Feedback improves the generalized degrees of freedom of the strong interference channel

Viveck R. Cadambe, Syed A. Jafar
Electrical Engineering and Computer Science
University of California Irvine,
Irvine, California, 92697, USA
Email: vcadambe@uci.edu, syed@uci.edu

***Abstract*— We provide inner and outer bounds on the generalized degrees of freedom (GDOF) of the two user symmetric interference channel. The bounds are tight in the moderately weak and strong interference regimes. Feedback is shown to provide unbounded improvements to the GDOF of the two user interference channel in the very strong interference regime. We also show that feedback does not improve the GDOF of the channel if the interference is moderately weak or moderately strong. Finally, we extend the outer and inner bounds to the symmetric MIMO interference channel with feedback.**

I. Introduction

Recent results [1]–[4] have reduced the gap between lower and upper bounds on the capacity of the interference channel which has been a long standing open problem in information theory. An approximation of the capacity of the interference channel within one bit was derived in [1]. An important contribution of [1] was the identification of various operating regimes of the interference channel using *generalized degrees of freedom* (GDOF). In the context of the *symmetric* Gaussian interference channel with unit variance noise at both receivers, the number of generalized degrees of freedom $d(\alpha)$ of the channel is defined as

$$d(\alpha) = \lim_{\text{SNR}\to\infty} \frac{C(\alpha, \text{SNR})}{\log(\text{SNR})}$$

where $\alpha \stackrel{\triangle}{=} \frac{\log(\text{INR})}{\log(\text{SNR})}$, SNR represents the signal-to-noise ratio of both the users, INR represents interference to noise ratio, $C(\alpha, \text{SNR})$ represents the sum-capacity of the interference channel as a function of α and SNR. The conventional degrees of freedom[1] (DOF) introduced in [5] is $d(\alpha)$ evaluated at $\alpha = 1$. Since GDOF is a more general version of DOF, it is a more precise approximation of the capacity of a channel. The various operating interference regimes in the *symmetric* interference channel identified in [1] are listed below

1) Very weak interference $(0 < \alpha \leq 2/3)$: $d(\alpha) = 2\max(\alpha, 1-\alpha)$
2) Moderately weak interference $(2/3 < \alpha \leq 1)$: $d(\alpha) = 2 - \alpha$
3) Moderately strong interference $(1 < \alpha \leq 2)$: $d(\alpha) = \alpha$
4) Very strong interference $(2 < \alpha)$: $d(\alpha) = 2$

[1]Also known as multiplexing gain

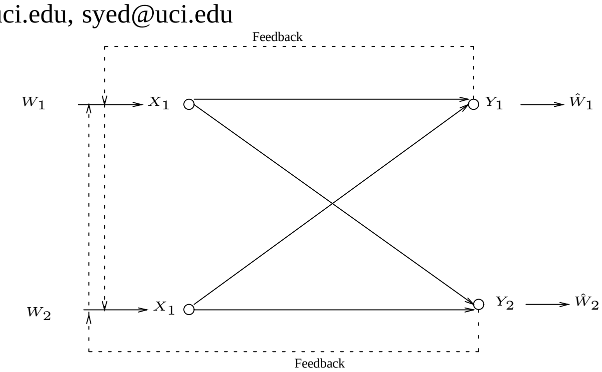

Fig. 1. The 2 user interference channel with feedback

Also, following standard terminology, we use 'strong interference channel' to mean that INR $>$ SNR or equivalently $\alpha > 1$. Similarly, by 'weak interference channel', we mean $\alpha \leq 1$. Note that the interference channel has $d(1) = 1$ degree of freedom. While the degrees of freedom of the interference channel may be improved by increasing the number of antennas at each node [6], the degrees of freedom approximation of capacity is too coarse to capture the benefits of techniques such as feedback, noisy co-operation and relays [7]. In this paper, we use the *generalized* degrees of freedom (GDOF) metric to study the benefits of perfect feedback on the interference channel. Note that the advantages offered by multiple antennas have been characterized in terms of GDOF in [8]

In this paper, we derive inner and outer bounds for the symmetric interference channel with feedback (see figure 2). For $\alpha \leq 2/3$, we bound GDOF as

$$\max(2\alpha, 2-2\alpha) \leq d(\alpha) \leq 2 - \alpha$$

For $\alpha > 2/3$, the inner and outer bounds are tight and we obtain a GDOF characterization of the interference

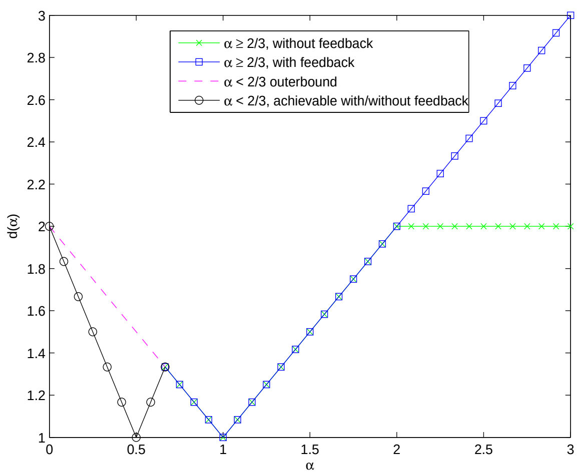

Fig. 2. Generalized degrees of freedom of the interference channel - the effect of feedback

channel with feedback as follows

$$d(\alpha) = \begin{cases} 2-\alpha & 2/3 < \alpha \leq 1 \\ \alpha & 1 < \alpha \end{cases}$$

Note that if $2/3 < \alpha \leq 2$, the GDOF of the interference channel does not improve with feedback. Therefore, the GDOF optimal achievable scheme does not even require feedback - the Han Kobayashi scheme presented in [1] is sufficient. For the very strong interference regime i.e. $\alpha > 2$, we present a co-operative achievable scheme in which each transmitter uses the feedback only from its from its corresponding receiver i.e. transmitter 1 needs feedback only from receiver 1 and transmitter 2 from receiver 2. In the very strong interference regime, while the GDOF of the interference channel saturates to 2 in absence of feedback, the GDOF grows linearly with α if feedback is present (Figure 2). This implies that feedback can provide unbounded GDOF improvement in this regime. We also finally present, without proof, bounds for the GDOF of the symmetric MIMO interference channel with feedback. Proofs for the MIMO case may be found in the extended paper [9]. We now proceed to formally introduce the system model.

II. System Model

We consider the 2 user symmetric interference channel (Figure 1)described by

$$Y_1(\tau) = \sqrt{\text{SNR}}\ H_{11}X_1(\tau) + \sqrt{\text{INR}}\ H_{12}X_2(\tau) + Z_1(\tau)$$
$$Y_2(\tau) = \sqrt{\text{INR}}\ H_{21}X_1(\tau) + \sqrt{\text{SNR}}\ H_{22}X_2(\tau) + Z_2(\tau)$$

where at the τ^{th} channel use, $X_i(\tau)$ is the complex symbol transmitted by transmitter i. Similarly, at receiver i, $Y_i(\tau)$ and $Z_i(\tau)$ represent the received symbol and the additive noise term respectively corresponding to the τ^{th} use of the channel. The noise process $Z_i \sim \mathcal{N}(0,1)$ is i.i.d and independent of other variables in the system. H_{ij} satisfies $|H_{ij}| = 1$. In other words, H_{ij} essentially represents the argument of the complex channel gain between transmitter j and receiver i. For a code spanning T uses of the channel, the codeword transmitted by transmitter i satisfies an average power constraint that may be expressed as $\frac{1}{T}E\left[\sum_{\tau=1}^{T}|X_i(\tau)|^2\right] \leq 1, i = 1,2$. The power constraint assumption is made in this manner since, SNR represents the actual signal-to-noise ratio between transmitter i and receiver i. The transmitters receive feedback from both receivers so that, the encoding function f_i at transmitter $i \in \{1,2\}$ may be expressed as

$$X_i(\tau) = f_i(W_i, Y_1^{[\tau-1]}, Y_2^{[\tau-1]})$$

where W_i represents the message corresponding to user i and $Y_i^{[\tau]} = (Y_i(1), Y_i(2)\ldots Y_i(\tau))$. In remaining parts of this paper i.e $A^{[\tau]}$ is used to indicate the tuple $(A(1), A(2)\ldots A(\tau))$. We will also use the following quantities later in the paper.

$$\begin{aligned} S_{11}(\tau) &\triangleq \sqrt{\text{SNR}}\ H_{11}X_1(\tau) + Z_1(\tau) \\ S_{12}(\tau) &\triangleq \sqrt{\text{INR}}\ H_{12}X_2(\tau) + Z_1(\tau) \\ S_{21}(\tau) &\triangleq \sqrt{\text{INR}}\ H_{21}X_1(\tau) + Z_2(\tau) \\ S_{22}(\tau) &\triangleq \sqrt{\text{SNR}}\ H_{22}X_2(\tau) + Z_2(\tau) \end{aligned}$$

A. Generalized Degrees of Freedom

Let $C_\Sigma(\alpha)$ represent the sum-capacity of the interference channel whose SNR and INR satisfy

$$\alpha = \frac{\log(\text{INR})}{\log(\text{SNR})}$$

Then, the generalized degrees of freedom (GDOF) of the interference channel $d(\alpha)$ is defined as

$$d(\alpha) = \lim_{\text{SNR}\to\infty} \frac{C_\Sigma(\alpha)}{\log(\text{SNR})}$$

III. Generalized Degrees of Freedom of the Interference Channel with Feedback

A. Outerbound

Consider any achievable coding scheme. Since X_1 represents the symbol transmitted by transmitter 1, this transmitter is aware of $X_1^{[\tau-1]}$ before the τ^{th} channel use. It can therefore cancel the effect of $X_1^{[\tau-1]}$ from the feedback received to obtain $S_{12}^{[\tau-1]}$ and $S_{22}^{[\tau-1]}$. This leads us to the following observation :

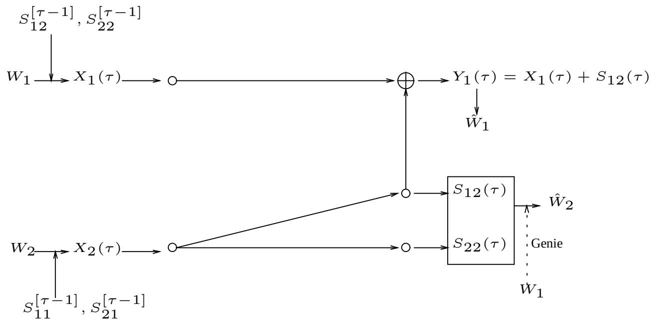

Fig. 3. The genie aided channel in the outerbound argument

Observation 1: The encoding function at transmitter 1 can equivalently be written in the following forms

$$\begin{aligned} X_1 &= f_1(W_1, Y_1^{[\tau-1]}, Y_2^{[\tau-1]}) \\ X_1 &= f_1^{'}(W_1, S_{12}^{[\tau-1]}, S_{22}^{[\tau-1]}) \\ X_1 &= f_1^{''}(W_1, S_{12}^{[\tau-1]}, Y_2^{[\tau-1]}) \end{aligned}$$

Now we convert the original channel to the channel in Figure 3 by letting a genie provide W_1, S_{12}^T to receiver 2. Now, note that observation 1 implies that using information of $S_{12}^{[T]}$, $Y_2^{[T]}$ and W_1, receiver 2 can construct $X_1^{[T]}$ and therefore cancel the signal from transmitter 1. The genie can only enhance the capacity region and therefore does not affect the outerbound argument. The genie aided channel is shown in in Figure 3. In this channel, we can use Fano's inequality to bound rates.

$$\begin{aligned}
& TR_2 - T\epsilon && (1)\\
& \leq I(S_{12}^{[T]}, S_{22}^{[T]}, W_1; W_2) && (2)\\
& = I(S_{12}^{[T]}, S_{22}^{[T]}; W_2|W_1) && (3)\\
& = h(S_{22}^{[T]}, S_{12}^{[T]}|W_1) - h(S_{22}^{[T]}, S_{12}^{[T]}|W_1, W_2) && (4)\\
& \leq \sum_{\tau=1}^{T} h(S_{22}(\tau), S_{12}(\tau)|W_1, S_{22}^{[\tau-1]}, S_{12}^{[\tau-1]}) \\
& \quad - \sum_{\tau=1}^{T} h(S_{22}(\tau), S_{12}(\tau)|W_1, W_2, X_1(\tau), X_2(\tau)) && (5)\\
& \leq \sum_{\tau=1}^{T} h(S_{12}(\tau)|W_1, S_{22}^{[\tau-1]}, S_{12}^{[\tau-1]}) \\
& \quad + \sum_{\tau=1}^{T} h(S_{22}(\tau)|W_1, S_{22}^{[\tau-1]}, S_{12}^{[\tau-1]}) \\
& \quad - \sum_{\tau=1}^{T} h(Z_2(\tau), Z_1(\tau)|W_1, W_2, X_1(\tau), X_2(\tau)) && (6)\\
& \leq \sum_{\tau=1}^{T} h(S_{12}(\tau)|W_1, S_{22}^{[\tau-1]}, S_{12}^{[\tau-1]}, X_1^{[\tau]}) \\
& \quad + \sum_{\tau=1}^{T} h(S_{22}(\tau)|S_{12}(\tau)) - 2T\log(\pi e) && (7)\\
& \leq \sum_{\tau=1}^{T} h(Y_1(\tau)|W_1, S_{22}^{[\tau-1]}, Y_1^{[\tau-1]}, X_1^{[\tau]}) \\
& \quad + \sum_{\tau=1}^{T} h(S_{22}(\tau)|S_{12}(\tau)) - 2T\log(\pi e) && (8)\\
& = \sum_{\tau=1}^{T} h(Y_1(\tau)|W_1, Y_1^{[\tau-1]}) \\
& \quad + \sum_{\tau=1}^{T} h(S_{22}(\tau)|S_{12}(\tau)) - 2T\log(\pi e) && (9)\\
& = h(Y_1^{[\tau]}|W_1) + \sum_{\tau=1}^{T} h(S_{22}(\tau)|S_{12}(\tau)) \\
& \quad - 2T\log(\pi e) && (10)
\end{aligned}$$

where we have used the fact that conditioning reduces entropy in (5) ,(7) and (9). In (7) we have also used observation 1 in the first summand term. In the last summand of (7), we have used the fact that the instantaneous noise $Z_i(\tau)$ is independent of $X_i(\tau)$. Inequality (8) uses the fact that

$$Y_1(\tau) = S_{12}(\tau) + \sqrt{\text{SNR}}\ H_{11} X_1(\tau)$$

i.e. given $X_1(\tau)$, uncertainty in $Y_1(\tau)$ is equal to the uncertainty in $S_{21}(\tau)$.

Now, we bound R_1 using Fano's inequality as well

$$\begin{aligned} TR_1 - T\epsilon &\leq I(Y_1^{[T]}; W_1) \\ &= h(Y_1^{[T]}) - h(Y_1^{[T]}|W_1) \qquad (11) \end{aligned}$$

Adding (10) and (11), we get

$$\begin{aligned} TR_1 + TR_2 - 2T\epsilon &\leq h(Y_1^{[T]}) + \sum_{\tau=1}^{T} h(S_{22}(\tau)|S_{12}(\tau)) \\ & \quad -2T\log(\pi e) \end{aligned}$$

Using the fact that Gaussian variables maximize entropy and conditional entropy, we can write

$$\begin{aligned} h(Y_1^{[T]}) &\leq T\log\left(\pi e\left(1 + \text{SNR}\ |H_{11}|^2 + \text{INR}\ |H_{12}|^2\right)\right) \\ h(Y_1^{[T]}) &\leq T\log\left(1 + \text{SNR}\ + \text{INR}\ \right) + \mathcal{O}(1) \end{aligned}$$

where, above, we have used the fact that $|H_{ij}|^2 = 1, i, j \in \{1, 2\}$.

$$\begin{aligned} & \sum_{\tau=1}^{T} h\left(S_{22}(\tau)|S_{12}(\tau)\right) \\ & \leq T\log\left(\pi e\left(E\left[|S_{22}|^2\right] - \frac{E\left[|S_{22}S_{12}^*|\right]^2}{E\left[|S_{12}|^2\right]}\right)\right) \qquad (12) \end{aligned}$$

$$\leq \log\left(1+\text{SNR }|H_{22}|^2-\text{SNR INR }\frac{|H_{12}H_{22}|^2}{1+\text{INR}|H_{12}|^2}\right) + \mathcal{O}(1) \quad (13)$$

$$= \log(1+\text{INR}+\text{SNR}) - \log(1+\text{INR }) + \mathcal{O}(1) \quad (14)$$

Therefore, letting $T \to \infty$ in (11), we get

$$R_1 + R_2 \quad \leq \quad \Gamma_1 + \Gamma_2 + \mathcal{O}(1) \quad (15)$$

where

$$\begin{aligned} \Gamma_1 &\triangleq \log(1+\text{SNR}+\text{INR}) + \mathcal{O}(1) \\ \Gamma_2 &\triangleq \log(1+\text{INR}+\text{SNR}) \\ &\quad - \log(1+\text{INR}) \end{aligned}$$

It can be clearly seen that

$$\begin{aligned} \lim_{\text{SNR}\to\infty} \frac{\Gamma_1}{\text{SNR}} &= \max(\alpha, 1) \\ \lim_{\text{SNR}\to\infty} \frac{\Gamma_2}{\text{SNR}} &= \max(\alpha, 1) - \alpha \end{aligned}$$

Using the above equations in (15), the following outerbound can be shown

Theorem 1: The generalized degrees of freedom of the 2 user interference channel with feedback is bounded as

$$d(\alpha) \leq \begin{cases} 2-\alpha & \alpha \leq 1 \\ \alpha & \alpha > 1 \end{cases}$$

B. Inner Bound : Co-operative Achievable scheme

We provide an innerbound to the GDOF of the *strong* interference channel using through a simple two-stage achievable scheme. The scheme achieves a GDOF of $d(\alpha) = \alpha$. Combined with the outerbound of the previous section this achievable scheme is optimal. Note that in contrast to the interference channel without feedback whose GDOF performance is bounded by 2 (Figure 2), the of the interference channel in presence of feedback grows linearly with increasing α in the strong interference regime. We now describe our achievable scheme.

Stage 1 : In the first stage of the achievable scheme, each transmitter learns the other user's message using the feedback channel. For example, note that the feedback to transmitter 1 from receiver 1 is equivalent to

$$S_{12}(\tau) = \sqrt{\text{INR}}\; H_{12}X_2(\tau) + Z_1(\tau)$$

In other words, feedback effectively provides an AWGN channel from transmitter 2 to transmitter 1. Using this effective AWGN channel transmitter 1 decodes message W_2. Similarly transmitter 2 learns W_1 using the feedback from receiver 2. The first stage ends when both transmitters are aware of both messages.

Stage 2 : In the second stage, the two transmitters co-operate to broadcast the messages W_1 and W_2 to their respective destinations.

Let us assume a message of B bits i.e. $H(W_1) = B = H(W_2)$. Now, note that, at high SNR, the first stage lasts $\frac{B}{C_c}$ symbols, where $C_c = \alpha\log(\text{SNR}) + o(\log(\text{SNR}))$ represents the capacity of the point-to-point channel described by S_{12} (or S_{21}) . Also, note that the capacity of the broadcast channel in the second stage is $C_b = 2\max(\alpha,1)\log(\text{SNR}) + o(\log(\text{SNR}))$. Therefore to transfer a total of $2B$ bits i.e. B bits for each user, the total time taken is $\frac{B}{C_c} + \frac{2B}{C_b}$ symbols. Therefore, assuming $\alpha > 1$, the sum-rate may be written as

$$R_{sum} \quad = \quad \frac{2B}{\frac{B}{C_c} + \frac{2B}{C_b}}$$

The degrees of freedom achieved may be expressed as

$$\begin{aligned} d(\alpha) &= \lim_{\text{SNR}\to\infty} \frac{R_{sum}}{\log(\text{SNR})} \\ d(\alpha) &= \frac{2}{\frac{1}{\alpha} + \frac{2}{2\alpha}} \\ &= \alpha \end{aligned}$$

We can now proceed to the following achievability result

Theorem 2: The degrees of freedom of the 2 user interference channel with feedback maybe bounded as

$$d(\alpha) \geq \begin{cases} \alpha & \alpha > 1 \\ \min(\max(2\alpha, 2-2\alpha), 2-\alpha) & \alpha \leq 1 \end{cases}$$

The achievable scheme for $\alpha > 1$ is described earlier in this section. For $\alpha \leq 1$, the achievable scheme is simply the Han Kobayashi scheme described in [1]. Note that the achievable scheme for the above theorem does not use feedback if $\alpha \leq 1$. Combining Theorems 2 and 1 we obtain the GDOF characterization of the interference channel with feedback of $\alpha > 2/3$.

Corollary 1: If $\alpha > 2/3$, then the number of GDOF of the interference channel with feedback is given by

$$d(\alpha) = \max(2-\alpha, \alpha)$$

Note that feedback increases the GDOF performance if $\alpha > 2$, but does not improve the performance if $2/3 < \alpha \leq 2$. The effect of feedback on the GDOF for $\alpha < 2/3$ is an open problem.

IV. Generalized Degrees of Freedom of the Symmetric MIMO Interference Channel with Feedback

Consider the 2 user symmetric interference channel with M antennas at each transmitter and N antennas at each receiver. This channel is described by

$$\begin{aligned}\mathbf{Y}_1(\tau) &= \sqrt{\text{SNR}}\ \mathbf{H}_{11}\mathbf{X}_1(\tau) + \sqrt{\text{INR}}\ \mathbf{H}_{12}\mathbf{X}_2(\tau) \\ &\quad +\mathbf{Z}_1(\tau) \\ \mathbf{Y}_2(\tau) &= \sqrt{\text{INR}}\ \mathbf{H}_{12}\mathbf{X}_1(\tau) + \sqrt{\text{SNR}}\ \mathbf{H}_{22}\mathbf{X}_2(\tau) \\ &\quad +\mathbf{Z}_2(\tau)\end{aligned}$$

where at the τ^{th} channel use, $\mathbf{X}_i(\tau)$ is a $M \times 1$ column vector representing the transmitted (vector) symbol at transmitter i. Similarly $\mathbf{Y}_i(\tau)$ and $\mathbf{Z}_i(\tau)$ are $N \times 1$ vectors which represent the received symbol and the additive noise term respectively. The noise process $\mathbf{Z}_i \sim \mathcal{N}(0, \mathbf{I}_N)$ is i.i.d and independent of other variables in the system.The $N \times M$ channel matrix $\mathbf{H}_{ij}$ satisfies $||\mathbf{H}_{ij}||_F^2 = 1$, where $||\mathbf{A}||_F^2$ represents the frobenius norm of matrix $\mathbf{A}$. Furthermore, we assume that the channel matrix H_{ij} is full rank for all $i, j \in \{1, 2\}$. The power constraint, rates and the generalized degrees of freedom are defined in a manner similar to the single antenna case.

Similar to [8], we can extend the ideas of the outer-bound of Theorem 1 to the MIMO case. Furthermore, the achievable scheme can also be extended to the MIMO case and we can therefore bounds the GDOF of the symmetric MIMO interference channel. We summarize our result in the theorem below. The reader is referred to the extended paper [9] for a proof of the theorem.

Theorem 3: If $d(\alpha)$ represents the generalized degrees of freedom of the MIMO interference channel with feedback, then we can bound it as follows

$$\frac{d(\alpha)}{\min(M,N)} \geq \begin{cases} \min(\max(2\alpha, 2-2\alpha), 2-\alpha) & \alpha \leq 1 \\ \alpha & \alpha > 1 \end{cases}$$

$$d(\alpha) \leq \begin{cases} M + M\max(\alpha,1) & N > 2M \\ M(\alpha-1)+N & M < N \leq 2M, \alpha > 1 \\ 2M - 2M\alpha + N\alpha & M < N \leq 2M, \alpha \leq 1 \\ N(\alpha-1)+M & N < M \leq 2N, \alpha > 1 \\ 2N - 2N\alpha + M\alpha & N < M \leq 2N, \alpha \leq 1 \\ N + N\max(\alpha,1) & M > 2N \end{cases}$$

V. Conclusion

We have derived bounds on the generalized degrees of freedom of the symmetric interference channel with feedback. The inner and outerbounds are tight for strong interference and moderately weak interference channels (i.e $\alpha \geq 2/3$). In the very strong interference regime, we observe that the presence of feedback can cause unbounded improvement on the *generalized* degrees of freedom of the interference channel, We have also provided bounds for the generalized degrees of freedom for the symmetric MIMO interference channel. Future work involves finding tighter bounds for $\alpha \leq 2/3$, the MIMO case and extension of the results to general (i.e. not symmetric) interference channels. Another important extension of this work is the exploration of the benefits of other techniques such as relays and noisy co-operation on the generalized degrees of freedom of interference and other wireless networks.

References

[1] R. Etkin, D. Tse, and H. Wang, "Gaussian interference channel capacity to within one bit," *arxiv.org*, Feb 2007. eprint - cs/0702045.

[2] X. Shang, G. Kramer, and B. Chen, "A new outer bound and the noisy-interference sum-rate capacity for gaussian interference channels," 2007. http://www.citebase.org/abstract?id=oai:arXiv.org:0712.1987.

[3] V. S. Annapureddy and V. V. Veeravalli, "Gaussian interference networks: Sum capacity in the low interference regime and new outer bounds on the capacity region," 2008. http://www.citebase.org/abstract?id=oai:arXiv.org:0802.3495.

[4] A. S. Motahari and A. K. Khandani, "Capacity bounds for the gaussian interference channel," 2008. http://www.citebase.org/abstract?id=oai:arXiv.org:0801.1306.

[5] L. Zheng and D. N. Tse, "Packing spheres in the grassmann manifold: A geometric approach to the non-coherent multi-antenna channel," *IEEE Trans. Inform. Theory*, vol. 48, Feb 2002.

[6] S. A. Jafar and M. Fakhereddin, "Degrees of freedom of the mimo interference channel," *IEEE Transactions on Information Theory*, vol. 7, pp. 2637–2642, July 2007.

[7] V. R. Cadambe and S. A. Jafar, "Capacity of wireless networks within o(log(snr)) - the impact of relays, feedback, cooperation and full-duplex operation," *arxiv.org*, Feb 2008. eprint - arxiv:0802.0534.

[8] P. Parker, D. W. Biss, and V. Tarokh, "On the degrees of freedom of the mimo interference channel," *Conference of Information Sciences and Systems, Princeton*, Mar 2008.

[9] V. R. Cadambe and S. A. Jafar, "The generalized degrees of freedom of the mimo interference channel with feedback," 2008. Preprint available on author's website.