nature of these results. Both M and Q are good classical observables near a cut of $\mathscr{I}^+$. This follows directly from Eq. (5.2.1), and from the analogous surface integral for the Bondi mass, Eq. (5.3.5). Both expressions are gauge-invariant and require no data extrinsic to the near-cut region $\mathcal{R}$ for their evaluation. This constrasts with certain other quantities appearing in the Bondi metric expansion, Eq. (5.1.1), which are prohibited by the equivalence principle from being observable already at the classical level [257].

Let us try to gain some intuition for the divergence of $\langle Q^2\rangle$ and $\langle M^2\rangle$ that we found. To understand the physical origin of the fluctuations, suppose, for simplicity, that Bob remains at fixed radius throughout his measurement, so that $\alpha = 0$ and $\delta u = \delta t = \delta\tau$. Consider Q as a surface integral over E_r, rather than a volume integral. An observation restricted to a finite time interval leads to approximately thermal quantum noise of characteristic energy $1/\delta\tau$. This noise arises in the region causally accessible to the observer; here, this would be a shell of width δt around the sphere r_B. Since $r_B \gg \delta\tau$, there will be a large number $N \sim r_B^2/\delta\tau^2$ of "cells" just inside and outside of Bob's sphere. Each cell contains $O(1)$ quanta of any massless field the detectors couple to, which includes the charges. This contributes to E_r an additional field strength of order $1/\delta\tau^2$ and random sign. The contribution to Q from one cell, in Eq. (5.2.1), is thus of order ± 1. The fluctuations in different cells are uncorrelated, so the total fluctuation of Q is given by $\langle Q^2\rangle^{1/2} \sim \sqrt{N} \sim r_B/\delta\tau$. This agrees with Eq. (5.2.15) for this special case, $\alpha = 0$.[6]

Note that neither infrared nor ultraviolet physics alone can explain the divergent fluctuations of Q and M. Rather, they arise from a combination of both. The fixed duration δu of Bob's measurement sets a characteristic "ultraviolet" energy scale for the fluctuations. The infrared effect comes from taking the limit as $r_B \to \infty$, which creates an ever larger region over which those fluctuations can contribute.

Our work lends some insight on the structure of operator algebras of gauge theories and gravity when quantizing at $\mathscr{I}^+$. We emphasize that the paradox noted in Section 5.1 would arise for any quantity associated to a subset of $\mathscr{I}^+$ that is not tied to energy flux arriving in that subset. For example, the BMS group at $\mathscr{I}^+$ yields an infinite set of supertranslation charges [98], which essentially correspond to the Bondi mass aspect (whose integral yields the Bondi mass) [266, 267, 268, 25]. We thus find that these supertranslation charges are not observable in a neighborhood of any cut of $\mathscr{I}^+$ in the quantum theory[7].

[5] The study of fluctuation of electric charge (in finite regions) dates back to the early days of QED (see e.g. [264] and [265]).

[6] It would be nice to extend this heuristic argument to the optimal case, when Bob is expanding outward during the measurement according to Eq. (5.2.16). But using Eq. (5.2.13), the above argument would appear to imply $\langle Q^2\rangle \sim r_B^2/\delta\tau^2 \sim (r_B/\delta u)^{3/2}$, in conflict with Eq. (5.2.17).

[7] We established that a certain operator $\hat{O}$ does not belong to the algebra of observables by showing that $\langle \hat{O}^2\rangle = \infty$. This is not a perfect criterion, since there are contrived examples of observables in quantum mechanics with $\langle \hat{O}^2\rangle = \infty$ but well-defined spectrum. However, we do expect all *reasonable* operators to

The absence of such observables also has potential significance for understanding the holographic principle. There has been considerable interest in trying to construct a holographic theory dual to asymptotically flat spacetimes (see [269, 270, 271] for recent examples). By analogy to AdS/CFT, one expects that such a putative holographic dual should be defined on the conformal boundary of the spacetime, and that limits of bulk observables that are defined as they approach $\mathscr{I}^+$ should correspond to local operators in the putative boundary theory. Since we have shown that conserved charges are not in fact well-defined operators on any finite portion of $\mathscr{I}^+$, we expect that no such operators should exist in a dual boundary theory either.

have finite fluctuations.

Part II

From Gravity to Quantum Field Theory

Chapter 6

The Quantum Null Energy Condition, Entanglement Wedge Nesting, and Quantum Focusing

6.1 Introduction and Summary

The Quantum Focusing Conjecture (QFC) is a new principle of semiclassical quantum gravity proposed in [13]. Its formulation is motivated by classical focusing, which states that the expansion θ of a null congruence of geodesics is nonincreasing. Classical focusing is at the heart of several important results of classical gravity [272, 273, 274, 275], and likewise quantum focusing can be used to prove quantum generalizations of many of these results [276, 277, 278, 279].

One of the most important and surprising consequences of the QFC is the Quantum Null Energy Condition (QNEC), which was discovered as a particular nongravitational limit of the QFC [13]. Subsequently the QNEC was proven for free fields [280] and for holographic CFTs on flat backgrounds [17] (and recently extended in [281] in a similar way as we do here). The formulation of the QNEC which naturally comes out of the proofs we provide here is as follows.

Consider a codimension-two Cauchy-splitting surface Σ, which we will refer to as the entangling surface. The Von Neumann entropy $S[\Sigma]$ of the interior (or exterior) of Σ is a functional of Σ, and in particular is a functional of the embedding functions $X^i(y)$ that define Σ. Choose a one-parameter family of deformed surfaces $\Sigma(\lambda)$, with $\Sigma(0) = \Sigma$, such that (i) $\Sigma(\lambda)$ is given by flowing along null geodesics generated by the null vector field k^i normal to Σ for affine time λ , and (ii) $\Sigma(\lambda)$ is either "shrinking" or "growing" as a function of λ, in the sense that the domain of dependence of the interior of Σ is either shrinking or growing. Then for any point on the entangling surface we can define the combination

$$T_{ij}(y)k^i(y)k^j(y) - \frac{1}{2\pi}\frac{d}{d\lambda}\left(\frac{k^i(y)}{\sqrt{h(y)}}\frac{\delta S_{\mathrm{ren}}}{\delta X^i(y)}\right). \tag{6.1.1}$$

Here $\sqrt{h(y)}$ is the induced metric determinant on Σ. Writing this down in a general curved background requires a renormalization scheme both for the energy-momentum tensor T_{ij} and the renormalized entropy S_{ren}. Assuming that this quantity is scheme-independent (and hence well-defined), the QNEC states that it is positive. Our main task is to determine the necessary and sufficient conditions we need to impose on Σ and the background spacetime at the point y in order that the QNEC hold.

In addition to a proof through the QFC, the holographic proof method of [17] is easily adaptable to answering this question in full generality. The backbone of that proof is Entanglement Wedge Nesting (EWN), which is a consequence of subregion duality in AdS/CFT [279]. A given region on the boundary of AdS is associated with a particular region of the bulk, called the entanglement wedge, which is defined as the bulk region spacelike-related to the extremal surface [4, 282, 283, 284] used to compute the CFT entropy on the side toward the boundary region. This bulk region is dual to the given boundary region, in the sense that there is a correspondence between the algebra of operators in the bulk region and that of the operators in the boundary region which are good semiclassical gravity operators (i.e., they act within the subspace of semiclassical states) [285, 137, 136]. EWN is the statement that nested boundary regions must be dual to nested bulk regions, and clearly follows from the consistency of subregion duality.

While the QNEC can be derived from both the QFC and EWN, there has been no clear connection between these derivations.[1] As it stands, there are apparently two QNECs, the QNEC-from-QFC and the QNEC-from-EWN. We will show in full generality that these two QNECs are in fact the same, at least in $d \leq 5$ dimensions.

Here is a summary of our results:

- The holographic proof of the QNEC from EWN is extended to CFTs on arbitrary curved backgrounds. In $d = 5$ we find that the necessary and sufficient conditions for the ordinary QNEC to hold at a point are that[2]

$$\theta_{(k)} = \sigma^{(k)}_{ab} = D_a\theta_{(k)} = D_a\sigma^{(k)}_{bc} = R_{ka} = 0 \tag{6.1.2}$$

  at that point. For $d < 5$ only a subset of these conditions are necessary. This is the subject of §6.2.

- We also show holographically that under the weaker set of conditions

$$\sigma^{(k)}_{ab} = D_a\theta_{(k)} + R_{ka} = D_a\sigma^{(k)}_{bc} = 0 \tag{6.1.3}$$

  the Conformal QNEC holds. The Conformal QNEC was introduced in [17] as a conformally-transformed version of the QNEC. This is the strongest inequality that we can get out of EWN. This is the subject of §6.2

[1] In [279] it was shown that the QFC in the bulk implies EWN, which in turn implies the QNEC. This is not the same as the connection we are referencing here. The QFC which would imply the boundary QNEC in the sense that we mean is a *boundary* QFC, obtained by coupling the boundary theory to gravity.

[2] Here $\sigma^{(k)}_{ab}$ and $\theta_{(k)}$ are the shear and expansion in the k^i direction, respectively, and D_a is a surface covariant derivative. Our notation is further explained in Appendix A.24.

- By taking the non-gravitational limit of the QFC we are able to derive the QNEC again under the same set of conditions as we did for EWN. This is the subject of §6.3.
- We argue in §6.3 that the statement of the QNEC is scheme-independent whenever the conditions that allow us to prove it hold. This shows that the two proofs of the QNEC are actually proving the same, unambiguous field–theoretic bound.

We conclude in §9.6 with a discussion and suggest future directions. A number of technical Appendices are included as part of our analysis.

Relation to other work While this work was in preparation, [281] appeared which has overlap with our discussion of EWN and the scheme-independence of the QNEC. The results of [281] relied on a number of assumptions about the background: the null curvature condition and a positive energy condition. From this they derive certain sufficient conditions for the QNEC to hold. We do not assume anything about our backgrounds a priori, and include all relevant higher curvature corrections. This gives our results greater generality, as we are able to find both necessary and sufficient conditions for the QNEC to hold.

6.2 Entanglement Wedge Nesting

Subregion Duality

The statement of AdS/CFT includes a correspondence between operators in the semiclassical bulk gravitational theory and CFT operators on the boundary. Moreover, it has been shown [286, 136] that such a correspondence exists between the operator algebras of subregions in the CFT and certain associated subregions in the bulk as follows: Consider a spatial subregion A in the boundary geometry. The extremal surface anchored to ∂A, which is used to compute the entropy of A [4, 282], bounds the so-called entanglement wedge of A, $\mathcal{E}(A)$, in the bulk. More precisely $\mathcal{E}(A)$ is the codimension-zero bulk region spacelike-related to the extremal surface on the same side of the extremal surface as A. Subregion duality is the statement that the operator algebras of $\mathcal{D}(A)$ and $\mathcal{E}(A)$ are dual, where $\mathcal{D}(A)$ denotes the domain of dependence of A.

Entanglement Wedge Nesting The results of this section follow from EWN, which we now describe. Consider two boundary regions A_1 and A_2 such that $\mathcal{D}(A_1) \subseteq \mathcal{D}(A_2)$. Then consistency of subregion duality implies that $\mathcal{E}(A_1) \subseteq \mathcal{E}(A_2)$ as well, and this is the statement of EWN. In particular, EWN implies that the extremal surfaces associated to A_1 and A_2 cannot be timelike-related.

We will mainly be applying EWN to the case of a one-paramter family of boundary regions, $A(\lambda)$, where $\mathcal{D}(A(\lambda_1)) \subseteq \mathcal{D}(A(\lambda_2))$ whenever $\lambda_1 \leq \lambda_2$. Then the union of the one-parameter family of extremal surfaces associated to $A(\lambda)$ forms a codimension-one surface

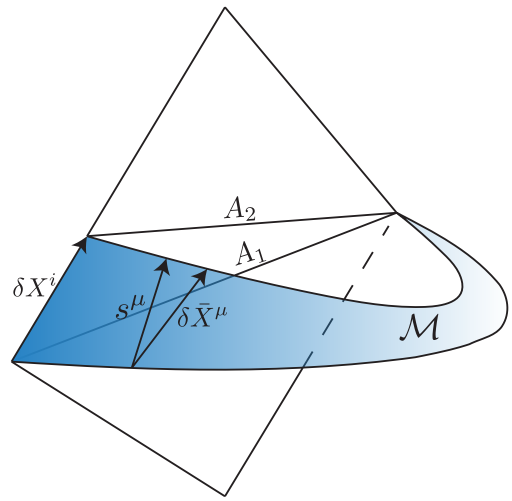


Figure 6.1: Here we show the holographic setup which illustrates Entanglement Wedge Nesting. A spatial region A_1 on the boundary is deformed into the spatial region A_2 by the null vector δX^i. The extremal surfaces of A_1 and A_2 are connected by a codimension-one bulk surface $\mathcal{M}$ (shaded blue) that is nowhere timelike by EWN. Then the vectors $\delta\overline{X}^\mu$ and s^μ, which lie in $\mathcal{M}$, have nonnegative norm.

in the bulk that is nowhere timelike. We denote this codimension-one surface by $\mathcal{M}$. See Fig. 6.1 for a picture of the setup.

Since $\mathcal{M}$ is nowhere timelike, every one of its tangent vectors must have nonnegative norm. In particular, consider the embedding functions $\overline{X}^\mu$ of the extremal surfaces in some coordinate system. Then the vectors $\delta\overline{X}^\mu \equiv \partial_\lambda \overline{X}^\mu$ is tangent to $\mathcal{M}$, and represents a vector that points from one extremal surface to another. Hence we have $(\delta\overline{X})^2 \geq 0$ from EWN, and this is the inequality that we will discuss for most of the remainder of this section.

Before moving on, we will note that $(\delta\overline{X})^2 \geq 0$ is not necessarily the strongest inequality we get from EWN. At each point on $\mathcal{M}$, the vectors which are tangent to the extremal surface passing through that point are known to be spacelike. Therefore if $\delta\overline{X}^\mu$ contains any components which are tangent to the extremal surface, they will serve to make the inequality $(\delta\overline{X})^2 \geq 0$ weaker. We define the vector s^μ at any point of $\mathcal{M}$ to be the part of $\delta\overline{X}^\mu$ orthogonal to the extremal surface passing through that point. Then $(\delta\overline{X})^2 \geq s^2 \geq 0$. We will discuss the $s^2 \geq 0$ inequality in §6.2 after handling the $(\delta\overline{X})^2 \geq 0$ case.

Near-Boundary EWN

In this section we explain how to calculate the vector $\delta\overline{X}^\mu$ and s^μ near the boundary explicitly in terms of CFT data. Then the EWN inequalities $(\delta\overline{X})^2 > 0$ and $s^2 > 0$ can be given a CFT meaning. The strategy is to use a Fefferman-Graham expansion of both the metric and extremal surface, leading to equations for $\delta\overline{X}^\mu$ and s^μ as power series in the bulk coordinate z (including possible log terms). In the following sections we will analyze the inequalities that are derived in this section.

Bulk Metric We work with a bulk theory in AdS_{d+1} that consists of Einstein gravity plus curvature-squared corrections. For $d \leq 5$ this is the complete set of higher curvature corrections that have an impact on our analysis. The Lagrangian is[3]

$$\mathcal{L} = \frac{1}{16\pi G_N}\left(\frac{d(d-1)}{\tilde{L}^2} + \mathcal{R} + \ell^2\lambda_1\mathcal{R}^2 + \ell^2\lambda_2\mathcal{R}_{\mu\nu}^2 + \ell^2\lambda_{\rm GB}\mathcal{L}_{\rm GB}\right), \tag{6.2.1}$$

where $\mathcal{L}_{GB} = \mathcal{R}_{\mu\nu\rho\sigma}^2 - 4\mathcal{R}_{\mu\nu}^2 + \mathcal{R}^2$ is the Gauss–Bonnet Lagrangian, ℓ^2 is the cutoff scale, and $\tilde{L}^2$ is the scale of the cosmological constant. The bulk metric has the following near boundary expansion in Fefferman-Graham gauge [287]:

$$ds^2 = \frac{L^2}{z^2}(dz^2 + \overline{g}_{ij}(x,z)dx^i dx^j), \tag{6.2.2}$$

$$\overline{g}_{ij}(x,z) = g_{ij}^{(0)}(x) + z^2 g_{ij}^{(2)}(x) + z^4 g_{ij}^{(4)}(x) + \ldots + z^d \log z\, g_{ij}^{(d,\log)}(x) + z^d g_{ij}^{(d)}(x) + o(z^d). \tag{6.2.3}$$

Note that the length scale L is different from $\tilde{L}$, but the relationship between them will not be important for us. Demanding that the above metric solve bulk gravitational equations of motion gives expressions for all of the $g_{ij}^{(n)}$ for $n < d$, including $g_{ij}^{(d,\log)}(x)$, in terms of $g_{ij}^{(0)}(x)$. This means, in particular, that these terms are all state-independent. One finds that $g_{ij}^{(d,\log)}(x)$ vanishes unless d is even. We provide explicit expressions for some of these terms in Appendix A.26.

The only state-dependent term we have displayed, $g_{ij}^{(d)}(x)$, contains information about the expectation value of the energy-momentum tensor T_{ij} of the field theory. In odd dimensions we have the simple formula [288][4]

$$g_{ij}^{(d=\text{odd})} = \frac{16\pi G_N}{\eta d L^{d-1}}\langle T_{ij}\rangle, \tag{6.2.4}$$

with

$$\eta = 1 - 2\left(d(d+1)\lambda_1 + d\lambda_2 + (d-2)(d-3)\lambda_{\rm GB}\right)\frac{\ell^2}{L^2} \tag{6.2.5}$$

In even dimensions the formula is more complicated. For $d = 4$ we discuss the form of the metric in Appendix A.28

Extremal Surface EWN is a statement about the causal relation between entanglement wedges. To study this, we need to calculate the position of the extremal surface. We parametrize our extremal surface by the coordinate (y^a, z), and the position of the surface

[3]For simplicity we will not include matter fields explicitly in the bulk, but their presence should not alter any of our conclusions.

[4]Even though [288] worked with a flat boundary theory, one can check that this formula remains unchanged when the boundary is curved.

is determined by the embedding functions $\overline{X}^{\mu}(y^a, z)$. The intrinsic metric of the extremal surface is denoted by $\overline{h}_{\alpha\beta}$, where $\alpha = (a, z)$. For convenience we will impose the gauge conditions $\overline{X}^{z} = z$ and $\overline{h}_{az} = 0$.

The functions $\overline{X}(y^a, z)$ are determined by extremizing the generalized entropy [283, 284] of the entanglement wedge. This generalized entropy consists of geometric terms integrated over the surface as well as bulk entropy terms. We defer a discussion of the bulk entropy terms to §6.4 and write only the geometric terms, which are determined by the bulk action:

$$S_{\text{gen}} = \frac{1}{4G_N}\int \sqrt{\overline{h}}\left[1 + 2\lambda_1 \ell^2 \mathcal{R} + \lambda_2 \ell^2 \left(\mathcal{R}_{\mu\nu}\mathcal{N}^{\mu\nu} - \frac{1}{2}\mathcal{K}_{\mu}\mathcal{K}^{\mu}\right) + 2\lambda_{\text{GB}}\ell^2 \overline{r}\right]. \tag{6.2.6}$$

We discuss this entropy functional in more detail in Appendix A.26. The Euler-Lagrange equations for S_{gen} are the equations of motion for $\overline{X}^{\mu}$. Like the bulk metric, the extremal surface equations can be solved at small-z with a Fefferman–Graham-like expansion:

$$\overline{X}^{i}(y, z) = X^i_{(0)}(y) + z^2 X^i_{(2)}(y) + z^4 X^i_{(4)}(y) + \ldots + z^d \log z\, X^i_{(d,\log)}(y) + z^d X^i_{(d)}(y) + o(z^d), \tag{6.2.7}$$

As with the metric, the coefficient functions $X^i_{(n)}$ for $n < d$, including the log term, can be solved for in terms of $X^i_{(0)}$ and $g^{(0)}_{ij}$, and again the log term vanishes unless d is even. The state-dependent term $X^i_{(d)}$ contains information about variations of the CFT entropy, as we explain below.

The z-Expansion of EWN By taking the derivative of (6.2.7) with respect to λ, we find the z-expansion of $\delta\overline{X}^{i}$. We will discuss how to take those derivatives momentarily. But given the z-expansion of $\delta\overline{X}^{i}$, we can combine this with the z-expansion of $\overline{g}_{ij}$ in (6.2.3) to get the z-expansion of $(\delta\overline{X})^2$:

$$\frac{z^2}{L^2}(\delta\overline{X})^2 = g^{(0)}_{ij}\delta X^i_{(0)}\delta X^j_{(0)} + z^2\left(2g^{(0)}_{ij}\delta X^i_{(0)}\delta X^j_{(2)} + g^{(2)}_{ij}\delta X^i_{(0)}\delta X^j_{(0)} + X^m_{(2)}\partial_m g^{(0)}_{ij}\delta X^i_{(0)}\delta X^j_{(0)}\right) + \cdots \tag{6.2.8}$$

EWN implies that $(\delta\overline{X})^2 \geq 0$, and we will spend the next few sections examining this inequality using the expansion (6.2.8). From the general arguments given above, we can get a stronger inequality by considering the vector s^{μ} and its norm rather than $\delta\overline{X}^{\mu}$. The construction of s^{μ} is more involved, but we would similarly construct an equation for s^2 at small z. We defer further discussion of s^{μ} to §6.2.

Now we return to the question of calculating $\delta\overline{X}^{i}$. Since all of the $X^i_{(n)}$ for $n < d$ are known explicitly from solving the equation of motion, the λ-derivatives of those terms can be taken and the results expressed in terms of the boundary conditions for the extremal surface. The variation of the state-dependent term, $\delta X^i_{(d)}$, is also determined by the boundary conditions in principle, but in a horribly non-local way. However, we will now show that $X^i_{(d)}$ (and hence $\delta X^i_{(d)}$) can be re-expressed in terms of variations of the CFT entropy.

Variations of the Entropy The CFT entropy S_{CFT} is equal to the generalized entropy S_{gen} of the entanglement wedge in the bulk. To be precise, we need to introduce a cutoff at $z = \epsilon$ and use holographic renormalization to properly define the entropy. Then we can use the calculus of variations to determine variations of the entropy with respect to the boundary conditions at $z = \epsilon$. There will be terms which diverge as $\epsilon \to 0$, as well as a finite term, which is the only one we are interested in at the moment. In odd dimensions, the finite term is given by a simple integral over the entangling surface in the CFT:

$$\delta S_{\mathrm{CFT}}|_{\mathrm{finite}} = \eta d L^{d-1} \int d^{d-2}y \sqrt{h} g_{ij} X^i_{(d)} \delta X^j. \tag{6.2.9}$$

This finite part of S_{CFT} is the renormalized entropy, S_{ren}, in holographic renormalization. Eventually we will want to assure ourselves that our results are scheme-independent. This question was studied in [289], and we will discuss it further in §6.3. For now, the important take-away from (6.2.9) is

$$\frac{1}{\sqrt{h}} \frac{\delta S_{\mathrm{ren}}}{\delta X^i(y)} = -\frac{\eta d L^{d-1}}{4G_N} X^i_{(d,\mathrm{odd})}. \tag{6.2.10}$$

The case of even d is more complicated, and we will cover the $d = 4$ case in Appendix A.28.

State-Independent Inequalities

The basic EWN inequality is $(\delta \overline{X})^2 \geq 0$. The challenge is to write this in terms of boundary quantities. In this section we will look at the state-independent terms in the expansion of (6.2.8). The boundary conditions at $z = 0$ are given by the CFT entangling surface and background geometry, which we denote by X^i and g_{ij} without a (0) subscript. The variation vector of the entangling surface is the null vector $k^i = \delta X^i$. We can use the formulas of Appendix A.27 to express the other $X^i_{(n)}$ for $n < d$ in terms of X^i and g_{ij}. This allows us to express the state-independent parts of $(\delta \overline{X})^2 \geq 0$ in terms of CFT data. In this subsection we will look at the leading and subleading state-independent parts. These will be sufficient to fully cover the cases $d \leq 5$.

Leading Inequality From (6.2.8), we see that the first term is actually $k_i k^i = 0$. The next term is the one we call the leading term, which is

$$L^{-2} (\delta \overline{X})^2 \Big|_{z^0} = 2k_i \delta X^i_{(2)} + g^{(2)}_{ij} k^i k^j + X^m_{(2)} \partial_m g_{ij} k^i k^j. \tag{6.2.11}$$

From (A.26.10), we easily see that this is equivalent to

$$L^{-2} \, (\delta \overline{X}^i)^2 \Big|_{z^0} = \frac{1}{(d-2)^2} \theta^2_{(k)} + \frac{1}{d-2} \sigma^2_{(k)}, \tag{6.2.12}$$

where $\sigma^{(k)}_{ab}$ and $\theta_{(k)}$ are the shear and expansion of the null congruence generated by k^i, and are given by the trace and trace-free parts of $k_i K^i_{ab}$, with K^i_{ab} the extrinsic curvature of the entangling surface. This leading inequality is always nonnegative, as required by EWN. Since we are in the small-z limit, the subleading inequality is only relevant when this leading inequality is saturated. So in our analysis below we will focus on the $\theta_{(k)} = \sigma_{ab}^{(k)} = 0$ case, which can always be achieved by choosing the entangling surface appropriately. Note that in $d = 3$ this is the only state-independent term in $(\delta\overline{X})^2$, and furthermore we always have $\sigma_{ab}^{(k)} = 0$ in $d = 3$.

Subleading Inequality The subleading term in $(\delta\overline{X})^2$ is order z^2 in $d \geq 5$, and order $z^2 \log z$ in $d = 4$. These two cases are similar, but it will be easiest to focus first on $d \geq 5$ and then explain what changes in $d = 4$. The terms we are looking for are

$$L^{-2}(\delta\overline{X})^2\big|_{z^2} = 2k_i\delta X^i_{(4)} + 2g^{(2)}_{ij}k^i\delta X^j_{(2)} + g_{ij}\delta X^i_{(2)}\delta X^j_{(2)} + g^{(4)}_{ij}k^ik^j + X^m_{(4)}\partial_m g_{ij}k^ik^j + 2X^m_{(2)}\partial_m g_{ij}k^i\delta X^j_{(2)} + X^m_{(2)}\partial_m g^{(2)}_{ij}k^ik^j + \frac{1}{2}X^m_{(2)}X^n_{(2)}\partial_m\partial_n g_{ij}k^ik^j. \tag{6.2.13}$$

This inequality is significantly more complicated than the previous one. The details of its evaluation are left to Appendix A.27. The result, assuming $\theta_{(k)} = \sigma_{ab}^{(k)} = 0$, is

$$\begin{aligned} L^{-2}(\delta\overline{X})^2\big|_{z^2} = &\frac{1}{4(d-2)^2}(D_a\theta_{(k)} + 2R_{ka})^2 \\ &+ \frac{1}{(d-2)^2(d-4)}(D_a\theta_{(k)} + R_{ka})^2 + \frac{1}{2(d-2)(d-4)}(D_a\sigma^{(k)}_{bc})^2 \\ &+ \frac{\kappa}{d-4}\left(C_{kabc}C_k{}^{abc} - 2C_k{}^c{}_{ca}C_k{}^b{}_b{}^a\right). \end{aligned} \tag{6.2.14}$$

where κ is proportional to $\lambda_{\mathrm{GB}}\ell^2/L^2$ and is defined in Appendix A.27. Aside from the Gauss–Bonnet term we have a sum of squares, which is good because EWN requires this to be positive when $\theta_{(k)}$ and $\sigma_{(k)}$ vanish. Since $\kappa \ll 1$, it cannot possibly interfere with positivity unless the other terms were zero. This would require $D_a\theta_{(k)} = D_a\sigma^{(k)}_{bc} = R_{ka} = 0$ in addition to our other conditions. But, following the arguments of [290], this cannot happen unless the components C_{kabc} of the Weyl tensor also vanish at the point in question. Thus EWN is always satisfied. Also note that the last two terms in middle line of (6.2.14) are each conformally invariant when $\theta_{(k)} = \sigma_{ab}^{(k)} = 0$, which we have assumed. This will become important later.

Finally, though we have assumed $d \geq 5$ to arrive at this result, we can use it to derive the expression for $L^{-2}(\delta\overline{X})^2\big|_{z^2\log z}$ in $d = 4$. The rule, explained in Appendix A.28, is to multiply the RHS by $4 - d$ and then set $d = 4$. This has the effect of killing the conformally non-invariant term, leaving us with

$$L^{-2}(\delta\overline{X})^2\big|_{z^2\log z, d=4} = -\frac{1}{4}(D_a\theta_{(k)} + R_{ka})^2 - \frac{1}{4}(D_a\sigma^{(k)}_{bc})^2. \tag{6.2.15}$$

The Gauss–Bonnet term also disappears because of a special Weyl tensor identity in $d = 4$ [289]. The overall minus sign is required since $\log z < 0$ in the small z limit. In addition, we no longer require that R_{ka} and $D_a\theta_{(k)}$ vanish individually to saturate the inequality: only their sum has to vanish. This still requires that $C_{kabc} = 0$, though.

The Quantum Null Energy Condition

The previous section dealt with the two leading state-independent inequalities that EWN implies. Here we deal with the leading state-*dependent* inequality, which turns out to be the QNEC.

At all orders lower than z^{d-2}, $(\delta\overline{X})^2$ is purely geometric. At order z^{d-2}, however, the CFT energy-momentum tensor enters via the Fefferman–Graham expansion of the metric, and variations of the entropy enter through $X^i_{(d)}$. In odd dimensions the analysis is simple and we will present it here, while in general even dimensions it is quite complicated. Since our state-independent analysis is incomplete for $d > 5$ anyway, we will be content with analyzing only $d = 4$ for the even case. The $d = 4$ calculation is presented in Appendix A.28. Though is it more involved that the odd-dimensional case, the final result is the same.

Consider first the case where d is odd. Then we have

$$L^{-2}(\delta\overline{X})^2\big|_{z^{d-2}} = g^{(d)}_{ij}k^ik^j + 2k_i\delta X^i_{(d)} + X^m_{(d)}\partial_m g_{ij}k^ik^j = g^{(d)}_{ij}k^ik^j + 2\delta\left(k_i\delta X^i_{(d)}\right). \quad (6.2.16)$$

From (6.2.4) and (6.2.10), we find that

$$L^{-2}(\delta\overline{X})^2\big|_{z^{d-2}} = \frac{16\pi G_N}{\eta d L^{d-1}}\left[\langle T_{kk}\rangle - \delta\left(\frac{k^i}{2\pi\sqrt{h}}\frac{\delta S_{\mathrm{ren}}}{\delta X^i}\right)\right]. \quad (6.2.17)$$

The nonnegativity of the term in brackets is equivalent to the QNEC. The case where d is even is more complicated, and we will go over the $d = 4$ case in Appendix A.28.

The Conformal QNEC

As mentioned in §6.2, we can get a stronger inequality from EWN by considering the norm of the vector s^μ, which is the part of $\delta\overline{X}^\mu$ orthogonal to the extremal surface. Our gauge choice $\overline{X}^z = z$ means that $s^\mu \neq \delta\overline{X}^\mu$, and so we get a nontrivial improvement by considering $s^2 \geq 0$ instead of $(\delta\overline{X})^2 \geq 0$.

We can actually use the results already derived above to compute s^2 with the following trick. We would have had $\delta\overline{X}^\mu = s^\mu$ if the surfaces of constant z were already orthogonal to the extremal surfaces. But we can change our definition of the constant-z surfaces with a coordinate transformation in the bulk to make this the case, apply the above results to $(\delta\overline{X})^2$ in the new coordinate system, and then transform back to the original coordinates. The coordinate transformation we are interested in performing is a PBH transformation [291], since it leaves the metric in Fefferman–Graham form, and so induces a Weyl transformation on the boundary.

So from the field theory point of view, we will just be calculating the consequences of EWN in a different conformal frame, which is fine because we are working with a CFT.

With that in mind it is easy to guess the outcome: the best conformal frame to pick is one in which all of the non-conformally-invariant parts of the state-independent terms in $(\delta\overline{X})^2$ are set to zero, and when we transform the state-dependent term in the new frame back to the original frame we get the so-called Conformal QNEC first defined in [17]. This is indeed what happens, as we will now see.

Orthogonality Conditions First, we will examine in detail the conditions necessary for $\delta\overline{X}^\mu = s^\mu$, and their consequences on the inequalities derived above. We must check that

$$\overline{g}_{ij}\partial_\alpha\overline{X}^i\delta\overline{X}^j = 0. \tag{6.2.18}$$

for both $\alpha = z$ and $\alpha = a$. As above, we will expand these conditions in z. When $\alpha = z$, at lowest order in z we find the condition

$$0 = k_i X^i_{(2)}, \tag{6.2.19}$$

which is equivalent to $\theta_{(k)} = 0$. When $\alpha = a$, the lowest-order in z inequality is automatically satisfied because k^i is defined to be orthogonal to the entangling surface on the boundary. But at next-to-lowest order we find the condition

$$\begin{aligned} 0 &= k_i\partial_a X^i_{(2)} + e_{ai}\delta X^i_{(2)} + g^{(2)}_{ij} e^i_a k^j + X^m_{(2)}\partial_m g_{ij} e^i_a k^j && (6.2.20)\\ &= -\frac{1}{2(d-2)}\left[(D_a - 2w_a)\theta_{(k)} + 2R_{ka}\right]. && (6.2.21)\end{aligned}$$

Combined with the $\theta_{(k)} = 0$ condition, this tells us that that $D_a\theta_{(k)} = -2R_{ka}$ is required. When these conditions are satisfied, the state-dependent terms of $(\delta\overline{X})^2$ analyzed above become[5]

$$L^{-2}(\delta\overline{X})^2 = \frac{1}{d-2}\sigma^2_{(k)} + \left[\frac{1}{(d-2)^2(d-4)}(R_{ka})^2 + \frac{1}{2(d-2)(d-4)}(D_a\sigma^{(k)}_{bc})^2\right]z^2 + \cdots \tag{6.2.22}$$

Next we will demonstrate that $\theta_{(k)} = 0$ and $D_a\theta_{(k)} = -2R_{ka}$ can be achieved by a Weyl transformation, and then use that fact to write down the $s^2 \geq 0$ inequality that we are after.

Achieving $\delta\overline{X}^\mu = s^\mu$ with a Weyl Transformation Our goal now is to begin with a generic situation in which $\delta\overline{X}^\mu \neq s^\mu$ and use a Weyl transformation to set $\delta\overline{X}^\mu \to s^\mu$. This means finding a new conformal frame with $\hat{g}_{ij} = e^{2\phi(x)}g_{ij}$ such that $\hat{\theta}_{(k)} = 0$ and $\hat{D}_a\hat{\theta}_{(k)} = -2\hat{R}_{ka}$, which would then imply that $\delta\hat{X}^\mu = s^\mu$ (we omit the bar on $\delta\hat{X}^\mu$ to avoid cluttering the notation, but logically it would be $\delta\hat{\overline{X}}^\mu$).

[5]We have not included some terms at order z^2 which are proportional to $\sigma^{(k)}_{ab}$ because they never play a role in the EWN inequalities.

Computing the transformation properties of the geometric quantities involved is a standard exercise, but there is one extra twist involved here compared to the usual prescription. Ordinarily a vector such as k^i would be invariant under the Weyl transformation. However, for our setup is it is important that k^i generate an affine-parameterized null geodesic. Even though the null geodesic itself is invariant under Weyl transofrmation, k^i will no longer be the correct generator. Instead, we have to use $\hat{k}^i = e^{-2\phi}k^i$. Another way of saying this is that $k_i = \hat{k}_i$ is invariant under the Weyl transformation. With this in mind, we have

$$e^{2\phi}\hat{R}_{ka} = R_{ka} - (d-2)\left[D_a\partial_k\phi - w_a\partial_k\phi - k_j K^j_{ab}\partial^b\phi - \partial_k\phi\partial_a\phi\right], \tag{6.2.23}$$

$$e^{2\phi}\hat{\theta}_{(k)} = \theta_{(k)} + (d-2)\partial_k\phi, \tag{6.2.24}$$

$$e^{2\phi}\hat{D}_a\hat{\theta}_{(k)} = D_a\theta_{(k)} + (d-2)D_a\partial_k\phi - 2\theta_{(k)}\partial_a\phi - 2(d-2)\partial_k\phi\partial_a\phi, \tag{6.2.25}$$

$$\hat{\sigma}^{(k)}_{ab} = \sigma^{(k)}_{ab}, \tag{6.2.26}$$

$$\hat{D}_c\hat{\sigma}^{(k)}_{ab} = D_c\sigma^{(k)}_{ab} - 2\left[\sigma^{(k)}_{c(b}\partial_{a)}\phi + \sigma^{(k)}_{ab}\partial_c\phi - g_{c(a}\sigma^{(k)}_{b)d}\nabla^d\phi\right], \tag{6.2.27}$$

$$\hat{w}_a = w_a - \partial_a\phi. \tag{6.2.28}$$

So we may arrange $\hat{\theta}_{(k)} = 0$ at a given point on the entangling surface by choosing $\partial_k\phi = -\theta_{(k)}/(d-2)$ that that point. Having chosen that, and assuming $\sigma^{(k)}_{ab}$ =0 at the same point, one can check that

$$e^{2\phi}\left(\hat{D}_a\hat{\theta}_{(k)} + 2\hat{R}_{ka}\right) = D_a\theta_{(k)} - 2w_a\theta_{(k)} + 2R_{ka} - (d-2)D_a\partial_k\phi \tag{6.2.29}$$

So we can choose $D_a\partial_k\phi$ to make the combination $\hat{D}_a\hat{\theta}_{(k)} + 2\hat{R}_{ka}$ vanish. Then in the new frame we have $\delta\hat{X}^\mu = s^\mu$.

The $s^2 \geq 0$ Inequality Based on the discussion above, we were able to find a conformal frame that allows us to compute the s^2. For the state-independent parts we have

$$L^{-2}s^2 = \frac{1}{d-2}\hat{\sigma}^2_{(k)} + \left[\frac{1}{(d-2)^2(d-4)}(\hat{R}_{ka})^2 + \frac{1}{2(d-2)(d-4)}(\hat{D}_a\hat{\sigma}^{(k)}_{bc})^2\right]\hat{z}^2 + \cdots \tag{6.2.30}$$

Here we also have a new bulk coordinate $\hat{z} = ze^\phi$ associated with the bulk PBH transformation. All we have to do now is transform back into the original frame to find s^2. Since $\hat{\theta}_{(k)} = \hat{D}_a\hat{\theta}_{(k)} + 2\hat{R}_{ka} = 0$, we actually have that

$$\hat{R}_{ka} = \hat{D}_a\hat{\theta}_{(k)} - \hat{w}_a\hat{\theta}_{(k)} - \hat{R}_{ka}, \tag{6.2.31}$$

which transforms homogeneously under Weyl transformations when $\sigma^{(k)}_{ab} = 0$. Thus, up to an overall scaling factor, we have

$$\begin{aligned} L^{-2}s^2 = &\frac{1}{d-2}\sigma^2_{(k)} \\ &+ \left[\frac{1}{(d-2)^2(d-4)}(D_a\theta_{(k)} - w_a\theta_{(k)} - R_{ka})^2 + \frac{1}{2(d-2)(d-4)}(D_a\sigma^{(k)}_{bc})^2\right]z^2 + \cdots, \end{aligned} \tag{6.2.32}$$

where we have dropped terms of order z^2 which vanish when $\sigma^{(k)}_{ab} = 0$. As predicted, these terms are the conformally invariant contributions to $(\delta\overline{X})^2$.

In order to access the state-dependent part of s^2 we need the terms in (6.2.32) to vanish. Note that in $d = 3$ this always happens. In that case there is no z^2 term, and $\sigma^{(k)}_{ab} = 0$ always. Though our expression is singular in $d = 4$, comparing to (6.2.22) shows that actually the term in brackets above is essentially the same as the $z^2 \log z$ term in $\delta\overline{X}$. We already noted that this term was conformally invariant, so this is expected. The difference now is that we no longer need $\theta_{(k)} = 0$ in order to get to the QNEC in $d = 4$. In $d = 5$ the geometric conditions for the state-independent parts of s^2 to vanish are identical to those for $d = 4$, whereas in the $(\delta\overline{X})^2$ analysis we found that extra conditions were necessary. These were relics of the choice of conformal frame. Finally, for $d > 5$ there will be additional state-independent terms that we have not analyzed, but the results we have will still hold.

Conformal QNEC Now we analyze the state-dependent part of s^2 at order z^{d-2}. When all of the state-independent parts vanish, the state-dependent part is given by the conformal transformation of the QNEC. This is easily computed as follows:

$$L^{-2}\, s^2\big|_{z^{d-2}} = \frac{16\pi G_N}{\eta d L^{d-1}}\left[2\pi\langle \hat{T}_{ij}\rangle k^i k^j - \delta\left(\frac{k^i}{\sqrt{h}}\frac{\delta \hat{S}_{\rm ren}}{\delta X^i(y)}\right) - \frac{d}{2}\theta_{(k)}\left(\frac{k^i}{\sqrt{h}}\frac{\delta \hat{S}_{\rm ren}}{\delta X^i(y)}\right)\right]. \qquad (6.2.33)$$

Of course, one would like to replace $\hat{T}_{ij}$ with T_{ij} and $\hat{S}_{\rm ren}$ with $S_{\rm ren}$. When d is odd this is straightforward, as these quantities are conformally invariant. However, when d is even there are anomalies that will contribute, leading to extra geometric terms in the conformal QNEC [292, 17].

6.3 Connection to Quantum Focusing

The Quantum Focusing Conjecture

We start by reviewing the statement of the QFC [13, 290] before moving on to its connection to EWN and the QNEC. Consider a codimension-two Cauchy-splitting (i.e. entangling) surface Σ and a null vector field k^i normal to Σ. Denote by $\mathcal{N}$ the null surface generated by k^i. The generalized entropy, $S_{\rm gen}$, associated to Σ is given by

$$S_{\rm gen} = \langle S_{\rm grav}\rangle + S_{\rm ren} \qquad (6.3.1)$$

where $S_{\rm grav}$ is a state-independent local integral on Σ and $S_{\rm ren}$ is the renormalized von Neumann entropy of the interior (or exterior of Σ. The terms in $S_{\rm grav}$ are determined by the low-energy effective action of the theory in a well-known way [293]. Even though $\langle S_{\rm grav}\rangle$ and $S_{\rm ren}$ individually depend on the renormalization scheme, that dependence cancels out between them so that $S_{\rm gen}$ is scheme-independent.

The generalized entropy is a functional of the entangling surface Σ, and the QFC is a statement about what happens when we vary the shape of Σ by deforming it within the surface $\mathcal{N}$. Specifically, consider a one-parameter family $\Sigma(\lambda)$ of cuts of $\mathcal{N}$ generated by deforming the original surface using the vector field k^i. Here λ is the affine parameter along the geodesic generated by k^i and $\Sigma(0) \equiv \Sigma$. To be more precise, let y^a denote a set of intrinsic coordinates for Σ, let h_{ab} be the induced metric on Σ, and let $X^i(y, \lambda)$ be the embedding functions for $\Sigma(\lambda)$. With this notation, $k^i = \partial_\lambda X^i$. The change in the generalized entropy is given by

$$\left.\frac{dS_{\text{gen}}}{d\lambda}\right|_{\lambda=0} = \int_\Sigma d^{d-2}y \; \frac{\delta S_{\text{gen}}}{\delta X^i(y)} \partial_\lambda X^i(y) \equiv \frac{1}{4G_N}\int_\Sigma d^{d-2}y\sqrt{h}\,\Theta[\Sigma, y] \tag{6.3.2}$$

This defines the quantum expansion $\Theta[\Sigma, y]$ in terms of the functional derivative of the generalized entropy:

$$\Theta[\Sigma, y] = 4G_N \frac{k^i(y)}{\sqrt{h}} \frac{\delta S_{\text{gen}}}{\delta X^i(y)}. \tag{6.3.3}$$

Note that we have suppressed the dependence of Θ on k^i in the notation, but the dependence is very simple: if $k^i(y) \to f(y)k^i(y)$, then $\Theta[\Sigma, y] \to f(y)\Theta[\Sigma, y]$.

The QFC is simple to state in terms of Θ. It says that Θ is non-increasing along the flow generated by k^i:

$$0 \geq \frac{d\Theta}{d\lambda} = \int_\Sigma d^{d-2}y \; \frac{\delta\Theta[\Sigma, y]}{\delta X^i(y')} k^i(y'). \tag{6.3.4}$$

Before moving on, let us make two remarks about the QFC.

First, the functional derivative $\delta\Theta[\Sigma, y]/\delta X^i(y')$ will contain local terms (i.e. terms proportional to δ-functions or derivatives of δ-functions with support at $y = y'$) as well as non-local terms that have support even when $y \neq y'$. S_{grav}, being a local integral, will only contribute to the local terms of $\delta\Theta[\Sigma, y]/\delta X^i(y')$. The renormalized entropy S_{ren} will contribute both local and non-local terms. The non-local terms can be shown to be nonpositive using strong subadditivity of the entropy [13], while the local terms coming from S_{ren} are in general extremely difficult to compute.

Second, and more importantly for us here, the QFC as written in (6.3.4) does not quite make sense. We have to remember that S_{grav} is really an operator, and its expectation value $\langle S_{\text{grav}}\rangle$ is really the thing that contributes to Θ. In order to be well-defined in the low-energy effective theory of gravity, this expectation value must be smeared over a scale large compared to the cutoff scale of the theory. Thus when we write an inequality like (6.3.4), we are implicitly smearing in y against some profile. The profile we use is arbitrary as long as it is slowly-varying on the cutoff scale. This extra smearing step is necessary to avoid certain violations of (6.3.4), as we will see below [290].

QNEC from QFC

In this section we will explicitly evaluate the QFC inequality, (6.3.4), and derive the QNEC in curved space from it as a nongravitational limit. We consider theories with a gravitational action of the form

$$I_{\rm grav} = \frac{1}{16\pi G_N}\int \sqrt{g}\left(R + \ell^2\lambda_1 R^2 + \ell^2\lambda_2 R_{ij}R^{ij} + \ell^2\lambda_{\rm GB}\mathcal{L}_{\rm GB}\right) \tag{6.3.5}$$

where $\mathcal{L}_{GB} = R^2_{ijmn} - 4R^2_{ij} + R^2$ is the Gauss-Bonnet Lagrangian. Here ℓ is the cutoff length scale of the effective field theory, and the dimensionless couplings λ_1, λ_2, and $\lambda_{\rm GB}$ are assumed to be renormalized.

The generalized entropy functional for these theories can be computed using standard replica methods [293] and takes the form

$$S_{\rm gen} = \frac{A[\Sigma]}{4G_N} + \frac{\ell^2}{4G_N}\int_\Sigma \sqrt{h}\left[2\lambda_1 R + \lambda_2\left(R_{ij}N^{ij} - \frac{1}{2}K_iK^i\right) + 2\lambda_{\rm GB} r\right] + S_{\rm ren}. \tag{6.3.6}$$

Here $A[\Sigma]$ is the area of the entangling surface, N^{ij} is the projector onto the normal space of Σ, K^i is the trace of the extrinsic curvature of Σ, and r is the intrinsic Ricci scalar of Σ.

We can easily compute Θ by taking a functional derivative of (6.3.6), taking care to integrate by parts so that the result is proportional to $k^i(y)$ and not derivatives of $k^i(y)$. One finds

$$\Theta = \theta_{(k)} + \ell^2\Bigg[2\lambda_1(\theta_{(k)}R + \nabla_k R) + \lambda_2\Big((D_a - w_a)^2\theta_{(k)} + K_iK^{iab}K^k_{ab} \tag{6.3.7}$$

$$+ \theta_{(k)}R_{klkl} + \nabla_k R - 2\nabla_l R_{kk} + \theta_{(k)}R_{kl} - \theta_{(l)}R_{kk} + 2K^{kab}R_{ab}\Big)$$

$$- 4\lambda_{GB}\left(r^{ab}K^k_{ab} - \frac{1}{2}r\theta_{(k)}\right)\Bigg] + 4G_N\frac{k^i}{\sqrt{h}}\frac{\delta S_{\rm ren}}{\delta X^i} \tag{6.3.8}$$

Now we must compute the λ-derivative of Θ. When we do this, the leading term comes from the derivative of $\theta_{(k)}$, which by Raychaudhuri's equation contains the terms $\theta^2_{(k)}$ and $\sigma^2_{(k)}$. Since we are ultimately interested in deriving the QNEC as the non-gravitational limit of the QFC, we need to set $\theta_{(k)} = \sigma^{(k)}_{ab} = 0$ so that the nongravitational limit is not dominated by those terms. So for the rest of this section we will set $\theta_{(k)} = \sigma^{(k)}_{ab} = 0$ at the point of evaluation (but not globally!). Then we find

$$\begin{aligned}\frac{d\Theta}{d\lambda} &= -R_{kk} + 2\lambda_1\ell^2\left(\nabla_k^2 R - RR_{kk}\right)\\ &+ \lambda_2\ell^2\Big[2D_a(w^a R_{kk}) + \nabla_k^2 R - D_a D^a R_{kk} - \frac{d}{d-2}(D_a\theta_{(k)})^2 - 2R_{kb}D^b\theta_{(k)} - 2(D_a\sigma_{bc})^2\\ &- 2\nabla_k\nabla_l R_{kk} - 2R_{kakb}R^{ab} - \theta_{(l)}\nabla_k R_{kk}\Big] - 2\lambda_{\rm GB}\ell^2\Bigg[\frac{d(d-3)(d-4)}{(d-1)(d-2)^2}RR_{kk}\\ &- 4\frac{(d-4)(d-3)}{(d-2)^2}R_{kk}R_{kl} - \frac{2(d-4)}{d-2}C_{klkl}R_{kk} - \frac{2(d-4)}{d-2}R^{ab}C_{akbk} + 4C^{kalb}C_{kakb}\Bigg]\\ &+ 4G_N\frac{d}{d\lambda}\left(\frac{k^i}{\sqrt{h}}\frac{\delta S_{\rm ren}}{\delta X^i}\right)\end{aligned} \tag{6.3.9}$$

This expression is quite complicated, but it simplifies dramatically if we make use of the equation of motion coming from (6.3.5) plus the action of the matter sector. Then we have $R_{kk} = 8\pi G T_{kk} - H_{kk}$ where [294]

$$\begin{aligned}H_{kk} &= 2\lambda_1\left(RR_{kk} - \nabla_k^2 R\right) + \lambda_2\Big(2R_{kikj}R^{ij} - \nabla_k^2 R + 2\nabla_k\nabla_l R_{kk} - 2R_{klki}R^i_k\\ &+ D_c D^c R_{kk} - 2D_c(w^c R_{kk}) - 2(D_b\theta_{(k)} + R_{bmkj}P^{mj})R^b_k + \theta_{(l)}\nabla_k R_{kk}\Big)\\ &+ 2\lambda_{\rm GB}\left(\frac{d(d-3)(d-4)}{(d-1)(d-2)^2}RR_{kk} - 4\frac{(d-4)(d-3)}{(d-2)^2}R_{kk}R_{kl} - 2\frac{d-4}{d-2}R^{ij}C_{kikj} + C_{kijm}C_k{}^{ijm}\right)\end{aligned} \tag{6.3.10}$$

For the Gauss-Bonnet term we have used the standard decomposition of the Riemann tensor in terms of the Weyl and Ricci tensors. Using similar methods to those in Appendix A.27, we have also exchanged $k^i k^j \Box R_{ij}$ in the R_{ij}^2 equation of motion for surface quantities and ambient curvatures.

After using the equation of motion we have the relatively simple formula

$$\begin{aligned}\frac{d\Theta}{d\lambda} &= -\lambda_2\ell^2\left(\frac{d}{d-2}(D_a\theta_{(k)})^2 + 4R^b_k D_b\theta_{(k)} + 2R_{bk}R^b_k + 2(D_a\sigma^{(k)}_{bc})^2\right)\\ &+ 2\lambda_{\rm GB}\ell^2\left(C_{kabc}C_k{}^{abc} - 2C_{kba}{}^b C_{kc}{}^{ac}\right) + 4G_N\frac{d}{d\lambda}\left(\frac{k^i}{\sqrt{h}}\frac{\delta S_{\rm ren}}{\delta X^i}\right) - 8\pi G_N\langle T_{kk}\rangle\end{aligned} \tag{6.3.11}$$

The Gauss-Bonnet term agrees with the expression derived in [289]. However unlike [289] we have not made any perturbative assumptions about the background curvature.

At first glance it seems like (6.3.11) does not have definite sign, even in the non-gravitational limit, due to the geometric terms proportional to λ_2 and $\lambda_{\rm GB}$. The difficulty posed by the Gauss-Bonnet term, in particular, was first pointed out in [281]. However, this is where we have to remember the smearing prescription mentioned in §6.3. We must integrate (6.3.11) over a region of size larger than ℓ before testing its nonpositivity. The crucial point, used in

[290], is that we must also remember to integrate the terms $\theta_{(k)}^2$ and $\sigma_{(k)}^2$ that we dropped earlier over the same region. When we integrate $\theta_{(k)}^2$ over a region of size ℓ centered at a point where $\theta_{(k)} = 0$, the result is $\xi \ell^2 (D_a \theta_{(k)})^2 + o(\ell^2)$, where $\xi \gtrsim 10$ is a parameter associated with the smearing profile. A similar result holds for $\sigma_{ab}^{(k)}$. Thus we arrive at

$$\begin{aligned}\frac{d\Theta}{d\lambda} &= -\frac{\xi}{d-2}\ell^2 (D_a\theta_{(k)})^2 - \xi \ell^2 (D_a \sigma_{bc}^{(k)})^2 \\ &- \lambda_2 \ell^2 \left(\frac{d}{d-2}(D_a\theta_{(k)})^2 + 4R_k^b D_b \theta_{(k)} + 2R_{bk}R_k^b + 2(D_a\sigma_{bc}^{(k)})^2 \right) \\ &+ 2\lambda_{\rm GB}\ell^2 \left(C_{kabc}C_k{}^{abc} - 2C_{kba}{}^b C_{kc}{}^{ac} \right) \\ &+ 4G_N \frac{d}{d\lambda}\left(\frac{k^i}{\sqrt{h}} \frac{\delta S_{\rm ren}}{\delta X^i} \right) - 8\pi G_N \langle T_{kk} \rangle + o(\ell^2) \end{aligned} \tag{6.3.12}$$

Since the size of ξ is determined by the validity of the effective field theory, by construction the terms proportional to ξ in (6.3.12) dominate over the others. Thus in order to take the non-gravitational limit, we must eliminate these smeared terms.

Clearly we need to be able to choose a surface such that $D_a\theta_{(k)} = D_a \sigma_{bc}^{(k)} = 0$. Then smearing $\theta_{(k)}^2$ and $\sigma_{(k)}^2$ would only produce terms of order ℓ^4 (terms of that order would also show up from smearing the operators proportional to λ_2 and $\lambda_{\rm GB}$). As explained in [290], this is only possible given certain conditions on the background spacetime at the point of evaluation. We must have

$$C_{kabc} = \frac{1}{d-2} h_{ab} R_{kc} - \frac{1}{d-2} h_{ac} R_{kb}. \tag{6.3.13}$$

This can be seen by using the Codazzi equation for Σ. Imposing this condition, which allows us to set $D_a\theta_{(k)} = D_a\sigma_{bc}^{(k)} = 0$, we then have.

$$\begin{aligned}\frac{d\Theta}{d\lambda} &= -2\ell^2 \left(\lambda_2 + 2\frac{(d-3)(d-4)}{(d-2)^2}\lambda_{\rm GB} \right) R_{bk}R_k^b \\ &+ 4G_N \frac{d}{d\lambda}\left(\frac{k^i}{\sqrt{h}} \frac{\delta S_{\rm ren}}{\delta X^i} \right) - 8\pi G_N \langle T_{kk} \rangle + o(\ell^3). \end{aligned} \tag{6.3.14}$$

This is the quantity which must be negative according to the QFC. In deriving it, we had to assume that $\theta_{(k)} = \sigma^{(k)} = D_a\theta_{(k)} = D_a\sigma_{bc}^{(k)} = 0$.

We make two observations about (6.3.14). First, if we assume that $R_{ka} = 0$ as an additional assumption and take $\ell \to 0$, then we arrive at the QNEC as long as $G_N > o(\ell^3)$. This is the case when ℓ scales with the Planck length and $d \leq 5$. These conditions are similar to the ones we found previously from EWN, and below in §6.3 we will discuss that in more detail.

The second observation has to do with the lingering possibility of a violation of the QFC due to the terms involving the couplings. In order to have a violation, one would need the linear combination

$$\lambda_2 + 2\frac{(d-3)(d-4)}{(d-2)^2}\lambda_{\rm GB} \tag{6.3.15}$$

to be negative. Then if one could find a situation where the first line of (6.3.14) dominated over the second, there would be a violation. It would be interesting to interpret this as a bound on the above linear combination of couplings coming from the QFC, but it is difficult to find a situation where the first line of (6.3.14) dominates. The only way for R_{ka} to be large compared to the cutoff scale is if T_{ka} is nonzero, in which case we would have $R_{ka} \sim G_N T_{ka}$. Then in order for the first line of (6.3.14) to dominate we would need

$$G_N \ell^2 T_{ka} T_k^a \gg T_{kk}. \tag{6.3.16}$$

As an example, for a scalar field Φ this condition would say

$$G_N \ell^2 (\partial_a \Phi)^2 \gg 1. \tag{6.3.17}$$

This is not achievable within effective field theory, as it would require the field to have super-Planckian gradients. We leave a detailed and complete discussion of this issue to future work.

Scheme-Independence of the QNEC

We take a brief interlude to discuss the issue of the scheme-dependence of the QNEC, which will be important in the following section. It was shown in [289], under some slightly stronger assumptions than the ones we have been using, that the QNEC is scheme-independent under the same conditions where we expect it to hold true. Here we will present our own proof of this fact, which actually follows from the manipulations we performed above involving the QFC.

In this section we will take the point of view of field theory on curved spacetime without dynamical gravity. Then each of the terms in I_{grav}, defined above in (6.3.5), are completely arbitrary, non-dynamical terms we can add to the Lagrangian at will.[6] Dialing the values of those various couplings corresponds to a choice of *scheme*, as even though those couplings are non-dynamical they will still contribute to the definitions of quantities like the renormalized energy-momentum tensor and the renormalized entropy (as defined through the replica trick). The QNEC is scheme-independent if it is insensitive to the values of these couplings.

To show the scheme-independence of the QNEC, we will begin with the statement that S_{gen} is scheme-independent. We remarked on this above, when our context was a theory with dynamical gravity. But the scheme-independence of S_{gen} does not require use of the equations of motion, so it is valid even in a non-gravitational theory on a fixed background. In fact, only once in the above discussion did we make use of the gravitational equations of motion, and that was in deriving (6.3.11). Following the same steps up to that point, but

[6] We should really be working at the level of the quantum effective action, or generating functional, for correlation functions of T_{ij} [281]. The geometrical part has the same form as the classical action I_{grav} and so does not alter this discussion.

without imposing the gravitational equations of motion, we find instead

$$\frac{d\Theta}{d\lambda} = -\lambda_2 \ell^2 \left(\frac{d}{d-2} (D_a \theta_{(k)})^2 + 4 R_k^b D_b \theta_{(k)} + 2 R_{bk} R_k^b + 2 (D_a \sigma_{bc})^2 \right) + 2\lambda_{\mathrm{GB}} \ell^2 \left(C_{kabc} C_k{}^{abc} - 2 C_{kba}{}^b C_{kc}{}^{ac} \right) + 4 G_N \frac{d}{d\lambda} \left(\frac{k^i}{\sqrt{h}} \frac{\delta S_{\mathrm{ren}}}{\delta X^i} \right) - k_i k_j \frac{16\pi G_N}{\sqrt{g}} \frac{\delta I_{\mathrm{grav}}}{\delta g_{ij}}. \tag{6.3.18}$$

Since the theory is not gravitational, we would not claim that this quantity has a sign. However, it is still scheme-independent.

To proceed, we will impose all of the additional conditions that are necessary to prove the QNEC. That is, we impose $D_b \theta_{(k)} = R_k^b = D_a \sigma_{bc} = 0$, as well as $\theta_{(k)} = \sigma_{ab}^{(k)} = 0$, which in turn requires $C_{kabc} = 0$. Under these conditions, we learn that the combination

$$\frac{d}{d\lambda} \left(\frac{k^i}{\sqrt{h}} \frac{\delta S_{\mathrm{ren}}}{\delta X^i} \right) - k_i k_j \frac{4\pi}{\sqrt{g}} \frac{\delta I_{\mathrm{grav}}}{\delta g_{ij}} \tag{6.3.19}$$

is scheme-independent. The second term here is one of the contributions to the renormalized $2\pi \langle T_{kk} \rangle$ in the non-gravitational setup, the other contribution being $k_i k_j \frac{4\pi}{\sqrt{g}} \frac{\delta I_{\mathrm{matter}}}{\delta g_{ij}}$. But I_{matter} is already scheme-independent in the sense we are discussing, in that it is independent of the parameters appearing in I_{grav}. So adding that to the terms we have above, we learn that

$$\frac{d}{d\lambda} \left(\frac{k^i}{\sqrt{h}} \frac{\delta S_{\mathrm{ren}}}{\delta X^i} \right) - 2\pi \langle T_{kk} \rangle \tag{6.3.20}$$

is scheme-independent. This is what we wanted to show.

QFC vs EWN

As we have discussed above, by taking the non-gravitational limit of (6.3.14) under the assumptions $D_b \theta_{(k)} = R_k^b = D_a \sigma_{bc} = \theta_{(k)} = \sigma_{ab}^{(k)} = 0$ we find the QNEC as a consequence of the QFC (at least for $d \leq 5$). And under the same set of geometric assumptions, we found the QNEC as a consequence of EWN in (6.2.17). The discussion of the previous section demonstrates that these assumptions also guarantee that the QNEC is scheme-independent. So even though these two QNEC inequalities were derived in different ways, we know that at the end of the day they are the same QNEC. It is natural to ask if there is a further relationship between EWN and the QFC, beyond the fact that they give the same QNEC. We will begin to investigate that question in this section.

The natural thing to ask about is the state-independent terms in the QFC and in $(\delta\overline{X})^2$. We begin by writing down all of the terms of $(\delta\overline{X})^2$ in odd dimensions that we have computed:

$$\begin{aligned}(d-2)L^{-2}(\delta\overline{X}^i)^2 =& \frac{1}{(d-2)}\theta_{(k)}^2 + \sigma_{(k)}^2 \\ &+ z^2\frac{1}{4(d-2)}(D_a\theta_{(k)} + 2R_{ka})^2 \\ &+ z^2\frac{1}{(d-2)(d-4)}(D_a\theta_{(k)} + R_{ka})^2 + z^2\frac{1}{2(d-4)}(D_a\sigma_{bc}^{(k)})^2 \\ &+ z^2\frac{\kappa}{d-4}\left(C_{kabc}C_k^{\ abc} - 2C_{k\ ca}^{\ c}C_{k\ b}^{\ b\ a}\right) \\ &+ \cdots + z^{d-2}\frac{16\pi(d-2)G_N}{\eta d L^{d-1}}\left[\langle T_{kk}\rangle - \delta\left(\frac{k^i}{2\pi\sqrt{h}}\frac{\delta S_{\text{ren}}}{\delta X^i}\right)\right]. \end{aligned} \quad (6.3.21)$$

The first line looks like $-\dot{\theta}$, which would be the leading term in $d\Theta/d\lambda$, except it is missing an R_{kk}. Of course, we eventually got rid of the R_{kk} in the QFC by using the equations of motion. Suppose we set $\theta_{(k)} = 0$ and $\sigma_{ab}^{(k)} = 0$ to eliminate those terms, as we did with the QFC. Then we can write $(\delta\overline{X})^2$ suggestively as

$$\begin{aligned}(d-2)L^{-2}(\delta\overline{X}^i)^2 =& z^2\tilde{\lambda}_2\Big(\frac{d}{(d-2)}(D_a\theta_k)^2 + 4R_k^a D_a\theta + \frac{4(d-3)}{(d-2)}R_{ka}R_k^a + 2(D_a\sigma_{bc}^{(k)})^2\Big) \\ &- 2z^2\tilde{\lambda}_{\text{GB}}\left(C_{kabc}C_k^{\ abc} - 2C_{k\ ca}^{\ c}C_{k\ b}^{\ b\ a}\right) \\ &+ \cdots + 8\pi\tilde{G}_N\langle T_{kk}\rangle - 4\tilde{G}_N\delta\left(\frac{k^i}{\sqrt{h}}\frac{\delta S_{\text{ren}}}{\delta X^i}\right). \end{aligned} \quad (6.3.22)$$

where

$$\tilde{G}_N = G_N\frac{2(d-2)z^{d-2}}{\eta d L^{d-1}}, \quad (6.3.23)$$

$$\tilde{\lambda}_2 = \frac{1}{4(d-4)}, \quad (6.3.24)$$

$$\tilde{\lambda}_{\text{GB}} = -\frac{\kappa}{2(d-4)}. \quad (6.3.25)$$

Written this way, it almost seems like $(d-2)L^{-2}(\delta\overline{X}^i)^2 \sim -d\Theta/d\lambda$ in some kind of model gravitational theory. One discrepancy is in the coefficient of the $R_{ka}R^{ka}$ term, unless $d = 4$. It is also intriguing that the effective coefficients $\tilde{G}_N$, $\tilde{\lambda}_2$, and $\tilde{\lambda}_{\text{GB}}$ are close to, but not exactly the same as, the effective braneworld induced gravity coefficients found in [295]. This is clearly something that deserves further study.

6.4 Discussion

We have displayed a strong similarity between the state-independent inequalities in the QFC and the state-independent inequalities from EWN. We now discuss several possible future directions and open questions that follow naturally from these results.

Bulk Entropy Contributions

We ignored the bulk entropy S_{bulk} in this work, but we know that it produces a contribution to CFT entropy [296] and plays a role in the position of the extremal surface [283, 284]. The bulk entropy contributions to the entropy are subleading in N^2 and do not interfere with the gravitational terms in the entropy. We could include the bulk entropy as a source term in the equations determining $\overline{X}$, which could lead to extra contributions to the $X_{(n)}$ coefficients. However, it does not seem possible for the bulk entropy to have an effect on the state-independent parts of the extremal surface, namely on $X_{(n)}$ for $n < d$, which means the bulk entropy would not affect the conditions we derived for when the QNEC should hold.

Another logical possibility is that the bulk entropy term could affect the statement of the QNEC itself, meaning that the schematic form $T_{kk} - S''$ would be altered. This would be problematic, especially given that the QFC always produces a QNEC of that same form. It was argued in [279] that this does not happen, and that argument holds here as well.

Smearing of EWN

We were careful to include a smearing prescription for defining the QFC, and it was an important ingredient in the analysis of §6.3. But what about smearing of EWN? Of course, the answer is that we *should* smear EWN appropriately, but as we will see now it would not make a difference to our analysis.

The issue is that the bulk theory is a low-energy effective theory of gravity with a cutoff scale ℓ, and the quantities that we use to probe EWN, like $(\delta\overline{X})^2$, are operators in that theory. As such, these operators need to be smeared over a region of proper size ℓ on the extremal surface. Of course, due to the warp factor, such a region has coordinate size $z\ell/L$. We can ask what effect such a smearing would have on the inequality $(\delta\overline{X})^2$.

When we performed our QNEC derivation, we assumed that $\theta_{(k)} = 0$ at the point of evaluation, so that the $\theta_{(k)}^2$ term in $(\delta\overline{X})^2\big|_{z^0}$ would not contribute. However, after smearing this term would contribute a term of the form $\ell^2(D_a\theta_{(k)})^2/L^2$ to $(\delta\overline{X})^2\big|_{z^2}$. But we already had such a term at this order, so all this does is shift the coefficient. Furthermore, the coefficient is shifted only by an amount of order ℓ^2/L^2. If the cutoff ℓ is of order the Planck scale, then this is suppressed in powers of N^2. In other words, this effect is negligible for the analysis. A similar statement applies for $\sigma_{ab}^{(k)}$. So in summary, EWN should be smeared, but the analysis we performed was insensitive to it.

Future Work

There are a number of topics that merit investigation in future work. We will touch on a few of them to finish our discussion.

Relevant Deformations Perhaps the first natural extension of our work is to include relevant deformations in the EWN calculation. There are a few reasons why this is interesting. First, one would like to test the continued correspondence between the QFC and EWN when it comes to the QNEC. The QFC arguments do not care whether relevant deformations are turned on, so one would expect that the same is true in EWN. This is indeed the case when the boundary theory is formulated on flat space [17], and one would expect similar results to hold when the boundary is curved.

Another reason to add in relevant deformations is to test the status of the Conformal QNEC when the theory is not a CFT. To be more precise, the $(\delta\overline{X})^2$ and s^2 calculations we performed differed by a Weyl transformation on the boundary, and since our boundary theory was a CFT this was a natural thing to do. When the boundary theory is not a CFT, what is the relationship between $(\delta\overline{X})^2$ and s^2? One possibility, perhaps the most likely one, is that they simply reduce to the same inequality, and the Conformal QNEC no longer holds.

Finally, and more speculatively, having a relevant deformation turned on when the background is curved allows for interesting state-independent inequalities from EWN. We saw that for a CFT the state-independent terms in both $(\delta\overline{X})^2$ and s^2 were trivially positive. Perhaps when a relevant deformation is turned on more nontrivial results uncover themselves, such as the possibility of a c-theorem hiding inside of EWN. We are encouraged by the similarity of inequalities used in recent proofs of the c-theorems to inequalities obtained from EWN [297].

Higher Dimensions Another pressing issue is extending our results to $d = 6$ and beyond. This is an algebraically daunting task using the methods we have used for $d \leq 5$. Considering the ultimate simplicity of our final expressions, especially compared to the intermediate steps in the calculations, it is likely that there are better ways of formulating and performing the analyses we performed here. It is hard to imagine performing the full $d = 6$ analysis without such a simplification.

Further Connections Between EWN and QFC Despite the issues outlined in §6.3, we are still intrigued by the similarities between EWN and the QFC. It is extremely natural to couple the boundary theory in AdS/CFT to gravity using a braneworld setup [298, 299, 300, 295]. Upon doing this, one can formulate the QFC on the braneworld. However, at the same time near-boundary EWN becomes lost, or at least changes form: extremal surfaces anchored to a brane will in general not be orthogonal to the brane, and in that case a null deformation on the brane will induce a timelike deformation of the extremal surface in the vicinity of the brane. Of course, one has to be careful to take into account the uncertainty in the position of the brane since we are dealing with expectation values of operators, which complicates things. We hope that such an analysis could serve to unify the QFC with EWN, or at least illustrate their relationship with each other.

Conformal QNEC from QFC While we emphasized the apparent similarity between the EWN-derived inequality $(\delta\overline{X})^2 \geq 0$ and the QFC, the stronger EWN inequality $s^2 \geq 0$ is nowhere to be found in the QFC discussion. It would be interesting to see if there is a direct QFC calculation that yields the Conformal QNEC (rather than first deriving the ordinary QNEC and then performing a Weyl transformation). In particular, the Conformal QNEC applies even in cases where $\theta_{(k)}$ is nonzero, while in those cases the QFC is dominated by classical effects. Perhaps there is a useful change of variables that one can do in the semiclassical gravity when the matter sector is a CFT which makes the Conformal QNEC manifest from the QFC point of view. This is worth exploring.

Chapter 7

Entropy Variations and Light Ray Operators from Replica Defects

7.1 Introduction

Despite much progress in understanding entanglement entropy using bulk geometric methods in holographic field theories [4, 236, 282], significantly less progress has been made on the more difficult problem of computing entanglement entropy directly in field theory. Part of what makes entanglement entropy such a difficult object to study in field theory is its inherently non-local and state-dependent nature.

One way to access the structure of entanglement in field theories is to study its dependence on the shape of the entangling surface. Such considerations have led to important results on the nature of entanglement in quantum field theories [301, 280, 17, 302, 303, 15, 304, 305, 306]. To study the shape dependence of entanglement entropy for QFTs in $d > 2$ dimensions, consider a Cauchy slice Σ containing a subregion $\mathcal{R}$ with entangling surface $\partial\mathcal{R}$ in a general conformal field theory. By unitary equivalence of Cauchy slices which intersect the same surface $\partial\mathcal{R}$, the entanglement entropy for some fixed global state can be viewed as a functional of the entangling surface embedding coordinates $X^\mu(y^i)$ where the y^i with $i = 1, ..., d-2$ are internal coordinates on $\partial\mathcal{R}$. We write:

$$S_{\mathcal{R}} = S[X(y)]. \tag{7.1.1}$$

The shape dependence of the entanglement entropy can then be accessed by taking functional derivatives. In particular, we can expand the entanglement entropy about some background entangling surface $X(y) = X_0(y) + \delta X(y)$ as

$$\begin{aligned} S[X] = S[X_0] &+ \int d^{d-2}y \, \frac{\delta S_{\mathcal{R}}}{\delta X^\mu(y)}\bigg|_{X_0} \delta X^\mu(y) \\ &+ \int d^{d-2}y d^{d-2}y' \, \frac{\delta^2 S_{\mathcal{R}}}{\delta X^\mu(y)\delta X^\nu(y')}\bigg|_{X_0} \delta X^\mu(y)\delta X^\nu(y') + \dots . \end{aligned} \tag{7.1.2}$$

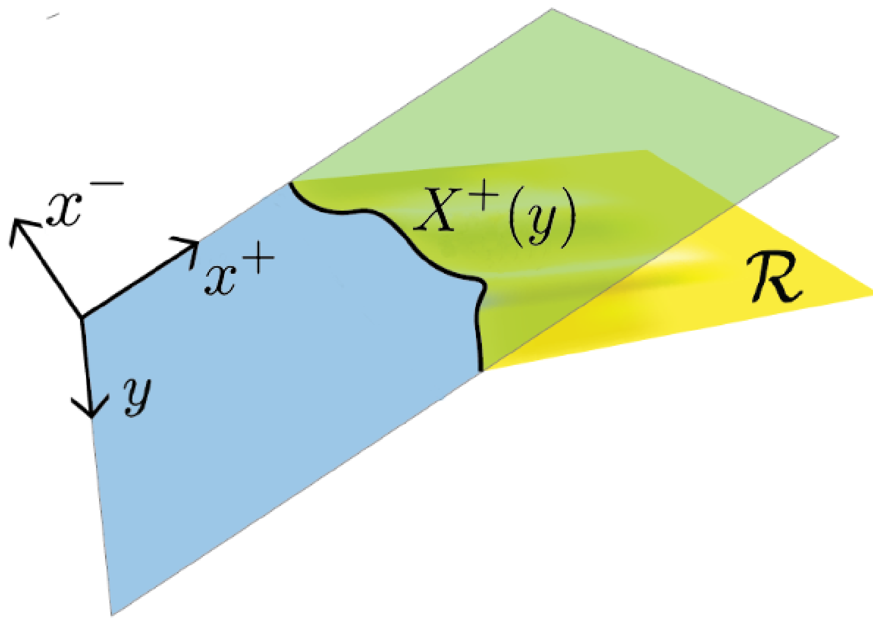


Figure 7.1: We consider the entanglement entropy associated to a spatial subregion $\mathcal{R}$. The entangling surface lies along $x^- = 0$ and $x^+ = X^+(y)$. In this work, we study the dependence of the entanglement entropy on the profile $X^+(y)$.

This second variation has received a lot of attention in part because it is an essential ingredient in defining the *quantum null energy condition* (QNEC) [13, 303]. The QNEC bounds the null-null component of the stress tensor at a point by a specific contribution from the second shape variation of the entanglement entropy. More specifically, this second variation can be naturally split into two pieces - the *diagonal* term which is proportional to a delta function in the internal coordinates y^i and the *off-diagonal* terms[1]

$$\frac{\delta^2 S_{\mathcal{R}}}{\delta X^+(y)\delta X^+(y')} = S''(y)\delta^{(d-2)}(y-y') + (\text{off-diagonal}). \tag{7.1.3}$$

where (X^+, X^-) are the null directions orthogonal to the defect. The QNEC states that the null energy flowing past a point must be lower bounded by the diagonal second variation

$$\langle T_{++}(y)\rangle \geq \frac{\hbar}{2\pi}S''(y), \tag{7.1.4}$$

where we are taking $\mathcal{R}$ to be a Rindler wedge. This inequality was first proposed as the $G_N \to 0$ limit of the quantum focussing conjecture [13], and was first proven in free and

[1]Note that the entanglement entropy, being UV divergent, will typically have divergent contributions that are local to the entangling surface. These will show up as a limited set of diagonal/contact terms in (7.1.3). For deformations about a sufficiently flat entangling surface these terms do not contribute to the contact term that is the subject of the QNEC. The divergent terms will not be the subject of investigation here.

super-renormalizable field theories in [280]. The proof for general QFTs with an interacting UV fixed point was given in [15]. More recently, yet another proof was given using techniques from algebraic quantum field theory [16].

The method of proof in the free case involved explicitly computing S''_{++} where it was found that

$$S'' = \frac{2\pi}{\hbar} \langle T_{++} \rangle - Q \tag{7.1.5}$$

where for general states $Q \geq 0$. In contrast, the proof in general QFTs relied on relating the inequality (7.1.4) to the causality of a certain correlation function involving modular flow. This left open the question of whether S'' could be explicility computed in more general field theories.

In [307] the diagonal term S'' was computed in large N QFTs in states with a geometric dual. Remarkably, the result was

$$S''(y) = 2\pi \langle T_{++}(y) \rangle \tag{7.1.6}$$

where we have now set $\hbar = 1$. In other words, $Q = 0$ for such theories. In that work, it was argued that neither finite coupling nor finite N corrections should affect this formula. This led the authors of [307] to conjecture (7.1.6) for all interacting CFTs. The main goal of this paper is to provide evidence for (7.1.6) in general CFTs with a twist gap.

The method of argument will follow from the replica trick for computing entanglement entropy. The replica trick uses the formula

$$S[\mathcal{R}] = \lim_{n \to 1} (1 - n\partial_n) \log \mathrm{Tr}[\rho_{\mathcal{R}}^n] \tag{7.1.7}$$

to relate the entanglement entropy to the partition function of the CFT on a replicated manifold [308, 309] (see also [310, 311, 312, 313])

$$\mathrm{Tr}[\rho_{\mathcal{R}}^n] = Z_n/(Z_1)^n. \tag{7.1.8}$$

At integer n, Z_n can be computed via a path integral on a branched manifold with n-sheets. Alternatively, one can compute this as a path integral on an unbranched manifold but in the presence of a twist defect operator Σ_n of co-dimension 2 that lives at the entangling surface [314]. Doing so allows us to employ techniques from defect CFTs. See [315, 316, 317, 318] for a general introduction to these tools.

In particular, shape deformations of the defect are controlled by a defect operator, namely the displacement operator, with components $\hat{D}_+, \hat{D}_-$. This operator is universal to defect CFTs. Its importance in entanglement entropy computations was elucidated in [315, 15, 314]. Consequently, the second variation of the entanglement entropy is related to the two-point function of displacement operators

$$\frac{\delta^2 S}{\delta X^+(y) \delta X^+(y')} = \lim_{n \to 1} \frac{-2\pi}{n-1} \langle \Sigma_n^\psi \hat{D}_+(y) \hat{D}_+(y') \rangle , \tag{7.1.9}$$

where the notation Σ_n^ψ will be explained in the next section.

Since we are interested in the delta function contribution to this second variation, we can take the limit where the two displacement operators approach each other, $y \to y'$. This suggests that we should study the OPE of two displacement operators and look for terms which produce a delta function, at least as $n \to 1$.

It might seem strange to look for a delta function in an OPE since the latter, without further input, results in an expansion in powers of $|y - y'|$. We will find a delta function can emerge from a delicate interplay between the OPE and the replica limit $n \to 1$.

An obvious check of our understanding of (7.1.6) is to explain how this formula can be true for interacting theories while there exist states for which $Q > 0$ in free theories. This is a particularly pertinent concern in, for example, $\mathcal{N} = 4$ super-Yang Mills where one can tune the coupling to zero while remaining at a CFT fixed point. We will find that in the free limit certain terms in the off-diagonal contributions of (7.1.3) become more singular and "condense" into a delta function in the zero coupling limit. In a weakly interacting theory it becomes a question of resolution as to whether one considers Q to be zero or not.

In fact this phenomenon is not unprecedented. The authors of [319] studied energy correlation functions in a so called conformal collider setup. The statistical properties of the angular distribution of energy in excited states collected at long distances is very different for free and interacting CFTs. We conjecture that these situations are controlled by the same physics. Explicitly, in certain special "near vacuum" states, there is a contribution to the second variation of entanglement that can be written in terms of these energy correlation functions.

Schematically, we will find

$$\frac{\delta^2 S}{\delta X^+(y)\delta X^+(y')} - \frac{2\pi}{\hbar} \langle T_{++} \rangle \, \delta^{(d-2)}(y - y') \sim \int ds e^s \, \langle \mathcal{O} \hat{\mathcal{E}}_+(y) \hat{\mathcal{E}}_+(y') e^{iKs} \mathcal{O} \rangle \tag{7.1.10}$$

where

$$\hat{\mathcal{E}}_+(y) = \int_{-\infty}^{\infty} d\lambda \, \langle T_{++}(x^+ = \lambda, x^- = 0, y) \rangle \tag{7.1.11}$$

is the averaged null energy operator discussed in [319] and the $\mathcal{O}$'s should be thought of as state-creation operators. The operator K is the boost generator about the undeformed entangling surface.

The singularities in $|y - y'|$ of the correlator in (7.1.10) are then understood by taking the OPE of two averaged null energy operators. This OPE was first discussed in [319] where a new non-local "light ray" operator of spin 3 was found to control the small $y - y'$ limit.

In the free limit, we will show that this non-local operator has the correct scaling dimension to give rise to a new delta function term in (7.1.10). In the interacting case this operator picks up an anomalous dimension and thus lifts the delta function.

In other words, the presence of an extra delta function in the second variation of the entanglement entropy in free theories can be viewed as a manifestation of the singular behavior of the conformal collider energy correlation functions in free theories. This is just another manifestation of the important relationship between entanglement and energy density in QFT.

The presence of this spin-3 light ray operator in the shape variation of entanglement in specific states however points to an issue with our defect OPE argument. In particular one can show that this contribution cannot come directly from one of the local defect operators that we enumerated in order to argue for saturation. Thus one might worry that there are other additional non-trivial contributions to the OPE that we miss by simply analyzing this local defect spectrum. The main issue seems to be that the $n \to 1$ limit does not commute with the OPE limit. Thus in order to take the limit in the proper order we should first re-sum a subset of the defect operators in the OPE before taking the limit $n \to 1$. For specific states we can effectively achieve this resummation (by giving a general expression valid for finite $|y - y'|$) however for general states we have not managed to do this. Thus, we are not sure how this spin-3 light ray operator will show up for more general states beyond those covered by (7.1.10). Nevertheless we will refer to these non-standard contributions as arising from "nonlocal defect operators."

The basic reason it is hard to make a general statement is that entanglement can be thought of as a state dependent observable. This state dependence shows up in the replica trick as a non-trivial n dependence in the limit $n \to 1$ so the order of limits issue discussed above is linked to this state dependence. We are thus left to compute the OPE of two displacement operators for some specific states and configurations. This allows us to check the power laws that appear in the $|y_1 - y_2|$ expansion for possible saturation violations. Given this we present two main pieces of evidence that the nonlocal defect operators do not lead to violations of QNEC saturation. The first is the aforementioned near vacuum state calculation. The second is a new calculation of the fourth shape variation of *vacuum* entanglement entropy which is also sensitive to the displacement operator defect OPE. In both cases we find that the only new operator that shows up is the spin-3 light ray operator. The outline of the paper is as follows.

- In Section 7.2, we begin by reviewing the basics of the replica trick and the relevant ideas from defect conformal field theory. We review the spectrum of local operators that are induced on the defect, including the infinite family of so-called higher spin displacement operators. We show that, in an interacting theory, these higher spin operators by themselves cannot contribute to the diagonal QNEC. We also present a present a certain conjecture about the nonlocal defect operators.

- In Section 7.3, we discuss how a delta function appears in the OPE of two displacement operators. We focus on a specific defect operator that limits to T_{++} as $n \to 1$. For this defect operator we derive a prediction for the ratio of the $D_+ D_+$ OPE coefficient and its anomalous defect dimension. In Section 7.4, we check this prediction by making use of a modified Ward identity for the defect theory. In Appendix A.18-A.19 we also explicitly compute the anomalous dimension and the OPE coefficient to confirm this prediction.

- In Section 7.5, we take up the concern that there could be other operators which lead to delta functions even for interacting CFTs. To do this, we compute the defect four point function $\mathcal{F}_n := \langle \Sigma_n^0 \hat{D}_+(y_1)\hat{D}_+(y_2)\hat{D}_-(y_3)\hat{D}_-(y_4)\rangle$ in the limit $n \to 1$. From this we can read off the spectrum by analyzing the powers of $|y_1 - y_2|$ that appear as $y_1 \to y_2$. We will find that these powers arise from the light-ray OPE of two averaged null energy operators.

- Finally, in Section 7.6, we check our results by explicitly computing the entanglement entropy second variation in near-vacuum states. By using null quantiation for free theories, we show that our results agree with that of [303].

- In Section 7.7, we end with a discussion of our results.

7.2 Replica Trick and the Displacement Operator

In this section, we will review the replica trick and discuss the connection between entanglement entropy and defect operators. This naturally leads to the displacement operator, which will be the key tool for studying (7.1.6).

As outlined in the introduction, the replica trick instructs us to compute the partition function $Z_n/(Z_1)^n = \mathrm{Tr}[\rho_{\mathcal{R}}^n]$, which can be understood as a path integral on a branched manifold $\mathcal{M}_n(\mathcal{R})$, where taking the product of density matrices acts to glue each consecutive sheet together. Using the state operator correspondence, a general state can be represented by the insertion of of a scalar operator in the Euclidean section, so that

$$Z_n = \langle \psi^{\dagger\otimes n} \psi^{\otimes n} \rangle_{\mathcal{M}_n(\mathcal{R})} \tag{7.2.1}$$

where each ψ is inserted on cyclicly consecutive sheets. Alternatively, we can view this $2n$-point correlation function as being computed not on an n-sheeted manifold but on a manifold with trivial topology in the presence of a codimension 2 twist defect operator

$$Z_n = \langle \Sigma_n^0 \psi^{\dagger\otimes n} \psi^{\otimes n} \rangle_{\mathrm{CFT}^{\otimes n}/\mathbb{Z}_n} \equiv \langle \Sigma_n^\psi \rangle \tag{7.2.2}$$

where we have used a compact notation for the twist operator that includes the state operator insertions: $\Sigma_n^\psi \equiv \Sigma_n^0 \psi^{\dagger\otimes n}\psi^{\otimes n}$. It is convenient (and possible) to orbifold the $\mathrm{CFT}^{\otimes n}$ which projects onto states in the singlet of $\mathbb{Z}_n$. This allows us to work with a CFT that for example has only one conserved stress tensor.

We take the defect Σ_n^0 to be associated to a flat cut of a null plane in Minkowski space. We take the metric to be

$$ds^2 = dzd\bar{z} + d\vec{y}^2 \tag{7.2.3}$$

where z and $\bar{z}$ are complexified lightcone coordinates. That is, on the Lorentzian section we have $z = -x^- = x + i\tau$ and $\bar{z} = x^+ = x - i\tau$. Thus, we take the defect to lie at $x^- = X^-(y) = 0$ and $x^+ = X^+(y) = 0$.

For the case of a flat defect, the operator Σ_n^0 breaks the conformal symmetry group down to $SO(2) \times SO(d-1,1)$, with the $SO(2)$ corresponding to the rotations of the plane orthogonal to the defect. This symmetry group suggests that a bulk dimension-d CFT descends to a dimension $d-2$ defect CFT, which describes the excitations of the defect. We can thus use the language of boundary CFTs to analyze this problem. We will only give a cursory overview of this rich subject. For a more thorough review of the topic see [15, 314, 315], and for additional background see [320, 321, 322, 323]. The important aspect for us will be the spectrum of operators that live on the defect.

The spectrum of operators associated to the twist defect was studied in [15]. In that work, techniques were laid out to understand how bulk primary operators induce operators on the defect. This can be quantitatively understood by examining the two-point function of bulk scalar operators in the limit that they both approach the defect. We imagine that as a bulk operator approaches the defect, we can expand in the transverse distance $|z|$ in a bulk to defect OPE so that

$$\lim_{|z|\to 0} \sum_{k=0}^{n-1} \mathcal{O}^{(k)}(z,\bar{z},y)\Sigma_n^0 = z^{-(\Delta_{\mathcal{O}}+\ell_{\mathcal{O}})}\bar{z}^{-(\Delta_{\mathcal{O}}-\ell_{\mathcal{O}})} \sum_j C_{\mathcal{O}}^j z^{(\hat{\Delta}_j+\ell_j)/2}\bar{z}^{(\hat{\Delta}_j-\ell_j)/2}\hat{\mathcal{O}}_j(y)\Sigma_n^0 \tag{7.2.4}$$

where $\Delta_{\mathcal{O}}$ is the dimension of the bulk operator, while $\hat{\Delta}_j$ is the dimension of the jth defect operator $\hat{\mathcal{O}}_j$. Every operator is also now labeled by its spin, ℓ, under the $SO(2)$ rotations $z \to ze^{-i\phi}$. From the defect CFT point of view, the $SO(2)$ spin is an internal symmetry and the ℓ_j's are the defect operators' associated quantum numbers. Notice that the $\mathbb{Z}_n$ symmetry has the effect of projecting out operators of non-integer spin. This is another reason for why the $\mathbb{Z}_n$ orbifolding is needed for treating the theory on the defect as a normal Euclidean CFT.

Equation (7.2.4) suggests an easy way to obtain defect operators in terms of the bulk operators. Consider the lowest dimension defect operator $\hat{\Delta}_\ell$ of a fixed spin ℓ. Then we can extract the defect operator via a residue projection,

$$\hat{\mathcal{O}}_\ell(0)\Sigma_n^0 = \lim_{|z|\to 0} \frac{|z|^{-\hat{\tau}_\ell+\tau_\alpha}}{2\pi i} \oint \frac{dz}{z} z^{-\ell+\ell_\alpha} \sum_{k=0}^{n-1} \mathcal{O}_\alpha^{(k)}(z,|z|^2/z,0)\Sigma_n^0 \tag{7.2.5}$$

where $\hat{\tau}_\ell$ and τ_α are the twists of the defect and bulk operators respectively. Note that these leading twist operators are necessarily defect primaries.

Note that in general, due to the breaking of full conformal symmetry, $\hat{\Delta}_\ell$ will contain an anomalous dimension $\gamma_\ell(n)$. In this paper we will mainly be interested in the defect spectrum near $n=1$ so after analytically continuing in n we can expand $\gamma_\ell(n)$ around $n=1$ as $\gamma(n) = \gamma^{(0)} + \gamma^{(1)}(n-1) + \mathcal{O}((n-1)^2)$. We now give a brief review of the various defect operators discovered in [15].[2]

[2]See [324] for a complementary method for computing the defect spectrum from the bootstrap and an appropriate Lorentzian inversion formula. It would be interesting to derive some of the results presented here in that language.

Operators induced by bulk scalars or spin one primaries

Associated to each bulk scalar ϕ, or spin-one primary V_μ, of dimension Δ_ϕ, Δ_V, the authors of [15] found a family of defect operators of dimension $\hat{\Delta}^\ell_{\phi,V} = \Delta_{\phi,V} - J_{\phi,V} + \ell + \gamma^{(1)}_{\phi,V}(n-1) + \mathcal{O}((n-1)^2)$ with $SO(2)$ spin ℓ along with their defect descendants. Here $J_{\phi,V} = 0, 1$ for ϕ and V respectively and importantly $\ell \geq J$. The anomalous dimensions for the operators induced by bulk scalars, γ_ϕ, are given in formula (3.25) of [15]. We will not be concerned with these two families in this paper.

Operators induced by bulk primaries of spin $J \geq 2$

For primary operators of spin $J \geq 2$, the authors of [15] again found a similar family of operators with dimensions $\hat{\Delta}^\ell_J = \Delta_J - J + \ell + \gamma^{(1)}_{J,\ell}(n-1) + \mathcal{O}((n-1)^2)$ where $\ell \geq J$.

For a primary of spin $J \geq 2$, there are also $J-1$ "new" operators with $SO(2)$ charge $J-1 \geq \ell \geq 1$. These "displacement operators" can be written at integer n as

$$\hat{D}^J_\ell = i \oint d\overline{z} \frac{\overline{z}^{J-\ell-1}}{|z|^{\gamma_{J,\ell}(n)}} \sum_{k=0}^{n-1} \mathcal{J}^{(k)}_{+\dots+}(|z|^2/\overline{z}, \overline{z}) \tag{7.2.6}$$

where J is the spin of the bulk primary $\mathcal{J}_{+\dots+}$ and $1 \leq \ell \leq J-1$ is the $SO(2)$ spin of the defect operator. The power of $|z|^\gamma$ accounts for the dependence of the defect operator dimension on n.

We will primarily be interested in the spectrum of T_{++} on the defect for which there is only one displacement operator, $\hat{D}_+$. The displacement operator can also be equivalently defined in terms of the diffeomorphism Ward identity in the presence of the defect [315]

$$\nabla^\mu \langle \Sigma^\psi_n T_{\mu\nu} \rangle = \delta(z, \overline{z}) \langle \Sigma^\psi_n \hat{D}_\nu \rangle. \tag{7.2.7}$$

This implies that $\hat{D}_+$ corresponds to a null deformation of the orbifold partition function with respect to the entangling surface. In particular, entropy variations are given by $\hat{D}_+$ insertions in the limit $n \to 1$:

$$\langle \Sigma^\psi_n \hat{D}_+(y) \rangle = (n-1) \langle \Sigma^\psi_n \rangle \frac{\delta S_\psi}{\delta x^+(y)} + \mathcal{O}((n-1)^2) \tag{7.2.8}$$

The generalization to two derivatives is then just

$$\langle \Sigma^\psi_n \hat{D}_+(y) \hat{D}_+(y') \rangle = (n-1) \langle \Sigma^\psi_n \rangle \frac{\delta^2 S_\psi}{\delta X^+(y) X^+(y')} + \mathcal{O}((n-1)^2). \tag{7.2.9}$$

We see importantly that statements about entropy variations can be related directly to displacement operator correlation functions.

7.3 Towards saturation of the QNEC

With the displacement operator in hand, we can now describe an argument for QNEC saturation. As just described, second derivatives of the entanglement entropy can be computed via two point functions of the defect CFT displacement operator. Thus, we are interested in proving the following identity:

$$\lim_{n\to 1}\frac{1}{n-1}\langle \Sigma_n^\psi \hat{D}_+(y)\hat{D}_+(y')\rangle = 2\pi\,\langle \hat{T}_{++}(y)\rangle_\psi\,\delta^{d-2}(y-y') \\ + (\text{less divergent in } |y-y'|) \tag{7.3.1}$$

where $|\psi\rangle$ is any well-defined state in the CFT.

Since we are only interested in the short distance behavior of this equality - namely the delta function piece - we can examine the OPE of the displacement operators

$$\frac{1}{n-1}\hat{D}_+(y)\hat{D}_+(y') = \frac{1}{n-1}\sum_\alpha \frac{c_\alpha(n)\hat{\mathcal{O}}^\alpha_{++}(y)}{|y-y'|^{2(d-1)-\Delta_\alpha+\gamma_\alpha(n)}} + \text{descendants} \tag{7.3.2}$$

where Δ_α is the dimension of the defect primary $\hat{\mathcal{O}}_\alpha$ at $n=1$ and $\gamma_\alpha(n)$ gives the n dependence of the dimension away from $n=1$. We will refer to $\gamma_\alpha(n)$ as an anomalous dimension. Note that this is an OPE defined purely in the defect CFT. The $++$ labels denote the $SO(2)$ spin of the defect operator, which must match on both sides of the equation. The dimension of the displacement operators themselves are independent of n and fixed by a Ward identity to be $d-1$.

At first glance, this equation would suggest that there are no delta functions in the OPE, only power law divergences. In computing the entanglement entropy, however, we are interested in the limit as $n\to 1$. In this limit, it is possible for a power law to turn into a delta function as follows:

$$\lim_{n\to 1}\frac{n-1}{|y-y'|^{d-2-\gamma^{(1)}(n-1)}} = \frac{S_{d-3}}{\gamma^{(1)}}\delta^{(d-2)}(y-y') \tag{7.3.3}$$

where $\gamma = \gamma^{(1)}(n-1)+\mathcal{O}((n-1)^2)$ and S_{d-3} in the area of the $d-3$ sphere. Comparison of equations (7.3.3) and (7.3.2) shows that a delta function can "condense" in the $\hat{D}_+\times\hat{D}_+$ OPE only if the OPE coefficient and anomalous dimension obey

$$c_\alpha(n)/\gamma_\alpha(n) \sim (n-1)+\mathcal{O}((n-1)^2) \tag{7.3.4}$$

as n approaches 1.

This is, however, not sufficient for a delta function to appear in (7.3.2) as $n\to 1$. We also need to have

$$\Delta_\alpha = d \tag{7.3.5}$$

at $n = 1$. In other words, the defect operators we are looking for must limit to an operator of $SO(2)$ spin two and dimension d as the defect disappears. Clearly, the $\ell = 2$ operator induced by the bulk stress tensor, $\hat{T}_{++}$, satisfies these conditions. Indeed, the first law of entanglement necessitates the appearance of $\hat{T}_{++}$ in the $\hat{D}_+ \times \hat{D}_+$ OPE with a delta function (see Section 7.4 below).

Our main claim, (7.3.1), is the statement that no other operator can show up in (7.3.2) whose contribution becomes a delta function in the $n \to 1$ limit. In the rest of this section, we enumerate all the possible operators that could appear in the $\hat{D}_+ \times \hat{D}_+$ OPE (7.3.2).

Defect operators induced by low-dimension scalars

If there exists a scalar operator of dimension $\Delta = d - 2$, then the associated defect operator with $SO(2)$ spin $\ell = 2$ will have dimension $\Delta = d$ at leading order in $n - 1$. This possibility was discussed in [307]. The contribution of such an operator was found to drop out of the final quantity $\langle T_{++} \rangle - \frac{1}{2\pi} S''_{++}$ for holographic CFTs. We expect the same thing to happen in general CFTs in the presence of such an operator, so we ignore this possibility.

$\ell = 2$ operators induced by spin one primaries

As discussed earlier, these defect operators have dimension $\hat{\Delta} = \Delta_V + 1 + \mathcal{O}(n-1)$. We see that for spin one primaries not saturating the unitarity bound, i.e. $\Delta_V > d-1$, these cannot contribute delta functions. Actually, since these operators exist in the CFT at $n = 1$, we will argue in the next section that the first law of entanglement forces their OPE coefficients to be of order $(n - 1)^2$.

For spin-one primaries saturating the unitarity bound, V_μ is then the current associated to some internal symmetry. The entropy is uncharged under all symmetries, so such operators cannot contribute to $\hat{D}_+ \times \hat{D}_+$.

$\ell = 2$ higher spin displacement operators

The most natural candidate for contributions to the $\hat{D}_+ \times \hat{D}_+$ OPE are the $\ell = 2$ higher spin displacement operators discussed in the previous section. These operators are given by equation (7.2.6).

To show that such operators do not contribute delta functions to $\hat{D}_+ \times \hat{D}_+$, we need to argue that their dimensions $\Delta_n(\ell = 2, J)$ do not limit to d as $n \to 1$. As discussed in the previous section, the dimensions of the higher spin displacement operators are given by

$$\Delta_n(\ell, J) = \Delta_J - J + \ell + \mathcal{O}(n-1). \tag{7.3.6}$$

The anomalous dimensions have not yet been computed but we expect them to be of order $n - 1$, although we will not need this calculation here. The important point for us will be that in a CFT with a twist gap, the leading order dimension of these operators is

$$\Delta_n(2, J) = \tau_J + 2 + \mathcal{O}(n-1) > d \tag{7.3.7}$$

assuming the twist of the bulk primaries satisfies $\tau_J > d-2$. Here we are using a result on the convexity of twist on the leading Regge trajectory for all J proven in [325]. We see that the bulk higher spin operators would need to saturate the unitarity bound to contribute a delta function. Furthermore, there could be defect descendants of the form $(\partial_y^i \partial_y^i)^k \hat{D}_{++}^J(y)$. But such operators will necessarily contribute to the OPE with larger, positive powers of $|y-y'|$, hence they cannot produce delta functions.

Nonlocal defect operators

So far we have focused on the individual contribution of local defect operators and by power counting we see that these operators cannot appear in the diagonal QNEC. At fixed n, it is reasonable to conjecture that this list we just provided is complete. However we have not fully concluded that something more exotic does not appear in the OPE. As discussed in the introduction this possibility arises because the $n \to 1$ limit may not commute with the OPE.

Indeed, we will find evidence that something non-standard does appear in the displacement OPE. In Section 7.5 and Section 7.6 we will present some computations of correlation functions of the displacement operator for particular states and entangling surfaces. In these specific cases we will be able to make the analytic continuation to $n \to 1$ before taking the OPE. In both cases, we find that the power laws as $y_1 \to y_2$ are controlled by the dimensions associated to non-local spin-3 light ray operators [326]. In the discussion section we will come back to the possibility that these contributions come from an infinite tower of the local defect operators that we have thus far enumerated. We conjecture that when this tower is appropriately re-summed, we will find these non-standard contributions to the entanglement entropy.

We will refer to these operators as *nonlocal defect operators*, and we further conjecture that a complete list of such operators and dimensions is determined by the nonlocal $J=3$ lightray operators that appear in the lightray OPE of two averaged null energy operators as studied in [319, 327] for the CFT *without* a defect. In order to give further evidence for this conjecture, in Section 7.5 we will compute the analytic continuation of the spectrum of operators appearing around $n=1$ in the $\hat{D}_+ \times \hat{D}_+$ OPE by computing a fourth order shape variation of vacuum entanglement. Our answer is consistent with the above conjecture. While this relies on a specific continuation in n (a specific choice of "state dependence") we think this is strong evidence that we have not missed anything.

Before studying this nonlocal contribution further, we return to the local defect contribution where we would like to check that the ratio of $c(n)/\gamma(n)$ for $\hat{T}_{++}$ obeys (7.3.4).

7.4 Contribution of $\hat{T}_{++}$

In this section, we will review the first law argument which fixes the coefficient of the stress tensor defect operator to leading order in $n-1$. We will then use defect methods to demonstrate that the stress tensor does contribute with the correct ratio of $c(n)$ and $\gamma(n)$ to produce a delta function with the right coefficient demanded by the first law. To do this, we will make use of a slightly modified form of the usual diffeomorphism Ward identity in the presence of a twist defect that will compute $c(n)/\gamma(n)$. In Appendices A.18 and A.19, we also explicitly calculate $c(n)$ and $\gamma(n)$ separately for the stress tensor and show that they agree with the result of this sub-section.

The First Law

A powerful guiding principle for constraining which defect operators can appear in the OPE (7.3.2) is the first law of entanglement entropy. The entanglement entropy $S(\rho) = -\operatorname{Tr}[\rho\log\rho]$, when viewed as the expectation value of the operator $-\log\rho$, is manifestly non-linear in the state. The first law of entanglement says that if one linearizes the von Neumann entropy about a reference density matrix - σ - then the change in the entropy is just equal to the change in the expectation value of the vacuum modular Hamiltonian. Specifically it says that

$$\delta\operatorname{Tr}[\rho\log\rho] = \operatorname{Tr}[\delta\rho\log\sigma] \tag{7.4.1}$$

where $\rho = \sigma + \delta\rho$.

The case we will be interested in here is when σ is taken to be the vacuum density matrix for the Rindler wedge. The first law then tells us that the *only* contributions to $\langle\Sigma_n^\psi \hat{D}_+(y)\hat{D}_+(y')\rangle$ that are linear in the state as $n \to 1$ must come from the shape variations of the vacuum modular Hamiltonian.

The second shape derivative of the Rindler wedge modular Hamiltonian is easy to compute from the form of the vacuum modular Hamiltonian associated to generalized Rindler regions [12, 301, 328, 329]. Defining $\Delta\langle H_{\mathcal{R}}^\sigma\rangle_\psi = -\operatorname{Tr}[\rho_{\mathcal{R}}\log\sigma_{\mathcal{R}}] + \operatorname{Tr}[\sigma_{\mathcal{R}}\log\sigma_{\mathcal{R}}]$ to be the vacuum subtracted modular Hamiltonian for a general region $\mathcal{R}$ bounded by a cut of the $x^- = 0$ null plane, then we have the simple universal formula

$$\frac{\delta^2\Delta\langle H_{\mathcal{R}}^\sigma\rangle_\psi}{\delta X^+(y)\delta X^+(y')} = \frac{2\pi}{\hbar}\langle T_{++}\rangle_\psi\,\delta^{(d-2)}(y-y'). \tag{7.4.2}$$

This is a simple but powerful constraint on the displacement operator OPE; it tells us that the only operator on the defect which is manifestly linear in the state as $n \to 1$ and appears in $\hat{D}_+ \times \hat{D}_+$ at $n = 1$ is the stress tensor defect operator

$$\hat{T}_{++} = \oint \frac{d\bar{z}}{\bar{z}|z|^{\gamma_n}}\sum_{j=0}^{n-1} T_{++}^{(j)}(|z|^2/\bar{z},\bar{z}). \tag{7.4.3}$$

Thus, any other operator which appears in the OPE around $n = 1$ must contribute in a manifestly non-linear fashion. Examining the list of local defect operators discussed in Section 7.3 the only operators that are allowed by the above argument, aside from $\hat{T}_{++}$, are the higher spin displacement operators. As shown in [15] the limit $n \to 1$ of the expectation value of these operators give a contribution that is non-linear in the state.

We will return to these state dependent operators in later sections. Now we check that indeed the stress tensor contributes with the correct coefficient.

Using the modified Ward identity

In Appendix A.16, we prove the following intuitive identity:

$$\int d^{d-2}y' \langle \Sigma_n^0 \hat{D}_+(y')\hat{D}_+(y)T_{--}(w,\overline{w},0)\rangle = -\partial_{\overline{w}}\langle \Sigma_n^0 \hat{D}_+(y)T_{--}(w,\overline{w},0)\rangle. \tag{7.4.4}$$

We now show that the identity (7.4.4) allows us to compute the stress tensor contribution to the $\hat{D}_+ \times \hat{D}_+$ OPE, which can be written as:

$$\hat{D}_+(y)\hat{D}_+(y') \supset \frac{c(n)}{|y-y'|^{d-2-\gamma(n)}}\hat{T}_{++}(y) + \dots \tag{7.4.5}$$

where we have focused on the $\hat{T}_{++}$ contribution and the ellipsis stand for the defect descendants of $\hat{T}_{++}$. We are free to ignore other defect primaries since they get projected out by the $T_{--}(w,\overline{w},0)$ insertion in (7.4.4). Of course, since (7.4.4) involves a y integral, one might worry that we are using the OPE outside its radius of convergence. For now, we will follow through with this heuristic computation using the OPE. At the end of this subsection, we will say a few words about why this is justified.

Inserting (7.4.5) into (7.4.4) and ignoring the descendants, we find

$$\int d^{d-2}y' \frac{c(n)}{|y-y'|^{d-2-\gamma(n)}} \langle \Sigma_n^0 \hat{T}_{++}(y)T_{--}(w,\overline{w},0)\rangle = \frac{c(n)}{\gamma(n)} S_{d-3} \langle \Sigma_n^0 \hat{T}_{++}(y)T_{--}(w,\overline{w},0)\rangle \tag{7.4.6}$$

where S_n is the area of the unit n-sphere. We can write $\hat{T}_{++}(y)$ in terms of T_{++} integrated around the defect:

$$\hat{T}_{++}(y) = -\frac{1}{2\pi i}\sum_{k=0}^{n-1}\oint \frac{d\overline{z}}{\overline{z}|z|^{\gamma(n)}} T^{(k)}_{++}(|z|^2/\overline{z},\overline{z},y) \tag{7.4.7}$$

We now take the $n \to 1$ limit of equation (7.4.4). Since the right hand side starts at order $(n-1)$, we see that $c(n)$ must begin at one higher order in $n-1$ than $\gamma(n)$. Generically we expect $\gamma(n)$ to begin at order $n-1$ and in Appendix A.19 we will see that it does. We thus get the relation

$$\frac{c^{(2)}}{\gamma^{(1)}} \langle \Sigma_1^0 \hat{T}_{++}(y)T_{--}(w,\overline{w},0)\rangle = -\partial_n\big|_{n=1}\partial_{\overline{w}} \langle \Sigma_n^0 \hat{D}_+(y)T_{--}(w,\overline{w},0)\rangle \tag{7.4.8}$$

where $c(n) = c^{(1)}(n-1) + c^{(2)}(n-1)^2 + ...$ and $\gamma(n) = \gamma^{(1)}(n-1) + ...$.

At $n = 1$, $\langle \Sigma_1^0 \hat{T}_{++}(y) T_{--}(w, \overline{w}, 0)\rangle$ is just the usual stress tensor 2-point function. Moreover, we can evaluate the right hand side of (7.4.4) at order $(n-1)$ by following the steps leading up to eq. (3.31) of [15]. This leads to

$$\begin{aligned}\partial_{\overline{w}} \langle \hat{D}_+(y) T_{--}(w, \overline{w}, 0)\rangle\Big|_{|w|\to 0} &= i(n-1) \oint d\overline{z}\; \partial_{\overline{w}} \left(\int_0^{-\infty} \frac{d\lambda\; \lambda^2}{(\lambda-1)^2} \frac{c_T y^4}{4(w\overline{w} - w\overline{z}\lambda + y^2)^{d+2}} \right)\Bigg|_{|w|,|z|\to 0} \\ &= -2\pi(n-1)\frac{c_T}{4} y^{-2d} \end{aligned} \tag{7.4.9}$$

We are then left with the following expressions for c_1 and c_2:

$$c^{(2)} = \frac{2\pi\gamma^{(1)}}{S_{d-3}},\; c^{(1)} = 0 \tag{7.4.10}$$

This is exactly what is needed in order to write (7.4.5) near $y = y'$ as $\hat{D}_+(y)\hat{D}_+(y') \supset \delta^{(d-2)}(y-y')\hat{T}_{++}(y)$.

We now comment on the justification for using the $\hat{D}_+ \times \hat{D}_+$ OPE. Since the left hand side of (7.4.4) involves a y integral over the whole defect, one might worry that the we have to integrate outside the radius of convergence for the $\hat{D}_+ \times \hat{D}_+$ OPE. We see, however, that the y integral produces an enhancement in $(n-1)$ only for the T_{++} primary. In particular, this enhancement does not happen for the descendants of T_{++}. This suggests that if we were to plug in the explicit form of the defect-defect-bulk 3 point function into equation (7.4.4) we would have seen that the $(n-1)$ enhancement comes from a region of the y integral where $\hat{D}_+$ and $\hat{D}_+$ approach each other. We could then effectively cap the integral over y so that it only runs over regions where the OPE is convergent and still land on the same answer. As a check of our reasoning, in Appendices A.18 and A.19, we also compute the $c(n)$ and $\gamma(n)$ coefficients separately and check that they have the correct ratio.

7.5 Higher order variations of vacuum entanglement

In this section, we return to the possibility mentioned in Section 7.3 that something non-standard might appear in the displacement operator OPE. The authors of [15] argued that they had found a complete list of all local defect operators. This leaves open the possibility that the $n \to 1$ limit behaves in such a way that forces us to re-sum an infinite number of defect operators. In this Section and the next, we will find evidence that indeed this does occur. We will also give evidence that we have found a complete list of such nonlocal operators important for the $\hat{D}_+ \times \hat{D}_+$ OPE. In interacting theories with a twist gap this list does not include an operator with the correct dimension and spin that would contribute a delta function and violate saturation.

To get a better handle on what such a re-summed operator might be, we turn to explicitly computing the spectrum of operators in the $\hat{D} \times \hat{D}$ OPE. To do this, we consider the defect four point function

$$\mathcal{F}_n(y_1, y_2, y_3, y_4) = \langle \Sigma_n^0 \hat{D}_+(y_1)\hat{D}_+(y_2)\hat{D}_-(y_3)\hat{D}_-(y_4)\rangle \, . \tag{7.5.1}$$

We will consider configurations where $|y_1 - y_2| = |y_3 - y_4|$ are small but $|y_1 - y_4|$ is large. With these kinematics, we can use the $\hat{D} \times \hat{D}$ OPE twice and re-write the four point function as a sum over defect two point functions

$$\mathcal{F}_n = \sum_{\mathcal{O},\mathcal{O}'} \frac{c^{\mathcal{O}}_{++}(n) c^{\mathcal{O}'}_{--}(n) \, \langle \Sigma_n^0 \hat{\mathcal{O}}_{++}(y_2) \hat{\mathcal{O}}'_{--}(y_4)\rangle}{|y_1 - y_2|^{2(d-1)+\hat{\Delta}_n^{\mathcal{O}}} |y_3 - y_4|^{2(d-1)+\hat{\Delta}_n^{\mathcal{O}'}}} \tag{7.5.2}$$

where $\mathcal{O}, \mathcal{O}'$ denote the local defect primaries and their descendants appearing in $\hat{D} \times \hat{D}$. We immediately see that by examining the powers of $|y_1 - y_2|$ appearing in $\mathcal{F}_n$, we can read off the spectrum of operators we are after. That is, at least before taking the limit $n \to 1$. We have not attempted to compute the OPE coefficients explicitly for all the local defect operators. This is left as an important open problem that would greatly clarify some of our discussion, but this is beyond the scope of this paper.

If we assume that the $n \to 1$ limit commutes with the OPE limit $y_1 \to y_2$ we can now find a contradiction. To see this contradiction, we can compute $\lim_{n\to 1} \mathcal{F}_n$ in an alternate manner holding y_1, y_2 fixed and compare to (7.5.2). The main result we will find is that the divergences in $|y_1 - y_2|$ appear to arise from defect operators of dimension $\Delta_{J_*} - J_* + 2$ where $J_* = 3$ and Δ_{J_*} is defined by analytically continuing the dimensions in (7.3.6) to odd J (recall that (7.3.6) was only considered for even spins previously.) Generically we do not expect these particular dimensions to appear in the list of operator dimensions of the local defect operators that we enumerated. However we conjecture that by including such operator dimensions we complete the list of possible powers that can appear in the displacement OPE at $n = 1$.

This discussion further suggests that the final non-local defect operator that makes the leading contribution beside T_{++} should be an analytic continuation in spin of the local higher spin displacement operators. We will come back to this possibility in the discussion.

We now turn to computing $\mathcal{F}_n$ without using the defect OPE. In Appendix A.20, we explicitly do the analytic continuation of $\mathcal{F}_n$, but here we simply state the answer. We find that $\mathcal{F}_n$ takes the form

$$\begin{aligned}\mathcal{F}_n \sim (n-1) \int ds e^{-s} \Big\langle T_{--}(x^+ = 0, x^- = -1, y_3) \hat{\mathcal{E}}_+(y_1) \hat{\mathcal{E}}_+(y_2) T_{--}(x^+ = 0, x^- = -e^{-s}, y_4) \Big\rangle \\ + \mathcal{O}\left((n-1)^2\right),\end{aligned} \tag{7.5.3}$$

which can also be written as:

$$\mathcal{F}_n \sim (n-1) \frac{\Big\langle \mathcal{E}_-(y_3) \hat{\mathcal{E}}_+(y_1) \hat{\mathcal{E}}_+(y_2) \mathcal{E}_-(y_4) \Big\rangle}{\mathrm{vol}\, SO(1,1)}. \tag{7.5.4}$$

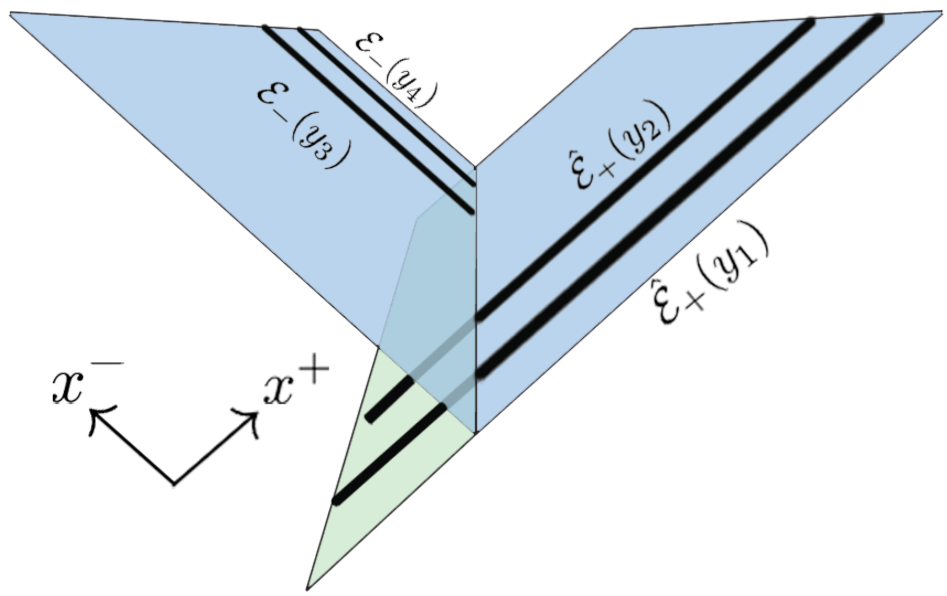


Figure 7.2: The answer for the defect four point function $\mathcal{F}_n$ upon analytic continuation to $n = 1$. We find that there are two insertions of half-averaged null energy operators, $\mathcal{E}_-$, as well as two insertions of $\hat{\mathcal{E}}_+$. Note that strictly speaking, in (7.5.3), the half-averaged null energy operators are inserted in the right Rindler wedge, but by CRT invariance of the vacuum, we can take the half-averaged null energy operators to lie in the left Rindler wedge instead, as in the figure.

The later division by the infinite volume of the 1 dimensional group of boosts is necessary to remove an infinity arising from an overall boost invariance of the four light-ray integrals. See for example [330]. The un-hatted $\mathcal{E}_-$ operators represent half averaged null energy operators, integrated from the entangling surface to infinity. Similar modifications to light-ray operators were used in [327] in order to define their correlation functions and it is necessary here since otherwise the full light-ray operator would annihilate the vacuum.

We see that the effect of two $\hat{D}_+$ insertions was to create two $\hat{\mathcal{E}}_+$ insertions in the limit $n \to 1$. Thus considering the OPE of two displacement operators leads us to the OPE of two null energy operators. This object was studied in [319] and more recently [327]. These authors found that the two averaged null energy insertions can be effectively replaced by a sum over spin 3 “light-ray" operators, one for each Regge trajectory. In other words,

$$\hat{\mathcal{E}}_+(y_1)\hat{\mathcal{E}}_+(y_2) \sim \sum_i \frac{c_i \hat{\mathbb{O}}_i(y_2)}{|y_1 - y_2|^{2(d-2)-\tau^i_{\text{even},J=3}}} \tag{7.5.5}$$

where $\tau^i_{\text{even},J=3}$ is the twist of the even J primaries on the ith Regge trajectory analytically continued down to $J = 3$. A delta function can appear in this expression if $\tau^i_{\text{even},J=3} = d-2$, i.e. if the dimensions saturate the unitarity bound.

Using the recent results in [325] again, we know that the twists on the leading Regge trajectory obey $\frac{d\tau(J)}{dJ} \geq 0$ and $\frac{d^2\tau(J)}{dJ^2} \leq 0$. Since the stress tensor saturates the unitarity bound, for a theory with a twist gap we know that $\tau^i_{\text{even},J=3} > d-2$, therefore there cannot be a delta function in $y_1 - y_2$. By the previous discussion then, formula (7.5.3) suggests that there are no extra operators besides the stress tensor that produce a delta function. To give further evidence for this we next explicitly work out another case where we can compute the $n \to 1$ limit before we do the OPE and we find the same spectrum of operators.

7.6 Near Vacuum States

We have just seen that the OPE of two displacement operators appears to be controlled by defect operators of dimension $\Delta_{J=3} - 1$. As a check of this result, we will now independently compute the second variation of the entanglement entropy for a special class of states. In these states, we will again see the appearance of the OPE of two null energy operators $\hat{\mathcal{E}}_+(y)\hat{\mathcal{E}}_+(y')$. This again implies a lack of a delta function for theories with a twist gap.

This computation is particularly illuminating in the case of free field theory where we can use the techniques of null quantization (see Appendix A.21 for a brief review). Null quantization allows us to reduce a computation in a general state of a free theory to a near-vacuum computation. In this way we will also reproduce the computations in [303] using a different method.

The state we will consider is a near vacuum state reduced to a right half-space

$$\rho(\lambda) = \sigma + \lambda\delta\rho + \mathcal{O}(\lambda^2) \tag{7.6.1}$$

where σ is the vacuum reduced to the right Rindler wedge. We can imagine $\rho(\lambda)$ as coming from the following pure state reduced to the right wedge

$$|\psi(\lambda)\rangle = \left(1 + i\lambda \int drd\theta d^{d-2}y g(r,\theta,y)\mathcal{O}(r,\theta,y)\right)|\Omega\rangle + \mathcal{O}(\lambda^2) \tag{7.6.2}$$

where (r,θ,y) are euclidean coordinates centered around the entangling surface and

$$\mathcal{O}(r,\theta,y) = \exp\left(iH_R^\sigma\theta\right)\mathcal{O}(r,0,y)\exp\left(-iH_R^\sigma\theta\right) \tag{7.6.3}$$

where H_R^σ is the Rindler Hamiltonian for the right wedge.

From this expression for $|\Psi(\lambda)\rangle$, we have the formula

$$\delta\rho = \sigma \int drd\theta d^{d-2}y f(r,\theta,y)\mathcal{O}(r,\theta,y) \tag{7.6.4}$$

where

$$f(r,\theta,y) = i\left(g(r,\theta,y) - g(r,2\pi-\theta,y)^*\right). \tag{7.6.5}$$

Note that f obeys the reality condition $f(r,\theta,y) = f(r,2\pi-\theta,y)^*$.

We are interested in calculating the shape variations of the von-Neumann entropy. To this aim, since the vacuum has trivial shape variations we can compute the vacuum-subtracted entropy ΔS instead. We start by using the following identity

$$\Delta S = \mathrm{Tr}\left((\rho(\lambda) - \sigma)\, H^\sigma\right) - S_{\mathrm{rel}}(\rho(\lambda)|\sigma). \tag{7.6.6}$$

We can now obtain ΔS to second order in λ. The vacuum modular Hamiltonian of the Rindler wedge is just the boost energy

$$\mathrm{Tr}\left[(\rho(\lambda) - \sigma) H^\sigma\right] = \int d^{d-2}y \int dv v\, \mathrm{Tr}\left[\rho(\lambda) T_{++}(u=0, v, y)\right] \tag{7.6.7}$$

where the computation of $S_{\mathrm{rel}}(\rho(\lambda)|\sigma)$ was done in Appendix B of [331]. There it was demonstrated that

$$S_{\mathrm{rel}}(\rho(\lambda)|\sigma) = -\frac{\lambda^2}{2} \int \frac{ds}{4\sinh^2(\frac{s+i\epsilon}{2})}\, \mathrm{Tr}\left[\sigma^{-1}\delta\rho\sigma^{\frac{is}{2\pi}}\delta\rho\sigma^{\frac{-is}{2\pi}}\right] + \mathcal{O}(\lambda^3) \tag{7.6.8}$$

For a pure state like (7.6.2), we can instead write the above expression as a correlation function

$$S_{\mathrm{rel}}(\rho|\sigma) = -\frac{\lambda^2}{2} \int d\mu \int \frac{ds}{4\sinh^2(\frac{s+i\epsilon}{2})} \langle \mathcal{O}(r_1, \theta_1, y_1) e^{is\hat{K}} \mathcal{O}(r_2, \theta_2, y_2)\rangle \tag{7.6.9}$$

where we have used the shorthand

$$\int d\mu = \int dr_{1,2} d\theta_{1,2} d^{d-2}y_{1,2} f(r_1, \theta_1, y_1) f(r_2, \theta_2, y_2) \tag{7.6.10}$$

and $\hat{K} = H^\sigma_R - H^\sigma_L$ is the full modular Hamiltonian associated to Rindler space. This formula (7.6.9) and generalizations has been applied and tested in various contexts [332, 333, 334, 335]. Most of these papers worked with perturbations about a state and a cut with associated to a modular Hamiltonian with a local flow such as the Rindler case. However it turns out that this formula can be applied more widely where $\hat{K}$ need not be local.[3]

We can thus safely replace the Rindler Hamiltonian in (7.6.9) with the Hamiltonian associated to an arbitrary cut of the null plane. This allows us to take shape deformations directly from (7.6.9); by using the algebraic relation for arbitrary-cut modular Hamiltonians [329]

$$e^{-i\hat{K}(X^+)s} e^{i\hat{K}(0)s} = e^{i(e^s-1)\int dy \int dx^+ X^+(y) T_{++}(x^+)} \tag{7.6.11}$$

[3]The only real subtlety is the angular ordering of the insertion of $\mathcal{O}$ in Euclidean. This can be dealt with via an appropriate insertion of the modular conjugation operator - a detail that does not affect the final result. We plan to work out these details in future work.

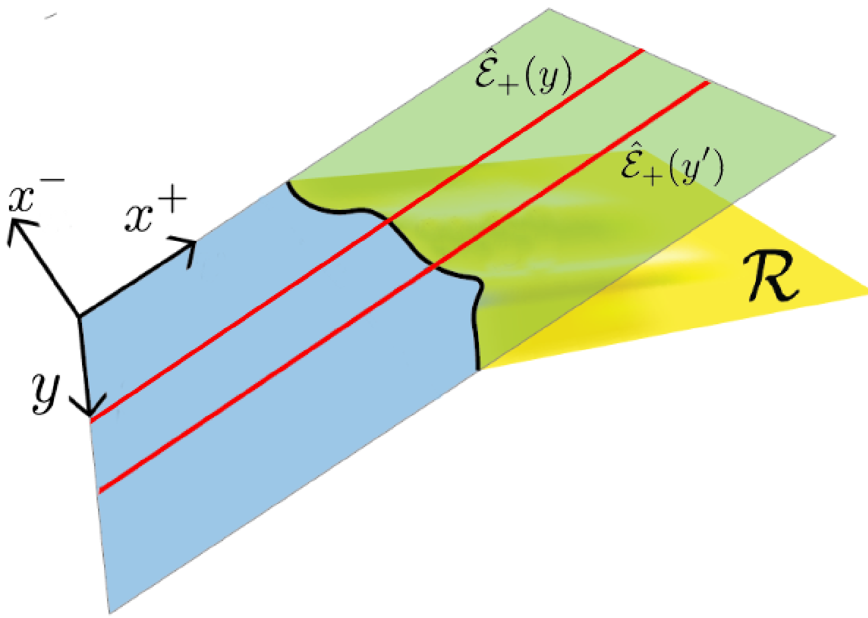


Figure 7.3: For near vacuum states, the insertions of displacement operators limit to two insertions of the averaged null energy operators $\hat{\mathcal{E}}_+$.

we have

$$\frac{\delta^2 S_{\rm rel}(\rho|\sigma)}{\delta X^+(y)\delta X^+(y')} = \frac{\lambda^2}{2}\int d\mu \int ds e^s \langle \mathcal{O}(r_1,\theta_1,y_1)\mathcal{E}_+(y)\mathcal{E}_+(y')e^{is\hat{K}(X^+)}\mathcal{O}(r_2,\theta_2,y_2)\rangle \quad (7.6.12)$$

where the states ρ, σ depend implicitly on $X^+(y)$.[4] Notice that upon taking the variations the double poles in the $1/\sinh^2(s/2)$ kernel of (7.6.8) were precisely canceled by the factors of $e^s - 1$ in the exponent of equation (7.6.11).

This equation is the main result of this section. We see that taking shape derivatives of the entropy can for this class of states be accomplished by insertions of averaged null energy operators. This helps to explain the appearance and disappearance of extra delta functions as we change the coupling in a CFT continuously connected to a free theory. For example, in a free scalar theory, one can show that the OPE contains a delta function,

$$\hat{\mathcal{E}}_+(y)\hat{\mathcal{E}}_+(y') \supset \delta^{d-2}(y-y'). \quad (7.6.13)$$

This is consistent with the findings of [280] where this extra delta function contribution to the QNEC was computed explicitly. To this aim, in Appendix A.21, we explicitly reproduce the answer in [280] using the above techniques.

[4] Note the similarity between (7.6.12) and (A.20.6). This is because one can view the defect four point function in (7.5.3) as going to second order in a state-deformation created by stress tensors with a particular smearing profile.

7.7 Discussion

In this discussion, we briefly elaborate on the possible origin of the non-local operators whose dimensions we found in the displacement operator OPE considered in Sections 7.5 and 7.6. As mentioned in the main text, the appearance of new operators is a bit puzzling since the authors in [15] found a complete set of defect operators as $n \to 1$. In other words, at fixed $n > 1$, it should in principle be possible to expand these new operators as a (perhaps infinite) sum of $\ell = 2$ defect operators.

In particular, we expect them to be representable as an infinite sum over the higher spin displacement operators. We believe that it is necessary to do such an infinite sum before taking the $n \to 1$ limit, which entails that the OPE and replica limits do not commute. This is why [15] did not find such operators. It also seems, given the non-trivial re-derivation of the results in [15] using algebraic tecniques in [16], that these new non-local defect operators are not necessary for the limit $n \to 1$ limit of the bulk to defect OPE used in [15] to compute modular flow correlation functions.

We give the following speculative picture for how the nonlocal defect operators might arise:

$$\hat{D}_+(y_1)\hat{D}_+(y_2) = \frac{c_{J=2}(n)\hat{T}_{++}}{|y_1 - y_2|^{2(d-1)-\Delta_n^{J=2}}} + \sum_{J=3}^{\infty} \frac{c_J(n)\hat{D}_{++}^{(J)}}{|y_1 - y_2|^{2(d-1)-\Delta_n^J}} \tag{7.7.1}$$

where we have suppressed the contribution of defect descendants. The latter sum in (7.7.1) comes from the spin 2 displacement operators that come from the spin J CFT operator. This is a natural infinite class of operators that one could try to re-sum should that prove necessary.

In our calculations, we did not see any powers in $|y_1 - y_2|$ that could be associated to any individual higher spin displacement operator (as in the second term in (7.7.1)). Instead, in Section 7.5 and Section 7.6 after taking the $n \to 1$ limit we observed dimensions that did not belong to any of the known local defect operators. One possibility is that the higher spin operators in (7.7.1) re-sum into a new term that has a non-trivial interplay with the $n \to 1$ limit. One way this might happen is if the OPE coefficients of the higher spin displacement operators take the form

$$c_{J=2k}(n) \sim \frac{1}{(J-3)(n-1)^{J-3}} \tag{7.7.2}$$

so that they diverge as n approaches 1. Such a divergent expansion is highly reminiscent of the Regge limit for four point functions where instead the divergence appears from the choice of kinematics. This pattern of divergence where the degree increases linearly with spin can be handled using the Sommerfeld-Watson trick for re-summing the series. The basic idea is to re-write the sum as a contour integral in the complex J-plane. One then unwraps the contour and picks up various other features depending on the correlator.

Our conjecture in (7.7.2) is that the other features which one encounters upon unwrapping the J contour is quite simple: there is just one pole at $J = 3$. Upon unwrapping the contour in the J-plane, we pick up the pole at $J = 3$, which suggests that indeed these new divergences in $|y_1 - y_2|$ are associated to operators which are analytic continuations in spin of the higher spin displacement operators. In this way we would reproduce the correct power law in $|y_1 - y_2|$ as predicted for near vacuum states.

Note that this needs to be true for *any* CFT - not just at large N or large coupling. The universality of this presumably comes from the universality of three point functions. Indeed, one can try to compute these OPE coefficients. We should consider the following three point function:

$$\langle \Sigma_n^0 \hat{D}_+(y_1)\hat{D}_+(y_2)\hat{D}_{--}^{(J)}(y_3)\rangle \sim \frac{c_J(n)\,\langle \Sigma_n^0 \hat{D}_{++}^{(J)}(y_2)\hat{D}_{--}^{(J)}(y_3)\rangle}{|y_1 - y_2|^{2(d-1)-\hat{\Delta}_n(J)}} \tag{7.7.3}$$

Via calculations based on the results in Appendix A.17, we find the three point function above in the the replica limit is:

$$\sim (n-1)\oint dw w^{J-3}\,\langle \mathcal{J}_{-\ldots-}(w,\overline{w}=0,y_3)\hat{\mathcal{E}}_+(y_1)\mathcal{E}_+(y_2)\rangle + \mathcal{O}((n-1)^2). \tag{7.7.4}$$

Naively, the full null energy operator $\hat{\mathcal{E}}_+(y_1)$ commutes with the half null energy operator $\mathcal{E}_+(y_2)$ and one can use the fact that $\hat{\mathcal{E}}_+(y_1)\,|\Omega\rangle = 0$ to conclude that $c_J(n=1)$ vanishes. This seems to be incorrect however due to a divergence that arrises in the null energy integrals. Rather we claim that this coefficient diverges. The way to see this is to write

$$\begin{aligned}&\langle \mathcal{J}_{-\ldots-}(w,\overline{w}=0,y_3)\hat{\mathcal{E}}_+(y_1)\mathcal{E}_+(y_2)\rangle = \\ &\int_{-\infty}^{\infty} dx_1^+ \int_0^{\infty} dx_2^+\,\langle \mathcal{J}_{-\ldots-}(w,\overline{w}=0,y_3)T_{++}(0,x_1^+,y_1)T_{++}(0,x_2^+,y_2)\rangle\,.\end{aligned} \tag{7.7.5}$$

We can now attempt to apply the bulk OPE between the two T_{++}'s which in these kinematics must become[5]

$$T_{++}(x^-=0,x_1^+,y_1)T_{++}(x^-=0,x_2^+,y_2) = \sum_{J=2}^{\infty}\frac{(x_{12}^+)^{J-4}\mathcal{J}_{+\ldots+}^J(x_2^+,y_2)}{|y_1-y_2|^{2(d-1)-\hat{\Delta}_1(J)}} + (\text{descendants}). \tag{7.7.6}$$

where $\hat{\Delta}_1(J) = \Delta_J - J + 2$. Plugging (7.7.6) into (7.7.5) and re-labeling $x_1 \to \lambda_1 x_2$, we see that for even $J \geq 3$, the λ_1 integral has an IR divergence

[5] To get the exact answer, one needs to account for all of the $SO(2)$ descendants in this OPE as well since they contribute equally to the higher spin displacement operator. We expect all of these descendants to have the same scaling behavior with $n-1$ and $J-3$.

One can cut-off the integral over λ_1 at some cutoff Λ. The answer will then diverge like

$$\frac{\left(\int_{-\Lambda}^{\Lambda} d\lambda_1\, \lambda_1^{J-4}\right)}{|y_1-y_2|^{2(d-1)-\hat{\Delta}_1(J)}} \times \int_0^\infty dx_2 x_2^{J-3} \left\langle \mathcal{J}_{-\ldots-}(w,\overline{w}=0,y_3)\mathcal{J}_{+\ldots+}(z=0,\overline{z}=x_2^+,y_2)\right\rangle$$
$$\sim \frac{\Lambda^{J-3}}{J-3}\int_0^\infty dx_2\, x_2^{J-3} \left\langle \mathcal{J}_{-\ldots-}(w,\overline{w}=0,y_3)\mathcal{J}_{+\ldots+}(z=0,\overline{z}=x_2^+,y_2)\right\rangle \times \frac{1}{|y_1-y_2|^{2(d-1)-\hat{\Delta}_1(J)}}. \tag{7.7.7}$$

The $\mathcal{J}-\mathcal{J}$ correlator on the right is precisely the order $n-1$ piece in $\langle \Sigma_n^0 \hat{D}^J_{++}\hat{D}^{(J)}_{--}\rangle$ so we find that the OPE coefficient scales like $c(n=1) \sim \frac{\Lambda^{J-3}}{J-3}$.

Since Λ is some auxiliary parameter, it is tempting to assign $\Lambda \sim 1/(n-1)$; we then find the conjectured behavior in (7.7.2). This is ad hoc and we do not have an argument for this assignmennt, except to say that the divergence is likely naturally regulated by working at fixed n close to 1. This is technically difficult so we leave this calculation to future work.

Chapter 8

Ignorance is Cheap: From Black Hole Entropy To Energy-Minimizing States In QFT

8.1 Introduction and Summary

There is a remarkable interplay between testable low-energy properties of quantum field theory (QFT), and certain conjectures about quantum gravity, in which the area of surfaces is associated to an entropy. For example, the classical focussing theorem in General Relativity relies on the Null Energy Condition and so can fail in the presence of quantum matter. A Quantum Focussing Conjecture (QFC) was proposed to hold in the semiclassical regime; it implements a quantum correction to the classical statement by replacing the area with the area plus exterior entropy, i.e., the "generalized entropy." This was a guess about quantum gravity, but it led to a new result in QFT. Namely, the Quantum Null Energy Condition (QNEC) was discovered as the QFT limit of the QFC [13].

The QNEC has since been laboriously proven within relativistic quantum field theory [303, 15, 16]. The fact that the QNEC arises more directly and simply from a hypothesis about quantum gravity is striking. Experimental tests of the QNEC may be viable and should be regarded as test of this hypothesis.

Here we will discover a related but distinct connection of this type. We begin again with a classical gravity construction, though one motivated by quantum gravity. The notion that black holes carry Bekenstein-Hawking entropy (proportional to their area) has been fruitful and widely explored, but we stress here that it is a hypothesis that has not been experimentally tested. This hypothesis leads to a puzzle: if the black hole was formed from a pure state, then the entropy should vanish. Thus the Bekenstein-Hawking entropy must be the von Neumann entropy of another quantum state, presumably one that is obtained by an appropriate coarse-graining of the original state. What characterizes this coarse-grained state?

This question was the subject of a recent conjecture by Engelhardt and Wall (EW) [18]. The EW conjecture applies to a class of surfaces that may lie on or inside the event horizon. The Bekenstein-Hawking entropy associated with a "minimar" surface σ is the area of the extremal (Ryu-Takayanagi [4] or HRT [282]) surface, maximized over all spacetimes that agree with the given solution outside of σ. (The input spacetime may have no such surface and thus no entropy.) Engelhardt and Wall showed that the coarse-grained entropy so defined does indeed agree with the area of σ. The interpretation of extremal surface area as an entropy in the quantum gravity theory is well-motivated by the success of the RT proposal in asymptotically Anti-de Sitter spacetimes. We review the EW coarse-graining procedure in Sec. 8.2.

However, the EW construction and proof are purely classical. In particular, the construction fails when quantum matter is included, because it relies on the Null Energy Condition. Moreover, there is considerable evidence that in semi-classical gravity, it is the generalized entropy [115] (and not the area) that is naturally associated with thermal states of the underlying quantum gravity theory [296, 336].

Here, we will formulate a semi-classical extension of the EW coarse-graining proposal for black hole states; that is, we include effects that are suppressed by one power of $G\hbar$ compared to the classical construction. In Sec. 8.3, we consider a suitably defined quantum version of a "minimar" surface. At this order, we must hold fixed not only its exterior geometry but also the exterior state of the quantum fields. We conjecture a construction that explains the generalized entropy of the quantum minimar surface σ in terms of a suitably coarse-grained state: one can find an interior completion of the geometry and quantum state that contains a quantum stationary surface [296, 336, 284] with equal generalized entropy, but none with larger generalized entropy. Moreover, we propose that saturation is obtained by extending σ along a stationary null hypersurface whose classical and quantum expansions both vanish.

Unlike the classical EW construction, we cannot prove our conjecture. But in Sec. 8.4, following the example of the QFC $\to$ QNEC derivation, we are able to extract a pure quantum field theory limit. We apply our construction to states on a fixed background black hole spacetime with a complete Killing horizon. In this limit, coarse-graining requires the existence of QFT states with specific and somewhat surprising properties, which we list. The most striking property of the coarse-grained state is that the energy flux across the horizon has delta-function support on σ; and that it vanishes at all earlier times on the horizon. (At later times the state agrees with the input state by construction.) The strength of the delta function is set by the derivative of the von Neumann entropy along the horizon in the input state, $\hbar S'/2\pi$.

In particular, the existence of a quantum state with these properties would imply a new result in QFT, Wall's "ant conjecture" [337] concerning the minimum energy of global completions of a half-space quantum state. (We review the ant conjecture in Appendix A.22. The QNEC follows from this conjecture, but it has also been directly proven.) Our proposal thus implies that a state that maximizes the generalized entropy minimizes the nongravitational energy inside of a cut of a Killing horizon, subject to holding fixed the state on the outside. Roughly speaking, ignorance saves energy.

In fact, Wall's ant conjecture was recently proven by Ceyhan and Faulkner (CF) [16]. The CF construction takes as input a state on a Killing horizon and a cut at some surface σ on the horizon. Connes cocycle flow then generates a family of states that differ only to the past of the cut. In the limit of infinite flow, a state is approached whose properties prove the ant conjecture.

In greater than 1+1 dimensions, the requirements we derive appear to be stronger than those demanded by the ant conjecture; see Appendix A.22. Thus it is not immediately obvious that the quantum states required for our coarse-graining proposal exist. However, in Sec. 8.5 we show that the CF family of states attains all of the properties required by our conjecture. In particular, a delta function shock appears at the cut, with precisely the predicted strength. It is interesting that this feature arises in an algebraic construction whereas in the black hole setting, it arose geometrically from requiring a source for a discontinuity in the metric derivative. Thus, the CF construction proves the QFT limit of our conjecture, even though it was originally designed to prove the ant conjecture.

We briefly discuss some future directions in Sec. 8.6.

8.2 Classical coarse-graining of black hole states

In this section we review a classical geometric construction by Engelhardt and Wall (EW) [18, 338]. In Sec. 8.2, we provide definitions of (classically) marginally trapped, "minimar", stationary, and HRT surfaces.

In Sec. 8.2, we summarize the EW proposal for the outer entropy of a "minimar" surface, a marginally trapped surface σ that satisfies certain addition conditions. EW define this entropy in terms of geometries that agree with in the exterior of σ but differ in the interior. For any such auxiliary geometry, inspired by the Ryu-Takayanagi proposal, the von Neumann entropy is assumed to be given by the area of a stationary surface. Maximizing this area over all possible auxiliary geometries, EW show that it agrees with the area of σ, which thus represents a coarse-grained entropy in agreement with the Bekenstein-Hawking formula.

Classical marginal, minimar, and stationary surfaces

We begin by fixing some notations and conventions; see Sec. 2 of [338] for details. Let σ be a *Cauchy splitting surface*, that is, σ is an achronal codimension two compact surface that divides a Cauchy surface Σ into two sides, Σ_{in} and Σ_{out}.

Let k^a, l^a be the two future-directed null vector fields orthogonal to σ, normalized so that $k_a l^a = -1$; and let θ_k, θ_l be their expansions.

If exactly one null expansion vanishes, we shall take this to be the k-expansion. Then σ is called *marginally outer trapped*, with k defining the "outside." If $\theta_l < 0$ everywhere on a marginally outer trapped σ, we call σ *marginally trapped*.

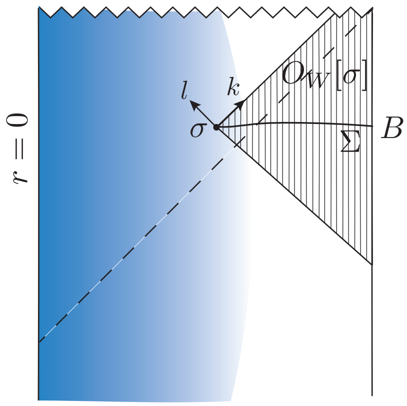


Figure 8.1: Penrose diagram of a black hole formed from collapse in Anti-de Sitter space, showing a minimar surface σ and its outer wedge $\mathcal{O}_W[\sigma]$ with Cauchy surface Σ.

The *outer wedge* $O_W[\sigma]$ of a marginally trapped surface σ is the set of spacelike separated events on the outside of σ (the side that k points towards, see above): $O_W[\sigma] \equiv D[\Sigma_{\text{out}}]$, where D denotes the domain of dependence. See Fig. 8.1.

A *minimar surface* is a marginally trapped surface σ that satisfies two additional restrictions:

- $O_W[\sigma]$ contains a connected component B of an asymptotic conformal boundary (as would be the case, for example, if σ lies in a single black hole formed from collapse in asymptotically anti-de Sitter or flat spacetime). Moreover, $O_W[\sigma]$ admits a Cauchy surface on which σ is the surface homologous to B that minimizes the area; see Fig. 8.1.

- $k^a \nabla_a \theta_{(l)} < 0$

A *stationary surface* X is a surface whose expansion vanishes in both null directions, k and l:[1]

$$\theta_k = \theta_l = 0 \text{ everywhere on } X \; . \tag{8.2.1}$$

A *Hubeny-Rangamani-Takayanagi (HRT) surface* X is a stationary surface that satisfies additional requirements: it is the stationary surface with the smallest area, subject to a homology condition [282, 4]. Here, we will require that X be homologous to a minimar surface σ, and hence to a connected component B of a conformal boundary.

Bekenstein-Hawking entropy from coarse-graining behind minimar surfaces

Engelhardt and Wall [338] argued that the area of a minimar surface σ can be understood as a coarse-grained entropy. For geometries with a conformal field theory (CFT) dual, an

[1] In an abuse of language, this is sometimes referred to as extremal rather than stationary.

explicit prescription for this coarse-graining can be formulated in the CFT. Here, we will be interested in the bulk definition of this coarse-graining, which can be discussed in more general geometries.

In the bulk, the coarse-graining consists of holding fixed the outer wedge of σ, $O_W[\sigma]$, while erasing the spatial interior of σ and replacing it with an auxiliary geometry. One seeks the auxiliary geometry with the largest possible HRT surface X behind σ. The coarse-grained entropy of σ is defined as $A[X]/4G\hbar$.

So far, we have reviewed the definition of the outer entropy. The EW proposal is the conjecture that

- $S_{\rm outer} \equiv A[X]/4G\hbar$ represents the von Neumann entropy of a well-defined state in a quantum gravity theory; and
- $A[X] = A[\sigma]$.

EW proved the first part of the conjecture for the special case where B lies on the conformal boundary of an asymptotically AdS spacetime, and σ lies on a perturbed Killing horizon; moreover the proof assumes the Ryu-Takayanagi [4] and HRT [282] proposals for the von Neumann entropy of the boundary CFT. In this case, it is possible to construct the dual CFT state explicitly, and to show that its entropy agrees with $S_{\rm outer}$.

The second part of the conjecture was proven more generally [338]. Using the maximin definition of the HRT surface [339], it can be shown that

$$A[X] \leq A[\sigma] \ . \tag{8.2.2}$$

This argument assumes the Null Energy Condition (NEC), that the stress tensor satisfies

$$T_{ab}k^a k^b \geq 0 \tag{8.2.3}$$

for any null vector k^a.

EW explicitly construct an interior geometry that saturates the inequality (8.2.2). This implies

$$S_{\rm outer}[\sigma] \equiv \frac{A[X]}{4G\hbar} = \frac{A[\sigma]}{4G\hbar} \ . \tag{8.2.4}$$

The interior geometry with $A[X] = A[\sigma]$ is constructed by specifying initial conditions on the null hypersurface N_k^- orthogonal to σ towards the interior and past. Appropriate initial data is generated by null-translating the intrinsic geometry of σ, thus generating a stationary null hypersurface:

$$\theta_k = 0 \ \text{ on } N_k^- \ . \tag{8.2.5}$$

This ensures that all cross sections of N_k^-—in particular, X—have the same intrinsic metric and area as σ. This construction is consistent with the relevant constraint, the Raychaudhuri equation,

$$k^a \nabla_a \theta_k = -\frac{1}{2}\theta_k^2 - \varsigma^2 - 8\pi G\, T_{kk} \ , \tag{8.2.6}$$

if one sets

$$\varsigma = 0 \text{ and } T_{kk} = 0 \ \text{ on } N_k^- \ . \tag{8.2.7}$$

on N_k^-. EW [338] show that this choice is always possible. Since θ_k vanishes on σ, Eqs. (8.2.6) and (8.2.7) ensure that the entire extrinsic curvature tensor in the k-direction vanishes everywhere on N_k^-, achieving the desired stationarity of N_k^-.

Moreover, it is important to show that there exists a stationary (HRT) surface X on N_k^-. The outgoing expansion θ_k vanishes on any cut of N_k^-, by the above construction. The question is whether there exists a cut X on which the ingong expansion θ_l vanishes as well. This is accomplished in the following sequence of steps.

The minimar assumption dictates that on σ, $\theta_l < 0$ and $k^a\nabla_a\theta_l < 0$. One can choose initial conditions on N_k^- such that along every null generator of N_k^-, $k^a\nabla_a\theta_l$ is constant and equal to its value on σ: by the cross-focussing equation,

$$k^a\nabla_a\theta_l = -\frac{1}{2}\mathcal{R} - \theta_k\theta_l + \chi^2 + \nabla\cdot\chi + 8\pi G\, T_{kl} \ , \tag{8.2.8}$$

this can be accomplished by choosing all terms on the right hand side to be constant on N_k^-. This is already ensured for the intrinsic curvature scalar $\mathcal{R}$ and for the (vanishing) $\theta_k\theta_l$ term, by stationarity of N_k^-. The twist, or normal 1-form, is defined by

$$\chi_a = h^c{}_a l^d \nabla_c k_d \ , \tag{8.2.9}$$

where $h_{ab} = g_{ab} + 2l_{(a}k_{b)}$ is the induced metric on a cut. The twist evolves according to

$$k^a\nabla_a\chi_i = 8\pi T_{ik}(+\text{ terms that vanish when } \theta_k = \varsigma = 0) \ . \tag{8.2.10}$$

To summarize, one can accomplish $k^a\nabla_a\theta_l = k^a\nabla_a\theta_l|_\sigma$ on N_k^- by choosing Eqs. (8.2.5) and (8.2.7) and in addition, along each null generator of N_k^-,

$$T_{kl} = T_{kl}|_\sigma \ \text{ and } T_{ik} = 0 \ \text{ on } N_k^- \ . \tag{8.2.11}$$

Again, EW argue that these choices are always possible.

Let v be the affine parameter associated to k^a, and let y be the transverse coordinates (angular coordinates) on σ. The location of a stationary surface X, $v = f(y)$, is determined by the differential equation

$$L^a[f] = -\theta_l|_\sigma \ , \tag{8.2.12}$$

where L^a is the stability operator (see Ref. [338] for details). This can be shown to have a solution with $-\infty < f < 0$, so the HRT surface exists and lies on N_k^-.

EW then glue the geometry exterior to X (that is, N_k^- and the outer wedge) to its CPT image across X. This constructs a "two-sided" geometry in which X functions as a kind of bifurcation surface of a two-sided black hole/white hole pair. (However, the stationary auxiliary portion N_k^- does not in general correspond to the horizon of a Kerr-Newman black hole, as its intrinsic metric can differ.)

In a final step, EW show that X is not just stationary but is an HRT surface, i.e., that X is the smallest-area stationary surface homologous to σ. This step uses the NEC as well as the second part of the minimar property of σ.

This concludes our summary of the EW coarse-graining prescription. Again, we refer the interested reader to Ref. [338] for more detailed definitions and arguments.

8.3 Semiclassical coarse-graining of black hole states

In this section, we formulate a semiclassical extension of the Engelhardt-Wall construction, starting from a quantum marginally trapped surface σ. We conjecture that the semiclassical state invoked in our construction exists in the full quantum gravity theory; and that in this theory this state has a von Neumann entropy given by the generalized entropy of σ.

In Sec. 8.3, we introduce relevant concepts such as generalized entropy, quantum expansion, quantum marginally trapped surfaces, and quantum HRT surfaces.

In Sec. 8.3, we state our quantum extension of the EW coarse-graining proposal.

In Sec. 8.3, we refine our conjecture by describing key properties that the coarse-grained state is expected to satisfy at the level of semiclassical gravity. (These properties will be shown to have an interesting nongravitational limit in Sec. 8.4. In Sec. 8.5 we will show that a recent construction by Ceyhan and Faulkner [16] generates quantum field theory states which achieve these properties in a certain limit.)

Quantum marginal, minimar, and stationary surfaces

Before we turn to the question of why and how the EW construction should be extended to the semiclassical regime, we introduce here the relevant concepts: generalized entropy, quantum expansion, quantum (marginally) trapped surfaces, and quantum extremal surfaces. More details can be found, e.g., in Refs. [277, 336, 303, 12].

The notion of *generalized entropy* was originally introduced by Bekenstein [115] as an extension of ordinary entropy that includes the contribution from black holes, $S_{\text{out}} \to S_{\text{out}} + \frac{A}{4G\hbar}$. But in an expansion in $G\hbar$, it is the exterior entropy that should be regarded as a quantum correction:

$$S_{\text{gen}} = \frac{A}{4G\hbar} + S_{\text{out}} + \dots \ , \tag{8.3.1}$$

Equivalently, $4G\hbar S_{\text{gen}}$ represents a quantum-corrected area.

In Bekenstein's original proposal, A represented the area of a cut of a black hole event horizon; and S_{out} represented the entropy in the black hole's exterior. However, the generalized entropy can be defined for any Cauchy-splitting surface σ, with S_{out} the von Neumann entropy of the quantum fields restricted to one side of σ. $A/4G\hbar$ should be regarded as the leading counterterm that cancels divergences in the entropy; we suppress subleading terms here. Given its wide applicability, the notion of generalized entropy can be used to define quantum-corrected notions of trapped, stationary, etc., as follows.

Recall that the classical expansion of a surface $\hat{\sigma}$ at a point $y \in \hat{\sigma}$ is the trace of the null extrinsic curvature at y. It can also be defined as a functional derivative,

$$\theta[\hat{\sigma}; y] = h(y)^{-1/2} \frac{\delta A[V]}{\delta V(y)} , \tag{8.3.2}$$

where h is the area element on $\hat{\sigma}$. Here $V(y)$ defines a surface that lies an affine parameter distance V from $\hat{\sigma}$ along the null geodesic emanating from $\hat{\sigma}$ at y.

The above definition is overkill, as the classical expansion depends only on the local geometry near y. But it generalizes directly to the *quantum expansion*, Θ, which depends on $\hat{\sigma}$ nonlocally:

$$\Theta[\hat{\sigma}; y] = \frac{4G\hbar}{\sqrt{h(y)}} \frac{\delta S_{\text{gen}}[V]}{\delta V(y)} . \tag{8.3.3}$$

A *quantum marginally outer trapped surface* is a surface whose quantum expansion in one of the two null directions (say, k) vanishes at every point. Let σ be such a surface:

$$\Theta_k[\sigma; y] \equiv 0 . \tag{8.3.4}$$

It follows that

$$\theta_k(y) = -\frac{4G\hbar}{\sqrt{h(y)}} \frac{\delta S_{\text{out}}}{\delta V(y)} \tag{8.3.5}$$

at every point on σ.

A *quantum marginally trapped surface* is a quantum marginally outer trapped surface for which in addition

$$\Theta_l[\sigma; y] < 0 . \tag{8.3.6}$$

(As usual, *anti-trapped* corresponds to the opposite inequality on the l-expansion.)

The *outer wedge* $O_W[\sigma]$ of a quantum marginal surface σ is the set of spacelike separated events on the "marginal" side of σ, i.e., the side that k points towards: $O_W[\sigma] = D[\Sigma_{\text{out}}]$; see Fig. 8.1.

A *quantum minimar surface*, is a quantum marginally trapped surface σ that satisfies two additional restrictions:

- $O_W[\sigma]$ contains a connected component of an asymptotic conformal boundary (as would be the case, for example, if σ lies in a single black hole formed from collapse in asymptotically anti-de Sitter or flat spacetime). Moreover, $O_W[\sigma]$ admits a Cauchy surface on which σ is the surface homologous to B that minimizes the generalized entropy; see Fig. 8.1.

- $k^a \nabla_a \theta_l < 0$.

Note that we impose the second condition on the classical expansion, not the quantum expansion. Since the inequality is strict, the classical expansion θ_l will dominate in the semiclassical expansion in $G\hbar$.

A *quantum stationary surface* is a surface whose quantum expansions vanish in both null directions, k and l. We will demand that X be such a surface:

$$\Theta_k[X;y] \equiv 0 \ , \quad \Theta_l[X;y] \equiv 0 \ . \tag{8.3.7}$$

A *quantum HRT surface* satisfies additional requirements: it is the quantum stationary surface with the smallest generalized entropy; and it must obey a homology condition [336]. Here, we will require that it be homologous to a quantum minimar surface σ, and and hence to a connected component B of a conformal boundary.

Generalized entropy from coarse-graining behind quantum marginally trapped surfaces

We will now motivate and formulate a quantum extension of the EW proposal. To see that such an extension is needed, note that the classical EW construction relies on the Null Energy Condition, Eq. (8.2.3). The NEC guarantees that no HRT surface with area greater than that of the marginally trapped surface can be constructed. It also guarantees that the stationary surface with equal area is an HRT surface. But the NEC is known to fail in any relativistic quantum field theory, so none of these conclusions survive at the semiclassical level.

Indeed, one does not expect any quantum state of the full quantum gravity theory to correspond to just the area of a surface (as is implicit in the classical EW construction). Rather, one expects its von Neumann entropy to match the generalized entropy. That is, to the extent that a quantum state corresponds to a surface, one expects it to also describe the surface's exterior.

There is significant evidence supporting this expectation from the AdS/CFT correspondence [3]. Consider the quantum state ρ_B on a region B, where B can be all or part of the boundary. This state is expected [296, 285] to describe the entire *entanglement wedge* of B, i.e., the spacetime region enclosed by B and the HRT surface $X[B]$. The $1/N$ expansion on the boundary (with N the rank of the CFT's gauge group) corresponds to the $G\hbar$ expansion in the bulk. In particular, the von Neumann entropy $S(\rho_B)$ can be expanded in this way, with the leading $O(N^2)$ piece corresponding to the area of $X[B]$, and the subleading $O(1)$ piece corresponding to the exterior bulk entropy S_{out}. When expanding to higher orders, X_B should be taken to be the quantum HRT surface of B [336].

We thus seek a proposal in which the generalized entropy of a surface σ is explained as a coarse-grained entropy. The coarse-graining should correspond to maximizing the generalized entropy of a quantum HRT surface X, subject to holding fixed the outer wedge $O_W[\sigma]$ (now including the quantum state of bulk fields in $O_W[\sigma]$). The coarse-graining prescription will be successful if $S_{\text{gen}}[X] = S_{\text{gen}}[\sigma]$.

The remaining question is what characterizes a surface σ that we may consider for coarse-graining. In the classical case, the appropriate criterion was that σ be minimar. In the quantum case, the natural candidates are minimar surfaces or quantum minimar surfaces.

In the EW construction of the maximally coarse-grained state, the HRT surface X of the coarse-grained state lies on a stationary null surface N_k^- extended to the past and inwards from σ. Our construction will share this feature. This excludes (classically) minimar as the relevant criterion for σ. The variation of $S_{\rm out}$ does not have definite sign on such surfaces, and so their quantum expansion would not have a definite sign. However, if $\Theta[\sigma] > 0$ then by the quantum focussing conjecture, it would be impossible to find an X with $\Theta[X] = 0$ on $N_k^-[\sigma]$. Therefore, we will require that σ be quantum minimar; in particular, $\Theta[\sigma] = 0$.

We now state our **proposal**. Let σ be a quantum minimar surface homologous to a boundary region B, with generalized entropy $S_{\rm gen}[\sigma]$ and outer wedge $O_W[\sigma]$. Let $\overline{X}$ be a quantum HRT surface in any geometry such that:

- $O_W[\overline{X}] \supset O_W[\sigma]$.
- $\overline{X}$ is homologous to σ.
- Both the geometry and the quantum state of $O_W[\overline{X}]$ agree with that of $O_W[\sigma]$ upon restriction of $O_W[\overline{X}]$ to $O_W[\sigma]$. (To be precise, let $\Sigma_{\rm out}[\overline{X}]$ be a Cauchy surface of $O_W[\overline{X}]$ such that $\Sigma_{\rm out}[\overline{X}] \cap O_W[\sigma]$ is a Cauchy surface of $O_W[\sigma]$, $\Sigma_{\rm out}[\sigma]$, and let $\rho_{\overline{X}}$ and ρ_σ be the state of the quantum fields on $\Sigma_{\rm out}[\overline{X}]$ and $\Sigma_{\rm out}[\sigma]$, respectively. We require that $\mathrm{Tr}_{\Sigma_{\rm out}[\overline{X}]-\Sigma_{\rm out}[\sigma]}\, \rho_{\overline{X}} = \rho_\sigma$.)

We claim that

$$\sup_{\overline{X}} S_{\rm gen}[\overline{X}] = S_{\rm gen}[\sigma] \ . \tag{8.3.8}$$

Moreover, let X be a surface $\overline{X}$ that achieves the supremum. (This should be taken as a limiting statement if no such X exists.) Then $O_W[X]$ represents a coarse-graining of the original geometry, with respect to the quantum minimar surface σ. In particular, in AdS/CFT the quantum state on B dual to the entanglement wedge $O_W[X]$ has von Neumann entropy $S_{\rm gen}[\sigma]$.

Unlike the classical case, we will not prove this conjecture, but we will provide some evidence supporting its plausibility. We proceed in two steps as in the classical case: first, we will argue that

$$S_{\rm gen}[\overline{X}] \leq S_{\rm gen}[\sigma] \tag{8.3.9}$$

for any $\overline{X}$ satisfying the conditions in our proposal. We then refine our conjecture by detailing the properties of a semiclassical geometry and quantum state that would achieve equality.

In order to show Eq. (8.3.9), we generalize the result in [338] to the quantum case. This involves two main assumptions. The first assumption is the quantum focusing conjecture [13] which asserts that in the semi-classical limit the derivative of the quantum expansion of codimension 2 surfaces under any null deformation is non-negative:

$$\frac{\delta\Theta_k[X;y]}{\delta V(y)} \leq 0 \ . \tag{8.3.10}$$

The second assumption is a slightly weaker quantum generalization of the classical maximin construction [339]. More precisely, we assume that the quantum extremal surface $\overline{X}$ is also the surface of minimal generalized entropy on some Cauchy slice Σ.

By global hyperbolicity, the congruence of null geodesics orthogonal to σ in the $\pm k$ directions intersect Σ at some Cauchy splitting surface $\overline{\sigma}$. (The congruence should be terminated at conjugate points or self-intersections [340, 341]. Since σ is a quantum marginally trapped surface, quantum focusing ensures that

$$S_{\text{gen}}[\overline{\sigma}] \leq S_{\text{gen}}[\sigma] \ . \tag{8.3.11}$$

The quantum maximin assumption further implies

$$S_{\text{gen}}[\overline{X}] \leq S_{\text{gen}}[\overline{\sigma}] \ , \tag{8.3.12}$$

which establishes Eq. (8.3.9).

Properties of a Generalized Entropy Maximizing Bulk State

We will now describe a geometry and quantum state with a quantum extremal surface X whose generalized entropy saturates the inequality (8.3.9). The existence of a state with the properties we describe would imply our conjecture, Eq. (8.3.8).

By asserting the existence of this semiclassical state, we are refining our conjecture. In Sec. 8.4, we will explore the implications of this refinement in a pure field theory limit. In Sec. 8.5, we will show that these implications are realized in a recent construction by Ceyhan and Faulkner [16].

Our construction will be analogous to the classical one, in that we will approach X along the null hypersurface $N_k^-[\sigma]$. Since we require $\Theta_k[\sigma] = \Theta_k[X] = 0$, the quantum focussing conjecture ($\Theta_k' \leq 0$) requires that $\Theta_k = 0$ everywhere on N_k^-. That is, S_{gen} must be constant along N_k^-. (This is analogous to classical focussing and the null energy condition requiring that N_k^- have constant area in the classical case.)

In the classical case, all relevant quantities could be chosen to be constant on N_k^-. In other words, the surface N_k^- is truly stationary. This would not be the case if θ and the derivative of the entropy varied along N_k^-, with only their sum Θ_k vanishing. Motivated by this observation, we conjecture that a state can be found such that the two terms in Θ_k vanish *separately* on N_k^-:

$$\theta_k = 0 \quad \text{and} \quad \frac{\delta S_{\text{out}}}{\delta V(y)} = 0 \ . \tag{8.3.13}$$

In analogy with the classical construction we also take the shear tensor to vanish at all orders in $\hbar$ along N_k^-:

$$\varsigma = 0 \ . \tag{8.3.14}$$

These considerations place nontrivial constraints on the limit state we seek. For θ_k and ς to vanish everywhere on N_k^-, the stress tensor component T_{kk} must vanish on N_k^-. Moreover, note that θ_k need not vanish on σ, where only $\Theta_k = 0$ is required. It follows that generically, θ_k must jump discontinuously, by an amount

$$\Delta\theta_k|_\sigma = -\frac{4G\hbar}{\sqrt{h(y)}} \frac{\delta S_{\text{out}}}{\delta V(y)}\bigg|_\sigma \ . \tag{8.3.15}$$

By Raychaudhuri's equation, this implies the presence of a delta function term in the stress tensor, at σ. Combining these results, we conclude that

$$T_{vv} = \frac{\hbar}{2\pi} \frac{\delta S_{\text{out}}}{\delta V(y)}\bigg|_\sigma \delta(v) \ \ , v \leq 0 \ , \tag{8.3.16}$$

i.e., in the region $N_k^- \cup \sigma$.

To summarize, we conjecture the existence of a state with

$$T_{vv} = \frac{\hbar}{2\pi} \frac{\delta S_{\text{out}}}{\delta V(y)}\bigg|_\sigma \delta(v) \ \ , \ v \leq 0 \ , \tag{8.3.17}$$

$$\varsigma = 0 \ \ , \ v < 0 \ , \tag{8.3.18}$$

$$\frac{\delta S_{\text{out}}}{\delta V(y)} = 0 \ \ , \ v < 0. \tag{8.3.19}$$

Eq. (8.3.17) trivially implies that

$$\int_{-\infty}^{v} dv \, T_{vv} = 0 \ , \tag{8.3.20}$$

and we will use this property in Sec. 8.4.[2] In addition, we assume that the remaining EW conditions listed in Eq. (8.2.11) can be met at the classical level.

With these assumptions, the existence of a classical HRT surface on N_k^- is guaranteed by the argument summarized around Eq. (8.2.12). This surface satisfies $\theta_l = 0$. A quantum stationary surface X can be found nearby (in the $G\hbar \to 0$ limit), by solving iteratively for $\theta_l = -\frac{4G\hbar}{\sqrt{h(y)}} \frac{\delta S}{\delta U(y)}$, where the functional derivative refers to the shape deformation along the l-congruence.

Finally, we need to show that X is quantum HRT, i.e., that it is the quantum stationary surface homologous to σ with smallest generalized entropy. This proceeds in exact analogy with the classical argument [338], with the QFC replacing the NEC, so we will not spell out the argument here. See [342] for details.

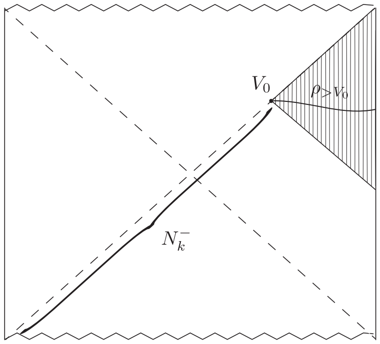


Figure 8.2: Coarse-graining behind a Killing horizon. Any cut V_0 can be viewed as a quantum marginally trapped surface in the limit as $G \to 0$. The state $\rho_{>V_0}$ on the Cauchy surface Σ of the outer wedge is held fixed. The coarse-grained geometry is the original geometry. The stationary null surface N_k^- is the past of V_0 on the Killing horizon. The coarse-grained quantum state demanded by our proposal lives on $N_k^- \cup \sigma \cup \Sigma$. We identify the properties the state must have, and we show that the Ceyhan and Faulkner "ant states" satisfy these.

8.4 Quantum field theory limit of coarse-grained quantum gravity states

In this section, we study the implications of our conjecture for quantum field theory decoupled from gravity. We will apply our proposal to input states that are small perturbations of the Killing horizon of a maximally extended vacuum solution such as Kruskal; see Fig. 8.2.

In the perturbative setting, any quantum marginally trapped surface σ will be at a distance of order G from the Killing horizon, and so will lie on the horizon as $G \to 0$. We can think of the area and null expansion of σ as fields defined on the unperturbed Killing horizon whose changes are sourced by the state of the matter fields on the horizon. Thus, every cut of the Killing horizon can be viewed as quantum marginally trapped, and our conjecture can be applied.

We will first establish notation and review some standard results in Sec. 8.4. In Sec. 8.4, we will derive some interesting additional properties of the coarse-graining states that must hold in the perturbative setting. In the limit as $G \to 0$, our conjecture thus implies the existence of states with both the properties established in the previous section, and the additional properties derived here, in quantum field theory on a fixed background. This is an in-principle testable conjecture about quantum field theory.

[2]Strictly, we must allow for the possibility that a state with the properties we conjecture does not itself exist. It suffices that the properties we require can be arbitrarily well approximated by some family or sequence of states (as in the example of Sec. 8.5). In this case, Eq. (8.3.17) need not imply Eq. (8.3.20), so the latter property should be considered explicitly as part of our refined conjecture.

Notation, definitions, and standard results

Consider a quantum field theory on a background with a Killing horizon and an arbitrary global state ρ defined on the horizon. Let v be the affine parameter on the Killing horizon, u the affine parameter that moves off of the Killing horizon (associated with null vectors k and l respectively), and take y to be the transverse coordinates on a cut $V(y)$ of the horizon. The cut defines a surface σ, which we assume to be Cauchy-splitting as usual.

Let the *right half-space state* $\rho_{>V_0}$ be the restriction of ρ to the half-space $v > V_0(y)$ as in Fig. 8.2:

$$\rho_{>V_0} \equiv \mathrm{Tr}_{\leq V_0}\rho \ . \tag{8.4.1}$$

where the trace is over the algebra associated with the complement region. Let us denote the *von Neumann entropy* of $\rho_{>V_0}$ by

$$S(V_0) = -\mathrm{Tr}\,\rho_{>V_0}\log\rho_{>V_0} \ . \tag{8.4.2}$$

Let $\sigma \equiv |\Omega\rangle\langle\Omega|$ be the global vacuum, which can be reduced to the *right vacuum* $\sigma_{>V_0} = \mathrm{Tr}_{\leq V_0}\sigma$. The *vacuum-subtracted von Neumann entropy* of $\rho_{>V_0}$ is

$$\Delta S(V_0) = S(V_0) + \mathrm{Tr}\,\sigma_{>V_0}\log\sigma_{>V_0} \ . \tag{8.4.3}$$

The *right (half-)modular Hamiltonian* K is defined by the relation

$$\sigma_{>V_0} = \frac{e^{-K(V_0)}}{\mathrm{Tr}\,e^{-K(V_0)}} \ . \tag{8.4.4}$$

The *right modular energy* in a global state ρ is $\langle K(V_0)\rangle \equiv \mathrm{Tr}\,[K(V_0)\rho_{>V_0}]$, and the *vacuum-subtracted right modular energy* is

$$\begin{aligned}\Delta K(V_0) &\equiv \langle K(V_0)\rangle - \mathrm{Tr}\,[\sigma_{>V_0}K(V_0)] &(8.4.5)\\ &= \frac{2\pi}{\hbar}\int dy \int_{V_0(y)}^{\infty} dv\,[v - V_0(y)]\langle T_{vv}\rangle \ , &(8.4.6)\end{aligned}$$

where the explicit expression is due to Bisognano and Wichmann [343] and its generalization to arbitrary cuts of Killing horizons [301, 45]. The *relative entropy* of $\rho_{>v_0}$ with respect to the reduced global vacuum, $\sigma_{>v_0}$, is defined as

$$\begin{aligned}S_{\mathrm{rel}}(V_0) &\equiv S(\rho_{>V_0}|\sigma_{>V_0}) &(8.4.7)\\ &\equiv \mathrm{Tr}\,\rho_{>V_0}\log\rho_{>V_0} - \mathrm{Tr}\,\rho_{>V_0}\log\sigma_{>V_0} \ . &(8.4.8)\end{aligned}$$

It follows from this definition that

$$S_{\mathrm{rel}}(V) = \Delta K(V) - \Delta S(V) \ . \tag{8.4.9}$$

We will often be interested in derivatives, where the vacuum-subtraction drops out. For example,

$$\frac{\delta K}{\delta V(y)} = \frac{\delta \Delta K}{\delta V(y)} = -\frac{2\pi}{\hbar}\int_v^{\infty} d\tilde{v}\ \langle T_{vv}(y)\rangle \ . \tag{8.4.10}$$

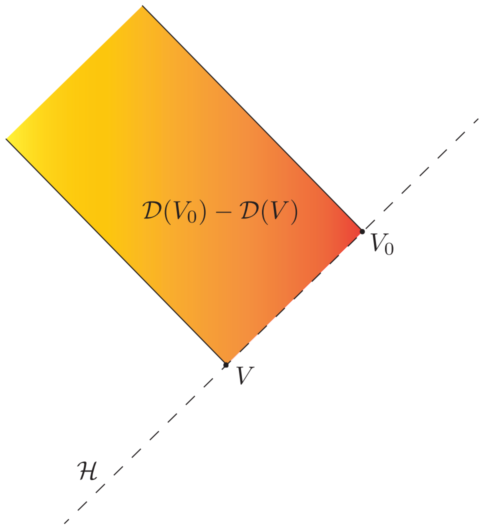


Figure 8.3: The spacetime region associated to the interval $V < v < V_0$ on the null surface for which all observables in the algebra should register vacuum values in the coarse-graining state.

Similar definitions apply to the region $v < V_0$; we denote the associated "left" quantities with an overbar. Strictly, we define the left and right quantities in terms of the limit as $\epsilon \to 0$ of the open intervals $(-\infty, V_0(y) + \epsilon)$ and $(V_0(y) + \epsilon, \infty)$, respectively. The small shift ensures that any distributional sources at $V_0(y)$ contribute asymmetrically to the left but not to the right quantities. (We will see that in the minimum energy states of interest in this paper, the stress tensor generically has a delta function at $V_0(y)$. Our choice resolves an associated ambiguity, attributing this energy entirely to the left.)

The relative entropy satisfies positivity and monotonicity:

$$S_{\rm rel} \geq 0 \;, \quad \frac{\delta S_{\rm rel}}{\delta V} \leq 0 \;. \tag{8.4.11}$$

Via Eq. (8.4.9), monotonicity implies

$$\frac{\delta \overline{K}}{\delta V} \geq \frac{\delta \overline{S}}{\delta V} \geq \frac{\delta S}{\delta V} \;. \tag{8.4.12}$$

The second inequality follows from the strong subadditivity of the von Neumann entropy,

$$S_{BC} + S_{CD} \geq S_B + S_D \;, \tag{8.4.13}$$

applied to the intervals $B = (-\infty, v_0), C = [v_0, v_0 + \delta], D = (v_0 + \delta, \infty)$ in the limit as $\delta \to 0$ [337].

Additional properties of the coarse-graining states

Our conjecture says that the coarse-grained state will have vanishing T_{vv} and constant right entropy in the left region:

$$\langle T_{vv}\rangle = \frac{\hbar}{2\pi}\left.\frac{\delta S}{\delta V(y)}\right|_{\sigma}\delta(v-V_0(y)) \ , \ v\le V_0 \ , \tag{8.4.14}$$

$$\frac{\delta S}{\delta V(y)} = 0 \ , \ v<0 \ . \tag{8.4.15}$$

In particular, in the strong form of Eq. (8.3.20), these properties imply the ant conjecture (see Appendix A.22).

But additionally, on the Killing horizon, the nested inequalities (8.4.12) hold. Combined with the above equations, this implies that the left von Neumann entropy is also constant:

$$0=\int_{-\infty}^{V(y)}\langle T_{vv}\rangle \ge \frac{\delta \overline{S}}{\delta V(y)} \ge \frac{\delta S}{\delta V(y)} = 0 \tag{8.4.16}$$

$$\Longrightarrow \ \frac{\delta \overline{S}}{\delta V(y)} = 0, \ v<V_0(y) \ . \tag{8.4.17}$$

By Eqs. (8.3.20), (8.4.9) and (8.4.10), it follows that the left relative entropy is constant:

$$\frac{\delta \overline{S}_{\rm rel}(\rho_{<V}|\sigma_{<V})}{\delta V(y)} = 0, \ v<V_0(y) \ . \tag{8.4.18}$$

But the relative entropy is a measure of the distinguishability of the state $\rho_{<V}$ from the vacuum $\sigma_{<V}$. Suppose that by moving up the cut V, i.e., by gaining access to a larger region, one could perform some measurement that would better distinguish $\rho_{<V}$ from the vacuum. Then the relative entropy of the larger region would have to be greater. Thus, Eq. (8.4.18) implies that all observables restricted to the difference between the left domains of dependence associated to cuts $V_0(y)$ and $V(y)$ (as in Fig. 8.3) need to register vacuum values. In particular, the stress tensor one-point function must vanish:

$$\langle T_{\mu\nu}(x)\rangle = 0, \quad x\in \mathcal{D}(V_0)-\mathcal{D}(V) \ . \tag{8.4.19}$$

It is more subtle to draw conclusions about $\langle T_{\mu\nu}(x)\rangle$ when x is on the boundary of the region (marked by red in Fig. 8.3), $u=0, v<V_0$. Because $T_{\mu\nu}$ does not exist as an operator unless it is smeared to both sides of this boundary, it will not be in the left operator algebra, and it cannot be used to distinguish $\rho_{<V}$ from the vacuum $\sigma_{<V}$.

We will now give a rough physical argument that certain components of $\langle T_{\mu\nu}(x)\rangle$ must vanish also on the Killing horizon below the cut, $u=0, v<V_0$. We emphasize that this argument is not rigorous, as it borrows from classical intuition. (In forthcoming work we will explore a more detailed coarse-graining proposal involving a family of states; in that setting a rigorous argument can be given.)

Physically, $\langle T_{vv}\rangle$ can be thought of as the momentum orthogonal to an observer's worldline in the (u, v) plane, in the limit as the observer moves at the speed of light in the v-direction. Similarly, T_{iv} is the transverse momentum seen by such an observer. Since all observables in the algebra associated to $\mathcal{D}(V_0) - \mathcal{D}(V)$ have to register vacuum values, no excitations can enter this region. By causality, therefore, the state on the null surface $u = 0, v < V_0$ can only differ from the vacuum by matter moving *along* it, i.e., purely in the v-direction. This implies $\langle T_{vv}\rangle = 0$, consistent with Eq. (8.4.15) above. It also implies the new result

$$\langle T_{iv}\rangle = 0 \ , \quad v < V_0 \ . \tag{8.4.20}$$

Conservation of the stress tensor,

$$-\partial_v\langle T_{uv}\rangle - \partial_u\langle T_{vv}\rangle + \partial_i\langle T_{iv}\rangle = 0 \ , \tag{8.4.21}$$

combined with (8.4.14) then yields

$$\langle T_{uv}\rangle = \text{const} \ . \tag{8.4.22}$$

We conclude that coarse-grained states on Killing horizons must satisfy not only Eqs. (8.4.14) and (8.4.15) but also Eqs. (8.4.17), (8.4.18), (8.4.20), and (8.4.22).

Crucially, these results pertain to quantum field theory on a fixed background, so they can be checked in a rigorous setting. In the next section we will see that all of the above properties are indeed satisfied by the "ant states" constructed by Ceyhan and Faulkner [16]. This proves our conjecture in the Killing horizon limit.

8.5 Existence of coarse-graining states in QFT limit

In this section we show that the "predictions" of the previous section have already been confirmed. We consider a recent explicit construction of states in QFT by Ceyhan and Faulkner (CF) [16]. CF constructed these states in order to prove a conjecture by Wall [344] that we will discuss in detail in Appendix A.22 below. For now, we merely verify that they satisfy the properties we found for the coarse-graining state on Killing horizons in the non-gravitational limit: Eqs. (8.4.14), (8.4.15), (8.4.17), (8.4.18), (8.4.20), and (8.4.22).

Consider a cut $V_0(y)$ of the Rindler horizon $u = 0$ and let $\mathcal{A}_{V_0}, \mathcal{A}'_{V_0}$ be the algebra of operators associated to the region $\{u = 0, v > V_0(y)\}$ and its complement respectively. Given a global state $|\psi\rangle$ we can consider its restriction to $\mathcal{A}_{V_0}$. One can then purify this restriction in different ways, including the trivial purification. We will be interested in the purification introduced in [16], which is based on modular flow.

For the global vacuum $|\Omega\rangle$ recall that the full modular Hamiltonian associated to the cut V_0 defines a modular operator via $K_{V_0} = -\log \Delta_{\Omega;\mathcal{A}_{V_0}}$ and that $\Delta^{is}_{\Omega;\mathcal{A}_{V_0}}$ simply acts as the boost that fixes V_0. We note that $\Delta_{\Omega;\mathcal{A}_{V_0}}$ is related to the reduced density matrix in Eq. (8.4.4) by $\Delta_{\Omega;\mathcal{A}_{V_0}} = \log \sigma_{>V_0} \otimes \mathbb{1}_{<V_0} - \mathbb{1}_{>V_0} \otimes \log \sigma_{<V_0}$.

For a general state $|\psi\rangle$ that is cyclic and separating, one can define the relative modular operator as [345, 343, 346]

$$\Delta_{\psi|\Omega;\mathcal{A}_{V_0}} = S^\dagger_{\psi|\Omega;\mathcal{A}_{V_0}} S_{\psi|\Omega;\mathcal{A}_{V_0}} \ , \tag{8.5.1}$$

where

$$S_{\psi|\Omega;\mathcal{A}_{V_0}}\alpha|\psi\rangle = \alpha^\dagger|\Omega\rangle, \ \forall\alpha \in \mathcal{A}_{V_0} \tag{8.5.2}$$

defines the Tomita operator.

We then purify $|\psi\rangle$ restricted to $\mathcal{A}_{V_0}$ using the Connes cocycle

$$|\psi_s\rangle = u_s'|\psi\rangle, \ u_s' = (\Delta_\Omega')^{is}(\Delta'_{\Omega|\psi})^{-is} \in \mathcal{A}'_{V_0} \ . \tag{8.5.3}$$

The Connes cocycle can roughly be thought of as a half-sided boost that fixes the state restricted to $\mathcal{A}_{V_0}$ but stretches all of the excited modes in the complement region. Specifically, expectation values of operators in $\mathcal{A}_{V_0}$ are left invariant whereas expectation values of operators in $\mathcal{A}'_{V_0}$ are equivalent to those evaluated in the state $\Delta_\Omega^{-is}|\psi\rangle$. This follows (restricting to cyclic and separating states for simplicity) from the relation $(\Delta')^{is}_{\psi|\Omega}\Delta^{-is}_{\Omega|\psi} = 1$, which implies

$$|\psi_s\rangle = \Delta_\Omega^{-is}u_s|\psi\rangle \ . \tag{8.5.4}$$

If we consider an operator $\mathcal{O}' \in \mathcal{A}'_{V_0}$ then $[u_s, \mathcal{O}'] = 0$ so

$$\langle\psi_s|\mathcal{O}'|\psi_s\rangle = \langle\psi|\Delta_\Omega^{is}\mathcal{O}'\Delta_\Omega^{-is}|\psi\rangle \ . \tag{8.5.5}$$

Note that $v = V_0(y)$ is a fixed point of the boost.

In the limit $s \to \infty$ all of these excitations become soft. More specifically,

$$\begin{aligned}\langle T_{vv}\rangle_s|_{v<V_0(y)} &\equiv \langle\psi_s|T_{vv}(v)|\psi_s\rangle|_{v<V_0(y)} \\ &= e^{-4\pi s}\langle\psi|T_{vv}(V_0 + e^{-2\pi s}(v - V_0))|\psi\rangle|_{v<V_0(y)}\end{aligned} \tag{8.5.6}$$

which just follows from the usual algebra of half-sided modular inclusions. Hence $\langle T_{vv}\rangle_s \to 0$ as $s \to \infty$ for $v < V_0(y)$.

Not only that but also

$$\lim_{s\to\infty}\int_{-\infty}^{v} dv \ \langle T_{vv}\rangle_s \to 0, \ v < V_0(y) \ . \tag{8.5.7}$$

To see what this implies about the energy of the boosted side, we make use of the sum rule derived in [16] for null derivatives of the relative entropy:

$$2\pi\left(P_s - e^{-2\pi s}P\right) = \left(e^{-2\pi s} - 1\right)\frac{\delta S_{\rm rel}(\psi|\Omega;\mathcal{A}_V)}{\delta V}\Big|_{V_0} \ , \tag{8.5.8}$$

where

$$P = \int_{-\infty}^{\infty} dv \ \langle T_{vv} \rangle_\psi \tag{8.5.9}$$

is the average null energy of the original state, P_s is the average null energy of $|\psi_s\rangle$, and

$$S_{\rm rel}(\psi|\Omega; \mathcal{A}_V) = -\langle \psi | \log \Delta_{\psi|\Omega;\mathcal{A}_V} |\psi\rangle \tag{8.5.10}$$

is the relative entropy of the original state for some general cut $V(y)$.

The relative entropy can also be written as

$$S_{\rm rel}(\psi|\Omega; \mathcal{A}_V) = \langle K_V \rangle_\psi - S(V) \tag{8.5.11}$$

and moreover [16]

$$\frac{\delta \langle K_V \rangle_\psi}{\delta V}\Big|_{V_0} = -2\pi \int_{V_0(y)}^{\infty} dv \ \langle T_{vv} \rangle_\psi \ . \tag{8.5.12}$$

Thus in the limit $s \to \infty$ we find, using Eq. (8.5.7),

$$\langle T_{vv} \rangle|_{v \le V_0(y)} = -\frac{1}{2\pi} \frac{\delta S}{\delta V}\Big|_{V_0} \delta(v - V_0(y)) \tag{8.5.13}$$

as desired. This reproduces both Eq. (8.4.14) and Eq. (8.4.15).

As a final point, note that under the Connes cocycle we also have the following properties:

$$\langle T_{uv} \rangle_{s\to\infty} = \langle T_{uv}(V_0) \rangle_\psi \ , \tag{8.5.14}$$
$$\langle T_{iv} \rangle_{s\to\infty} \to 0 \ . \tag{8.5.15}$$

This very easily reproduces the properties Eq. (8.4.20) and Eq. (8.4.22).

8.6 Discussion

We end by discussing the boundary interpretation of the generalized entropy of a QMT surface. We will also briefly describe future work on a systematic algorithm for constructing the states we conjectured in Sec. 8.3.

Boundary dual

Within AdS/CFT, it is natural to ask whether the coarse-graining prescription for $S_{\rm outer}$ in Sec. 8.3 has a boundary dual. In other words, there must exist a boundary state dual to the bulk coarse-grained semiclassical state of Sec. 8.3. Based on Eq. (8.3.12), we know that the boundary dual to this state is a mixed state that maximizes the boundary von

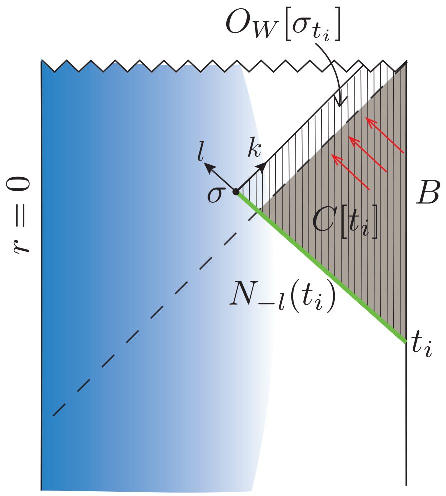


Figure 8.4: We would like to fix the data on $N_{-l}(t_i)$ (green thick line), while coarse-graining in the interior of the QMT surface. Simple data in the boundary region $t > t_i$ fixes the causal wedge $C[t_i]$ and thus fixes only a portion of $N_{-l}(t_i)$. In order to fix all of $N_{-l}(t_i)$ one must allow for sources that remove the excitations (red arrows) that enter the black hole after σ; this can cause the causal wedge to grow to include N_{-l}. In the coarse-graining set $\mathcal{F}$, the simple data must agree for all allowed sources.

Neumann entropy subject to fixing the semiclassical state in $O_W[\mu]$. Since in Sec. 8.3 we only considered a case where we have reflecting boundary conditions at infinity, fixing $O_W[\mu]$ amounts to fixing the past boundary of $O_W[\mu]$, labelled $N_{-l}(t_i)$ in Fig. 8.4.

Therefore, the question of whether there is a natural boundary dual to our bulk coarse-graining prescription reduces to that of whether fixing the semiclassical state on $N_{-l}(t_i)$ has a natural interpretation in the boundary. Our answer to this question is very similar to the simple entropy S_{simple} prescription of [18, 338].

Since we would like to refer to the bulk as little as possible, we define the QMT surface μ associated to a time slice t_i of the boundary by constructing an ingoing null surface from t_i and marking the first QMT surface on it. In general, this surface could reach caustics before reaching μ; Ref. [338] deals with this technicality. Here we ignore this issue by restricting to special classes of states (e.g. perturbations to Killing horizons).

Let $\rho(t_i)$ be the original boundary state at time t_i. We would like to construct a boundary state with maximum von Neumann entropy, which agrees with the semiclassical bulk state on $N_{-l}(t_i)$. In order to accomplish this, we must find a boundary definition of $\mathcal{F}$, the set of density matrices dual to the semiclassical state on $N_{-l}(t_i)$.

Let us first consider $\mathcal{F}$ to be the states that agree with $\rho(t_i)$ on simple boundary observables $\mathcal{A}$ on $t > t_i$. Simple observables are defined to be boundary operators whose associated excitations propagate causally in the bulk [18, 338], so this data fixes the bulk causal wedge of $t > t_i$ ($C[t_i]$ in Fig. 8.4). However, $C[t_i] \subseteq O_W[\sigma_{t_i}]$, so in general this set $\mathcal{F}$ would not be constrained enough to fix all of the data on $N_{-l}(t_i)$.

The discrepancy between $C[t_i]$ and $O_W[\sigma_{t_i}]$ arises from matter that enters the black hole to the future of σ_{t_i}. This causes the event horizon to grow and lie properly inside of the outer wedge. To fix all of $O_W[\sigma_{t_i}]$ given $\rho(t_i)$, one must turn on boundary sources that will absorb the future infalling excitations and achieve $C[t_i] = O_W[\sigma_{t_i}]$. This may seem acausal, but so is the definition of simple operator as an operator that can be represented by local boundary operators smeared over space *and time*.

Therefore, the coarse-graining set $\mathcal{F}$ should consist of the states such that the simple boundary observables $\mathcal{A}$ agree with those of $\rho(t_i)$ even after both states have been subject to turning on various simple sources on the boundary:

$$S_{\text{simple}}(t_i) = \max_{\tilde{\rho}\in\mathcal{F}} \; S(\tilde{\rho}) \tag{8.6.1}$$

with

$$\mathcal{F} = \{\tilde{\rho} : \langle E\mathcal{A}E^\dagger\rangle_{\tilde{\rho}(t)} = \langle E\mathcal{A}E^\dagger\rangle_\rho,\; t \geq t_i;\; \forall E\} \tag{8.6.2}$$

where $\mathcal{A}$ is the set of simple observables and E denotes unitaries associated with turning on various simple boundary sources.

Note that $C[t_i] \subseteq O_W[\sigma_{t_i}]$ in all semiclassical states [277]. Therefore, subjecting the states to various simple sources is never going to make a slice larger than $N_{-l}(t_i)$ causally accessible from the boundary. Given the state $\rho(t_i)$, there exists a fine-tuned choice of sources that will make $C[t_i] = O_W[\sigma_{t_i}]$. But since this choice is state-dependent and difficult to specify from a pure boundary perspective, we choose the boundary coarse-graining family $\mathcal{F}$ to agree with $\rho(t_i)$ on simple data subject to *all* simple sources turned on.

So far we have defined $\mathcal{A}$ as the set of boundary observers that correspond to bulk excitations that propagate causally. The classical analysis of Refs. [18, 338] further specified $\mathcal{A}$ to consist only of one point functions of all local operators on the boundary. This will fix the states of the classical fields in the bulk that are causally determined by the boundary region $t \geq t_i$. Since here we are interested in fixing the quantum state of the bulk fields on $N_{-l}(t_i)$, our set $\mathcal{A}$ needs to include higher point function of local bulk operators.

However, we are still interested in maintaining locality in the bulk and therefore want to disallow a large density of local probes in any bulk region. This is following the expectation that such excitations would cause large backreaction and therefore a breakdown of locality [347]. From a boundary perspective, a local bulk operator in the causal wedge is dual to a smeared boundary operator [348]. Therefore, our set $\mathcal{A}$ needs to include all products of smeared boundary operator as long as there is not an $\mathcal{O}(N)$ number of overlap in the support of the smeared operators. This choice of $\mathcal{A}$ in Eq. (8.6.2) is a natural candidate for fixing the quantum state on $N_{-l}(t_i)$; we leave a thorough investigation of this issue to future work.

We refer the reader to [338] for a careful demonstration of $S_{\text{simple}} = S_{\text{outer}}$ in the bulk classical limit.

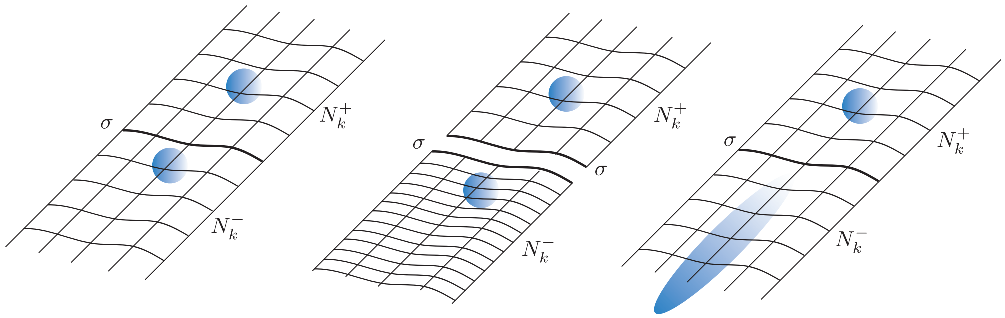


Figure 8.5: The left stretch is a classical analogue of the CF flow that generalizes it to nontrivial geometries. Left: The null surface N_k split by the marginally trapped surface σ. Middle: The affine parameter is rescaled on N_k^- but held fixed on N_k^+. This is the same initial data in nonaffine parametrization. Right: The two pieces are glued back together, treating the new parameter as affine. This yields inequivalent initial data.

Semiclassical Stretched States

In this paper, we started from a classical construction in general relativity, whose quantum interpretation is the coarse-graining of a quantum state so that its entropy matches the area of a marginally trapped surface. We elevated this to a semi-classical conjecture that we interpret as a coarse-graining that will match the generalized entropy of a quantum marginally trapped surface, while holding fixed the exterior quantum state. In the QFT limit, our conjecture is confirmed by the limit of the CF sequence of states [16].

Thus, we were able to derive a nontrivial, testable property of QFT from a hypothetical assumption about quantum gravity. This is similar to how the QNEC was derived from the QFC, a hypothetical extension of the classical focussing property of general relativity. This is a satisfying connection. QFT has not been directly probed in this limit, and a direct verification of the CF limit or of the QNEC would constitute a test of our ideas about quantum gravity.

Interestingly, there appears to be a larger set of relations of the type we explored here. Our starting point, the EW construction, is essentially unique. However, the CF construction produces a one-parameter family of states, given an input state and a cut on a Killing horizon. Here we only made use of the limit approached by these states as the flow parameter diverges. But we expect that there exists a classical construction (which may limit to the EW construction) that matches the entire one-parameter CF family.

In the special case where the cut is a bifurcation surface of the Killing horizon, the CF construction admits an interesting intuitive interpretation: all correlators of operators restricted to the left (or to the right) behave as if we had boosted the state on the left side of the cut (but not on the right). In QFT, a one-sided boost would result in a divergent-energy shock at the cut, because it would destroy the vacuum. But the CF flow is more subtle; in a sense it boosts only the "excited part" of the state on N_k^-, while leaving the vacuum entanglement across the cut intact.

This suggests a simple classical analogue of CF flow. At the classical level, a half-sided boost is innocuous. It can be applied to initial data on the null surface N_k^- with no ill consequences at the cut. However, a generic cut of a Killing horizon is not a bifurcation surface and hence is not a fixed point of the Killing flow.

Nonetheless, one can construct a sequence of geometries by a construction we will call the *left stretch*. Given the state and affine parameter v on the entire Killing horizon N_k, rescale $V \to V' = e^s V$ on the left side N_k^-, and do nothing on the right: $V \to V'$ on N_k^+. This will rescale all v-derivatives of classical fields by e^{-s}. To preserve the inner product $k^a l_a \equiv g_{ab}(\partial_v)^a(\partial_u)^b = -1$, rescale the u-derivatives at constant (v, y) by e^s. Then glue the two halves back together, treating V' as a true affine parameter, see Fig. 8.5.

For the full initial data on N, we need to know not only the intrinsic geometry but also θ_l, the expansion in the null direction off of N_k. This is obtained by holding θ_l fixed on N_k^+ and integrating the cross-focussing equation,

$$k^a \nabla_a \theta_l = -\frac{1}{2}\mathcal{R} - \theta_k \theta_l + \chi^2 + \nabla \cdot \chi + 8\pi G\, T_{kl} \ , \tag{8.6.3}$$

to obtain θ_l on N_k^-. Since all terms on the right hand side scale trivially, this rescales the difference $\theta_l - \theta_l|_{V_0}$ by e^s.

Because θ_l is not given by a simple rescaling unless $\theta_l|_{V_0} = 0$, the left stretch results in physically inequivalent initial data *even in the left exterior of σ alone*. The intrinsic data on N_k^- are stretched, as measured by a ruler defined by the evolution of the extrinsic curvature θ_l.

Interestingly, the left stretch is physically sensible if and only if the cut satisfies $\theta_k = 0$. This is because the expansion θ_k along N_k is determined not only by the left stretch itself, but also by the Raychaudhuri equation, and the two methods must agree. Let the inaffinity κ be defined by $k^b \nabla_b k^a = \kappa k^a$. Affine parametrization corresponds to $\kappa = 0$ everywhere. The left stretch implements

$$V(y) \to e^{sH[-V(y)+V_0(y)]} V(y) \ , \tag{8.6.4}$$

where $H(v)$ is the Heaviside step function and $v = V_0(y)$ is the marginally trapped surface σ. This generates a non-zero inaffinity

$$\kappa = (1 - e^{-s})\delta[V(y)] \ . \tag{8.6.5}$$

The Raychaudhuri equation for non-affine parametrization reads

$$k^a \nabla_a \theta_k = -\frac{1}{2}\theta_k^2 - \varsigma_k^2 - \kappa\theta_k - 8\pi G\ T_{kk} \ . \tag{8.6.6}$$

We insist that the new parameter V' be treated as affine, which means we are demanding that the inaffinity term $\kappa\theta$ vanishes even after the left stretch. By Eq. (8.6.5), this will be the case if and only if $\theta_k = 0$ at the cut.

Importantly, Eqs. (8.2.5), (8.2.7) and (8.2.11) become satisfied in the limit as $s \to \infty$. These are precisely the conditions imposed by EW for the classical coarse-graining construction. In this sense the left stretch can be viewed as generating a one-parameter interpolation from the original initial data to the coarse-grained data.[3]

We close with two brief remarks. At the level of semiclassical gravity, the left stretch should naturally combine with the CF construction, so that not only the geometric and classical data, but also the quantum initial data are stretched. Moreover, we expect that the left stretch (applied classically to the RT or semiclassically to the quantum RT surface) is the gravity dual of the CF flow applied to the boundary of Anti-de Sitter space.

[3] However, there are interesting differences to the EW analysis. For example, the left stretch yields divergent T_{uu}, as does the CF limit. Yet, EW argue that this can be avoided. There may be a larger family of relevant states.

Chapter 9

Gravity Dual of Connes Cocycle Flow

9.1 Introduction

The AdS/CFT duality [3, 349, 350] has led to tremendous progress in the study of quantum gravity. However, our understanding of the holographic dictionary remains limited. In recent years, quantum error correction was found to play an important role in the emergence of a gravitating ("bulk") spacetime from the boundary theory [347, 286, 351]. The study of modular operators led to the result that the boundary relative entropy in a region A equals the bulk relative entropy in its entanglement wedge $\mathrm{EW}(A)$ [137]. The combination of these insights was used to derive subregion duality: bulk operators in $\mathrm{EW}(A)$ can in principle be reconstructed from operators in the subregion A [136].

The relation between bulk modular flow in $\mathrm{EW}(A)$ and boundary modular flow in A has been used to explicitly reconstruct bulk operators both directly [352, 353], and indirectly via the Petz recovery map and its variants [354, 355, 356]. Thus, modular flow has shed light on the emergence of the bulk spacetime from entanglement properties of the boundary theory.

Modular flow has also played an important role in proving various properties of quantum field theory (QFT), such as the averaged null energy condition (ANEC) and quantum null energy condition (QNEC) [301, 357]. Tomita-Takesaki theory, the study of modular flow in algebraic QFT, puts constraints on correlation functions that are otherwise hard to prove directly [358].

Recently, an alternate proof of the QNEC was found using techniques from Tomita-Takesaki theory [16]. The key ingredient was Connes cocyle (CC) flow. Given a subregion A and global pure state ψ, Connes cocycle flow acts with a certain combination of modular operators to generate a sequence of well-defined global states ψ_s. In the limit $s \to \infty$, these states saturate Wall's "ant conjecture" [344] on the minimum amount of energy in the complementary region A'. This proves the ant conjecture, which, in turn, implies the QNEC.

CC flow also arises from a fascinating interplay between quantum gravity, quantum information, and QFT. Recently, the classical black hole coarse-graining construction of Engelhardt and Wall [18] was conjecturally extended to the semiclassical level [139]. In the non-gravitational limit, this conjecture requires the existence of flat space QFT states with properties identical to the $s \to \infty$ limit of CC flowed states. This is somewhat reminiscent of how the QNEC was first discovered as the nongravitational limit of the quantum focusing conjecture [280]. Clearly, CC flow plays an important role in the connection between QFT and gravity. Our goal in this paper is to investigate this connection at a deeper level within the setting of AdS/CFT.

In Sec. 9.2, we define CC flow and discuss some of its properties. If ∂A lies on a null plane in Minkowski space, operator expectation values and subregion entropies within the region A remain the same, whereas those in A' transform analogously to a boost [16]. Further, CC flowed states ψ_s exhibit a characteristic stress tensor shock at the cut ∂A, controlled by the derivative of the von Neumann entropy of the region A in the state ψ under shape deformations of ∂A [139].

As is familiar from other examples in holography, bulk duals of complicated boundary objects are often much simpler [4, 17]. Motivated by the known properties of CC flow, we define a bulk construction in Sec. 9.3, which we call the "kink transform." This is a one-parameter transformation of the initial data of the bulk spacetime dual to the original boundary state ψ. We consider a Cauchy surface Σ that contains the Ryu-Takayanagi (RT) surface $\mathcal{R}$ of the subregion A. The kink transform acts as the identity except at $\mathcal{R}$, where an s-dependent shock is added to the extrinsic curvature of Σ. We show that this is equivalent to a one-sided boost of Σ in the normal bundle to $\mathcal{R}$. We prove that the new initial data satisfies the gravitational constraint equations, thus demonstrating that the kink transform defines a valid bulk spacetime $\mathcal{M}_s$. We show that $\mathcal{M}_s$ is independent of the choice of Σ.

We propose that $\mathcal{M}_s$ is the holographic dual to the CC-flowed state ψ_s, if the boundary cut ∂A is (conformally) a flat plane in Minkowski space.

In Sec. 9.4, we provide evidence for this proposal. The kink transform separately preserves the entanglement wedges of A and A', but it glues them together with a relative boost by rapidity $2\pi s$. This implies the one-sided expectation values and subregion entropies of the CC flowed state ψ_s are correctly reproduced when they are computed holographically in the bulk spacetime $\mathcal{M}_s$. We then perform a more nontrivial check of this proposal. By computing the boundary stress tensor holographically in $\mathcal{M}_s$, we reproduce the stress tensor shock at ∂A in the CC-flowed state ψ_s.

Having provided evidence for kink transform/CC flow duality, we use the duality to make a novel prediction for CC flow in Sec. 9.5. The kink transform fully determines all independent components of the shock at ∂A in terms of shape derivatives of the entanglement entropy. Strictly, our results only apply only to the CC flow of a holographic CFT across a planar cut. However, their universal form suggests that they will hold for general QFTs under CC flow. Moreover, the shocks we find agree with properties required to exist in quantum states under the coarse-graining proposal of Ref. [359]. Thus, our new results may also hold for CC flow across general cuts of a null plane.

In Sec. 9.6, we discuss the relation of our construction to earlier work on the role of modular flow in AdS/CFT [360, 137, 361]. The result of Jafferis *et al.* (JLMS) [137] has conventionally been understood as a relation that holds for a small code subspace of bulk states on a fixed background spacetime. However, results from quantum error correction suggest that this code subspace could be made much larger to include different background geometries [286, 362, 363, 364]. Our proposal then follows from such an extended version of the JLMS result which includes non-perturbatively different background geometries. Equipped with this understanding, we can distinguish our proposal from the closely related bulk duals of one-sided modular flowed states [360, 361]. We provide additional evidence for our proposal based on two sided correlation functions of heavy operators, and we discuss generalizations and applications of the proposed kink transform/CC flow duality.

In Appendix A.23 we derive the null limit of the kink transform, and show that it generates a Weyl shock, which provides intuition for how the kink transform modifies gravitational observables.

9.2 Connes Cocycle Flow

In this section, we review Connes cocycle flow and its salient properties; for more details see [16, 139]. We then reformulate Connes cocycle flow in as a simpler map to a state defined on a "precursor" slice. This will prove useful in later sections.

Definition and General Properties

Consider a quantum field theory on Minkowski space $\mathbb{R}^{d-1,1}$ in standard Cartesian coordinates $(t, x, y_1, \ldots, y_{d-2})$. Consider a Cauchy surface $\mathcal{C}$ that is the disjoint union of the open regions A_0, A_0' and their shared boundary ∂A_0. Let $\mathcal{A}_0, \mathcal{A}_0'$ denote the associated algebras of operators. Let $|\psi\rangle$ be a cyclic and separating state on $\mathcal{C}$, and denote by $|\Omega\rangle$ the global vacuum (the assumption of cyclic and separating could be relaxed for $|\psi\rangle$, at the cost of complicating the discussion below). The Tomita operator is defined by

$$S_{\psi|\Omega;\mathcal{A}_0}\alpha|\psi\rangle = \alpha^\dagger|\Omega\rangle, \forall\alpha \in \mathcal{A}_0 \ . \tag{9.2.1}$$

The relative modular operator is defined as

$$\Delta_{\psi|\Omega} \equiv S^\dagger_{\psi|\Omega;\mathcal{A}_0} S_{\psi|\Omega;\mathcal{A}_0} \ , \tag{9.2.2}$$

and the vacuum modular operator is

$$\Delta_\Omega \equiv \Delta_{\Omega|\Omega} \ . \tag{9.2.3}$$

Note that we do not include the subscript $\mathcal{A}_0$ on Δ; instead, for modular operators, we indicate whether they were constructed from $\mathcal{A}_0$ or $\mathcal{A}_0'$ by writing Δ or Δ'.

Connes cocycle (CC) flow of $|\psi\rangle$ generates a one parameter family of states $|\psi_s\rangle$, $s \in \mathbb{R}$, defined by

$$|\psi_s\rangle = (\Delta'_\Omega)^{is}(\Delta'_{\Omega|\psi})^{-is}|\psi\rangle \ . \tag{9.2.4}$$

Thus far the definitions have been purely algebraic. In order to elucidate the intuition behind CC flow, let us write out the modular operators in terms of the left and right density operators, $\rho^{\psi}_{A_0} = \mathrm{Tr}_{A'_0}|\psi\rangle\langle\psi|$ and $\rho^{\psi}_{A'_0} = \mathrm{Tr}_{A_0}|\psi\rangle\langle\psi|$:[1]

$$\Delta_{\psi|\Omega} = \rho^{\Omega}_{A_0} \otimes (\rho^{\psi}_{A'_0})^{-1} \ . \tag{9.2.5}$$

One finds that the CC operator acts only in $\mathcal{A}'_0$:

$$(\Delta'_{\Omega})^{is}(\Delta'_{\Omega|\psi})^{-is} = (\rho^{\Omega}_{A'_0})^{is}(\rho^{\psi}_{A'_0})^{-is} \in \mathcal{A}'_0 \ . \tag{9.2.6}$$

It follows that the reduced state on the right algebra satisfies

$$\rho^{\psi_s}_{A_0} = \rho^{\psi}_{A_0} \ . \tag{9.2.7}$$

Therefore, expectation values of observables $\mathcal{O} \in \mathcal{A}_0$ remain invariant under CC flow. These heuristic arguments would be valid only for finite-dimensional Hilbert spaces [365]; but Eq. (9.2.6) can be derived rigorously [16].

It can also be shown that $(\Delta'_{\psi|\Omega})^{is}\Delta^{is}_{\Omega|\psi} = 1$. Hence for operators $\mathcal{O}' \in \mathcal{A}'_0$, one finds that CC flow acts as Δ^{is}_{Ω} inside of expectation values:

$$\begin{aligned}\langle\psi_s|\mathcal{O}'|\psi_s\rangle &= \mathrm{Tr}_{\mathcal{A}'_0}\left[\rho^{\psi}_{A'_0}(\Delta^{-is}_{\psi|\Omega}\Delta^{is}_{\Omega})\mathcal{O}'(\Delta^{-is}_{\Omega}\Delta^{is}_{\psi|\Omega})\right] \ , \\ &= \mathrm{Tr}_{\mathcal{A}'_0}\left(\rho^{\psi}_{A'_0}(\rho^{\Omega}_{A'_0})^{-is}\mathcal{O}'(\rho^{\Omega}_{A'_0})^{is}\right) && (9.2.8) \\ &= \mathrm{Tr}\left[|\psi\rangle\langle\psi|\Delta^{is}_{\Omega}(\mathbf{1}\otimes\mathcal{O}')\Delta^{-is}_{\Omega}\right] \ , && (9.2.9)\end{aligned}$$

where we have used the cyclicity of the trace.

To summarize, expectation values of one-sided operators transform as follows:

$$\langle\psi_s|\mathcal{O}|\psi_s\rangle = \langle\psi|\mathcal{O}|\psi\rangle \ , \tag{9.2.10}$$
$$\langle\psi_s|\mathcal{O}'|\psi_s\rangle = \langle\psi|\Delta^{is}_{\Omega}\mathcal{O}'\Delta^{-is}_{\Omega}|\psi\rangle \ . \tag{9.2.11}$$

There is no simple description of two-sided correlators in $|\psi_s\rangle$ such as $\langle\psi_s|\mathcal{O}\mathcal{O}'|\psi_s\rangle$; we discuss such objects in Sec. 9.6.

CC Flow from Cuts on a Null Plane

Let us now specialize to the case where ∂A_0 corresponds to a cut $v = V_0(y)$ of the Rindler horizon $u = 0$. We have introduced null coordinates $u = t - x$ and $v = t + x$ and denoted the transverse coordinates collectively by y. It can be shown that the modular operator Δ^{is}_{Ω}

[1] We follow the conventions in [365] where complement operators are written to the right of the tensor product.

acts locally on each null generator y of $u = 0$ as a boost about the cut $V_0(y)$ [45]. More explicitly, one can define the *full* vacuum modular Hamiltonian $\widehat{K}_{V_0}$ by

$$\widehat{K}_{V_0} = -\log \Delta_{\Omega;\mathcal{A}_{V_0}} \ . \tag{9.2.12}$$

We can write the full modular Hamiltonian as

$$\widehat{K}_{V_0} = K_{V_0} \otimes \mathbf{1}' - \mathbf{1} \otimes K'_{V_0} \ . \tag{9.2.13}$$

Let Δ denote vacuum subtraction, $\Delta\langle\mathcal{O}\rangle = \langle\mathcal{O}\rangle_\psi - \langle\mathcal{O}\rangle_\Omega$. Then, for arbitrary cuts of the Rindler horizon, we have [45]

$$\Delta\langle K'_{V_0}\rangle = -2\pi \int dy \int_{-\infty}^{V_0} dv[v - V_0(y)]\langle T_{vv}\rangle_\psi \ , \tag{9.2.14}$$

and similarly for K_{V_0}. Thus K'_{V_0} is simply the boost generator about the cut $V_0(y)$ in the left Rindler wedge. That is, it generates a y-dependent dilation,

$$v \to V_0(y) + [v - V_0(y)]e^{2\pi s} \ . \tag{9.2.15}$$

This allows us to evaluate Eq. (9.2.11) explicitly for local operators at $u = 0$. For example, the CC flow of the stress tensor is

$$\langle\psi_s|T_{vv}|\psi_s\rangle|_{v<V_0} = e^{-4\pi s}\langle\psi|T_{vv}\left(V_0 + e^{-2\pi s}(v - V_0)\right)|\psi\rangle|_{v<V_0} \ , \tag{9.2.16}$$

and similarly for the other components of $T_{\mu\nu}$. There is a slight caveat here since Δ_Ω^{is} only acts as a boost strictly at $u = 0$. This would be sufficient for free theories, where T_{vv} can be defined through null quantization on the Rindler horizon with a smearing that only needs support on $u = 0$ [12]. More generally, $T_{\mu\nu}$ must be smeared in an open neighborhood of $u = 0$. However, if $V_0(y)$ is a perturbation of a flat cut then one can show that inside correlation functions Δ_Ω^{is} approximately acts as a boost with subleading errors that vanish as $u \to 0$, to all orders in the perturbation [357, 366]. In the non-perturbative case, evidence comes from the fact that classically the vector field on the Rindler horizon which generates boosts about $V_0(y)$ can be extended to an approximate Killing vector field in a neighborhood of the horizon [39, 74]. Therefore we expect Eq. (9.2.16) to hold on the null surface even after smearing.

Now consider a second cut $V(y)$ of the Rindler horizon which lies entirely below $V_0(y)$, so $V < V_0$ for all y. The cut defines a surface ∂A_V that splits a Cauchy surface $\mathcal{C}_V = A'_V \cup \partial A_V \cup A_V$; we take A'_V to be the "left" side ($v < V$), with operator algebra $\mathcal{A}'_V$. The Araki definition of relative entropy is [365]

$$S'_{\text{rel}}(\psi|\Omega;V) = -\langle\psi|\log \Delta_{\psi|\Omega;\mathcal{A}'_V}|\psi\rangle \ . \tag{9.2.17}$$

It has the following transformation properties [16]:

$$S_{\rm rel}(\psi_s|\Omega;V) = S_{\rm rel}(\psi|\Omega;V_0+e^{-2\pi s}(V-V_0)) \ , \tag{9.2.18}$$

$$\frac{\delta S_{\rm rel}(\psi_s|\Omega;V)}{\delta V} = e^{-2\pi s}\frac{\delta S_{\rm rel}(\psi|\Omega;V_0+e^{-2\pi s}(V-V_0))}{\delta V} \ . \tag{9.2.19}$$

Moreover, the “left" von Neumann entropy is defined as

$$S'(\psi,V) = -\mathrm{tr}_{A'_V}\rho^{\psi}_{A'_V}\log\rho^{\psi}_{A'_V} \ . \tag{9.2.20}$$

With these definitions in hand, one can decompose the relative entropy as

$$S'_{\rm rel}(\psi|\Omega;V) = \Delta\langle K'_V\rangle - \Delta S'(V) \ . \tag{9.2.21}$$

At this point we drop the explicit vacuum subtractions, as we will only be interested in shape derivatives of the vacuum subtracted quantities, which automatically annihilate the vacuum expectation values. In particular, one can directly compute shape derivatives of K'_V:

$$\frac{\delta\langle K'_V\rangle_\psi}{\delta V}\Big|_{V_0} = 2\pi\int_{-\infty}^{V_0} dv\ \langle T_{vv}\rangle_\psi \ . \tag{9.2.22}$$

Hence the transformations of both K'_V and its derivative simply follow from Eq. (9.2.16).

Combining Eq. (9.2.18) and Eq. (9.2.14), as well as Eq. (9.2.19) and Eq. (9.2.22), we see that $S'(\psi,V)$ and its derivative transform as

$$S'(\psi_s,V) = S'(\psi,V_0+e^{-2\pi s}(V-V_0)) \ , \tag{9.2.23}$$

$$\frac{\delta S'}{\delta V}\Big|_{\psi_s,V} = e^{-2\pi s}\frac{\delta S'}{\delta V}\Big|_{\psi,V_0+e^{-2\pi s}(V-V_0)} \ . \tag{9.2.24}$$

The respective properties of the complement entropy follow from purity.

Stress Tensor Shock at the Cut

CC flow generates a stress tensor shock at the cut V_0, proportional to the jump in the variation of the one-sided von Neumann entropy under deformations, at the cut [139]. To see this, let us start with the sum rule derived in [16] for null variations of relative entropy:[2]

$$2\pi(P_s - e^{-2\pi s}P_0) = (e^{-2\pi s}-1)\frac{\delta S'_{\rm rel}(\psi|\Omega;V)}{\delta V}\Big|_{V_0} \ , \tag{9.2.25}$$

where

$$P \equiv \int_{-\infty}^{\infty} dv\ T_{vv} \tag{9.2.26}$$

[2]For type I algebras, one can derive the analogous sum rule from simpler arguments [323].

is the averaged null energy operator at $u = 0$, and $P_s \equiv \langle\psi_s|P|\psi_s\rangle$, so in particular $P_0 \equiv \langle\psi|P|\psi\rangle$. (There is one such operator for every generator, *i.e.*, for every y.)

Inserting Eq. (9.2.21) and Eq. (9.2.22) into Eq. (9.2.25), and making use of Eq. (9.2.16), we see that there must exist a shock at $v = V_0(y)$:

$$\langle\psi_s|T_{vv}|\psi_s\rangle = (1 - e^{-2\pi s})\frac{1}{2\pi}\frac{\delta S'}{\delta V}\Big|_{V_0}\delta(v - V_0) + o(\delta) \ . \tag{9.2.27}$$

Here $o(\delta)$ designates the finite (non-distributional) terms. These are determined by Eq. (9.2.16), and by its trivial counterpart in the $v > V_0$ region.

This s-dependent shock is a detailed characteristic of the CC flowed state. As such, reproducing it through the holographic dictionary will be the key test of our proposal of the bulk dual of CC flow (see Sec. 9.4).

Flat Cuts and the Precursor Slice

For the remainder of the paper we further specialize to flat cuts of the Rindler horizon, so that ∂A_0 corresponds to $u = v = 0$. We therefore set $V_0 = 0$ in what follows. We take $\mathcal{C}$ to be the Cauchy surface $t = 0$, so that A_0 ($t = 0$, $x > 0$) and A_0' ($t = 0$, $x < 0$) are partial Cauchy surfaces for the right and left Rindler wedges.

In this case Δ_Ω^{is} is a global boost by rapidity s about ∂A_0 [367]. Thus, it has a simple geometric action not only on the null plane $u = 0$, but everywhere. CC flow transforms observables in $\mathcal{A}_0'$ by Δ_Ω^{is} and leaves invariant those in $\mathcal{A}_0$. For a flat cut, this action can be represented as a geometric boost in the entire left Rindler wedge. This allows us to characterize the CC flowed state $|\psi(s)\rangle$ on $\mathcal{C}$ very simply in terms of a different state on a different Cauchy surface which we call the "precursor slice". This description will motivate the formulation of our bulk construction in Sec. 9.3.

By Eq. (9.2.11), the CC flowed state on the slice $\mathcal{C}$,

$$|\psi_s(\mathcal{C})\rangle = (\Delta_\Omega')^{is}(\Delta_{\Omega|\psi}')^{-is}\,|\psi(\mathcal{C})\rangle \ , \tag{9.2.28}$$

satisfies

$$\langle\psi_s(\mathcal{C})|\,\mathcal{O}_A\,|\psi_s(\mathcal{C})\rangle = \langle\psi(\mathcal{C})|\,\mathcal{O}_A\,|\psi(\mathcal{C})\rangle \ , \tag{9.2.29}$$

$$\langle\psi_s(\mathcal{C})|\,\Delta_\Omega^{-is}\mathcal{O}_{A'}\Delta_\Omega^{is}\,|\psi_s(\mathcal{C})\rangle = \langle\psi(\mathcal{C})|\,\mathcal{O}_{A'}\,|\psi(\mathcal{C})\rangle \ , \tag{9.2.30}$$

where $\mathcal{O}_A$ and $\mathcal{O}_{A'}$ denote an arbitrary collection of local operators that act on spacelike half-slices A and A' of $\mathcal{C}$ respectively.[3] In the second equality above, we used the fact that Δ_Ω^{is} acts as a global boost to move it to the other side of the equality, compared to Eq. (9.2.11).

[3] More precisely, one would have to smear the operator in a codimension 0 neighborhood of points on the slices.

We work in the Schrödinger picture where the argument $\mathcal{C}$ should be interpreted as the time variable. The fact that Δ_Ω^{is} acts as a boost around ∂A_0 motivates us to consider the time slice

$$\mathcal{C}_s = A'_s \cup \partial A_0 \cup A_0 \ , \tag{9.2.31}$$

where

$$A'_s = \{t = (\tanh 2\pi s)x,\ x < 0\} \ . \tag{9.2.32}$$

By Eqs. (9.2.29) and (9.2.30), each side of the CC-flowed state $|\psi_s(\mathcal{C}_s)\rangle$ is simply related to the left and right restrictions of the original state on the original slice:

$$\langle\psi_s(\mathcal{C}_s)|\, \mathcal{O}_A \,|\psi_s(\mathcal{C}_s)\rangle = \langle\psi(\mathcal{C})|\, \mathcal{O}_A \,|\psi(\mathcal{C})\rangle \ , \tag{9.2.33}$$

$$\langle\psi_s(\mathcal{C}_s)|\, \mathcal{O}_{A'_s} \,|\psi_s(\mathcal{C}_s)\rangle = \langle\psi(\mathcal{C})|\, \mathcal{O}_{A'} \,|\psi(\mathcal{C})\rangle \ . \tag{9.2.34}$$

In the second equation, $\mathcal{O}_{A'_s}$ denotes local operators on A'_s which are analogous to $\mathcal{O}_{A'}$ on A'. More precisely, because the intrinsic metric of A' and A'_s are the same, there exists a natural map between local operators on A' and A'_s.

In words, Eqs. (9.2.33) and (9.2.34) say that correlation functions in each half of $\mathcal{C}$ in the state $|\psi(\mathcal{C})\rangle$ are equal to the analogous correlation functions on each half of $\mathcal{C}_s$ in the state $|\psi_s(\mathcal{C}_s)\rangle$. This justifies calling $\mathcal{C}_s$ the precursor slice since the CC flowed state on $\mathcal{C}$ arises from it by time evolution.

We find it instructive to repeat this point in the less rigorous language of density operators. In the density operator form of CC flow,

$$|\psi_s(\mathcal{C})\rangle = (\rho^\Omega_{A'_0})^{is}(\rho^\psi_{A'_0})^{-is}\,|\psi(\mathcal{C})\rangle \ , \tag{9.2.35}$$

it is evident that the action of $(\rho^\Omega_{A'_0})^{is}$ can be absorbed into a change of time slice $\mathcal{C} \to \mathcal{C}_s$:

$$|\psi_s(\mathcal{C}_s)\rangle = (\rho^\psi_{A'_0})^{-is}\,|\psi(\mathcal{C})\rangle \ . \tag{9.2.36}$$

Tracing out each side of ∂A_0 implies

$$\rho^{\psi_s}_{A_0} = \rho^\psi_{A_0} \ , \tag{9.2.37}$$

$$\rho^{\psi_s}_{A'_s} = \rho^\psi_{A'_0} \ . \tag{9.2.38}$$

The first equality is trivial and was already discussed in Eq. (9.2.7). The second equality follows because $(\rho^\psi_{A'_0})^{is}$ commutes with $(\rho^\psi_{A'_0})$. This is the density operator version of Eqs. (9.2.33) and (9.2.34).

Eq. (9.2.36) should be contrasted with the one-sided modular-flowed state $|\phi(\mathcal{C})\rangle = (\rho^\psi_{A'_0})^{-is}\,|\psi(\mathcal{C})\rangle$. The latter state would live on the original slice $\mathcal{C}$, but it is not well-defined since it would have infinite energy at the entangling surface.

It will be useful to define new coordinates adapted to the precursor slice $\mathcal{C}_s$. Let

$$\tilde{v} = v\,\Theta(v) + e^{-2\pi s}\,v\,(1 - \Theta(v)) \ , \tag{9.2.39}$$

$$\tilde{u} = e^{2\pi s}u\,\Theta(u) + \,u\,(1 - \Theta(u)) \ , \tag{9.2.40}$$

where $\Theta(.)$ is the Heaviside step function. Let $\tilde{t} = \frac{1}{2}(\tilde{v} + \tilde{u})$ and $\tilde{x} = \frac{1}{2}(\tilde{v} - \tilde{u})$. In these coordinates, the Minkowski metric takes the form

$$\begin{aligned} ds^2 = &\left[\Theta(\tilde{t}+\tilde{x}) + e^{2\pi s}(1-\Theta(\tilde{t}+\tilde{x}))\right]\left[e^{-2\pi s}\Theta(\tilde{t}-\tilde{x}) + (1-\Theta(\tilde{t}-\tilde{x}))\right](-d\tilde{t}^2 + d\tilde{x}^2) \\ &+ d^{d-2}y \ , \end{aligned} \tag{9.2.41}$$

and the precursor slice corresponds to $\tilde{t} = 0$.

In these "tilde" coordinates, the stress tensor shock of Eq. (9.2.27) takes the form[4]

$$\langle\psi_s|T_{\tilde{v}\tilde{v}}|\psi_s\rangle = \frac{1}{2\pi}\left(\frac{\partial v}{\partial \tilde{v}}\right)^2 (1-e^{-2\pi s})\frac{\delta S}{\delta V}\Big|_{V=0}\delta(v) + o(\delta) \ . \tag{9.2.42}$$

Recall that the entropy variation is evaluated in the state $|\psi\rangle$. By Eq. (9.2.24),

$$\frac{\delta S}{\delta V}\Big|_{\psi} = \frac{\delta S}{\delta \widetilde{V}}\Big|_{\psi_s} \ , \tag{9.2.43}$$

where $\widetilde{V}(y)$ is a cut of the Rindler horizon in the $\tilde{v}$ coordinates. Thus we may instead evaluate the entropy variation in the state $|\psi_s\rangle$ on the precursor slice. This will be convenient when matching the bulk and boundary.

The Jacobian in Eq. (9.2.42) has a step function in it, as will the Jacobian coming from $\delta(v)$. A step function multiplying a delta function is well-defined if one averages the left and right derivatives:

$$\left(\frac{\partial v}{\partial \tilde{v}}\right)^2 \delta(v) = \frac{1}{2}\left(\frac{\partial v}{\partial \tilde{v}}\Big|_{0^-} + \frac{\partial v}{\partial \tilde{v}}\Big|_{0^+}\right)\delta(\tilde{v}) \ . \tag{9.2.44}$$

Thus Eq. (9.2.42) becomes

$$\langle\psi_s|T_{\tilde{v}\tilde{v}}(\tilde{v})|\psi_s\rangle = \frac{1}{2\pi}\sinh(2\pi s)\frac{\delta S}{\delta \widetilde{V}}\Big|_{\psi_s,\widetilde{V}=0}\delta(\tilde{v}) + o(\delta). \tag{9.2.45}$$

Since we are dealing with a flat cut, the symmetry $s \leftrightarrow -s, v \leftrightarrow u$ implies that CC flow also generates a T_{uu} shock in the state $|\psi_s\rangle$ at $u = v = 0$:

$$\langle\psi_s|T_{\tilde{u}\tilde{u}}|\psi_s\rangle = \frac{1}{2\pi}\sinh(2\pi s)\frac{\delta S}{\delta \widetilde{U}}\Big|_{\psi_s,\widetilde{V}=0}\delta(\tilde{u}) + o(\delta) \ . \tag{9.2.46}$$

(Note that $\delta/\delta V$ goes to $-\delta/\delta U$.) The linear combination

$$\langle\psi_s|T_{\tilde{t}\tilde{x}}(\tilde{t},\tilde{x})|\psi_s\rangle = \frac{1}{2\pi}\sinh(2\pi s)\frac{\delta S}{\delta \widetilde{X}}\Big|_{\psi_s,\widetilde{X}=0}\delta(\tilde{x}) + \langle\psi|T_{tx}(t=\tilde{t},x=\tilde{x})|\psi\rangle \ . \tag{9.2.47}$$

will be useful in Sec. 9.4. The last term was obtained from Eqs. (9.2.37) and (9.2.38); it makes the finite piece explicit. Note that these equations are valid in the entire left and right wedges, not just on $\mathcal{C}_s$.

[4]We remind the reader that $o(\delta)$ refers to any finite (non-distributional) terms.

9.3 Kink Transform

In this section, we introduce a novel geometric transformation called the kink transform. The construction is motivated by thinking about what the bulk dual of the boundary CC flow would be in the context of AdS/CFT. As we discussed in Sec. 9.2, CC flow boosts observables in $D(A')$ and leaves observables in $D(A)$ unchanged. Subregion duality in AdS/CFT then implies that the bulk dual of the state $|\psi_s\rangle$ has to have the property that the entanglement wedges of $D(A)$ and $D(A')$ will be diffeomorphic to those of the state $|\psi\rangle$, but are glued together with a "one-sided boost" at the Hubeny-Rangamani-Takayanagi (HRT) surface. In a general geometry, a boost Killing symmetry need not exist. The kink transform appropriately generalizes the notion of a one-sided boost to any extremal surface.

In Sec. 9.3, we formulate the kink transform. In Sec. 9.3, we describe a different but equivalent formulation of the kink transform and show that the kink transform results in the same new spacetime, regardless of which Cauchy surface containing the extremal surface is used for the construction. In Sec. 9.4, we will describe the duality between the bulk kink transform and the boundary CC flow in AdS/CFT and provide evidence for it.

Formulation

Consider a $d+1$ dimensional spacetime $\mathcal{M}$ with metric $g_{\mu\nu}$ satisfying the Einstein field equations. (We will discuss higher curvature gravity in Sec. 9.6.) Let Σ be a Cauchy surface of $\mathcal{M}$ that contains an extremal surface $\mathcal{R}$ of codimension 1 in Σ. (That is, the expansion of both sets of null geodesics orthogonal to $\mathcal{R}$ vanishes.)

Initial data on Σ consist of [368] the intrinsic metric $(h_\Sigma)_{ab}$ and the extrinsic curvature,

$$(K_\Sigma)_{ab} = P^\mu_a P^\nu_b \nabla_{(\mu} t_{\nu)} \ . \tag{9.3.1}$$

Here P^μ_a is the projector from $\mathcal{M}$ onto Σ, and t^μ is the unit norm timelike vector field orthogonal to Σ. Indices $a, b, \ldots$ are reserved for directions tangent to Σ. For matter fields, initial data consist of the fields and normal derivatives, for example $\phi(w^a)$ and $[t^\mu \nabla_\mu \phi](w^a)$, where ϕ is a scalar field and w^a are coordinates on Σ.

By the Einstein equations, the initial data on Σ must satisfy the following constraints:

$$r_\Sigma + K_\Sigma^2 - (K_\Sigma)_{ab}(K_\Sigma)^{ab} = 16\pi G\, T_{\mu\nu} t^\mu t^\nu \ , \tag{9.3.2}$$

$$D^a (K_\Sigma)_{ab} - D_b K_\Sigma = 8\pi G\, T_{b\nu} t^\nu \ , \tag{9.3.3}$$

where $D_a = P^\mu_a \nabla_\mu$ is the covariant derivative that Σ inherits from $(\mathcal{M}, g_{\mu\nu})$; r_Σ is the Ricci scalar intrinsic to Σ; and K_Σ is the trace of the extrinsic curvature: $K_\Sigma = (h_\Sigma)^{ab}(K_\Sigma)_{ab}$.

Let Σ be a Cauchy slice of $\mathcal{M}$ containing $\mathcal{R}$ and smooth in a neighborhood of $\mathcal{R}$. The kink transform is then a map of the initial data on Σ to a new initial data set, parametrized by a real number s analogous to boost rapidity. The transform acts as the identity on all data except for the extrinsic curvature, which is modified only at the location of the extremal

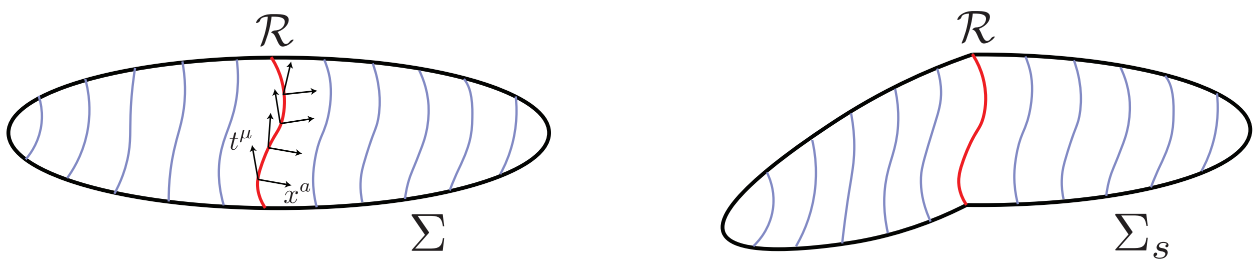


Figure 9.1: Kink transform. Left: a Cauchy surface Σ of the original bulk $\mathcal{M}$. An extremal surface $\mathcal{R}$ is shown in red. The orthonormal vector fields t^a and x^a span the normal bundle to $\mathcal{R}$; x^a is tangent to Σ. Right: The kink transformed Cauchy surface Σ_s. As an initial data set, Σ_s differs from Σ only in the extrinsic curvature at $\mathcal{R}$ through Eq. (9.3.4). Equivalently, the kink transform is a relative boost in the normal bundle to $\mathcal{R}$, Eq. (9.3.21).

surface $\mathcal{R}$, as follows:

$$(K_\Sigma)_{ab} \to (K_{\Sigma_s})_{ab} = (K_\Sigma)_{ab} - \sinh(2\pi s)\; x_a x_b\; \delta(\mathcal{R}) \; . \tag{9.3.4}$$

Here x^a is a unit norm vector field orthogonal to $\mathcal{R}$ and tangent to Σ, and we define

$$\delta(\mathcal{R}) \equiv \delta(x) \; , \tag{9.3.5}$$

where x is the Gaussian normal coordinate to $\mathcal{R}$ in Σ ($\partial_x = x^a$). Thus, the only change in the initial data is in the component of the extrinsic curvature normal to $\mathcal{R}$. An equivalent transformation exists for initial choices of Σ that are not smooth around $\mathcal{R}$ though the transformation rule will be more complicated than Eq. (9.3.4). We will discuss this later in the section.

Let Σ_s be a time slice with this new initial data, as in Fig. 9.1, and let $\mathcal{M}_s$ be the Cauchy development of Σ_s. That is, $\mathcal{M}_s$ is the new spacetime resulting from the evolution of the kink-transformed initial data. Since the intrinsic metric of Σ_s and Σ are the same, they can be identified as d-manifolds with metric; the subscript s merely reminds us of the different extrinsic data they carry. In particular the surface $\mathcal{R}$ can be so identified; thus $\mathcal{R}_s$ has the same intrinsic metric as $\mathcal{R}$. It also trivially has identical extrinsic data with respect to Σ_s. In fact, we will find below that like $\mathcal{R}$ in $\mathcal{M}$, $\mathcal{R}_s$ is an extremal surface in $\mathcal{M}_s$. However, the trace-free part of the extrinsic curvature of $\mathcal{R}_s$ in $\mathcal{M}_s$ may have discontinuities.

We will now show that the constraint equations hold on Σ_s; that is, the kink transform generates valid initial data. This need only be verified at $\mathcal{R}$ since the transform acts as the identity elsewhere. Here we will make essential use of the extremality of $\mathcal{R}$ in $\mathcal{M}$, which we express as follows.

The extrinsic curvature of $\mathcal{R}$ with respect to $\mathcal{M}$ has two independent components. Often these are chosen to be the two orthogonal null directions, but we find it useful to consider

$$(B^{(t)}_{\mathcal{R}})_{ij} = P^\mu_i P^\nu_j \nabla_{(\mu} t_{\nu)} \; , \tag{9.3.6}$$

$$(B^{(x)}_{\mathcal{R}})_{ij} = P^\mu_i P^\nu_j \nabla_{(\mu} x_{\nu)} \; . \tag{9.3.7}$$

Here i, j represent directions tangent to $\mathcal{R}$, and P_i^μ is the projector from $\mathcal{M}$ to $\mathcal{R}$. Extremality of $\mathcal{R}$ in $\mathcal{M}$ is the statement that the trace of each extrinsic curvature component vanishes:

$$B_{\mathcal{R}}^{(t)} = (\gamma_{\mathcal{R}})^{ij}(B_{\mathcal{R}}^{(t)})_{ij} = 0 \ , \tag{9.3.8}$$

$$B_{\mathcal{R}}^{(x)} = (\gamma_{\mathcal{R}})^{ij}(B_{\mathcal{R}}^{(x)})_{ij} = 0 \ , \tag{9.3.9}$$

where $(\gamma_{\mathcal{R}})_{ij} = P_i^a P_j^b (h_\Sigma)_{ab}$ is the intrinsic metric on $\mathcal{R}$.

Orthogonality of t^μ and x^μ implies that $P_i^\mu = P_i^a P_a^\mu$, and hence

$$(B_{\mathcal{R}}^{(t)})_{ij} = P_i^a P_j^b (K_\Sigma)_{ab} \ . \tag{9.3.10}$$

Since x^a is the unit norm orthogonal vector field at $\mathcal{R}$, the trace of $(K_\Sigma)_{ab}$ at $\mathcal{R}$ can be written as:

$$K_\Sigma|_{\mathcal{R}} = x^a x^b (K_\Sigma)_{ab} + (\gamma_{\mathcal{R}})^{ij}(B_{\mathcal{R}}^{(t)})_{ij} = x^a x^b (K_\Sigma)_{ab} \ . \tag{9.3.11}$$

A little algebra then implies

$$(K_{\Sigma_s})^2 - (K_{\Sigma_s})_{ab}(K_{\Sigma_s})^{ab} = (K_\Sigma)^2 - (K_\Sigma)_{ab}(K_\Sigma)^{ab} \ . \tag{9.3.12}$$

Moreover, we have $r_\Sigma = r_{\Sigma_s}$ since the two initial data slices have the same intrinsic metric. Thus Eq. (9.3.2) implies that the kink-transformed slice satisfies the scalar constraint equation:

$$r_{\Sigma_s} + (K_{\Sigma_s})^2 - (K_{\Sigma_s})_{ab}(K_{\Sigma_s})^{ab} = 16\pi G T_{\mu\nu} t^\mu t^\nu \ . \tag{9.3.13}$$

To check the vector constraint Eq. (9.3.3), we separately consider the two cases of $b = x$ and $b = i$ where i, j represent directions tangent to $\mathcal{R}$:

$$\begin{aligned} D_a(K_{\Sigma_s})_x^a - D_x K_{\Sigma_s} &= D_a(K_\Sigma)_x^a - D_x K_\Sigma + B_{\mathcal{R}}^{(x)} \sinh(2\pi s)\delta(x) \\ &= D_a(K_\Sigma)_x^a - D_x K_\Sigma = 8\pi G\, T_{x\nu} t^\nu \ , \end{aligned} \tag{9.3.14}$$

$$D_a(K_{\Sigma_s})_i^a - D_i K_{\Sigma_s} = D_a(K_\Sigma)_i^a - D_i K_\Sigma = 8\pi G\, T_{i\nu} t^\nu \ , \tag{9.3.15}$$

where the second line of the first equation follows from the extremality of $\mathcal{R}$.

We conclude that the kink transform is a valid modification to the initial data. For both constraints to be satisfied after the kink, it was essential that $\mathcal{R}$ is an extremal surface. Thus the kink transform is only well-defined across an extremal surface. Note also that $\mathcal{R}_s \subset \Sigma_s$ is an extremal surface in $\mathcal{M}_s$. By Eq. (9.3.10),

$$(B_{\mathcal{R}_s}^{(t)})_{ij} = P_i^a P_j^b \ (K_{\Sigma_s})_{ab}|_{\mathcal{R}_s} = (B_{\mathcal{R}}^{(t)})_{ij} \implies B_{\mathcal{R}_s}^{(t)} = 0 \ . \tag{9.3.16}$$

In the second equality we used Eq. (9.3.4) as well as the fact that all relevant quantities are intrinsic to Σ_s, so $\mathcal{R}_s$ can be identified with $\mathcal{R}$. Moreover,

$$(B_{\mathcal{R}_s}^{(x)})_{ij} = (B_{\mathcal{R}}^{(x)})_{ij} \implies B_{\mathcal{R}_s}^{(x)} = 0 \ , \tag{9.3.17}$$

since this quantity depends only on the intrinsic metrics of Σ and Σ_s, which are identical.

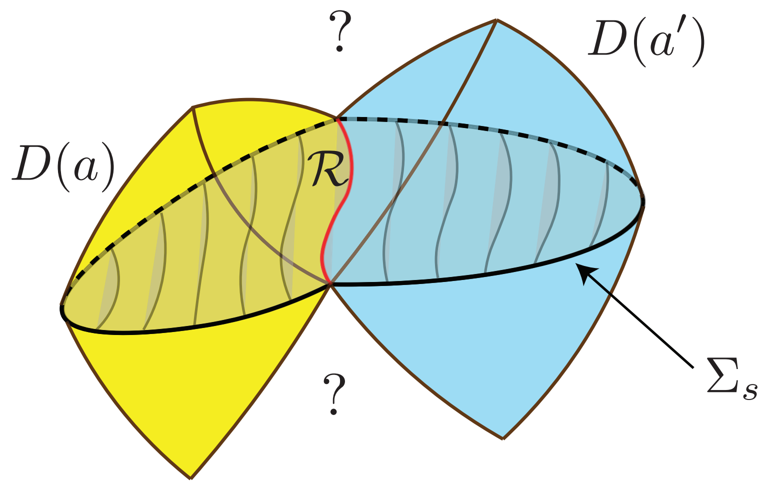


Figure 9.2: The kink-transformed spacetime $\mathcal{M}_s$ is generated by the Cauchy evolution of the kinked slice Σ_s. This reproduces the left and right entanglement wedges $D(a)$ and $D(a')$ of the original spacetime $\mathcal{M}$. The future and past of the extremal surface $\mathcal{R}$ are in general not related to the original spacetime.

Properties

We will now establish important properties and an equivalent formulation of the kink transform.

Let us write Σ as the disjoint union

$$\Sigma = a' \cup \mathcal{R} \cup a \ . \tag{9.3.18}$$

The spacetime $\mathcal{M}$ contains $D(a)$ and $D(a')$ where $D(.)$ denotes the domain of dependence. The kink transformed slice Σ_s contains regions a and a' with identical initial data, so $\mathcal{M}_s$ also contains $D(a)$ and $D(a')$. Because Σ_s has different extrinsic curvature at $\mathcal{R}$, the two domains of dependence will be glued to each other differently in $\mathcal{M}_s$, so the full spacetime will differ from $\mathcal{M}$ in the future and past of $\mathcal{R}$. This is depicted in Fig. 9.2.

We will now derive an alternative formulation of the kink transform as a one-sided local Lorentz boost at $\mathcal{R}$. The unit vector field $t^\mu_{\Sigma_s}$ normal to Σ_s is discontinuous at $\mathcal{R}$ due to the kink. Let

$$(t^\mu_{\Sigma_s})_R = \lim_{x\to 0^+} t^\mu_{\Sigma_s} \ , \tag{9.3.19}$$

$$(t^\mu_{\Sigma_s})_L = \lim_{x\to 0^-} t^\mu_{\Sigma_s} \tag{9.3.20}$$

be the left and right limits to $\mathcal{R}$. The metric of $\mathcal{M}_s$ is continuous since it arises from valid initial data on Σ_s. Therefore, the normal bundle of 1+1 dimensional normal spacetimes to points in $\mathcal{R}$ is well-defined. The above vector fields $(t^\mu_{\Sigma_s})_R$ and $(t^\mu_{\Sigma_s})_L$ belong to this normal

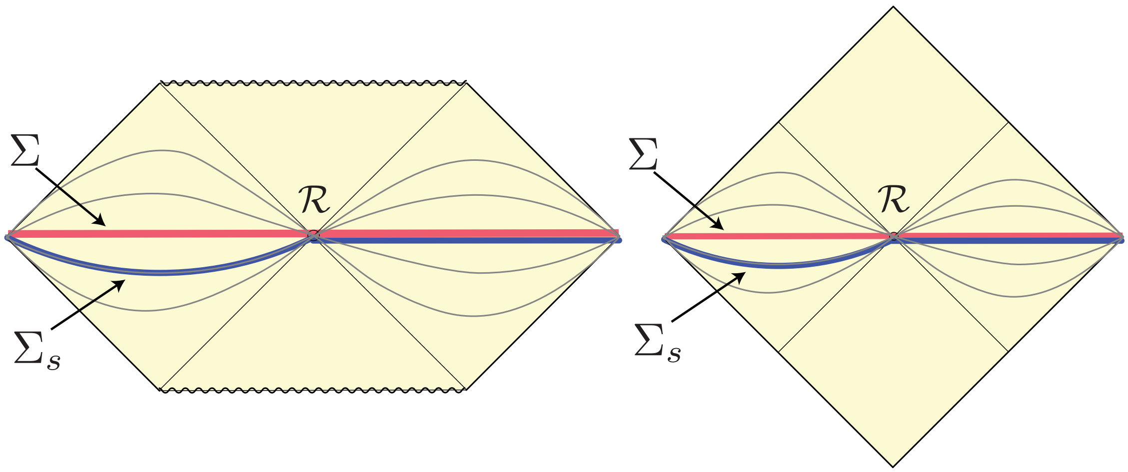


Figure 9.3: Straight slices Σ (red) in a maximally extended Schwarzschild (left) and Rindler (right) spacetime get mapped to kinked slices Σ_s (blue) under the kink transform about $\mathcal{R}$.

bundle. Therefore at each point on $\mathcal{R}$, the two vectors can only differ by a Lorentz boost acting in 1+1 dimensional Minkowski space. The kink transform, Eq. (9.3.4), implies:

$$(t^{\mu}_{\Sigma_s})_R = (\Lambda_{2\pi s})^{\mu}_{\nu}(t^{\nu}_{\Sigma_s})_L \ , \tag{9.3.21}$$

where $(\Lambda_{2\pi s})^{\mu}_{\nu}$ is a Lorentz boost of rapidity $2\pi s$. In this sense, the kink transform resembles a local boost around $\mathcal{R}$. Alternatively, we can view Eq. (9.3.21) as the definition of the kink transform. This definition can be applied to Cauchy slices that are not smooth around $\mathcal{R}$, but it reduces to Eq. (9.3.4) in the smooth case.

This observation applies equally to any other vector field ξ^{μ} in the normal bundle to $\mathcal{R}$, if ξ^{μ} has a smooth extension into $D(a')$ and $D(a)$ in $\mathcal{M}$. The norm of ξ^{μ} and its inner products with $(t^{\mu}_{\Sigma_s})_L$ and $(t^{\mu}_{\Sigma_s})_R$ are unchanged by the kink transform. Hence, in $\mathcal{M}_s$, the left and right limits of ξ^{μ} to $\mathcal{R}$ will satisfy

$$\xi^{\mu}_R = (\Lambda_{2\pi s})^{\mu}_{\nu}\xi^{\nu}_L \ . \tag{9.3.22}$$

Now let $\Xi \supset \mathcal{R}$ be another Cauchy slice of $D(\Sigma)$. Since Ξ contains $\mathcal{R}$, its timelike normal vector field ξ^{μ} (at $\mathcal{R}$) lies in the normal bundle to $\mathcal{R}$. We have shown that Eq. (9.3.21) is equivalent to the kink transform of Σ; that Eq. (9.3.22) is equivalent to the kink transform of Ξ; and that Eq. (9.3.21) is equivalent to Eq. (9.3.22). Hence the kink transform of Σ is equivalent to the kink transform of Ξ. In other words, the spacetime resulting from a kink transform about $\mathcal{R}$ does not depend on which Cauchy surface containing $\mathcal{R}$ we apply the kink transform to.

The kink transform (with $s \neq 0$) always generates physically inequivalent initial data. However $\mathcal{M}_s$ need not differ from $\mathcal{M}$. They will be the same if and only if Σ_s is an initial data set in $\mathcal{M}$. There is an interesting special case where this holds for all values of s. Namely,

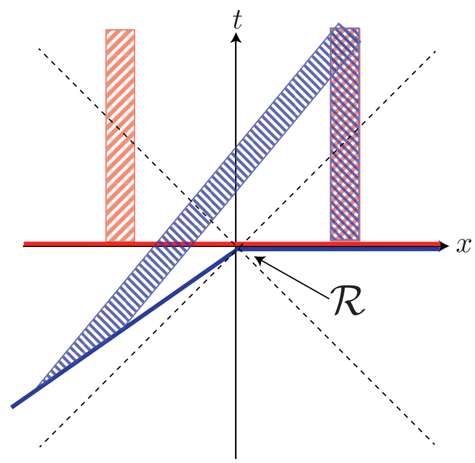


Figure 9.4: On a fixed background with boost symmetry, the kink transform changes the initial data of the matter fields. In this example, $\mathcal{M}$ is Minkowski space with two balls relatively at rest (red).The kink transform is still Minkowski space, but the balls collide in the future of $\mathcal{R}$ (blue).

suppose $\mathcal{M}$ has a Killing vector field that reduces to a boost in the normal bundle to $\mathcal{R}$. Then $\Sigma_s \subset \mathcal{M}$ (as a full initial data set), for all s. For example, the kink transform maps straight to kinked slices in the Rindler or maximally extended Schwarzschild spacetimes (see Fig. 9.3).

We can also consider the kink transform of matter fields on a fixed background spacetime with the above symmetry. Geometrically, $\mathcal{M} = \mathcal{M}_s$ for all s, but the matter fields will differ in $\mathcal{M}_s$ by a one-sided action of the Killing vector field. For example, let $\mathcal{M}$ be Minkowski space, with two balls at rest at $x = \pm 1$, $y = z = 0$ (see Fig. 9.4); and let $\mathcal{R}$ given by $x = t = 0$. In the spacetime $\mathcal{M}_s$ obtained by a kink transform, the two balls will approach with velocity $\tanh 2\pi s$ and so will collide. The right and left Rindler wedge, $D(a)$ and $D(a')$, are separately preserved; the collision happens in the past or future of $\mathcal{R}$.

9.4 Bulk Kink Transform = Boundary CC Flow

In this section, we will argue that the kink transform is the bulk dual of boundary CC flow. We will show that the kink transform satisfies two nontrivial necessary conditions. First, in Sec. 9.4, we show that the left and right bulk region are the subregion duals to the left and right boundary region, respectively. In Sec. 9.4 we show that the bulk kink transform leads to precisely the stress tensor shock at the boundary generated by boundary CC flow, Eq. (9.2.47). (In Sec. 9.5 we will show that the kink transform predicts additional shocks in the CC flowed state, which have not been derived previously purely from QFT methods.)

Matching Left and Right Reduced States

The entanglement wedge of a boundary region A in a (pure or mixed) state ρ_A,

$$\mathrm{EW}(\rho_A) = D[a(\rho_A)] \tag{9.4.1}$$

is the domain of dependence of a bulk achronal region a satisfying the following properties [4, 282, 296, 283]:

(1) The topological boundary of a (in the unphysical spacetime that includes the conformal boundary of AdS) is given by $\partial a = A \cup \mathcal{R}$.
(2) $S_{\mathrm{gen}}(a)$ is stationary under small deformations of $\mathcal{R}$.
(3) Among all regions that satisfy the previous criteria, $\mathrm{EW}(\rho_A)$ is the one with the smallest $S_{\mathrm{gen}}(a)$.

We neglect end-of-the-world branes in this discussion [369, 370]. The generalized entropy is given by

$$S_{\mathrm{gen}} = \frac{\mathrm{Area}(\mathcal{R})}{4G\hbar} + S(a) + \dots \ , \tag{9.4.2}$$

where $S(a)$ is the von Neumann entropy of the region a and the dots indicate subleading geometric terms. The entanglement wedge is also referred to as the Wheeler-DeWitt patch of A.

There is significant evidence [136, 351] that $\mathrm{EW}(\rho_A)$ represents the entire bulk dual to the boundary region A. That is, all bulk operators in $\mathrm{EW}(\rho_A)$ have a representation in the algebra of operators $\mathcal{A}$ associated with A; and all simple correlation functions in A can be computed from the bulk. In other words, the entanglement wedge appears to be the answer [283] to the question [371, 372, 285, 373] of "subregion duality." A bulk surface $\mathcal{R}$ is called quantum extremal (with respect to A in the state ρ) if it satisfies the first two criteria, and quantum RT if it satisfies all three. When the von Neumann entropy term in Eq. (9.4.2) is neglected, $\mathcal{R}$ is called an extremal or RT surface, respectively. This will be the case everywhere in this paper except in Sec. 9.6.

We now specialize to the setting in which CC flow was considered in Sec. 9.2. Recall that the pure boundary state $|\psi(\mathcal{C})\rangle$ is given on a boundary slice $\mathcal{C}$ corresponding to $t = 0$ in standard Minkowski coordinates; and that we regard $\mathcal{C}$ as the disjoint union of the left region A'_0 ($x < 0$), with reduced state $\rho^{\psi}_{A_0}$; the cut ∂A_0 ($x = 0$); and the right region A_0 ($x > 0$), with reduced state $\rho^{\psi}_{A'_0}$. Let a'_0 and a_0 be arbitrary Cauchy surfaces of the associated entanglement wedges $\mathrm{EW}(\rho^{\psi}_{A'_0})$ and $\mathrm{EW}(\rho^{\psi}_{A_0})$.

The entanglement wedges of non-overlapping regions are always disjoint, so

$$\mathrm{EW}(\rho^{\psi}_{A'_0}) \cap \mathrm{EW}(\rho^{\psi}_{A_0}) = \varnothing \ . \tag{9.4.3}$$

For the bipartition of a pure boundary state ψ, entanglement wedge complementarity holds:

$$a[|\psi(\mathcal{C})\rangle] = a'_0 \cup \mathcal{R} \cup a_0 \ , \tag{9.4.4}$$

where $a[|\psi(\mathcal{C})\rangle]$ is a Cauchy surface of $\mathrm{EW}(|\psi(\mathcal{C})\rangle)$. In particular, the left and right entanglement wedge share the same HRT surface $\mathcal{R}$.

Crucially, the classical initial data on $a[|\psi(\mathcal{C})\rangle]$ is almost completely determined by the data on a_0' and a_0; however the data on $\mathcal{R}$ are not contained in a_0' nor in a_0. In the semiclassical regime, the quantum state on $a[|\psi(\mathcal{C})\rangle]$ also includes global information (through its entanglement structure) that neither subregion contains on its own. Hence in general

$$\mathrm{EW}(|\psi(\mathcal{C})\rangle) = D\left[\mathrm{EW}(\rho^{\psi}_{A_0'}) \cup \mathcal{R} \cup \mathrm{EW}(\rho^{\psi}_{A_0})\right] \tag{9.4.5}$$

is a proper superset of $\mathrm{EW}(\rho^{\psi}_{A_0'}) \cup \mathrm{EW}(\rho^{\psi}_{A_0})$ that also includes some of the past and future of $\mathcal{R}$.

Now consider the CC-flowed state on the precursor slice $|\psi_s(\mathcal{C}_s)\rangle$. By Eqs. (9.2.37) and (9.2.38), we have

$$\mathrm{EW}(\rho^{\psi_s}_{A_s'}) = \mathrm{EW}(\rho^{\psi}_{A_0'}) = D(a_0') \ , \tag{9.4.6}$$

$$\mathrm{EW}(\rho^{\psi_s}_{A_0}) = \mathrm{EW}(\rho^{\psi}_{A_0}) = D(a_0) \ , \tag{9.4.7}$$

Since $|\psi_s\rangle$ is again a pure state, $\mathrm{EW}[|\psi_s(\mathcal{C}_s)\rangle] = D\left(a[|\psi_s(\mathcal{C}_s)\rangle]\right)$ where

$$a[|\psi_s(\mathcal{C}_s)\rangle] = a_0' \cup \mathcal{R} \cup a_0 \ . \tag{9.4.8}$$

We see that this initial data slice has the same intrinsic geometry as that of the original bulk dual. Indeed, by the remarks following Eq. (9.4.4), the full classical initial data for the bulk dual to $|\psi_s\rangle$ will be identical on $a_0' \cup a_0$ and can only differ from the initial data for the original bulk at $\mathcal{R}$.

We pause here to note that a kink transform of $a[|\psi(\mathcal{C})\rangle]$ centered on $\mathcal{R}$ satisfies this necessary condition and hence becomes a candidate for $a[|\psi_s(\mathcal{C}_s)\rangle]$. However, this does not yet constrain the value of s. In order to go further, we would now like to show that a kink transform of $a[|\psi(\mathcal{C})\rangle]$ with parameter s yields a bulk slice whose boundary is geometrically the precursor slice $\mathcal{C}_s$.

The bulk metric takes the asymptotic form [374]:[5]

$$ds^2 = \frac{1}{z^2}\left[dz^2 + \eta_{AB}dx^A dx^B + O(z^d)\right] \ , \tag{9.4.9}$$

where η_{AB} is the metric of Minkowski space. Consider a stationary bulk surface $\mathcal{R}$ anchored on the boundary cut $u = v = 0$. At leading order, $\mathcal{R}$ will reside at $u = v = 0$ in the asymptotic bulk, in the above metric [17]. (The first subleading term, which appears at order z^d, will be crucial in our derivation of the boundary stress tensor shock in Sec. 9.4.)

Let Σ be a bulk surface that contains $\mathcal{R}$ and satisfies $t = 0 + O(z^d)$ in the metric of Eq. (9.4.9). Since the initial data on each side of $\mathcal{R}$ are separately preserved (see Sec. 9.3),

[5] We set $\ell_{\mathrm{AdS}} = 1$.

Eq. (9.3.21) dictates that the kink transform Σ_s of Σ satisfies $t = 0$ ($x > 0$) and $t = x \tanh 2\pi s$ ($x < 0$), again up to corrections of order z^d. The corrections all vanish at $z = 0$, where Σ is bounded by $\mathcal{C}$ and Σ_s is bounded by $\mathcal{C}_s$ (see Eq. (9.2.32)). Recall also that the kink transform is slice-independent. Thus we have established that the kink transform of any Cauchy surface $a[|\psi(\mathcal{C})\rangle]$, by s along $\mathcal{R}$, yields a Cauchy surface bounded by the precursor slice $\mathcal{C}_s$.

The above arguments establish that

$$\mathrm{EW}[|\psi_s(\mathcal{C}_s)\rangle] = D\left(a[|\psi_s(\mathcal{C}_s)\rangle]\right) \ , \tag{9.4.10}$$

where $a[|\psi_s(\mathcal{C}_s)\rangle]$ is given by Eq. (9.4.8). In words, the bulk dual of the CC-flowed boundary state is the Cauchy development of the kink-transform of a Cauchy slice containing the HRT surface $\mathcal{R}$. Note that the classical initial data on this Cauchy surface is fully determined by the initial data on a_0' and a_0 inherited from the bulk dual of $|\psi(\mathcal{C})\rangle$, combined with the distributional geometric initial data consisting of the extrinsic curvature shock at $\mathcal{R}$. The full spacetime geometry will differ from $\mathrm{EW}[|\psi(\mathcal{C})\rangle]$ because of the different gluing at $\mathcal{R}$.

Matching Bulk and Boundary Shocks

In Sec. 9.3, we gave a prescription for generating bulk geometries in AdS by inserting a kink on the Cauchy surface, at the HRT surface. With the standard holographic dictionary, the resulting geometry manifestly yields the correct behavior of one-sided boundary observables under CC flow. This was shown in the previous subsection.

Another characteristic aspect of the CC flowed state $|\psi_s\rangle$ is the presence of a stress tensor shock at the cut (Sec. 9.2), proportional to shape derivatives of the von Neumann entropy; see Eq. (9.2.47). We will now verify that this shock is reproduced by the kink transform in the bulk, upon applying the AdS/CFT dictionary. Notably, the shock is not localized to either wedge. Verifying kink/CC duality for this observable furnishes an independent, nontrivial check of our proposal.

We will now keep the first subleading term in the Fefferman-Graham expansion of the asymptotic bulk metric [374, 17]:

$$ds^2 = \frac{1}{z^2}\left(dz^2 + g_{AB}(x,z)dx^A dx^B\right) \ , \tag{9.4.11}$$

$$g_{AB}(x,z) = \eta_{AB} + z^d \frac{16\pi G}{d}\langle T_{AB}\rangle + o(z^d) \ , \tag{9.4.12}$$

where indices $A, B, \ldots$ correspond to directions along $z = $ const. surfaces.

The location of the RT surface $\mathcal{R}$ in the bulk can be described by a collection of $(d-1)$ embedding functions

$$X^\mu(y,z) = (z, X^A(y,z)) \ , \tag{9.4.13}$$

where (y, z) are intrinsic coordinates on $\mathcal{R}$. The expansion in z takes the simple form

$$X^A(y,z)) = z^d X^A_{(d)} + o(z^d) \ , \tag{9.4.14}$$

because the boundary anchor is the flat cut $u = v = 0$ of the Rindler horizon [17]. Stationarity of $\mathcal{R}$ can be shown to imply [17]

$$X^A_{(d)} = -\frac{4G}{d}\,\frac{\delta S}{\delta X^A}\bigg|_{\mathcal{R}} \ . \tag{9.4.15}$$

We consider a bulk Cauchy slice $\Sigma \supset \mathcal{R}$ for which $\partial\Sigma$ corresponds to the $t = 0$ slice on the boundary. Since the subleading terms in Eqs. (9.4.12) and (9.4.14) start at z^d, we are free to choose Σ so that it is given by

$$t = z^d \varsigma(x) + o(z^d) \ , \tag{9.4.16}$$

Recall that the vector fields t^μ and x^μ are defined to be orthogonal to $\mathcal{R}$, and respectively orthogonal and tangent to Σ_s. In FG coordinates one finds:

$$t^A = z\left(t^A_{(0)} + z^d t^A_{(d)} + o(z^d)\right) \ , \tag{9.4.17}$$
$$x^A = z\left(x^A_{(0)} + z^d x^A_{(d)} + o(z^d)\right) \ , \tag{9.4.18}$$
$$t^z = z\left(z^{d-1} t^z_{(d-1)} + o(z^{d-1})\right) \ , \tag{9.4.19}$$
$$x^z = z\left(z^{d-1} x^z_{(d-1)} + o(z^{d-1})\right) \ . . \tag{9.4.20}$$

The overall factor of z is due to normalization. Note that $t^\mu_{(0)}$ is a coordinate vector field but in general, t^μ is not. Individual coordinate components of vectors and tensors are defined by contractions with $t^\mu_{(0)}$ and $x^\mu_{(0)}$ respectively, for example $t^t \equiv t_\mu t^\mu_{(0)}$.

We now consider a contraction of the extrinsic curvature tensor on Σ,

$$(K_\Sigma)_{ab} x^b = P^\mu_a x^\nu \nabla_{(\mu} t_{\nu)} \ . \tag{9.4.21}$$

We would like to further project the a index onto the z direction. Deep in the bulk the z direction does not lie entirely in Σ. However, note that $g_{\mu z} t^\mu \to 0$ in the limit $z \to 0$ due to Eq. (9.4.19). Therefore, at leading order in z, the z direction does lie entirely in Σ; moreover, $P^\mu_z \to \delta^\mu_z$ as $z \to 0$. We will only be interested in evaluating Eq. (9.4.21) at leading order in z so we may freely set $a = z$, which yields:[6]

$$(K_\Sigma)_{z\nu} x^\nu - x^\nu \partial_{(z} t_{\nu)} = x^\nu t_\gamma \, \Gamma^\gamma_{\nu z} \tag{9.4.22}$$
$$= z^2 \Gamma_{txz} + z x^t \Gamma_{ttz} + z t^x \Gamma_{xxz} + x^z t^z \Gamma_{zzz} + o(z^{d-1}) \tag{9.4.23}$$
$$= \frac{(d-2)}{2} z^{d-1} \frac{16\pi G}{d} \langle T_{tx} \rangle - z^{-3} t^z x^z - z^{d-1}(t^x_{(d)} - x^t_{(d)}) + o(z^{d-1}) \ . \tag{9.4.24}$$

The condition $x_\mu t^\mu = 0$ implies that

$$z^{d-1} \frac{16\pi G}{d} \langle T_{tx} \rangle + z^{-3} x^z t^z + z^{d-1}(t^x_{(d)} - x^t_{(d)}) + o(z^{d-1}) = 0 \ . \tag{9.4.25}$$

[6] In $d > 2$ the terms involving $x^z t^z$ will be higher order, by Eqs. (9.4.19) and (9.4.20), and need not be included. Since they cancel out either way, we include them here to avoid an explicit case distinction.

Hence we find

$$(K_\Sigma)_{z\nu} x^\nu - x^\nu \partial_{(z} t_{\nu)} = z^{d-1}\, 8\pi G \, \langle T_{tx} \rangle + o(z^{d-1}) \ . \tag{9.4.26}$$

We now apply the kink transform to Σ (viewed as an initial data set). This yields a new initial data set on a slice Σ_s in a new spacetime $\mathcal{M}_s$. We again expand in Fefferman-Graham coordinates:

$$ds^2 = \frac{1}{z^2}\left(dz^2 + \tilde{g}_{AB}(\tilde{x}, z) d\tilde{x}^A d\tilde{x}^B\right) \ , \tag{9.4.27}$$

$$\tilde{g}_{AB}(x, z) = \tilde{\eta}_{AB} + z^d \frac{16\pi G}{d} \langle \tilde{T}_{AB} \rangle + o(z^d) \ . \tag{9.4.28}$$

Here $\tilde{\eta}_{AB}$ is still Minkowski space; any change in the bulk geometry will be encoded in the subleading term.

The notation $\tilde{\eta}_{AB}$ indicates that we will be using the specific coordinates in which the metric of d-dimensional Minkowski space takes the nonstandard form given by Eq. (9.2.41). This has the advantage that the *coordinate* form of all vectors, tensors, and embedding equations in $D(a') \cup \mathcal{R} \cup D(a)$ will be unchanged by the kink transform, if we use standard Cartesian coordinates before the transform and the tilde coordinates afterwards.

For example, the invariance of the left and right bulk domains of dependence under the kink transform implies that Σ_s is given by

$$\tilde{t} = z^d \varsigma(\tilde{x}) + o(z^d) \ , \tag{9.4.29}$$

with the *same* ς that appeared in Eq. (9.4.16). (In fact, this extends to at all orders in z.) As already shown in the previous subsection, $\partial\Sigma_s$ lies at $\tilde{t} = 0$, $z = 0$.

As another example, the coordinate components of the unit normal vector to Σ_s in $\mathcal{M}_s$, $\tilde{t}^\mu$, will be the same as the components of the normal vector to Σ in $\mathcal{M}$, t^μ, and therefore

$$\partial_{(z} t_{\nu)}\big|_{\mathcal{M}} = \partial_{(z} \tilde{t}_{\nu)}\big|_{\mathcal{M}_s} \ . \tag{9.4.30}$$

Below we will use the convention that any quantity with a tilde is evaluated in $\mathcal{M}_s$, in the coordinates of Eq. (9.4.28). Any quantity without a tilde is evaluated in $\mathcal{M}$, in the coordinates of Eq. (9.4.12). The only exception is the extrinsic curvature tensor, where the corresponding distinction is indicated by the subscript Σ_s or Σ, for consistency with Sec. 9.3.

We now consider the extrinsic curvature of Σ_s. A calculation analogous to the derivation of Eq. (9.4.26) implies

$$(K_{\Sigma_s})_{z\nu} \tilde{x}^\nu - \tilde{x}^\nu \partial_{(z} \tilde{t}_{\nu)} = z^{d-1}\, 8\pi G \, \langle \tilde{T}_{\tilde{t}\tilde{x}} \rangle + o(z^{d-1}) \ . \tag{9.4.31}$$

From Eqs. (9.4.26) and (9.4.30) we find

$$z^{d-1} \langle \tilde{T}_{\tilde{t}\tilde{x}} \rangle = z^{d-1} \langle T_{tx} \rangle + \frac{1}{8\pi G} \left[(K_{\Sigma_s})_{z\nu} - (K_\Sigma)_{z\nu} \right] x^\nu + o(z^{d-1}) \ , \tag{9.4.32}$$

$$= z^{d-1} \langle T_{tx} \rangle - \frac{\sinh(2\pi s)}{8\pi G} \delta(\mathcal{R}) x_z + o(z^{d-1}) \ . \tag{9.4.33}$$

In the first equality, we used the fact that x^μ and $\tilde{x}^\mu$ can be identified as vector fields, and the extrinsic curvature tensors can be compared, in the submanifold $\Sigma = \Sigma_s$. The second equality follows from the definition of the kink transform, Eq. (9.3.4).

By Eq. (9.4.18), $\delta(\mathcal{R}) = \delta(z^{-1}\tilde{x}) = z\delta(\tilde{x})$. The condition $x_\mu \partial_z X^\mu = 0$ yields

$$x_z = -d\, z^{d-2} \widetilde{X}_{(d)} + o(z^{d-2}) \ , \tag{9.4.34}$$

where $\widetilde{X}_{(d)}$ is the $A = \tilde{x}$ component of $X^A_{(d)}$. Taking $z \to 0$ and using Eq. (9.4.15), we thus find

$$\langle \tilde{T}_{\tilde{t}\tilde{x}} \rangle = \langle T_{tx} \rangle + \frac{1}{2\pi} \sinh(2\pi s) \frac{\delta S}{\delta \widetilde{X}}\Big|_{\widetilde{X}=0} \delta(\tilde{x}) \ , \tag{9.4.35}$$

which agrees precisely with Eq. (9.2.47).

Note that this derivation applies to any boosted coordinate system $(\check{t}, \check{x})$ as well. Linear combinations of Eq. (9.4.35) with its boosted version reproduces both the $T_{\tilde{u}\tilde{u}}$ shock of Eq. (9.2.46) and the $T_{\tilde{v}\tilde{v}}$ shock of Eq. (9.2.45) holographically.

9.5 Predictions

Having found nontrivial evidence for kink transform/CC flow duality, we now change our viewpoint and assume the duality. In this section, we will derive a *novel* property of CC flow from the kink transform: a shock in the $\langle T_{xx} \rangle$ component of the stress tensor in the CC flowed state. We do not yet know of a way to derive this directly in the quantum field theory, so this result demonstrates the utility of the kink transform in extracting nontrivial properties of CC flow. We further argue that $\langle T_{xx} \rangle$ and $\langle T_{tx} \rangle$ constitute all of the independent, nonzero stress tensor shocks in the CC flowed state.

Our holographic derivation only depends on near boundary behavior, and the value of the shock takes a universal form similar to Eq. (9.4.35). Thus, we expect that the properties we find in holographic CC flow hold in non-holographic QFTs as well.

To derive the $\langle T_{xx} \rangle$ shock, we use the Gauss-Codazzi relation [375]

$$P^\mu_a P^\nu_b P^\alpha_c P^\beta_d R_{\mu\nu\alpha\beta} = K_{ac}K_{bd} - K_{bc}K_{ad} + r_{abcd} \ , \tag{9.5.1}$$

where r_{abcd} is the intrinsic Riemann tensor of Σ. It is important to note that this relation is purely intrinsic to Σ. Since $\Sigma = \Sigma_s$ as submanifolds, we can not only evaluate Eq. (9.5.1) in both $\mathcal{M}$ and $\mathcal{M}_s$ but also meaningfully subtract the two. We emphasize that the following calculation is only nontrivial in $d > 2$ (in $d = 2$, the Gauss-Codazzi relation is trivial). We comment on $d = 2$ at the end.

First we evaluate Eq. (9.5.1) in $\mathcal{M}$. We will only be interested in evaluating it to leading order in z in the Fefferman-Graham expansion. As argued in Sec. 9.4, when working at leading order we can freely set $a = c = z$. We then compute the following at leading order in z:

$$R_{zxzx} = K_{zz}K_{xx} - (K_{xz})^2 + r_{zxzx} \ . \tag{9.5.2}$$

We start by computing K_{zz}. First we note that $\Gamma^{\alpha}_{zz} t_\alpha = 0$ identically. Therefore,

$$K_{zz} = \partial_z t_z = 4G(d-2) z^{d-3} \frac{\delta S}{\delta T}\Big|_{\mathcal{R}} + o(z^{d-3}) \ . \tag{9.5.3}$$

We have made use of

$$t_z = 4G z^{d-2} \frac{\delta S}{\delta T}\Big|_{\mathcal{R}} + o(z^{d-2}) \ , \tag{9.5.4}$$

which follows from $t_\mu \partial_z X^\mu = 0$.

Next we compute

$$R_{zxzx} = \partial_x \Gamma^x_{zz} - \partial_z \Gamma^x_{xz} + \Gamma^x_{x\mu}\Gamma^\mu_{zz} - \Gamma^x_{z\mu}\Gamma^\mu_{xz} \ . \tag{9.5.5}$$

One finds

$$\partial_z \Gamma^x_{xz} = \frac{1}{2}(d-2)(d-1) z^{d-2} \frac{16\pi G}{d} \langle T_{xx} \rangle + o(z^{d-2}) \ , \tag{9.5.6}$$

$$\Gamma^x_{xz}\Gamma^z_{zz} = -\frac{1}{2}(d-2) z^{d-2} \frac{16\pi G}{d} \langle T_{xx} \rangle + o(z^{d-2}) \ , \tag{9.5.7}$$

with all other terms either subleading in z or identically vanishing, and hence

$$R_{zxzx} = -8\pi G(d-2) z^{d-2} \langle T_{xx} \rangle + o(z^{d-2}) \ . \tag{9.5.8}$$

Putting all this together, we have

$$-8\pi G(d-2) z^{d-2} \langle T_{xx} \rangle = 4G z^{d-3} \frac{\delta S}{\delta \widetilde{T}}\Big|_{\mathcal{R}} K_{xx} - (K_{xz})^2 + r_{zxzx} + o(z^{d-2}) \ . \tag{9.5.9}$$

The analogous relation evaluated in $\mathcal{M}_s$ reads

$$-8\pi G(d-2) z^{d-2} \langle \tilde{T}_{\tilde{x}\tilde{x}} \rangle = 4G z^{d-3} \frac{\delta S}{\delta \widetilde{T}}\Big|_{\mathcal{R}} \tilde{K}_{\tilde{x}\tilde{x}} - (\tilde{K}_{\tilde{x}z})^2 + \tilde{r}_{z\tilde{x}z\tilde{x}} + o(z^{d-2}) \ , \tag{9.5.10}$$

where we have made use of Eq. (9.4.30) to set $K_{zz} = \tilde{K}_{zz}$. We can now subtract these two relations. First note that $\tilde{r}_{abcd} = r_{abcd}$ since it is purely intrinsic to Σ. Next, recall from the definition of the kink transform Eq. (9.3.4) that

$$\tilde{K}_{\tilde{x}\tilde{x}} - K_{xx} = -z \sinh(2\pi s) \delta(\tilde{x}) \ . \tag{9.5.11}$$

Lastly, it is easy to check that $K_{xz} \sim o(z^{d-2})$ hence its contribution to Eq. (9.5.9) is subleading, and similarly for Eq. (9.5.10). Thus, subtracting Eq. (9.5.10) from Eq. (9.5.9) yields

$$\langle \tilde{T}_{\tilde{x}\tilde{x}} \rangle - \langle T_{xx} \rangle = \frac{1}{2\pi} \sinh(2\pi s) \frac{\delta S}{\delta \widetilde{T}}\Big|_{\widetilde{X}=0} \delta(\tilde{x}) \ . \tag{9.5.12}$$

The above calculation only works in $d > 2$. In $d = 2$, since the boundary theory is a CFT, tracelessness of the boundary stress tensor further implies that $\langle T_{tx}\rangle$ is the *only* independent component of the stress tensor shock so there is no need for a calculation analogous to the one above. We expect that this argument is robust under relevant deformations of the CFT since the shock is highly localized and should universally depend only on the UV fixed point.

Together with the $\langle T_{tx}\rangle$ shock we reproduced in the previous section, and using Lorentz invariance of the boundary, this result determines the transformation of the the stress tensor contracted with any pair of linear combinations of t and x, such as $\langle T_{tt}\rangle$. This linear space contains all of the independent nonvanishing components of the shock. To see this, note that

$$x^\nu \left(\nabla_\nu \tilde{y}_\mu - \nabla_\nu y_\mu\right) = 0 \ , \tag{9.5.13}$$

$$y^\nu \left(\nabla_\nu \tilde{y}_\mu - \nabla_\nu y_\nu\right) = 0 \ , \tag{9.5.14}$$

$$y^\nu \left(\nabla_\nu \tilde{t}_\mu - \nabla_\nu t_\mu\right) = 0 \ , \tag{9.5.15}$$

where $y^\mu = P^\mu_i y^i$ for any vector field y^i in the tangent bundle of $\mathcal{R}$. Eqs. (9.5.13) and (9.5.14) follow trivially from the fact that the prescription Eq. (9.3.22) only introduces a discontinuity in vector fields in the normal bundle of $\mathcal{R}$, while Eq. (9.5.15) simply follows from Eq. (9.3.4). Evaluating the $\mu = z$ components in the same way as in Sec. 9.4, we find,

$$\langle \tilde{T}_{\tilde{\mu}\tilde{y}}\rangle - \langle T_{\mu y}\rangle = 0 \ . \tag{9.5.16}$$

For $s \to \infty$, the shocks derived in the previous two sections agree with those found to be required for the existence of certain coarse grained bulk states in Ref. [139]. In that work, the cut was allowed to be a wiggly or flat cut of a bifurcate horizon such as a Rindler horizon, and the state could belong to any quantum field theory. Interpolation of these results suggests that the shocks we have derived here generalize to the case of CC flow for a wiggly cut of the Rindler horizon, in general QFTs with a conformal fixed point.

9.6 Discussion

Relation to JLMS and One Sided Modular Flow

The bulk dual of one-sided modular flow [360, 361] resembles the kink transform. CC flow yields a well defined state, however, whereas a one sided modular flowed state is singular in QFT. Correspondingly, the kink transform defined here yields a smooth bulk solution whereas the version implicitly defined in Ref. [361] results in a singular spacetime (see also Ref. [16], footnote 4). We will now explain this in detail.

Consider a boundary region A_0 with reduced state ρ_{A_0}, dual to a semi-classical state ρ_a in the bulk entanglement wedge a associated to A_0 as seen in Fig. 9.5. We denote by $K_A = -\log \rho_A$ and $K_a = -\log \rho_a$ the boundary and bulk modular Hamiltonians, respectively. The JLMS result [137] states that

$$\hat{K}_{A_0} = \frac{\hat{A}[\mathcal{R}]}{4G} + \hat{K}_a \ , \tag{9.6.1}$$

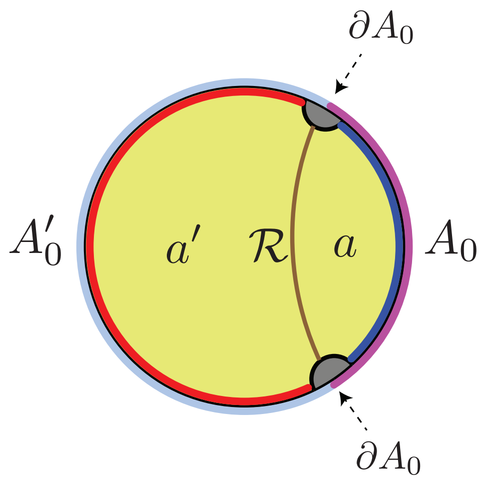


Figure 9.5: A boundary subregion A_0 (pink) has a quantum extremal surface denoted $\mathcal{R}$ (brown) and an entanglement wedge denoted a. The complementary region A_0' (light blue) has the entanglement wedge a'. CC flow generates valid states, but one-sided modular flow is only defined with a UV cutoff. For example, one can consider regulated subregions $A^{(\epsilon)}$ (deep blue) and $A'^{(\epsilon)}$ (red). In the bulk, this amounts to excising an infrared region (gray) from the joint entanglement wedge resulting in a regulated entanglement wedge $D(\Sigma_\epsilon)$ (yellow).

where $\hat{A}[\mathcal{R}]$ is the area operator that formally evaluates the area of the quantum extremal surface $\mathcal{R}$ [283].

Suppose now that A_0 has a nonempty boundary ∂A_0. Then there is an interesting asymmetry in Eq. (9.6.1). The one-sided boundary modular operator appearing on the left hand side is well-defined only with a UV cutoff. On the other hand, at least the leading (area) term in the bulk modular operator on the right hand side has a well-defined action. Let us discuss each side in turn.

In Einstein gravity, the area operator $\hat{A}$ is the generator of one-sided boosts. To see this, let us restrict the gravitational phase space to a truncated bulk region $D(\Sigma_\epsilon)$. Σ_ϵ is a partial bulk Cauchy slice that excludes asymptotic portions of the bulk near the entangling surface ∂A as seen in Fig. 9.5. There exists a (non-unique) vector field ξ^a in $D(a') \cup \mathcal{R}$ such that ξ^a generates an infinitesimal one-sided boost at $\mathcal{R}$ [376, 142]. This boost can be quantified by a parameter s in the normal bundle to $\mathcal{R}$, as described in Sec. 9.3. The area functional $A[\mathcal{R}]/4G$ is the Noether charge at $\mathcal{R}$ associated to ξ^a, given by the expectation value of the area operator in the semi-classical bulk state:

$$A[\mathcal{R}] = \langle \hat{A}[\mathcal{R}] \rangle \ . \tag{9.6.2}$$

Each point in the gravitational phase space can be specified by the metric in $D(a')$, the metric in $D(a)$, and the boost angle s at $\mathcal{R}$ with which the two domains of dependence are glued together [361, 142, 176, 363]. The action of

$$\left\langle e^{2\pi i s \hat{A}[\mathcal{R}]/4G} \right\rangle \tag{9.6.3}$$

on points in the gravitational phase space is to simply shift the conjugate variable, *i.e.*, the relative boost angle between the left and right domains of dependence, by s. Note that the metrics in the left and right domains of dependence are unchanged since the area functional acts purely on the phase space data at $\mathcal{R}$. This is the classical analogue of the statement that the area operator is in the center of the algebras of the domains of dependence [286]. Comparing with Sec. 9.3, we see that this action is equivalent to the kink transform of Σ_ϵ about $\mathcal{R}$ by s. We stress that this action is well-defined even if $\mathcal{R}$ extends all the way out to the conformal boundary, *i.e.*, in the far ultra-violet from the boundary perspective.

We turn to the left hand side of Eq. (9.6.1), still assuming that A_0 has a nonempty boundary ∂A. Since the algebra of a QFT subregion A_0 is a Type-III$_1$ von Neumann algebra, the Hilbert space does not factorize across ∂A_0 [365]. A reduced density matrix ρ_{A_0}, and hence $\hat{K}_{A_0}$, do not exist. Physically, the action of $\hat{K}_{A_0}$ on a fixed boundary time slice would break the vacuum entanglement of arbitrarily short wavelength modes across ∂A_0; this would create infinite energy.

Therefore, any discussion of $\hat{K}_{A_0}$ requires introducing a UV regulator. Consider the regulated subregions $A_0^{(\epsilon)}$ and $A_0'^{(\epsilon)}$ shown in Fig. 9.5. The split property in algebraic QFT [365, 377] guarantees the existence of a (non-unique) Type-I von Neumann algebra $\mathcal{N}$ nested between the algebras of subregion $A_0^{(\epsilon)}$ and the complementary algebra of $A_0'^{(\epsilon)}$, *i.e.*,

$$\mathcal{A}_0^{(\epsilon)} \subset \mathcal{N} \subset \left(\mathcal{A}_0'^{(\epsilon)}\right)' \,. \tag{9.6.4}$$

With this prescription, one can define a regulated version of the reduced density matrix ρ_A by using the Type-I factor $\mathcal{N}$ [**Doplicher:1984zz**]. It has been suggested that there exists an $\mathcal{N}$ consistent with the geometric cutoff shown in Fig. 9.5 [378, 377]: the quantum extremal surface $\mathcal{R}$ in the bulk is regulated by a cutoff brane B demarcating the entanglement wedge of the subregion $A_0^{(\epsilon)} \cup A_0'^{(\epsilon)}$. The regulated area operator $\hat{A}[\mathcal{R}]/4G$ is well defined once boundary conditions on B are specified.

Let us now specialize to the case for which we have conjectured kink transform/CC-flow duality: the boundary slice $\mathcal{C} = A_0 \cup A_0'$ is a Cauchy surface of Minkowski space, and ∂A_0 is the flat cut $u = v = 0$ of the Rindler horizon. We have just argued that the kink transformation is generated by the area operator through Eq. (9.6.3). By Eq. (9.6.1), the boundary dual of this action should be one-sided modular flow, not CC flow. By Eqs. (9.2.4) and (9.2.6), these are manifestly different operations. Indeed, unlike one-sided modular flow, Connes cocycle flow yields a well-defined boundary state for all s, without any UV divergence at the cut ∂A_0: $|\psi(\mathcal{C})\rangle \to |\psi_s(\mathcal{C})\rangle$.

In fact there is no contradiction. For both modular flow and CC flow on the boundary, a bulk-dual Cauchy surface Σ_s is generated by the kink transform. The difference is in how Σ_s is glued back to the boundary.

For modular flow, Σ_s is glued back to the original slice $\mathcal{C}$. Generically, this would violate the asymptotically AdS boundary conditions, necessitating a regulator such as the excision of the grey asymptotic region in Fig. 9.5 and interpolation by a brane B. The boundary dual is an appropriately regulated modular flowed state with energy concentrated near the cut ∂A_0. This construction is possible even if ∂A_0 is not a flat plane, but the regulator is ambiguous and cannot be removed.[7]

For CC flow, Σ_s is glued to the precursor slice $\mathcal{C}_s$ as discussed in Sec. 9.3. This yields $|\psi_s(\mathcal{C}_s)\rangle$. Time evolution on the boundary yields $|\psi_s(\mathcal{C})\rangle$, the CC-flowed state on the original slice $\mathcal{C}$.

On the boundary, we can use the one-sided modular operator in two ways. As a map between states on $\mathcal{C}$ [137, 352] it requires a UV regulator. As a map that takes a state on $\mathcal{C}$ to a state on the precursor slice $\mathcal{C}_s$, $|\psi(\mathcal{C})\rangle \to |\psi_s(\mathcal{C}_s)\rangle$, it is equivalent to CC flow on $\mathcal{C}$ by Eqs. (9.2.35) and (9.2.36). This is a more natural choice due to its UV-finiteness. But it is available only if the vacuum modular operator for cut ∂A is geometric, so that the precursor slice is well-defined.

Quantum Corrections

It is natural to include semiclassical bulk corrections to all orders in G to our proposal. The natural guess would be to perform the kink transform operation about the quantum extremal surface along with a CC flow for the bulk state. In general, it is difficult to describe this procedure within EFT. In states far from the vacuum, the background spacetime changes under the kink transform, and it is unclear how to map states from one spacetime to another. However, we will find some evidence that suggests that the bulk operation relating the two states is a generalized version of CC flow in curved spacetime.

To see this, note that the quantum extremal surface $\mathcal{R}$ satisfies the equations

$$\mathcal{B}^{(t)}_{\mathcal{R}} + 4G\hbar\,\frac{\delta S}{\delta T} = 0\ , \tag{9.6.5}$$

$$\mathcal{B}^{(x)}_{\mathcal{R}} + 4G\hbar\,\frac{\delta S}{\delta X} = 0\ , \tag{9.6.6}$$

where $(\mathcal{B}^{(t)}_{\mathcal{R}})$ and $(\mathcal{B}^{(x)}_{\mathcal{R}})$ denote the trace of the extrinsic curvature (expansion) in the two normal directions to $\mathcal{R}$, *i.e.*, t^μ and x^μ respectively. Similarly $\frac{\delta S}{\delta T}$ and $\frac{\delta S}{\delta X}$ are the entropy variations in the t^μ and x^μ directions respectively.

The classical kink transform involves an extrinsic curvature shock at the classical RT surface. As shown in Sec. 9.3, extremality of the surface ensures that the constraint equations continue to be satisfied after the kink transform in this case. However, the quantum extremal surface has non-vanishing expansion, the constraint equations are not automatically satisfied when an extrinsic curvature shock is added at the quantum RT surface.

More precisely, the left hand side of the constraint equations on a slice Σ are modified

[7] There is evidence that a code subspace can be defined with an appropriate regulator such that one-sided modular flow keeps the state within the code subspace [363, 362, 364].

by the kink transform by

$$\Delta\left(r_\Sigma - (K_\Sigma)_{ab}(K_\Sigma)^{ab} + (K_\Sigma)^2\right) = 8G\hbar\,\sinh(2\pi s)\,\frac{\delta S}{\delta T}\,\delta(X)\ , \tag{9.6.7}$$

$$\Delta\left(D_a(K_\Sigma)^a_x - D_x K_\Sigma\right) = 4G\hbar\,\sinh\left(2\pi s\right)\frac{\delta S}{\delta X}\,\delta(X)\ , \tag{9.6.8}$$

$$\Delta\left(D_a(K_\Sigma)^a_i - D_i K_\Sigma\right) = 0\ , \tag{9.6.9}$$

where Δ represents the difference in the constraint equations between the original spacetime $\mathcal{M}$ and the kink transformed spacetime $\mathcal{M}_s$, and we have used Eqs. (9.6.5) and (9.3.4). These are essentially the analogs of Eqs. (9.3.13) and (9.3.14), and we have simplified the notation slightly.

For the constraint equations to be solved, the kink transform would have to generate the same change on the right hand side of the constraints. It would thus have to induce an additional stress tensor shock of the form

$$\Delta T_{TT} = \frac{1}{2\pi}\sinh(2\pi s)\,\frac{\delta S}{\delta T}\,\delta(X)\ , \tag{9.6.10}$$

$$\Delta T_{TX} = \frac{1}{2\pi}\sinh(2\pi s)\,\frac{\delta S}{\delta X}\,\delta(X)\ . \tag{9.6.11}$$

Formally, these conditions agree precisely with the properties of CC flow discussed in Sec. 9.2. Thus, we might expect a generalized bulk CC flow to result in shocks of precisely this form.

In fact, the existence of semiclassical states satisfying the above equations was conjectured in [139]; the fact that CC flow generates such states in the non-gravitational limit was interpreted as non-trivial evidence in support of the conjecture. Thus, we expect a kink transform at the quantum extremal surface with a suitable modification of the state to provide the bulk dual of CC flow to all orders in G.

At a more speculative level, we can also discuss the bulk dual of CC flow in certain special states called fixed area states, which serve as a natural basis for modular flow [362, 363, 364]. These are approximate eigenstates of the area operator and are therefore unlike smooth semiclassical states which are analogous to coherent states. The Lorentzian bulk dual of such states potentially involves superpositions over geometries [379].

However, by construction, the reduced density matrix is maximally mixed at leading order in G. Thus, the state $|\psi\rangle$ is unaffected by one sided modular flow, and the only effect of CC flow is that we describe the state on a kinematically related slice $\mathcal{C}_s$. Thus, the dual description must be invariant under CC flow up to a diffeomorphism.

In such states, one could apply the semiclassical prescription using Eq. (9.6.1). As discussed above, the action of the area operators results in a diffeomorphism of the geometric description, if it exists. From Eq. 9.6.1, the remaining action of the boundary CC flow is to simply induce a bulk CC flow.

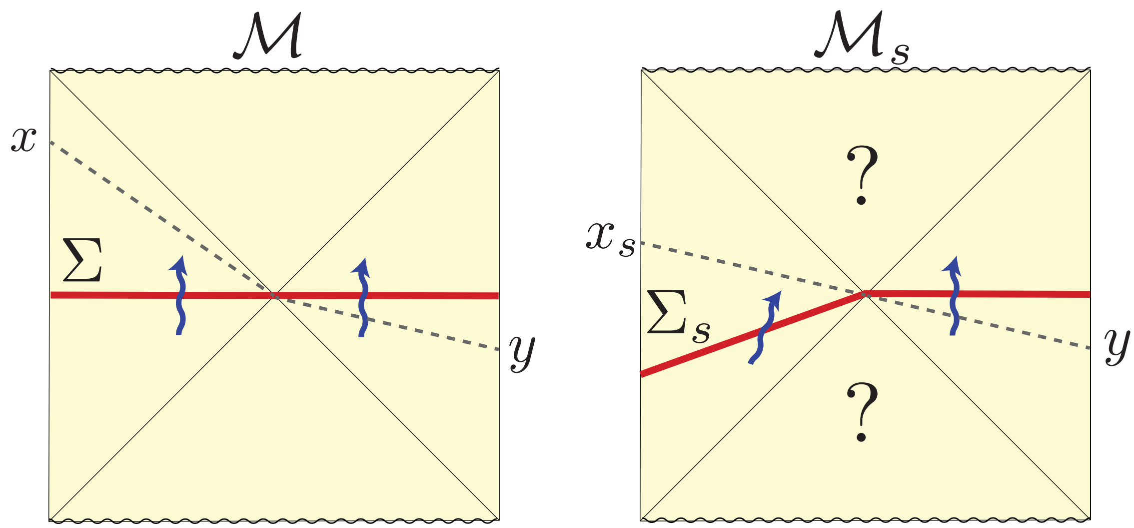


Figure 9.6: An arbitrary spacetime $\mathcal{M}$ with two asymptotic boundaries is transformed to a physically different spacetime $\mathcal{M}_s$ by performing a kink transform on the Cauchy slice Σ. A piecewise geodesic (dashed gray line) in $\mathcal{M}$ connecting x and y with boost angle $2\pi s$ at $\mathcal{R}$ becomes a geodesic between x_s and y in $\mathcal{M}_s$.

Beyond Flat Cuts

Kink transform/CC flow duality can be generalized to other choices of boundary subsystems, so long as a precursor slice can be defined. The precursor slice is generated by acting on the original slice with the vacuum modular Hamiltonian; this is well-defined only if this action is geometric. In Sec. 9.2, we ensured this by taking the boundary to be Minkowski space and choosing a planar cut. Precursor slices also exist in any conformally related choice, such as a spherical cut.

But there are other settings where the vacuum modular Hamiltonian acts geometrically. This includes multiple asymptotically AdS boundaries where the boundary manifold has a time translation symmetry. For example, consider a two-sided black hole geometry $\mathcal{M}$ with a compact RT surface $\mathcal{R}$ as seen in Fig. 9.6. The boundary manifold is of the form $\mathcal{C} \times \mathbb{R}$, where the first factor corresponds to the spatial geometry and the second corresponds to the time direction. The boundary Hilbert spaces factorizes; each boundary algebra is a Type-I factor. Thus, the version of CC flow defined in terms of density matrices in Eq. (9.2.6) becomes rigorous in this situation. A natural choice of vacuum state is the thermofield double [380, 381]. The reduced state on each side is thermal, $\rho_{A_0} \sim \exp(-\beta H)$. Thus the modular Hamiltonian is proportional to the ordinary Hamiltonian on each boundary. This generates time translations and so is geometric.

Now, in any such geometry $\mathcal{M}$ one can pick a Cauchy slice Σ that ends on boundary time slices on both sides and contains $\mathcal{R}$. In obvious analogy with Sec. 9.3, we conjecture that the domain of dependence of the kink transformed slice Σ_s in a modified geometry $\mathcal{M}_s$

is dual to the boundary state:

$$|\psi_s(\mathcal{C}_s)\rangle_{LR} = \rho_L^{-is} |\psi(\mathcal{C})\rangle_{LR} \ , \tag{9.6.12}$$

where we have used the notation of Eq. (9.2.36).

In such a situation, it is again manifest that the Wheeler-DeWitt patches dual to either side are preserved by arguments similar to those made in Sec. 9.4. However, since there is no portion of $\mathcal{R}$ that reaches the asymptotic boundary, there is no analog of the shock matching done in Sec. 9.4. Notably, since $\partial A = \varnothing$ in this case, there is no subtlety regarding boundary conditions for JLMS and thus, one-sided modular flow makes sense without any regulator. Thus, our construction is simply kinematically related to the construction in [361].

An interesting situation arises for wiggly cuts of the Rindler horizon, *i.e.*, $u = 0$ and $v = V(y)$. The modular Hamiltonian acts locally, but only when restricted to the null plane [45]. Its action becomes non-local when extended to the rest of the domain of dependence. The properties of CC flow described in Sections 9.2-9.2 all hold for this choice of cut. This constrains one-sided operator expectation values on the null plane, subregion entropies for cuts entirely to one side of $V(y)$, and even the T_{vv} shock at the cut. Interestingly, all of them are matched by the kink transform, by the arguments given in Sec. 9.3. Even the expected stress tensor shock can still be derived, by taking a null limit of our derivation as described in Appendix A.23. One might then guess that the kink transform is also dual to CC flow for arbitrary wiggly cuts.

Even in the vacuum, however, the kink transform across a wiggly cut results in a boundary slice that cannot be embedded in Minkowski space, due to the absence of a boost symmetry that preserves the entangling surface. Thus, the kink transform would have to be modified to work for wiggly cuts. The transformation of boundary observables off the null plane is quite complicated for wiggly cut CC flow. Thus, we also expect that regions of the entanglement wedge probed by such observables should be drastically modified, unlike the case where the entangling surface is a flat Rindler cut.

However, the wiggly-cut boundary transformation remains simple for observables restricted to the null plane. Thus one could try to formulate a version of the kink transform on Cauchy slices anchored to the null plane on the boundary and the RT surface in the bulk. Perhaps a non-trivial transformation of the entanglement wedge arises from the need to ensure that the kink transformed initial data be compatible with corner conditions at the junction where the slice meets the asymptotic boundary [382]. We leave this question to future work.

Other Probes of CC Flow

In Sec. 9.4, we provided evidence for kink transform/CC flow duality. The preservation of the left and right entanglement wedges under the kink transform ensures that all one-sided correlation functions transform as required. It would be interesting to consider two sided correlation functions. However, these do not change universally and are difficult to compute in general. In the bulk, this is manifested by the fact that the future and past wedges do not change simply and need to be solved for.

However, because of the shared role of the kink transform, we can take advantage of the modular toolkit for one-sided modular flow [361]. Let $|\widetilde{\psi}_s\rangle = \rho_A^{-is}\,|\psi\rangle$ be a family of states generated by one-sided modular flow as discussed in Sec. 9.6. Then certain two sided correlation functions $\langle\widetilde{\psi}_s|\,O(x)\,O(y)\,|\widetilde{\psi}_s\rangle$ can be computed as follows.

Suppose $O(x)$ is an operator dual to a "heavy" bulk field with mass m such that $1/\ell_{\mathrm{AdS}} \ll m \ll 1/\ell_{\mathrm{P}}, 1/\ell_{\mathrm{s}}$. Correlation functions for such an operator can then be computed using the geodesic approximation,

$$\langle O(x)O(y)\rangle \approx \exp(-mL) \ , \tag{9.6.13}$$

where L is the length of the bulk geodesic connecting boundary points x and y. Now consider boundary points x and y such that there is a piecewise bulk geodesic of length $L(x,y)$ joining them in the spacetime dual to the state $|\psi\rangle$.

This kinked geodesic is required to pass through the RT surface of subregion A with a specific boost angle $2\pi s$ as seen in Fig. 9.6. (This is a fine-tuned condition on the set of points x, y.) Since single sided modular flow behaves locally as a boost at the RT surface, it straightens out the kinked geodesic such that it is now a true geodesic in the spacetime dual to the state $|\widetilde{\psi}_s\rangle$. Thus, we have

$$\langle\widetilde{\psi}_s|\,O(x)\,O(y)\,|\widetilde{\psi}_s\rangle \approx \exp(-mL(x,y)) \ . \tag{9.6.14}$$

As discussed in Sec. 9.2, the CC flowed state can equivalently be thought of as the single sided modular flowed state $|\widetilde{\psi}_s\rangle$ on a kinematically transformed slice $\mathcal{C}_s$. Thus, the above rules can still be used to compute two sided correlation functions in the CC flowed state $|\psi_s\rangle = u_s\,|\psi\rangle$ as

$$\langle\psi_s|\,O(x_s)O(y)\,|\psi_s\rangle \approx \exp(-mL(x,y)) \ , \tag{9.6.15}$$

where x_s is the point related to x by the vacuum modular flow transformation.

We also note that the shock matching performed in Sec. 9.4 was a near boundary calculation. However, a bulk shock exists everywhere on the RT surface. One could solve for the position of the RT surface to further subleading orders and relate the bulk shock to the boundary stress tensor. This would yield a sequence of relations that the stress tensor must satisfy in order to be dual to the kink transform. In general these relations may be highly theory-dependent, but it would be interesting to see if some follow directly from CC flow or make interesting universal predictions for CC flow in holographic theories.

Higher Curvature Corrections

In Sec. 9.4, we argued that the bulk kink transform in a theory of Einstein gravity satisfies properties consistent with the boundary CC flow. However, this result is robust to the addition of higher curvature corrections in the bulk theory. The preservation of the two entanglement wedges, *i.e.*, Eq. (9.2.38), is a geometric fact that remains unchanged.[8]

Further, the matching of the stress tensor shock crucially depended on two ingredients. Firstly, Eq. (9.4.12), the holographic dictionary between the boundary stress tensor and the bulk metric perturbation and secondly, Eq. (9.4.15), the relation between the boundary entropy variation and the shape of the RT surface. Both of these relations are modified once higher curvature corrections are included [288, 307]. However, it follows generally from dimension counting arguments that

$$g_{ij}^{(d)} = \eta_1 \frac{16\pi G}{d} \langle T_{ij} \rangle \ , \tag{9.6.16}$$

$$X_{(d)}^A = -\eta_2 \frac{4G}{d} \left. \frac{\delta S}{\delta X^A} \right|_{\mathcal{R}} \ , \tag{9.6.17}$$

where η_1 and η_2 are constants that depend on the higher curvature couplings. Using the first law of entanglement, it can be shown that in fact $\eta_1 = \eta_2$ [288, 307]. Hence, the boundary stress tensor shock obtained from the kink transform is robust to higher curvature corrections.

Holographic proof of QNEC

A recent proof of the QNEC from the ANEC [16] considers CC flow for a subregion A on the null plane $u = 0$ with entangling surfaces $v = V_1(y)$ and $v = V_2(y)$ surrounding a given point p. From the transformation properties of the stress tensor under CC flow described in Sec. 9.2, $T_{vv} \to 0$ as $s \to \infty$. In addition, there are stress tensor shocks at ∂A, as described in Sec. 9.2, of weight $f(s) \frac{\delta S}{\delta V(y)}\Big|_{\psi,\partial A}$. In the limit $V_1(y) \to V_2(y)$, computing the ANEC in the CC flowed state, one obtains contributions from the stress tensor $T_{vv}(p)$ in subregion A, and a contribution proportional to $\frac{\delta^2 S}{\delta V(y_1) \delta V(y_2)}\Big|_{\psi,p}$ from the shocks. Positivity of the averaged null energy in the CC flowed state then implies the QNEC in the original state.

Prior to the QFT proofs, both the ANEC and QNEC had been proved holographically [384, 17]. The guiding principle behind both of these proofs was the fact that consistency of the holographic duality requires bulk causality to respect boundary causality as we demonstrate in Fig. 9.7. In the case of the ANEC, one considers an infinitely long curve connecting points on past null infinity to future null infinity through the bulk and demands that it respect boundary causality [384]. In the proof of the QNEC, one requires that curves joining the RT surfaces of subregions $v < V_1(y)$ and $v > V_2(y)$, denoted $\mathcal{R}_1$ and $\mathcal{R}_2$, respect boundary causality [17, 357]. There are two contributions to the lightcone tilt of this bulk curve coming from the metric perturbation h_{vv} in the near boundary geometry, and the shape of

[8] Here we assume that the initial value formulation of Einstein gravity can be perturbatively adjusted to include higher curvature corrections despite the fact that a non-perturbative classical analysis of higher curvature theories is often problematic due to the Ostrogradsky instability [383].

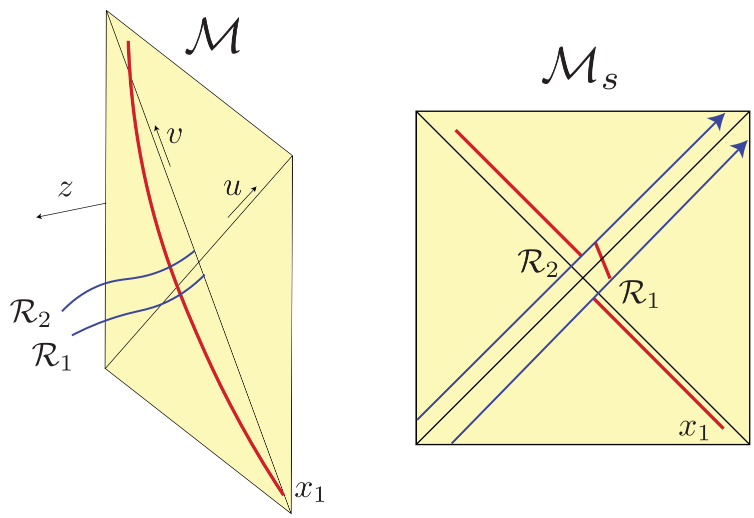


Figure 9.7: Holographic proofs. *Left:* Boundary causality is respected by the red curve that goes through the bulk in a spacetime $\mathcal{M}$; this is used in proving the ANEC. The RT surfaces $\mathcal{R}_1$ and $\mathcal{R}_2$ must be spacelike separated; this is used in proving the QNEC. *Right:* In the kink transformed spacetime $\mathcal{M}_s$ as $s \to \infty$, the QNEC follows from causality of the red curve, which only gets contributions from the Weyl shocks (blue) at $\mathcal{R}_1$ and $\mathcal{R}_2$, and the metric perturbation in the region between them.

the RT surface $X^\mu(y, z)$. By the holographic dictionary, these contributions can be related to the boundary stress tensor T_{vv} and the entropy variations $\frac{\delta S}{\delta V}$ as discussed in Sec. 9.4.

Now performing the kink transform removes the contribution coming from the shape of the RT surface and puts it into a time advance/delay coming from shocks in the bulk Weyl tensor that we compute in Appendix A.23. Considering the extended curve from past to future null infinity, we see that whether or not it respects boundary causality is determined entirely by the region between the entangling surfaces $\mathcal{R}_1$ and $\mathcal{R}_2$ since the bulk solution approaches the vacuum everywhere else in the limit $s \to \infty$. Requiring causality of the ANEC curve then results in the QNEC, making the connection to the boundary proof manifest.

Appendix A

Appendix

A.1 Orthonormal basis formalism for null surfaces

In this appendix we translate the definitions and formalism described in Sec. 2.3, and some of the results of Sec. 2.6, into the language of components on an orthonormal basis. This specialization is often useful in computations, although it does depend on arbitrary choices. We first describe the specializations that occur when one chooses an auxiliary null vector, and then the specializations associated with a complete orthonormal basis.

Review of structures associated with a choice of auxiliary null vector

We choose an auxiliary null vector field n^a on $\mathcal{N}$ which satisfies

$$n_a n^a = 0, \tag{A.1.1a}$$
$$n_a \ell^a = -1. \tag{A.1.1b}$$

The pullback of the covector field n_a yields a covector on $\mathcal{N}$

$$n_i = \Pi^a_i n_a \tag{A.1.2}$$

which from Eqs. (2.3.7) and (A.1.1b) satisfies $n_i \ell^i = -1$ [1]. We define the projection tensor

$$\pi^a{}_b = \delta^a{}_b + n^a \ell_b. \tag{A.1.3}$$

At a given point p the mapping $v^a \to \pi^a{}_b v^b$ maps vectors into the space of vectors orthogonal to ℓ_a, *i.e.*, into the tangent space $T_p(\mathcal{N})$. We write this mapping from $T_p(M)$ to $T_p(\mathcal{N})$ as

$$v^a \to \Upsilon^i_a v^a. \tag{A.1.4}$$

[1] Given a covector n_i on $\mathcal{N}$ with $n_i \ell^i = -1$, n_a is uniquely determined by the conditions (A.1.1a) and (A.1.2).

The quantities Υ^i_a and Π^a_i satisfy

$$\delta^i_j = \Upsilon^i_a \, \Pi^a_j, \qquad \pi^a{}_b = \Pi^a_i \, \Upsilon^i_b. \tag{A.1.5}$$

We can now define spacetime tensors that correspond to the induced metric

$$q_{ab} = \Upsilon^i_a \Upsilon^j_b q_{ij} = g_{ab} + 2\ell_{(a} n_{b)}, \tag{A.1.6}$$

and shear tensor

$$\sigma_{ab} = \Upsilon^i_a \Upsilon^j_b \sigma_{ij}. \tag{A.1.7}$$

These quantities depend on the choice of auxiliary null vector n_a. We can also define a derivative operator D_i on $\mathcal{N}$ by, for a given vector field v^i on $\mathcal{N}$,

$$D_i v^j = \Pi^a_i \Upsilon^j_b \nabla_a v^b. \tag{A.1.8}$$

Here, on the right hand side, v^a is any choice of vector field on M for which $v^a = \Pi^a_i v^i$ when evaluated on $\mathcal{N}$. It can be checked that this prescription yields a well defined derivative operator, which depends on the choice of n_a.

We define the rotation one-form ω_i by

$$\omega_i = -n_j \mathcal{K}_i{}^j. \tag{A.1.9}$$

From Eq. (2.3.18) this satisfies

$$\omega_i \ell^i = \kappa. \tag{A.1.10}$$

As noted in Sec. 2.3 above the rotation one-form depends on n_i except when $K_{ij} = 0$.

Geometric fields on an orthonormal basis

We choose on $\mathcal{N}$ a set of basis vectors

$$\vec{e}_{\hat{\alpha}} = (\vec{e}_{\hat{0}}, \vec{e}_{\hat{1}}, \vec{e}_{\hat{A}}) = \left(\vec{\ell}, \vec{n}, \vec{e}_{\hat{A}}\right), \tag{A.1.11}$$

where $\hat{A} = 2, 3$, and where $\vec{\ell}^2 = \vec{n}^2 = \vec{\ell} \cdot \vec{e}_{\hat{A}} = \vec{n} \cdot \vec{e}_{\hat{A}} = 0$, $\vec{\ell} \cdot \vec{n} = -1$, $\vec{e}_{\hat{A}} \cdot \vec{e}_{\hat{B}} = \delta_{\hat{A}\hat{B}}$ on $\mathcal{N}$. We extend the definition of these vectors off $\mathcal{N}$ but do not require them to be orthonormal off $\mathcal{N}$.

We can decompose the covariant derivative of the normal on this basis as

$$\begin{aligned}\nabla_a \ell_b = {} & \gamma \ell_a \ell_b + \eta \ell_a n_b + \tau_{\hat{A}} \ell_a e^{\hat{A}}_b + \epsilon n_a \ell_b + \zeta n_a n_b + \kappa_{\hat{A}} n_a e^{\hat{A}}_b + \alpha_{\hat{A}} e^{\hat{A}}_a \ell_b + \iota_{\hat{A}} e^{\hat{A}}_a n_b \\ & + \left(\frac{1}{2}\theta \delta_{\hat{A}\hat{B}} + \sigma_{\hat{A}\hat{B}}\right) e^{\hat{A}}_a e^{\hat{B}}_b. \end{aligned} \tag{A.1.12}$$

where $\sigma_{\hat{A}\hat{B}}$ is traceless. Imposing the orthonormality of the basis on the hypersurface gives $\zeta = \iota_{\hat{A}} = 0$, while imposing (2.3.2) gives $\epsilon = -\kappa$, $\kappa_{\hat{A}} = 0$. The induced metric, second fundamental form, Weingarten map and rotation one-form in terms of these quantities are

$$q_{ij} = \delta_{\hat{A}\hat{B}} e_i^{\hat{A}} e_j^{\hat{B}}, \tag{A.1.13a}$$

$$K_{ij} = \left(\frac{1}{2}\theta\delta_{\hat{A}\hat{B}} + \sigma_{\hat{A}\hat{B}}\right) e_i^{\hat{A}} e_j^{\hat{B}}, \tag{A.1.13b}$$

$$\mathcal{K}_i{}^j = -\kappa n_i \ell^j + \alpha_{\hat{A}} e_i^{\hat{A}} \ell^j + \left(\frac{1}{2}\theta\delta_{\hat{A}\hat{B}} + \sigma_{\hat{A}\hat{B}}\right) e_i^{\hat{A}} e^{\hat{B}\,j}, \tag{A.1.13c}$$

$$\omega_i = -\kappa n_i + \alpha_{\hat{A}} e_i^{\hat{A}}. \tag{A.1.13d}$$

Expressions for charges

A simple expression for the Noether charge in terms of the orthonormal basis can be found by combining Eqs. (2.6.7), (2.3.16), (2.4.11a), (2.3.7) and (A.4.2):

$$Q_\xi(\mathcal{S}) = \frac{1}{8\pi}\int_{\mathcal{S}} \varepsilon_{ij}\left[n_b \ell^a \nabla_a \xi^b\right]. \tag{A.1.14}$$

Here the null vector n_a has been chosen so that its pullback n_i to $\mathcal{N}$ is normal to the cross section $\mathcal{S}$. A similar calculation starting from the localized charge (2.6.27) gives

$$\mathcal{Q}_\xi^{\mathrm{loc}}(\mathcal{S}) = \frac{1}{8\pi}\int_{\mathcal{S}} \varepsilon_{ij}\left[n_b \ell^a \nabla_a \xi^b - \theta \xi^a n_a\right]. \tag{A.1.15}$$

For diff$(\mathcal{Z})$ generators we have $\xi^a n_a \mathrel{\hat{=}} 0$, and this charge can be rewritten as

$$\mathcal{Q}_X^{\mathrm{loc}}(\mathcal{S}) = -\frac{1}{8\pi}\int_{\mathcal{S}} \varepsilon_{ij}\left[\ell^a \xi^b \nabla_a n_b\right]. \tag{A.1.16}$$

A.2 Gauge fixing in the definition of field configuration space

In this appendix we show that the field configuration space $\mathscr{F}_{\mathfrak{p}}$ that we defined is obtained from the larger space $\mathscr{F}_0$ by a gauge fixing. Specifically, given a manifold M with boundary $\mathcal{N}$, a complete boundary structure $\mathfrak{p}$ on $\mathcal{N}$, and a metric g_{ab} on M for which $\mathcal{N}$ is null and for which the boundary structure induced by g_{ab} is complete, we show that one can find a diffeomorphism $\psi : M \to M$ which takes $\mathcal{N}$ into $\mathcal{N}$ for which $\psi_* g_{ab}$ lies in $\mathscr{F}_{\mathfrak{p}}$.

Let $\mathfrak{u}$ be the intrinsic structure induced by $\mathfrak{p}$, and $\mathfrak{u}'$ be the intrinsic structure induced by the metric g_{ab}. By hypothesis, both $\mathfrak{u}$ and $\mathfrak{u}'$ are complete. Hence by the argument given in Sec. 2.4 there exists a diffeomorphism $\varphi : \mathcal{N} \to \mathcal{N}$ which takes $\mathfrak{u}$ to $\mathfrak{u}'$. Now choose a diffeomorphism $\psi : M \to M$ whose restriction to $\mathcal{N}$ is φ. By acting with ψ on the metric we can without loss of generality assume that $\mathfrak{u} = \mathfrak{u}'$.

Now let $\mathfrak{p}'$ be the boundary structure induced by g_{ab}, and choose representatives $(\ell^a, \kappa, \hat{\ell}_a)$ and $(\ell'^a, \kappa', \hat{\ell}'_a)$ of $\mathfrak{p}$ and $\mathfrak{p}'$. Since $\mathfrak{u} = \mathfrak{u}'$ we can, by adjusting the choice of representative if necessary, take $\ell^a = \ell'^a$ and $\kappa = \kappa'$. The two normal covectors must be related by some rescaling of the form $\hat{\ell}_a = e^\lambda \hat{\ell}'_a$ for some smooth function λ on $\mathcal{N}$. We thus have

$$g^{ab}\hat{\ell}_a \mathrel{\hat{=}} e^\lambda \ell^b, \tag{A.2.1}$$

and we want to show that there exists a diffeomorphism ψ that preserves $\mathfrak{u}'$ so that

$$(\psi_* g^{ab})\hat{\ell}_a \mathrel{\hat{=}} \ell^b. \tag{A.2.2}$$

By applying ψ_*^{-1} to both sides of Eq. (A.2.2), specializing the diffeomorphism so that the induced diffeomorphism φ on $\mathcal{N}$ is the identity, and using (A.2.1), we find that a sufficient condition for (A.2.2) is that

$$\psi_*^{-1}\hat{\ell}_a \mathrel{\hat{=}} e^{-\lambda}\hat{\ell}_a. \tag{A.2.3}$$

To find a diffeomorphism ψ satisfying (A.2.3), we need only specify its action to linear order in deviation off the surface $\mathcal{N}$. We can parameterize points near $\mathcal{N}$ to linear order by specifying a point $\mathcal{P}$ on $\mathcal{N}$ and a vector v^a at $\mathcal{P}$. We define ψ to be the mapping that takes

$$\psi : (\mathcal{P}, v^a) \to (\mathcal{P}, v^a + \zeta^a(\mathcal{P})\hat{\ell}_b v^b), \tag{A.2.4}$$

where ζ^a is some vector field defined on $\mathcal{N}$. This mapping is well defined despite the fact that representing points near $\mathcal{N}$ as pairs $(\mathcal{P}, v^a)$ is not unique, since components of v^a along the surface are annihilated by the term proportional to ζ^a. Now computing the pullback of the mapping (A.2.4) we find that the condition (A.2.3) will be satisfied if we choose the vector field ζ^a to satisfy

$$1 + \zeta^a\hat{\ell}_a = e^\lambda. \tag{A.2.5}$$

A.3 Characterization of trivial diffeomorphisms at a null boundary

In this appendix we show that the charge variation (2.2.13) vanishes for all cross sections $\mathcal{S}$ of a null boundary, and for all solutions and variations of solutions, if and only if the symmetry ξ^a satisfies $\chi^i = 0$ and $\gamma(\xi^a) = 0$, where χ^i is defined by Eq. (2.5.15) and γ by Eq. (2.5.16).

The charge variation is given by Eq. (2.6.19), but with β replaced by $(\beta + \gamma)/2$ from Eq. (2.6.5):

$$\begin{aligned}\delta\mathcal{Q}_\xi &= \frac{1}{16\pi}\int_{\mathcal{S}} \varepsilon_{ijk}\Big[h\chi^l\mathcal{K}_l{}^k - h\beta(\chi^i)\ell^k/2 - h\gamma(\xi^a)\ell^k/2 - \chi^l\Gamma_l\ell^k + \chi^l \pounds_\ell h_l{}^k + 2\chi^l h^m{}_l\mathcal{K}_m{}^k \\ &\quad -2\chi^l h^k{}_m\mathcal{K}_l{}^m - \chi^k\pounds_\ell h - \chi^k h_i{}^j\mathcal{K}_j{}^i\Big].\end{aligned} \tag{A.3.1}$$

This expression vanishes if χ^i and γ vanish, from Eq. (2.4.11a). Conversely, we want to show that the vanishing of the expression (A.3.1) for all solutions and variations of solutions forces $\chi^i = \gamma = 0$.

Fix a cross section $\mathcal{S}$. We make use of the explicit form of the general solution to the vacuum Einstein equations on a null surface given by Hayward [385]. It follows from this solution that, on shell, we can freely specify $h_i{}^j$ on $\mathcal{S}$ subject to the constraint (2.5.26), $\pounds_\ell h_i{}^j$ subject to the constraint

$$\ell^i \pounds_\ell h_i{}^j = 0, \tag{A.3.2}$$

and the quantity Γ_i defined by Eq. (2.5.27) subject to the constraint (2.5.29). We now choose $h_i{}^j = 0$ and $\pounds_\ell h_i{}^j = 0$. In this case the charge variation (A.3.1) reduces to

$$\delta \mathcal{Q}_\xi \;\; = \;\; \frac{1}{16\pi} \int_{\mathcal{S}} \varepsilon_{ij} \, \chi^l \Gamma_l. \tag{A.3.3}$$

Since Γ_l can be chosen arbitrarily on $\mathcal{S}$ subject to Eq. (2.5.29), this forces $\chi^i = f\ell^i$ on $\mathcal{S}$ for some function f. Returning now to Eq. (A.3.1), choosing $h_i{}^j = 0$, and making use of the constraint (A.3.2) gives the charge variation

$$\delta \mathcal{Q}_\xi \;\; = \;\; \frac{1}{16\pi} \int_{\mathcal{S}} \varepsilon_{ij} \, f \pounds_\ell h. \tag{A.3.4}$$

Since $\pounds_\ell h$ can be chosen arbitrarily on $\mathcal{S}$, this forces f to vanish on $\mathcal{S}$. Since $\mathcal{S}$ was chosen arbitrarily, f (and therefore χ^i) must vanish on all of $\mathcal{N}$, and so $\beta = 0$ from Eq. (2.4.19). Now reverting to a general $h_i{}^j$ and $\pounds_\ell h_i{}^j$ in Eq. (A.3.1), we obtain the charge variation

$$\delta \mathcal{Q}_\xi \;\; = \;\; \frac{1}{32\pi} \int_{\mathcal{S}} \varepsilon_{ij} \, h\gamma. \tag{A.3.5}$$

Since h can be chosen arbitrarily on $\mathcal{S}$, this forces $\gamma = 0$ on $\mathcal{S}$. Finally, since the choice of $\mathcal{S}$ was arbitrary, if follows that χ^i and γ vanish on all of $\mathcal{N}$.

A.4 Consistency check of symmetry algebra

In this appendix we verify that for vector fields ξ^a satisfying the conditions (2.4.11) and (2.5.18) of the symmetry algebra, the corresponding metric perturbation (2.5.23) satisfies the boundary conditions (2.5.22) derived in Sec. 4.3.

Taking the Lie derivative of Eq. (2.5.4a) with respect to ξ^a gives

$$\pounds_\xi \hat{\ell}_a \;\hat{=}\; \pounds_\xi g_{ab} \ell^b + g_{ab} \pounds_\xi \ell^b. \tag{A.4.1}$$

Making use of Eqs. (2.5.15), (2.5.23), (2.4.11a), (2.5.16) and (2.5.18) gives

$$h_{ab}\ell^b \;\hat{=}\; (\gamma - \beta)\hat{\ell}_a \;\hat{=}\; 0, \tag{A.4.2}$$

which establishes the condition (2.5.22a).

Next for simplicity and without loss of generality we specialize to a representative of the boundary structure with $\kappa = 0$. We write the definition (2.5.16) in the form, using (2.5.14) and (2.5.4a),

$$\xi^b \nabla_b \ell^a + \ell^b \nabla^a \xi_b \mathrel{\hat{=}} \gamma \ell^a, \tag{A.4.3}$$

and take the Lie derivative with respect to ℓ^a. The right hand side becomes $(\pounds_\ell \gamma)\ell^a$, which vanishes by Eqs. (2.5.18) and (2.4.11b). Writing v^a for the expression on the left hand side, the left hand side becomes $\ell^c \nabla_c v^a - v^c \nabla_c \ell^a$, and the second term can be written as $\gamma \ell^c \nabla_c \ell^a \mathrel{\hat{=}} \gamma \kappa \ell^a \mathrel{\hat{=}} 0$. We thus obtain

$$0 \mathrel{\hat{=}} \ell^c \nabla_c \xi^b \nabla_b \ell^a + \ell^c \xi^b \nabla_c \nabla_b \ell^a + \ell^c \ell^b \nabla_c \nabla^a \xi_b. \tag{A.4.4}$$

The first term can be written using the definition (2.4.11a) as $-\beta \ell^b \nabla_b \ell^a + \xi^c \nabla_c \ell^b \nabla_b \ell^a \mathrel{\hat{=}} -\xi^c \ell^b \nabla_c \nabla_b \ell^a$, where we have used (2.3.2) and $\kappa = 0$. It follows that

$$0 \mathrel{\hat{=}} -\ell^c \xi^b R_{cbda} \ell^d + \ell^c \ell^b \nabla_c \nabla_a \xi_b \mathrel{\hat{=}} \ell^c \ell^b \nabla_a \nabla_c \xi_b, \tag{A.4.5}$$

from which the condition (2.5.22b) follows.

A.5 Choice of reference solution

As explained in Sec. 2.2, the dynamics of a theory fix the symmetry generator charges on phase space only up to an overall "constant of integration". To fix that constant of integration, following Wald and Zoupas [24], we choose a reference solution and demand that the charges vanish on that solution. There are two different cases, complete intrinsic structures, and incomplete intrinsic structures associated with nontrivial boundaries $\partial\mathcal{N}$ of the null hypersurface $\mathcal{N}$, as discussed in Sec. 2.4

In the first case of complete intrinsic structures, we choose a one-parameter family of reference solutions $g_{ab}(\varepsilon)$ and demand the the limit $\varepsilon \to 0$ of the charges evaluated on the reference solution vanish. (We use a one parameter family rather than a single solution since our chosen family of solutions does not have a continuous limit as $\varepsilon \to 0$.) The reference solution is maximally extended Schwarzschild written in Kruskal coordinates

$$ds^2 = -2e^{2\mu(s)} dU dV + m^2 \rho(s)^2 d\Omega^2, \tag{A.5.1}$$

where $s = UV/m^2$, m is the mass, and $\mu(s)$ and $\rho(s)$ are functions whose exact forms are unimportant for what follows. We also need to specify how this manifold is to be identified with our given boundary structure $(M, \mathcal{N}, \mathfrak{p})$. We identify $\mathcal{N}$ with the horizon $U = 0$, and pick $\mathfrak{p}$ to be determined by the representative $(\ell^a, \hat{\ell}_a, \kappa)$ where

$$\hat{\ell}_a = (dU)_a, \tag{A.5.2a}$$

$$\ell^a = -e^{-2\mu(0)} \left(\frac{\partial}{\partial V}\right)^a, \tag{A.5.2b}$$

$$\kappa = 0. \tag{A.5.2c}$$

We identify the parameter ε with the mass m and will take the $m \to 0$ limit.

We now show that the charge (2.6.27) integrated over a fixed cross section $\mathcal{S}$ vanishes for the reference solution, in the limit $m \to 0$, as claimed in Sec. 2.6. The expansion θ and Weingarten map $\mathcal{K}_i^{\ j}$ vanish for this solution with the choice (A.5.2) of normal. The charge therefore reduces to

$$\mathcal{Q}_\xi^{\rm loc}(\mathcal{S}) = -\frac{1}{8\pi}\int_{\mathcal{S}} \varepsilon_{ijk}\beta(\chi^i)\ell^i. \tag{A.5.3}$$

The only quantity that depends on the metric in this expression is the volume form ε_{ijk}, which from Eq. (A.5.1) is of the form $\varepsilon_{ijk} = m^2\varepsilon^0_{ijk}$ where ε^0_{ijk} is independent of m. Hence $\mathcal{Q}_\xi^{\rm loc}(\mathcal{S}) \to 0$ as $m \to 0$ as required.

Note that this conclusion is unchanged if we replace the reference solution $g_{ab}(m)$ with $\psi_* g_{ab}(m)$ for any diffeomorphism $\psi : M \to M$ which preserves the boundary structure $\mathfrak{p}$. The only effect of this change on the argument is to replace ε^0_{ijk} with $\varphi_*\varepsilon^0_{ijk}$, where φ is the restriction of ψ to $\mathcal{N}$, which does not affect the conclusion. Thus the consistency condition[2] discussed by Wald and Zoupas [24] is satisfied.

Turn now to the second case of a nontrivial boundary $\partial\mathcal{N}$. If the boundary $\partial\mathcal{N}$ is a twosphere, we take the reference solution to be the Schwarzschild solution (A.5.1), with the hypersurface $\mathcal{N}$ now being restricted to $U > 0$, so that the boundary $\partial\mathcal{N}$ is identified with the bifurcation twosphere of Schwarzschild. Apart from this modifications the analysis and conclusions are unchanged.

The case where the boundary $\partial\mathcal{N}$ is a single point $\{\mathcal{P}\}$ is slightly more complicated. We choose the reference solution to be the Schwarzschild solution (A.5.1) for $U > U_0(m)$, and a spherically symmetric ingoing Vaidya solution at earlier advanced times, so that the origin of the event horizon is mapped onto $\mathcal{P}$. The reference boundary structure is chosen to satisfy Eqs. (A.5.2) in the Schwarzschild region, which determines its definition everywhere. If we choose the function $U_0(m)$ to go to zero as $m \to 0$, then the charge (2.6.27) integrated over a fixed cross section $\mathcal{S}$ is evaluated entirely in the Schwarzschild region for sufficiently small m, and the rest of the argument follows as before. Roughly speaking, we are taking the limit of small black holes formed in the distant past to define the reference solution in this case.

Of course, we could dispense with the reference solutions and simply say that we are picking the constant of integration to enforce the expression (2.6.27) starting from its variation. The reference solutions clarify the physical interpretation of that assumption.

A.6 Consistency of two expressions for flux of localized charge

In this appendix we show explicitly that the two expressions (2.6.28) and (2.6.29) for the flux of the localized charge coincide, as they must from the general Wald-Zoupas framework

[2] The charges need not vanish in the $m \to 0$ limit for the transformed reference solution $\psi(m)_* g_{ab}(m)$ which allows the diffeomorphism ψ to depend on m. However, there is no physical argument for imposing this more stringent requirement.

reviewed in Sec. 2.2.

The expression (2.6.28) was derived from Eq. (2.6.22). The variation in the second term in (2.6.22) can be replaced with a Lie derivative with respect to ξ, from Eq. (2.5.23), giving from the expression (2.6.24) a contribution to Θ_{ijk} of

$$-\frac{1}{8\pi}\pounds_{\chi}(\theta\varepsilon_{ijk}). \tag{A.6.1}$$

Using Cartan's formula $\pounds_v \boldsymbol{\omega} = i_v d\boldsymbol{\omega} + d(i_v\boldsymbol{\omega})$ and the definition (2.3.25) of the divergence operator shows that this contribution matches the second term in Eq. (2.6.29). Hence, using the expression (2.6.16) for θ_{ijk}, it remains to show that

$$\ell^f(\nabla_f h - \nabla_e h_f{}^e) = 2\hat{D}_i(\chi^j \mathcal{K}_j{}^i - \beta\ell^i). \tag{A.6.2}$$

Inserting the expression (2.5.23) for the metric perturbation h_{ab} into the left hand side of Eq. (A.6.2), commuting derivatives and making use of the vacuum equation of motion $R_{ab} = 0$ gives the expression

$$\ell^f\nabla_f(\nabla_a\xi^a) - \ell^f\nabla_e\nabla^e\xi_f. \tag{A.6.3}$$

It follows from Eqs. (2.5.22) and (2.3.27) that

$$\nabla_a\left[(\nabla^a\xi^b + \nabla^b\xi^a)\ell_b\right] = 0, \tag{A.6.4}$$

and simplifying by once again commuting derivatives acting on ξ^a and inserting into (A.6.3) gives that the left hand side of Eq. (A.6.2) is

$$2\pounds_\ell(\nabla_a\xi^a) + (\nabla^a\xi^b + \nabla^b\xi^a)\nabla_a\ell_b. \tag{A.6.5}$$

Note that this expression is independent of the definition of ℓ^a off $\mathcal{N}$, by Eq. (2.5.22a).

We now turn to evaluating the right hand side of Eq. (A.6.2). We define the vector

$$v^a = \xi^a\nabla_a\ell^b - \beta\ell^b, \tag{A.6.6}$$

which satisfies $\hat{\ell}_a v^a = 0$, in terms of which the right hand side can be written as $2\hat{D}_i v^i$. We now make use of the relation (2.3.27) between the three dimensional and four dimensional divergence operators, and the definition (2.3.26), which yields for the right hand side

$$2\nabla_a v^a + 2n^b(v^a\nabla_a\hat{\ell}_b + \hat{\ell}_a\nabla_b v^a). \tag{A.6.7}$$

Here n^b is any null vector field which satisfies $n_a\ell^a = -1$. Now using the definition (2.4.11a) of β in Eq. (A.6.6) we obtain $v^a = \ell^c\nabla_c\xi^a$, and substituting into (A.6.7) gives

$$2\nabla_a\ell^c\nabla_c\xi^a + 2\ell^c\nabla_a\nabla_c\xi^a + 2n^b\ell^c\nabla_a\hat{\ell}_b\nabla_c\xi^a + 2n^b\hat{\ell}_a\nabla_b\ell^c\nabla_c\xi^a + 2n^b\hat{\ell}_a\ell^c\nabla_b\nabla_c\xi^a. \tag{A.6.8}$$

It remains to show that the expressions (A.6.5) and (A.6.8) coincide. Commuting the derivatives in the second term in (A.6.8) and using the vacuum equation of motion $R_{ab} = 0$

shows that this term matches the first term in Eq. (A.6.5). The last term in (A.6.8) vanishes by Eqs. (2.5.22). In the third term, the derivative acting on $\hat{\ell}_a$ is entirely along the surface, since $\ell^c \ell_a \nabla_c \xi^a = 0$ by Eq. (2.5.22a). Hence we can replace $\hat{\ell}_a$ with ℓ_a in this term, and also in the fourth term. Next, we have that ℓ^a is hypersurface orthogonal on $\mathcal{N}$, so $\ell_{[a} \nabla_b \ell_{c]} \mathrel{\hat{=}} 0$. It follows that $\nabla_a \ell_b \mathrel{\hat{=}} \nabla_{(a} \ell_{b)} + w_{[a} \ell_{b]}$ for some w_a with $w_a n^a = 0$. Substituting this into the first, third and fourth terms in Eq. (A.6.8) we find that the dependence on w_a cancels out, so that $\nabla_a \ell_b$ can be replaced in these terms with $\nabla_{(a} \ell_{b)}$. The first term in (A.6.8) then matches the second term in (A.6.5). The third and fourth terms can be written as $4 p_a \ell_b \nabla^{(a} \xi^{b)}$ where $p_a = n^b \nabla_{(a} \ell_{b)}$, which vanishes by Eq. (2.5.22a). Thus the expressions (A.6.5) and (A.6.8) coincide as desired.

A.7 Symplectic currents on black holes horizons

Our explicit expressions for the symplectic current and charges for general null surfaces allow us to establish a number of results about black hole horizons.

First, in vacuum general relativity, the obstruction (2.2.15) to defining the contribution to a global symmetry generator charge $\mathcal{Q}_\xi$ from an integral over a future horizon $\mathcal{H}^+$ vanishes,

$$\int_{\mathcal{H}^+_\pm} i_\xi \boldsymbol{\omega} = 0, \tag{A.7.1}$$

as discussed in Sec. 2.2 above, assuming certain fall off conditions on the shear along the horizon at the future boundary $\mathcal{H}^+_+$, which we now discuss. Consider a cross section $\mathcal{S}$ of the horizon that approaches $\mathcal{H}^+_+$. The integrand in Eq. (A.7.1) is given explicitly for a null surface in Eq. (2.6.20), and scales as a product of a symmetry generator χ^i, times the expansion or shear of the background, times two factors of metric perturbation h_{ij}. Denoting an affine parameter along the horizon by v, the symmetry generator scales $\sim v$ as $v \to \infty$, by Eq. (2.4.8). If the shear of the background and perturbations scales as

$$\sigma_{ij} \sim v^{-p} \tag{A.7.2}$$

for some $p > 1$ as $v \to \infty$, then it follows from Eq. (9.2.32) of Wald [67] that the expansion θ is negligible. Also from Eq. (2.3.15) it follows $h_{ij} \sim v^{-(p-1)} + (\text{const})$, and hence the condition (A.7.1) will be satisfied at $\mathcal{H}^+_+$.

Is the condition (A.7.2) on the late time decay of the shear physically realistic? Consider first linear gravitational perturbations of a Kerr black hole with initial data of compact spatial support. For this case Barack showed that the Weyl scalars Ψ_0 and Ψ_4 decay along the horizon at late times like v^{-7} or smaller [386]. It then follows from Eqs. (9.2.32) and (9.2.33) of Wald [67] that $\sigma_{ij} \sim v^{-6}$. For more general solutions with incoming radiation at $\mathscr{I}^-$, we conjecture that imposing that the News tensor fall off along $\mathscr{I}^-$ as $\sim v^{-p}$ with $p > 1$ in the limit $v \to \infty$ towards $\mathscr{I}^-_+$ will be sufficient to ensure the fall off condition (A.7.2) along the event horizon, both linearly and nonlinearly. This conjecture is based on the intuition that backscattering should serve to decrease rather than increase the incoming flux at late advanced times v.

For eternal black holes with a bifurcation two-sphere $\mathcal{H}^+_-$, the condition (A.7.1) will be satisfied at $\mathcal{H}^+_-$ from Eqs. (2.4.40) and (2.6.20).

Second, we show that the contribution to any global symmetry generator charge $\mathcal{Q}_\xi$ from the integral over a future event horizon $\mathcal{H}^+$ can be expressed in terms of corresponding localized charges $\mathcal{Q}^{\rm loc}_\xi$ evaluated on the components $\mathcal{H}^+_\pm$ of $\partial\mathcal{H}^+$, as discussed in Sec. 2.2 above. This requires the vanishing of the correction term $i_\xi\boldsymbol{\Theta}$ in the definition (2.2.25) of the localized charge:

$$\int_{\mathcal{H}^+_\pm} i_\xi \boldsymbol{\Theta} = 0. \tag{A.7.3}$$

Using the explicit expression (2.6.26), an argument analogous to that given in the last paragraph shows that the quantity (A.7.3) vanishes at $\mathcal{H}^+_+$, under the same assumptions on the shear as above. For eternal black holes with a bifurcation two-sphere $\mathcal{H}^+_-$, the corresponding integral (A.7.3) vanishes by the condition (2.4.40).

A.8 Alternative definition of field configuration space and associated symmetry algebra

In the body of this paper we have presented a specific definition of a field configuration space $\mathscr{F}$ for general relativity in the presence of a null boundary, and derived from that definition a symmetry algebra and various types of charges. A natural question is whether there is any freedom in the choice of definition of $\mathscr{F}$. In this appendix, we explore a modification of the definition of $\mathscr{F}$, in which we allow a larger set of metrics. A key motivation for this exploration is the fact is that the new metric variations which are now allowed do not correspond to degeneracies of the symplectic form, and so can be regarded as physical degrees of freedom. We will show that our analysis of the symmetry algebra can be straightforwardly generalized, but that it is not possible to implement the Wald-Zoupas prescription described in Sec. 2.2 to compute localized charges in this context. One can obtain expressions for localized charges but they are not unique.

The starting point for the modified field configuration space definition is to omit the non-affinity κ in the definition (2.4.2) of intrinsic structure $\mathfrak{u}$. Thus, $\mathfrak{u}$ consists of an equivalence class of normals ℓ^i that are related by rescalings of the form (4.3.1). The symmetry group is modified by replacing the transformation (2.4.7a) with an arbitrary smooth mapping $\overline{u} = \overline{u}(u, \theta^A)$, and the algebra (2.4.11) is modified by dropping the requirement (2.4.11b). The definition of the boundary structure $\mathfrak{p}$ in Sec. 2.5 is correspondingly modified by omitting the non-affinity κ from the definition (2.5.3), and omitting the requirement (2.5.2b) from the definition of the equivalence relation. The definition of the field configuration space $\mathscr{F}_\mathfrak{p}$ is modified by omitting the requirement (2.5.4b). The conclusions (2.5.18), (2.5.19) and (2.5.20) then continue to hold. In particular, a key point is that the arguments of Appendix

A.3 continue to apply, and so none of the new symmetry generators χ^i on the null surface correspond to degeneracies of the symplectic form.

In the following subsection 4.3, the conditions (2.5.22b) and (2.5.29) on variations of the metric are no longer valid. Also the non-affinity κ is no longer preserved under variation of the metric, its variation is given by $\delta\kappa = -\Gamma_a \ell^a/2$, from Eqs. (2.5.27), (2.6.10) and (2.3.18).

The computation of charges in Sec. 2.6 is modified as follows. The expression (2.6.7) for the Noether charge is still valid, as is its variation (2.6.13). In Sec. 2.6, the expression (2.6.16) for the presymplectic potential θ_{ijk} is valid, but the subsequent expression (2.6.17) acquires the extra term $-\varepsilon_{ijk}\Gamma_a \ell^a/(16\pi)$, and there is the corresponding correction $\varepsilon_{ijk}\chi^k\Gamma_a\ell^a/(16\pi)$ to Eq. (2.6.19). In the computation of localized charges, we are unable to find a presymplectic potential $\boldsymbol{\Theta}$ satisfying all the requirements listed in Sec. 2.2. Specifically, if we use the choice (2.6.24) of the 3-form $\boldsymbol{\alpha}$, then the the extra term in $\boldsymbol{\theta}$ implies that $\boldsymbol{\Theta}$ no longer vanishes on stationary backgrounds. One could cancel this extra term by adding a term proportional to $\kappa\varepsilon_{ijk}$ to α_{ijk}, but this term is not invariant under the rescaling (4.3.1) as it must be. The expression $\kappa - \pounds_\ell \ln\theta$ is invariant under rescaling, but from Raychaudhuri's equation in vacuum it is equivalent to $\theta/2 + \sigma_{AB}\sigma^{AB}/\theta$ which is not well defined in the limit $\theta \to 0$. It does not appear to be possible to find a presymplectic potential $\boldsymbol{\Theta}$ satisfying all the requirements.

Of course, one can drop the requirements related to stationarity, and choose the same expression (2.6.24) for the 3-form $\boldsymbol{\alpha}$ as before. Then the argument of Appendix A.5 shows that, assuming the localized charges $\boldsymbol{\mathcal{Q}}^{\rm loc}_\xi$ vanish on the reference solution, the expressions (2.6.23) and (2.6.27) for the localized charge are still valid. However, since we are no longer imposing any assumptions related to stationarity, the relation (2.2.29) between the flux $d\boldsymbol{\mathcal{Q}}^{\rm loc}_\xi$ and presymplectic potential $\boldsymbol{\Theta}$ need not hold, and the flux will not vanish on stationary backgrounds. In addition, one could have picked other expressions for $\boldsymbol{\alpha}$, so the expression for the localized charge is not unique. It may be possible in this context to find some other criterion that could be used to determine a unique charge expression.

A.9 Covariant phase space formalism and the Wald-Zoupas charges

The computation of boundary charges and their fluxes makes use of the covariant phase space formalism [60] and the Wald-Zoupas prescription [24]. We quickly review the main ingredients needed for null boundaries, and refer the reader to [119] for details.

Consider a diffeomorphism covariant theory of the metric g_{ab} as a dynamical field which is described by a Lagrangian 4-form $\boldsymbol{L}(g)$ that depends locally and covariantly on g_{ab}. Under perturbations $g \mapsto g + \delta g$ the Lagrangian changes as

$$\delta\boldsymbol{L} = \boldsymbol{E}^{ab}\delta g_{ab} + d\boldsymbol{\theta}(g;\delta g) \tag{A.9.1}$$

where $\boldsymbol{E}^{ab}$ is a 4-form presenting the equations of motion of the theory and the 3-form $\boldsymbol{\theta}(g, \delta g)$ is the presymplectic potential. The 3-form presymplectic current is defined by

$$\boldsymbol{\omega}(g; \delta_1 g, \delta_2 g) = \delta_1 \boldsymbol{\theta}(g; \delta_2 g) - \delta_2 \boldsymbol{\theta}(g; \delta_1 g) \tag{A.9.2}$$

where $\delta_1 g$ and $\delta_2 g$ are any two independent perturbations.

Given a vector field ξ^a, one can then show that

$$\boldsymbol{\omega}(g; \delta g, \pounds_\xi g) = d[\delta \boldsymbol{Q}_\xi - i_\xi \boldsymbol{\theta}(\delta g)] \tag{A.9.3}$$

where the 2-form $\boldsymbol{Q}_\xi$ is the Noether charge [60, 68, 24].

Consider now a null boundary N in the spacetime and a spacelike hypersurface Σ that intersects N at some cross-section S. Integrating eq. (A.9.3) we get

$$\int_\Sigma \boldsymbol{\omega}(g; \delta g, \pounds_\xi g) = \int_S \delta \boldsymbol{Q}_\xi - i_\xi \boldsymbol{\theta}(\delta g) \tag{A.9.4}$$

Two vector fields ξ^a and $\tilde{\xi}^a$ are equivalent representatives of symmetries on N if $\xi^a|_N = \tilde{\xi}^a|_N$ and the right-hand-side of eq. (A.9.4) evaluated with ξ^a and $\tilde{\xi}^a$ are equal for all backgrounds $g \in \overline{\mathscr{F}}$, all perturbations δg within $\overline{\mathscr{F}}$ and all cross-sections S of N. The boundary symmetries on N are then given by vector fields ξ^a factored out by the above equivalence relation.

From the above identity it would be "natural" to define a charge at S associated to a symmetry ξ^a as a function $Q_\xi[S]$ on phase space so that

$$\delta Q_\xi[S] = \int_S \delta \boldsymbol{Q}_\xi - i_\xi \boldsymbol{\theta}(\delta g) \tag{A.9.5}$$

for all perturbations δg within $\overline{\mathscr{F}}$ and all cross-sections S. Unfortunately, in general, the right-hand-side of eq. (A.9.5) is not integrable in phase space and no function Q_ξ satisfying eq. (A.9.5) exists on the phase space. As shown in [24] the integrability condition for the existence of a charge $Q_\xi[S]$ for some symmetry ξ^a is

$$0 = (\delta_1 \delta_2 - \delta_2 \delta_1) Q_\xi = -\int_S i_\xi \boldsymbol{\omega}(g, \delta_1 g, \delta_2 g) \tag{A.9.6}$$

for all perturbations $\delta_1 g, \delta_2 g$ within $\overline{\mathscr{F}}$ and all cross-sections S. This above criteria is not satisfied, even at null infinity in general relativity, except in very special cases [24].

Nevertheless, Wald and Zoupas [24] developed a prescription for defining a modified charge which is always integrable. Define a boundary presymplectic potential $\boldsymbol{\Theta}(g; \delta g)$ for the pullback to N of the presymplectic current,

$$\underline{\boldsymbol{\omega}}(g, \delta_1 g, \delta_2 g) = \delta_1 \boldsymbol{\Theta}(g, \delta_2 g) - \delta_2 \boldsymbol{\Theta}(g, \delta_1 g) \tag{A.9.7}$$

where $\underline{\boldsymbol{\omega}}$ denotes the pullback to N. Then define the *Wald-Zoupas charge* (WZ charge) by

$$\delta \mathcal{Q}_\xi[S] = \int_S \delta \boldsymbol{Q}_\xi - i_\xi \boldsymbol{\theta}(\delta g) + i_\xi \boldsymbol{\Theta}(\delta g) \tag{A.9.8}$$

It can be shown using eqs. (A.9.4) and (A.9.7) that $\delta \mathcal{Q}_\xi[S]$ is integrable in phase space. Thus eq. (A.9.8) determines a function $\mathcal{Q}_\xi[S]$ up to a constant of integration on $\overline{\mathscr{F}}$, which can be fixed by choosing a reference solution g_0 such that $\mathcal{Q}_\xi[S]\big|_{g_0} = 0$ for all symmetries ξ^a and all cross-sections S.

The prescription given by Wald and Zoupas is to choose the 3-form $\boldsymbol{\Theta}$ such that $\boldsymbol{\Theta}(g;\delta g)$ vanishes for all perturbation δg_{ab} for any background g_{ab} which is stationary, and to choose the reference solution g_0 to also be stationary. The consistency conditions for such choices and the ambiguities in them are detailed in [24].

If the above choices can be made then the flux $\mathcal{F}_\xi[\Delta N]$ of the WZ charge $\mathcal{Q}_\xi$ through a part of the null boundary N is given by [24]

$$\mathcal{F}_\xi[\Delta N] = \int_{\Delta N} \boldsymbol{\Theta}(g; \pounds_\xi g) \tag{A.9.9}$$

It can also be shown that the perturbed flux $\delta\mathcal{F}_\xi$ for any symmetry ξ^a and any perturbation δg_{ab} satisfies (see Eq. 29 [24])

$$\delta\mathcal{F}_\xi = \int_N \boldsymbol{\omega}(g; \delta g, \pounds_\xi g) + \int_{\partial N} i_\xi \boldsymbol{\Theta}(\delta g) \tag{A.9.10}$$

If $i_\xi \boldsymbol{\Theta}(\underline{\delta g}) \to 0$ on ∂N for *all* perturbations δg_{ab} then $\mathcal{F}_\xi$ is a function on the covariant phase space $\overline{\mathscr{F}}$ satisfying

$$\delta\mathcal{F}_\xi = \int_N \boldsymbol{\omega}(g; \delta g, \pounds_\xi g) \tag{A.9.11}$$

for all perturbations $\delta \underline{g_{ab}}$, that is, $\mathcal{F}_\xi$ defines a Hamiltonian which generates the flow on the covariant phase space $\overline{\mathscr{F}}$ associated to the symmetry ξ^a.

In [24] this procedure was applied to the asymptotic symmetries at null infinity in general relativity to derive the charges and symmetries for the BMS algebra. The case of finite null boundaries in vacuum general relativity was handled in [119], where it as shown that the notion of symmetries defined below eq. (A.9.4) coincides with those defined in section 3.2 and the Wald-Zoupas prescription gives the charges and fluxes described in eqs. (3.2.20) and (3.2.21). In [119] the reference solution (used in the Wald-Zoupas prescription) was chosen to be the horizon of a Schwarzschild black hole in the limit that the mass tends to zero. It was shown that this reference solution satisfies all the criteria given by Wald and Zoupas [24]. In this paper, we simply adopt the formulae for the charges and fluxes from [119] and do not analyze the choice of reference solution in detail.

A.10 Structure of $\mathfrak{g}_{\mathrm{CD}}$ as a central extension

In this section we explore the structure of the summetry algebra $\mathfrak{g}_{\mathrm{CD}}$ of the causal diamond as a non-trivial extension of $\mathfrak{g}_{\mathrm{CD}}/\mathfrak{b}_0$ by the boosts $\mathfrak{b}_0$.

Recall from section 3.3 that the elements of $\mathfrak{g}_{\rm CD} \cong \mathrm{diff}(\mathbb{S}^2) \ltimes \mathfrak{b}$ are of the form (β, X^A) where β is a smooth function and X^A a vector feld on $\mathbb{S}^2$. The central elements (i.e. those which commute with all other elements in $\mathfrak{g}_{\rm CD}$) of boosts in $\mathfrak{b}_0$ are the ones given by $(\beta = \text{constant}, 0)$. Consider the quotient $\mathfrak{g}_{\rm CD}/\mathfrak{b}_0$ which consists of equivalence classes given by the relation $(\beta, X^A) \sim (\beta + \text{constant}, X^A)$. Thus, $\mathfrak{g}_{\rm CD}$ is a central extension of $\mathfrak{g}_{\rm CD}/\mathfrak{b}_0$ by the abelian algebra $\mathfrak{b}_0$. We now show that this central extension is, in fact, a non-trivial central extension. What this means is the following:

Does the bracket of two representative elements belonging to $\mathfrak{g}_{\rm CD}/\mathfrak{b}_0$, computed in $\mathfrak{g}_{\rm CD}$, have a non-vanishing $\mathfrak{b}_0$-part?

If the answer is 'no' then the central extension is trivial and $\mathfrak{g}_{\rm CD}$ will be a direct product of $\mathfrak{g}_{\rm CD}/\mathfrak{b}_0$ and $\mathfrak{b}_0$. If the answer is 'yes' then $\mathfrak{g}_{\rm CD}$ has the structure of a non-trivial central extension of $\mathfrak{g}_{\rm CD}/\mathfrak{b}_0$ by $\mathfrak{b}_0$.[3]

Since the symmetry algebra $\mathfrak{g}_{\rm CD}$ is independent of the metric q_{AB} on N the null boundary of the causal diamond, we can deduce its structure in any choice of metric, in particular it is convenient to choose q_{AB} to be the metric of a unit-sphere in the standard (θ, ϕ) coordinates on $\mathbb{S}^2$. We now compute the bracket of any two elements in $\mathfrak{g}_{\rm CD}/\mathfrak{b}_0$, with the only relevant case being the bracket between an element of $\mathfrak{b}/\mathfrak{b}_0$ with an element of $\mathrm{diff}(\mathbb{S}^2)$ which gives (see eq. (3.3.7))

$$[(\beta_1, 0), (0, X_2^A)] = (\beta, 0) \quad \text{with} \quad \beta = -X_2^A \partial_A \beta_1 \tag{A.10.1}$$

To answer the above question we expand in terms of spherical harmonics $Y_{l,m}(\theta, \phi)$. Note that elements of $\mathfrak{b}_0$ are purely $l = 0$ spherical harmonics. Now let β_1 be a l_1-harmonic with $l_1 \geq 1$ and X_2^A be a l_2-vector harmonic. We can write $X_2^A = \partial^A X + \epsilon^{AB}\partial_B \tilde{X}$ where $X, \tilde{X}$ are l_2-harmonics with $l_2 \geq 1$. We want determine whether β can have a non-trivial $l = 0$ part, that is a non-vanishing constant piece.

Before considering the general case, we note the following example

$$\beta_1 = \cos\theta \quad ; \quad X_2^A \equiv -\sin\theta \partial_\theta \implies \beta = \sin^2\theta \tag{A.10.2}$$

Thus, β has non-vanishing $l = 0, 2$ parts in terms of spherical harmonics. This already shows that $\mathfrak{g}_{\rm CD}$ is a non-trivial central extension of $\mathfrak{g}_{\rm CD}/\mathfrak{b}_0$ by $\mathfrak{b}_0$.

For the general situation, first consider the case $X = 0$, $X_2^A = \epsilon^{AB}\partial_B \tilde{X}$. Then we have, by integrating-by-parts on the unit-sphere (we leave the area element implicit for notational convenience)

$$\int \beta \, \overline{Y}_{l=0,m} \propto \int \beta = -\int \epsilon^{AB}\partial_B \tilde{X} \partial_A \beta_1 = -\int \partial_B(\epsilon^{AB}\tilde{X}\partial_A\beta_1) = 0 \tag{A.10.3}$$

Thus, β always has a vanishing $l = 0$ component for $X_2^A = \epsilon^{AB}\partial_B \tilde{X}$.

[3] In a more mathematical language, every central extension of $\mathfrak{g}_{\rm CD}/\mathfrak{b}_0$ by the abelian algebra $\mathfrak{b}_0$ corresponds to a 2-cocycle in the cohomology group $H^2(\mathfrak{g}_{\rm CD}/\mathfrak{b}_0, \mathfrak{b}_0)$ (see Sec. IV.2 [387]). Our computation in this section amounts to showing that the cocycle which gives the Lie algebra structure of $\mathfrak{g}_{\rm CD}$ is non-trivial.

Next consider the case $\tilde{X} = 0$, $X^A = \partial^A X$ we have

$$\begin{aligned}\int \beta\, \overline{Y}_{l=0,m} \propto \int \beta = -\int \partial^A X \partial_A \beta_1 = \int X \partial^2 \beta_1 \\ = -l_1(l_1+1)\int X\beta_1\end{aligned} \tag{A.10.4}$$

Expanding the functions X and β_1 in terms of the corresponding spherical harmonics, and using the orthonormality and completeness of the spherical harmonic basis we conclude that, the right-hand-side is non-vanishing if and only if

$$l_1 = l_2 \geq 1 \quad ; \quad m_2 = -m_1 \tag{A.10.5}$$

Thus, β has a non-vanishing constant ($l = 0$) part whenever eq. (A.10.5) is satisfied, which has many solutions; taking $l_1 = l_2 = 1$ and $m_1 = m_2 = 0$ gives the above example eq. (A.10.2). Thus, $\mathfrak{g}_{\rm CD}$ is a non-trivial central extension of $\mathfrak{g}_{\rm CD}/\mathfrak{b}_0$ by $\mathfrak{b}_0$ as we wished to show.

A.11 Commutation relation for anomaly operator

Here, we give a proof of the relation (4.2.2) satisfied by the anomaly operator $\Delta_{\hat{\xi}}$. By writing out the commutator, we find

$$[\Delta_{\hat{\xi}}, \Delta_{\hat{\zeta}}] = [L_{\hat{\xi}}, L_{\hat{\zeta}}] + [\pounds_\xi, \pounds_\zeta] - [L_{\hat{\xi}}, \pounds_\zeta] - [\pounds_\xi, L_{\hat{\zeta}}] = L_{[\hat{\xi},\hat{\zeta}]_{\mathcal{S}}} + \pounds_{[\xi,\zeta]}. \tag{A.11.1}$$

Here, $[\hat{\xi}, \hat{\zeta}]_{\mathcal{S}}$ is the Lie bracket of vector fields on $\mathcal{S}$, and to arrive at the second equality, we use the fact that $[L_{\hat{\xi}}, \pounds_\zeta] = 0$, since ζ^a is field-independent, $\delta\zeta^a = 0$. The field space bracket can be related to the spacetime bracket simply by contracting with a covariant field δg_{ab},

$$\begin{aligned}I_{[\hat{\xi},\hat{\zeta}]_{\mathcal{S}}}\delta g_{ab} &= L_{\hat{\xi}} I_{\hat{\zeta}}\delta g_{ab} - I_{\hat{\zeta}} L_{\hat{\xi}}\delta g_{ab} = L_{\hat{\xi}}\pounds_\zeta g_{ab} - I_{\hat{\zeta}}\pounds_\xi \delta g_{ab} = \pounds_\zeta\pounds_\xi g_{ab} - \pounds_\xi\pounds_\zeta g_{ab} = -\pounds_{[\xi,\zeta]} g_{ab} \\ &= -I_{\widehat{[\xi,\zeta]}}\delta g_{ab},\end{aligned} \tag{A.11.2}$$

and hence we derive

$$[\hat{\xi}, \hat{\zeta}]_{\mathcal{S}} = -\widehat{[\xi,\zeta]} \tag{A.11.3}$$

for field-independent generators. Applying this to (A.11.1) yields the desired identity

$$[\Delta_{\hat{\xi}}, \Delta_{\hat{\zeta}}] = -\Delta_{\widehat{[\xi,\zeta]}}. \tag{A.11.4}$$

It is also useful to note the commutators with $L_{\hat{\zeta}}$ and $\pounds_\zeta$,

$$[\Delta_{\hat{\xi}}, L_{\hat{\zeta}}] = -L_{\widehat{[\xi,\zeta]}} \tag{A.11.5}$$

$$[\Delta_{\hat{\xi}}, \pounds_\zeta] = -\pounds_{[\xi,\zeta]}. \tag{A.11.6}$$

A.12 Derivation of the bracket identity

Here, we derive the main identity for the Barnich-Troessaert bracket and the resulting extension $K_{\xi,\zeta}$. To be completely general, we do not assume that $\Delta_{\hat{\xi}}\beta = 0$. We first work with the definition (4.2.20) of the quasilocal charges, so that all of $\Delta_{\hat{\xi}}\beta$ is contained in the flux. The Barnich-Troessaert bracket is then

$$\{H_\xi, H_\zeta\} = I_{\hat{\xi}}\delta H_\zeta - \int_{\partial\Sigma}\left(i_\xi I_{\hat{\zeta}}\mathcal{E} - I_{\hat{\zeta}}\Delta_{\hat{\xi}}\beta\right) \equiv \int_{\partial\Sigma} m_{\xi,\zeta}, \tag{A.12.1}$$

where we have written the final result in terms of a local 2-form $m_{\xi,\zeta}$ to be integrated. We can calculate the expression for $m_{\xi,\zeta}$ on $\mathcal{N}$ as follows:

$$\begin{aligned}
m_{\xi,\zeta} &= I_{\hat{\xi}}\delta Q_\zeta + I_{\hat{\xi}} i_\zeta \delta\ell - I_{\hat{\xi}}\delta I_{\hat{\zeta}}\beta - i_\xi I_{\hat{\zeta}}\boldsymbol{\theta} - i_\xi I_{\hat{\zeta}}\delta\ell + I_{\hat{\zeta}} i_\xi d\beta + I_{\hat{\zeta}}\Delta_{\hat{\xi}}\beta \\
&= -Q_{[\xi,\zeta]} + i_\xi dQ_\zeta - i_\xi I_{\hat{\zeta}}\boldsymbol{\theta} + i_\zeta\Delta_{\hat{\xi}}\ell - i_\xi\Delta_{\hat{\zeta}}\ell + i_\zeta \pounds_\xi \ell - i_\xi \pounds_\zeta \ell - L_{\hat{\xi}} I_{\hat{\zeta}}\beta + I_{\hat{\zeta}} L_{\hat{\xi}}\beta \\
&\qquad\qquad + d\left(i_\xi Q_\zeta - i_\xi I_{\hat{\zeta}}\beta\right) \\
&= -Q_{[\xi,\zeta]} - i_{[\xi,\zeta]}\ell + I_{\widehat{[\xi,\zeta]}}\beta - i_\xi\Delta_{\hat{\zeta}}\ell + i_\zeta\Delta_{\hat{\xi}}\ell - i_\xi i_\zeta(L + d\ell) + d\left(i_\xi Q_\zeta + i_\xi i_\zeta \ell - i_\xi I_{\hat{\zeta}}\beta\right)
\end{aligned} \tag{A.12.2}$$

where the first equality used the relation (4.2.14) for $\mathcal{E}$, the second equality expanded the variation of Q_ζ via

$$I_{\hat{\xi}}\delta Q_\zeta = L_{\hat{\xi}} Q_\zeta = \pounds_\xi Q_\zeta + \Delta_{\hat{\xi}} Q_\zeta = i_\xi dQ_\zeta + d i_\xi Q_\zeta - Q_{[\xi,\zeta]}, \tag{A.12.3}$$

the third equality employed the identities

$$i_\zeta \pounds_\xi \ell - i_\xi \pounds_\zeta \ell = -i_{[\xi,\zeta]}\ell + i_\xi i_\zeta d\ell + d i_\xi i_\zeta \ell \tag{A.12.4}$$

and

$$-L_{\hat{\xi}} I_{\hat{\zeta}}\beta + I_{\hat{\zeta}} L_{\hat{\xi}}\beta = -I_{[\hat{\xi},\hat{\zeta}]_{\mathcal{S}}}\beta = I_{\widehat{[\xi,\zeta]}}\beta \tag{A.12.5}$$

where the $\mathcal{S}$ Lie bracket $[\hat{\xi},\hat{\zeta}]_{\mathcal{S}}$ is related to the spacetime Lie bracket for field-independent generators by a minus sign according to (A.11.3). By integrating (A.12.2) over $\partial\Sigma$, we arrive at the desired identity for the bracket,

$$\{H_\xi, H_\zeta\} = -\left[H_{[\xi,\zeta]} + \int_{\partial\Sigma}\left(i_\xi\Delta_{\hat{\zeta}}\ell - i_\zeta\Delta_{\hat{\xi}}\ell\right)\right] \tag{A.12.6}$$

noting that the exact term in (A.12.2) integrates to zero and $i_\xi i_\zeta(L + d\ell)$ pulls back to zero since ξ^a and ζ^a are tangent to the hypersurface $\mathcal{N}$, so their transverse components to $\partial\Sigma$ must be parallel to each other.

Note that if we examine the steps leading to (A.12.2), we see that the terms involving β do not mix with the other terms, i.e. we have an independent identity involving only β,

$$-I_{\hat{\xi}}\delta I_{\hat{\zeta}}\beta + I_{\hat{\zeta}} i_\xi d\beta + I_{\hat{\zeta}}\Delta_{\hat{\xi}}\beta = I_{\widehat{[\xi,\zeta]}}\beta - d i_\xi I_{\hat{\zeta}}\beta. \tag{A.12.7}$$

This immediately implies that different choices of β in the decomposition (4.2.14) of $\boldsymbol{\theta}$ do not affect the algebra or extension $K_{\xi,\zeta}$. Stated differently, different choices of how to separate off the corner term β from the flux $\mathcal{E}$ correspond to changes in the charges associated with trivial extensions, $H_\xi \to H_\xi + \int_{\partial\Sigma} I_{\hat{\xi}}(\beta - \beta')$. This explains why the choice of corner term did not enter into the results for the central charges reported in [149, 171].

When utilizing the corner improvement described in appendix A.13, the modification of the charges and bracket amounts to shifting $\{H_\xi, H_\zeta\}$ by the term,

$$-\int_{\partial\Sigma}\Big(I_{\hat{\xi}}\delta\Delta_{\hat{\zeta}}c - I_{\hat{\zeta}}\delta\Delta_{\hat{\xi}}c\Big) \tag{A.12.8}$$

with c defined by equation (A.13.1). Then noting that the integrand can be written

$$\begin{aligned}-L_{\hat{\xi}}\Delta_{\hat{\zeta}}c + L_{\hat{\zeta}}\Delta_{\hat{\xi}}c &= -\pounds_\xi\Delta_{\hat{\zeta}}c - \Delta_{\hat{\xi}}\Delta_{\hat{\zeta}}c + \pounds_\zeta\Delta_{\hat{\xi}}c + \Delta_{\hat{\zeta}}\Delta_{\hat{\xi}}c \\ &= \Delta_{\widehat{[\xi,\zeta]}}c - i_\xi\Delta_{\hat{\zeta}}dc + i_\zeta\Delta_{\hat{\xi}}dc - d(i_\xi\Delta_{\hat{\zeta}}c - i_\zeta\Delta_{\hat{\xi}}c)\end{aligned} \tag{A.12.9}$$

where we have applied the relation (A.11.4). The first term is the contribution to improved charge $-H_{[\xi,\zeta]}$, while the second and third terms correct $K_{\xi,\zeta}$, and the last term integrates to zero. This then leads to the expression (A.13.7) for the central charge using the corner improvement.

Finally, we verify the cocycle identity (4.2.35) that must be satisfied by $K_{\xi,\zeta}$. Using the expression (4.2.36) for $K_{\xi,\zeta}$ as a trivial field-dependent cocycle, we have

$$I_{\hat{\chi}}\delta K_{\xi,\zeta} = \int_{\partial\Sigma}\Big(i_\xi L_{\hat{\chi}}L_{\hat{\zeta}}\ell - i_\zeta L_{\hat{\chi}}L_{\hat{\xi}}\ell - i_{[\xi,\zeta]}I_{\hat{\chi}}\delta\ell\Big) \tag{A.12.10}$$

Then adding cyclic permutations we get

$$I_{\hat{\chi}}\delta K_{\xi,\zeta} + \text{cyclic} = \int_{\partial\Sigma}\Big(i_\zeta I_{\widehat{[\chi,\xi]}}\delta\ell - i_{[\chi,\xi]}I_{\hat{\zeta}}\delta\ell - i_{[\zeta,[\chi,\xi]]}\ell\Big) + \text{cyclic}, \tag{A.12.11}$$

where we note that the cyclic contributions of the form $i_{[\zeta,[\chi,\xi]]}\ell$ actually sum to zero by the Jacobi identity. They are included to put the right hand side into the form $K_{[\chi,\xi],\zeta}$ + cyclic, which verifies the cocycle identity (4.2.35).

A.13 Corner improvement

In deriving the expression (4.2.20) for the quasilocal charges, we assumed that the corner term was covariant, $\Delta_{\hat{\xi}}\beta = 0$. Although we will find that for a null surface this condition is satisfied, it is still interesting to consider the case where the corner term is not covariant, as it leads to a useful improvement to the expression for the quasilocal charges and the extensions $K_{\xi,\zeta}$. Another reason to consider this case is to resolve an additional ambiguity that arises in the decomposition (4.2.14) of θ. Fixing the form of $\mathcal{E}$ still allows us to make the shifts

$\ell \to \ell + da$, $\beta \to \beta + \delta a$. Under this transformation, the quasilocal charge transforms as $H_\xi \to H_\xi - \int_{\partial\Sigma} \Delta_{\hat{\xi}} a$, and hence H_ξ is sensitive to this ambiguity if a is not covariant. Since we are allowing for noncovariance in ℓ, there is no reason to assume that β and a cannot similarly be constructed from noncovariant objects.

To handle the case where β is not covariant, we return to equation (4.2.19) and find that we need a way to separate $\Delta_{\hat{\xi}}\beta$ into a contribution to the charge and a contribution to the flux. Similar to how we handled θ, we look for a decomposition of β at $\partial\Sigma$ of the form

$$\beta = -\delta c + \varepsilon. \tag{A.13.1}$$

Note that this decomposition should be made on $\mathcal{N}$ without pulling back β to $\partial\Sigma$. In principle we could also include an exact contribution $d\gamma$ in the decomposition, but these will always end up integrating to zero on $\partial\Sigma$.[4] This decomposition allows us to identify ε with a corner contribution to the flux, while c is the contribution to the charge.

The improved quasilocal charge can then be written

$$H_\xi = \int_{\partial\Sigma} \Big(Q_\xi + i_\xi \ell - I_{\hat{\xi}}\beta - \Delta_{\hat{\xi}} c \Big) \tag{A.13.2}$$

$$= \int_{\partial\Sigma} \Big(Q_\xi - I_{\hat{\xi}}\varepsilon + i_\xi(\ell + dc) \Big) \tag{A.13.3}$$

and its variation satisfies an equation similar to (4.2.21),

$$\delta H_\xi = -I_{\hat{\xi}}\Omega + \int_{\partial\Sigma} \Big(i_\xi \mathcal{E} - \Delta_{\hat{\xi}}\varepsilon \Big). \tag{A.13.4}$$

The continuity equation for the change in the charges between two cuts of $\mathcal{N}$ is

$$H_\xi(S_2) - H_\xi(S_1) = \int_{N_1^2} \Big(I_{\hat{\xi}}\mathcal{E} - \Delta_{\hat{\xi}}(\ell + dc) \Big), \tag{A.13.5}$$

with $F_\xi = \int_{\mathcal{N}_1^2} I_{\hat{\xi}}\mathcal{E}$ still interpreted as the flux, but with an anomalous source now given by $\int_{\mathcal{N}_1^2} \Delta_{\hat{\xi}}(\ell + dc)$. Finally, the Barnich-Troessaert bracket is defined for these charges as

$$\{H_\xi, H_\zeta\} = -I_{\hat{\xi}} I_{\hat{\zeta}}\Omega + \int_{\partial\Sigma} \Big(I_{\hat{\xi}}(i_\zeta\mathcal{E} - \Delta_{\hat{\zeta}}\varepsilon) - I_{\hat{\zeta}}(i_\xi\mathcal{E} - \Delta_{\hat{\xi}}\varepsilon) \Big) \tag{A.13.6}$$

which again satisfies (4.2.31) with the extension given by

$$K_{\xi,\zeta} = \int_{\partial\Sigma} \Big(i_\zeta \Delta_{\hat{\xi}}(\ell + dc) - i_\xi \Delta_{\hat{\zeta}}(\ell + dc) \Big). \tag{A.13.7}$$

[4]However, this type of contribution may be relevant when considering surfaces with codimension-3 defects, such as caustics on a null surface, or when considering singular symmetry generators, such as superrotations [233, 388].

As before, the ambiguities in the decomposition are fixed once we have specified the form of the corner flux term ε. We expect in this case a Dirichlet condition would fix the form of ε, and arguments based on the variational principle should relate the matching to codimension-2 junction conditions, such as those considered in [237]. Once this is done, the shift, $\beta \to \beta + \delta a$ causes $c \to c - a$, while ε is invariant. Hence, the combination $\ell + dc$ is also insensitive to this shift, and can be viewed as the improvement to the boundary Lagrangian ℓ by a contribution from a corner Lagrangian c. We see that many of the improved formulas are obtained from those of previous sections by merely replacing ℓ with its invariant form, $\ell + dc$.

Note that the formula for the improved quasilocal charges (A.13.3) can be used even in the case that β is already covariant. This could be useful in cases where one wishes for the corner flux to depend on the geometry of $\partial\Sigma$, in which case it will not be covariant with respect to transformations that move $\partial\Sigma$, even if β originally was.

A.14 Checking extension is central

As discussed in section 4.2, the Barnich-Troessaert bracket of quasilocal charges generically produces an abelian extension of the associated algebra of vector fields. We found that for the generators ξ^a_n and $\bar{\xi}^a_n$, all of the extensions $K_{m,n}$ vanished in the Killing horizon background except for $K_{m,-m}$. However, the quantities $K_{m,n}$ have nonzero variations, so in principle their brackets with the L_n generators could show that the algebra is a nontrivial abelian extension of the Witt algebra.[5] Here we will demonstrate that in fact the extension is central, verifying that the resulting algebra is the Virasoro algebra.

The quantity to compute for χ^c, ξ^c and ζ^c three of the ξ^a_n generators is (ignoring factors of $8\pi G$)

$$I_{\hat{\chi}}\delta\Big(i_\xi \Delta_{\hat{\zeta}}\ell\Big) = -i_\xi I_{\hat{\chi}}\delta\big(\eta l^c\big)\nabla_c w_\zeta \tag{A.14.1}$$

since $\delta w_\zeta = 0$, which follows from $\delta w_\zeta l_a = -\delta\Delta_{\hat{\zeta}} l_a = \Delta_{\hat{\zeta}}\delta l_a = 0$. Then we have

$$\begin{aligned} I_{\hat{\chi}}\delta\big(\eta l^c\big) &= \pounds_\chi\big(\eta l^c\big) + \Delta_{\hat{\chi}}\big(\eta l^c\big) = (\pounds_\chi\eta)l^c + \eta\pounds_\chi l^c = (I_{\hat{\chi}}\delta\eta)l^c - (\Delta_{\hat{\chi}}\eta)l^c + \eta[\chi, l]^c \\ &= \eta\Big(-w_\chi l^c + [\chi, l]^c\Big) \\ &= -\eta\left(w_\chi l^c + in\frac{\kappa}{\alpha}\chi^c\right) \end{aligned} \tag{A.14.2}$$

using that $\Delta_{\hat{\chi}}(\eta l^c) = 0$ for any vector that preserves the horizon, and $I_{\hat{\chi}}\delta\eta \mathrel{\widehat{=}} 0$ for the Virasoro vector fields. The last line uses that $l^c = \frac{\kappa}{\alpha}\xi^c_0 + \frac{\kappa}{\bar{\alpha}}\bar{\xi}^c_0$ to compute the bracket with χ^c, and has chosen $\chi^c = \xi^c_n$.

[5] See [389] for a classification of these abelian extensions.

Now setting $\zeta^a = \chi_m^a$, and using the expression (4.4.34) for w_ζ, w_χ, we have that

$$\left(w_\chi l^c + in\frac{\kappa}{\alpha}\chi^c\right)\nabla_c w_\zeta = -inm^2\frac{\kappa}{\alpha}\left(W^+\right)^{\frac{i(m+n)}{\alpha}} + inm^2\frac{\kappa}{\alpha}\left(W^+\right)^{\frac{i(m+n)}{\alpha}} = 0 \tag{A.14.3}$$

using that $l^c = \kappa V\partial_V^c$ on $\mathcal{H}^+$ in Kruskal coordinates (4.4.10), and $\chi^c = \alpha\left(W^+\right)^{\frac{in}{\alpha}}\left(W^+\partial_+^c + \frac{in}{2\alpha}y\partial_y^c\right)$ in conformal coordinates (4.4.22). This shows that the integrand in $I_{\bar{\chi}}\delta K_{\xi,\zeta}$ vanishes. According to the definition (4.2.33) for the Barnich-Troessaert bracket of $K_{\xi,\zeta}$ with the other generators, we see that this implies that $K_{\xi,\zeta}$ commutes with all generators, and hence must be central. Thus we arrive at the advertised result, that we have the Virasoro algebra as our extension, as opposed to some other abelian extension. The analysis on the past horizon for the $\bar{\xi}_n^a$ generator is analogous, and similarly confirms that the $\bar{L}_n$ generators form a Virasoro algebra.

A.15 Calculation of $\langle Q^2\rangle$ and $\langle M^2\rangle$

In this appendix we describe in more detail the calculations of $\langle Q^2\rangle$ described in section 5.2 and of $\langle M^2\rangle$ described in section 5.3.

For the QED calculation, we start with eq.(5.2.14). Inserting the expression for the current 2-point function, Eq. (5.2.2), and evaluating the $d^3\vec{x}$ integral, we get

$$\langle Q^2\rangle = \kappa\int dx^0 d\Delta^0 4\pi^2 d\Delta\frac{\left(\Delta^2+(\Delta^0)^2\right)\mathrm{Vol}(r_1(t),r_2(t),\Delta)}{\left(\Delta^2-(\Delta^0)^2\right)^2}w(x^0)w(x^0-\Delta^0), \tag{A.15.1}$$

Note that here $\Delta = |\vec{\Delta}|$, whereas in the main text Δ denoted a four-vector.

The radii, as functions of time, are specified by

$$\begin{aligned} r_1(t) &= r_B + \alpha(t-t_B), \\ r_2(t) &= r_B + \alpha(t-\Delta^0-t_B), \end{aligned} \tag{A.15.2}$$

$w(t)$ is given in Eq.(5.2.10) and $\mathrm{Vol}(r_1,r_2,\Delta)$ is the volume of the intersection of two spheres of radii r_1 and r_2 whose centers are separated a distance Δ. Explicitly, the volume formula is

$$\mathrm{Vol}(r_1,r_2,\Delta) = \frac{\pi(-\Delta+r_1+r_2)^2\left(\Delta^2-3\left({r_1}^2+{r_2}^2\right)+2\Delta(r_1+r_2)+6r_1r_2\right)}{12\Delta}, \tag{A.15.3}$$

for $|r_1-r_2|\le\Delta\le r_1+r_2$. For $\Delta > r_1+r_2$, the spheres do not intersect, and so $\mathrm{Vol}(r_1,r_2,\Delta) = 0$. For $\Delta < |r_1-r_2|$, one ball is inside the other and so $\mathrm{Vol}(r_1,r_2,\Delta) = \frac{4}{3}\pi\min(r_1,r_2)^3$. Evaluating the Δ integral in Eq. (A.15.1), we get

$$\begin{aligned}\langle Q^2\rangle = \frac{16\pi^6\kappa}{15}\int dx^0 d\Delta^0 &\frac{{r_1}^3{r_2}^3\left(-5(\Delta^0)^4+2(\Delta^0)^2\left({r_1}^2+{r_2}^2\right)+3\left({r_1}^2-{r_2}^2\right)^2\right)}{(\Delta^0+r_1-r_2)^3(\Delta^0+r_1+r_2)^3(-\Delta^0+r_1-r_2)^3(-\Delta^0+r_1+r_2)^3} \\ &\times\frac{\delta t^2/\pi^2}{\left(\delta t^2+x^{02}\right)\left(\delta t^2+(x^0-(\Delta^0)^2)\right)}.\end{aligned} \tag{A.15.4}$$

We now choose contours to evaluate, in turn, the Δ^0 and the x^0 integrals. Keep in mind that at this step the expressions for r_1 and r_2, Eq.(A.15.2), need to be explicitly inserted. Seen as a function on the complex Δ^0 plane, the integrand in Eq. (A.15.4) has four branch points, all on the real axis, and two simple poles, at $\Delta^0 = x^0 \pm i\delta t$. We choose a contour that goes along the real axis, with infinitesimal deformations around the branch points to avoid them, and then close along a semi-circle on the upper half-plane (See Figure A.1). This contour picks up a residue at $\Delta^0 = x^0 + i\delta t$, thus yielding

$$
\begin{aligned}
\langle Q^2\rangle = -\int & dx^0 \pi^4 \delta t \kappa \times 8((\alpha-1)r_B + i\delta t\alpha)((\alpha-1)r_B - \alpha x^0) \\
& \times -\left(\alpha^2-1\right)\delta t^2 - 2i\delta t(x^0 - (\alpha-1)\alpha r_B) + (\alpha-1) \\
& \times \left(2(\alpha-1)r_B^2 - 2\alpha r_B x^0 + (\alpha+1)(x^0)^2\right) \\
& \times \frac{1}{(\alpha-1)(\alpha+1)(\delta t(\alpha+1) - i(\alpha-1)(2r_B - x^0))} \\
& \times \frac{1}{(\delta t(\alpha-1) - i(2(\alpha-1)r_B - (\alpha+1)x^0))} \\
& \times \frac{1}{12\pi^3(\delta t - ix^0)^3(\delta t + ix^0)} \\
& \frac{(\delta t - ix^0)^2 \log\left(\frac{\left(\delta t^2 + (-2(\alpha-1)r_B + \alpha(x^0 - i\delta t))^2 - 2i\delta t x^0 - (x^0)^2\right)^2}{\left(\alpha^2-1\right)^2 (x^0 + i\delta t)^4}\right)}{12\pi^3(\delta t - ix^0)^3(\delta t + ix^0)}
\end{aligned} \tag{A.15.5}
$$

Looking at the integrand above as a function of x^0 on the complex plane we see that the branch points, in the limit of interest ($\alpha \to 1_-$), do not lie above the real line. Thus, the same contour prescription can be used to evaluate the x^0 integral, which now picks up a residue only at the simple pole at $x^0 = i\delta t$. Doing so, and using Eq.(5.2.12) to replace δt to δu, gives

$$
\kappa\pi^2 \frac{\begin{aligned} -\alpha^2\left(\alpha^2-2\right)\delta u^4 + 8(\alpha-1)^4\delta u^2 {r_B}^2 - \left(1-\alpha^2\right)\delta u^2\left(\delta u^2 + 4(\alpha-1)^4 {r_B}^2\right) \\ \times \log\left(\frac{\left(\delta u^2 + 4(\alpha-1)^4 {r_B}^2\right)^2}{\left(\alpha^2-1\right)^2 \delta u^4}\right) + 16(\alpha-1)^8 {r_B}^4 \end{aligned}}{12(\alpha-1)(\alpha+1)\delta u^2\left(\delta u^2 + 4(\alpha-1)^4 {r_B}^2\right)}. \tag{A.15.6}
$$

The series expansion of this at large r_B gives the result in Eq. (5.2.15). We have also checked that this agrees with the result of numerically integrating Eq. (A.15.4).

The calculation of energy fluctuation in the null infinity limit parallels the calculation above. For concreteness, let's consider a scalar field, and take as our starting point Eq. (5.3.4). Inserting Eq. (5.3.3), and evaluating the $d^3\vec{x}$ integral, we get

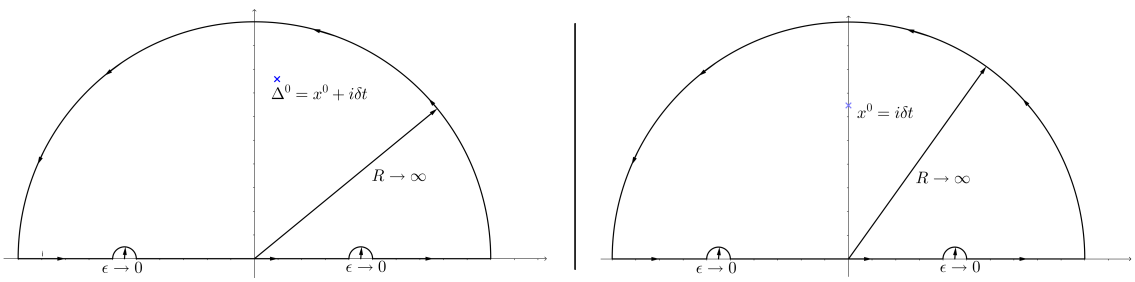


Figure A.1: In the Δ^0 integral (left diagram), the contour avoids the branch points on the real axis and picks up a residue at the simple pole at $\Delta^0 = x^0 + i\delta t$. In the x^0 integral (right diagram), a similar contour is used. It now picks up a residue at the simple pole at $x^0 = i\delta t$.

$$\langle M^2 \rangle = \int dx^0 d\Delta^0 4\pi^2 d\Delta \, \mathrm{Vol}(r_1, r_2, \Delta) 8 \left(30\xi^2 - 10\xi + 1\right) \frac{3\vec{\Delta}^4 + 10\Delta_0^2\vec{\Delta}^2 + 3\Delta_0^4}{\left(\Delta_0^2 - \vec{\Delta}^2\right)^6} w(x^0) w(\Delta^0). \tag{A.15.7}$$

Evaluating the Δ integral gives

$$\langle M^2 \rangle = -\frac{\left(30\xi^2 - 10\xi + 1\right) 128\delta t^2 {r_1}^3 {r_2}^3 \left(-5(\Delta^0)^4 + 2(\Delta^0)^2 \left({r_1}^2 + {r_2}^2\right) + 3\left({r_1}^2 - {r_2}^2\right)^2\right)}{\begin{array}{c} 15\left(\delta t^2 + (x^0)^2\right)\left(\delta t^2 + (x^0 - \Delta^0)^2\right)\left(-\Delta^0 + r_1 - r_2\right)^3 \\ \times \left(-\Delta^0 + r_1 + r_2\right)^3 (\Delta^0 + r_1 - r_2)^3 (\Delta^0 + r_1 + r_2)^3 \end{array}} \tag{A.15.8}$$

Following the same contour prescription as before (see Figure A.1), the Δ^0 integral picks up the residue at $\Delta^0 = t + ia$ and evaluates to

$$\begin{aligned} \langle M^2 \rangle = {} & 128\left(30\xi^2 - 10\xi + 1\right)\pi\delta t((\alpha - 1)r_B + i\delta t\alpha)^3(-\alpha r_B + r_B + \alpha x^0)^3 \\ & \times \frac{1}{15(\alpha - 1)^3(\alpha + 1)^3(\delta t - ix^0)^5(\delta t + ix^0)} \\ & \times \frac{1}{(\delta t(\alpha + 1) - i(\alpha - 1)(2r_B - x^0))^3(\delta t(\alpha - 1) - i(2(\alpha - 1)r_B - (\alpha + 1)x^0))^3} \\ & \times [\left(3\alpha^4 + 2\alpha^2 - 5\right)\delta t^2 + 2i\delta t\left(\left(3\alpha^4 + 5\right)x^0 - 2\alpha\left(3\alpha^3 - 3\alpha^2 + \alpha - 1\right)r_B\right) \\ & (\alpha - 1)\left(4\left(3\alpha^3 - 3\alpha^2 + \alpha - 1\right)r_B^2 - 4\left(3\alpha^3 + \alpha\right)r_B x^0 + \left(3\alpha^3 + 3\alpha^2 + 5\alpha + 5\right)(x^0)^2\right)] \end{aligned} \tag{A.15.9}$$

A similar contour can be used for the x^0 integral now, which picks up a residue at $t = ia$, and gives the answer in Eq. (5.3.7).

A.16 Modified Ward identity

In this Appendix, we prove the following identity:

$$\int d^{d-2}y' \langle \Sigma_n^0 \hat{D}_+(y')\hat{D}_+(y)T_{--}(w,\overline{w},0)\rangle = -\partial_{\overline{w}}\langle \Sigma_n^0 \hat{D}_+(y)T_{--}(w,\overline{w},0)\rangle. \tag{A.16.1}$$

This is similar to the defect CFT ward identity of [315] except there is another insertion of the displacement operator. A priori it is not obvious that some form of the Ward identity carries through in the case where more than one operator is a defect operator. We will argue essentially that the second insertion of a $\hat{D}_+$ just comes along for the ride.

To show this, first we write the displacement operator as a stress tensor integrated around the defect:

$$\hat{D}_+(y) = i\oint d\overline{z}\; T_{++}(0,\overline{z},y) \tag{A.16.2}$$

where we have suppressed the sum over replicas to avoid clutter. We will then argue that the following equality holds

$$\begin{aligned} &i\lim_{\varepsilon\to 0}\oint_{\varepsilon>|\overline{z}|} d\overline{z}\int_{|y-y'|>\epsilon} d^{d-2}y' \langle \Sigma_n^0 \hat{D}_+(y')T_{++}(0,\overline{z},y)T_{--}(w,\overline{w},0)\rangle \\ &\quad= \int d^{d-2}y' \langle \Sigma_n^0 \hat{D}_+(y')\hat{D}_+(y)T_{--}(w,\overline{w},0)\rangle \end{aligned} \tag{A.16.3}$$

for some appropriate $\varepsilon > 0$ that acts as the cutoff $|y'-y| > \varepsilon$.

To see this, simply note that we can replace $T_{++}(0,\overline{z},y)$ by a sum over local defect operators at y using the bulk-defect OPE. The important point is that this OPE converges because the $\overline{z}$ contour is always inside of the sphere of size ε (by construction). We can take $|\overline{z}|$ to be arbitrarily small by making the size of the $\overline{z}$ contour as small as we like. The $\overline{z}$ integral outside now simply projects the sum onto the displacement operator since we only consider the leading twist $d-2$ operators in the lightcone limit. Explicitly, we will be left with

$$\begin{aligned} &i\lim_{\varepsilon\to 0}\oint_{\varepsilon>|\overline{z}|} d\overline{z}\int_{|y-y'|>\epsilon} d^{d-2}y' \langle \Sigma_n^0 \hat{D}_+(y')T_{++}(0,\overline{z},y)T_{--}(w,\overline{w},0)\rangle \\ &\quad= \lim_{\epsilon\to 0}\int_{|y-y'|>\epsilon} d^{d-2}y' \langle \Sigma_n^0 \hat{D}_+(y')\hat{D}_+(y)T_{--}(w,\overline{w},0)\rangle. \end{aligned} \tag{A.16.4}$$

Note that perturbatively around $n=1$, the integral over $|y-y'|>\epsilon$ will miss the delta function contribution to the $\hat{D}_+ \times \hat{D}_+$ OPE. Non-perturbatively away from $n=1$, however, there are no delta-function singularities in $|y-y'|$ present in the $\hat{D}_+ \times \hat{D}_+$ OPE. In what follows, we must be careful to take $\epsilon \to 0$ *before* taking $n \to 1$.

Using this identity, we can view the displacement-displacement-bulk three point function as the contour integral of a displacement-bulk-bulk three point function. We can then use the regular displacement operator Ward identity on the latter three point function. This Ward identity follows from general diffeomorphism invariance [315]. To do this, define the deformation vector field

$$\xi(y') = f(y')\partial_+ \text{ with } f(y') = \Theta(|y'-y| - \varepsilon). \tag{A.16.5}$$

For this deformation, the Ward identity takes the form

$$\begin{aligned}
&i \oint_{\varepsilon > |\overline{z}|} d\overline{z} \int_{|y-y'|>\epsilon} d^{d-2}y' \langle \Sigma_n^0 \hat{D}_+(y') T_{++}(0,\overline{z},y) T_{--}(w,\overline{w},0)\rangle \\
&= -f(0)\partial_{\overline{w}} \langle \Sigma_n^0 \hat{D}_+(y) T_{--}(w,\overline{w},0)\rangle - i \oint d\overline{z} f(y) \partial_{\overline{z}} \langle \Sigma_n^0 T_{++}(0,\overline{z},y) T_{--}(w,\overline{w},0)\rangle \\
&- i \int_{\mathcal{M}_n} d^d x' \oint d\overline{z}\ \langle T_{++}(0,\overline{z},y) T_{--}(w,\overline{w},0) T^{\mu\nu}(x') \partial_\mu \xi_\nu(x')\rangle
\end{aligned} \tag{A.16.6}$$

where $\mathcal{M}_n$ is the full replica manifold.

The second term on the right hand side of the equality vanishes because $f(y) = 0$. Since $f(0) = 1$ by construction we just need to argue that the last term in (A.16.6) vanishes.

Arguing the last term vanishes

It is tempting at this stage to integrate by parts on the last term and conclude that this vanishes as one sends $\varepsilon \to 0$. Unfortunately, the last term in (A.16.6) can produce $1/\varepsilon$ enhancements due to T_{i+} operator coming ε close to T_{++}. Therefore one must take care to first do the x' integral and then take the $\varepsilon \to 0$ limit when evaluating this term.

To do so, note that

$$T^{\mu\nu}(x')\partial_\mu \xi_\nu(x') = \frac{1}{2} T_{i+}(x') \hat{n}^i \delta(|y'-y| - \varepsilon) \tag{A.16.7}$$

where $\hat{n}^i = (y'-y)^i/|y'-y|$. We then have the following

$$\begin{aligned}
&\int_{\mathcal{M}_n} d^d x' \oint d\overline{z}\ \langle T_{++}(0,\overline{z},y) T_{--}(w,\overline{w},0) T^{\mu\nu}(x') \partial_\mu \xi_\nu(x')\rangle \\
&= \frac{1}{2}\varepsilon^{d-3} \int \rho' d\rho' d\theta' \oint d\overline{z} \int d^{d-3}\vartheta'\ \hat{n}^i \langle T_{++}(0,\overline{z},y) T_{--}(w,\overline{w},0) T_{i+}(|\vec{y}+\vec{\varepsilon}|, \vartheta'_{\vec{\varepsilon}}, \rho' e^{-i\theta'}, \rho' e^{-i\theta'})\rangle
\end{aligned} \tag{A.16.8}$$

where $|\vec{\varepsilon}| = \varepsilon$. In going to the second line we have done the coordinate transformation $x'^+ = \rho' e^{-i\theta'}$, $x'^- = \rho' e^{i\theta'}$ because we are in the Euclidean section, and in going to the last line we have written y' in spherical coordinates on the defect. At this point we can safely send $w, \overline{w} \to 0$ so that T_{--} is simply fixed at the origin. Then, in particular, let us focus on

$$\int d\theta' \oint d\overline{z}\ \langle T_{++}(0,\overline{z},y) T_{--}(0) T_{i+}(|\vec{y}+\vec{\varepsilon}|, \vartheta'_{\vec{\varepsilon}}, \rho' e^{-i\theta'}, \rho' e^{-i\theta'})\rangle. \tag{A.16.9}$$

It is easy to see that this identically vanishes from the boost weights of the quantities involved. Specifically, T_{++} will yield a factor of $e^{2i\theta'}$, T_{i+} will yield a factor of $e^{i\theta'}$, T_{--} does not have a boost weight since it is fixed at the origin, and the measure $d\bar{z}$ will yield a factor of $e^{-i\theta'}$ so overall we will have $\int_0^{2\pi} d\theta' e^{i\theta'} = 0$. Therefore (A.16.8) is zero for any ε.

Thus, the identity in (A.16.6) becomes

$$\begin{aligned} &i \lim_{\epsilon\to 0} \oint_{\varepsilon>|\bar{z}|} d\bar{z} \int_{|y-y'|>\epsilon} d^{d-2}y' \langle \Sigma_n^0 \hat{D}_+(y') T_{++}(0,\bar{z},y) T_{--}(w,\overline{w},0)\rangle \\ &= -\partial_{\overline{w}} \langle \Sigma_n^0 \hat{D}_+(y) T_{--}(w,\overline{w},0)\rangle \end{aligned} \tag{A.16.10}$$

which, using (A.16.3), proves (A.16.1).

A.17 Analytic Continuation of a Replica Three Point Function

In this section, we analytically continue a general $\mathbb{Z}_n$-symmetrized three point function of the form[6]

$$\mathcal{A}_n^{(3)} = n \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} \mathrm{Tr}\left[e^{-2\pi n H} \mathcal{T} \mathcal{O}_a(0) \mathcal{O}_b(\tau_{ba} + 2\pi j) \mathcal{O}_c(\tau_{ca} + 2\pi k)\right] \tag{A.17.1}$$

where H is the vacuum modular Hamiltonian for the Rindler wedge and $\mathcal{T}$ denotes Euclidean time ordering with respect to this Hamiltonian.

Following [332], we begin by rewriting the the j-sum as as a contour integral

$$\frac{n}{2\pi i} \sum_{k=0}^{n-1} \oint_{C_b} ds_b \frac{\mathrm{Tr}\left[e^{-2\pi n H} \mathcal{T} \mathcal{O}_a(0) \mathcal{O}_b(-is_b) \mathcal{O}_c(2\pi k + \tau_{ca})\right]}{(e^{s_b - i\tau_{ba}} - 1)} \tag{A.17.2}$$

where the contour C_b wraps the n poles at $s_b = i(2\pi j + \tau_{ba})$ for $j = 0, ..., n-1$. We will now unwrap the s_b contour integral in the complex plane, but will need to be careful as the analytic structure of the integrand in (A.17.2) is non-trivial as a function of s_b; the integrand has poles at $s_b = i(2\pi j + \tau_{ba})$ and light-cone branch cuts lying along the lines $\Im s_b = 0, 2\pi n$ and $\Im s_b = 2\pi k + \tau_{ca}$ for a fixed k. The first two branch cuts were discussed in [332]. The third (middle in the figure) branch cut arises from singularities due to $\mathcal{O}_b$ and $\mathcal{O}_c$ lying on the same light-cone.

[6]Note that we are writing this as a thermal three point function on $\mathbb{H}_{d-1} \times S_1$, which is related to the flat space replica answer via conformal transformation. For a review of the relevant conformal factors, which we suppress for convenience, see [332].

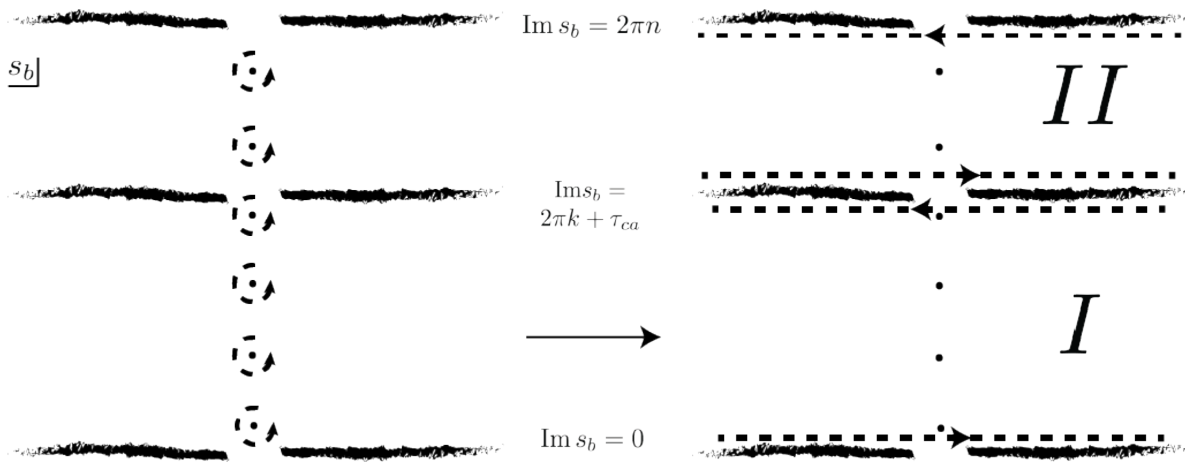


Figure A.2: The analytic structure of the integral in equation (A.17.2) represented in the s_b plane for fixed $s_k = i(2\pi k + \tau_{ca})$ for $n = 6$. The dots represent poles at $s_b = i(2\pi j + \tau_{ba})$ and the fuzzy lines denote light-cone branch cuts. The bottom and top branch cuts (which are identified by the KMS condition) arise from $\mathcal{O}_b$ becoming null separated from $\mathcal{O}_a$ and the middle branch cut arises from $\mathcal{O}_b$ becoming null separated from $\mathcal{O}_c$. Note that in this figure, $k = 3$ and $\tau_{ca} > \tau_{ba} > 0$. We start with the contour C_b represented by the dashed lines encircling the poles at $s_b = i(2\pi j + \tau_{ba})$ and unwrap so that it just picks up contributions from the branch-cuts. Region I corresponds to the ordering $\mathcal{O}_a\mathcal{O}_b\mathcal{O}_c$ whereas region II corresponds to $\mathcal{O}_a\mathcal{O}_c\mathcal{O}_b$.

.

We can unwrap the C_b contour now so that it hugs the branch cuts as in the right-hand panel of Figure A.2. We will then be left with a sum of four Lorentzian integrals

$$\begin{aligned}
&\frac{n}{2\pi i}\sum_{k=0}^{n-1}\mathrm{Tr}\Bigg[e^{-2\pi n H}\int_{-\infty}^{\infty} ds_b \times \\
&\frac{\mathcal{O}_a(0)\mathcal{O}_b(-is_b+\epsilon_j)\mathcal{O}_c(2\pi k+\tau_{ca})}{(e^{s_b - i\tau_{ba}}-1)} - \frac{\mathcal{O}_a(0)\mathcal{O}_b(-is_b+2\pi i k+\tau_{ca}-\epsilon)\mathcal{O}_c(2\pi k+\tau_{ca})}{(e^{s_b+2\pi i k+\tau_{ca}-i\epsilon-i\tau_{ba}}-1)} \\
&+\frac{\mathcal{O}_a(0)\mathcal{O}_c(2\pi k+\tau_{ca})\mathcal{O}_b(-is_b+2\pi k+\tau_{ca}+\epsilon)}{(e^{s_b+2\pi i k+\tau_{ca}+i\epsilon-i\tau_{ba}}-1)} - \frac{\mathcal{O}_a(0)\mathcal{O}_c(2\pi k+\tau_{ca})\mathcal{O}_b(-is_b+2\pi n-\epsilon)}{(e^{s_b+i2\pi n-i\epsilon-i\tau_{ba}}-1)}\Bigg],
\end{aligned} \tag{A.17.3}$$

where we have set $2\pi k + \tau_{ca} = -is_c$ since the C_c contour still wraps the poles at these values.

We now need to make a choice about how to do the analytic continuation in n. The usual prescription, which was advocated for in [332], is to set $e^{2\pi i n} = 1$ in the last term of (A.17.3). We will follow this but also make one other choice. In the second and third terms in the integrand of (A.17.3) we make the choice to set $e^{2\pi i k} = 1$ for all $k = 0, ..., n-1$.

Making this analytic continuation, we can now re-write the k-sum as a contour integral over s_c along some contour C_c. Unwrapping this s_c contour into the Lorentzian section, and after repeated use of the KMS condition to push operators back around the trace, we land on the relatively simple formula

$$\begin{aligned}&\mathcal{A}_n^{(3)} = \\ &\frac{-n}{4\pi^2}\int_{-\infty}^{\infty} ds_c ds_b \,\mathrm{Tr}[e^{-2\pi n H}(\frac{[[\mathcal{O}_a(0),\mathcal{O}_b(-is_b)],\mathcal{O}_c(-is_c)]}{(e^{s_b-i\tau_{ba}}-1)(e^{s_c-i\tau_{ca}}-1)} \\ &-\frac{[\mathcal{O}_a(0),[\mathcal{O}_b(-is_b-is_c),\mathcal{O}_c(-is_c)]]}{(e^{s_b+i\tau_{ca}-i\tau_{ba}}-1)(e^{s_c-i\tau_{ca}}-1)})]\end{aligned} \tag{A.17.4}$$

In deriving this formula, we have assumed $\tau_{ba} > 0$ and $\tau_{ca} > 0$ but we have not yet assumed any relationship between τ_{ba} and τ_{ca}. This formula is the full answer. One could stop here, but we will massage this formula into a slightly different form for future convenience. Instead of following [332] and applying ∂_n at this stage, which drops down powers of H, we will use a slightly different (although equivalent) technique.

We first focus on re-writing the two Lorentzian integrals in region I of Figure A.2 as one double integral.

Region I

Before re-writing the k-sum as a contour integral, the integrals in region I are[7]

$$\begin{aligned}&\frac{n}{2\pi i}\sum_{k=0}^{n-1}\int_{-\infty}^{\infty} ds_b(\frac{\langle\mathcal{O}_a(0)\mathcal{O}_b(-is_b)\mathcal{O}_c(2\pi k+\tau_{ca})\rangle_n}{(e^{s_b-i\tau_{ba}}-1)} \\ &-\frac{\langle\mathcal{O}_a(0)\mathcal{O}_b(-is_b+2\pi k+\tau_{ca}-\epsilon)\mathcal{O}_c(2\pi k+\tau_{ca})\rangle_n}{(e^{s_b+i\tau_{ca}-i\tau_{ba}}-1)})\end{aligned} \tag{A.17.5}$$

where as before we have set $e^{2\pi i k} = 1$ in the second term. The goal will be to make the denominators in these two terms the same so that we may combine their numerators. We will try to shift the s_b contour in the second term by an amount $-i\tau_{ca}$, making sure not to cross any poles or branch cuts. To make our lives easier, we will assume a fixed ordering of the operators. For now, we will pick $\tau_{ca} > \tau_{ba} > 0$. Note that any other ordering can be reached just by exchanging the a, b, c labels.

In this ordering, sending $s_b \to s_b - i\tau_{ca}$ crosses a pole at $\Im s_b = 2\pi k + \tau_{ba}$. This contour shift is illustrated in Figure A.3. After doing this shift, we get

$$\begin{aligned}&\frac{n}{2\pi i}\sum_{k=0}^{n-1}\int_{-\infty}^{\infty} ds_b\left(\frac{\langle\mathcal{O}_a(0)\mathcal{O}_b(-is_b)\mathcal{O}_c(2\pi k+\tau_{ca})\rangle_n-\langle\mathcal{O}_a(0)\mathcal{O}_b(-is_b+2\pi k)\mathcal{O}_c(2\pi k+\tau_{ca})\rangle_n}{(e^{s_b-i\tau_{ba}}-1)}\right) \\ &+\theta(\tau_{cb})\times(\text{terms with } j=k).\end{aligned} \tag{A.17.6}$$

where we will mostly neglect the extra term coming from picking up the pole since it will not be important for most calculations we are interested in. We will refer to these terms as the

[7] For ease of notation, we have switched to $\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\rangle_n = \mathrm{Tr}[e^{-2\pi n H}\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3]$.

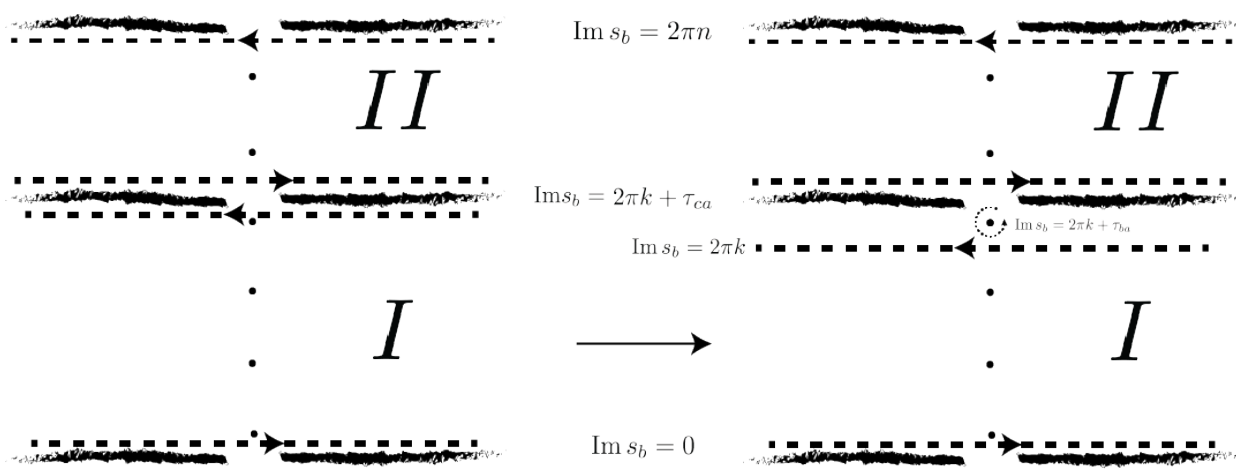


Figure A.3: This figure illustrates the contour shift $s_b \to s_b - i\tau_{ca}$ done at the cost of picking up the pole at $s = i(2\pi k + \tau_{ba})$ when $\tau_{cb} = \tau_{ca} - \tau_{ba} > 0$.

"replica diagonal terms" since they arise from terms in the double sum over j, k in (A.17.1) where $j = k$.

The numerator for the first term in equation (A.17.6) then looks like the integral of a total derivative in some auxiliary parameter t_b which we write as

$$\frac{-n}{2\pi i}\sum_{k=0}^{n-1}\int_{-\infty}^{\infty} ds_b \int_0^{i2\pi k} dt_b \left(\frac{\frac{d}{dt_b}\langle \mathcal{O}_a(0)\mathcal{O}_b(-is_b - it_b)\mathcal{O}_c(2\pi k + \tau_{ca})\rangle_n}{(e^{s_b - i\tau_{ba}} - 1)}\right). \tag{A.17.7}$$

Since t_b shows up on equal footing with s_b in the numerator, we see we can re-write the derivative in t_b as one in s_b. Integrating by parts and dropping the boundary terms[8], we get

$$\frac{-n}{2\pi i}\sum_{k=0}^{n-1}\int_{-\infty}^{\infty} ds_b \int_0^{i2\pi k} dt_b \frac{\langle \mathcal{O}_a(0)\mathcal{O}_b(-is_b - it_b)\mathcal{O}_c(2\pi k + \tau_{ca})\rangle_n}{4\sinh^2((s_b - i\tau_{ba})/2)}. \tag{A.17.8}$$

We are now ready, as above, to turn the sum over k into a contour integral over some Lorentzian parameter s_c. We can then execute the same trick as before: we re-write two terms as the boundary terms of one integral in some new auxiliary parameter t_c. After all of this, the answer we find is the relatively simple result for region I

$$\begin{aligned}\text{region I} =& \frac{-n}{4\pi^2}\int_{-\infty}^{\infty} ds_c ds_b \int_0^{i2\pi(n-1)} dt_c \int_0^{s_c + t_c} dt_b \frac{\langle \mathcal{O}_a(0)\mathcal{O}_b(-is_b - it_b)\mathcal{O}_c(-is_c - it_c + \tau_{ca})\rangle_n}{16\sinh^2((s_b - i\tau_{ba})/2)\sinh^2((s_c - i\epsilon)/2)} \\ &+ \theta(\tau_{cb}) \times (\text{terms with } j = k).\end{aligned} \tag{A.17.9}$$

Note that the quadruple integral term is manifestly order $n - 1$ because of the limits on the t_c integral.

[8]We will drop boundary terms at large Lorentzian time everywhere throughout this discussion, as we expect thermal correlators to fall off sufficiently quickly [332].

Region II

In region II of Figure A.2, the calculations are exactly analogous, except now the ordering of the operators is different. We find that (up to terms that again come from picking up specific poles) the answer for region II is

$$\begin{aligned}&\text{region II} = \\ &\frac{-n}{4\pi^2}\int_{-\infty}^{\infty} ds_c ds_b \int_0^{i2\pi(n-1)} dt_c \int_{s_c+t_c+i2\pi}^{i2\pi n} dt_b \frac{\langle \mathcal{O}_a(0)\mathcal{O}_c(-is_c - it_c + \tau_{ca})\mathcal{O}_b(-is_b - it_b)\rangle_n}{16\sinh^2((s_b - i\tau_{ba})/2)\sinh^2((s_c - i\epsilon)/2)} \\ &+ \theta(\tau_{bc}) \times (\text{terms with } j = k).\end{aligned} \tag{A.17.10}$$

Combining Regions I and II

Adding the Region I and Region II contributions, we get for the non-replica diagonal contributions to $\mathcal{A}_n^{(3)}$

$$\begin{aligned}&\frac{n}{4\pi^2}\int_{-\infty}^{\infty} ds_c ds_b \int_0^{i2\pi(n-1)} dt_c \int_0^{s_c+t_c} dt_b \frac{\langle [\mathcal{O}_b(-is_b - it_b), \mathcal{O}_a(0)]\mathcal{O}_c(-is_c - it_c + \tau_{ca})\rangle_n}{16\sinh^2((s_b - i\tau_{ba})/2)\sinh^2((s_c - i\epsilon)/2)} \\ &+\frac{n}{4\pi^2}\int_{-\infty}^{\infty} ds_c ds_b \int_0^{i2\pi(n-1)} dt_c \int_{s_c+t_c}^{s_c+t_c+i2\pi(1-n)} dt_b \frac{\langle \mathcal{O}_b(-is_b - it_b)\mathcal{O}_a(0)\mathcal{O}_c(-is_c - it_c + \tau_{ca})\rangle_n}{16\sinh^2((s_b - i\tau_{ba})/2)\sinh^2((s_c - i\epsilon)/2)}\end{aligned} \tag{A.17.11}$$

where we used the KMS condition to push $\mathcal{O}_b$ around to the left of $\mathcal{O}_a$ in (A.17.10). We then split the t_b contour in (A.17.10) into two pieces, one purely Lorentzian integral from $t_b = 0$ to $t_b = s_c + t_c$ and another purely Euclidean integral from $t_b = s_c + t_c$ to $t_b = s_c + t_c + 2\pi i(n-1)$. Again, this is the full answer for the replica three point function, $\mathcal{A}_n^{(3)}$, at all n excluding the replica diagonal terms.

From this we can compute the leading order in n correction to the three-point function (dropping the diagonal terms). Taking an n-derivative and setting $n \to 1$, the total correction is

$$\begin{aligned}\mathcal{A}_n^{(3)} \sim \frac{i(n-1)}{2\pi}\int_{-\infty}^{\infty} ds_c ds_b \int_0^{s_c} dt_b \frac{\langle [\mathcal{O}_b(-is_b - it_b), \mathcal{O}_a(0)]\mathcal{O}_c(-is_c + \tau_{ca})\rangle_1}{16\sinh^2((s_b - i\tau_{ba})/2)\sinh^2((s_c - i\epsilon)/2)} \\ + (\text{replica diagonal terms}) + \mathcal{O}\left((n-1)^2\right).\end{aligned} \tag{A.17.12}$$

Replica Diagonal Terms

For future reference, we now list the replica diagonal (or $j = k$) terms that we have suppressed. In the order we considered above, we have

$$
\begin{aligned}
&n\theta(\tau_{cb})\theta(\tau_{ba})\sum_{k=0}^{n-1}\langle\mathcal{O}_a(0)\mathcal{O}_b(2\pi k+\tau_{ba})\mathcal{O}_c(2\pi k+\tau_{ca})\rangle_n\\
&= n\theta(\tau_{cb})\theta(\tau_{ba})\Bigg(\langle\mathcal{O}_a(0)\mathcal{O}_b(\tau_{ba})\mathcal{O}_c(\tau_{ca})\rangle_n -\\
&\frac{1}{2\pi i}\int_{i2\pi}^{i2\pi n}dt_c\int_{-\infty}^{\infty}ds_c\frac{\langle\mathcal{O}_a(0)\mathcal{O}_b(-is_c-it_c-\tau_{cb})\mathcal{O}_c(-is_c-it_c)\rangle_n}{4\sinh^2((s_c-i\tau_{ca})/2)}\Bigg).
\end{aligned}
\tag{A.17.13}
$$

Again, other orderings can be found just by swapping the a, b, c labels accordingly. Note that at $n = 1$, the integral term vanishes and the answer reduces to the angular ordered three-point function as expected.

A.18 Explicit Calculation of $c^{(2)}$

In this section, we compute the OPE coefficient of $\hat{T}_{++}$ in the $\hat{D}_+ \times \hat{D}_+$ OPE. This requires us to compute the twist defect three point function $\langle\Sigma_n^0\hat{D}_+\hat{D}_+\hat{T}_{--}\rangle$. As described around equation (A.16.3), the appearence of a delta function in the $\hat{D}_+ \times \hat{D}_+$ OPE requires that the coefficient c_n for $\hat{T}_{--}$ must be at least of order $(n-1)^2$ near $n = 1$. We now show that this is indeed true. In the next section, we will explicitly compute the anomalous dimension of $\hat{T}_{--}$ and show that it behaves as $g_n \sim \gamma^{(1)}(n-1) + \mathcal{O}((n-1)^2)$. We will finally show that their ratio obeys the relation

$$
c^{(2)}/\gamma^{(1)} = 2\pi/S_{d-3} \tag{A.18.1}
$$

as required by the first law of entanglement entropy.

The three point function we are after, at integer n, takes the form

$$
\begin{aligned}
&\langle\Sigma_n^0\hat{T}_{--}(y')\hat{D}_+(y)\hat{D}_+(y=0)\rangle\\
&= -\oint d\bar{z}\oint d\bar{w}\oint\frac{du}{2\pi iu}\langle\Sigma_n^0T_{--}(u,\bar{u}=0,y')T_{++}(z=0,\bar{z},y)T_{++}(w=0,\bar{w},0)\rangle
\end{aligned}
\tag{A.18.2}
$$

where it is understood that all the stress tensor operators should be $\mathbb{Z}_n$ symmetrized. Our goal is now to analytically continue this expression in n and then expand around $n = 1$. We can turn to the previous section for this result, letting $\mathcal{O}_a = T_{++}(w = 0, \bar{w}, 0)$, $\mathcal{O}_b = T_{++}(z = 0, \bar{z}, y)$ and $\mathcal{O}_c = T_{--}(u, \bar{u} = 0, 0)$.

Just as in Section 7.5, a major simplification occurs for this correlator; the two displacement operators are space-like separated from each other, so they commute even upon analytic continuation. Thus, any terms with commutators between $\mathcal{O}_a$ and $\mathcal{O}_b$ in the previous section drop out.

Furthermore, the so-called "replica diagonal" terms in the previous section will also vanish. This is because they do not contain enough s-integrals that produce necessary poles in $\overline{z}$ and $\overline{w}$. Thus, these terms vanish upon the contour integration over $\overline{z}$ and $\overline{w}$ in (A.18.2).

These considerations together with equation (A.17.11) of the previous section make it clear that the correlator in (A.18.2) vanishes up to order $(n-1)^2$. Indeed, the only surviving contribution is the second term in (A.17.11). Expanding that to second order while being careful to account for the spin of the stress tensors, we find

$$\begin{aligned}&\langle \Sigma_n^0 \hat{T}_{--} \hat{D}_+ \hat{D}_+ \rangle_n = \\ &\frac{-(n-1)^2}{2} \oint d\overline{z} d\overline{w} \frac{du}{2\pi i u} \int_0^\infty \int_0^\infty d\lambda_b d\lambda_c \lambda_b^2 \lambda_c^2 \frac{\langle T_{++}(\overline{z}\lambda_b, y) T_{++}(\overline{w}\lambda_c) T_{--}(u, y')\rangle}{(\lambda_b - 1 - i\epsilon)^2 (\lambda_c - 1 + i\epsilon)^2} + \mathcal{O}((n-1)^3).\end{aligned} \tag{A.18.3}$$

Rescaling $\lambda_b \to \lambda_b/\overline{z}$ and $\lambda \to \lambda_c/\overline{w}$, we can then expand the denominators in small $\overline{z}, \overline{w}$ and perform the residue projections in $\overline{z}, \overline{w}$ and u. The final answer is the simple result

$$\langle \Sigma_n^0 \hat{T}_{--} \hat{D}_+ \hat{D}_+ \rangle = 2\pi^2 (n-1)^2 \langle \mathcal{E}_+(y) \mathcal{E}_+(y=0) T_{--}(u=0, y')\rangle + \mathcal{O}((n-1)^3). \tag{A.18.4}$$

where $\mathcal{E}_+(y)$ is the half-averaged null energy operator

$$\mathcal{E}_+(y) = \int_0^\infty d\lambda \, T_{++}(z=0, \lambda, y) \tag{A.18.5}$$

We now set about computing this correlator. Expanding the stress tensor three point function in a general CFT into the free field basis, we have

$$\langle TTT \rangle = n_s \langle TTT \rangle_s + n_f \langle TTT \rangle_f + n_v \langle TTT \rangle_v \tag{A.18.6}$$

where n_s, n_f and n_v are charges characterizing the specific theory.

One can demonstrate that the only non-vanishing contribution from these three terms is from the scalar three point function. The way to see this is as follows. The fermion term can be computed by considering a putative free Dirac fermion theory with field ψ. The stress tensor looks like $T_{++} \sim \overline{\psi} \Gamma_+ \partial_+ \psi$. Then we can compute the $\langle TTT \rangle$ three point function via Wick contractions. There will always be at least one Wick contraction between operators in each T_{++}. The kinematics of these operators ensure that such a contraction vanishes because they are both on the same null plane.[9]

The same argument can be made for the vector fields. In fact, the *only* reason that the scalar contribution doesn't vanish is because of the presence of a total derivative term in the

[9]Actually these contractions will be proportional to a delta function $\delta^{d-2}(y)$ but we are assuming the three stress tensors sit at different y's.

conformal stress tensor, namely $T_{++} \supset -\frac{d-2}{4(d-1)}\partial_+^2 :\phi^2:$. One can then show that the only non-vanishing term is

$$\langle \mathcal{E}_+(y)\mathcal{E}_+(0)T_{--}(y')\rangle = \frac{4n_s(d-2)}{(d-1)^3}\frac{1}{|y|^{d-2}|y'|^{2d}}. \tag{A.18.7}$$

Dividing by the two point function $\langle T_{++}(0)T_{--}(y')\rangle = \frac{c_T}{4|y'|^{2d}}$, we find

$$c^{(2)} = \frac{32\pi^2 n_s(d-2)}{c_T(d-1)^3}. \tag{A.18.8}$$

We now turn to computing the anomalous dimension $\gamma^{(1)}$ for the stress tensor operator $\hat{T}$ on the defect.

A.19 Explicit Calculation of $\gamma^{(1)}$

In this section, we will follow the steps laid out in [15] for computing the spectrum of defect operators and associated anomalous dimension induced by the bulk stress tensor. To do this, we must compute

$$n\sum_{j=0}^{n-1}\langle \Sigma_n^0 T_{--}(w,0,y)T_{++}(0,\overline{z},0)\rangle\,. \tag{A.19.1}$$

To leading order in $n-1$ this expression takes the form of a sum of two terms, a "modular energy" piece and a "relative entropy" piece

$$\begin{aligned}(\partial_n - 1)\langle\Sigma_n^0\hat{T}_{--}\hat{T}_{++}\rangle|_{n=1} = \Big(&-2\pi\langle HT_{--}(w,0,y)T_{++}(0,\overline{z},0)\rangle \\ &-\int_0^{-\infty}d\lambda\frac{\lambda^2}{(\lambda-1+i\epsilon)^2}\langle T_{--}(w,0,y)T_{++}(0,\overline{z}\lambda,0)\rangle\Big)\end{aligned} \tag{A.19.2}$$

We will try to extract the anomalous dimensions and spectra of operators by examining the two point function of the defect stress tensor. In this framework, the signal of an anomalous dimension is a logarithmic divergence. As explained in [15], the log needs to be cutoff by $\overline{z}w/y^2$ or $z\overline{w}/y^2$. In fact, there will be two such logarithms that will add to make the final answer single-valued on the Euclidean section.

We are thus tasked with looking for all of the terms containing log divergences in (A.19.2). Since the modular Hamiltonian is just a local integral of the stress tensor

$$H = \int d^{d-2}y'\int_0^\infty dx^+ x^+ T_{++}(x^- = 0, x^+, y') \tag{A.19.3}$$

then the first term on the r.h.s. of (A.19.2) is a stress tensor three point function. Following the method of the previous section, we can then break up (A.19.2) into the free field basis.

This determines both terms on the r.h.s of (A.19.2) in terms of charges n_s, n_f and n_v. This allows us to instead compute the answer in a theory of free massless scalars, fermions and vectors. While this might seem like three times the work, it actually illuminates why g_n is only dependent on n_s. We start by examining the case of a free scalar and will see why the free fermion and free vector terms do not contribute to g_n.

Spectrum induced by free scalar

This spectrum of $\phi(z, \bar{z}, y)$ was analyzed in [314]. The authors found that the leading twist defect primaries are all twist one (in $d = 4$) and have dimension independent of n. As noted in Appendix C of that work, this can be understood in any dimension from the fact that ϕ is annihilated by the bulk Laplacian. This constraint - for defect primaries - enforces holomorphicity in $z, \bar{z}$ of the bulk-defect OPE which translates to a lack of anomalous dimensions. For free fermions and vectors, the same argument goes through since their two point functions are also annihilated by the Laplacian.

One might be confused because the anomalous dimension for scalar operators of dimension Δ was computed in [15] and found to be non-zero for operators of dimension $\Delta = \frac{d-2}{2}$. This discrepancy has to do with a subtlety related to the extra boundary term in the modular Hamiltonian for free scalars. This discrepancy is related to the choice of the stress tensor - the traceless, conformal stress tensor vs. the canonical stress tensor.

The authors of [314] worked with *canonical* free fields, for which the stress tensor is just $T_{++}^{\text{canonical}} = \partial_+\phi\partial_+\phi$. Indeed if one inserts the canonical stress tensor into the modular Hamiltonian in equation (3.20) of [15], then the anomalous dimension vanishes. On the other hand, if one uses the conformal stress tensor, $T_{++}^{\text{conformal}} = \,:\partial_+\phi\partial_+\phi: - \frac{(d-2)}{4(d-1)}\partial_+^2 :\phi^2:$, then anomalous dimension for ϕ is given by [15].

This discrepancy thus amounts to a choice of the stress tensor. Note that this is special to free scalars and does not exist for free fermions and vectors since there are no dimension $d-2$ scalar primaries in these CFTs. This proves that if one works with canonical free fields, there should be no anomalous dimension for the defect operators induced by the fundamental fields ϕ, ψ and A_μ. This is enough to prove that the defect primary induced by the *canonical* bulk stress tensor must also have zero anomalous dimension since this is just formed by normal-ordered products of the defect primaries induced by the bulk fundamental fields.

Back to the stress tensor

The upshot is that we only need to worry about the terms in (A.19.2) proportional to n_s. Furthermore, we only need to worry about terms in the $\langle HTT\rangle$ term that involve the boundary term of the modular Hamiltonian. This reduces the expression down to the term

$$\langle HTT\rangle \supset -\frac{(d-2)}{4(d-1)} \int d^{d-2}y \, \langle :\phi^2: T_{++}(0, \bar{z}, y) T_{--}(w, 0, 0)\rangle \, . \tag{A.19.4}$$

A simple calculation shows that the only contractions that give log divergences come from

$$\begin{aligned}\langle HTT\rangle \supset \frac{n_s(d-2)^2}{4(d-1)^2}\int d^{d-2}y'\,\langle\phi(0,0,y')\phi(0,0,0)\rangle\,\langle\phi(0,0,y')\partial_{\overline{z}}^2\phi(0,\overline{z},0)T_{--}(0,0,y)\rangle\\ =-\frac{n_s c_{\phi\phi}^3 d(d-2)^4}{16(d-1)^3}\int d^{d-2}y'\frac{1}{|y'|^{d-2}|y-y'|^{d-2}|y|^{d+2}}.\end{aligned} \tag{A.19.5}$$

This integral has two log divergences coming from $y'=0$ and $y'=y$, however they can be regulated by fixing $z,\overline{z}$ and $w,\overline{w}$ away from zero. The two singularities just add to make the final answer single valued under rotations by 2π about the defect as in [15]. We thus find

$$\begin{aligned}\langle HTT\rangle \supset &-n_s\frac{c_{\phi\phi}^3 d(d-2)^4}{32(d-1)^3}S_{d-3}\log(w\overline{w}z\overline{z}/|y|^4)\frac{1}{|y|^{2d}}\\ &=-\frac{2n_s(d-2)}{(d-1)^3}S_{d-3}\log(w\overline{w}z\overline{z}/|y|^4)\frac{1}{|y|^{2d}}.\end{aligned} \tag{A.19.6}$$

Dividing by $\langle T_{++}T_{--}\rangle$ gives

$$\gamma^{(1)}=\frac{16\pi n_s(d-2)}{c_T(d-1)^3}S_{d-3}. \tag{A.19.7}$$

Comparing with (A.18.8), we see that

$$\frac{c^{(2)}}{\gamma^{(1)}}=\frac{2\pi}{S_{d-3}} \tag{A.19.8}$$

as required by the first law of entanglement.

A.20 Calculating $\mathcal{F}_n$

At first glance, $\mathcal{F}_n$ seems difficult to calculate; we would like a method to compute this correlation function at leading order in $n-1$ without having to analytically continue a $\mathbb{Z}_n$ symmetrized four point function. The method for analytic continuation is detailed in Appendix A.17.

As detailed in Appendix A.17, part of what makes the analytic continuation in n difficult is the analytic structure (branch cuts) due to various operators becoming null separated from each other in Lorentzian signature. One might naively worry that we have to track this for four operators in the four point function $\mathcal{F}_n$.

We will leverage the fact that the two stress tensors in $\hat{D}_+(y_1)$ and $\hat{D}_+(y_2)$ are in the lightcone limit with respect to the defect since

$$\hat{D}_+(y_1)=\lim_{|z|\to 0} i\oint d\overline{z}\sum_{j=0}^{n-1}\,T_{++}^{(j)}(z=0,\overline{z},y_1). \tag{A.20.1}$$

Thus, the stress tensors at y_1 and y_2 commute with each other even after a finite amount of boost. This means that these two operators do not see each other in the analytic continuation. In other words, the analytic structure for each of these operators is just that of a $\mathbb{Z}_n$ symmetrised *three* point function. This was computed in Appendix A.17.

We can thus jump straight to (A.17.12) but now with two $\mathcal{O}_b$ operators. The final replica four point function assuming $[\mathcal{O}_{b_1}, \mathcal{O}_{b_2}] = 0$ is given by[10]

$$\begin{aligned}
&\frac{(n-1)}{8\pi^2}\int_{-\infty}^{\infty} ds_c ds_{b_1} ds_{b_2} \\
&\times \int_0^{s_c} dt_{b_1} dt_{b_2} \frac{\langle [\mathcal{O}_{b_2}(-is_{b_2}-it_{b_2}), [\mathcal{O}_{b_1}(-is_{b_1}-it_{b_1}), \mathcal{O}_a(0)]]\mathcal{O}_c(-is_c+\tau_{ca})\rangle_1}{64\sinh^2((s_{b_1}-i\tau_{b_1a})\sinh^2((s_{b_2}-i\tau_{b_2a})/2)\sinh^2((s_c-i\epsilon)/2)} \\
&+\mathcal{O}((n-1)^2).
\end{aligned} \tag{A.20.2}$$

To make contact with $\mathcal{F}_n$, we assign

$$\begin{aligned}
\mathcal{O}_{b_1}(-is_1) &= \lim_{|z|\to 0} i\oint d\overline{z}\, e^{2s_1-2i\tau_{b_1a}} T_{++}(x^-=0, x^+=r_{\overline{z}}e^{s_1}, y_1) \\
\mathcal{O}_{b_2}(-is_2) &= \lim_{|w|\to 0} i\oint d\overline{w}\, e^{2s_2-2i\tau_{b_2a}} T_{++}(x^-=0, x^+=r_{\overline{w}}e^{s_2}, y_2) \\
\mathcal{O}_c(-is_c) &= \lim_{|u|\to 0} i\oint du\, e^{-2s_c+2i\tau_{ca}} T_{--}(x^-=-r_u e^{-s_c}, x^+=0, y_4) \\
\mathcal{O}_a(0) &= \lim_{|v|\to 0} i\oint \frac{dv}{2\pi i} T_{--}(x^-=-r_v, x^+=0, y_3)
\end{aligned} \tag{A.20.3}$$

with $\overline{z}, \overline{w} = r_{\overline{z},\overline{w}}e^{i\tau_{b_1,b_2}}$ and $u, v = r_{u,v}e^{-i\tau_{a,c}}$. The funny factors of $e^{2s-2i\tau}$ are to account for the spin of the stress tensor.

Shifting $s_{b_{1,2}} \to s_{b_{1,2}} - t_{b_{1,2}} - \log(r_{1,2})$ and moving to null coordinates $\lambda = e^s$, we find the expression

$$\begin{aligned}
&\mathcal{F}_n = \lim_{|z|,|w|,|u|,|v|\to 0}\oint d\overline{z}\, d\overline{w}\, du\, dv \times \\
&\frac{(n-1)}{8\pi^2}\int_{-\infty}^{\infty} ds_c \int_0^{\infty}\frac{d\lambda_{b_{1,2}}\lambda_{b_1}^2\lambda_{b_2}^2}{\overline{z}^3\overline{w}^3}\int_0^{s_c} dt_{b_1}dt_{b_2} e^{-s_c}e^{-t_{b_1}-t_{b_2}}e^{6i\tau_a} \times \\
&\frac{\langle [T_{++}(x^+=\lambda_{b_1}), [T_{++}(x^+=\lambda_{b_2}), T_{--}(x^-=-r_v)]]T_{--}(x^-=-r_u e^{-s_c-i\tau_{ca}})\rangle_1}{\left(\frac{\lambda_{b_1}e^{i\tau_a}}{\overline{z}e^{t_{b_1}}}-1\right)^2\left(\frac{\lambda_{b_2}e^{i\tau_a}}{\overline{w}e^{t_{b_2}}}-1\right)^2(e^{s_c-i\epsilon}-1)^2}.
\end{aligned} \tag{A.20.4}$$

The first line in (A.20.4) comes from the residue projections in the definitions of the displacement operators. Expanding the integrand at small $|\overline{z}|$ and $|\overline{w}|$, we can perform the residue integrals over $\overline{z}$ and $\overline{w}$ leaving us with

[10] We have dropped the so-called "replica diagonal" terms in (A.17.12) since they will drop out of the final answer after the residue projection in (A.20.1).

$$\mathcal{F}_n = \lim_{|u|,|v|\to 0} \oint du\, dv \times$$
$$\frac{1-n}{2} \int_{-\infty}^{\infty} ds_c \int_0^{s_c} dt_{b_1} dt_{b_2} e^{-s_c+2i\tau_a} e^{t_{b_1}+t_{b_2}}$$
$$\times \frac{\langle [\mathcal{E}_+(y_1), [\mathcal{E}_+(y_2), T_{--}(x^- = -r_v)]] T_{--}(x^- = -ue^{-s_c+i\tau_a}) \rangle_1}{(e^{s_c-i\epsilon}-1)^2} \tag{A.20.5}$$

where $\mathcal{E}_+(y_1)$ is a half-averaged null energy operator, $\int_0^\infty dx^+ T_{++}(x^+)$.

We can now do the t_{b_1} and t_{b_2} integrals which produce two factors of $e^{s_c}-1$ precisely cancelling the denominator. Note that a similar cancellation occurred in equation (7.6.12). We can then replace commutators of half-averaged null energy operators with commutators of full averaged null energy operators. Using the fact that $\hat{\mathcal{E}}_+ |\Omega\rangle = 0$, we are left with the expression

$$\mathcal{F}_n = \lim_{|v|,|u|\to 0} \oint dudv \times$$
$$\frac{(1-n)}{2} \int_{-\infty}^{\infty} ds_c\, e^{-s_c+2i\tau_a}$$
$$\times \left\langle T_{--}(x^- = -r_v, x^+ = 0, y_3) \hat{\mathcal{E}}_+(y_1) \hat{\mathcal{E}}_+(y_2) T_{--}(x^- = -ue^{-s_c+i\tau_a}, x^+ = 0, y_4) \right\rangle_1 . \tag{A.20.6}$$

Using boost invariance, we can also write this as

$$\mathcal{F}_n = 4\pi^2(n-1) \int_{-\infty}^{\infty} ds_c\, e^{-s_c}$$
$$\times \left\langle T_{--}(x^- = -1, x^+ = 0, y_3) \hat{\mathcal{E}}_+(y_1) \hat{\mathcal{E}}_+(y_2) T_{--}(x^- = -e^{-s_c}, x^+ = 0, y_4) \right\rangle_1 \tag{A.20.7}$$

where we have performed the projection over v, u.

This is precisely the formula we were after. From here, one can just insert the $\hat{\mathcal{E}}_+ \times \hat{\mathcal{E}}_+$ OPE as described in the main text.

A.21 Free Field Theories and Null Quantization

In this section we review the basics of null quantization (see [390, 280]). We then show that our computations in Section 7.6 can reproduce the results of [280]. In free (and super-renormalizable) quantum field theories, one can evolve the algebra of operators on some space-like slice up to the null plane $x_- = 0$ and quantize using the null generator $P_+ = \int d^{d-2}y\; dx^+\; T_{++}(x^+, y)$ as the Hamiltonian. One can show that for free scalar fields, the algebra on the null plane factorizes across each null-generator (or "pencil") of the $x^- = 0$ plane. For each pencil, the algebra $\mathcal{A}_{p_y}$ is just the algebra associated to a 1+1-d chiral CFT. Accordingly, the vacuum state factorizes as an infinite tensor product of $1+1$-d chiral CFT vacua:

$$|\Omega\rangle = \bigotimes_y |\Omega\rangle^{p_y} \tag{A.21.1}$$

where $|0\rangle_{p_y}$ is the vacuum for the chiral $1+1$-d CFT living on the pencil at transverse coordinate y.

Thus, if we trace out everything to the past of some (possibly wiggly) cut of the null plane defined by $x^+ = X^+(y)$, we will be left with an infinite product of reduced vacuum density matrices for a $1+1$-d CFT on the pencil

$$\sigma_{X^+(y)} = \bigotimes_y \sigma^{p_y}_{x^+>X^+(y)}. \tag{A.21.2}$$

As discussed in [280], a general excited state on the null plane $|\Psi\rangle$ can also be expanded in the small transverse size of $\mathcal{A}$ of a given pencil. For any p_y, the full reduced density matrix above some cut of the null plane takes the form

$$\rho = \sigma^{p_y}_{X^+(y)} \otimes \rho^{(0)}_{\text{aux}} + \mathcal{A}^{1/2} \sum_{ij} \sigma^{p_y}_{X^+(y)} \int dr d\theta f_{ij}(r,\theta) \partial\phi(re^{i\theta}) \otimes E_{ij}(\theta) \tag{A.21.3}$$

where $\partial\phi$ is an operator acting on the pencil Hilbert space and $E_{ij}(\theta) = e^{\theta(K_i - K_j)} |i\rangle \langle j|$, with $|i\rangle$ eigenvectors for the auxiliary modular Hamiltonian, K_{aux}. Note that E_{ij} parameterizes our ignorance about the rest of the state on the null plane which is not necessarily the vacuum.

As a consistency check of (7.6.12), we now demonstrate agreement with the result of [280]. In null quantization, the delta function piece of the shape deformation corresponds to a shape deformation of the pencil while keeping the auxiliary system fixed. Note that the ansatz A.21.3 is analogous to the λ expansion in Section 7.6 even though we are now considering a general excited state

$$\rho = \sigma + \mathcal{A}^{1/2} \delta\rho + \mathcal{O}(\mathcal{A}). \tag{A.21.4}$$

We now just plug in our expression of $\delta\rho$ into (7.6.8) and find that the relative entropy second variation is

$$\begin{aligned} \frac{d^2}{dX^+(y)^2} S_{\text{rel}}(\rho|\rho_0) = \frac{1}{2} \sum_{ij} \int \int & (dr d\theta)_1 (dr d\theta)_2 (f_{ij}(r,\theta))_1 (f_{ji}(r,\theta))_2 \\ & \int ds\; e^s \langle (\partial\phi)_1 \mathcal{E}_+ \mathcal{E}_+ (\partial\phi)_2(s) \rangle_{\text{p}} \langle E_{ij}(\theta_1) E_{ji}(\theta_2 - is) \rangle_{\text{aux}}. \end{aligned} \tag{A.21.5}$$

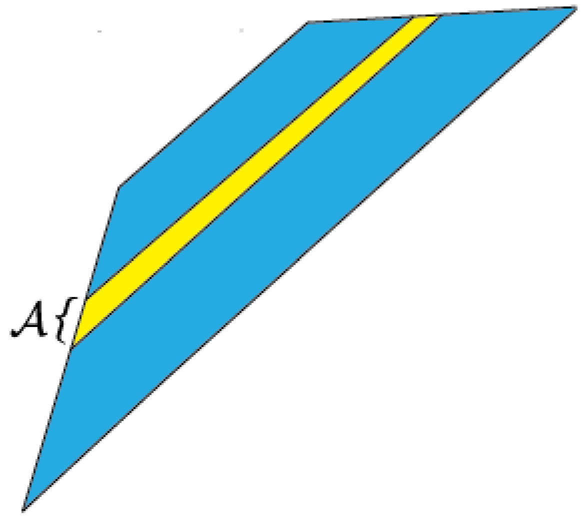


Figure A.4: The Hilbert space on a null hypersurface of a free (or superrenormalizable) quantum field theory factorizes across narrow pencils of width $\mathcal{A}$. One pencil is shown above in yellow. The neighboring pencils then can be thought of as an auxiliary system (shown in blue). In the vacuum, the state between the pencil and the auxiliary system factorizes, but in an excited state there could be nontrivial entanglement between the two systems.

Now on the pencil, $\mathcal{E}_+$ is the translation generator so we can use the commutator $i[\mathcal{E}_+, \partial\phi] = \partial^2\phi$ and the fact that $\mathcal{E}_+ |0\rangle = 0$ to get

$$\frac{d^2}{dX^+(y)^2} S_{\rm rel}(\rho|\rho_0) = \frac{1}{2}\sum_{ij}\int\int (drd\theta)_1 (drd\theta)_2 (f_{ij}(r,\theta))_1 (f_{ji}(r,\theta))_2$$
$$\int dse^s \langle(\partial^3\phi)_1(\partial\phi)_2(s)\rangle_{\rm p}\langle E_{ij}(\theta_1)E_{ji}(\theta_2 - is)\rangle_{\rm aux}. \quad (A.21.6)$$

Using the chiral two-point function we have

$$\langle(\partial^3\phi)_1(\partial\phi)_2(s)\rangle_{\rm p} = \frac{e^s}{(r_1 e^{i\theta_1} - r_2 e^{i\theta_2 + s})^4}. \quad (A.21.7)$$

Moreover, the auxiliary correlator is given by

$$\langle E_{ij}(\theta_1)E_{ji}(\theta_2 - is)\rangle = e^{-2\pi K_i} e^{\nu_{ij}(\theta_1-\theta_2+is)},\ \nu_{ij} = K_i - K_j \quad (A.21.8)$$

We now shift the integration contour by $s \to s + i(\theta_1 - \theta_2) + i\pi + \log(r_1/r_2)$. Putting this all together we are left with evaluating

$$e^{-\pi(K_i+K_j)} e^{-2i(\theta_1+\theta_2)} \left(\frac{r_1}{r_2}\right)^{i\nu_{ij}} \frac{1}{(r_1 r_2)^2}\int_{-\infty}^{\infty} ds \frac{e^{is\nu_{ij}} e^{2s}}{(1+e^s)^4}. \quad (A.21.9)$$

The θ integrals project us onto the $m = 2$ Fourier modes of f_{ij}, $f_{ij}^{(m=2)}(r)$, and we find the final answer

$$\frac{d^2}{dX^+(y)^2} S_{\rm rel}(\rho|\rho_0) = \frac{1}{2}\sum_{ij} |F_{ij}^{(2)}|^2 e^{-\pi(K_i+K_j)} g(\nu_{ij}) \tag{A.21.10}$$

where

$$F_{ij}^{(m)} = \int \frac{dr}{r^m} r^{i\nu_{ij}} f_{ij}^{(m)}(r),\ \ g(\nu) = \frac{\pi\nu(1+\nu^2)}{\sinh(\pi\nu)}. \tag{A.21.11}$$

This is precisely the answer that was found by different methods in [280]. Note that the right hand side of (A.21.10) is manifestly positive as required by the QNEC.

A.22 Ant Conjecture and Properties of Energy Minimizing States

In Sec. 8.5, we showed that the Ceyhan-Faulkner construction proves our conjecture in the pure-QFT limit. The original purpose of the CF construction, however, was to prove Wall's "ant conjecture" [337] (and thus, the Quantum Null Energy Condition [303]). It is therefore of interest to ask how closely related our coarse-graining conjecture is to the ant conjecture on Killing horizons.It is easy to see that Eqs. (8.4.14) and (8.4.15) imply the ant conjecture. Conversely, we will show in this section that the ant conjecture implies Eqs. (8.4.14) and (8.4.15), but only in 1+1 dimensions.

In Appendix A.22, we will review the ant conjecture. In Appendices A.22 and A.22, we establish some general properties that energy-minimizing states must satisfy. We show that the minimum energy completion has vanishing stress tensor on the unconstrained half-space, with all of the remaining energy appearing as a shock immediately on the cut. We also show that for a pure minimum energy state, the von Neumann entropy of semi-infinite regions is constant so long as the region's boundary lies on the unconstrained side. In 1+1 dimensions, we can also show that the integrated left stress tensor vanishes. Thus the ant conjecture implies Eqs. (8.4.14) and (8.4.15), the key properties of the field theory limit of our coarse-graining conjecture. In higher dimensions, we are unable to establish this result.

Ant Conjecture

Wall's "ant argument" for the Quantum Null Energy Condition in 1+1 dimensions invokes an ant that has walked left from $+\infty$ to v_0. (See Fig. A.5.) That is, given a global state ρ, the ant has knowledge only of the right half-space state $\rho_{>v_0}$. Pausing for rest, the ant contemplates how much energy it might still encounter in the remainder of its path, the interval $(-\infty, v_0]$. Because of global energy conditions, this amount is bounded from below.

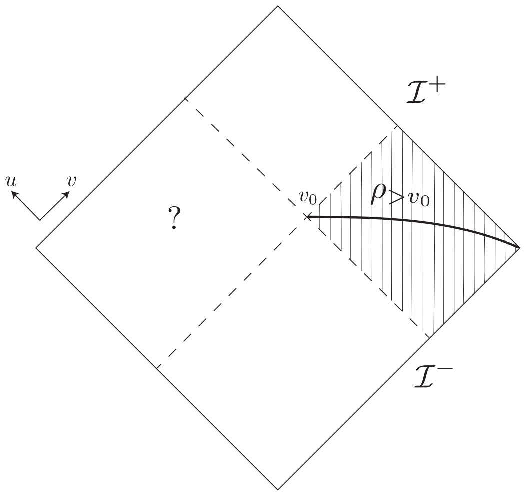


Figure A.5: The ant conjecture in 1+1 dimensions. A left-walking ant has access to all the information in the right wedge. It asks what is the least amount of additional energy it might still encounter to the left of v_0. The conjecture states that this is $\hbar S'/2\pi$, where S' is the right derivative of the von Neumann entropy of the reduced state on the right, evaluated at the cut. We show that this statement is equivalent to the nongravitational limit of our coarse-graining conjecture.

Let $M(v_0)$ be the lowest energy of any global state that reduces to the same $\rho_{>v_0}$.[11] More precisely,

$$M(v_0) \equiv \inf_{\hat{\rho}} \left[\int_{-\infty}^{\infty} d\tilde{v} \, \langle T_{vv} \rangle \big|_{\hat{\rho}} \right] \; . \tag{A.22.1}$$

The infimum is over all global states $\hat{\rho}$ that agree with ρ in the region $v > v_0$: $\mathrm{Tr}_{\leq v_0} \hat{\rho} = \rho_{>v_0}$. A strictly larger set of global states will agree with ρ on a smaller region, ρ_{v_1}, $v_1 > v_0$, so the infimum can only decrease with v:

$$\partial_v M(v) \leq 0 \; . \tag{A.22.2}$$

One can readily establish a lower bound on $M(v)$. The global energy appearing in the infimum can be written as $(\hbar/2\pi)(\partial_v \overline{K} - \partial_v K)$, by Eq. (8.4.10) and its left analogue. Moreover, Eq. (8.4.12) must hold for all states appearing in the infimum, so by adding $\partial_v K$ to it one finds that

$$M(v_0) \geq -\frac{\hbar}{2\pi} \partial_v S_{\mathrm{rel}}|_{v_0} \; , \tag{A.22.3}$$

[11] We should point out two differences in our conventions compared to [337]. First, we have switched the side on which the state is held fixed, from left to right. Secondly, in [337], M was the infimum of the energy density integrated only over the complement of that fixed half-space, whereas here it is the infimum the global energy. This choice is more convenient as otherwise the presence of distributional sources at the cut v_0 would lead to ambiguities and require a more elaborate definition. In this respect, our conventions agree with [16].

where we have used Eq. (8.4.9). Note that the lower bound is determined solely by the input state ρ.

Wall conjectured [337] that this inequality is saturated:

$$M(v_0) = -\frac{\hbar}{2\pi}\partial_v S_{\rm rel}|_{v_0} \ . \tag{A.22.4}$$

This conjecture is equivalent to the existence of a sequence of states $\hat{\rho}^{(n)}$, all of which reduce to $\rho_{>v_0}$ on the right, such that

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} d\tilde{v}\langle T_{vv}\rangle|_{\hat{\rho}^{(n)}} = -\frac{\hbar}{2\pi}\partial_v S_{\rm rel}|_{v_0} \ . \tag{A.22.5}$$

Here we will assume the conjecture to be true. We will be interested in certain universal properties of the states in this sequence that emerge in the limit as $n \to \infty$.

Properties of the minimum energy completion in 1+1 dimensions

For compactness of notation, we will ascribe any limiting properties of the states $\hat{\rho}^{(n)}$ as $n \to \infty$ to a "limit state" $\hat{\rho}^{\infty}$. We stress that such a state need not exist. Rather, $\hat{\rho}^{\infty}$ is shorthand for $\lim_{n\to\infty}\hat{\rho}^{(n)}$, where the limit should be moved outside of any maps of the state to other quantitities. Moreover, we indicate $\hat{\rho}^{\infty}$ as the argument of a map by the superscript ∞. For example,

$$S^{\infty}(v_0) = \lim_{n\to\infty}\left[-\mathrm{Tr}\,\hat{\rho}^{(n)}_{>v_0}\log\hat{\rho}^{(n)}_{>v_0}\right] \ . \tag{A.22.6}$$

By Eq. (A.22.4) and the discussion leading to Eq. (A.22.3), the state $\hat{\rho}^{\infty}$ must saturate both inequalities in Eq. (8.4.12):

$$\partial_v\overline{K}^{\infty}|_{v_0} = \partial_v\overline{S}^{\infty}|_{v_0} = \partial_v S^{\infty}|_{v_0} \ . \tag{A.22.7}$$

The first equality implies

$$\partial_v\overline{S}^{\infty}_{\rm rel}|_{v_0} = 0 \ . \tag{A.22.8}$$

Applying the left analogues of Eqs. (A.22.2) and (A.22.4) to $\overline{M}$ (with $\hat{\rho}^{\infty}$ as the input state!), we have

$$\partial_v^2\overline{S}^{\infty}_{\rm rel} \geq 0 \ , \tag{A.22.9}$$

for all v. The above two consequences of Wall's conjecture, combined with positivity and monotonicity of the left relative entropy,

$$\overline{S}^{\infty}_{\rm rel} \ \geq \ 0 \ , \tag{A.22.10}$$

$$\partial_v\overline{S}^{\infty}_{\rm rel} \ \geq \ 0 \ , \tag{A.22.11}$$

imply that

$$\partial_v\overline{S}^{\infty}_{\rm rel} = 0 \text{ for all } v < v_0 \ . \tag{A.22.12}$$

This is a very strong condition and it intuitively suggests that for $v < v_0$ we have a vacuum-like state. In particular all local observable in the region between v and v_0 for $v < v_0$ need to register vacuum values otherwise we would have $S^{\infty}_{\rm rel}(v_0) > S^{\infty}_{\rm rel}(v)$. This is particular tells us that

$$\langle T_{vv}(v)\rangle|_{\hat{\rho}^{\infty}} = 0 \text{ for } v < v_0 \ . \tag{A.22.13}$$

The above equation combined with Eq. (A.22.12) implies

$$\partial_v^2 \overline{S}^{\infty} = 0 \implies \partial_v \overline{S}^{\infty} = \alpha \text{ for } v \leq v_0 \ , \tag{A.22.14}$$

In fact, in 1+1 CFTs we can argue that $\alpha = 0$ by invoking the strengthened version of the QNEC [391, 392] [12]:

$$\langle T_{vv}\rangle \geq \frac{\hbar}{2\pi}\partial_v^2 \overline{S} + \frac{6\hbar}{c}(\partial_v \overline{S})^2 \ . \tag{A.22.15}$$

Now, Eq. (A.22.12) implies that

$$\langle T_{vv}\rangle = \frac{\hbar}{2\pi}\partial_v^2 \overline{S} \text{ for } v < v_0 \ , \tag{A.22.16}$$

which together with Eq. (A.22.15) implies that $\partial_v \overline{S} = 0$. So, we conclude that for $v < v_0$,

$$\partial_v \overline{S}^{\infty} = 0 \text{ and } \partial_v \overline{S}^{\infty}_{\rm rel} = 0 \implies \tag{A.22.17}$$

$$\lim_{\epsilon\to 0}\int_{-\infty}^{v_0-\epsilon} d\tilde{v}\, \langle T_{vv}\rangle\big|_{\hat{\rho}^{(\infty)}} = 0 \ . \tag{A.22.18}$$

We also know that

$$\lim_{\epsilon\to 0}\left[\int_{-\infty}^{v_0+\epsilon} d\tilde{v}\, \langle T_{vv}\rangle\big|_{\hat{\rho}^{\infty}}\right] = \frac{\hbar}{2\pi}\partial_v S\big|_{v_0} \ . \tag{A.22.19}$$

This along with Eq. (A.22.17) implies that the minimum energy state contains a shock (a delta function in energy density) at v_0, and vanishing energy to its left:

$$\langle T_{vv}\rangle = \left(\frac{\hbar}{2\pi}\partial_v S\big|_{v_0}\right)\delta(v - v_0) \text{ for } v \leq v_0 \ . \tag{A.22.20}$$

If $\hat{\rho}^{\infty}$ is a pure state[13] this further implies that

$$\partial_v S = 0 \text{ for } v < v_0 \ . \tag{A.22.21}$$

[12] We thank Aron Wall for suggesting the use of strengthened QNEC here

[13] The conclusion would extend to mixed states under the assumption that $\Delta S(v)$ remains bounded from below for any v in the limit as $n \to \infty$. The status of this assumption is not clear to us, however.

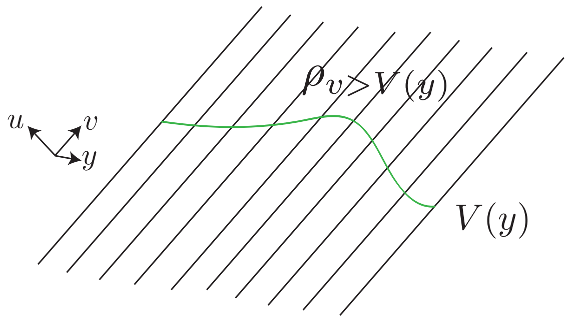


Figure A.6: A general cut of the Rindler horizon in $d > 2$. An army of ants marches down along the null direction towards the cut. Given the state above the cut, they ask what is the minimum energy still to come.

In fact, we expect that $\hat{\rho}^\infty$ can always taken to be pure. The basic idea is that any density operator can be purified by a suitable auxiliary system. In general the auxiliary system has to be external, but we now argue it can be taken to be distant soft modes in the quantum field itself.

Suppose we had identified a sequence $\hat{\rho}^{(n)}$ that limits to a mixed $\hat{\rho}^\infty$. Finiteness of the energy requires that each state in the sequence looks like the vacuum in some sufficiently distant left region $v < v^{(n)}$ with $v^{(n)} < v_0$. We can take $v^{(n)} \to -\infty$ as $n \to \infty$. We can add a purification of the state $\hat{\rho}^{(n)}$ in soft wavepackets localized to the region $v < v^{(n)}$. This results in a new, pure state and we redefine $\hat{\rho}^{(n)}$ to be that state. Since we have not modified the state in the region $v > v_0$, it will still reduce to the given right state $\rho_{>v_0}$; and since the region $v < v^{(n)}$ is semi-infinite, we can take the purifying wave-packets to have arbitrarily small energy. In particular, we can take their contribution to the energy to vanish in the limit as $n \to \infty$.

Higher-dimensional case

The generalization of the above result to higher dimensions is straightforward. We can consider any Killing horizon $N = \mathbf{R} \times \mathcal{B}$, with $v \in \mathbf{R}$ an affine parameter along light-rays orthogonal to the $d-2$ dimensional spatial surface $\mathcal{B}$ with collective coordinates y.

The analogue of the 1+1 dimensional ant is now an army of ants that have walked along the null generators from $v = +\infty$ to the position $v = V(y)$, so that they know the state $\rho_{>V(y)}$. (See Fig. A.6.) The ants again ask about the minimum global energy consistent with this knowledge, $M[V(y)]$. This quantity can only decrease under deformations of $V(y)$ that are everywhere positive:

$$\frac{\delta M}{\delta V(y)} \le 0 \ . \tag{A.22.22}$$

The definition of M differs from the 1+1 case only through an additional transverse integral over $d^{d-2}y$. It can be shown [328, 329] that the modular Hamiltonian, too, is simply the sum of the local Rindler energies associated with the individual null generators, Eq. (8.4.6):

$$\Delta K(V_0(y)) = \frac{2\pi}{\hbar}\int d^{d-2}y \int_{V_0(y)}^{\infty} dv\,(v - V_0(y))T_{vv}\ , \tag{A.22.23}$$

By the analogue of Eq. (8.4.12),

$$\frac{\delta \overline{K}}{\delta V(y)} \geq \frac{\delta \overline{S}}{\delta V(y)} \geq \frac{\delta S}{\delta V(y)}\ , \tag{A.22.24}$$

one finds

$$M \geq -\frac{\hbar}{2\pi}\frac{\delta S_{\text{rel}}}{\delta V(y)}\ . \tag{A.22.25}$$

The ant conjecture again demands that this be an equality. That is, there exists a global state $\hat{\rho}^{\infty}$ that saturates Eq. (A.22.25) (or if not, saturation can at least be approached, in the limit of a sequence of global states). The same arguments as in the 1+1 dimensional case imply that $\hat{\rho}^{\infty}$ satisfies

$$\frac{\delta \overline{S}_{\text{rel}}^{\infty}}{\delta V(y)} = 0 \text{ for all } v < V_0(y)\ . \tag{A.22.26}$$

Exactly as in the 1+1 case, the above condition implies

$$\langle T_{vv}(v)\rangle|_{\hat{\rho}^{\infty}} = 0 \quad \text{for} \quad v < V_0(y)\ , \tag{A.22.27}$$

$$\frac{\delta^2 \overline{S}^{\infty}}{\delta V(y_1)\delta V(y_2)} = 0 \implies \frac{\delta \overline{S}^{\infty}}{\delta V(y)} = \alpha \ \text{ for all } v < V_0(y)\ . \tag{A.22.28}$$

where α is some constant. As was discussed at the end of the previous section, we can take ρ^{∞} to be a limit of pure states where we additionally have

$$\frac{\delta S^{\infty}}{\delta V(y)} = \alpha \ \text{ for all } v < V_0(y)\ , \tag{A.22.29}$$

At this point, it would be nice to argue that $\alpha = 0$ as in the 1+1 dimensional case, but we will leave this to future work. If we assume that $\alpha = 0$, then the purity of the global state implies

$$\frac{\delta \overline{S}^{\infty}}{\delta V(y)} = 0 \ \text{ for all } v < V_0(y)\ , \tag{A.22.30}$$

and together with Eq. (A.22.26) one obtains

$$\lim_{\epsilon\to 0}\left[\int_{-\infty}^{V_0(y)-\epsilon} d\tilde{v}\,\langle T_{vv}\rangle\big|_{\hat{\rho}^{\infty}}\right] = 0\ , \tag{A.22.31}$$

for all y. Note that Eq. (A.22.31) does not otherwise follow from Eq. (A.22.27): because $\hat{\rho}^\infty$ is defined as a limit of a sequence, it would be possible for $\langle T_{vv}\rangle$ to approach zero while its integral approaches a finite value. Assuming the ant conjecture, that Eq. (A.22.25) is an equality, it follows that

$$\langle T_{vv}(v,y)\rangle\big|_{\hat{\rho}^\infty} = \left(\frac{\hbar}{2\pi}\frac{\delta S}{\delta V(y)}\Big|_{V_0}\right)\delta(v-V_0(y)) \quad \text{for } v \leq V_0(y) \ . \tag{A.22.32}$$

To summarize, in 1+1 dimensions, the ant conjecture implies the key properties of the coarse-graining states we conjectured: Eqs. (8.4.14) and (8.4.15) hold on a Killing horizon. In greater than 1+1 dimensions, this implication obtains only with the unproven assumption that $\alpha = 0$ above.

A.23 Null Limit of the Kink Transform

In this appendix we apply the kink transform to a Cauchy slice Σ that has null segments. In the null limit we express the kink transform in terms of the null initial value problem. We then show that this leads to a shock in the Weyl tensor for $d > 2$. From this Weyl shock we extract the boundary stress tensor shock. This serves a two-fold purpose. The first is that it provides direct intuition for how the kink transform modifies the geometry. The second is that, as will be evident from the calculation below, the derivation of the stress tensor shock from the Weyl shock works even for wiggly cuts of the Rindler horizon on the boundary.[14]

Let N_k be a null segment of Σ in a neighborhood of $\mathcal{R}$ and let k^a be the null generator of N_k. We now allow the boundary anchor of $\mathcal{R}$ to be an arbitrary cut $V_0(y)$ of the Rindler horizon, as considered in Sec. 9.2. Lastly, denote by P^a_μ and P^i_μ the projectors onto N_k and cross-sections of N_k (including the RT surface $\mathcal{R}$), respectively. We can compose these to obtain the projector P^i_a.

By Eq. (9.3.4), when Σ is spacelike in a neighborhood of $\mathcal{R}$ the kink transform can be contracted as follows:

$$x^a(K_\Sigma)_{ab} \to x^a(K_\Sigma)_{ab} - \sinh(2\pi s)x_b\ \delta(\mathcal{R}) \ . \tag{A.23.1}$$

In the null limit both x^a and t^μ approach k^a. Therefore, the quantity in the LHS of Eq. (A.23.1) has the following null limit:

$$x^a(K_\Sigma)_{ab} \overset{\text{null}}{\to} k^a\nabla_a k_b \ . \tag{A.23.2}$$

The transformation of Eq. (A.23.1) then becomes

$$\kappa \to \kappa - \sinh(2\pi s)\delta(\lambda) \ , \tag{A.23.3}$$

[14]The results of this section do not apply when $d = 2$, as the shear and the Weyl tensor vanish identically. However in $d = 2$ there is no distinction between flat and wiggly cuts on the boundary so we gain neither additional intuition nor generality compared with the analysis in Sec. 9.4.

where λ is a null parameter adapted to k^a and κ is the inaffinity defined by

$$k^b \nabla_b k^a = \kappa k^a \ . \tag{A.23.4}$$

We refer to this transformation as the *left stretch*, as it arises from a one-sided dilatation along N_k. This transformation was originally described in [139] in the context of black hole coarse-graining.

We now show that the left stretch generates a Weyl tensor shock at the RT surface. The shear of a null congruence is defined by

$$\sigma_{ij} = P_i^a P_j^b \nabla_{(a} k_{b)} \ . \tag{A.23.5}$$

It satisfies the evolution equation [375]

$$\mathcal{L}_k \sigma_{ij} = \kappa \sigma_{ij} + \sigma_i{}^k \sigma_{kj} - P_i^\mu P_j^\mu k^a k^b C_{a\mu b\nu} \ . \tag{A.23.6}$$

Now let λ be a parametrization of N_k adapted to k^a, with $\lambda = 0$ corresponding to $\mathcal{R}$. In terms of λ, the evolution equation can be written as

$$\partial_\lambda \sigma_{ij} = \kappa \sigma_{ij} + \sigma_i{}^k \sigma_{kj} - C_{\lambda i \lambda j} \ . \tag{A.23.7}$$

Consider now the new spacetime $\mathcal{M}_s$ generated by the left stretch. As in Sec. 9.4, we denote quantities in $\mathcal{M}_s$ with tildes. We can then write the evolution equation in $\mathcal{M}_s$,

$$\partial_{\tilde{\lambda}} \tilde{\sigma}_{ij} = \tilde{\kappa} \tilde{\sigma}_{ij} + \tilde{\sigma}_i{}^k \tilde{\sigma}_{kj} - \tilde{C}_{\lambda i \lambda j} \ . \tag{A.23.8}$$

Since k^a is tangent to N_k, and $(N_k)_s = N_k$ as submanifolds, we can identify k^a with $\tilde{k}^a$. Thus we can use the same parameter λ in both spacetimes. Since σ_{ij} is intrinsic to N_k, we can identify σ_{ij} and $\tilde{\sigma}_{ij}$ for the same reason. Comparing Eqs. (A.23.7) and (A.23.8), and inserting Eq. (A.23.3), we find that there is a Weyl shock

$$\tilde{C}_{\lambda i \lambda j} = C_{\lambda i \lambda j} - \sinh(2\pi s) \sigma_{ij} \delta(\lambda) \ . \tag{A.23.9}$$

We now show that the Weyl shock Eq. (A.23.9) reproduces the near boundary shock Eq. (9.2.44), but now for wiggly cuts of the Rindler horizon. To do this, we evaluate both σ_{ij} and $C_{\lambda i \lambda j}$ in Fefferman-Graham coordinates to leading non-trivial order. The Fefferman-Graham coordinates for $\mathcal{M}$ and $\mathcal{M}_s$ are defined exactly as in Sec. 9.4, except we now use null coordinates (u, v) and $(\tilde{u}, \tilde{v})$ on the boundary as defined in Sec. 9.2. To start with, we note that $k_a \partial_z \overline{X}^a = 0$ since $\partial_z \overline{X}^a$ is tangent to the RT surface. Evaluating this at leading order yields the relation

$$k_z = -d z^{d-3} \mathcal{U}_{(d)} + \mathcal{O}(z^{d-4}) \ . \tag{A.23.10}$$

We recall that

$$\mathcal{U}_{(d)} = -\frac{4G}{d} \frac{\delta S}{\delta V}\Big|_{V_0} \ . \tag{A.23.11}$$

Moreover, the projector is given by

$$P_i^\mu = \partial_i \overline{X}^\mu \ . \tag{A.23.12}$$

From this definition, one can check that

$$P_i^z = \delta_i^z + \mathcal{O}(z^{d-1}) \ , \tag{A.23.13}$$
$$P_i^A = \mathcal{O}(z^{d-1}) \ . \tag{A.23.14}$$

Furthermore,

$$\begin{aligned}
&\nabla_z k_A,\ \nabla_A k_z \sim \mathcal{O}(z^{-1}) \ ,\\
&\nabla_A k_B \sim \mathcal{O}(1) \ ,\\
&\nabla_z k_z = -d(d-2)\mathcal{U}_{(d)} z^{d-4} + \mathcal{O}(z^{d-5}) \ ,
\end{aligned} \tag{A.23.15}$$

where we have used that $k^A \sim \mathcal{O}(1)$. Hence to leading order we simply have

$$\sigma_{ij} = -d(d-2)\mathcal{U}_{(d)} z^{d-4} \delta_i^z \delta_j^z + \mathcal{O}(z^{d-5}) \ . \tag{A.23.16}$$

Finally, a straightforward but tedious calculation of the Weyl tensor yields

$$\tilde{C}_{\tilde{v}i\tilde{v}j} = C_{vivj} - 8\pi G(d-2)\left(\langle \tilde{T}_{\tilde{v}\tilde{v}}\rangle - \langle T_{vv}\rangle\right) z^{d-4}\delta_i^z\delta_j^z \delta(\tilde{v} - V_0) + \mathcal{O}(z^{d-5}) \ , \tag{A.23.17}$$

where we have used that $\lambda \to v, \tilde{v}$ as $z \to 0$ in $\mathcal{M}, \mathcal{M}_s$ respectively. Putting this together yields the desired shock for wiggly cuts of the Rindler horizon.

A.24 Notation and Definitions

Basic Notation

Notation for basic bulk and boundary quantities

- Bulk indices are $\mu, \nu, \ldots$.
- Boundary indices are $i, j, \ldots$. Then $\mu = (z, i)$.
- We assume a Fefferman–Graham form for the metric: $ds^2 = \frac{L^2}{z^2}(dz^2 + \overline{g}_{ij} dx^i dx^j)$.
- The expansion for $\overline{g}_{ij}(x, z)$ at fixed x is

  $$\overline{g}_{ij} = g_{ij}^{(0)} + z^2 g_{ij}^{(2)} + z^4 g_{ij}^{(4)} + \cdots + z^d \log z g_{ij}^{(d,\log)} + z^d g_{ij}^{(d)} + \cdots \ . \tag{A.24.1}$$

  The coefficients $g_{ij}^{(n)}$ for $n < d$ and $g_{ij}^{(d,\log)}$ are determined in terms of $g_{ij}^{(0)}$, while $g_{ij}^{(d)}$ is state-dependent and contains the energy-momentum tensor of the CFT. If d is even, then $g_{ij}^{(d,\log)} = 0$. To avoid clutter we will often write $g_{ij}^{(0)}$ simply as g_{ij}. Unless otherwise indicated, i, j indices are raised and lowered by $g_{ij}^{(0)}$.
- We use $\mathcal{R}$, $\mathcal{R}_{\mu\nu}$, $\mathcal{R}_{\mu\nu\rho\sigma}$ to denote bulk curvature tensors, and R, R_{ij}, R_{ijmn} to denote boundary curvature tensors.

Notation for extremal surface and entangling surface quantities

- Extremal surface indices are $\alpha, \beta, \ldots$.
- Boundary indices are $a, b, \ldots$. Then $\alpha = (z, a)$.
- The extremal surface is parameterized by functions $\overline{X}^\mu(z, y^a)$. We choose a gauge such that $X^z = z$, and expand the remaining coordinates as

$$\overline{X}^i = X^i_{(0)} + z^2 X^i_{(2)} + z^4 X^i_{(4)} + \cdots + z^d \log z X^i_{(d,\log)} + z^d X^i_{(d)} + \cdots . \tag{A.24.2}$$

  The coefficients $X^i_{(n)}$ for $n < d$ and $X^i_{(d,\log)}$ are determined in terms of $X^i_{(0)}$ and $g^{(0)}_{ij}$, while $X^i_{(d)}$ is state-dependent and is related to the renormalized entropy of the CFT region.

- The extremal surface induced metric will be denoted $\overline{h}_{\alpha\beta}$ and gauge-fixed so that $\overline{h}_{za} = 0$.
- The entangling surface induced metric will be denoted h_{ab}.
- Note that we will often want to expand bulk quantities in z at fixed y instead of fixed x. For instance, the bulk metric at fixed y is

$$\begin{aligned} \overline{g}_{ij}(y, z) &= \overline{g}_{ij}(\overline{X}(z, y), z) = \overline{g}_{ij}(X_{(0)}(y) + z^2 X_{(2)}(y) + \cdots, z) \\ &= g^{(0)}_{ij} + z^2 \left(g^{(2)}_{ij} + X^m_{(2)} \partial_m g^{(0)}_{ij} \right) + \cdots \end{aligned} \tag{A.24.3}$$

  Similar remarks apply for things like Christoffel symbols. The prescription is to always compute the given quantity as a function of x first, the plug in $\overline{X}(y, z)$ and expand in a Taylor series.

Intrinsic and Extrinsic Geometry

Now will introduce several geometric quantities, and their notations, which we will need. First, we define a basis of surface tangent vectors by

$$e^i_a = \partial_a X^i. \tag{A.24.4}$$

We will also make use of the convention that ambient tensors which are not inherently defined on the surface but are written with surface indices (a, b, etc.) are defined by contracting with e^i_a. For instance:

$$g^{(2)}_{aj} = e^i_a g^{(2)}_{ij}. \tag{A.24.5}$$

We can form the surface projector by contracting the surface indices on two copies of e^i_a:

$$P^{ij} = h^{ab} e^i_a e^j_b = e^i_a e^{ja}. \tag{A.24.6}$$

We introduces a surface covariant derivative D_a that acts as the covariant derivative on both surface and ambient indices. So it is compatible with both metrics:

$$D_a h_{bc} = 0 = D_a g_{ij}. \tag{A.24.7}$$

Note also that when acting on objects with only ambient indices, we have the relationship

$$D_a V^{ij\cdots}_{pq\cdots} = e^m_a \nabla_m V^{ij\cdots}_{pq\cdots}, \tag{A.24.8}$$

where ∇_i is the ambient covariant derivative compatible with g_{ij}.

The extrinsic curvature is computed by taking the D_a derivative of a surface basis vector:

$$K^i_{ab} = -D_a e^i_b = -\partial_a e^i_b + \gamma^c_{ab} e^i_b - \Gamma^i_{ab}. \tag{A.24.9}$$

Note the overall sign we have chosen. Here γ^c_{ab} is the Christoffel symbol of the metric h_{ab}, and the lower indices on the Γ symbol were contracted with two basis tangent vectors to turn them into surface indices. Note that K^i_{ab} is symmetric in its lower indices. It is an exercise to check that it is normal to the surface in its upper index:

$$e_{ic} K^i_{ab} = 0. \tag{A.24.10}$$

The trace of the extrinsic curvature is denoted by K^i:

$$K^i = h^{ab} K^i_{ab}. \tag{A.24.11}$$

Below we will introduce the null basis of normal vectors k^i and l^i. Then we can define expansion $\theta_{(k)}$ ($\theta_{(l)}$) and shear $\sigma^{(k)}_{ab}$ ($\sigma^{(l)}_{ab}$) as the trace and traceless parts of $k_i K^i_{ab}$ ($l_i K^i_{ab}$), respectively.

There are a couple of important formulas involving the extrinsic curvature. First is the Codazzi Equation, which can be computed from the commutator of covariant derivatives:

$$\begin{aligned} D_c K^i_{ab} - D_b K^i_{ac} &= (D_b D_c - D_c D_b) e^i_a \\ &= R^i{}_{abc} - r^d{}_{abc} e^i_d. \end{aligned} \tag{A.24.12}$$

Here $R^i{}_{abc}$ is the ambient curvature (appropriately contracted with surface basis vectors), while $r^d{}_{abc}$ is the surface curvature. We can take traces of this equation to get others. Another useful thing to do is contract this equation with e^i_d and differentiate by parts, which yields the Gauss–Codazzi equation:

$$K_{cdi} K^i_{ab} - K_{bdi} K^i_{ac} = R_{dabc} - r_{dabc}. \tag{A.24.13}$$

Various traces of this equation are also useful.

Null Normals k and l

A primary object in our analysis is the bull vector k^i, which is orthogonal to the entangling surface and gives the direction of the surface deformation. It will be convenient to also introduce the null normal l^i, which is defined so that $l_i k^i = +1$. This choice of sign is different from the one that is usually made in these sorts of analysis, but it is necessary to avoid a proliferation of minus signs. With this convention, the projector onto the normal space of the surface is

$$N^{ij} \equiv g^{ij} - P^{ij} = k^i l^j + k^j l^i = 2k^{(i}l^{j)}. \tag{A.24.14}$$

As we did with the tangent vectors e_a^i, we will introduce a shorthand notation to denote contraction with k^i or l^i: any tensor with k or l index means it has been contracted with k^i or l^i. As such we will avoid using the letters k and l as dummy indices. For instance.

$$R_{kl} \equiv k^i l^j R_{ij}. \tag{A.24.15}$$

Another quantity associated with k^i and l^i is the normal connection w^a, defined through

$$w_a \equiv l_i D_a k^i. \tag{A.24.16}$$

With this definition, the tangent derivative of k^i can be shown to be

$$D_a k^i = w_a k^i + K_{ab}^k e^{bi}, \tag{A.24.17}$$

which is a formula that is used repeatedly in our analysis.

At certain intermediate stages of our calculations it will be convenient to define extensions of k^i and l^i off of the entangling surface, so here we will define such an extension. Surface deformations in both the QNEC and QFC follow geodesics generated by k^i, so it makes sense to define k^i to satisfy the geodesic equation:

$$\nabla_k k^i = 0. \tag{A.24.18}$$

However, we will *not* define l^i by parallel transport along k^i. It is conceptually cleaner to maintain the orthogonality of l^i to the surface even as the surface is deformed along the geodesics generated by k^i. This means that l^i satisfies the equation

$$\nabla_k l^i = -w^a e_a^i. \tag{A.24.19}$$

These equations are enough to specify l^i and k^i on the null surface formed by the geodesics generated by k^i. To extend k^i and l^i off of this surface, we specify that they are both parallel-transported along l^i. In other words, the null surface generated by k^i forms the initial condition surface for the vector fields k^i and l^i which satisfy the differential equations

$$\nabla_l k^i = 0, \qquad \nabla_l l^i = 0\ . \tag{A.24.20}$$

This suffices to specify k^i an l^i completely in a neighborhood of the original entangling surface. Now that we have done that, we record the commutator of the two fields for future use:

$$[k, l]^i = \nabla_k l^i - \nabla_l k^i = -w^c e_c^i. \tag{A.24.21}$$

A.25 Surface Variations

Most of the technical parts of our analysis have to do with variations of surface quantities under the deformation $X^i \to X^i + \delta X^i$ of the surface embedding coordinates. Here δX^i should be interpreted a vector field defined on the surface. In principle it can include both normal and tangential components, but since tangential components do not actually correspond to physical deformations of the surface we will assume that δX^i is normal. The operator δ denotes the change in a quantity under the variation. In the case where $\delta X^i = \partial_\lambda X^i$, which is the case we are primarily interested in, δ can be identified with ∂_λ. With this in mind, we will always impose the geodesic equation on k^i whenever convenient. In terms of the notation we are introducing here, this is

$$\delta k^i = -\Gamma^i_{kk}. \tag{A.25.1}$$

To make contact with the main text, we will use the notation $k^i \equiv \delta X^i$, and assume that k^i is null since that is ultimately the case we care about. Some of the formulas we discuss below will not depend on the fact that k^i is null, but we will not make an attempt to distinguish them.

Ambient Quantities For ambient quantities, like curvature tensors, the variation δ can be interpreted straightforwardly as $k^i\partial_i$ with no other qualification. Thus we can freely use, for instance, the ambient covariant derivative ∇_k to simplify the calculations of these quantities. Note that δ itself is not the covariant derivative. As defined, δ is a coordinate dependent operator. This may be less-than-optimal from a geometric point of view, but it has the most conceptually straightforward interpretation in terms of the calculus of variations. In all of the variational formulas below, then, we will see explicit Christoffel symbols appear. Of course, ultimately these non-covariant terms must cancel out of physical quantities. That they do serves as a nice check on our algebra.

Tangent Vectors The most fundamental formula is that of the variation of the tangent vectors $e^i_a \equiv \partial_a X^i$. Directly from the definition, we have

$$\delta e^i_a = \partial_a k^i = D_a k^i - \Gamma^i_{ak} = w_a k^i + K^k_{ab} e^{bi} - \Gamma^i_{ak}. \tag{A.25.2}$$

This formula, together with the discussion of how ambient quantities transform, can be used together to compute the variations of many other quantities.

Intrinsic Geometry and Normal Vectors The intrinsic metric variation is easily computed from the above formula as

$$\delta h_{ab} = 2K^k_{ab}. \tag{A.25.3}$$

From here we can find the variation of the tangent projector, for instance:

$$\begin{aligned}\delta P^{ij} &= \delta h^{ab} e_a^i e_b^j + 2h^{ab} e_a^{(i} \partial_b k^{j)} \\ &= -2K_k^{ab} e_a^i e_b^j + 2h^{ab} e_a^{(i} D_b k^{j)} - 2h^{ab} e_a^{(i} \Gamma^{j)}_{bk} \\ &= 2w^a e_a^{(i} k^{j)} - 2h^{ab} e_a^{(i} \Gamma^{j)}_{bk}. \end{aligned} \tag{A.25.4}$$

Notice that the second line features a derivative of $k^i = \delta X^i$. In a context where we are taking functional derivatives, such as when computing equations of motion, this term would require integration by parts. We can write the last line covariantly as

$$\nabla_k P^{ij} = 2w^a e_a^{(i} k^{j)}. \tag{A.25.5}$$

Earlier we saw that l^i satisfied the equation $\nabla_k l^i = -w^a e_a^i$ as a result of keeping l^i orthogonal to the surface even as the surface is deformed. In the language of this section, this is seen by the following manipulation:

$$e_a^i \delta l_i = -l_i \partial_a k^i = -w_a - \Gamma^l_{ak}. \tag{A.25.6}$$

Again, note the derivative of k^i. It is easy to confirm that represents the only nonzero component of $\nabla_k l^i$.

The normal connection $w_a = l^i D_a k_i$ makes frequent appearances in our calculations, and we will need to know its variation. We can calculate that as follows:

$$\begin{aligned}\delta w_a &= \delta l^i D_a k_i + l^i \partial_a \delta k_i - l^i \delta \Gamma^n_{ji} e_a^j k_n - l^i \Gamma^n_{ji} \partial_a k^j k_n - l^i \Gamma^n_{ji} e_a^j \delta k_n \\ &= \nabla_k l^i D_a k_i + R_{klak} \\ &= -w^c K_{ac} + R_{klak}. \end{aligned} \tag{A.25.7}$$

Extrinsic Curvatures The simplest extrinsic curvature variation is that of the trace of the extrinsic curvature

$$\delta K^i = -K^m \Gamma^i_{mk} - D_a D^a k^i - R^i_{mkj} P^{mj} + \left(2D^a (K^k_{ad}) - D_d (K^k)\right) e^{di} - 2K_k^{ab} K^i_{ab} \tag{A.25.8}$$

Note that the combination $\delta K^i + K^k \Gamma^i_{km} k^m$ is covariant, so it makes sense to write

$$\nabla_k K^i = -D_a D^a k^i - R^i_{mkj} P^{mj} + \left(2D^a (K^k_{ad}) - D_d (K^k)\right) e^{di} - 2K_k^{ab} K^i_{ab} \tag{A.25.9}$$

This formula is noteworthy because of the first term, which features derivatives of $k^i = \delta X^i$. This is important because when K^i occurs inside of an integral and we want to compute the functional derivative then we have to first integrate by parts to move those derivatives off of k^i. This issue arises when computing Θ as in the QFC, for instance.

We can contract the previous formulas with l^i and k^i to produce other useful formulas. For instance, contracting with k^i leads to

$$\delta K^k = -K^{kab} K^k_{ab} - R_{kk}, \tag{A.25.10}$$

which is nothing but the Raychaudhuri equation.

The variation of the full extrinsic curvature K^i_{ab} is quite complicated, but we will not needed. However, its contraction with k^i will be useful and so we record it here:

$$k_i \delta K^i_{ab} = -K^j_{ab}\Gamma^m_{jn} k_m k^n - k_i D_a D_b k^i - R_{kakb}. \tag{A.25.11}$$

A.26 z-Expansions

Bulk Metric

We are focusing on bulk theories with gravitational Lagrangians

$$\mathcal{L} = \frac{1}{16\pi G_N}\left(\frac{d(d-1)}{\tilde{L}^2} + \mathcal{R} + \ell^2\lambda_1\mathcal{R}^2 + \ell^2\lambda_2\mathcal{R}^2_{\mu\nu} + \ell^2\lambda_{\rm GB}\mathcal{L}_{\rm GB}\right). \tag{A.26.1}$$

where $\mathcal{L}_{GB} = \mathcal{R}^2_{\mu\nu\rho\sigma} - 4\mathcal{R}^2_{\mu\nu} + \mathcal{R}^2$ is the Gauss-Bonnet Lagrangian, ℓ is the cutoff length scale of the bulk effective field theory, and the couplings λ_1, λ_2, and $\lambda_{\rm GB}$ are defined to be dimensionless. We have decided to include $\mathcal{L}_{GB}$ as part of our basis of interactions rather than $\mathcal{R}^2_{\mu\nu\rho\sigma}$ because of certain nice properties that the Gauss-Bonnet term has, but this is not important.

We recall that the Fefferman–Graham form of the metric is defined by

$$ds^2 = \frac{1}{z^2}(dz^2 + \overline{g}_{ij}dx^i dx^j), \tag{A.26.2}$$

where $\overline{g}_{ij}(x,z)$ is expanded as a series in z:

$$\overline{g}_{ij} = g^{(0)}_{ij} + z^2 g^{(2)}_{ij} + z^4 g^{(4)}_{ij} + \cdots + z^d \log z\, g^{(d,\log)}_{ij} + z^d g^{(d)}_{ij} + \cdots . \tag{A.26.3}$$

In principle, one would evaluate the equation of motion from the above Lagrangian using the Fefferman–Graham metric form as an ansatz to compute these coefficients. The results of this calculation are largely in the literature, and we quote them here. To save notational clutter, in this section we will set $g_{ij} = g^{(0)}_{ij}$.

The first nontrivial term in the metric expansion is independent of the higher-derivative couplings, and in fact is completely determined by symmetry [291]:

$$g^{(2)}_{ij} = -\frac{1}{d-2}\left(R_{ij} - \frac{1}{2(d-1)}Rg_{ij}\right). \tag{A.26.4}$$

The next term is also largely determined by symmetry, except for a pair of coefficients [291]. We are only interested in the kk-component of $g^{(4)}_{ij}$, and where one of the coefficients drops out. The result is

$$\begin{aligned} g^{(4)}_{kk} = \frac{1}{d-4}\Bigg[&\kappa C_{kijm}C_k{}^{ijm} + \frac{1}{8(d-1)}\nabla^2_k R - \frac{1}{4(d-2)}k^i k^j \Box R_{ij} \\ &- \frac{1}{2(d-2)}R^{ij}R_{kikj} + \frac{d-4}{2(d-2)^2}R_{ki}R^i_k + \frac{1}{(d-1)(d-2)^2}RR_{kk}\Bigg], \end{aligned} \tag{A.26.5}$$

where C_{ijmn} is the Weyl tensor and

$$\kappa = -\lambda_{GB}\frac{\ell^2}{L^2}\left(1 + O\left(\frac{\ell^2}{L^2}\right)\right). \tag{A.26.6}$$

In $d = 4$ we will need an expression for $g_{kk}^{(4,\log)}$ as well. One can check that this is obtainable from $g_{kk}^{(4)}$ by first multiplying by $4-d$ and then setting $d \to 4$. We record the answer for future reference:

$$g_{kk}^{(4,\log)} = -\left[\kappa C_{kijm}C_k{}^{ijm} + \frac{1}{24}\nabla_k^2 R - \frac{1}{8}k^i k^j \Box R_{ij} - \frac{1}{4}R^{ij}R_{kikj} + \frac{1}{12}RR_{kk}\right]. \tag{A.26.7}$$

Extremal Surface Coordinates

The extremal surface position is determined by extremizing the generalized entropy functional [283, 284]:

$$S_{\text{gen}} = \frac{1}{4G_N}\int\sqrt{h}\left[1 + 2\lambda_1\ell^2\mathcal{R} + \lambda_2\ell^2\left(\mathcal{R}_{\mu\nu}\mathcal{N}^{\mu\nu} - \frac{1}{2}\mathcal{K}_\mu\mathcal{K}^\mu\right) + 2\lambda_{\text{GB}}\ell^2\overline{r}\right] + S_{\text{bulk}}. \tag{A.26.8}$$

Here we are using $\mathcal{K}^i$ to denote the extrinsic curvature and $\overline{r}$ the intrinsic Ricci scalar of the surface.

The equation of motion comes from varying S_{gen} and is (ignoring the S_{bulk} term for simplicity)

$$\begin{aligned} 0 = \mathcal{K}^\mu &\left[1 + 2\lambda_1\ell^2\mathcal{R} + \lambda_2\ell^2\left(\mathcal{R}_{\rho\nu}\mathcal{N}^{\rho\nu} - \frac{1}{2}\mathcal{K}_\rho\mathcal{K}^\rho\right) + 2\lambda_{\text{GB}}\ell^2\overline{r}\right] + 2\lambda_1\ell^2\nabla^\mu\mathcal{R} \\ &+ \lambda_2\ell^2\Big(\mathcal{N}^{\rho\nu}\nabla^\mu\mathcal{R}_{\rho\nu} + 2\mathcal{P}^{\rho\nu}\nabla_\rho\mathcal{R}^\mu_\nu - 2\mathcal{R}^\mu_\rho\mathcal{K}^\rho + 2\mathcal{K}^{\mu\alpha\beta}\mathcal{R}_{\alpha\beta} + D_\alpha D^\alpha\mathcal{K}^\mu \\ &+ \mathcal{K}^\rho\mathcal{R}_{\mu\sigma\rho\nu}\mathcal{P}^{\nu\sigma} + 2\mathcal{K}^{\mu\alpha\beta}\mathcal{K}_\nu\mathcal{K}^\nu_{\alpha\beta}\Big) - 4\lambda_{\text{GB}}\ell^2\overline{r}^{\alpha\beta}\mathcal{K}^\mu_{\alpha\beta}. \end{aligned} \tag{A.26.9}$$

This equation is very complicated, but since we are working in $d \leq 5$ dimensions we only need to solve perturbatively in z for $X^i_{(2)}$ and $X^i_{(4)}$[15]. Furthermore, $X^i_{(2)}$ is fully determined by symmetry to be [393]

$$X^i_{(2)} = \frac{1}{2(d-2)}D^a\partial_a X^i_{(0)} = -\frac{1}{2(d-2)}K^i, \tag{A.26.10}$$

where K^i denotes the extrinsic curvature of the $X^i_{(0)}$ surface, but we are leaving off the (0) in our notation to save space.

[15] It goes without saying that these formulas are only valid for $d > 2$ and $d > 4$, respectively.

The computation of $X^i_{(4)}$ is straightforward but tedious. We will only need to know $k_i X^i_{(4)}$ (where indices are being raised and lowered with $g^{(0)}_{ij}$), and the answer turns out to be

$$\begin{aligned}4(d-4)X^k_{(4)} &= 2X^k_{(2)}\left(P^{jm}g^{(2)}_{jm} - 4(X_{(2)})^2\right)\\&+ K^k_{ab}g^{ab}_{(2)} + 4g^{(2)}_{km}X^m_{(2)} + 2X^{(2)}_j K^j_{ab}K^{kab} + k_i D_a D^a X^i_{(2)}\\&+ k^j(\nabla_n g^{(2)}_{jm} - \frac{1}{2}\nabla_j g^{(2)}_{mn})P^{mn} + X^n_{(2)}R_{kmnj}P^{jm}\\&+ 8\kappa\sigma^{ab}_{(k)}C_{kalb} - 2(d-4)\Gamma^k_{jm}X^j_{(2)}X^m_{(2)}.\end{aligned} \tag{A.26.11}$$

Here κ depends on λ_{GB} as in (A.26.6). Notice that the last term in this expression is the only source of noncovariant-ness. One can confirm that this noncovariant piece is required from the definition of $X^i_{(4)}$—despite its index, $X^i_{(4)}$ does not transform like a vector under boundary diffeomorphisms.

We also note that the terms in $X^k_{(4)}$ with covariant derivatives of $g^{(2)}_{ij}$ can be simplified using the extended k^i and l^i fields described §A.24 and the Bianchi identity:

$$k^j(\nabla_n g^{(2)}_{jm} - \frac{1}{2}\nabla_j g^{(2)}_{mn})P^{mn} = -\frac{1}{4(d-1)}\nabla_k R + \frac{1}{d-2}\nabla_l R_{kk}. \tag{A.26.12}$$

Finally, we record here the formula for $X^k_{(4,\log)}$ which is obtained from $X^k_{(4)}$ by multiplying by $4-d$ and sending $d\to 4$:

$$\begin{aligned}-4X^k_{(4,\log)} &= 2X^k_{(2)}\left(P^{jm}g^{(2)}_{jm} - 4(X_{(2)})^2\right)\\&+ K^k_{ab}g^{ab}_{(2)} + 4g^{(2)}_{km}X^m_{(2)} + 2X^{(2)}_j K^j_{ab}K^{kab} + k_i D_a D^a X^i_{(2)}\\&+ k^j(\nabla_n g^{(2)}_{jm} - \frac{1}{2}\nabla_j g^{(2)}_{mn})P^{mn} + X^n_{(2)}R_{kmnj}P^{jm}\\&+ 8\kappa\sigma^{ab}_{(k)}C_{kalb}.\end{aligned} \tag{A.26.13}$$

We will not bother unpacking all of the definitions, but the main things to notice is that the noncovariant part disappears.

A.27 Details of the EWN Calculations

In this section we provide some insight into the algebra necessary to complete the calculations of the main text, primarily regarding the calculation of the subleading part of $(\delta\overline{X})^2$ in §6.2. The task is to simplify (6.2.13),

$$\begin{aligned}L^{-2}(\delta\overline{X})^2\big|_{z^2} &= 2k_i\delta X^i_{(4)} + 2g^{(2)}_{ij}k^i\delta X^j_{(2)} + g_{ij}\delta X^i_{(2)}\delta X^j_{(2)} + g^{(4)}_{ij}k^ik^j + X^m_{(4)}\partial_m g_{ij}k^ik^j\\&+ 2X^m_{(2)}\partial_m g_{ij}k^i\delta X^j_{(2)} + X^m_{(2)}\partial_m g^{(2)}_{ij}k^ik^j + \frac{1}{2}X^m_{(2)}X^n_{(2)}\partial_m\partial_n g_{ij}k^ik^j.\end{aligned} \tag{A.27.1}$$

After some algebra, we can write this as

$$L^{-2}(\delta\overline{X})^2\big|_{z^2} = g^{(4)}_{kk} + 2\delta(X^k_{(4,\mathrm{cov})}) + 2g^{(2)}_{ik}\nabla_k X^i_{(2)} + \nabla_k X^{(2)}_j \nabla_k X^j_{(2)} - \frac{1}{d-2}(X^l_{(2)})\nabla_k R_{kk}. \tag{A.27.2}$$

Here we have defined

$$X^i_{(4,\mathrm{cov})} = X^i_{(4)} + \frac{1}{2}\Gamma^i_{lm} X^l_{(2)} X^m_{(2)}, \tag{A.27.3}$$

which transforms like a vector (unlike $X^i_{(4)}$). From here, the algebra leading to (6.2.14) is mostly straightforward, though tedious. The two main tasks which require further explanation are the simplification of one of the terms in $g^{(4)}_{kk}$ and one of the terms in $\delta X^k_{(4,\mathrm{cov})}$. We will explain those now.

$g^{(4)}_{kk}$ Simplification We recall the formula for $g^{(4)}_{kk}$ from (A.26.5):

$$\begin{aligned} g^{(4)}_{kk} = \frac{1}{d-4}\Bigg[&\kappa C_{kijm}C_k{}^{ijm} + \frac{1}{8(d-1)}\nabla^2_k R - \frac{1}{4(d-2)}k^i k^j \Box R_{ij} \\ &-\frac{1}{2(d-2)}R^{ij}R_{kikj} + \frac{d-4}{2(d-2)^2}R_{ki}R^i_k + \frac{1}{(d-1)(d-2)^2}RR_{kk}\Bigg]. \end{aligned} \tag{A.27.4}$$

The main difficulty is with the term $k^i k^j \Box R_{ij}$. We will rewrite this term by making use of the geometric quantities introduced in the other appendices, and in particular we make use of the extended k and l field from §A.24. We first separate it into two terms:

$$k^i k^j \Box R_{ij} = k^i k^j N^{rs}\nabla_r\nabla_s R_{ij} + k^i k^j P^{rs}\nabla_r\nabla_s R_{ij}. \tag{A.27.5}$$

Now we compute each of these terms individually:

$$\begin{aligned} k^i k^j N^{rs}\nabla_r\nabla_s R_{ij} &= 2k^i k^j l^s \nabla_k\nabla_s R_{ij} + 2R_{kmlk}R^m_k \\ &= 2\nabla_k\nabla_l R_{kk} + 2w^c k^i k^j D_c R_{ij} + 2R_{kmlk}R^m_k \\ &= 2\nabla_k\nabla_l R_{kk} + 2w^c D_c R_{kk} - 4w^c w_c R_{kk} - 4w^c K^a_{ck}R_{ka} + 2R_{kmlk}R^m_k \\ &= 2\nabla_k\nabla_l R_{kk} + 2w^c D_c R_{kk} - 4w^c w_c R_{kk} + 2R_{kmlk}R^m_k. \end{aligned} \tag{A.27.6}$$

In the last line we assumed that $\sigma_{(k)} = 0$ and $\theta_{(k)} = 0$, which is the only case we will need to worry about. The other term is slightly messier, becoming

$$\begin{aligned}
&k^i k^j P^{rs}\nabla_r\nabla_s R_{ij} = k^i k^j e^{sc} D_c \nabla_s R_{ij} \\
&= D_c(k^i k^j D^c R_{ij}) - D_c(k^i k^j e^{sc})\nabla_s R_{ij} \\
&= D_c(k^i k^j D^c R_{ij}) - 2w_c D^c R_{kk} + 4w_c w^c R_{kk} + 6w_c K_k^{ca} R_{ak} \\
&\quad - 2K_k^{ca} D_c R_{ka} + 2K_k^{ca} K_{ca}^i R_{ik} + 2K_k^{ca} K_c^{bk} R_{ab} + K^s \nabla_s R_{kk} \\
&= D_c D^c R_{kk} - 2D_c(w^c R_{kk}) - 2D_c(K^{cak} R_{ka}) - 2w_c D^c R_{kk} + 4w_c w^c R_{kk} + 6w_c K_k^{ca} R_{ak} \\
&\quad - 2K_k^{ca} D_c R_{ka} + 2K_k^{ca} K_{ca}^i R_{ik} + 2K_k^{ca} K_c^{bk} R_{ab} + K^s \nabla_s R_{kk} \\
&= D_c D^c R_{kk} - 2D_c(w^c R_{kk}) - 2D_c(K^{cak}) R_{ka} - 2w_c D^c R_{kk} + 4w_c w^c R_{kk} + K^s \nabla_s R_{kk}.
\end{aligned} \tag{A.27.7}$$

In the last line we again assumed that $\sigma_{(k)} = 0$ and $\theta_{(k)} = 0$. Putting the two terms together leads to some canellations:

$$\begin{aligned}
k^i k^j \Box R_{ij} &= 2\nabla_k \nabla_l R_{kk} + 2R_{kmlk} R_k^m + D_c D^c R_{kk} - 2D_c(w^c R_{kk}) \\
&\quad - 2(D_a \theta_{(k)} + R_{kcac}) R_k^a + K^s \nabla_s R_{kk}.
\end{aligned} \tag{A.27.8}$$

$\delta X^k_{(4,\text{cov})}$ **Simplification** The most difficult term in (A.26.11), which also gives the most interesting results, is

$$k_i D_a D^a X^i_{(2)} = -\frac{1}{2(d-2)}(D_a - w_a)^2 \theta_{(k)} + \frac{1}{2(d-2)} K_{ab} K^{abi} K_i. \tag{A.27.9}$$

The interesting part here is the first term, so we will take the rest of this section to discuss its variation. The underlying formula is (A.25.7),

$$\delta w_a = -w^c K_{ac} + R_{klak}. \tag{A.27.10}$$

From this we can compute the following related variations, assuming that $\theta_{(k)} = 0$ and $\sigma_{(k)} = 0$:

$$\delta(D^a w_a) = D^a R_{klak} + w^a \partial_a \theta_{(k)} - 3D_a(K_k^{ab} w_b) \tag{A.27.11}$$
$$\delta(w^a D_a \theta_{(k)}) = -3K_k^{ab} w_a D_b \theta_{(k)} + R_{klak} D^a \theta_{(k)} + w^a D_a \dot{\theta}_{(k)} \tag{A.27.12}$$
$$\delta(D^a D_a \theta_{(k)}) = D^a D_a \dot{\theta} - \partial_a \theta_{(k)} \partial^a \theta_{(k)} - 2P^{jm} R_{kjbm} D^b \theta_{(k)}. \tag{A.27.13}$$

Here $\dot{\theta}_{(k)} \equiv \delta\theta_{(k)}$ is given by the Raychaudhuri equation. We can combine these equations to get

$$\begin{aligned}
\delta\left((D_a - w_a)^2 \theta_{(k)}\right) &= \delta\left(D^a D_a \theta_{(k)}\right) - 2\delta\left(w^a D_a \theta_{(k)}\right) - \delta\left((D_a w^a)\theta_{(k)}\right) + \delta\left(w_a w^a \theta_{(k)}\right) \\
&= -D^a D_a R_{kk} + 2w^a D_a R_{kk} + (D_a w^a) R_{kk} - w_a w^a R_{kk} \\
&\quad - \frac{d}{d-2}(D_a \theta_{(k)})^2 - 2R_{kb} D^b \theta_{(k)} - 2(D\sigma)^2.
\end{aligned} \tag{A.27.14}$$

A.28 The $d = 4$ Case

As mentioned in the main text, many of our calculations are more complicated in even dimensions, though most of the end results are the same. The only nontrivial even dimension we study is $d = 4$, so in this section we record the formulas and special derivations necessary for understanding the $d = 4$ case. Some of these have been mentioned elsewhere already, but we repeat them here so that they are all in the same place.

Log Terms In $d = 4$ we get log terms in the extremal surface, the metric, and the EWN inequality. By looking at the structure of the extremal surface equation, it's easy to see that the log term in in the extremal surface is related to $X^i_{(4)}$ in $d \neq 4$ by first multipling by $4-d$ and then setting $d \to 4$. The result was recorded in (A.26.13), and we repeat it here:

$$\begin{aligned}-4X^k_{(4,\log)} = {} & 2X^k_{(2)}\left(P^{jm}g^{(2)}_{jm} - 4(X_{(2)})^2\right) \\ & + K^k_{ab}g^{ab}_{(2)} + 4g^{(2)}_{km}X^m_{(2)} + 2X^{(2)}_j K^j_{ab}K^{kab} + k_i D_a D^a X^i_{(2)} \\ & + k^j(\nabla_n g^{(2)}_{jm} - \frac{1}{2}\nabla_j g^{(2)}_{mn})P^{mn} + X^n_{(2)}R_{kmnj}P^{jm} \\ & + 8\kappa\sigma^{ab}_{(k)}C_{kalb}. \end{aligned} \tag{A.28.1}$$

There is a similar story for $g^{(4,\log)}_{kk}$, which was recorded earlier in (A.26.7):

$$g^{(4,\log)}_{kk} = -\left[\kappa C_{kijm}C_k{}^{ijm} + \frac{1}{24}\nabla^2_k R - \frac{1}{8}k^i k^j \Box R_{ij} - \frac{1}{4}R^{ij}R_{kikj} + \frac{1}{12}RR_{kk}\right]. \tag{A.28.2}$$

From these two equations, it is easy to see that the log term in $(\delta\overline{X})^2$ has precisely the same form as the subleading EWN inequality (6.2.14) in $d \geq 5$, except we first multiply by $4-d$ and then set $d \to 4$. This results in

$$L^{-2}(\delta\overline{X})^2\big|_{z^2\log z, d=4} = -\frac{1}{4}(D_a\theta_{(k)} + R_{ka})^2 - \frac{1}{4}(D_a\sigma^{(k)}_{bc})^2. \tag{A.28.3}$$

Note that the Gauss-Bonnet term drops out completely due to special identities of the Weyl tensor valid in $d = 4$ [289]. The overall minus sign is important because $\log z$ should be regarded as negative.

QNEC in Einstein Gravity For simplicity we will only discuss the case of Einstein gravity for the QNEC in $d = 4$, so that the entropy functional is just given by the extremal surface area divided by $4G_N$. At order z^2, the norm of $\delta\overline{X}^\mu$ is formally the same as the expression in other dimensions:

$$L^{-2}(\delta\overline{X})^2\big|_{z^2} = g^{(4)}_{kk} + 2g^{(2)}_{ik}\nabla_k X^i_{(2)} + \nabla_k X^{(2)}_j \nabla_k X^j_{(2)} - \frac{1}{2}X^l_{(2)}\nabla_k R_{kk} + 2\delta(k_i X^i_{(4)\mathrm{cov}}). \tag{A.28.4}$$

Now, though, $X^k_{(4)}$ and $g^{(4)}_{kk}$ are state-dependent and must be related to the entropy and energy-momentum, respectively.

We begin with the entropy. From the calculus of variations, we know that the variation of the extremal surface area is given by

$$\delta A = -\lim_{\epsilon\to 0}\frac{L^3}{\epsilon^3}\int\sqrt{h}\frac{1}{\sqrt{1+g_{nm}\partial_z\overline{X}^n\partial_z\overline{X}^m}}g_{ij}\partial_z\overline{X}^i\delta X^j. \tag{A.28.5}$$

A few words about this formula are required. The $\overline{X}^\mu$ factors appearing here must be expanded in ϵ, but the terms without any (n) in their notation do *not* refer to (0), unlike elsewhere in this paper. The reason is that we have to do holographic renormalization carefully at this stage, and that means the boundary conditions are set at $z=\epsilon$. So when we expand out $\overline{X}^\mu$ we will find its coefficients determined by the usual formulas in terms of $X^i_{(0)}$. We need to then solve for $X^i_{(0)}$ in term of $X^i \equiv \overline{X}^i(z=\epsilon)$ re-express the result in terms of X^i alone. Since we are not in a high dimension this task is relatively easy. An intermediate result is

$$\frac{k^i}{L^3\sqrt{h}}\frac{\delta A}{\delta X^i}\bigg|_{\epsilon^0} = -2\ X^k_{(2)}\big|_{\epsilon^2} - 4\left(X^k_{(4)} - (X_{(2)})^2X^k_{(2)}\right) - X^k_{(4,\log)}. \tag{A.28.6}$$

The notation on the first term refers to the order ϵ^2 part of $X^i_{(2)}$ that is generated when $X^i_{(2)}$ is written in terms of $\overline{X}^i(z=\epsilon)$. The result of that calculation is

$$\begin{aligned}-4\ X^k_{(2)}\big|_{\epsilon^2} &= 2X^{(2)}_jK^{jab}K^i_{ab}k_i + k_iD^bD_bX^i_{(2)} + K^m\Gamma^i_{ml}X^l_{(2)}k_i\\ &+ g^{ab}_{(2)}K^i_{ab}k_i + P^{kj}R^i_{jmk}X^m_{(2)}k_i + k^m\left(\nabla_jg^{(2)}_{mk} - \frac{1}{2}\nabla_mg^{(2)}_{jk}\right)P^{jk}\\ &= -4X^k_{(4,\log)} - 2X^k_{(2)}\left(P^{jm}g^{(2)}_{jm} - 4(X_{(2)})^2\right) - 4g^{(2)}_{km}X^m_{(2)} + K^m\Gamma^i_{ml}X^l_{(2)}k_i.\end{aligned} \tag{A.28.7}$$

We have dropped terms of higher order in ϵ. Thus we can write

$$\frac{k^i}{L^3\sqrt{h}}\frac{\delta A}{\delta X^i}\bigg|_{\epsilon^0} = -3X^k_{(\log)} - X^k_{(2)}P^{jm}g^{(2)}_{jm} + 8X^k_{(2)}(X_{(2)})^2 - 2g^{(2)}_{km}X^m_{(2)} - 4X^k_{(4)\mathrm{cov}}. \tag{A.28.8}$$

We will want to take one more variation of this formula so that we can extract $\delta X^k_{(4)\mathrm{cov}}$. We can get some help by demanding that the $z^2\log z$ part of EWN be saturated, which states

$$g^{(\log)}_{kk} + 2\delta X^k_{\log} = 0. \tag{A.28.9}$$

Then we have

$$\delta\left(\frac{k^i}{L^3\sqrt{h}}\frac{\delta A}{\delta X^i}\bigg|_{\epsilon^0}\right) = \frac{3}{2}g^{(\log)}_{kk} - \delta(X^k_{(2)}P^{jm}g^{(2)}_{jm}) + 8\delta(X^k_{(2)}(X_{(2)})^2) - 2\delta(g^{(2)}_{km}X^m_{(2)}) - 4\delta X^k_{(4)\mathrm{cov}}. \tag{A.28.10}$$

Assuming that $\theta_{(k)} = \sigma_{(k)} = 0$, we can simplify this to

$$\delta \left(\frac{k^i}{L^3\sqrt{h}} \frac{\delta A}{\delta X^i}\bigg|_{\epsilon^0} \right) = \frac{3}{2} g_{kk}^{(\log)} - \frac{1}{4} R_{kk} P^{jm} g_{jm}^{(2)} - \frac{1}{4}\nabla_k(\theta_{(l)} R_{kk}) - \frac{1}{2} g_{kl}^{(2)} R_{kk} - 4\delta X^k_{(4)\mathrm{cov}}. \tag{A.28.11}$$

We can combine this with the holographic renormalization formula [394]

$$\begin{aligned} g_{kk}^{(4)} &= 4\pi G_N L^{-3} T_{kk} + \frac{1}{2}(g_{(2)}^2)_{kk} - \frac{1}{4} g_{kk}^{(2)} g^{ij} g_{ij}^{(2)} - \frac{3}{4} g_{kk}^{(\log)} \\ &= 4\pi G_N L^{-3} T_{kk} + \frac{1}{8} R_k^i R_{ik} - \frac{1}{16} R_{kk} R - \frac{3}{4} g_{kk}^{(\log)} \end{aligned} \tag{A.28.12}$$

to get

$$L^{-2}(\delta \overline{X}^i)^2\Big|_{z^2} = 4\pi G_N L^{-3} T_{kk} - \frac{1}{2}\delta \left(\frac{k^i}{L^3\sqrt{h}} \frac{\delta A}{\delta X^i}\bigg|_{\epsilon^0} \right). \tag{A.28.13}$$

After dividing by $4G_N$, we recognize the QNEC.

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