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Dynamical approach to weakly dissipative granular collisions

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Granular systems present surprisingly complicated dynamics. In particular, nonlinear interactions and energy dissipation play important roles in these dynamics. Usually (but admittedly not always), constant coefficients of restitution are introduced phenomenologically to account for energy dissipation when grains collide. The collisions are assumed to be instantaneous and to conserve momentum. Here, we introduce the dissipation through a viscous (velocity-dependent) term in the equations of motion for two colliding grains. Using a first-order approximation, we solve the equations of motion in the low viscosity regime. This approach allows us to calculate the collision time, the final velocity of each grain, and a coefficient of restitution that depends on the relative velocity of the grains. We compare our analytic results with those obtained by numerical integration of the equations of motion and with exact ones obtained by other methods for some geometries.

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I. INTRODUCTION

The characterization of granular matter is extremely broad and includes essentially any conglomeration of discrete macroscopic particles. These can be as small as grains of sand and as large as asteroids, and they can be in a condensed or gas-like phase. The condensed phases may exhibit characteristics of solids or fluids or gases or various combinations thereof. Granular matter is important in more industrial applications than can be listed here and exhibits a huge variety of interesting behaviors that have provided food for thought for centuries. Behaviors such as the so-called jamming transition and the formation of patterns are frequent subjects of current research, as is the propagation of energy in granular materials.

A feature common to granular matter is the fact that energy is lost every time grains collide. Indeed, the grains have to be very hard and difficult to compress for a collision not to lead to a loss of energy. Yet it is usually the case that momentum is conserved in these inelastic collisions. The conservative limit, where only elastic collisions are involved, is famously illustrated by Newton's cradle, consisting of a row of very hard balls that just touch, each hanging on a string of the same length attached to a common support. When the ball at one end is picked up and released so that it collides with the next ball, the energy passes down the row until the last ball flies up to the same height as the first ball before it was released (the other balls remain at rest). The last ball then flies back, the energy is transferred across the row again, and the first ball flies up to the same height [1–3]. This continues, although not forever because of course some small amount of energy must be lost at each collision event.

The prototypical phenomenological description of energy loss involves the *coefficient of restitution ε* in the equation that describes a collision between two grains,

$$v_{f2} - v_{f1} = \varepsilon(v_{i1} - v_{i2}). \qquad (1)$$

Here the v's represent the velocities, the subscripts i and f stand for initial (before collision) and final (after collision), and

the numbers label particles 1 and 2. This description leads to an energy loss at each collision of $1 - \varepsilon^2$ of the kinetic energy of the center of mass before the collision. For a successfully built Newton's cradle, ε is exceedingly small.

The coefficient of restitution is most often treated as a parameter independent of the velocities. And yet, it is broadly recognized that this cannot be totally correct because it leads to problematic features in the asymptotic behavior such as the so-called inelastic collapse in a granular gas because there may be an infinite number of collisions in a finite time [4,5]. Indeed, when one considers realistic interaction models, it is in fact universally the case that interactions of any two compressible grains are nonlinear. For instance, the interactions between two spherical objects obey Hertz's law, where the repulsion is proportional to the compression to the power 3/2 rather than the more familiar Hook's law where the repulsion is simply proportional to the compression. The consequence of this nonlinearity is that the duration of a collision depends on the initial velocities of the particles before the collision. Therefore, when a dissipative collision occurs, the mechanism responsible for the dissipation of energy acts for different lengths of time depending on the initial velocities, leading to distinct energy losses, and consequently to a velocity-dependent coefficient of restitution. It is interesting to note that Hertz's law is frequently used together with a velocity-independent coefficient of restitution.

The history of the analysis of the effect of velocity-dependent frictional forces on the coefficient of restitution is long and varied. Here we only summarize some of its salient points. Hertz's law assumes that there are no attractive surface forces in lightly loaded spherical granules. Perhaps the earliest work to recognize that such forces lead to a finite contact area between surfaces under zero load (adhesion), and that this in turn modifies the external force required to separate two bodies of given surface energy and geometry, is that of Johnson *et al.* (commonly known as JKR theory) [6]. This in turn modifies the velocity dependence of the coefficient of restitution, as analyzed in detail by Brilliantov *et al.* [7]. While this correction

may be small, even negligible, between granules of high elastic modulus such as metals or glass, it is considerable in soft granules such as rubber and agrees well with experiments in these cases. Much of the literature on this topic is based on spherical granules, but other shapes are also discussed in this context. For instance, the work of Walton and Braun on frictional disks is noteworthy [8], as is the work of Herbold and Nesterenko [9]. In Ref. [10] the coefficient of restitution for spherical granules obeying Hertz's law for the elastic portion of the interactions plus a velocity-dependent frictional force was obtained exactly as an infinite series. The series cannot be summed analytically and converges very slowly, and truncation of the series leads to unphysical divergences. A compact Padé approximation to the series makes it possible to perform much more efficient event-driven molecular dynamics simulations, as well as direct Monte Carlo methods, than was possible with earlier methods, and provides excellent results when compared to those obtained from numerical integrations of the equations of motion [11]. Determination of the velocity dependence of the restitution coefficient has recently become increasingly detailed and addresses a more varied range of particle sizes and compositions. For instance, experimental determination [12] and molecular dynamics calculations [13] of the velocity dependence of the coefficient of restitution of argon nanoparticles have shown that these nanoparticles are hard and highly elastic at collision velocities smaller than the size-dependent yield velocity, while they progressively soften as the collision velocity increases beyond the yield velocity.

In this contribution, our goal is not to arrive at a more efficient approach for numerical simulations but rather to present a perturbative but very broadly applicable methodology to *analytically* analyze the consequences of a general viscoelastic model of dissipation on the outcome of a collision in the low dissipation regime. Our study provides an alternative approach to the problem. We arrive at approximate but accurate analytic expressions for the initial velocity dependence not only for the coefficient of restitution but also for the duration of a collision and for the velocities of the granules at the end of the collision for large ranges of parameter values. Our results have interesting implications for the understanding of a full range of realistic collisions, which should be useful for the study of granular gases and for pulse propagation in granular chains. Examples of granular systems with low intergranular friction include sand, glass beads, and steel beads.

We accomplish three objectives, all in the limit of low but nonzero dissipative forces. The first, presented in Sec. II, is to discuss the equations of motion that describe the collision of two not necessarily spherical grains. The scenario is this: the two grains are initially just touching. Grain 1 has velocity v_1 and grain 2 has velocity v_2. How these grains came to have these velocities is not important. If our two grains are elements in a granular gas, they may, for example, have arrived at this point due to previous collisions with other grains. Or, one might be preparing an experiment with just two grains, where one grain is given a kick of some kind at time $t = 0$ to cause it to start moving with velocity v_1 while the other is initially at rest ($v_2 = 0$). The equations of motion then determine the further evolution of the two granules. In particular, we are able to calculate the relative velocities of the granules

at the end of a collision as a function of the initial relative velocities.

Our second objective, detailed in Sec. III, is to use this result to calculate the duration of a collision as a function of the initial velocities. This makes explicit our assertion that collisions are in general not instantaneous and that their duration in fact depends on the initial velocities of the colliding granules. As pointed out above, this in turn leads to a velocity-dependent coefficient of restitution.

Our third objective is to calculate the coefficient of restitution and the velocities of the colliding granules at the end of the collision. This is presented in Sec. IV. In the absence of dissipative forces the coefficient of restitution is equal to unity. We are able to explicitly calculate the lowest-order corrections to this and thus to obtain an explicit form for the dependence of the coefficient of restitution on the initial velocity difference of the granules.

Finally, Sec. V contains a summary of the paper and comments on the possible generalizations of this study.

II. THE MODEL

Viscoelastic forces for the collision of two spheres include two terms. The first is due to the elastic repulsion between the two particles and has its origins in Hertz's theory [14,15]. The second term stands for the viscous dissipation via a dashpot [16,17]. Hence, the contact force can be written as

$$F = -r(x_1 - x_2)^{n-1} - \gamma(x_1 - x_2)^\alpha(\dot{x}_1 - \dot{x}_2). \qquad (2)$$

Here x is the displacement of a particle from its initial position at the beginning of a collision. A dot denotes a derivative with respect to time, and the subscripts on x label the two particles. The coefficient r is a constant dependent on Young's modulus and Poisson's ratio. For instance, for colliding spheres of radii R_1 and R_2, this constant is given by

$$r = \frac{2}{5D_{12}}\sqrt{\frac{2R_1 R_2}{R_1 + R_2}}, \qquad (3)$$

where

$$D_{12} = \frac{3}{4}\left(\frac{1 - \sigma_1^2}{E_1} + \frac{1 - \sigma_2^2}{E_2}\right), \qquad (4)$$

σ_i is Poisson's ratio, and E_i is the Young's modulus of sphere i. Returning to Eq. (2), γ is the coefficient of viscosity, α is a constant that defines the specific viscoelastic model, and n depends on the topology of the contact between the particles. For spheres, n is equal to 5/2 [14,15] and α to 1/2 [18,19], but we leave them as n and α for the sake of generality. We say that particle 1 is to the left of particle 2, and that our system of coordinates increases from left to right. When we work with two equal spheres so that $R_1 = R_2$, $\sigma_1 = \sigma_2$, and $E_1 = E_2$, we drop the subscript on D, that is, we set $D_{11} \equiv D$.

The equations of motion for two particles of mass m_1 and m_2 during a collision are

$$m_1\ddot{x}_1 = -r(x_1 - x_2)^{n-1} - \gamma(x_1 - x_2)^\alpha(\dot{x}_1 - \dot{x}_2),$$
$$m_2\ddot{x}_2 = r(x_1 - x_2)^{n-1} + \gamma(x_1 - x_2)^\alpha(\dot{x}_1 - \dot{x}_2). \qquad (5)$$

From Eq. (5), conservation of momentum immediately follows,

$$m_1 \ddot{x}_1 + m_2 \ddot{x}_2 = 0, \tag{6}$$

so that $m_1 \dot{x}_1 + m_2 \dot{x}_2 = \text{const}$. Equation (5) also leads to an uncoupled equation for the difference variable $z = x_1 - x_2$,

$$\ddot{z} = -\frac{r}{\mu} z^{n-1} - \frac{\gamma}{\mu} z^\alpha \dot{z}, \tag{7}$$

where μ is the reduced mass $\mu^{-1} = m_1^{-1} + m_2^{-1}$. For the latter equation, the initial conditions are $z(0) = 0$ because we deal with configurations where the granules are initially just touching each other, and $\dot{z}(0) = v_{i1} - v_{i2} \equiv v_0$. Here v_{i1} and v_{i2} are the initial velocities of the two colliding granules.

An analytic solution $z(t)$ of Eq. (7) for arbitrary n and α seems not to be available. However, we have been able to obtain the velocities at the end of the collision as a function of the initial velocities in the low-viscosity limit. We rewrite Eq. (7) as a first-order differential equation of the velocity as a function of the position. Defining $v = \dot{z}$, and noting that

$$\ddot{z} = \frac{dv}{dt} = \frac{dv}{dz}\frac{dz}{dt} = v\frac{dv}{dz}, \tag{8}$$

we rewrite Eq. (7) as

$$v\frac{dv}{dz} = -\frac{r}{\mu} z^{n-1} - \frac{\gamma}{\mu} z^\alpha v. \tag{9}$$

In the absence of dissipation ($\gamma = 0$), this equation admits two solutions for $v(z)$,

$$v_\pm(z) = \pm\sqrt{v_0^2 - \frac{2rz^n}{\mu n}}. \tag{10}$$

Obviously, the positive sign should be considered during compression, and the negative sign during decompression.

Our approximation in the low-dissipation regime starts by writing the velocity as a perturbation on the nondissipative solution. Consequently, during compression we have

$$v(z) = v_+(z) + \gamma v_{\text{comp}}(z), \tag{11}$$

where $v_{\text{comp}}(z)$ is a function to be determined. Substituting the trial solution Eq. (11) in Eq. (9), and collecting the terms of order γ, we have

$$\frac{r}{\mu} z^{n-1} v_{\text{comp}} + \left(\frac{2rz^n}{\mu n} - v_0^2\right)\left(\frac{z^\alpha}{\mu} + v'_{\text{comp}}\right) = 0, \tag{12}$$

where a prime denotes a derivative with respect to z. Furthermore, the condition $v_{\text{comp}}(0) = 0$ is necessary to satisfy the initial conditions. The solution of Eq. (12) with the initial conditions $v_{\text{comp}}(0) = 0$, when added to $v_+(z)$, then gives us the compression velocity to first order in the dissipation,

$$v(z) = \sqrt{v_0^2 - \frac{2rz^n}{\mu n}} - \gamma \frac{z^{1+\alpha}\left[2(1+\alpha) + n\,_2F_1\left(1, \frac{1}{2} + \frac{1+\alpha}{n}; 1 + \frac{1+\alpha}{n}; \frac{2rz^n}{n\mu v_0^2}\right)\right]}{(1+\alpha)(2+n+2\alpha)\mu}, \tag{13}$$

where $_2F_1$ is a hypergeometric function [20].

Next we move on to the decompression, which starts with $v = 0$ and $z = z_{\max}$. To calculate the maximum compression $z_{\max}$, we again look for a first-order correction in γ,

$$z_{\max} = \left(\frac{n\mu v_0^2}{2r}\right)^{1/n}(1 - \gamma z_c), \tag{14}$$

where z_c is a constant to be determined and $(n\mu v_0^2/2r)^{1/n}$ is the maximum compression in the absence of dissipation, obtained by setting $v_\pm(z) = 0$ in Eq. (10). The relative velocity must vanish for the maximum compression. This is the point at which the colliding grains stop moving and begin to separate. Therefore, we evaluate Eq. (13) at $z = z_{\max}$ as given in Eq. (14), expand to first order in γ, and set the left-hand side equal to zero. Solving the resulting equation for z_c and inserting the result in Eq. (14), we find for the maximum compression

$$z_{\max} = \left(\frac{n\mu v_0^2}{2r}\right)^{\frac{1}{n}}\left[1 - \gamma \frac{2^{-\frac{1+n+\alpha}{n}}\sqrt{\pi}\left(\frac{n\mu}{r}\right)^{\frac{1+\alpha}{n}} v_0^{\frac{2-n+2\alpha}{n}}\Gamma\left(\frac{1+\alpha}{n}\right)}{n^2\mu\,\Gamma\left(\frac{3}{2} + \frac{1+\alpha}{n}\right)}\right]. \tag{15}$$

While it might be tempting to assume that the collision ends when $z = 0$, that is, when the particles lose contact, this would neglect the fact that the particles have not recovered their original shape at this point [10]. The effect of neglecting this contribution leads to serious spurious attractive forces. The collision actually ends when the force is zero, $\ddot{z} = 0$, at which point the initial shape is recovered. Setting $\ddot{z} = 0$, $z = z_f$, and $\dot{z} = v_f$ in Eq. (7), where the subscript f stands for *final*, we have

$$z_f = \left(\frac{-\gamma v_f}{r}\right)^{\frac{1}{n-1-\alpha}}. \tag{16}$$

In what follows we restrict n to values $n > \alpha + 1$, so that z_f is small for low viscosity.

We write the perturbative solution for the decompression velocity in the low-viscosity case as

$$v(z) = v_-(z) + \gamma v_{\text{decomp}}(z). \tag{17}$$

Again substituting into Eq. (9) and expanding the latter to first order in γ, we find

$$v_{\text{decomp}}(z) = \frac{C}{\sqrt{n\mu v_0^2 - 2rz^n}} - \frac{4rz^{1+n+\alpha}}{(2+n+2\alpha)\mu(2rz^n - n\mu v_0^2)} - \frac{z^{1+\alpha}\left[2 + 2\alpha + n\,_2F_1\left(1,\frac{1}{2}+\frac{1+\alpha}{n};1+\frac{1+\alpha}{n};\frac{2rz^n}{n\mu v_0^2}\right)\right]}{(1+\alpha)(2+n+2\alpha)\mu}. \quad (18)$$

Here C is a constant to be determined by the continuity of the solution Eqs. (18) and (13) at $z_{\max}$. Remembering that $v(z_{\max}) = 0$, where $z_{\max}$ is given by Eq. (15), and expanding up to first order in γ, C is found to be

$$C = \frac{2^{-\frac{\alpha+1}{n}}\sqrt{\pi}(n\mu)^{\left(-\frac{1}{2}+\frac{\alpha+1}{n}\right)} r^{-\frac{\alpha+1}{n}} v_0^{\frac{n+2\alpha+2}{n}} \Gamma\left(\frac{\alpha+1}{n}\right)}{\Gamma\left(\frac{\alpha+1}{n}+\frac{3}{2}\right)}. \quad (19)$$

To use this result to calculate the leading contribution to the final relative velocity we write

$$v_f = v(z_f) = v(0) + v'(0)z_f, \quad (20)$$

since z_f is small. A prime denotes a derivative with respect to the argument z. Since the force is zero at the end of the collision, we have

$$\ddot{z} = 0 = d\dot{z}/dt = d\dot{z}/dz\, dz/dt = v'(z_f)v(z_f)$$
$$= [v(0) + v'(0)z_f][v'(0) + v''(0)z_f]$$
$$= v(0)v'(0) + \{v(0)v''(0) + [v'(0)]^2\} + \mathcal{O}(z_f^2). \quad (21)$$

This equation has two solutions for $v'(0)$: either $v' = -v(0)/z_f$ or $v' = 0$. The first leads to a final relative velocity that vanishes, which is not physical in our perturbative approach where we expect the final relative velocity to go to $-v_0$ when $\gamma \to 0$. We must thus choose $v'(0) = 0$ [or, more rigorously,

$v'(0) = \mathcal{O}(z_f^2)$]. Therefore,

$$v_f = v(0) + \mathcal{O}\left(z_f^3\right) = v(0) + \mathcal{O}[\gamma^{3/(n-1-\alpha)}]. \quad (22)$$

We choose values of n and α such that $0 < n - 1 - \alpha < 3$. For spheres with $n = 5/2$ and $\alpha = 1/2$ this condition is clearly satisfied. We later also consider $n = 5/2$, $\alpha = 0$ (used by others), for which the condition is also satisfied. The correction in the final velocity can then be neglected in our analysis to first order in γ. The first term, $v(0)$, is obtained by substituting Eq. (19) into Eq. (18), using this result together with Eq. (17), and taking the limit $z \to 0$. We then obtain

$$v_f = -v_0 + \gamma \frac{\left(\frac{n\mu v_0^2}{2r}\right)^{\frac{1+\alpha}{n}} \sqrt{\pi}\,\Gamma\left(1+\frac{1+\alpha}{n}\right)}{\mu\Gamma\left(\frac{3}{2}+\frac{1+\alpha}{n}\right)}. \quad (23)$$

III. COLLISION TIME

Our next objective is to calculate the collision time, which we do in two parts. First we calculate the compression time of the collision as the two grains compress one another and then, as the grains move apart, the attendant decompression time. The collision time is then the sum of the two.

First we calculate the compression time as

$$T_{\text{compression}} = \int_0^{z_{\max}} \frac{1}{v(z)}dz, \quad (24)$$

where, to order γ, $\frac{1}{v(z)}$ is obtained from Eq. (13) as

$$\frac{1}{v(z)} = \frac{1}{\sqrt{v_0^2 - \frac{2rz^n}{n\mu}}} + \gamma \frac{z^{1+\alpha}\left[2(1+\alpha) + n\,_2F_1\left(1,\frac{1}{2}+\frac{1+\alpha}{n};1+\frac{1+\alpha}{n};\frac{2rz^n}{n\mu v_0^2}\right)\right]}{(1+\alpha)(2+n+2\alpha)\mu\left(v_0^2 - \frac{2rz^n}{n\mu}\right)}. \quad (25)$$

Substituting this into Eq. (24) and integrating leads to a contribution of order γ^0 and a cancellation of two terms of order $\gamma^{1/2}$. Consequently, the compression time does not show any γ dependence up to first order:

$$T_{\text{compression}} = \frac{2^{-\frac{1}{n}} n^{-1+\frac{1}{n}} \sqrt{\pi} r^{-\frac{1}{n}} \mu^{\frac{1}{n}} v_0^{-1+\frac{2}{n}} \Gamma\left(\frac{1}{n}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{n}\right)} + \mathcal{O}(\gamma^{3/2}). \quad (26)$$

Next we move on to decompression, which starts with $v = 0$ and $z = z_{\max}$ (maximum compression) and ends when $\ddot{z} = 0$ (force is zero), at which point $z = z_f$ as given in Eq. (16). Hence, the decompression time is

$$T_{\text{decompression}} = \int_{z_{\max}}^{z_f} \frac{dz}{v(z)} = \int_{z_{\max}}^0 \frac{dz}{v(z)} + \int_0^{z_f} \frac{dz}{v(z)}. \quad (27)$$

We first deal with the first integral. Since the *zero*th-order term of the velocity during decompression, Eq. (17), is the

negative of the compression velocity in Eq. (13), and the limits of integration of the compression and the first term in the decompression times are switched, the latter time is the same as the compression time up to terms of order $\gamma^{3/2}$.

This still leaves the second term in the decompression time given in Eq. (27). Since $z_{\max}$ is small,

$$\int_0^{z_f} \frac{dz}{v(z)} \simeq \frac{z_f - 0}{v(z_f)} \simeq -\frac{z_f}{v_0} = -\frac{1}{v_0}\left(\frac{\gamma v_0}{r}\right)^{\frac{1}{n-1-\alpha}}. \quad (28)$$

For $n - 1 - \alpha > 0$, this correction is indeed small and our perturbative approach is valid. In particular, for spherical grains $n = 5/2$, and the particular values of $\alpha = 1/2$ (commonly used in the literature) and $\alpha = 0$ (used by others, see below), the value of the exponent of γ is 2/3 and 1, respectively. This in turn implies that this term is a small perturbation.

We have thus established that for the parameters used herein, the decompression time is equal to the compression

time up to order $\gamma^{2/3}$ or order γ depending on the value of α. Hence the total collision time to this order is twice the compression time plus an additional contribution to the decompression time,

$$T = \frac{2^{1-\frac{1}{n}} n^{-1+\frac{1}{n}} \sqrt{\pi} r^{-\frac{1}{n}} \mu^{\frac{1}{n}} v_0^{-1+\frac{2}{n}} \Gamma\left(\frac{1}{n}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{n}\right)} - \frac{1}{v_0}\left(\frac{\gamma v_0}{r}\right)^{\frac{1}{n-1-\alpha}},$$

(29)

and we have arrived at our second objective. We have not found any other calculation of the velocity-dependence of the collision time in the literature.

In a recent paper, the merits and problems of different choices of the parameter α, and even a generalization of the above model, were discussed [17]. Here, for $n = 5/2$ we consider two choices of this parameter that have been commonly used in the literature to test Eq. (2). The simplest case, $\alpha = 0$, was proposed in Ref. [21] and further developed in Ref. [22], but in our subsequent results below we see that this choice leads to problematic outcomes. The case $\alpha = 1/2$, proposed independently in Refs. [18,23], is more widespread in the granular gas community. Yet another combination of exponents was found experimentally for chains of o-rings in Ref. [9] and analyzed theoretically in Ref. [24]. In this model dissipation was not included, but the elastic force is a double power-law rather than a single one, with exponents 5/2 and 7. In the latter work it was shown that depending on the characteristics of the o-rings and the experimental setup, either the one or the other contribution can be dominant. In Fig. 1 we show the duration of the collision for two equal spherical grains ($n = 5/2$) as a function of the relative velocity at the beginning of the collision for both values of α. As can be seen in the figure, for small γ the data is very well

FIG. 2. Duration of collisions for several values of R_2 (from bottom to top, R_2 varies from 0.1 to 0.9 in steps of 0.1) obtained via numerical integration of the equations of motion. The lines represent the theoretical prediction Eq. (29). In this figure the parameters are as follows: $m_1 = 1$, $R_1 = 1$, $\alpha = 1/2$, $\gamma = 0.001$, and $2/(5D_{12}) = 1$. The densities of the two colliding spheres are equal.

predicted by our approximation, independently of the exponent α (the approximation only starts to deviate from the theoretical prediction for $\gamma = 0.1$, represented by the plus signs). This α-independence of the duration of the collision is one of the striking predictions of our theory. Another interesting characteristic of our solution is the power-law dependence of T on the initial relative velocity v_0, as evidenced in the inset of Fig. 1.

In Fig. 2 we again show T as a function of v_0, but this time for spheres of different sizes. Fixing the radius of granule 1 and varying the radius of granule 2 (assuming that they have the same density), we can see that the collision takes longer for larger values of R_2.
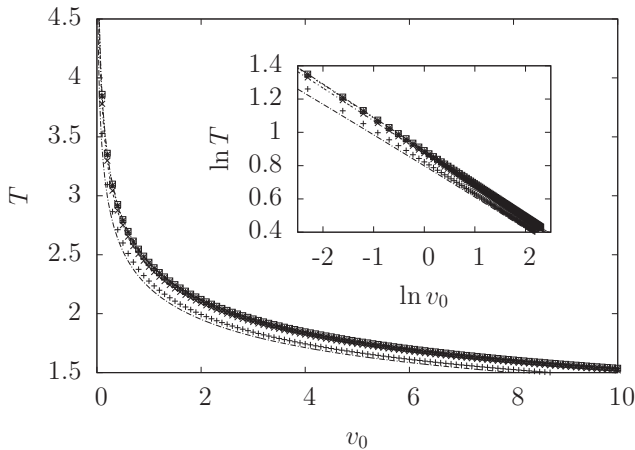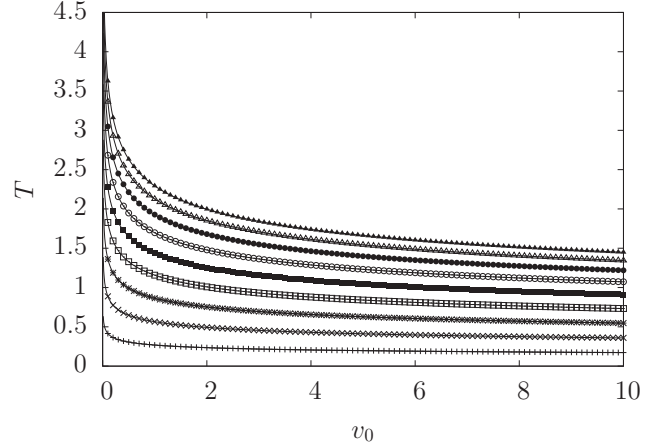
FIG. 1. Collision duration for several values of γ (from top to bottom, 0.0001, 0.001, 0.01, and 0.1) and two different values of α (0 for lower group and 1/2 for higher group), obtained via numerical integration of the equations of motion (symbols) and from our theory (line). The line represents the theoretical prediction of Eq. (29). The inset shows the same data on a log-log scale. In this figure, the parameters are as follows: $m_1 = m_2 = 1$, $R_1 = R_2 = 1$, and $2/(5D) = 1$. As predicted, for these values of γ and α the collision durations are essentially independent of these parameters.

IV. FINAL VELOCITIES AND COEFFICIENT OF RESTITUTION

We next turn our attention to the final velocities. From the conservation of momentum, we know that the total momentum at the beginning and end of the collision must be the same,

$$m_1 v_{i1} + m_2 v_{i2} = m_1 v_{f1} + m_2 v_{f2},$$

(30)

where the subscripts i and f once again label the initial and final velocities of the grains. On the other hand, we also know from the definition of $z(t)$ that

$$\dot{z}(T) = v_f = v_{f1} - v_{f2}.$$

(31)

Solving the set of the two equations given above for v_{f1} and v_{f2}, and using Eq. (17), we find

$$v_{f1} = \frac{m_1 - m_2}{m_1 + m_2} v_1 + \frac{2m_2}{m_1 + m_2} v_2$$
$$+ \frac{2^{-\frac{\alpha+1}{n}} m_2}{m_1 + m_2} \frac{\sqrt{\pi} \Gamma\left(\frac{\alpha+1}{n}\right)\left(\frac{r}{\mu n}\right)^{1-\frac{\alpha+1}{n}}}{r \Gamma\left(\frac{\alpha+1}{n}+\frac{3}{2}\right)} (v_{i1} - v_{i2})^{\frac{2(\alpha+1)}{n}} \gamma,$$
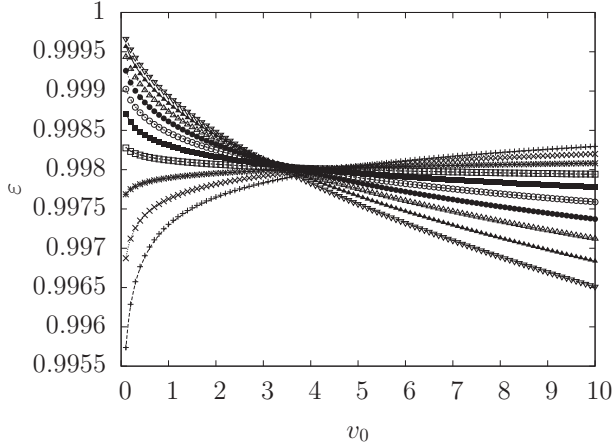
FIG. 3. Coefficient of restitution as a function of the initial relative velocity. Each curve corresponds to a different value of α (from bottom to top on the left side, α varies from 0 to 0.9 in steps of 0.1). The other parameters are: $\gamma = 0.001, m_1 = m_2 = 1, R_1 = R_2 = 1$, and $2/(5D) = 1$. The lines correspond to the theoretical prediction of Eq. (33).

$$v_{f2} = -\frac{m_1 - m_2}{m_1 + m_2}v_2 + \frac{2m_1}{m_1 + m_2}v_1$$
$$-\frac{2^{-\frac{\alpha+1}{n}}m_1}{m_1 + m_2}\frac{\sqrt{\pi}\,\Gamma\left(\frac{\alpha+1}{n}\right)\left(\frac{r}{\mu n}\right)^{1-\frac{\alpha+1}{n}}}{r\,\Gamma\left(\frac{\alpha+1}{n} + \frac{3}{2}\right)}(v_{i1} - v_{i2})^{\frac{2(\alpha+1)}{n}}\gamma. \quad (32)$$

As expected, the result for an elastic collision is recovered when $\gamma = 0$. Further, the influence of the dissipation is greater on the lighter particle, and the influence of the initial condition on the change in the final velocities due to dissipation depends only on the relative velocity.

We conclude this section by using the above results in Eq. (1) to calculate the coefficient of restitution and thus completing our third and principal objective:

$$\varepsilon = 1 - \gamma\frac{2^{-\frac{\alpha+1}{n}}\sqrt{\pi}\,\Gamma\left(\frac{\alpha+1}{n}\right)\left(\frac{r}{\mu n}\right)^{1-\frac{\alpha+1}{n}}}{r\,\Gamma\left(\frac{\alpha+1}{n} + \frac{3}{2}\right)}(v_{i1} - v_{i2})^{\frac{-n+2\alpha+2}{n}}. \quad (33)$$

In Fig. 3 we show the coefficient of restitution for several values of α. The agreement is equally good for all of them. An important characteristic of ε is that its qualitative dependence on the initial relative velocity is drastically different for α larger than or smaller than $(n-2)/2$ (in the case of spheres, this value is $1/4$): α smaller than this value leads to the unphysical situation of negative coefficients of restitution for very small relative velocities. For larger α, the collision approaches the elastic case for small relative velocities. Perhaps most importantly, the power $(-n + 2\alpha + 2)/n$ of the initial relative velocities of the two granules is equal to $1/5$ with the physically justified values $n = 5/2$ and $\alpha = 1/2$ for spheres; see, e.g., Eq. (12) in Ref. [25]. In addition to recovering this exponent, as in Eq. (14) of that reference, we also obtain the linear dependence of the correction of the coefficient of restitution on the coefficient of viscosity γ.
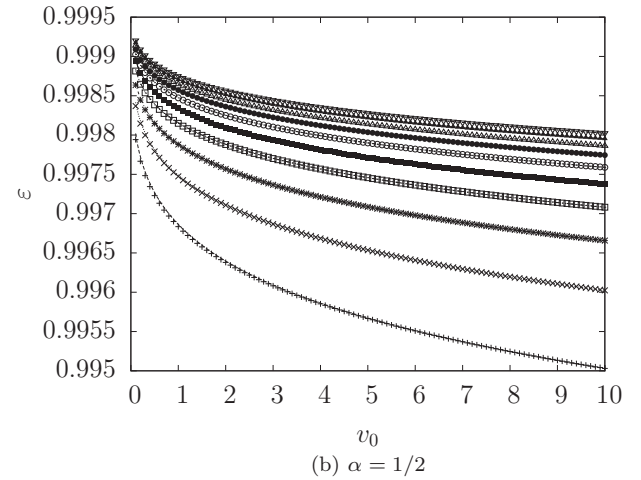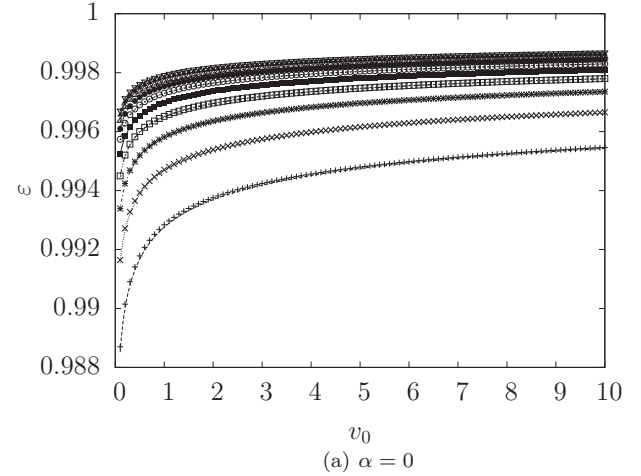

(a) $\alpha = 0$

(b) $\alpha = 1/2$

FIG. 4. Coefficient of restitution as a function of the initial relative velocity. Each curve corresponds to a different value of the radius of granule 2 (from bottom to top, R_2 varies from 0.5 to 1.4 in steps of 0.1). The other parameters are: $\gamma = 0.001$, $m_1 = 1$, $R_1 = 1$, and $2/(5D_{12}) = 1$. The densities of the two colliding spheres are equal. The lines correspond to the theoretical prediction of Eq. (33).

In Fig. 4, we show the coefficient of restitution as a function of v_0 for particles of different sizes (same density) for $\alpha = 0$ and $\alpha = 1/2$. It is evident from the figure that ε increases with the radius in both cases. However, the qualitative behavior is independent of the sizes of the grains.

V. CONCLUSIONS

We have succeeded in calculating quantities that characterize the collision of two granules that lead to the loss of energy (but not momentum) to the environment via viscous dissipation. We started with an equation of motion (Newton's Law) containing a kinetic energy contribution, a force due to the elastic repulsion between the two granules, and a dashpot viscous dissipation term. In addition to parameters related to the shape and size of the granules, the model contains two important parameters: a coefficient of viscosity γ, and a constant α that defines the specific viscoelastic model, cf. Eq (2). Our calculations are perturbative in the coefficient of viscosity; that is, we present lowest-order corrections to elastic (energy-conserving) collisions.

A collision begins with the two granules just touching head-on toward each other with a relative velocity v_0. This velocity and configuration define the collision strength. The collision begins at this initial moment with compression of the granules until their relative velocity is zero (at which point the compression is a maximum). Decompression then follows, until the force between the granules vanishes, at which point the collision ends.

Integration of the equations of motion leads to analytic results for several important quantities usually specified simply as phenomenological parameters. The first is the relative velocity of the granules during compression and during decompression. We calculate the final relative velocity as a function of the separation of the centers of the granules and find the dependence on initial relative velocity and on the parameters γ and α, cf. Eq. (23); if the collision were elastic, the final and initial relative velocities would of course just be the negatives of one another. Additional useful results are the final velocities of each grain, for which we obtain explicit expressions as a function of the parameters and of the initial velocities of each grain, cf. Eq. (32). These are important for simulations of granular gases.

The third quantity we calculate is the duration of the collision; cf. Eq. (29). In most phenomenologies, collisions are assumed to be instantaneous. Collisions are of course not instantaneous.

Finally, we find an analytic expression for the coefficient of restitution ε defined in Eq. (1). This coefficient, usually chosen phenomenologically, recognizes the inelasticity of granular collisions. We have found the dependence of this coefficient on particle shape (via the exponent in the force that determines the topology of the contact between the granules), its dependence on Young's modulus and Poisson's ratio, and most importantly, on the initial relative velocity and on the parameters γ (to lowest order) and α that define the viscoelastic model; cf. Eq. (33). If $\gamma = 0$ the coefficient of restitution is unity, that is, there is no energy loss in the collision. Similarly, if the initial velocities of the two granules are equal, the coefficient is trivially unity again. The dependencies on these quantities are nontrivial and, we submit, essentially impossible to arrive at phenomenologically. This then provides a physical basis for the usual phenomenological choice $\varepsilon < 1$. While our results are perturbative in γ and thus not as general for elastic spheres ($n = 5/2$, $\alpha = 1/2$) as is the infinite series developed in Ref. [10], our model allows different values of n and α and yields relatively simple explicit results for the initial velocity dependence of the final velocities of the colliding granules, of the duration of a collision, and of the coefficient of restitution.

In this paper we have only dealt with two colliding granules, taking into account the energy loss due to an explicit viscoelastic force in the equations of motion. This renders our results immediately applicable to granular gases where at low densities binary collisions are the most common interactions. The generalization to a granular chain or to even higher dimensional granular arrays is not trivial, cf. Ref. [25], but is now made considerably easier by the fact that we have explicitly found the principal ingredients of the problem. There is nevertheless a great deal of work to be done, especially toward higher-dimensional generalizations.

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