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# A perspective on quantum integrability in many-body-localized and Yang–Baxter systems

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Two of the most active areas in quantum many-particle dynamics involve systems with an unusually large number of conservation laws. Many-body-localized systems generalize ideas of Anderson localization by disorder to interacting systems. While localization still exists with interactions and inhibits thermalization, the interactions between conserved quantities lead to some dramatic differences from the Anderson case. Quantum integrable models such as the XXZ spin chain or Bose gas with delta-function interactions also have infinite sets of conservation laws, again leading to modifications of conventional thermalization. A practical way to treat the hydrodynamic evolution from local equilibrium to global equilibrium in such models is discussed. This paper expands upon a presentation at a discussion meeting of the Royal Society on 7 February 2017. The work described was carried out with a number of collaborators, including Jens Bardarson, Vir Bulchandani, Roni Ilan, Christoph Karrasch, Siddharth Parameswaran, Frank Pollmann and Romain Vasseur.

This article is part of the themed issue ‘Breakdown of ergodicity in quantum systems: from solids to synthetic matter’.

## 1. Introduction

One of the most fundamental distinctions in classical physics is between systems that are ‘integrable’ and those

that are ‘chaotic’. The simplest example of an integrable system is a set of decoupled harmonic oscillators: each oscillator evolves at its own frequency, and there is no meaningful relaxation or dissipation in the system, even if it is prepared or measured with a small amount of uncertainty. The energy of each oscillator is separately conserved. Most systems of many interacting particles are chaotic and behave in a different fashion: although the *total* energy is conserved, and the underlying physics is reversible, we observe an arrow of time and an increase of entropy yielding ultimately a thermalized state.

‘Thermalization’ for the purposes of this article means that measurements over length scales that are large compared with a microscopic cut-off, but small compared with the system size, become equivalent to those in the Gibbs ensemble with density matrix

$$\rho = \frac{e^{-\beta(H-\mu N)}}{Z}. \quad (1.1)$$

Note the privileged role of the energy and number operators, which are conserved quantities. For a system with additional conserved quantities, one might expect that this standard Gibbs ensemble will not be applicable and that at best a generalized Gibbs ensemble or ‘GGE’, with more conserved quantities and corresponding Lagrange multipliers generalizing  $\beta$  and  $\mu$ , will describe the system. Even the GGE concept is of debatable relevance to the long-time behaviour of the non-thermalizing systems to be discussed here.

Much effort in recent years has gone into understanding failures of thermalization in quantum systems. Two of the most studied failures result from the existence of an extensive number of local, independent conserved quantities, which can be taken as a working definition of integrability. The goal of this article is to summarize, at a level that is not very technical or rigorous, the consequences of two different kinds of integrability. The first kind appears in systems that exhibit many-body localization (MBL) [1–3]. A definition of a many-body-localized system of fermions is that it has an infinite number of conserved quantities that are local and adiabatically connected,<sup>1</sup> as the strength of interactions is reduced to zero, to the occupancies (i.e. the number operators  $n_i$  with eigenvalues 0 or 1) of orbitals in an Anderson-localized system,

$$H = \sum_i n_i \epsilon_i = \sum_i \epsilon_i c_i^\dagger c_i. \quad (1.2)$$

Anderson localization for our purposes means that the states created by the  $c_i^\dagger$  operators are spatially localized (fall off exponentially at spatial infinity).

A specific way to picture the failure of thermalization in the Anderson case is that particles move only a distance of the order of localization length, which is the typical size of the localized orbitals. With interactions, the form of the Hamiltonian becomes more complicated, but if there remain as many independent conserved quantities as before, one would expect that these would interfere with thermalization just as in the localized system. The first part of this perspective is concerned with understanding which physical phenomena are strongly modified in this process of introducing interactions. In other words, we would like to understand how MBL is different from Anderson localization, given that both have similar conservation laws. In many cases, one expects that, once the interactions reach some critical strength, the system begins to thermalize; the nature of the transition between MBL and thermalizing states [4], and the possibility of intermediate phases, will not be discussed here.

Even the above statements can be difficult to make precise. For example, there is already some subtlety in the requirement above that for integrability the conserved quantities need to be in some sense ‘local’. This requirement is there partly in order to rule out trivial constructions, such as allowing the projection operators onto individual energy eigenstates to count as conserved quantities. Precisely what qualifies as local depends on the context. Another important class of integrable systems, which are the second main subject of this perspective, does not involve quenched disorder. In these ‘Yang–Baxter’ integrable systems, conserved quantities

<sup>1</sup>One should be careful about adiabaticity: there are resonances across large spatial distances at any non-zero interaction strength, which could lead to conservation laws that are not equivalent to those of the Anderson-localized orbitals.

are translation-invariant sums of local or quasi-local operators, where quasi-local means with exponentially decaying support. For example, on a lattice a conserved quantity looks like

$$Q = \sum_i Q_i, \quad (1.3)$$

where the  $Q_i$  are local or quasi-local combinations of operators, and the effect of changing  $i$  is simply a translation.

Free particle systems without disorder are examples of integrable systems of this type, but a remarkable fact with deep implications across mathematical physics is that certain interactions can be added in one spatial dimension without destroying the infinite list of independent conserved quantities. Many physically important Hamiltonians are in this class: for example, spinless bosons interacting with a  $\delta$ -function interaction, or a spin-half chain with nearest-neighbour Heisenberg interactions. As in the many-body-localized case, the focus here will be on how the existence of an infinite number of conserved quantities modifies important dynamical phenomena in the system.

For the same reasons as before, we would expect the long-time state of the system to be a GGE rather than the normal Gibbs ensemble. In conventional thermalizing systems without disorder, hydrodynamics describes how a system evolves from local equilibrium to global equilibrium; assuming the equivalence of ensembles, this is the flow from a local Gibbs ensemble to a global one. Below we start from a simple far-from-equilibrium situation of two reservoirs initially prepared in different Gibbs ensembles and then joined at the origin. This model problem has been treated using a variety of methods, including a hydrodynamical approach [5,6] that turns out to provide a very general tool for the long-time, long-length-scale dynamics of integrable models [7,8]. The hydrodynamics of integrable models involves an infinite set of conserved quantities and has appeared before in classical models, including the problem of a dense gas of nonlinear Schrödinger solitons [9], which also describes the weak-coupling limit of the quantum Bose gas. These hydrodynamical approaches can be formulated in different ways. The kinetic theory form [9] of a single ‘Bethe–Boltzmann’ equation (for details and additional references, see [10]) is particularly tractable. Properly speaking, the final state is not, in general, a global GGE because there is no dissipation or irreversibility in the equation: densities keep streaming out to infinity.

The outline of this article is as follows. The following two sections briefly review recent works on MBL and integrable models with a specific focus on the existence and consequences of conserved quantities. (It should be noted that the work on conserved quantities we mention is a relatively small part of the recent work in both fields; for broader reviews, see, for example, [11] for MBL, or [12] for GGEs in Yang–Baxter integrable models.) The last section outlines some open questions for current and future work.

## 2. Dynamics in the many-body-localized state

Let us consider a particular model of spinless fermions with nearest-neighbour interactions that shows Anderson localization at zero interaction strength. After mapping the fermions to spin-half degrees of freedom via the Jordan–Wigner transformation, we obtain a disordered XXZ chain

$$H_{\text{XXZ}} = \sum_{i=1}^L \frac{J_{\perp}}{8} (\sigma_i^+ \sigma_{i+1}^- + \sigma_i^- \sigma_{i+1}^+) + \frac{J_z}{4} \sigma_i^z \sigma_{i+1}^z + \frac{h_i}{2} \sigma_i^z. \quad (2.1)$$

Here, the  $\sigma_i$  are Pauli matrices, and the random fields  $h_i$  are sampled independently and uniformly from the interval  $[-W, W]$ .

With  $J_z = 0$ , this Hamiltonian is quadratic in the fermionic representation and can be diagonalized in terms of orbitals as in (1.2). The orbitals are localized if the strength of disorder  $W$  is non-zero, with a localization length that decreases as  $W$  increases. When a small non-zero  $J_z$  is added, it is believed that the model goes into a many-body-localized phase; this belief



**Figure 1.** Schematic phase diagram of the one-dimensional disordered XXZ model (2.1). With no disorder, the model is integrable and transport is ballistic. With disorder but no interactions, the model is Anderson localized. In between the model is believed to have both many-body-localized and thermalizing phases, with a transition indicated by the solid circle. There are possibly other phases than conventional diffusive thermalization, including subdiffusive dynamics, in the region labelled by a question mark [14,15].

is based primarily on numerical evidence and experience with a different model for which MBL has been shown more rigorously [13]. The phase diagram of this Hamiltonian is shown in figure 1.

From the Introduction, we note that MBL can be defined via the presence of independent local conserved quantities that go continuously as  $J_z \rightarrow 0$  into the orbital occupancies of the Anderson localized case. While that is clearly sufficient to prevent ordinary thermalization, it raises the question of whether there are any differences between Anderson localization and MBL. Indeed, early work on MBL tended to focus on quantities such as ordinary linear-response conductivity that are not very different in the two phases. We note that the conserved quantities  $n_i$  commute not just with the single-electron Hamiltonian (1.2) but also with some possible extra terms, e.g.

$$H = \sum_i \epsilon_i n_i + \sum_{ij} U_{ij} n_i n_j + \sum_{ijk} V_{ijk} n_i n_j n_k + \dots \quad (2.2)$$

Here, the  $n_i$  in the many-body-localized phase should not be taken strictly as occupancies of some single-particle orbital, but rather as more general localized conserved quantities (sometimes called ‘ $l$ -bits’). A main conclusion of this section will be that a surprising amount of physics can be understood from the ‘real-space Fermi liquid’ form (2.2), thus called because the interaction terms are similar to those in the Landau–Silin theory of a Fermi liquid, where the quasi-particle occupancies in *momentum space* are good quantum numbers. This physics is readily visible even if the detailed form of the  $l$ -bits is unknown.

We note in passing that there are other possible definitions of the many-body-localized phase, including a useful one based on entanglement entropy [16] showing an area law for excited energy eigenstates rather than the volume law expected for a state satisfying the eigenstate thermalization hypothesis [17,18]. The focus here will be on dynamics starting from states that can in principle be readily prepared in the laboratory; how to prepare a large number of excited eigenstates is a challenging question, although some insight may be provided by recent tensor network approaches to this question [19,20].

As an example of such a preparation protocol, suppose that the system (2.1) is prepared with initial condition being an arbitrary simple product state in the  $S_z$  basis; clearly, this is the ground state of a trivial non-interacting Hamiltonian. Consider first  $J_z = 0$ , so all states are Anderson localized and the system consists of non-interacting fermions. We would expect each fermion to move a distance of the typical localization length from its initial location; for the box distribution mentioned above, the localization lengths have a well-defined statistical distribution. Indeed this is observed: particle density and energy density, once averaged over disorder realization and initial conditions, are found to move a distance of the order of the localization length and then stop. This is consistent with both charge and energy conductivities being zero. Entanglement entropy also behaves consistently with particles moving a finite distance on average and then stopping: the entanglement between two infinite reservoirs grows from zero (because the initial condition was a product state) for a finite period of time and then saturates [21].

When a small  $J_z$  is introduced, particle transport and energy transport behave similarly, again consistent with localization. However, there is a dramatic change in the behaviour of entanglement entropy: after a time that scales inversely with  $J_z$ , entanglement is found to increase *logarithmically* with time, and the slope of the logarithmic increase is nearly independent of the precise value of  $J_z$  [21]. (It had been noted a long time ago that disorder could reduce the entanglement growth in the XXZ model and hence make it more amenable to numerical simulation [22,23].) We now sketch the likely explanation of this behaviour [24–26] as follows: the local conserved quantities interact with each other in the many-body-localized case, unlike in the Anderson case, and this interaction leads to a dephasing-induced growth of entanglement.

To start, consider two localized conserved quantities ( $l$ -bits), each of which takes two possible values. Suppose that the interaction between these two conserved quantities, similar to the second term in (2.2), has an energy scale

$$J_{\text{eff}} = J_z \exp\left(-\frac{|x_1 - x_2|}{\xi_0}\right). \quad (2.3)$$

Here,  $\xi_0$  is the non-interacting localization length: for a nearest-neighbour interaction like  $J_z$ , the effective interaction requires overlap of the localized orbitals corresponding to the conserved quantities (we ignore ambiguities of order unity in the definition of the localization length). We then expect that, starting from a product state and treating these two  $l$ -bits in isolation, approximately one bit of entanglement entropy will be generated on a time scale  $\sim \hbar/J_{\text{eff}}$ . Note that neither  $l$ -bit has moved, i.e. no transport has occurred. We can estimate the distance out to which entanglement has taken place from

$$J_{\text{eff}} = J_z \exp -\frac{L}{\xi_0} = \frac{\hbar}{t} \Rightarrow \left(\frac{L}{\xi_0}\right) = \log t \left(\frac{J_z}{\hbar}\right). \quad (2.4)$$

Now, we make the somewhat drastic assumption that the total entanglement generated can be found by summing over all pairs of  $l$ -bits that have had sufficient time to become entangled.

$$S(t) \propto \xi_0 \log \left(\frac{J_z t}{\hbar}\right). \quad (2.5)$$

This functional form agrees well with the numerical results, in particular the prediction that the slope of the logarithmic growth is dominated by the non-interacting localization length.

A natural question is whether this logarithmic scaling from interactions between conserved quantities shows up in observables that are more experimentally accessible than entanglement entropy. An important idea in classical dynamical systems is of recurrences or revivals, and these are increasingly used to analyse quantum coherent dynamics in atomic systems. In quantum mechanical language, suppose that a system has certain oscillation frequencies in its dynamics. It will return close to its original state if all those frequencies are ‘in sync’, which becomes exponentially rare if a large number of frequencies are involved. The number of frequencies involved in the many-body-localized Hamiltonian (2.2) clearly depends on whether the additional interaction terms are included; as the dynamics are localized, one also has to consider which frequencies appear in the time dependence of a particular local observable.

As an example, consider adding a single probe spin (capital  $S$  variables) at the edge of the XXZ spin chain [27],

$$H = H_{\text{XXZ}}[\{\sigma_i\}] + \frac{\lambda}{4}(S^+ \sigma_1^- + S^- \sigma_1^+). \quad (2.6)$$

The same geometry was previously considered in the context of qubit decoherence [28]. Suppose that the probe spin is initially up and that the XXZ chain is in a random  $\sigma^z$  product state. We will show that the rate at which the probe spin returns to nearly its initial polarized state is a direct probe of the same logarithmic dynamics seen in entanglement.

Let us start with the non-interacting case where the frequencies of the many-body problem are just sums of  $N$  one-particle frequencies. We define a revival as an interval of time satisfying (e.g. [29])

$$\sum_{i=1}^N |1 - \cos(2\pi \omega_i t)| < \delta \quad (2.7)$$

with  $\delta$  a small parameter. Let the disorder-averaged revival rate  $\Gamma_0(t, N)$  be defined as the ratio of the number of such revivals in the time window  $[0, t]$  to the total time  $t$ . This decreases with increasing  $N$ , as the revival criterion becomes more difficult to satisfy, but the *time* dependence of  $\Gamma_0(t, N)$  depends on the statistics of the  $\omega_i$ . With weak interactions turned on, the frequencies experience Hartree-type level shifts as a consequence of the  $U_{ij}$  term in (2.2). As in the picture for entanglement, the modified energy splitting of levels  $\omega_i, \omega_j$  takes the form  $\delta\omega_{ij} \sim J_z e^{-|i-j|/\xi}$ . For times  $t \ll J_z^{-1}$ , the splitting is unimportant and does not significantly modify the revival criterion.

However, for  $t \gtrsim J_z^{-1}$ , the Hartree shift of each nearest-neighbour pair causes an additional frequency to enter the revival criterion. As  $t$  increases further, each pair separated by distance  $x$  leads to an additional frequency entering the revival criterion when  $t \gtrsim e^{x/\xi}/J_z$ , so that, at time  $t$ , the appropriate revival rate is roughly  $\Gamma_0(t, N + \alpha \log J_z t)$ . Indeed numerical data are found to collapse onto this form, and with some further analysis [27] one obtains the following characteristic form for the mean revival rate, in terms of the function  $\nu$  that gives the asymptotic revival rate for  $N$  independent frequencies once the initial condition has been forgotten:

$$\frac{\overline{N} - N_0}{T} \approx \nu(N + \alpha \log J_z t) - \nu(N). \quad (2.8)$$

It is somewhat surprising that, for a broad range of the many-body-localized phase in the disordered XXZ model, this picture based on only the one- and two-body terms in (2.2) seems to suffice; at longer times or close to the phase boundary, one expects the three- and higher-body terms to become important.

We conclude that the logarithmic scaling that results from interactions between conserved quantities in the many-body-localized phase appears in several contexts. However, it should be noted that the long-time behaviour, when the approximation of independent pairs of conserved quantities breaks down, remains difficult to calculate analytically. We avoided this question by looking at the unbounded increase of entanglement of an infinite subregion, but for a finite subregion it is currently not possible to calculate the long-time limit of entanglement and other quantities. For a thermalizing system, a key idea that enables the calculation of the long-time limit of a large finite subsystem is the Gibbs ensemble: we introduce a Lagrange multiplier, namely inverse temperature, that couples to the *extensive* conserved quantity (energy).

In a many-body-localized system, each individual conserved quantity is not extensive in the same way: the eigenvalues remain bounded (e.g. 0 and 1 for number operators, or up and down for spin operators), rather than scaling with system size-like energy. As a result, the equivalent ensemble that appears in dynamical properties is not well understood. It would be nice to have a practical way to use our relatively well-developed understanding of many-body-localized eigenstates to solve practical dynamical questions, and indeed there has been a lot of recent work in this area. There are also active research questions about to what extent this precisely defined MBL in terms of conservation laws survives in higher dimensions or with translational invariance, non-Abelian symmetries or long-ranged interactions.

### 3. Transport and hydrodynamics in quantum integrable models

Now, we turn to Yang–Baxter integrable models, which like MBL have infinitely many conserved quantities. However, the systems are translation invariant and the conserved quantities are sums of local operators rather than being properly local or quasi-local as in MBL. (The importance of the conserved quantities being written as sums of local operators, rather than arbitrarily non-local, is that this form leads to local notions of equilibration and continuity equations.) It is also believed

that Yang–Baxter integrability is generally unstable to small perturbations, while in at least one model [13] the conservation laws are stable in an open set. It is an interesting question whether MBL-like physics, and in particular integrability that is stable to small perturbations, can exist in a model with translation invariance [30–35].

Our main goal will be to understand how initial states that break translation invariance evolve in time, starting from the case of two semi-infinite ‘reservoirs’. A simple example of a Yang–Baxter integrable model is the XXZ model (2.1) with no disorder ( $W = 0$ ) and no interactions ( $J_z = 0$ ). Then the model is diagonalized by plane waves in the fermionic representation,

$$H = \sum_k \epsilon_k c_k^\dagger c_k = \sum_k \epsilon_k n_k. \quad (3.1)$$

(The reader can solve for  $\epsilon_k$  as an exercise.) Now, clearly the  $n_k$  operators commute with each other and with the Hamiltonian, but for the interacting case (and even for questions of equilibration in the free case [36]) a different basis is more useful. It is quite remarkable that infinitely many conserved quantities survive even with non-zero  $J_z$ , and this is only true for certain interactions—a generic two-body interaction, even in one spatial dimension, will not have this property.

Now, suppose that we prepare an initial condition for the infinite XX chain (i.e. XXZ with  $J_z = 0$ ) in which the set of sites  $i \leq 0$  are prepared at chemical potential and temperature  $\mu_L, T_L$ , and the set  $i > 0$  with other values  $\mu_R, T_R$ . (We deliberately refer to chemical potential rather than magnetic field to stress that this is a Lagrange multiplier used to prepare the initial state, not a term in the Hamiltonian used to evolve the system, as the latter would break translation invariance if different on the two sides.) One can calculate the steady-state charge and energy currents that result across the origin in this ‘two-reservoir quench’, after some initial transient, from Landauer-type expressions: for charge current, for example,

$$J = \int_{-\infty}^0 \frac{dk}{2\pi\hbar} f_R(\epsilon_k) v_k - \int_0^{\infty} \frac{dk}{2\pi\hbar} f_L(\epsilon_k) v_k, \quad (3.2)$$

where  $f_{R,L}$  is the Fermi function evaluated in the right and left reservoirs. A similar expression holds for energy current. In fact, the energy current for a free Fermi system at low temperature, assuming that the chemical potential is within a Fermi band, is one example of the result that a general conformal field theory has a steady-state energy current [37,38]

$$J = \frac{\pi c k_B^2}{12\pi\hbar} (T_R^2 - T_L^2). \quad (3.3)$$

For one band of spinless fermions, the central charge  $c = 1$ . This might seem to be a simple consequence of the decoupled propagation of left- and right-moving excitations in a conformal field theory, but conformal invariance also yields an interesting correction term in more general situations [37] related to the Schwarzian derivative of an initial locally thermal distribution.

Now, we turn to effects of interactions. A full review of integrability is beyond the scope of this perspective, but to get a sense of how an infinite number of conserved quantities could exist, it is useful to consider a different integrable model: the continuum Bose gas with delta-function interactions or Lieb–Liniger model [39,40]. In the coordinate Bethe ansatz approach to integrable models, there is a pseudomomentum  $k$  that generalizes ordinary momentum: the allowed wavevectors of particles in a periodic geometry are modified because particles interact with each other through phase shifts. The (Bethe) equations that determine the pseudomomenta are complicated, but in the thermodynamic limit, even at non-zero temperature (via the ‘thermodynamic Bethe ansatz’ [41,42]), there is a simple expression for the conserved quantities in terms of the pseudomomentum distribution  $\rho(k)$ ,

$$Q_n = \int dk \rho(k) k^n. \quad (3.4)$$

The first three of these are familiar:  $Q_0$  is proportional to particle number,  $Q_1$  to total momentum and  $Q_2$  to total energy. Looking ahead, in conventional hydrodynamics, these are the three quantities that are conserved, leading to three hydrodynamical equations as reviewed



below. Clearly, a system with an infinite number of conserved quantities should require an unconventional kind of hydrodynamics as well as a different Gibbs ensemble.

For XXZ, the conserved quantities are considerably more complicated and come in two classes. There is a long-known set with explicit expressions generated by standard methods, and then a new set of ‘quasi-local’ conservation laws found starting from work of Prosen [43]. This second set turns out to underlie spin currents and explain the observation in the gapless phase of non-zero Drude weight [44] using numerical density-matrix renormalization group time evolution, which is quite powerful for XXZ. The complexity of XXZ charges places a premium on methods that do not require explicit expressions for the conserved quantities and their continuity equations.

However, there is one useful and nearly unique property of the XXZ conserved quantities. The local energy current  $j_E$  is itself a conserved density (i.e. the local current density integrates over space to a conserved quantity or ‘charge’). Even though the XXZ model is certainly not Lorentz invariant, this is similar to the property that, in Lorentz-invariant theories, energy flux divided by  $c^2$  is equal to momentum density. With interactions, the problem of two reservoirs is much more complicated than in the free case. The fact that energy current is itself a charge in a continuity equation allows one to derive several exact far-from-equilibrium results for integrals of the energy current distribution in space

$$J_{\text{tot}}(t) = \int_{-\infty}^{\infty} j_E(x, t) dx \quad (3.5)$$

for the two-reservoir quench [45], which also explain some previously noted identities for XXZ [46]. The more general hydrodynamical methods described in a moment can calculate the full spatial distribution but are limited to long-time, long-distance behaviour.

We now return to the general properties of conserved charges in integrable and non-integrable models to try to develop some general principles for time evolution on long length and time scales. First, consider ordinary hydrodynamics in the dilute gas as studied for well over a century. The ‘zeroth-order’ hydrodynamical equations in three dimensions, which neglect dissipative behaviour such as viscosity, are

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{u}) = 0, \quad (3.6)$$

$$\left( \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) \mathbf{u} + \frac{1}{\rho} \nabla P = \frac{\mathbf{F}}{m} \quad (3.7)$$

and 
$$\left( \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) \tau + \frac{2}{3} (\nabla \cdot \mathbf{u}) \tau = 0. \quad (3.8)$$

Here,  $n$  is the density,  $\mathbf{u}$  is the mean velocity,  $\mathbf{F}$  is an external force,  $P$  is the scalar pressure and  $\tau$  is the kinetic energy density converted to a temperature via the ideal gas law. There are three equations because there are three quantities conserved in this system (particle number, momentum and energy). Hydrodynamics describes how local equilibrium, which establishes itself on the scale of a few collision times, evolves into global equilibrium (although strictly there are false equilibria of the zeroth-order hydrodynamics above, and first-order effects such as viscosity are needed for thermalization). For an integrable model, where standard Gibbs ensemble thermal equilibrium is replaced by a GGE, hydrodynamics should describe the flow starting from a local GGE, which it is hoped will appear rapidly from an arbitrary initial state.

One way of deriving the above equations is from the Boltzmann equation for the single-particle distribution function  $f(\mathbf{k}, \mathbf{r}, t)$ . The densities appearing in the hydrodynamical equations are simply integrals over this distribution function; for example,

$$n(\mathbf{r}, t) = \int f(\mathbf{k}, \mathbf{r}, t) d\mathbf{k}. \quad (3.9)$$

Clearly, (3.9) is similar to the expressions for conserved quantities in the Lieb–Liniger model in (3.4). In ordinary hydrodynamics, much information is lost in going from the Boltzmann equation of kinetic theory to hydrodynamics, as one has gone from an arbitrary momentum

distribution at a point in space–time to just three of its moments. In integrable models, there is a significant difference in how hydrodynamical equations are related to kinetic theory. Consider Lieb–Liniger as an example. As a distribution is uniquely determined by all of its moments under the Hamburger moment problem conditions, hydrodynamics for all the conserved charges is, at least possibly, equivalent to a kinetic theory description of the evolution of  $\rho(k, x, t)$ .

What form would we expect such an evolution equation for the pseudomomentum to take? In an integrable model, quasi-particles interact with each other through phase shifts: the initial and final momenta of two particles going through a collision are preserved, but there is a scattering phase shift. (In fact the Yang–Baxter equation, which we have used to name a class of integrable models, is an expression of how scattering phase shifts for three particles are determined by the two-particle scatterings and independent of their sequence.) According to Wigner, an energy-dependent phase shift can be interpreted as a time delay, i.e. particles of one momentum  $k$  are delayed by passing through particles of another momentum  $k'$ .

We thus should expect a Boltzmann-like equation with no collisions that rearrange momentum but with a velocity at each point that depends on the distribution of other particles,

$$\partial_t \rho(k, x, t) + \partial_x (v[\{\rho(k', x, t)\}] \rho(k, x, t)) = 0. \quad (3.10)$$

The only part of this ‘Bethe–Boltzmann’ equation requiring Bethe ansatz methods is the calculation of the self-consistent backflow that determines the velocity functional  $v[\rho]$  (for the case of the XXZ model, this was done in [6]). Hydrodynamic solutions (ignoring previous attempts that did not correctly capture the self-consistency) and derivations of (3.10) or closely related forms have been investigated for the two-reservoir quench for Lieb–Liniger [5] and XXZ [6,10,47,48] and shown to work exactly in the scaling limit. More recently, solutions have been investigated for general initial conditions in these models [7,8], including at  $T = 0$ , where interesting connections to normal hydrodynamics appear [49].

The two-reservoir quench is especially tractable in that there is no length scale in the initial condition, so solving to first order in time already gives the entire ‘scaling limit’, which is the most one could hope for from hydrodynamics. For more general initial conditions, solutions can be constructed numerically and there are also some interesting mathematical features that suggest routes to rigorous solutions [7,8]. The technical details of how equation (3.10) can be solved in practice are somewhat complicated and we refer the reader to the aforementioned papers.

Aside from techniques for solving dynamics, some remarkable properties of the thermodynamic Bethe ansatz only become clear with the recent understanding of hydrodynamics. This is particularly clear for XXZ where explicit representations of the charges are rather complicated. A general form of the continuity equation can be written down using the self-consistent velocity functional [6]. A stringent test that hydrodynamics passes is to compute the charge or spin ‘Drude weight’, the strength of the delta-function in a conductivity  $\sigma = D\delta(\omega)$  that describes ballistic transport in linear response. For XXZ the spin Drude weight is known to arise from the quasi-local conserved quantities [43,44]. It is nevertheless captured exactly by hydrodynamics [10,48]. In the final section of this perspective, we discuss some limitations of the hydrodynamical approach and a few open questions in both Yang–Baxter and many-body-localized integrability.

## 4. Comments and open problems

While in the discussion of Yang–Baxter integrability we have concentrated on the success of hydrodynamical approaches, there remain several limitations. Given an arbitrary initial pure state, it remains an assumption that this will evolve into a GGE to which hydrodynamics can be applied, and there are not in general practical methods to calculate which GGE a particular state evolves into. Related to this is the fact that the Bethe–Boltzmann equation describes ballistic propagation, while spin currents, for example, have a diffusively decaying part as well [50]. At low temperature the ‘coarse-graining’ length and time scales for a GGE to become established and for hydrodynamics to become valid for non-Bethe-state initial conditions are expected to diverge.

The most dramatic agreement so far with microscopic density-matrix renormalization group calculations is obtained for energy flows in XXZ, which are known to be somewhat special because of the energy current conservation mentioned above [45]. It would be very interesting to extend the approach to the gapped regime where transport is diffusive or superdiffusive [51]. As integrable models have applications all over mathematical physics, including in the theory of stochastic differential equations and also in topological phases in  $2 + 1$  dimensions, the hydrodynamical approach might have benefits there.

We close with a few general comments. Arguably the main surprise of MBL from a mathematical point of view is that it provides a type of ‘stable’ integrability, as an infinite set of conserved quantities survives even when one moves away from the Anderson-localized case [13]. The interactions between these conserved quantities lead to a logarithmically slow but very long-lived time evolution that may be visible in experiments. Entanglement was very useful in understanding the origin of this logarithmic time evolution, which provides a simple physical example of how the dynamics of entanglement in a system may be very different from that of more conventional quantities.

In Yang–Baxter-type integrable models, the Bethe ansatz was often cited as being simple for ground states and thermodynamics but much more challenging for dynamical correlation functions. Recent work shows, though, that, in addition of course to ‘quantum quenches’ [52], hydrodynamical evolution from general initial conditions (i.e. not Bethe states) can be obtained without significantly more difficulty than in standard thermodynamic Bethe ansatz calculations [7]. Even in one spatial dimension, which has long been known to be particularly amenable to calculations, it is remarkable how far the ability to understand and even solve quantum dynamics has advanced in the past decade or so.

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