Convergence to Perfect Competition of a Dynamic Matching and Bargaining Market with Two-sided Incomplete Information and Exogenous Exit Rate

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Abstract

Consider a decentralized, dynamic market with an infinite horizon in which both buyers and sellers have private information concerning their values for the indivisible traded good. Time is discrete, each period has length δ, and each unit of time a large number of new buyers and sellers enter the market to trade. Within a period each buyer is matched with a seller and each seller is matched with zero, one, or more buyers. Every seller runs a first price auction with a reservation price and, if trade occurs, both the seller and winning buyer exit the market with their realized utility. Traders who fail to trade either continue in the market to be rematched or exit at an exogenous rate. We characterize the steady-state, perfect Bayesian equilibria as δ becomes small and the market—in effect—becomes large. We show that, as δ converges to zero, equilibrium prices at which trades occur converge to the Walrasian price and the realized allocations converge to the competitive allocation.

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1 Introduction

Asymmetric information and strategic behavior interfere with efficient trade. Nevertheless economists have long believed that for private goods' economies the presence of many traders overcomes both these imperfections and results in convergence to perfect competition. This paper contributes to a burgeoning literature that shows the robust ability of simple market mechanisms to elicit cost and value information from buyers and sellers even as at it uses the information to allocate the available supply almost efficiently. In particular, we show how a completely decentralized market with two-sided incomplete information converges to a competitive outcome as each trader's ability to contact sequentially other traders increases. Thus a market that for each trader is big over time—as opposed to big at a moment in time—overcomes the difficulties of asymmetric information and strategic behavior. This is a step towards a full understanding of why price theory with its assumptions of complete information and price-taking works as well as it does even in markets where the validity of neither of these assumptions is self-evident.

These ideas may be made concrete by considering a bilateral bargaining situation in which the single buyer has a value $v \in [0,1]$ for an indivisible good and the single seller has a cost $c \in [0,1]$. They should trade only if $v \geq c$, but neither knows the other's value/cost. Instead each regards the other's value/cost as drawn from $[0,1]$ in accordance with a distribution $G(\cdot)$. Myerson and Satterthwaite (1983) showed that no individually rational, budget balanced mechanism exists that both respects the incentive constraints the asymmetric information imposes and prescribes trade only if $v \geq c$. Bilateral trade with two-sided incomplete information is intrinsically inefficient.

An instructive example of this phenomenon is the linear equilibrium Chatterjee and Samuelson (1983) derived for the bilateral $\frac{1}{2}$-double auction when G is the uniform distribution on $[0,1]$. The rules of this double auction are that buyer and seller simultaneously announce a bid $B(v)$ and offer $S(c)$ and they trade at price $p = \frac{1}{2}(B(v) + S(c))$ only if the buyer's bid is greater than the seller's offer. In their linear equilibrium trade occurs only if $v - c \geq \frac{1}{4}$, i.e., the asymmetric information and resulting misrepresentation of value/cost inserts an inefficient "wedge" of thickness $\frac{1}{4}$ into the double auction's outcome. Moreover the magnitude of this wedge is irreducible. Myerson and Satterthwaite (1983) showed that subject to budget balance, individual rationality, and incentive constraints this equilibrium maximizes the ex ante expected gains from trade and therefore is ex ante efficient.

A sequence of papers on the static, multi-lateral k-double auction in the independent private values environment have confirmed economists' intuition that increasing the number of traders causes this wedge to shrink and ultimately vanish in the limit. In the multilateral double auction there are n sellers each supplying one unit and n buyers each demanding one unit. Each trader's cost/value is private and, from the viewpoint of every other trader, independently drawn from [0,1] with distribution G. Sellers and buyers submit offers/bids simultaneously, a market clearing price p is computed, and the n units of supply are allocated at price p to those n traders who revealed through their offers/bids that they most value the available supply. Satterthwaite and Williams (1989) and Rustichini, Satterthwaite, and Williams (1994) established that as n increases the thickness of the wedge and the relative inefficiency associated with each equilibrium are $O(1/n)$ and $O(1/n^2)$ respectively. Relative inefficiency is the expected gains that the traders would realize if the market were perfectly competitive divided into the expected gains that the traders *fail* to realize in the equilibrium of the double auction market.

Thus, quite quickly, the static double auction market with independent private values converges to ex post efficiency—that is, perfect competition—as the number of traders grows.[1] This is despite dispensing with the technically important, but often unrealistic assumption of auction theory that the seller's cost is common knowledge among all participants. These results, however, are derived under three restrictive assumptions: costs/values are independently drawn private signals, sellers have unit supply and buyers have unit demand, and the timing of the market is a one-shot static game.

Papers by Fudenberg, Mobius, and Szeidl (2003), Cripps and Swinkels (2003), and Reny and Perry (2003) relax the first two assumptions. Specifically, Fudenberg, Mobius, and Szeidl show that for large markets in an environment with correlated private costs/values an equilibrium to the static double auction exists and traders misrepresentation of their true values is $O\left(\frac{1}{n}\right)$. Cripps and Swinkels, using a more general model of correlated private values, additionally dispense with the second assumption of unit supply/unit demand assumption and show that the relative inefficiency of the static double auction is $O\left(\frac{1}{n^{2-\varepsilon}}\right)$ where ε is arbitrarily small. Reny and Perry loosen the first assumption most dramatically, allowing traders' cost/values to have a common value component and their private signals to be affiliated. They show in a carefully crafted model that, if the market is large enough, an equilibrium exists, is almost ex post efficient, and almost fully aggregates the traders' private information, i.e., the double auction equilibrium is almost the unique, fully revealing rational expectations equilibrium that exists in the limit.

This paper, while retaining the independent private values and unit supply/unit demand assumptions, eliminates the third assumption that traders are playing a one-shot game in which, if they fail to trade now, they never have a later opportunity to trade. Commonly a trader who fails to trade now can enter into a new negotiation within a short time, perhaps even within minutes. To account for this possibility we consider a dynamic matching and bargaining model in which trades are consummated in a decentralized matter and traders who do not trade in the current period may rematch in the next period and try again.

A description of our model and result is this. An indivisible good is traded in a market in which time progresses in discrete periods of length δ and generations of traders overlap. The parameter δ is the exogenous friction in our model that

[1] Indeed Satterthwaite and Williams (2001) show that for this environment it converges as fast as possible in the sense of worst case asymptotic optimality.

we take to zero. Every active buyer is randomly matched with an active seller each period. Depending on the luck of the draw, a seller may end up being matched with several buyers, a single buyer, or even no buyers. Each seller solicits a bid from each buyer with whom she is matched. If the highest of the bids is satisfactory to her, she sells her single unit of the good and both she and the successful buyer exit the market. A buyer or seller who fails to trade remains in the market and seeks a new match the next period unless for exogenous reasons he elects to exit the market without trading.

Each unit of time a large number of potential sellers (formally, measure 1 of sellers) enters the market along with a large number of potential buyers (formally, measure a of sellers). Each potential seller independently draws a cost c in the unit interval from a distribution G_S and each potential buyer draws independently a value v in the unit interval from a distribution G_B. Individuals' costs and values are private to them. A potential trader only enters the market if, conditional on his private cost or value, his equilibrium expected utility is positive. Potential traders who have zero probability of profitable trade in equilibrium elect not to participate.

If trade occurs between a buyer and seller at price p, then they exit with utilities $v - p$ and $p - c$ respectively that they discount back at rate r to their times of entry. As in McAfee (1993) unsuccessful active traders face a risk of exiting whose source is exogenous. Specifically, each period each unsuccessful trader exits with probability $e^{-\delta\mu}$ where μ is the exit rate per unit of time. If δ is large (i.e., periods are long), then a trader who enters the market is impatient, seeking to consummate a trade and realize positive utility amongst the first few matches he realizes. If, however, δ is small (i.e., periods are short), then a trader can patiently wait through many matches looking for a good price with little concern about exiting with no gain.

Buyers with higher values find it worthwhile to submit higher bids than buyers with lower values. At the extreme, a buyer with a value 0.1 will certainly not submit a bid greater than 0.1 while a buyer with a value 0.95 certainly might. The same logic applies to sellers: low cost sellers may be willing to accept lower bids than are higher cost sellers. This means high value buyers and low cost sellers tend quickly to realize a match that results in trade and exit. Low value buyers and high cost sellers may take a much longer time on average to trade and are likely to exit without trading. Consequently, among the buyers and sellers who are active in the market in a given period t, low value buyers and high cost sellers may be overrepresented relative to the entering distributions G_B and G_S.

We characterize subgame perfect Bayesian equilibria for the steady state of this market and show that, as the period length goes to zero, all equilibria of the market converge to the Walrasian price and the competitive allocation. The Walrasian price p_W in this market is the solution to the equation

$$G_S(p_W) = a\left(1 - G_B(p_W)\right), \tag{1}$$

i.e., it is the price at which the measure of entering sellers with costs less than p_W equals the measure of entering buyers with values greater than p_W. If the market were completely centralized with every active buyer and seller participating in an enormous exchange that cleared each period's bids and offers simultaneously, then p_W would be the market clearing price each period. Our result is this. Given a $\delta > 0$, then each equilibria induces a trading range $\left[\underline{p}_\delta, \bar{p}_\delta\right]$. It is the range of offers that sellers of different types make, the range of bids that buyers make, and the range of prices at which trades are actually consummated. We show that $\lim_{\delta \to 0} \underline{p}_\delta = \lim_{\delta \to 0} \bar{p}_\delta = p_W$, i.e., the trading range converges to the competitive price. That the resulting allocations give traders the expected utility they would realize in a perfectly competitive market follows as a corollary.

This result, both intuitively and in its proof, is driven by two phenomena: local market size and global market clearing.[2] By local market size we mean the number of other traders with whom each individual trader interacts. This contrasts with global market size—the total number of traders active in the entire market—which is always large in our model. As the time period δ shrinks each trader expects he can match an increasing number of times seeking a profitable trade before some exogenous event in his live causes him to exit. Thus as δ becomes small each trader's local market becomes big over time rather than big at a point in time as is the case in the centralized k-double auction. This creates a strong option value effect for every trader. Even if a buyer has a high value, he has an increasing incentive as δ decreases to bid low and hold out for an offer near the low end of the offer distribution. Therefore all serious buyers bid within an increasingly narrow range just above the minimum offer any seller makes. A parallel argument applies to sellers, with the net effect being, as δ becomes small, all bids and offers concentrate within an interval of decreasing length, i.e., the trading range converges to a single price.

Local market size only forces the market to converge to a single price, not necessarily to the Walrasian price. It is global market clearing that forces convergence to the Walrasian price. To see this, suppose the market converges to a price p that is less than the Walrasian price. This low price attracts more buyers into the market than it does sellers. Buyers are therefore rationed randomly through exogenous exit, for then even a high value buyer has a substantial probability of exiting for exogenous reasons prior to being matched with a seller who is willing to sell at p. This, however, is inconsistent with equilibrium: the high value buyer can increase his bid above p so as to guarantee that he will trade instead of being rationed out. This increases his expected utility and contradicts the hypothesis that the equilibrium converges to the price p rather than the Walrasian price.

A substantial literature exists that investigates the non-cooperative foundations of perfect competition using dynamic matching and bargaining games.[3] This paper is most closely related to Gale (1987), Mortensen and Wright (2002),

[2] De Fraja and Sákovics (2001) introduced these distinctions.

[3] There is a related literature that we do not discuss here concerns the micro-structure of intermediaries in markets, e.g., Spulber (1999) and Rust and Hall (2002). These models allow entry of an intermediary who posts fixed ask and offer prices and is assumed to be large enough to honor any size buy or sell order without exhausting its inventory or financial resources.

and our companion paper, Satterthwaite and Shneyerov (2004).[4] These three papers show convergence to the Walrasian price and an ex post efficient allocation as the market friction vanishes. The primary difference between our papers and the papers of Gale and of Mortensen and Wright is that in their models when two traders meet they reciprocally observe each other's cost/value. This—full versus incomplete information—is fundamental, for the purpose of our papers is to determine if a decentralized market can elicit sufficient private valuation information at the same time it uses that information to assign the available supply almost efficiently. Their models, with their assumption of complete information, are silent on this important question.

The difference between this paper and our companion paper is that here traders have no cost of participating in the market while there each trader incurs a small participation cost each period he is active and never exits for exogenous reasons. As a consequence in the companion paper active traders only exit as the result of successful trade. There are two reasons—one substantive and one technical—why consideration of this, the exogenous exit rate variant of the model, is important.

First, substantively, the rise of Internet enabled markets increasingly is making the cost of participation in many markets trivial. Here we eliminate participation costs and substitute an exogenous exit rate. The idea is that participating in a market with trivial participation costs still requires the scarce resource of attention. A person when he decides to enter a market knows there is a significant probability that, if he is unsuccessful at trading quickly, his situation may change unexpectedly, preempt his attention, and force exit. The trader is therefore impatient to consummate the trade because exiting does not indicate that trade would no longer be of value. It only indicates that he can not give it attention now.[5]

Second, technically, these alternative assumptions—exogenous exit vs. positive participation costs—induce equilibria that have different structures yet share identical efficiency properties in the limit. As described above, given a positive exogenous exit rate, a trading interval $\left[\underline{p}_\delta, \bar{p}_\delta\right]$ characterizes each equilibrium. This interval is the range both of buyers' bids and of sellers' offers. Consequently trade does not occur each time a match takes place and the price at which trade does occur may fall anywhere within it. Equilibria in the presence of participation costs is quite different. The extreme case is "full trade

[4] The books of Osborne and Rubinstein (1990) and Gale (2000) contain excellent discussions of both their own and others' contributions to this literature. Papers, in addition to Gale (1987) and Mortensen and Wright (2002), that have been particularly influential include Mortensen (1982), Rubinstein and Wolinsky (1985, 1990), and Gale (1986).

[5] Scarce attention is not the only reason why making the exit rate μ a primitive of the model makes sense. Among the many decision biases psychologists have identified is overoptimism. One form this may take is that a trader may be optimistic as to how much time and attention consumating a trade in the market will take. For such a less than fully rational trader, participation in the market, if he is not fortunate in getting a good match early on, tends to disabuse him of this optimism and lead to a decision to exit. Two references to this literature, which Adam Galinsky and Keith Murnighan kindly brought to our attention, are Kahneman and Lovallo (1993) and Buelher, Griffin, and Ross (1994).

equilibria" in which the range of sellers' offers is an interval $\left[\underline{c}'_\delta, \underline{p}'_\delta\right]$ and the range of buyers' bids is an adjacent interval $\left[\underline{p}'_\delta, \bar{p}'_\delta\right]$ that shares only the point $\underline{p}'_\delta$ in common. As a consequence, every match for sure results in a trade being consummated. This ensures that if convergence to one price occurs, then that price must be the Walrasian price because if it were another price, traders from the long side of the market would accumulate and no steady state would exist. By contrast, in this paper's model, matches often fail to result in trade and consequently steady states can exist even if the market converges to a non-Walrasian price. Despite this, convergence to the Walrasian price and ex post efficiency still occurs. This illustrates the robustness of convergence to perfect competition and provide clues towards identifying a set of necessary and sufficient conditions for its occurrence.

Butters (circa 1979), Wolinsky (1988), De Fraja and Sákovics (2001), and Serrano (2002) are the most important dynamic bargaining and matching models that incorporate incomplete information, albeit one-sided in the cases of Wolinsky and of De Fraja and Sákovics.[6] Of these four papers, only Butters obtains robust convergence to perfect competition in the limit. Specifically, in an old, incomplete manuscript he analyzes almost the identical two-sided incomplete information model that we study here and makes a great deal of progress towards proving a variant of the convergence theorem that we prove here.

Without going into the details of Wolinsky (1988), De Fraja and Sákovics (2001), and Serrano (2002), the simplest explanation why they fail to converge robustly to the Walrasian price and allocation is that the information/allocation problem each attempts to solve is different than the problem that large, static double auctions solve robustly. Think of the baseline problem as being this. Each unit of time measure 1 sellers and measure a buyers enter the market, each of whom has a private cost/value for a single unit of the homogeneous good. The sellers' units of supply need to be reallocated to those traders who most highly value them. Whatever mechanism is employed, it must both induce the traders to reveal some degree of information about their costs/valuations and carry out the reallocation. Static double auctions with even a moderate number of traders solve this problem essentially perfectly by closely approximating the Walrasian price and then using that price to mediate trade.

Given this definition of the problem that both the static double auction and our matching and bargaining market solve, the reason why Wolinsky (1988), De Fraja and Sákovics (2001), and Serrano (2002) do not obtain competitive outcomes as the frictions in their models vanish is clear: the problems their models address are different and, as their results establish, not intrinsically perfectly competitive even when the market becomes almost frictionless. Wolinsky's model relaxes the homogeneous good assumption and does not fully analyze the effects of entry/exit dynamics. De Fraja and Sákovics' model's entry/exit dynamics do not specify fixed measures of buyers and sellers entering the market

[6] Another example of a centralized trading institution is the system of simultaneous ascending-price auctions, studied in Peters and Severinov (2002). They also find robust convergence to the competitive outcome.

each unit of time and therefore have no force moving the market towards a supply-demand equilibrium. Serrano's model is a market that may initially be large but, as buyers and sellers successfully trade, becomes small and non-competitive over time, an effect that the discreteness of its prices aggravates.

The next section formally states the model and our main result establishing that the Walrasian price robustly emerges as the market becomes frictionless. Section 3 derives basic properties of equilibria and presents a computed example illustrating our result. Section 4 proves our result and section 5 concludes with a discussion of possible extensions.

2 Model and theorem

We study the steady state of a market with two-sided incomplete information and an infinite horizon. In it heterogeneous buyers and sellers meet once per period ($t = \ldots, -1, 0, 1, \ldots$) and trade an indivisible, homogeneous good. Every seller is endowed with one unit of the traded good for which she has cost $c \in [0, 1]$. This cost is private information to her; to other traders it is an independent random variable with distribution G_S and density g_S. Similarly, every buyer seeks to purchase one unit of the good and has value $v \in [0, 1]$. This value is private; to others it is an independent random variable with distribution G_B and density g_B. Our model is therefore the standard independent private values model. We assume that the two densities are bounded away from zero: a $\underline{g} > 0$ exists such that, for all $c, v \in [0, 1]$, $g_S(c) > \underline{g}$ and $g_B(v) > \underline{g}$.

The length of each period is δ. Each unit of time a large number of potential sellers and a large number of potential buyers consider entering the market; formally each unit of time measure 1 of potential sellers and measure a of potential buyers consider entry where $a > 0$. This means that each period measure δ of potential sellers and measure $a\delta$ of potential buyers consider entry. Only those potential traders whose expected utility from entry is positive elect to enter and become active traders.[7] Active buyers and sellers who did not leave the market through either trade or exogenous exit the previous period carry over and remain active in the next period.

Let the strategy of a seller, $S : [0, 1] \to [0, 1] \cup \{\mathcal{N}\}$, map her cost c into either a decision $\mathcal{N}$ not to enter or the minimal bid that she is willing to accept. Similarly, let the strategy of a seller, $B : [0, 1] \to [0, 1] \cup \{\mathcal{N}\}$, map his value v into either a decision $\mathcal{N}$ not to enter or the bid that he places whenever he is matched with a seller. Denote with $\mu > 0$ the exogenous exit rate.[8] Finally let

[7] In an earlier version of this paper we assumed that potential traders whose expected utility is zero did enter the market and become active. These traders had zero probability of trading and exited the market at the exogenous rate $\mu\delta$ per period. Our convergence result (theorem 2 below) still holds under this alternative assumption, though the proofs of claims 16 and 17 are somewhat more complicated because of the presence of active traders who have zero probability of trading.

[8] The presence of a positive exit rate (or something similar) is necessary if particpation costs are zero. The reason is that every trader who enters must have a probability of either trading or exiting that, per unit of time, is bounded away from zero. Otherwise traders

ζ be the endogenous steady state ratio of active buyers to active sellers in the market. Given this notation, a period consists of four steps:

1. Each potential trader decides whether to enter and become an active trader as a function of his type, i.e., a potential seller declines entry if $S(c) = \mathcal{N}$ and a potential buyer declines entry if $B(v) = \mathcal{N}$.

2. Every active buyer is matched with one active seller. His match is equally likely to be with any active seller and is independent of the matches other buyers realize. Since there are a continuum of buyers and sellers the matching probabilities are Poisson: the probability that a seller is matched with $k = 0, 1, 2, \ldots$buyers is[9]

$$\xi_k = \frac{\zeta^k}{k!\, e^{\zeta}}. \tag{2}$$

   Consequently a seller may end up being matched with zero buyers, one buyer, two buyers, etc.

3. Traders within a match bargain in accordance with the rules of the buyers' bid double auction.

   (a) Simultaneously every buyer announces a take-it-or-leave-it offer to the seller. A type v buyer bids $B(v)$. At the time he submits his bid, he does not know how many other buyers he is bidding against; he only knows the endogenous steady-state probability distribution of how many buyers with whom he is competing.

   (b) The seller reviews the bids she has received and accepts the highest one provided it is at least as large as her reservation value, $S(c)$. If two or more buyers tie with the highest bid, then the seller uses a fair lottery to choose between them.

   (c) If trade occurs between a type c seller and a type v buyer at price p, then the seller leaves the market with utility $p - c$ and the buyer leaves the market with utility $v - B(v)$. Each seller, thus, runs an optimal auction; moreover their commitment to this auction is credible since the reservation value each sets stems from their dynamic optimization.[10]

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whose probability of trading is infinitesimal but positive would accumulate in the market and jeopardize the existence of a steady state. The presence of the exogenous exit rate does this directly. The presence of a small participation cost in Satterthwaite and Shneyerov (2004) does this indirectly, for it causes any potential trader who has a low or zero probability of trading to refuse entry because he can not in expectation recover those expected costs.

[9] In a market with M sellers and ζM buyers, the probability that a seller is matched with k buyers is $\xi_k^M = \binom{\zeta M}{k}\left(\frac{1}{M}\right)^k\left(1 - \frac{1}{M}\right)^{\zeta M - k}$. Poisson's theorem (see, for example, Shiryaev, 1995) shows that $\lim_{M\to\infty} \xi_k^M = \xi_k$.

[10] We do not know if these auctions are the equilibrium mechanism that would result if we tried to replicate McAfee's analysis (1993) within our model.

4. Every active trader who fails to trade remains active the next period with probability $e^{-\delta\mu}$ and, for exogenous reasons, exits with probability $1-e^{-\delta\mu}$. Traders who exit without trading leave with zero utility.

Traders' time preference cause them to discount their expected utility at the rate $r \geq 0$ per unit time. This, together with the exit risk μ per unit time, induces impatience. Section 3.1 shows that $e^{-\delta(\mu+r)}$ is the overall rate per period at which each trader discounts his utility.

A seller who has low cost tends to trade within a short number of periods of her entry because most buyers with whom she might be matched have a value higher than her cost and therefore tend to bid sufficiently high to obtain agreement. A high cost seller, on the other hand, tends not to trade as quickly or, perhaps, not at all. As a consequence, in the steady state among the population of sellers who are active, high cost sellers are relatively common and low cost seller are relatively uncommon. Exactly parallel logic implies that, in the steady state, low value buyers are relatively common and high value buyers are relatively uncommon. Moreover, this tendency of traders to wait several periods before trading or exiting implies that the total measure of traders active within the market may be larger—perhaps much larger—than the total measure $(1+a)\delta$ of potential traders who consider entry each period.

To formalize the fact that the distribution of trader types within the market's steady state is endogenous, let T_S be the measure of active sellers in the market at the beginning of each period, T_B be the measure of active buyers, F_S be the distribution of active seller types, and F_B be the distribution of active buyer types. The corresponding densities are f_S and f_B and, establishing useful notation, the right-hand distributions are $\bar{F}_S \equiv 1-F_S$ and $\bar{F}_B \equiv 1-F_B$. Let, in the steady state, the probability that in a given period a type c seller trades be $\rho_S\left[S(c)\right]$ and the let the probability that a type v buyer trades be $\rho_B\left[B(v)\right]$. Define $W_S(c)$ and $W_B(v)$ to be the beginning-of-period steady-state net payoffs to a seller of type c and the buyer of type v, respectively. Let

$$\begin{aligned} \underline{c} &\equiv S(0), \\ \bar{c} &\equiv \sup_c\{c \,|\, W_S(c) > 0\}, \\ \underline{v} &\equiv \inf_v\{v \,|\, W_B(v) > 0\}, \text{ and} \\ \bar{v} &\equiv B(1). \end{aligned} \tag{3}$$

No seller enters whose cost exceeds $\bar{c}$ and no buyer enters whose value is less than $\underline{v}$ because a trader only becomes active if his expected utility from participating is positive. We show in the next section that active sellers' equilibrium bids all fall in the interval $[\underline{c},\bar{c})$, active buyers' equilibrium offers all fall in $(\underline{v},\bar{v}]$, and that: $[\underline{c},\bar{c}] = [\underline{v},\bar{v}] \equiv \left[\underline{p},\bar{p}\right]$.

Our goal is to establish sufficient conditions for symmetric, steady state equilibria to converge to the Walrasian price and competitive allocation as the period length in the market goes to zero. By a steady state equilibrium we mean one in which every seller in every period plays a symmetric, time invariant strategy $S(\cdot)$, every buyer plays a symmetric, time invariant strategy $B(\cdot)$, and both these strategies are always optimal. Let $W_S(c)$ and $W_B(v)$ be the sellers and buyers' interim utilities for sellers of type c and the buyers of type v respectively, i.e, they are beginning-of-period, steady-state, equilibrium net payoffs conditional on their types. Given the friction δ, a market equilibrium M_δ consists of strategies $\{S, B\}$, traders' masses $\{T_S, T_B\}$, and distributions $\{F_S, F_B\}$ such that (i) $\{S, B\}$, $\{T_S, T_B\}$, and $\{F_S, F_B\}$ generate $\{T_S, T_B\}$ and $\{F_S, F_B\}$ as their steady state and (ii) no type of trader can increase his or her expected utility (including the continuation payoff from matching in future periods if trade fails in the current period) by a unilateral deviation from the strategies $\{S, B\}$, and (iii) equilibrium strategies $\{S, B\}$, masses $\{T_S, T_B\}$, and distributions $\{F_S, F_B\}$ are common knowledge among all active and potential traders.

We assume that:

A1. The equilibrium is subgame perfect Bayesian.

A2. For each $\delta > 0$ an equilibrium satisfying A1 exists in which each potential trader's ex ante probability of trade is positive.

Three points need emphasis concerning these assumptions. First, since within a given match buyers announce their bids simultaneously and only then does the seller decide to accept or reject the highest of the bids, Assumption A1 implies that a seller whose highest received bid is above her total dynamic opportunity cost of $c + e^{-\beta\delta} W_S(c)$ accepts that bid. In other words, a seller's strategy is her full dynamic opportunity cost:

$$S(c) = c + e^{-\beta\delta} W_S(c).$$

Second, beliefs are simple to handle because there are continuums of traders and all matching is anonymous and independent. Therefore off-the-equilibrium path actions do not cause any inference ambiguities. Third, Assumption A2 states that well behaved equilibria exist in which trade occurs. This is necessary for two reasons. First, a no-trade equilibrium always exists in which neither buyers nor sellers enter the market. Second, it is an open question that has not yet yielded to our efforts whether such non-trivial equilibria always exist, though numerical experiments (see section 3.5) suggest that they do for well behaved distribution G_S and G_B.[11]

In order to state our theorem we must define admissible sequences of equilibria. An admissible equilibrium sequence rules out sequences in which the buyer-seller ratio ζ_δ goes to either 0 or ∞ as δ goes to 0. Consider for example a sequence of equilibria indexed by δ such that $\delta_1, \delta_2, \ldots, \delta_n, \ldots \to 0$ and $\zeta_\delta \to \infty$.[12] Such a sequence is uninteresting because it violates the spirit of

[11] We have proved existence in our companion paper's model, provided δ is sufficiently small.

[12] We suspect such sequences can not exist, but have not been successful in proving that conjecture. Our expectation is that an existence proof will, as a by-product, rule out such sequences.

assumption A2's requirement that each trader's ex ante probability of trading is positive. Specifically, the number of buyers with which each seller is matched grows unboundedly. Therefore each seller is sure to sell to a buyer whose value v is arbitrarily close to 1 and each buyer whose value is significantly less than 1 is certain not to trade. In fact, if $P_{B\delta}^{\mathrm{EA}}$ denotes the ex ante probability that a potential buyer will trade, then in such sequences $\lim_{\delta\to 0} P_{B\delta}^{\mathrm{EA}} = 0$ because the probability of a buyer drawing value $v = 1$ is zero.

Definition 1 *A sequence of equilibria indexed by δ such that $\delta_1, \delta_2, \ldots, \delta_n, \ldots \to 0$ is admissible if a $\bar{\zeta} > 0$ exists such that the equilibrium for each δ_n exists, satisfies A1, gives each trader an ex ante positive probability of trade, and $\zeta_\delta \in (1/\bar{\zeta}, \bar{\zeta})$.*

We may now state our main result.

Theorem 2 *Fix a sequence of equilibria indexed by δ such that $\delta_1, \delta_2, \ldots, \delta_n, \ldots \to 0$. Let $\{S_\delta, B_\delta\}$ be the strategies associated with the equilibrium that δ indexes, let $[\underline{c}_\delta, \bar{c}_\delta] = [\underline{v}_\delta, \bar{v}_\delta]$ be the offer/bid ranges those strategies imply, and let $W_{S\delta}(c)$ and $W_{B\delta}(v)$ be the resulting interim expected utilities of the sellers and buyers respectively. Then both the bidding and offering ranges converge to p_W:*

$$\lim_{\delta\to 0} \underline{c}_\delta = \lim_{\delta\to 0} \bar{c}_\delta = \lim_{\delta\to 0} \underline{v}_\delta = \lim_{\delta\to 0} \bar{v}_\delta = p_W. \tag{4}$$

In addition, each trader's interim expected utility converges to the utility he would realize if the market were perfectly competitive:

$$\lim_{\delta\to 0} W_{S\delta}(c) = \max[0, p_W - c] \tag{5}$$

and

$$\lim_{\delta\to 0} W_{B\delta}(v) = \max[0, v - p_W]. \tag{6}$$

Observe that in setting up the model we assume that traders use symmetric, pure strategies. We do this for simplicity of exposition. At a cost in notation we could define trader-specific and mixed strategies and then prove that the anonymous nature of matching and the strict monotonicity of strategies implies they in fact must be symmetric and (essentially) pure. To see this, first consider the implication of anonymous matching for buyers. Even if different traders follow distinct strategies, every buyer would still draw his opponents from the same population of active traders.[13] Therefore, for a given value v, every buyer will have the identical best response correspondence. Second, as we show below, every selection from this correspondence is strictly increasing. This means that the best response is pure except at a measure zero set of values where jumps occur. Mixing can occur at these jump points, but does not affect other traders' strategies because the measure of the jump points is zero and therefore has no consequence for other traders' maximization problems.

Section 4 contains the theorem's proof.

[13] This is strictly true because we assume a continuum of traders.

3 Basic properties of equilibria

In this section we derive basic properties that equilibria satisfy. These properties—formulas for probabilities of trade, the strict monotonicity of strategies, and necessary conditions for a strategy pair (S, B) to be an equilibrium—enable us to compute examples of equilibria and provide the foundations for the proof of our main result. We assume throughout both this section and the next that δ and the equilibrium it indexes is an element of an admissible sequence.

3.1 Discounted ultimate probability of trade

An essential construct for our analysis is the discounted ultimate probability of trade. It allows a trader's expected gains from participating in the market to be written as simply as possible. Given any period, let $\rho_S(\lambda)$ be the probability that a seller who chooses reservation price λ trades that period and, similarly, let $\rho_B(\lambda)$ be the probability that a buyer who bids λ trades that period. Also, let $\bar{\rho}_S(\lambda) = 1 - \rho_S(\lambda)$ and $\bar{\rho}_B(\lambda) = 1 - \rho_B(\lambda)$.

Define recursively $P_B(\lambda)$ to be a buyer's discounted ultimate probability of trade if he bids λ:

$$\begin{aligned} P_B(\lambda) &= \rho_B(\lambda) + \bar{\rho}_B(\lambda) e^{-\mu\delta} e^{-r\delta} P_B(\lambda) \\ &= \rho_B(\lambda) + \bar{\rho}_B(\lambda) e^{-\beta\delta} P_B(\lambda) \end{aligned}$$

where $\beta = \mu + r$. Therefore

$$P_B(\lambda) = \frac{\rho_B(\lambda)}{1 - e^{-\beta\delta} + e^{-\beta\delta}\rho_B(\lambda)}. \tag{7}$$

Observe that the formula incorporates both the trader's risk of having to exit and his time discounting into the calculation. The parallel recursion for sellers implies that

$$P_S(\lambda) = \frac{\rho_S(\lambda)}{1 - e^{-\beta\delta} + e^{-\beta\delta}\rho_S(\lambda)}. \tag{8}$$

This construct is useful within a steady state equilibrium because it converts the buyer's dynamic decision problem into a static decision problem. Specifically, if successfully trading gives the type v buyer an expected gain $U = E_p(v - p)$, then his discounted expected utility W^B from following the stationary strategy of bidding λ is

$$\begin{aligned} W^B(\lambda, U) &= \rho_B(\lambda) U + \bar{\rho}_B(\lambda) e^{-\mu\delta} e^{-r\delta} W^B(\lambda, U) \\ &= \rho_B(\lambda) U + \bar{\rho}_B(\lambda) e^{-\beta\delta} W^B(\lambda, U). \end{aligned}$$

Solving this recursion gives the explicit formula:

$$W^B(\lambda, U) = P_B(\lambda) U. \tag{9}$$

Similarly,

$$W^S(\lambda, U) = P_S(\lambda) U. \tag{10}$$

In section 3.3 we derive explicit formulas for $\rho_B(\cdot)$ and $\rho_S(\cdot)$.

3.2 Strategies are strictly increasing

This subsection demonstrates the most basic property that our equilibria satisfy: equilibrium strategies are strictly increasing. As a preliminary, we first characterize the set of traders that are active in the market. We then turn to the monotonicity results.

Claim 3 *In any equilibrium* $\underline{v} < 1$*,* $\overline{c} > 0$*, and*

$$(\underline{v}, 1] \subseteq \{v | W_B(v) > 0\}, \tag{11}$$

$$[0, \overline{c}) \subseteq \{c | W_S(c) > 0\}. \tag{12}$$

Proof. If an equilibrium has positive ex ante probability of trade for each potential trader, then $T_B \int_{\underline{v}}^{1} \rho_B [B(v)] f_B(v) dv > 0$ and $T_S \int_0^{\overline{c}} \rho_S [S(c)] f_S(c) dc > 0$. This is true only if $\underline{v} < 1$ and $\overline{c} > 0$. By bidding $B(v)$ in every period, a buyer gets an equilibrium payoff $W_B(v) = vP_B[B(v)] - D_B(B(v))$ where $D_B(v)$ is his discounted expected equilibrium payment. By Milgrom and Segal's (2002) theorem 2,

$$W_B(v) = W_B(\underline{v}) + \int_{\underline{v}}^{v} P_B[B(x)] dx,$$

so $W_B(\cdot)$ is non-decreasing on $(\underline{v}, 1]$. Assume, contrary to (11), that $W_B[B(v')] = 0$ for some $v' \in (\underline{v}, 1]$. It then follows by the monotonicity of $W_B(\cdot)$ that $W_B(v) = 0$ for all $v \in (\underline{v}, v')$, contradicting the definition of $\underline{v}$. Therefore $W_B(v) > 0$ for all $v \in (\underline{v}, 1]$, establishing (11). The proof of (12) is exactly parallel and is omitted.■

Claim 4 B *is strictly increasing on* $(\underline{v}, 1]$*.*

Proof. $W_B(v) = \sup_{\lambda \geq 0} (v - \lambda) P_B(\lambda) = (v - B(v)) P_B(B(v))$ is the upper envelope of a set of affine functions . It follows that $W_B(\cdot)$ is a continuous, increasing, and convex function that is differentiable almost everywhere.[14] Convexity implies that $W_B'(\cdot)$ is non-decreasing on $[\underline{v}, 1]$. By the envelope theorem $W_B'(\cdot) = P_B[B(\cdot)]$; $P_B[B(\cdot)]$ is therefore non-decreasing on $[\underline{v}, 1]$ at all differentiable points. Milgrom and Segal's (2002) theorem 1 implies that at non-differentiable points $v' \in [\underline{v}, 1]$

$$\lim_{v \to v'-} W_B'(v) \leq P_B(B(v')) \leq \lim_{v \to v'+} W_B'(v).$$

Thus $P_B[B(\cdot)]$ is everywhere non-decreasing on $[\underline{v}, 1]$.

Pick any v, $v' \in (\underline{v}, 1]$ such that $v < v'$. Since $P_B[B(\cdot)]$ is everywhere non-decreasing, $P_B[B(v)] \leq P_B[B(v')]$ necessarily. We first show that B is non-decreasing on $(\underline{v}, 1]$. Suppose, to the contrary, that $B(v) > B(v')$. The rules of the buyer's bid double auction imply that $P_B(\cdot)$ is non-decreasing; therefore $P_B[B(v)] \geq P_B[B(v')]$. Consequently $P_B[B(v)] = P_B[B(v')]$. But this gives v' incentive to lower his bid to $B(v')$, since by doing so he will buy with

[14] An increasing function is differentiable almost everywhere.

the same positive probability but pay a lower price. This contradicts B being an optimal strategy and establishes that B is non-decreasing. If $B(v') = B(v)$ $(= \lambda)$ because B is not strictly increasing, then any buyer with $v'' \in (v, v')$ will raise his bid infinitesimally from λ to $\lambda' > \lambda$ to avoid the rationing that results from a tie. This proves that B is strictly increasing.[15] ■

Claim 5 *S is continuous and strictly increasing on $[0, \overline{c})$.*

Proof. Assumption A1 states that since sellers in the market do not affect price, they bid their total opportunity cost:

$$S(c) = c + e^{-\beta\delta} W_S(c) \tag{13}$$

for all $c \in [0, \overline{c})$ where $W_S(c)$ is the equilibrium payoff to a seller with cost c. In a stationary equilibrium $W_S(c) = D(S(c)) - cP_S(S(c))$ where $P_S[S(c)]$ is her discounted ultimate probability of trading when her offer is $S(c)$ and $D(S(c))$ is the expected equilibrium payment to the seller with cost c. Milgrom and Segal's theorem 2 implies that $W_S(\cdot)$ is continuous and can be written, for $c \in [0, \overline{c}]$, as

$$\begin{aligned} W_S(c) &= W_S(\bar{c}) + \int_c^{\bar{c}} P_S(S(x))dx \qquad (14) \\ &= \int_c^{\bar{c}} P_S(S(x))dx \qquad (15) \end{aligned}$$

where the second line follows from from the definition of $\bar{c}$ and the continuity of $W_S(\cdot)$. This immediately implies that $W_S(\cdot)$ is strictly decreasing (and therefore almost everywhere differentiable) because the definition of $\bar{c}$ implies that $P_S(S(c)) > 0$ for all $c \in [0, \overline{c})$. It, when combined with equation (13), also implies that $S(\cdot)$ is continuous. Therefore, for almost all $c \in [0, \overline{c})$,

$$S'(c) = 1 - e^{-\beta\delta} P_S[S(c)] > 0$$

because $W_S'(c) = -P_S[S(c)]$. Since $S(\cdot)$ is continuous, this is sufficient to establish that $S(\cdot)$ is strictly increasing for all $c \in [0, \overline{c})$. ■

Claim 6 *$\underline{p} = \underline{v} = \underline{c} = S(0) = B(\underline{v}+)$ and $\bar{p} = \overline{c} = \overline{v} = B(1) = S(\bar{c}-)$.*

Proof. Given that S is strictly increasing, $S(0) = \underline{c}$ is the lowest offer any seller ever makes. A buyer with valuation $v < \underline{c}$ does not enter the market because he can only hope to trade by submitting a bid at or above $\underline{c}$ and such a bid would be above his valuation. S is continuous by claim 5, so a buyer with valuation $v > \underline{c}$ will enter the market with a bid $B(v) \in (\underline{c}, v)$ because he can make profit with positive probability. Therefore $\lim_{v \to \underline{c}+} B(v) = \underline{v} = \underline{c}$.

By definition $\bar{c} \equiv \sup_c \{c \mid W_S(c) > 0\}$. Equation (13) therefore implies that $S(\bar{c}-) = \bar{c}$. A seller with cost $c > \overline{v} = B(1)$ will not enter the market, so

[15] Alternatively, one can use Theorem 2.2 in Satterthwaite and Williams (1989) with only trivial adaptations.

$\bar{c} \leq B(1)$. If $\bar{c} = S(\bar{c}-) < \bar{v} \equiv B(1)$, then a seller with cost $c' \in (\bar{c}, B(1))$ can enter and, with positive probability, earn a profit with an offer $S(c') \in (c', B(1))$ $.This, however, is a contradiction: \sup_c \{c \,|\, W_S(c) > 0\} \geq c' > \bar{c} \equiv \sup_c \{c \,|\, W_S(c) > 0\}$.Therefore $S(\bar{c}-) = \bar{c} = \bar{v} = B(1)$.■

These findings are summarized as follows.

Proposition 7 *Suppose that* $\{B, S\}$ *is a stationary equilibrium. Then* B *and* S *are strictly increasing over their domains. They also satisfy the boundary conditions* $\underline{p} = \underline{v} = \underline{c} = S(0) = B(\underline{v}+)$ *and* $\bar{p} = \bar{c} = \bar{v} = B(1) = S(\bar{c}-)$.

Note that strict monotonicity of B and S allows us to define their inverses, V and C: $V(\lambda) = \inf\{v : B(v) > \lambda\}$ and $C(\lambda) = \inf\{c : S(c) > \lambda\}$. These functions are used frequently below.

3.3 Explicit formulas for the probabilities of trading

Focus on a seller of type c who in equilibrium has a positive probability of trade. In a given period she is matched with zero buyers with probability ξ_0 and with one or more buyers with probability $\bar{\xi}_0 = 1 - \xi_0$. Suppose she is matched and v^* is the highest type buyer with whom she is matched. Since by Proposition 7 each buyer's bid function $B(\cdot)$ is increasing, she accepts his bid if and only if $B(v^*) \geq \lambda$ where λ is her reservation price. The distribution from which v^* is drawn is $F_B^*(\cdot)$: for $v \in [\underline{v}, 1]$,

$$\begin{aligned} F_B^*(v) &= \frac{1}{\bar{\xi}_0} \sum_{i=1}^{\infty} \xi_i \left[F_B(v)\right]^i \qquad (16) \\ &= \frac{1}{\bar{\xi}_0} \sum_{i=1}^{\infty} \frac{\zeta^i}{e^{\zeta} i!} \left[F_B(v)\right]^i \\ &= \frac{e^{\zeta F_B(v)} - 1}{e^{\zeta} - 1} \end{aligned}$$

where $F_B(\cdot)$ is the steady state distribution of buyer types and $\{\xi_0, \xi_1, \xi_{2,} \ldots\}$ are the probabilities with which each seller is matched with zero, one, two, or more buyers.[16] Note that this distribution is conditional on the seller being matched. Thus if a seller has reservation price λ, her probability of trading in a given period is

$$\rho_S(\lambda) = \bar{\xi}_0 \left[1 - F_B^*(V(\lambda))\right]. \qquad (17)$$

This formula takes into account the probability that she is not matched in the period.

[16] This formula follows from the facts that:

$$\sum_{i=1}^{\infty} \frac{x^i}{i!} = e^x - 1 \text{ and } \bar{\xi}_0 = \sum_{i=1}^{\infty} \xi_i = \frac{e^{\zeta} - 1}{e^{\zeta}}.$$

A similar expression obtains for $\rho_B(\lambda)$, the probability that a buyer submitting bid λ successfully trades in any given period. In order to derive this expression, we need a formula for ω_k, the probability that the buyer is matched with k rival buyers. If T_B is the mass of active buyers and T_S is the mass of active sellers, then $\omega_k T_B$, the mass of buyers participating in matches with k rival buyers, equals $k+1$ times $\xi_{k+1}T_S$, the mass of sellers matched with $k+1$ buyers:

$$\omega_k T_B = (k+1)\,\xi_{k+1}T_S.$$

Solving, substituting in the formula for ξ_{k+1}, and recalling that $\zeta = T_B/T_S$ shows that ω_k and ξ_k are identical:

$$\omega_k = \frac{(k+1)}{\zeta}\xi_{k+1} = \frac{(k+1)}{\zeta}\frac{\zeta^{k+1}}{(k+1)!\,e^{\zeta}} = \xi_k. \tag{18}$$

The striking implication of this is that the distribution of bids that a buyer must beat is exactly the same distribution of bids that each seller receives when she is matched with at least one buyer.

Turning back to ρ_B, a buyer who bids λ and is the highest bidder has probability $F_S(C(\lambda))$ of having his bid accepted. This is just the probability that the seller with whom the buyer is matched will have a low enough reservation price so as to accept his bid. If a total of $j+1$ buyers are matched with the seller with whom the buyer is matched, then he has j competitors and the probability that all j competitors will bid less than λ is $[F_B(V(\lambda))]^j$. Therefore the probability that the bid λ is successful in a particular period is

$$\begin{aligned}
\rho_B(\lambda) &= F_S(C(\lambda))\sum\nolimits_{j=0}^{\infty}\omega_j\,[F_B(V(\lambda))]^j \\
&= F_S(C(\lambda))\sum\nolimits_{j=0}^{\infty}\xi_j\,[F_B(V(\lambda))]^j \\
&= F_S(C(\lambda))\sum\nolimits_{j=0}^{\infty}\frac{\zeta^j}{e^{-\zeta}j!}\,[F_B(V(\lambda))]^j \\
&= F_S(C(\lambda))\,e^{-\zeta\overline{F}_B(V(\lambda))}.
\end{aligned}$$

where the fourth equality follows from $\Sigma_{j=0}^{\infty}x^j/j! = e^x$.

3.4 Necessary conditions for strategies and steady state distributions

In this subsection the goal is to write down a set of necessary conditions that are sufficiently complete so as to form a basis for calculating section 3.5's example and, also, to create a foundation for section 4's proof of theorem 2. We first derive fixed point conditions that traders' strategies must satisfy. Consider sellers first. Substituting (14),

$$W_S(c) = \int_c^{\bar{c}} P_S(S(x))dx \tag{19}$$

into (13) gives a fixed point condition sellers' strategies must satisfy:

$$S(c) = c + e^{-\beta\delta} \int_c^{\bar{c}} P_S(S(x))dx. \tag{20}$$

The parallel expression for a buyer's expected utility is[17]

$$W_B(v) = \int_{\underline{v}}^{v} P_B[B(x)]\,dx \tag{21}$$

for $v \in (\underline{v}, 1]$. Alternatively,

$$W_B(v) = \max_{\lambda \in [0,1]} (v - \lambda)P_B(\lambda) = (v - B(v))P_B(B(v)).$$

Substituting (21) into this and solving gives a fixed point condition buyers' strategies must satisfy:

$$B(v) = v - \frac{1}{P_B[B(v)]} \int_{\underline{v}}^{v} P_B[B(x)]\,dx \tag{22}$$

for $v \in [\underline{v}, 1]$.

In our model, the distributions $\{F_B, F_S\}$ are endogenously determined by traders' strategies. In any steady state, the numbers of entering and leaving traders must be equal. This gives rise to three necessary conditions. First, in the steady state, for each type $v \in [\underline{v}, 1]$, the density f_B must be such that the mass of buyers entering equals the mass of buyers leaving:

$$a\delta g_B(v) = T_B f_B(v) \left\{\rho_B[B(v)] + \bar{\rho}_B[B(v)]\left(1 - e^{-\mu\delta}\right)\right\} \tag{23}$$

where the left-hand side is the measure of type v buyers of who enter each period and the right-hand side is the measure of type v buyers who exit each period. Note that it takes into account that within each period successful traders exit prior to traders who exit for exogenous reasons. Second, the analogous steady state condition for the density f_S is, for $c \in [0, \bar{c}]$,

$$\delta g_S(c) = T_S f_S(c) \left\{\rho_S[S(c)] + \bar{\rho}_S[S(c)]\left(1 - e^{-\mu\delta}\right)\right\}. \tag{24}$$

Third, trade always occurs between pairs consisting of one seller and one buyer. Therefore, given a cohort of buyers and sellers who enter during a given unit of time, the mass of those buyers who ultimately end up trading must equal the mass of sellers who ultimately end up trading:

$$a \int_{\underline{v}}^{1} P_B(v)\,g_B(v)\,dv = \int_0^{\bar{c}} P_S(c)\,g_S(c)\,dc. \tag{25}$$

Together the fixed point conditions (20 and 22), the expected utility formulas (19 and 21), the steady state conditions (23 and 24), and the overall mass balance equation (25) form a useful set of necessary conditions for equilibria of our model.

[17] Formally, theorem 2 of Milgrom and Segal (2002) justifies this standard expression.

3.5 A computed example

These necessary conditions (19-25) supplemented with boundary conditions enable us to compute an illustrative example of an equilibrium for our model and to show how, as δ is reduced, the equilibrium converges towards the perfectly competitive limit. The boundary conditions are

$$\begin{aligned} S(0) &= \underline{c}, S(\bar{c}-) = \bar{c}, W_S(\bar{c}-) = 0 \\ B(\underline{v}+) &= \underline{v}, B(1) = \bar{v}, W_B(\underline{v}+) = 0 \end{aligned}$$

where $\bar{c} = \bar{v} = \bar{p}$ and $\underline{v} = \underline{c} = \underline{p}$. Our computation specifies that traders' private values are drawn from the uniform distribution ($g_S(c) = g_B(v) = 1$) on the unit interval, the mass of buyers entering each unit of time exceeds the mass of sellers entering by 10% ($a = 1.1$), the exit rate is one per unit time ($\mu = 1.0$), and the discount rate is zero ($r = 0.0$). The Walrasian price for these parameter values is $p_W = 0.524$. We computed the equilibrium by fitting sixth degree Chebyshev polynomials to the set of conditions using the method of collocation.

Figure 1 graphs equilibrium strategies S, B and steady state densities f_S, f_B for these parameter values.[18] The left column of the figure graphs strategies and densities for period length $\delta = 0.2$; the right column does the same for period length $\delta = 0.1$. Visual inspection of these equilibria shows the flattening of strategies that occurs as the period length shortens and each trader's option to wait another period for a better deal becomes more valuable. Thus, as δ is cut in half, the trading range $[\underline{p}, \bar{p}]$ narrows from [0.387, 0.570] down to [0.449, 0.550], which is almost a halving of its width from 0.182 to 0.100. In both equilibria the buyer-seller ratio is $\zeta = 1.570$. Observe that for both period lengths the trading range includes the Walrasian price. Inspection of the densities shows that, as the period length shortens, sellers with costs just below $\bar{c}$ and buyers with values just above $\underline{v}$ tend to accumulate within the market.

4 Proof of the theorem

4.1 A restriction on the shape of B

Our purpose in this subsection is to show that buyers' equilibrium strategies $B(\cdot)$ must be within $\delta^{1/3}$ of either $\underline{v}$ or $\bar{v}$ except within some interval contained in $[\underline{v}, 1]$ that has length no greater than $\delta^{1/3}$. The first claim we establish is a preliminary restriction on the shape of $B(\cdot)$.

Claim 8 *In equilibrium, for all $c \in [0, \bar{c}]$,*

$$\int_{V(S(c))}^{1} [B(x) - S(c)]\ f_B(x)\, dx \leq 2\bar{\zeta}\beta\delta. \tag{26}$$

[18] We do not know if this equilibrium is unique.

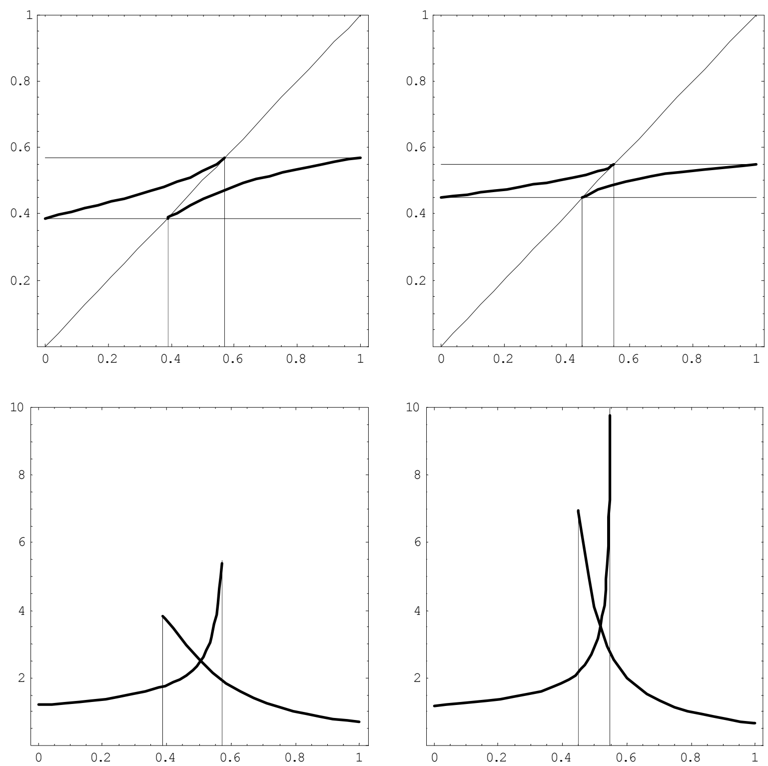

Figure 1: This figure graphs two equilibria for the case in which g_S and g_B are uniform, $a = 1.1$, $\mu = 1.0$, and $r = 0.0$. The upper panel exhibits bidding and acceptance strategies (the lower and upper curves, respectively). The lower panel exhibits the densities of types in the market (the left curve for the sellers and the right curve for the buyers). On the left side period length is $\delta = 0.20$.It has relative inefficiency $I = 0.095$ and masses of active traders $T_S = 0.201$ and $T_B = 0.316$. On the right side period length is $\delta = 0.10$. It has relative inefficiency $I = 0.0513$ and masses of active traders $T_S = 0.106$ and $T_B = 0.166$.

Proof. $W_S(c)$, a seller's expected utility can be written recursively as the sum of the seller's expected gains from trade in the current period plus her expected continuation value if she fails to trade in the current period:

$$\begin{aligned} W_S(c) &= \bar{\xi}_0 \int_{V(S(c))}^{1} B(x)\; f_B^*(x)\,dx - \bar{\xi}_0 \bar{F}_B^*(V(S(c))) \\ &\quad + \left\{\xi_0 + \bar{\xi}_0 F_B^*(V(S(c)))\right\} e^{-\beta\delta} W_S(c)\,. \end{aligned}$$

where $F_B^*(V(S(c)))$ is the probability that, conditional on at least one buyer being matched with her, she fails to trade in the current period. Move all terms involving $W_S(c)$ to the left-hand-side (LHS) and insert the expression $-S(c)+c+e^{-\beta\delta}W_S(c)=0$, which is equation (13) rewritten, into its RHS:

$$\begin{aligned} & W_S(c)\left\{1 - e^{-\beta\delta}\xi_0 - e^{-\beta\delta}\bar{\xi}_0 F_B^*(V(S(c)))\right\} = \\ & \qquad \bar{\xi}_0 \int_{V(S(c))}^{1} B(x)\; f_B^*(x)\,dx - \bar{\xi}_0 c \bar{F}_B^*(V(S(c))) \\ & \qquad\qquad + \bar{\xi}_0 \bar{F}_B^*(V(S(c)))\left\{-S(c) + c + e^{-\beta\delta} W_S(c)\right\}. \end{aligned}$$

Cancel two terms on the RHS and move terms to the LHS to get

$$\begin{aligned} & W_S(c)\left\{ \begin{array}{l} 1 - e^{-\beta\delta}\xi_0 \\ \qquad -e^{-\beta\delta}\bar{\xi}_0\left[F_B^*(V(S(c))) + \bar{F}_B^*(V(S(c)))\right] \end{array} \right\} = \\ & \qquad\qquad \bar{\xi}_0 \int_{V(S(c))}^{1} B(x)\; f_B^*(x)\,dx - \bar{\xi}_0 \bar{F}_B^*(V(S(c)))\,S(c)\,. \end{aligned}$$

Recall that $F_B^*(v) + \bar{F}_B^*(v) = 1$ and $\xi_0 + \bar{\xi}_0 = 1$. Then

$$\delta\beta W_S(c) = \bar{\xi}_0 \int_{V(S(c))}^{1} \left[B(x) - S(c)\right] f_B^*(x)\,dx, \tag{27}$$

i.e., in equilibrium, for a type c seller, the expected marginal cost of waiting an additional period to trade is equal to the expected marginal expected gain from waiting.

Since $1 - e^{-\beta\delta} \leq \beta\delta$, rearranging (27) gives

$$\int_{V(S(c))}^{1} \left[B(x) - S(c)\right] f_B^*(x)\,dx = \frac{\beta\delta}{\bar{\xi}_0} W_S(c) \leq \frac{\beta\delta}{\bar{\xi}_0}$$

because $W_S(c) \leq 1$. First order stochastic dominance implies that

$$\int_{V(S(c))}^{1} \left[B(x) - S(c)\right] f_B(x)\,dx \leq \int_{V(S(c))}^{1} \left[B(x) - S(c)\right] f_B^*(x)\,dx;$$

Therefore

$$\int_{V(S(c))}^{1} \left[B(x) - S(c)\right] f_B(x)\,dx \leq \frac{\beta\delta}{\bar{\xi}_0} \tag{28}$$

for all $c \in [0, \overline{c})$. The probability that a seller will not be matched with any buyer is

$$\xi_0 = \frac{\zeta^0}{0!\, e^{\zeta}} = \frac{1}{e^{\zeta}} < e^{-1/\bar{\zeta}} \tag{29}$$

because the equilibrium is an element of an admissible sequence and therefore $\zeta \in [1/\bar{\zeta}, \bar{\zeta}]$. A bound on the the complementary probability is

$$\bar{\xi}_0 > 1 - e^{-1/\bar{\zeta}} > \frac{1}{2\bar{\zeta}}$$

because $1 - e^{-x} = x - \frac{x^2}{2} + \dots$ and $\frac{1}{2\bar{\zeta}}$ is both small and positive. Using this observation, we conclude from (28):

$$\int_{V(S(c))}^{1} \left[B\left(x\right) - S\left(c\right)\right] \, f_B\left(x\right) dx \leq 2\bar{\zeta}\beta\delta. \blacksquare \tag{30}$$

The bound (30) does not have any bite if $f_B(x)$ becomes small as δ becomes small. Therefore in the next claim we establish a lower bound on $f_B(v)$ that is independent of δ.

Claim 9 *For all $v \in [\underline{v}, 1]$ and sufficiently small $\delta > 0$, $f_B(v) \geq \frac{\mu g}{2\beta}(\bar{c} - B(v))$.*

Proof. Consider the highest type buyer, $v = 1$. In equilibrium he bids $B(1)$ instead of some $\lambda < B(1)$. His expected gain from following this strategy is $P_B(B(1))(1 - B(1))$. If he bids $\lambda < B(1)$, then his expected gain is $P_B(\lambda)(1 - \lambda)$. Revealed preference implies $P_B(B(1))(1 - B(1)) \geq P_B(\lambda)(1 - \lambda)$. Therefore

$$P_B(\lambda) \leq \frac{P_B(B(1))(1 - B(1))}{1 - \lambda} = \frac{P_B(\bar{c})(1 - \bar{c})}{1 - \lambda}. \tag{31}$$

Note also that, for $\lambda < B(1)$, $\rho_B[B(1)] \geq \rho_B(\lambda)$ because ρ_B is a non-decreasing function.

Inequality (31) permits us to bound $\rho_B(\lambda)$ from above. It and formula (7) imply the following sequence of inequalities

$$P_B(\lambda) = \frac{\rho_B(\lambda)}{\rho_B(\lambda) + (1 - e^{-\beta\delta})[1 - \rho_B(\lambda)]} \leq \frac{P_B(\bar{c})(1 - \bar{c})}{1 - \lambda}, \tag{32}$$

$$\frac{\rho_B(\lambda)}{\rho_B(\lambda) + \beta\delta} \leq \frac{P_B(\bar{c})(1 - \bar{c})}{1 - \lambda},$$

$$\rho_B(\lambda) \leq \frac{\beta\delta}{\frac{1-\lambda}{P_B(\bar{c})(1-\bar{c})} - 1},$$

$$\rho_B(\lambda) \leq \frac{\beta\delta}{\frac{1-\lambda}{(1-\bar{c})} - 1}, \text{ and}$$

$$\rho_B(\lambda) \leq \frac{\beta\delta(1 - \bar{c})}{\bar{c} - \lambda}$$

where the second line follows from $1-e^{-\beta\delta} \leq \beta\delta$ and dropping the less than unity factor $(1-\rho_B(\lambda))$, the third line from solving the inequality, the fourth line from $P_B(\bar{c}) \leq 1$, and the fifth line from simplifying the fourth line.

Inequality (32) allows us to establish the desired lower bound on $f_B(v)$ provided we have an upper bound on T_B, the mass of buyers active in the market. Suppose all potential buyers (measure a each period) entered and became active, none successfully traded, and all ultimately left the market due to the workings of the exit rate μ. The total mass of active buyers in the market would then be $T_B = a\delta/\left(1-e^{-\mu\delta}\right)$, which follows from equating $T_B(1-e^{-\mu\delta})$, the measure of buyers who enter each period, with $a\delta$, the measure of buyers who enter each period, and solving. Since many buyers leave as a result of successful trade an upper bound on the mass of sellers in the market is $T_B \leq a\delta/(1-e^{-\mu\delta})$. To obtain the needed lower bound on $f_B(v)$ solve for it in equation (23) and then simplify:

$$\begin{aligned} f_B(v) &= \frac{1}{T_B}\frac{a\delta\, g_B(v)}{\rho_B[B(v)] + \bar{\rho}_B[B(v)](1-e^{-\mu\delta})} \qquad (33)\\ &\geq \frac{1}{T_B}\frac{a\delta\, g_B(v)}{\rho_B[B(v)] + \bar{\rho}_B[B(v)](1-e^{-\beta\delta})} \\ &\geq \frac{1}{T_B}\frac{a\delta\, g_B(v)}{(1-\beta\delta)\frac{\beta\delta(1-\bar{c})}{\bar{c}-\lambda} + \beta\delta} \\ &\geq \frac{\left(1-e^{-\mu\delta}\right)}{\beta\delta}\frac{g_B(v)}{(1-\beta\delta)\frac{1-\bar{c}}{\bar{c}-\lambda}+1} \\ &\geq \frac{\mu}{2\beta}\frac{g_B(v)}{(1-\beta\delta)\frac{1-\bar{c}}{\bar{c}-\lambda}+1} \\ &\geq \frac{\mu}{2\beta}\frac{\underline{g}}{(1-\beta\delta)\frac{1-\bar{c}}{\bar{c}-\lambda}+1} \\ &\geq \frac{\mu}{2\beta}\frac{\underline{g}}{\frac{(1-\bar{c})}{\bar{c}-\lambda}+1} = \frac{\mu}{2\beta}\frac{\underline{g}}{\lambda}(\bar{c}-\lambda) \\ &\geq \frac{\mu\underline{g}}{2\beta}(\bar{c}-B(v)) \end{aligned}$$

where $\beta = r+\mu \geq \mu$ implies the second line, (32) and $1-e^{-\beta\delta} \leq \beta\delta$ implies the third line, $T_B < a\delta/(1-e^{-\mu\delta})$ implies the fourth line, $1-e^{-\beta\delta} > \frac{1}{2}\beta\delta$ for sufficiently small δ implies the fifth line, $\underline{g}$ being the lower bound on the densities g_B implies the sixth line, $(1-\beta\delta) \leq 1$ implies the seventh line, and $\lambda \leq 1$ implies the last line.■

We now use the bounds established in claims 8 and 9 to place a strong restriction on the shape of B. Figure 1 shows the construction used in the next claim and shows how the claim's conclusion confines $B(\cdot)$ to a narrow band of width proportional to $\delta^{1/3}$.

Claim 10 *Suppose $\bar{c}-\ \underline{v} \geq 2\delta^{1/3}$. For given $\delta > 0$ sufficiently small, let $v^* =$*

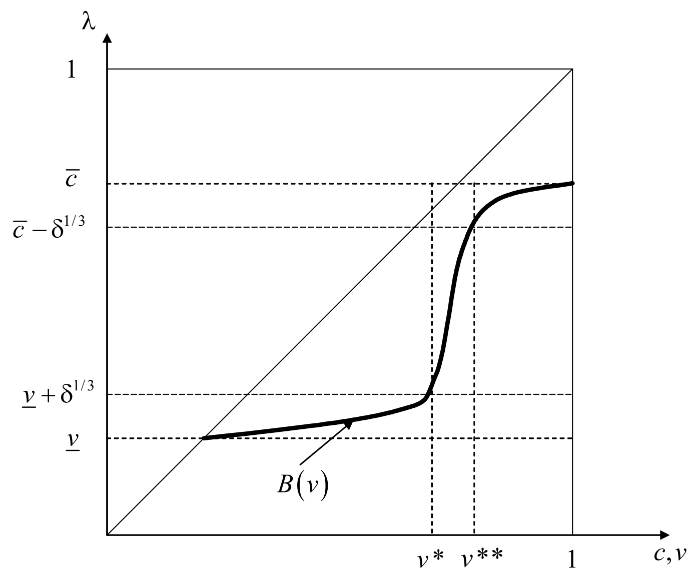

Figure 2: $\sqrt[3]{\delta}$ band that confines $B(\cdot)$

$V\left(\underline{v} + \delta^{1/3}\right)$ *and* $v^{**} = V\ \left(\bar{c} - \delta^{1/3}\right)$. *Then*

$$v^{**} - v^{*} \leq \frac{4\bar{\zeta}\beta^2}{\mu \underline{g}}\ \delta^{1/3}. \tag{34}$$

Proof. Substituting inequality (33) into (30) gives

$$\frac{\mu \underline{g}}{2\beta} \int_{V(S(c))}^{1} \left(B\left(x\right) - S\left(c\right)\right)\left(\bar{c} - B\left(x\right)\right)\, dx \leq 2\bar{\zeta}\beta\delta.$$

The special case of this inequality in which $c = 0$ gives the restriction on the buyers' strategy $B\left(\cdot\right)$:

$$\int_{\underline{v}}^{1} \left(B\left(x\right) - \underline{v}\right)\left(\bar{c} - B\left(x\right)\right)\, dx \leq \frac{4\bar{\zeta}\beta^2\delta}{\mu \underline{g}} \tag{35}$$

because $S\left(0\right) = \underline{c} = \underline{v}$ and $V\left(\underline{v}\right) = \underline{v}$.

Note that, for $x \in [v^{*}, v^{**}]$, the following inequalities are true: $B\left(x\right) - \underline{v} \geq \delta^{1/3}$ and $\left(\bar{c} - B\left(x\right)\right) \geq \delta^{1/3}$. Therefore

$$\int_{v^{*}}^{v^{**}} \left(B\left(x\right) - \underline{v}\right)\left(\bar{c} - B\left(x\right)\right)\, dx \geq \left(v^{**} - v^{*}\right)\delta^{2/3}.$$

This inequality together with the observation that the integrand of (35) is positive for the whole interval of integration $[\underline{v}, 1]$ implies

$$\begin{aligned} \frac{4\bar{\zeta}\beta^2}{\mu \underline{g}}\delta &\geq \int_{\underline{v}}^{1} \left(B\left(x\right) - \underline{v}\right)\left(\bar{c} - B\left(x\right)\right) dx \\ &\geq \int_{v^*}^{v^{**}} \left(B\left(x\right) - \underline{v}\right)\left(\bar{c} - B\left(x\right)\right) dx \\ &\geq \left(v^{**} - v^*\right)\delta^{2/3}. \end{aligned}$$

The first and last terms of this sequence of inequalities imply (34).■

4.2 The law of one price

In this subsection, we demonstrate that $\lim_{\delta\to 0}(\bar{c}_\delta - \underline{v}_\delta) = 0$. Since all trades occur at prices within the interval $[\underline{v}_\delta, \bar{c}_\delta]$ this means that as the period length approaches zero all trades occur at essentially one price. Intuitively this is driven by increasing local market size and the resulting option value, i.e., as δ becomes small each trader can safely wait for a very favorable offer/bid.

Proposition 11 *Consider any $\overline{\zeta}$-admissible sequence of equilibria $\delta_n \to 0$. Then*

$$\lim_{\delta\to 0}(\bar{c}_\delta - \underline{v}_\delta) = 0.$$

Proof. Suppose a sequence of equilibria indexed by δ exists such that $\delta_1, \delta_2, \ldots, \delta_n, \ldots \to 0$ and $\overline{\lim}_{\delta\to 0}(\bar{c}_\delta - \underline{v}_\delta) = \eta > 0$. We show that this is a contradiction: therefore, necessarily, $\lim_{\delta\to 0}(\bar{c}_\delta - \underline{v}_\delta) = 0$. From now on, fix a subsequence such that $\lim_{n\to\infty}(\bar{c}_\delta - \underline{v}_\delta) = \eta$ and $\bar{c}_\delta - \underline{v}_\delta > \eta$.

Pick a small δ from the subsequence and let the strategies $\{S, B\}$, probabilities $\{\xi_0, \xi_1, \xi_{2,} \ldots\}$, and distributions $[F_S, F_B]$ characterize the equilibrium associated with it. Recall that $S(0) = \underline{c} = \underline{v} = B(\underline{v}+)$ and $B(1) = \bar{v} = \bar{c} = S(\bar{c}-).Alsorecallabovefromabovethedefinitionsof$ v*and v^{**}.Define in addition

$$\tilde{v} = \bar{v} - \frac{1}{4}(\bar{v} - \underline{v}),\ b = B\left(\tilde{v}\right), b' = b + \delta^{1/3},\ \text{and}\ \tilde{v}' = V\left(b'\right)$$

as shown in figure 2. We prove the proposition with a sequence of four claims, the last of which has the proposition as a corollary. The first of these claims derives three intermediate inequalities.

Claim 12 *Given the construction of $\tilde{v}$, v^*, v^{**}, b, and b' and given that, by assumption, $\lim_{\delta\to 0}(\bar{c}_\delta - \underline{v}_\delta) = \eta > 0$, if δ is sufficiently small, then*

$$\tilde{v} \leq v^*, \tag{36}$$

$$\inf_{v\in[\tilde{v}, \tilde{v}']}(\bar{v} - B\left(v\right)) \geq \frac{1}{2}\eta, \tag{37}$$

$$\tilde{v}' - \tilde{v} \geq \frac{1}{8}\eta. \tag{38}$$

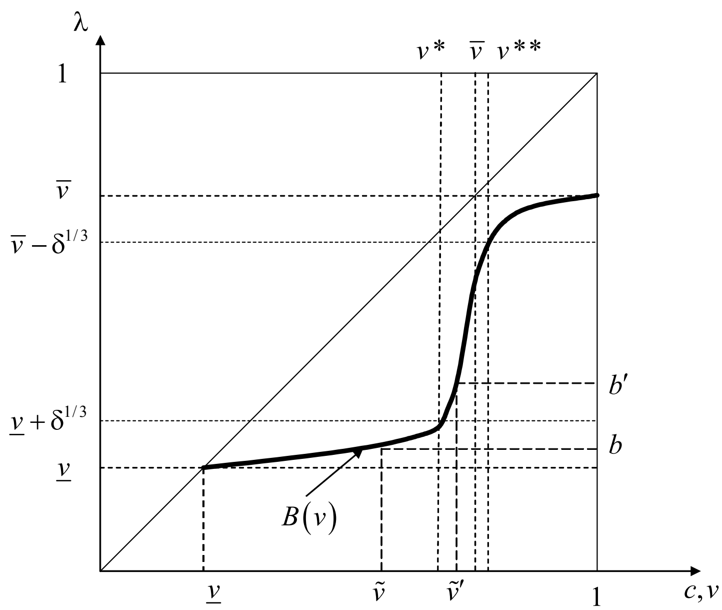

Figure 3: Construction of v^*, v^{**}, $\widetilde{v}$, $\widetilde{v}'$, b and b'.

Proof. We begin with three observations:

O1 The assumption that $\bar{v} - \underline{v} \geq \eta$ for all δ in the sequence and the definition $\tilde{v} = \bar{v} - \frac{1}{4}(\bar{v} - \underline{v})$ imply $\tilde{v} + \frac{1}{4}\eta < \bar{v}$.

O2 The definition $B(v^{**}) = \bar{v} - \delta^{1/3}$ and the inequality $B(v^{**}) \leq v^{**}$ imply that $\bar{v} - \delta^{1/3} \leq v^{**}$. That $B(v^{**}) \leq v^{**}$ follows from the fact that $v^{**} \in (\underline{v}, \bar{c})$ and therefore $\rho_B(v^{**}) > 0$; hence bidding $\lambda \in (\underline{v}, v^{**})$ generates a positive payoff and bidding $\lambda \in (v^{**}, 1)$ generates a negative payoff.

O3 Recall from (34) that $v^{**} \leq v^* + \frac{4\bar{\zeta}}{\mu \underline{g}} \beta^2 \delta^{1/3}$.

To derive (36), note that O2 and O3 imply

$$\bar{v} \leq v^* + \left(1 + \frac{4\bar{\zeta}\beta^2}{\mu \underline{g}}\right) \delta^{1/3}$$

Combining this with O1 gives

$$\tilde{v} \leq v^* - \frac{1}{4}\eta + \left(1 + \frac{4\bar{\zeta}\beta^2}{\mu \underline{g}}\right) \delta^{1/3}.$$

Thus, for small enough δ,

$$\tilde{v} \leq v^*. \tag{39}$$

Turning to (37), that $B(\cdot)$ is increasing, $\tilde{v} \le v^*$, $B(\tilde{v}) = b$, $B(v^*) = \underline{v} + \delta^{1/3}$, $b' = b + \delta^{1/3}$, and $B(v^{**}) = b'$ together imply that $b \in [\underline{v}, \underline{v} + \delta^{1/3}]$ and $b' \in [\underline{v} + \delta^{1/3}, \underline{v} + 2\delta^{1/3}]$. Consequently, for sufficiently small δ,

$$\begin{aligned} \inf_{v \in [\tilde{v}, \tilde{v}']} (\bar{v} - B(v)) &\ge \bar{v} - b' \qquad (40) \\ &\ge \bar{v} - \underline{v} - 2\delta^{1/3} \\ &\ge \eta - 2\delta^{1/3} \\ &\ge \frac{1}{2}\eta. \end{aligned}$$

This proves (37). Finally, to establish (38), note that by construction $v^* < \tilde{v}'$. Therefore

$$\begin{aligned} \tilde{v}' - \tilde{v} &> v^* - \tilde{v} \qquad (41) \\ &\ge v^{**} - \tilde{v} - \frac{4\bar{\zeta}\beta^2}{\mu \underline{g}}\,\delta^{1/3} \\ &\ge \bar{v} - \tilde{v} - \left(1 + \frac{4\bar{\zeta}\beta^2}{\mu \underline{g}}\right)\delta^{1/3} \\ &\ge \frac{1}{4}\eta - \left(1 + \frac{4\bar{\zeta}\beta^2}{\mu \underline{g}}\right)\delta^{1/3} \\ &\ge \frac{1}{8}\eta \end{aligned}$$

where line two follows from $v^{**} \le v^* + \frac{4\bar{\zeta}}{\mu \underline{g}}\,\beta^2\delta^{1/3}$, line three follows from $v^{**} > \bar{v} - \delta^{1/3}$, line four follows from $\bar{v} - \tilde{v} \ge \frac{1}{4}\eta$, and line five follows if δ is sufficiently small.■

Claim 13 *Given* $\lim_{\delta \to 0}(\bar{c}_\delta - \underline{v}_\delta) = \eta > 0$, *if* δ *is sufficiently small, then a* $\gamma > 0$ *exists such that*

$$\frac{\rho_B(b')}{\rho_B(b)} > 1 + \gamma.$$

Proof. Since $V(b') = \tilde{v}'$ and $V(b) = \tilde{v}$, the ratio of $\rho_B(b')$ and $\rho_B(b)$ is

$$\begin{aligned} \frac{\rho_B(b')}{\rho_B(b)} &= \frac{F_S(C(b'))\, e^{-\zeta \overline{F}_B(\tilde{v}')}}{F_S(C(b))\, e^{-\zeta \overline{F}_B(\tilde{v})}} \\ &\ge e^{\zeta[F_B(\tilde{v}') - F_B(\tilde{v})]} \\ &\ge 1 + \zeta\,[F_B(\tilde{v}') - F_B(\tilde{v})] \\ &\ge 1 + \frac{1}{\overline{\overline{\zeta}}} \int_{\tilde{v}}^{\tilde{v}'} f_B(x)\; dx, \end{aligned}$$

where the second line follows from $b' > b$ and both F_S and C being increasing, the third line follows by $e^x \ge 1 + x$ $(x \ge 0)$, and the last line follows from

$\zeta \geq 1/\overline{\zeta}$. Recall from (33) that $f_B(v) \geq \underline{g}(\overline{v} - B(v))$. Therefore

$$\begin{aligned} \frac{\rho_B(b')}{\rho_B(b)} &\geq 1 + \frac{\underline{g}}{\overline{\zeta}} \int_{\tilde{v}}^{\tilde{v}'} (\overline{v} - B(v)) \ dx \\ &\geq 1 + \frac{\underline{g}}{\overline{\zeta}} (\tilde{v}' - \tilde{v}) \inf_{v \in [\tilde{v}, \tilde{v}']} (\overline{v} - B(v)) \\ &\geq 1 + \frac{\underline{g}}{\overline{\zeta}} \left(\frac{1}{8}\eta\right) \left(\frac{1}{2}\eta\right) \\ &= 1 + \frac{1}{16} \frac{\underline{g}}{\overline{\zeta}} \eta^2 \\ &= 1 + \gamma \end{aligned}$$

where line three follows from (37) and (38) and line five follow from $\gamma = \frac{1}{16\overline{\zeta}} \underline{g} \eta^2$.■

Claim 14 *Given* $lim_{\delta \to 0}(\bar{c}_\delta - \underline{v}_\delta) = \eta > 0$, *if* δ *is sufficiently small, then*

$$\frac{P_B(b')}{P_B(b)} \geq 1 + \gamma^*$$

where $\gamma^* = \frac{1}{4}\gamma\eta > 0$.

Proof. Direct calculation proves this. Recall from (7) the formula for $P_B(b)$. Therefore

$$\frac{P_B(b')}{P_B(b)} = \frac{\rho_B(b')}{\rho_B(b)} \frac{\beta\delta + \rho_B(b) - \beta\delta\rho_B(b)}{\beta\delta + \rho_B(b') - \beta\delta\rho_B(b')}.$$

Define x and y so that $\rho_B(b') = \beta\delta x$ and $\rho_B(b) = \beta\delta y$. Then, after some manipulation,

$$\begin{aligned} \frac{P_B(b')}{P_B(b)} &= \frac{1 + \frac{1}{y} - \beta\delta}{1 + \frac{1}{x} - \beta\delta} \\ &\geq \frac{1 + \frac{1+\gamma}{x} - \beta\delta}{1 + \frac{1}{x} - \beta\delta} \\ &= 1 + \frac{\frac{\gamma}{x}}{1 + \frac{1}{x} - \beta\delta} \\ &\geq 1 + \frac{\frac{\gamma}{x}}{1 + \frac{1}{x}} \\ &= 1 + \frac{\gamma\beta\delta}{\rho_B(b') + \beta\delta} \\ &\geq 1 + \frac{1}{2}\gamma(\overline{v} - b') \end{aligned}$$

where line two follows from claim 13's implication that $\frac{1}{y} \geq \frac{1+\gamma}{x}$, line four follows from $\beta\delta \in (0,1)$, line five follows from the definition of x, and line six follows from inequality (32) and $1 - b' < 1$. By construction $b' \in (\underline{v} + \delta^{1/3}, \underline{v} + 2\delta^{1/3})$. Hence, for δ sufficiently small,

$$\begin{aligned}\frac{P_B(b')}{P_B(b)} &\geq 1 + \frac{1}{2}\gamma(\bar{v} - \underline{v} - \delta^{1/3}) \\ &\geq 1 + \frac{1}{4}\gamma\eta\end{aligned}$$

because $\bar{v} - \underline{v} \geq \eta$.■

Claim 15 *Given* $\lim_{\delta \to 0}(\bar{c}_\delta - \underline{v}_\delta) = \eta > 0$, *if* δ *is sufficiently small, then a type* $\tilde{v}$ *buyer has an incentive to deviate from bidding* $B(\tilde{v}) = b$ *to bidding* $b' > b$.

Proof. If we denote the expected utility of a type v buyer who bids λ as $\pi_B(\lambda, v)$, then to prove this we need to show that $\pi_B(b', \tilde{v}) - \pi_B(b, \tilde{v}) > 0$ for δ sufficiently small. First note that by construction $\tilde{v} = \bar{v} - \frac{1}{4}(\bar{v} - \underline{v})$ and $b < \underline{v} + \delta^{1/3}$. Therefore

$$\begin{aligned}\tilde{v} - b &\geq \bar{v} - \frac{1}{4}(\bar{v} - \underline{v}) - \underline{v} - \delta^{1/3} \\ &= \frac{3}{4}(\bar{v} - \underline{v}) - \delta^{1/3} \\ &\geq \frac{3}{4}\eta - \delta^{1/3} \\ &\geq \frac{1}{2}\eta\end{aligned}$$

for sufficiently small δ because $\bar{v} - \underline{v} > \eta$. Next observe that, for sufficiently small δ,

$$\begin{aligned}\pi_B(b', \tilde{v}) - \pi_B(b, \tilde{v}) &= (\tilde{v} - b')\, P_B(b') - (\tilde{v} - b)\, P_B(b) \\ &\geq [(1 + \gamma^*)(\tilde{v} - b') - (\tilde{v} - b)]\, P_B(b) \\ &= \left[(1 + \gamma^*)\left(\tilde{v} - b - \delta^{1/3}\right) - (\tilde{v} - b)\right] P_B(b) \\ &= \left[\gamma^*(\tilde{v} - b) - (1 + \gamma^*)\,\delta^{1/3}\right] P_B(b) \\ &\geq \left[\frac{1}{2}\eta\ \gamma^* - (1 + \gamma^*)\,\delta^{1/3}\right] P_B(b) \\ &> 0\end{aligned}$$

where line 2 follows from claim 14. ■

Claim 15 directly implies proposition 11 because it contradicts the maintained hypothesis that an admissible subsequence of equilibria exists such that $\lim_{\delta \to 0}(\bar{c}_\delta - \underline{v}_\delta) = \eta > 0$.

4.3 Convergence of the bidding and offering ranges to the Walrasian price

Recall that the Walrasian price p_W is the solution to the equation

$$G_S(p_W) = a\ \bar{G}_B(p_W); \tag{42}$$

it is just the price at which the measure of sellers entering the market with cost $c \leq p_W$ equals the measure of buyers entering the market with values $v \geq p_W$. This price would clear the market each period if there were a centralized market. In this subsection we prove our main result: as $\delta \to 0$ the bidding range $[\underline{v}, \bar{v}]$ collapses to the Walrasian price. More formally, for any sequence of equilibria indexed by δ such that $\delta_1, \delta_2, \ldots, \delta_n, \ldots \to 0$, both

$$\lim_{\delta \to 0} \underline{v}_\delta = p_W \text{ and } \lim_{\delta \to 0} \bar{v}_\delta = p_W. \tag{43}$$

We show this through the proof of two claims. Each of these claims uses the idea that if the price is not converging to the Walrasian price, then the market does not clear globally and an excess of traders builds up on one side or the other of the market. Traders on the long side then have an incentive to deviate from their prescribed bid/offer in order to trade before exiting for exogenous reasons.

Claim 16 $\underline{\lim}_{\delta \to 0} \bar{v}_\delta \geq p_W$.

Proof. Let $v_* = \underline{\lim}_{\delta \to 0} \bar{v}_\delta$ and assume, contrary to the statement in the claim, that $v_* < p_W$. For the remainder of this proof, fix a subsequence $\bar{v}_\delta \to v_*$. Let $\tilde{v}_\delta = \bar{v}_\delta + \sqrt{\bar{v}_\delta - \underline{v}_\delta}$ (note that $\tilde{v}_\delta \in (\bar{v}_\delta, 1]$ for all small enough δ, by proposition 11). Revealed preference implies that

$$\begin{aligned} \pi_B\left(B_\delta\left(\tilde{v}_\delta\right), \tilde{v}_\delta\right) &\geq \pi_B\left(B_\delta\left(1\right), \tilde{v}_\delta\right) \\ \left[\tilde{v}_\delta - B_\delta\left(\tilde{v}_\delta\right)\right] \; P_{B\delta}\left[B_\delta\left(\tilde{v}_\delta\right)\right] &\geq \left[\tilde{v}_\delta - B_\delta\left(1\right)\right] \; P_{B\delta}\left[B_\delta\left(1\right)\right]. \end{aligned}$$

Therefore

$$\begin{aligned} P_{B\delta}\left[B_\delta\left(\tilde{v}_\delta\right)\right] &\geq \frac{\tilde{v}_\delta - B_\delta\left(1\right)}{\tilde{v}_\delta - B_\delta\left(\tilde{v}_\delta\right)} P_{B\delta}\left[B_\delta\left(1\right)\right] \\ &\geq \frac{\tilde{v}_\delta - B_\delta\left(1\right)}{\tilde{v}_\delta - \underline{v}_\delta} P_{B\delta}\left[B_\delta\left(1\right)\right], \end{aligned} \tag{44}$$

where the second inequality follows from the fact that $B_\delta\left(\cdot\right)$ is strictly increasing and therefore $B_\delta\left(\tilde{v}_\delta\right) \geq B_\delta\left(\underline{v}_\delta\right) = \underline{v}_\delta$. Note that

$$\begin{aligned} \frac{\tilde{v}_\delta - B_\delta\left(1\right)}{\tilde{v}_\delta - \underline{v}_\delta} &= \frac{\tilde{v}_\delta - \bar{v}_\delta}{\tilde{v}_\delta - \underline{v}_\delta} \\ &= \frac{\sqrt{\bar{v}_\delta - \underline{v}_\delta}}{\sqrt{\bar{v}_\delta - \underline{v}_\delta} + \bar{v}_\delta - \underline{v}_\delta}, \end{aligned} \tag{45}$$

where the equality in the first line follows from $B_\delta\left(1\right) = \bar{v}_\delta$, and the equality in the second line is the substitution of the definition $\tilde{v}_\delta = \bar{v}_\delta + \sqrt{\bar{v}_\delta - \underline{v}_\delta}$. Combining (44) and (45) we get

$$P_{B\delta}\left[B_\delta\left(\tilde{v}_\delta\right)\right] \geq \frac{\sqrt{\bar{v}_\delta - \underline{v}_\delta}}{\sqrt{\bar{v}_\delta - \underline{v}_\delta} + \bar{v}_\delta - \underline{v}_\delta} P_{B\delta}\left[B_\delta\left(1\right)\right],$$

So in particular, $P_{B\delta}[B_\delta(1)] = 1$ and, by proposition 11, $\lim_{\delta\to 0}(\bar{v}_\delta - \underline{v}_\delta) = 0$ imply[19]

$$\lim_{\delta\to 0} P_{B\delta}[B_\delta(\tilde{v}_\delta)] = 1.$$

Mass balance, equation (25) above, states that

$$\int_{\underline{v}_\delta}^{1} a g_B(x) P_{B\delta}[B_\delta(x)]\,dx = \int_0^{\bar{v}_\delta} g_S(x) P_{S\delta}[S_\delta(x)]\,dx. \tag{46}$$

Given that $P_{B\delta}[B_\delta(\cdot)]$ is increasing and $\tilde{v}_\delta > \underline{v}$,

$$\int_{\underline{v}_\delta}^{1} a g_B(x) P_{B\delta}[B_\delta(x)] \geq P_{B\delta}[B_\delta(\tilde{v}_\delta)] \int_{\tilde{v}_\delta}^{1} a g_B(x) dx \geq P_{B\delta}[B_\delta(\tilde{v}_\delta)]\, a\overline{G}_B(\tilde{v}_\delta)$$

and

$$\int_0^{\bar{v}_\delta} g_S(x) P_{S\delta}[S_\delta(x)]\,dx \leq G_S[\bar{v}_\delta].$$

Therefore it follows from (46) that

$$P_{B\delta}[B_\delta(\tilde{v}_\delta)]\, a\overline{G}_B(\tilde{v}_\delta) \leq G_S[\bar{v}_\delta]$$

or, since $P_{B\delta}[B_\delta(\tilde{v}_\delta)] \leq 1$,

$$a\overline{G}_B(\tilde{v}_\delta) \leq G_S[\bar{v}_\delta]. \tag{47}$$

By taking limits in (47) as $\delta \to 0$ and invoking continuity of G_S and $\overline{G}_B$, we obtain

$$a\overline{G}_B\left(\lim_{\delta\to 0}\tilde{v}_\delta\right) \leq G_S\left(\lim_{\delta\to 0}\bar{v}_\delta\right). \tag{48}$$

The definition of v_* and proposition 11 imply $\lim_{\delta\to 0}\tilde{v}_\delta = \lim_{\delta\to 0}[\bar{v}_\delta + \sqrt{\bar{v}_\delta - \underline{v}_\delta}] = \bar{v}_\delta$ and, by hypothesis, $\lim_{\delta\to 0}\bar{v}_\delta = v_*$. Therefore we obtain from (48):

$$a\overline{G}_B(v_*) \leq G_S(v_*).$$

This, however, is a contradiction because the the maintained assumption that $v_* < p_W$ implies that $a\overline{G}_B(v_*) > a\overline{G}_B(p_W) = G_S(p_W) > G_S(v_*)$.■

Claim 17 $\overline{\lim}_{\delta\to 0}\underline{c}_\delta \leq p_W$.

Proof. Verification of this claim follows the same logic as that of claim 16. Define $c_* = \overline{\lim}_{\delta\to 0}\underline{c}_\delta$ and suppose, contrary to the statement in the claim, that $c_* > p_W$. For the remainder of this proof, fix a subsequence $\underline{c}_\delta \to c_*$. Let $\widetilde{c}_\delta = \underline{c}_\delta + \sqrt{\overline{c}_\delta - \underline{c}_\delta}$ noting that proposition 11 implies $\widetilde{c}_\delta \in [0, \overline{c}_\delta)$ for all small enough δ. A seller who offers and succeeds in trading does not realize $S_\delta(v)$ as her revenue. She realizes something more because the bid she accepts is at least as great as $S_\delta(v)$. Therefore, for each δ sufficiently small, a function

[19] A type 1 buyer always trades immediately because $B(1) = \bar{v} = \bar{c} = S(\bar{c}-)$.

$\phi_\delta(\cdot) : [0, \bar{c}_\delta] \to [\underline{c}_\delta, \bar{c}_\delta]$ exists that maps, conditional on consummating a trade, the seller's offer into her expected revenue from the sale. Thus $\phi_\delta[S_\delta(c)]$ is a type c seller's expected revenue given that she offers $S_\delta(c)$. Take note that $\phi_\delta[S_\delta(c)] \in [S_\delta(c), \bar{c}_\delta]$ because the expected revenue can not be less than the seller's offer $S_\delta(c)$. Revealed preference implies that

$$\begin{aligned} \pi_S(S_\delta(\tilde{c}_\delta), \tilde{c}_\delta) &\geq \pi_S(S_\delta(0), \tilde{c}_\delta) \\ [\phi_\delta[S_\delta(\tilde{c}_\delta)] - \tilde{c}_\delta]\ P_{S\delta}[S_\delta(\tilde{c}_\delta)] &\geq [\phi_\delta[S_\delta(0)] - \tilde{c}_\delta]\ P_{S\delta}[S_\delta(0)]. \end{aligned} \tag{49}$$

Solving,

$$\begin{aligned} P_{S\delta}[S_\delta(\tilde{c}_\delta)] &\geq \frac{\phi_\delta[S_\delta(0)] - \tilde{c}_\delta}{\phi_\delta[S_\delta(\tilde{c}_\delta)] - \tilde{c}_\delta}\ P_{S\delta}[S_\delta(0)] \qquad (50) \\ &\geq \frac{\underline{c}_\delta - \tilde{c}_\delta}{\bar{c}_\delta - \tilde{c}_\delta}\ P_{S\delta}[S_\delta(0)] \\ &= \frac{\sqrt{\bar{c}_\delta - \underline{c}_\delta}}{\bar{c}_\delta - \underline{c}_\delta + \sqrt{\bar{c}_\delta - \underline{c}_\delta}}\ P_{S\delta}[S_\delta(0)] \end{aligned}$$

where the second line follows from the fact that $\phi_\delta[S_\delta(0)] \geq S_\delta(0) = \underline{c}_\delta$ and the third line follows by substitution of the definition for $\tilde{c}_\delta$. So in particular, $P_{S\delta}[S_\delta(0)] = 1$ and, by proposition 11, $\lim_{\delta\to 0}(\bar{c}_\delta - \underline{c}_\delta) = 0$ together imply

$$\lim_{\delta\to 0} P_{S\delta}[S_\delta(\tilde{c}_\delta)] = 1. \tag{51}$$

As in the proof of claim 16, the mass balance equation (46) must hold:

$$\int_{\underline{c}_\delta}^{1} a g_B(x) P_{B\delta}[B_\delta(x)]\, dx = \int_0^{\bar{c}_\delta} g_S(x) P_{S\delta}[S_\delta(x)]\, dx. \tag{52}$$

Since

$$\int_0^{\bar{c}_\delta} g_S(x) P_{S\delta}[S_\delta(x)]\, dx \geq P_{S\delta}[S_\delta(\tilde{c}_\delta)]\, G_S(\tilde{c}_\delta)$$

and

$$\int_{\underline{c}_\delta}^{1} a g_B(x) P_{B\delta}[B_\delta(x)]\, dx \leq a\overline{G}_B(\underline{c}_\delta),$$

it follows from (52) that

$$a\overline{G}_B(\underline{c}_\delta) \geq P_{S\delta}[S_\delta(\tilde{c}_\delta)]\, G_S(\tilde{c}_\delta)$$

or, since $P_{S\delta}[S_\delta(\tilde{c}_\delta)] \leq 1$,

$$a\overline{G}_B(\underline{c}_\delta) \geq G_S(\tilde{c}_\delta). \tag{53}$$

By taking limits in (53) as $\delta \to 0$ and invoking continuity of G_S and $\overline{G}_B$, we obtain

$$a\overline{G}_B\left(\lim_{\delta\to 0} \underline{c}_\delta\right) \geq G_S\left(\lim_{\delta\to 0} \tilde{c}_\delta\right). \tag{54}$$

Since $\lim_{\delta\to 0} \tilde{c}_\delta = c_*$ and $\lim_{\delta\to 0} \underline{c}_\delta = c_*$ by proposition 11, (54) implies

$$a\overline{G}_B(c_*) \geq G_S(c_*) .$$

This, however, is a contradiction because the maintained assumption $c_* > p_W$ implies $a\overline{G}_B(c_*) < a\overline{G}_B(p_W) = G_S(p_W) < G_S(c_*)$.■

Proof of the main theorem. Claims 16 and 17 together with $\inf_{\delta\to 0}(\bar{c}_\delta - \underline{v}_\delta) = 0$ establishes (43): prices realized in the market converge to the Walrasian price. The proofs of those two claims together show that an arbitrarily small deviation upward in a buyer's equilibrium bid or an arbitrarily small deviation downward in a seller's equilibrium offer can guarantee trade, provided δ is sufficiently small. This, together with the result that realized prices converge to the Walrasian price, establishes (5) and (6): equilibrium expected utility for both buyers and sellers approaches what they would expect if the market were competitive.■

5 Conclusions

In this paper we consider a simple, dynamic matching and bargaining market in which both sellers and buyers have incomplete information and risk being forced to exit at any moment due to exogenous reasons. We show that this market converges to the Walrasian price and competitive allocation as the model's friction—the length of the matching period—goes to zero. This convergence is driven by the interaction of two forces within the model: local market size and global market clearing. The significance of our result is to show that in the presence of private information a fully decentralized market such as the one we model can deliver the same economic performance as a centralized market such as the k-double auction that Rustichini, Satterthwaite, and Williams (1994) studied.

This is an important extension of the full information dynamic matching and bargaining models because it shows that a decentralized market in which matching frictions are small can elicit sufficiently well private values and costs so as to allocate almost perfectly the market supply to the traders who most highly value that supply. Compared to Satterthwaite and Shneyerov (2004) this paper demonstrates that convergence to the Walrasian price does not depend on the entering number of buyers exactly matching the entering number of sellers each period. Equilibrium with small frictions appears to be robustly almost efficient.

The limitations of our model and results immediately raise further questions. Two stand out for future investigation. First, existence of admissible sequences of equilibria needs to be established. Second, our model assumes independent private values and costs. We would like to know if our results generalize to both correlated costs/values and to interdependent values with a common component and affiliated private signals. Showing this would be particularly interesting if the stochastic process generating traders' cost and values resulted in a time varying Walrasian price. Convergence to that price as the period length approached zero would establish that fully decentralized dynamic matching and bargaining markets can effectively follow—and reveal—an unknown and changing competitive price. Since economists appear to believe this to be true, it would be nice to have a theory showing how this can be.

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