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Rational Catalan Combinatorics

A dissertation submitted in partial satisfaction of the
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in

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by

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2018

The dissertation of Michelle Elizabeth Bodnar is approved, and it is acceptable in quality and form for publication on microfilm and electronically:

Chair

University of California, San Diego

2018

DEDICATION

To Andrew.

EPIGRAPH

Mathematics is like climbing a steep mountain. It is hard work, and often it is doubtful whether one can make it to the top, but very often one is rewarded by breathtaking views.

—Mario Bonk

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M. Bodnar, "Rational Noncrossing Partitions for all Coprime Pairs", *Journal of Combinatorics*, 2018. To appear. arXiv:1701.07198.

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ABSTRACT OF THE DISSERTATION

Rational Catalan Combinatorics

by

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Doctor of Philosophy in Mathematics

University of California San Diego, 2018

Professor Brendon Rhoades, Chair

Given a finite Coxeter group W and a Coxeter element c, the W-noncrossing partitions are given by $[1, c]$, the interval between 1 and c in W under the absolute order. When W is the symmetric group $\mathfrak{S}_a$, the noncrossing partitions turn out to be classical noncrossing partitions of $[a]$ counted by the Catalan numbers. By attaching an additional integral paramenter b which is coprime to a, we define a set $NC(a, b)$ of rational noncrossing partitions, a subset of the ordinary noncrossing partitions of $\{1, \ldots, b-1\}$. We study the poset structure this set inherits from the poset of classical noncrossing partitions, ordered by refinement. We prove that $NC(a, b)$ is closed under a dihedral action and that the rotation action on $NC(a,b)$ exhibits the cyclic sieving phenomenon. We also generalize noncrossing parking functions to the rational setting and provide a character formula for the action of $\mathfrak{S}_a \times \mathbb{Z}_{b-1}$ on $\mathsf{Park}^{NC}(a,b)$. Finally, we give a group-theoretic interpretation in type A for $NC(a,b)$ in terms of compatible sequences.

Chapter 1

Introduction and Background

There is a rich history of interplay between combinatorics and algebra, one often suggesting insights or generalizations in the other. Here we will examine various Catalan objects which can be viewed as combinatorial models for algebraic objects attached to the symmetric group.

1.1 Coxeter Groups

To understand the story of rational Catalan combinatorics in full generality, we must begin by considering some particularly nice groups. We'll review the basic properties of finite Coxeter groups here, but a more complete treatment can be found in [21]. Let V be a real vector space with inner product $\langle \cdot, \cdot \rangle$ and let s_α denote reflection across the hyperplane $H_\alpha = \{x \in V \mid \langle x, \alpha \rangle = 0\}$ orthogonal to α. Explicity, we can write

$$s_\alpha \lambda = \lambda - \frac{2\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle} \alpha$$

for any $\lambda \in V$. If Φ is a finite collection of nonzero vectors in V, then Φ is a *root system* if it satisfies the following properties:

1. $\Phi \cap \mathbb{R}\alpha = \{\alpha, -\alpha\}$ for all $\alpha \in \Phi$

2. $s_\alpha \Phi = \Phi$ for all $\alpha \in \Phi$.

We say Φ is the root system associated to the Coxeter group W generated by all reflections s_α such that $\alpha \in \Phi$. Moreover, every finite reflection group arises from some root system in this way, and the correspondence is unique up to lengths of roots. We say that a root system is *crystallographic* if

$$\frac{2\langle \alpha, \beta \rangle}{\langle \beta, \beta \rangle} \in \mathbb{Z} \ \ \forall \alpha, \beta \in \Phi.$$

For a crystallographic root system, define the *root lattice* to be the set of all integer linear combinations of elements of Φ. Reflection groups W generated by crystallographic root systems are also known as Weyl groups, and most notably they stabilize the root lattice.

For example, consider $\{e_i - e_j \mid 1 \leq i \neq j \leq n\}$, where e_i is the i^{th} standard basis vector in $\mathbb{R}^n$. These form a crystallographic root system spanning the space

$$V = \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n \,\middle|\, \sum_{i=1}^{n} x_i = 0 \right\}.$$

It is the root system of type A_{n-1}. In particular, reflection across the hyperplane orthogonal to $e_i - e_j$ acts on $(x_1, \ldots, x_n)$ by interchanging the i^{th} and j^{th} coordinates, so A_{n-1} is really just the symmetric group $\mathfrak{S}_n$ acting on V by permuting coordinates.

We call $\Delta \subset \Phi$ a *simple system* for Φ if Δ is linearly independent and each root in Φ is an $\mathbb{R}$-linear combination of elements of Δ whose coefficients are all of the same sign. Call the elements of $S = \{s_\alpha \mid \alpha \in \Delta\}$ the *simple reflections* of W. Let $c = s_1 \cdots s_n$ be the product of the simple reflections of some simple system for Φ in any order. We call c a *standard Coxeter element* of W, and its order is referred to as the *Coxeter number* of W. A *Coxeter element* is any conjugate of a standard Coxeter element in W. Let Φ^+ denote the set of positive roots, those which are a positive linear combination of elements of Δ. Similarly, Φ^- is the set of negative roots, those which are a negative linear combination of elements of Δ. Alternatively, if H is any hyperplane which doesn't intersect any root of Φ then H partitions Φ as $\Phi = \Phi^+ \sqcup \Phi^-$ and there is a unique set of roots Δ that is a simple system with respect to this decomposition. All simple systems have the same cardinality, which we refer to as the *rank* of W. Moreover, if Δ is a simple system then W is generated by just those reflections s_α where $\alpha \in \Delta$.

Continuing with our example, $\{e_i - e_j \mid 1 \leq i < j \leq n\}$ forms a positive system for $\mathfrak{S}_n$ and $\{\alpha_1, \alpha_2, \ldots, \alpha_{n-1}\} = \{e_i - e_{i+1} \mid 1 \leq i \leq n-1\}$ forms a simple system for $\mathfrak{S}_n = A_{n-1}$. The fact that $\text{rank}(\mathfrak{S}_n) = n - 1$ motivates the notation A_{n-1}. Let s_i denote reflection across $e_i - e_{i+1}$. We can think of the associated reflections across the hyperplanes orthogonal to these roots as the adjacent transpositions. In particular $s_i(x_1, \ldots, x_n) = (x_1, \ldots, x_{i-1}, x_{i+1}, x_i, x_{i+2}, \ldots, x_n)$. One Coxeter element for $\mathfrak{S}_n$ is $s_1 \cdots s_{n-1}$ which is the long cycle, sending i to $i+1$ for $i \in [n-1]$, and sending n to 1.

Given $w \in W$ with set of simple reflections S, the *length* of w, first studied in [14] and denoted $\ell_S(w)$, is the smallest r such that $w = s_1 \cdots s_r$ where $s_i \in S$. In this case, we say $s_1 \cdots s_r$ is a *reduced S-word* for w. There is a unique longest element, $w_0 \in W$, characterized by $w_0(\Phi^+) = \Phi^-$.

The *inversion number* of a permutation $\sigma \in \mathfrak{S}_n$ is given by

$$\mathrm{inv}(\sigma) = |\{(i,j) \mid i < j \text{ and } \sigma_i > \sigma_j\}|.$$

When $W = \mathfrak{S}_n$ the length of a permutation is its inversion number. The longest element is $n, n-1, \ldots, 2, 1$ in one-line notation.

Let $T = \{s_\alpha \mid \alpha \in \Phi\}$. By definition T is a set of reflections that generates W, but it turns out that T actually contains every reflection of W. For $w \in W$ we define the *absolute length* $\ell_T(w)$ of w to be the smallest r such that $w = t_1 \cdots t_r$ where $t_i \in T$. In this case, we say $t_1 \cdots t_r$ is a *reduced T-word* for w. Define the *absolute order* on W by

$$u \leq_T v \iff \ell_T(v) = \ell_T(u) + \ell_T(u^{-1}v).$$

This gives W the structure of a graded poset, with rank function ℓ_T, which we'll refer to as $\mathrm{Abs}(W)$. It has a unique minimal element, the identity $1 \in W$, but in general may have many maximal elements. See Figure 1.1 for a picture of $\mathrm{Abs}(\mathfrak{S}_2)$. Note that elements of $\mathfrak{S}_2$ are written in cycle notation and fixed points are omitted, so for instance $1 = (1)(2)(3)$ and $(12) = (12)(3)$.

Let W be a finite group and $S = \{s_1, s_2, \ldots, s_n\}$ be a set of generators for W. We call the pair (W, S) a *Coxeter system* if it has presentation

$$W = \langle s_1, \ldots, s_n \mid (s_i s_j)^{m(i,j)} = 1 \rangle$$

where 1 is the identity element in W, $m(i,i) = 1$ for all $1 \leq i \leq n$, and $m(i,j) = m(j,i) \geq 2$. When $m(i,j) = 2$ the generators s_i and s_j commute. It turns out that with

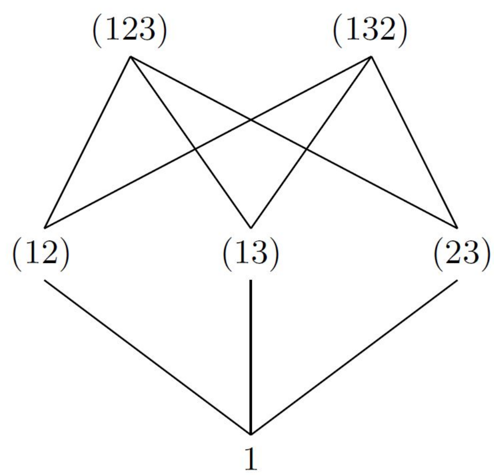


Figure 1.1: Abs($\mathfrak{S}_2$))

this presentation, W is a finite Coxeter group and S is a simple system for W. In the case of the symmetric group with simple reflections $S = \{s_1, \ldots, s_{n-1}\}$ we have

- $s_i^2 = e$ for $i \in [n-1]$
- $(s_i s_{i+1})^3 = e$ for $i \in [n-2]$
- $(s_i s_j)^2 = e$ for $|i - j| > 1$.

Let W be a finite Coxeter group with simple system Δ and set of simple reflections S. For any subset $I \subseteq S$ the *standard parabolic subgroup* of W, denoted W_I, is the subgroup of W generated by all $s_\alpha \in I$. A simple system for W_I is given by $\{\alpha \in \Delta \mid s_\alpha \in I\}$. We can represent W by its *Coxeter graph* Γ, a graph whose vertex set is S and which has an edge (s_i, s_j) if $m(i,j) \geq 3$. If $m(i,j) \geq 4$ then we label the edge by $m(i,j)$. We say W is *irreducible* if its Coxeter graph is connected. More generally, if (W, S) is a Coxeter system and $\Gamma_1, \ldots, \Gamma_r$ are the connected components of Γ with vertex sets $S_1, \ldots, S_r$ respectively, then W is the direct product of the parabolic subgroups $W_{S_1} \times \cdots \times W_{S_r}$ and each Coxeter system (W_{S_i}, S_i) is irreducible. Thus, it is enough to restrict our study to irreducible Coxeter systems. See Figure 1.2 for a complete list of all Coxeter graphs of finite irreducible Coxeter systems.

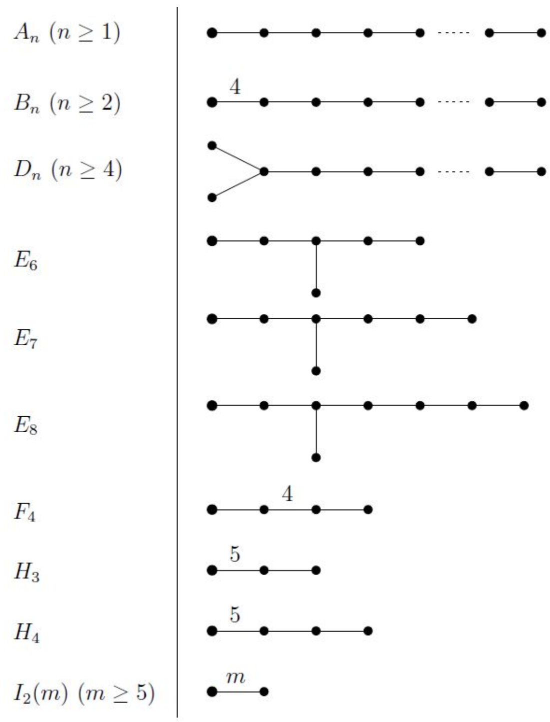


Figure 1.2: Coxeter graphs of the finite irreducible Coxeter systems

Let G be a finite subgroup of $GL_n(V)$ acting on a real n-dimensional vector space V. Let $\mathcal{S}$ denote the algebra of polynomial functions on V, which we can identify with $\mathbb{R}[x_1, \ldots, x_n]$. There is a natural action of G on $\mathcal{S}$ given by $(g.f)(v) = f(g^{-1}v)$ for $f \in \mathcal{S}$, $g \in G$ and $v \in V$. Moreover, this action preserves the natural grading of $\mathcal{S}$ by degree. We say f is G-invariant if $g \cdot f = f$ for all $g \in G$, and denote by $R = \mathcal{S}^G$ the *subalgebra of G-invariants.*

For example, we say that a polynomial $f \in \mathbb{R}[x_1, \ldots, x_n]$ is *symmetric* if it satisfies

$$f(x_1, \ldots, x_n) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$$

for all $\sigma \in \mathfrak{S}_n$. Let Λ^n denote the ring of symmetric polynomials in $x_1, \ldots, x_n$. If we identify elements of $\mathfrak{S}_n$ with permutation matrices and consider $\mathfrak{S}_n$ as permuting a basis of V, then Λ^n is the ring of polynomial invariants.

It had previously been known that S^G contained a set of n homogeneous, algebraically independent elements called *basic invariants* [15]. It was later shown [16] that when G is a finite real reflection group, the basic invariants are in fact a generating set for S^G. In other words, if $\{f_1, \ldots, f_n\}$ is a set of basic invariants then $S^G = \mathbb{R}[f_1, \ldots, f_n]$. Shephard and Todd [33] later showed that reflection groups are in fact all we need to consider. The main result is summarized in the following theorem, which has come to be known as the Chevalley-Shephard-Todd Theorem.

Theorem 1.1.1. *[21, Section 3.5] Let W be a finite subgroup of $GL(V)$. Let R be the subalgebra of $\mathbb{R}[x_1, \ldots, x_n]$ consisting of W-invariant polynomials. Then R is generated as an $\mathbb{R}$-algebra by n homogeneous, algebraically independent elements $f_1, \ldots, f_n$ of positve degree (together with 1) if and only if W is generated by reflections.*

This motivates the study of reflection groups from an algebraic standpoint, beyond their nice geometric properties. The basic invariants are not uniquely determined in general, but their degrees are, which we'll refer to as the *degrees of the basic invariants* of G. The degrees of the basic invariants of $\mathfrak{S}_n$ are $2, 3, \ldots, n$.

1.2 A generalization of classical noncrossing partitions

We are now ready to see how Coxeter groups provide a framework through which we may generalize many classical combinatorial objects. A *noncrossing partition* of $[n] = \{1, 2, \ldots, n\}$, first studied by Germain Kreweras in 1972 [24] is a set partition of $[n] = B_1 \sqcup \cdots \sqcup B_k$ into blocks with no crossings. That is, there do not exist $a < b < c < d$ such that $a, c \in B_i$ and $b, d \in B_j$ with $i \neq j$. The number of noncrossing partitions of $[n]$ are counted by one of the most famous sequences in all of combinatorics: the Catalan numbers, given by the formula

$$\mathsf{Cat}_n = \frac{1}{n+1}\binom{2n}{n}.$$

See [34] for a growing list of hundreds of distinct combinatorial objects each counted by the Catalan numbers.

Let W be a Weyl group and c a Coxeter element in W. Define the *poset of W-noncrossing partitions* [10] by

$$NC(W, c) = [1, c]_T = \{w \in W \mid 1 \leq_T w \leq_T c\}.$$

Since all Coxeter elements in W are conjugate to one another and conjugation by $w \in W$ is an automorphism of $\mathrm{Abs}(W)$, it turns out that $NC(W, c)$ is isomorphic to $NC(W, c')$ for any other Coxeter element c'. For this reason, we will generally omit the choice of c and just refer to $NC(W)$. For the rest of this dissertation, it will be most convenient for us to fix the long cycle, written $(12\cdots n)$ in cycle notation, as our Coxeter element in type A.

To see the connection between $NC(W)$ and noncrossing set partitions, let $W = \mathfrak{S}_n$. A simple bijection takes us from an element $w \in NC(W)$ to a noncrossing partition of $[n]$. Explicitly, write w in cycle notation and put i and j in the same block if and only if they are in the same cycle. For example, Figure 1.3 shows $\mathrm{Abs}(A_2)$. The subposet $NC(A_2)$ is indicated by bold lines. The identity 1 corresponds to the set partition $\{1\}, \{2\}, \{3\}$. The element (12) corresponds to the set partition $\{1, 2\}, \{3\}$. The element (123) corresponds to $\{1, 2, 3\}$. Collectively, we recover the 5 noncrossing partitions of [3], counted by Cat_3.

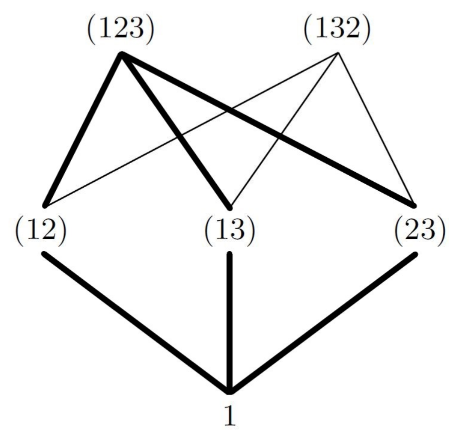


Figure 1.3: $\mathrm{Abs}(A_2)$ with $NC(A_2)$ in bold

More generally, let W be a Coxeter group with degrees $d_1, d_2, \ldots, d_r$ and Coxeter number h. Then we may define the *Coxeter-Catalan number* of W by

$$\mathsf{Cat}(W) := \prod_{i=1}^{r} \frac{h + d_i}{d_i}.$$

Note that $\mathsf{Cat}(\mathfrak{S}_n) = \mathsf{Cat}_n$, and Bessis and Reiner gave a case-by-case proof [10] that $|NC(W)| = \mathsf{Cat}(W)$, so this is indeed a generalization of the usual Catalan numbers and the noncrossing partitions to an algebraic setting. One can ask which other classical objects generalize in this way, and it turns out there are many. Triangulations of a convex $(n+2)$-gon, nonnesting partitions of $[n]$, Dyck paths of size n, and increasing parking functions of length n are all classical objects counted by Cat_n which have different interpretations on the W level.

A different sort of generalization of classical Catalan objects involves attaching an additional integral parameter k in such a way that these new objects are counted by the Fuss-Catalan numbers,

$$\mathsf{Cat}^{(k)}(n) := \frac{1}{kn+n+1}\binom{kn+n+1}{n}.$$

For instance, these count the number of *k-divisible noncrossing partitions*, the noncrossing partitions of $[kn]$ such that each block has size divisible by k. As with classical Catalan numbers, there are many objects counted by Fuss-Catalan numbers, each with their own interesting combinatorial properties. In the case of k-divisible noncrossing partitions, they survive an algebraic generalization. The *Fuss-Coxeter Catalan number* of W is given by

$$\mathsf{Cat}^{(k)}(W) = \frac{1}{|W|}\prod_{i=1}^{r}(kh+d_i),$$

where h is the Coxeter number for W and the d_i's are its degrees. This generalization is examined in great detail by Armstrong in [3], where he defines (W,k)-divisible noncrossing partitions in terms of length k multichains in $\mathrm{Abs}(W)$.

We now turn our attention to a different type of object. Let W be a Weyl group with root lattice Q, degrees $d_1 \leq d_2 \leq \cdots \leq d_\ell$, and Coxeter number $h = d_\ell$. Then W acts on the "finite torus" $Q/(h+1)Q$. Cosets in $Q/(h+1)Q$ give a model for parking functions attached to W [5]. It has been shown by Haiman [20] that the number of orbits of this action is given by the Coxeter-Catalan number $\mathtt{Cat}(W)$. More generally, if p is a positive integer which is coprime to the Coxeter number h, Haiman [20] showed that the number of orbits in the action of W on Q/pQ is

$$\mathtt{Cat}(W,p) = \prod_i \frac{p + d_i - 1}{d_i}.$$

This number has come to be known as the *rational Coxeter-Catalan number* of W at parameter p.

By taking $p = h+1$ and $mh+1$ respectively, we recover the Coxeter Catalan numbers and Fuss-Coxeter Catalan numbers. However, the combinatorics behind all p coprime to h is much less well understood. In fact, it wasn't until 2013 that Armstrong et. al [4, 6] undertook a systematic study of type A rational Catalan combinatorics.

For coprime positive integers a and b, the *rational Catalan number* is

$$\mathtt{Cat}(\mathfrak{S}_a, b) = \frac{1}{a+b}\binom{a+b}{a,b} = \mathtt{Cat}(a,b).$$

Observe that $\mathtt{Cat}(n, n+1) = \mathtt{Cat}(n)$, so that rational Catalan numbers are indeed a generalization of the classical Catalan numbers. The program of rational Catalan combinatorics seeks to generalize Catalan objects such as Dyck paths, the associahedron, noncrossing perfect matchings, and noncrossing partitions (each counted by the classical Catalan numbers) to the rational setting. For instance, $\mathtt{Cat}(a,b)$ counts the number of

a, b-*Dyck paths*, NE-lattice paths from the origin to (b, a) staying above the line $y = \frac{a}{b}x$ [1].

For coprime parameters $a < b$, Armstrong et. al [6] defined the a, b-*noncrossing partitions*, $NC(a, b)$, to be a subset of the collection of noncrossing partitions of $[b-1]$ arising from a laser construction involving rational Dyck paths. We will give a characterization of these rational noncrossing partitions, show that $NC(a, b)$ is closed under dihedral symmetries, and prove that the action of rotation on $NC(a, b)$ exhibits a cyclic sieving phenomenon. Additionally, a model for a, b-noncrossing parking functions will be given which carries an $\mathfrak{S}_a \times \mathbb{Z}_{b-1}$.

It is of intrinsic combinatorial interest to know whether such results hold in the case where $a > b$. Moreover, this seems reasonable as Haiman's formula holds for any coprime pair a, b. Furthermore, we are motivated by the favorable representation theoretic properties of the rational Cherednik algebra attached to the symmetric group $\mathfrak{S}_a$ at parameter b/a. Such properties persist even when $a > b$. It is thus desirable to remove the condition $a < b$ and define rational noncrossing partitions for all coprime pairs (a, b). We will do just that, providing the first type A combinatorial model for rational Catalan objects defined for all coprime a and b.

The rest of this dissertation is organized as follows: **Chapter 2** explains the construction and basic properties of $NC(a, b)$, and examines the poset of rational noncrossing partitions. In **Chapter 3** we prove closure of $NC(a, b)$ under rotation and reflection operations, and discuss the construction of rank sequences that provide a generalization of cardinality to blocks. **Chapter 4** introduces the cyclic sieving phenomenon and we prove various cyclic sieving results using d-modified rank sequences. In **Chapter 5**, we generalize a, b-noncrossing parking functions, $\mathsf{Park}^{NC}(a, b)$, to all coprime a and b and prove a character formula for the action of $\mathfrak{S}_a \times \mathbb{Z}_{b-1}$ on $\mathsf{Park}^{NC}(a, b)$. Finally in **Chapter 6** we consider a construction of rational Catalan objects in type A that is rooted in the symmetric group rather than rational Dyck paths, and offer conjectures and future directions for research.

This chapter contains material from "Rational Noncrossing Partitions for all Coprime Pairs", to appear in *Journal of Combinatorics*, 2018. The dissertation author was the primary investigator and author of this paper.

Chapter 2

Construction and Properties of $NC(a, b)$

2.1 Rational Dyck Paths

Let a and b be coprime positive integers. An a, b-*Dyck path* D is a lattice path in $\mathbb{Z}^2$ consisting of unit length north and east steps which starts at $(0, 0)$, ends at (b, a), and stays above the line $y = \frac{a}{b}x$. By coprimality, the path will never touch this line. For example, the 7,4-Dyck path $NNNENENENNENE$ is shown in Figure 2.1. The a, b-Dyck paths are counted by the *rational Catalan number* $\mathsf{Cat}(a, b) = \frac{1}{a+b}\binom{a+b}{a}$ [1]. A *vertical run* of D is a maximal contiguous sequence of north steps. The 7,4-Dyck path shown in Figure 2.1 has 4 vertical runs of lengths 3, 1, 2, and 1 respectively. Note: it is possible for a vertical run to have length 0. A *valley* of D is a lattice point p on D such that p is immediately preceded by an east step and succeeded by a north step. Figure 2.1

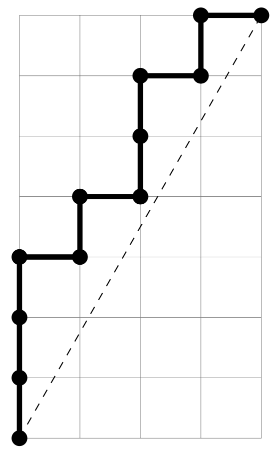


Figure 2.1: 7,4-Dyck Path NNNENENNENE with the dashed line $y = \frac{7}{4}x$

has three valley points. When $(a, b) = (n, n + 1)$, rational Dyck paths are in bijective correspondence with classical Dyck paths, NE-lattice paths from $(0, 0)$ to (n, n) which stay weakly above the line $y = x$, and are counted by the classical Catalan numbers.

2.2 Noncrossing Partitions

A set partition π of $[n] := \{1, 2, \ldots, n\}$ is *noncrossing* if its blocks do not cross when drawn on a disk whose boundary is labeled clockwise with the number 1, 2, . . ., n. Equivalently, π is noncrossing if there do not exist $a < b < c < d$ such that a and c are in the same block B, and b and d are in the same block $B' \neq B$. Let $NC(n)$ denote the set of noncrossing partitions of $[n]$. Such partitions are counted by the classical *Catalan numbers*

$$\mathtt{Cat}(n) = \frac{1}{n+1}\binom{2n}{n} = \frac{1}{2n+1}\binom{2n+1}{n} = |NC(n)|.$$

The *rotation* operator `rot` acts on the set $NC(n)$ by the permutation

$$\begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ 2 & 3 & \cdots & n & 1 \end{pmatrix}.$$

A *labeled noncrossing partition* is a noncrossing partition with a nonnegative integer called a *label* attached to each block. When we apply `rot` to a labeled noncrossing partition the elements of each block shift as in the unlabeled case, and blocks maintain their labels throughout the rotation.

2.3 Rational Pairs of Noncrossing Partitions

A simple bijection maps classical Dyck paths to noncrossing partitions. For $a < b$, Armstrong, Rhoades, and Williams [6] define a, b-*noncrossing partitions* to be the images of the a, b-Dyck paths under this map. We'll now define a more general version of this map, π, that makes sense for any a, b-Dyck path and use this map to define rational a, b-noncrossing partitions for any coprime a and b. Let D be an a, b-Dyck path and label the east ends of the nonterminal east steps of D from left to right with the numbers $1, 2, \ldots, b-1$. Let p be the label of a lattice point at the bottom of a vertical run (other than the first one) of D. The *laser* $\ell(p)$ is the line segment of slope $\frac{a}{b}$ which fires northeast from p and stops the next time it intersects D. By coprimality, $\ell(p)$ terminates on the interior of an east step of D. For instance, consider the 10,7-Dyck path shown on the left in Figure 2.2. We have that $\ell(3)$ hits D on the interior of the east step whose west endpoint is labeled 5. We define the *laser set* $\ell(D)$ to be the set of pairs (i, j) such that D contains a laser starting at label i and which terminates on an east step with west x-coordinate j. For the Dyck path in Figure 2.2 we have

$$\ell(D) = \{(1,1),(2,6),(3,5),(4,5),(6,6)\}.$$

Define a pair of labeled noncrossing partitions $\pi(D) = (P, Q)$ as follows: fire lasers from all labeled points which are also at the bottom of a north step. We define the partition P by the *visibility relation*

$$i \underset{P}{\sim} j \text{ if and only if the labels } i \text{ and } j \text{ are not separated by laser fire.}$$

We make the convention that the label i lies slightly below $\ell(i)$. Label each block of P by the length of the vertical run immediately preceding the minimal element of the block. We will refer to this label as the *rank* of the block. Call a vertical run a *P-rise* if it has length greater than $\frac{a}{b}$. We will now describe the creation of the blocks of Q, a genuinely new feature of this map.

We call a vertical run a *Q-rise* if it has length which is less than $\frac{a}{b}$, including zero. In the special case where $a < b$, there can only be Q-rises of length zero since $a/b < 1$. In Figure 2.2, the vertical runs with x-coordinates 0, 2, 3, and 4 are P-rises and with x-coordinates 1, 5, and 6 are Q-rises. We define the partition Q by the relation

$$i \underset{Q}{\sim} j$$

if and only if one of the following holds:

1. $\ell(i)$ and $\ell(j)$ hit the same east step immediately following a Q-rise

2. $(i, j) \in \ell(D)$

3. $(j, i) \in \ell(D)$.

We label the blocks of Q as follows: If B is a block of Q and $i \in B$, then we label B with the number of north steps beneath the west endpoint of the east step hit by $\ell(i)$. If i doesn't fire a laser, then we assign B rank 0. This is well-defined because different elements of a block of Q always touch or fire a laser which hits the same east step. As with P, we will call this block labeling the rank of the block. There will often be blocks of rank 0, which we will call the *trivial blocks* of Q. We will refer to blocks of Q whose ranks are positive as *nontrivial blocks*. Let $\pi(D)$ denote the labeled pair (P, Q) associated to D under this construction.

Figure 2.2 shows a 10,7-Dyck path with labels and lasers drawn in. The pair (P, Q) which results, also shown in Figure 2.2, is as follows:

$$P = \{\{1, 2\}, \{3, 6\}, \{4\}, \{5\}\}$$

where each block has rank 2.

$$Q = \{\{1\}, \{2, 6\}, \{3, 4, 5\}\}$$

with block ranks 1, 1, and 0 respectively. In particular, the block $\{3, 4, 5\}$ is a trivial block of Q. The ranks are written in smaller font near the lines indicating the block structure. We will often omit the trivial blocks of Q and simply write $Q = \{\{1\}, \{2, 6\}\}$, each with rank 1. As we will see later, the block structure of Q is uniquely determined

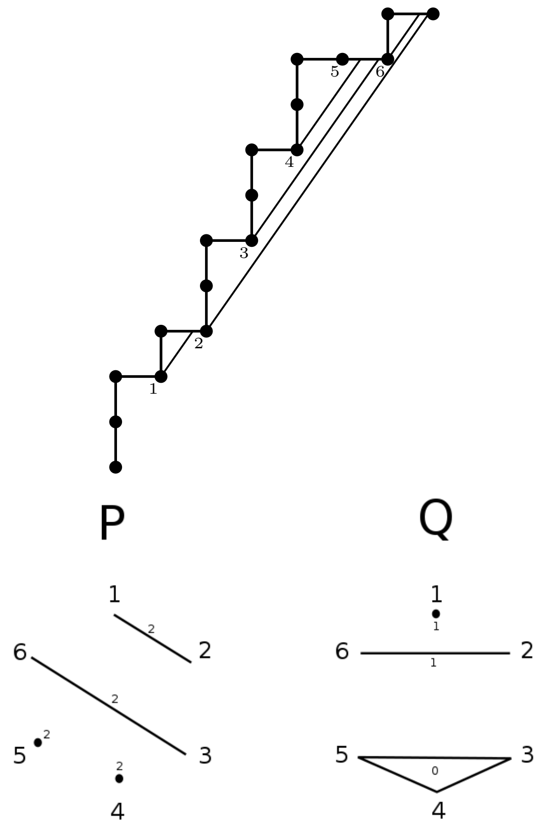


Figure 2.2: A 10,7-Dyck path with corresponding pair of labeled noncrossing partitions

by the block structure of P, so no information is lost in not recording trivial blocks of Q.

Each north step contributes to the rank of either a P block or a Q block, but not both. In particular, the length of a P-rise is the rank of a block of P, and the length of a Q-rise is the rank of a block of Q. This implies that the sum of the ranks of the P and Q blocks is a. Note that elements in the same block of Q are necessarily in different blocks of P, since elements in the same block of Q are always separated by at least one laser.

When $a < b$, Q contains only blocks of rank 0 and P is the rational noncrossing partition associated to D as described by the map in [13]. The ranks of blocks are uniquely determined in this case by the structure of P, which is why labeling blocks by rank had not previously been considered. When $a > b$, the ranks of blocks are no longer uniquely determined by the structure of P and Q. For instance, the 5,3-Dyck paths $NNNENNEE$ and $NNENNNEE$ both give rise to $P = \{\{1\}, \{2\}\}$ and only trivial Q blocks. Thus, the rank labels are a necessary feature of the construction of $\pi(D)$. Since ranks tell us precise vertical run lengths, the map π is injective. We are now ready to prove some useful properties of a, b-noncrossing partitions.

Proposition 2.3.1. *Let $(P, Q) = \pi(D)$ for an a, b Dyck path D. There cannot exist $1 \leq i < b - 1$ such that i is the maximal element of a block of Q and $i + 1$ is the minimal element of a block of P.*

Proof. If i is the maximal element of a block of Q then the lattice point labeled i in D is at the bottom of a Q-rise, whose the length is less than a/b. On the other, hand if $i + 1$ is also the minimal element of a block of P then the lattice point labeled i is at the bottom of a P-rise, whose length must be greater than a/b, a contradiction. □

At this point it will be useful to introduce the *Kreweras complement* of a noncrossing partition. Let P be a noncrossing partition of $[n]$. The Kreweras complement, denoted $\texttt{krew}(P)$, is computed as follows: Begin by drawing the $2n$ labels $1, 1', 2, 2', \ldots, n, n'$ clockwise on the boundary of a disk. Next, draw the blocks of P on the unprimed vertices. Then $\texttt{krew}(P)$ is the unique coarsest partition of the primed vertices which introduces no crossings. An example is shown in Figure 2.3. The map $\texttt{krew} : NC(n) \to NC(n)$ satisfies $\texttt{krew}^2 = \texttt{rot}$, so $\texttt{krew}$ is a bijection.

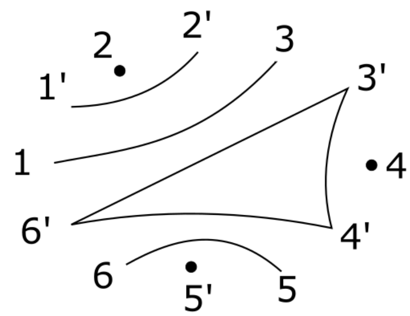


Figure 2.3: The partition $P = \{\{1,3\},\{2\},\{4\},\{5,6\}\}$ is drawn on $\{1,2,\ldots,6\}$ and $\texttt{krew}(P) = \{\{1,2\},\{3,4,6\},\{5\}\}$ is drawn on the primed vertices

By [13, Lemma 3.2] we can recover the laser set of an a, b noncrossing partition, where $a < b$, from its Kreweras complement. We have a similar result when we generalize to any coprime a and b:

Lemma 2.3.2. *Let a and b be coprime and $(P, Q) = \pi(D)$ have corresponding Dyck path D. If $\texttt{krew}(P)$ is the Kreweras complement of P then the laser set $\ell(D)$ is given by*

$$\ell(D) = \{(i, \max(B)) \mid B \in \texttt{krew}(P), i \in B, i \neq \max(B)\}$$

$$\cup \{(\max(B), \max(B)) \mid B \in Q, \operatorname{rank}(B) \neq 0\}$$

Proof. The first set consists of all lasers which determine blocks of P. The second set contains those additional lasers, unique to the the case $a > b$, which define nontrivial blocks of Q but not P, which are always of the form (p, p) where $p = \max(B)$ for some nontrivial block $B \in Q$. □

Lemma 2.3.3. *If $(P, Q) = \pi(D)$ for an a, b Dyck path D then, when viewed as unlabeled partitions, we have $Q = \texttt{krew}(P)$.*

Proof. First suppose that i and j are in the same block B of $\texttt{krew}(P)$ where $i \neq j$. If neither i nor j is equal to $\max(B)$ then by Lemma 2.3.2 we must have that $(i, \max(B))$ and $(j, \max(B))$ are both lasers in D. Since $\ell(i)$ and $\ell(j)$ hit the same east step, i and j are in the same block of Q. Now suppose $j = \max(B)$. Then (i, j) is a laser in D. Similarly, if $i = \max(B)$ then $(j, i) \in \ell(D)$. In all cases, i and j are in the same block of Q.

Conversely, suppose i and j are in the same block of Q. Let B_i denote the block in $\texttt{krew}(P)$ containing i and B_j denote the block in $\texttt{krew}(P)$ containing j. If $\ell(i)$ and $\ell(j)$ hit the same step immediately following a Q rise above label k then (i, k) and (j, k) are both lasers in $\ell(D)$ with $i \neq j \neq k$. By the characterization of the laser set given in Lemma 2.3.2, we must have that $k = \max(B_i)$ and $k = \max(B_j)$, so $B_i = B_j$. If $(i, j) \in \ell(D)$ then $j = \max(B_i)$. If $(j, i) \in \ell(D)$ then $i = \max(B_j)$. In all cases, i and j are in the same block of $\texttt{krew}(P)$. Thus, $Q = \texttt{krew}(P)$. □

Proposition 2.3.4. *Given a Dyck path D, if $\pi(D) = (P, Q)$ then Q is a noncrossing partition.*

Proof. By the definition of Kreweras complement, $Q = \texttt{krew}(P)$ is noncrossing. □

We say that two noncrossing partitions P_1 and P_2 of $\{1, 2, \ldots, n\}$ are *mutually noncrossing* if there do not exist $a < b < c < d$ such that a and c are in the same block of P_i and b and d are in the same block of P_j for $i, j \in \{1, 2\}$ and $i \neq j$. Equivalently, draw the numbers 1 through n on the boundary of a disk. Then P_1 and P_2 are mutually noncrossing if when we draw the boundary of the convex hulls of the blocks of P_1 with solid lines and the convex hulls of the blocks of P_2 in dashed lines, no solid line crosses the interior of a dashed line. Note that solid-dashed intersections at vertices are permissible. For example, the picture on the left of Figure 2.4 contains two noncrossing partitions, one whose blocks are indicated by solid lines, the other whose blocks are indicated by dashed lines. We see that there are intersections only at labels. On the other hand, the picture on the right in Figure 2.4 shows that if we superimpose a rotated version of the dashed line partition onto the solid line partition, then the partitions are no longer mutually noncrossing. In particular, the $\{1,4\}$ block of dashed line partition crosses both the $\{2,6\}$ and $\{3,5\}$ blocks of the solid line partition.

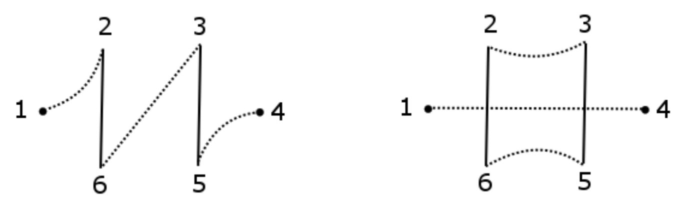


Figure 2.4: The pair on the left is mutually noncrossing. The pair on the right is not.

Proposition 2.3.5. *Given a Dyck path D, if $\pi(D) = (P, Q)$ then P and Q are mutually noncrossing. Moreover, if B is a block of P and B' is a block of Q, we can never have $a \leq b < c \leq d$ such that $a, c \in B$ and $b, c \in B'$ or $a < b \leq c < d$ such that $a, c \in B'$ and $b, d \in B$.*

Proof. This follows from the definition of the Kreweras complement and by viewing elements in blocks of Q as coming from the primed vertices. $\square$

It now makes sense to define the set $NC(a,b)$ of (a,b) noncrossing partitions by

$$NC(a,b) = \{\pi(D) \mid D \text{ is an } a,b\text{-Dyck path}\}.$$

2.4 The Poset of Rational Noncrossing Partitions

If $a < b$ and $(P,Q) \in NC(a,b)$ we have that Q consists only of blocks of rank 0. Thus, we need only consider P and moreover the structure of P uniquely determines the block ranks, so without loss of generality we may view P as an unlabeled noncrossing partition. For the remainder of this section we will assume that $a < b$. Let $NC(a,b)$ denote the poset of a,b-noncrossing partitions, ordered by refinement. Explicitly, $P_1 \leq P_2$ if and only if P_1 refines P_2. In other words, every block of P_1 is a subset of a block of P_2. Let $NC(n) = NC(n,n+1)$ denote the poset of noncrossing partitions of n.

The poset structure of $NC(n)$ is rich and many enumerative results exist, in part due to a beautiful labeling of Stanley [35]. Suppose P covers R in $NC(n)$. Then P is obtained from R by merging two blocks, call them B and B', in R. The *Stanley labeling* of the edge from R to P in the Hasse diagram is given as follows: Assume that $\min(B) < \min(B')$. Label the edge with the largest element of B which is smaller than all elements in B'. This labeling makes sense when we consider $NC(a,b)$ instead of $NC(n)$. See Figure 2.5 for $NC(3,5)$ with edges labeled according to the Stanley labeling. By starting at the minimal element, the partition into all singleton blocks, and reading the labels on a maximal chain up to the maximal element, one recovers a parking function of length $n-1$. For definitions and known results on parking functions, see

Chapter 5. Moreover, the labels read off from the maximal chains of $NC(n)$ give every parking function! Thus, there are n^{n-2} maximal chains in $NC(n)$. The poset is graded by assigning $\text{rank}(P) = n - b$ where b is the number of blocks in P, and the number of partitions with exactly i blocks is given by $\frac{1}{n}\binom{n}{i}\binom{n}{i-1}$. The number of multi-chains of length ℓ is given by $\frac{1}{n}\binom{(\ell+1)n}{n-1}$.

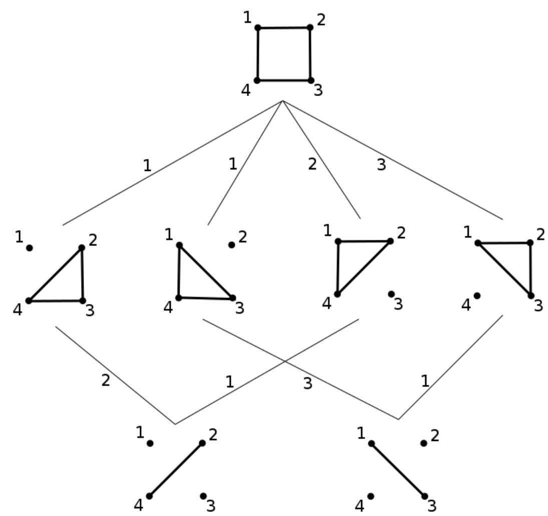


Figure 2.5: $NC(3,5)$ with Stanley labeling

Many similar results are known in the Fuss case [17]. In particular, let $NC^{(k)}(n)$ denote the poset of k-divisble noncrossing partitions, the partitions of $[kn]$ each with block size divisble by k. The number of maximal chains is given by $k(kn)^{n-2}$. The number of partitions with exactly i blocks is given by $\frac{1}{n}\binom{n}{i}\binom{kn}{i-1}$ and the number of multichains of length ℓ is given by $\frac{1}{n}\binom{(k\ell+1)n}{n-1}$.

Unfortunately things become more complicated in the rational case, and fewer results are known. The number of partitions with i blocks is given by $\frac{1}{a}\binom{a}{i}\binom{b-1}{i-1}$. However, it is unlikely that nice formulas (in the sense of product formulas) exist for the number of maximal chains or the number of multichains of length ℓ. Evidence for this comes from the fact that we quickly find large prime factors in their enumeration, even in small cases. For instance, there are $11184 = 233 \cdot 3 \cdot 2^4$ maximal chains in $NC(5, 17)$, and 233 is prime. There are 361421 multichiains of length 5 in $NC(5, 17)$, and this number is prime. For reference, Figure 2.6 records the number of maximal chains in $NC(a, b)$.

Furthermore, unlike the case of classical or k-divisible noncrossing partitions, not every minimal element in $NC(a, b)$ has the same rank. To see this, consider $NC(5, 7)$. The partitions

$$\{\{1\}, \{2\}, \{3, 6\}, \{4\}, \{5\}\} \text{ and } \{\{1\}, \{2, 4, 6\}, \{3\}, \{5\}\}$$

are each minimal elements but have different numbers of blocks, and thus different ranks.

Proposition 2.4.1. *The Stanley maximal chain labelings of $NC(a, b)$ are closed under permutation of label indices. In other words, if $\ell_1, \ell_2, \ldots \ell_r$ is the bottom-to-top labeling of a maximal chain in $NC(a, b)$, then there exists another maximal chain in $NC(a, b)$ with bottom-to-top labeling $\ell_{\sigma(1)}, \ell_{\sigma(2)}, \ldots, \ell_{\sigma(r)}$ for any $\sigma \in \mathfrak{S}_r$.*

Proof. Let P be a minimal element of $NC(a, b)$. The Stanley maximal chain labelings that start at P are the same as the Stanley labelings of all maximal chains in the principal filter $\{X \in NC(a, b) \mid X \geq P\}$. Since $NC(a, b)$ is closed under the merging of blocks, this is the same as the principal filter of P in $NC(b - 1)$, which is known to be closed under the permutation of label indices. $\square$

$a\backslash b$	1	2	3	4	5	6	7	8	9	10	11	12	13
1		1	1	1	1	1	1	1	1	1	1	1	1
2			1		2		3		4		5		6
3				3	4		12	14		27	30		48
4					16		30		128		180		432
5						125	174	252	336		2000	2310	2952
6							1296				5040		41472
7								16807	22640	32400	51840	71280	95040
8									262144		503040		1150560

Figure 2.6: Number of maximal chains in $NC(a, b)$

Proposition 2.4.2. *Let P be a noncrossing partition of $b-1$ and $b_1, b_2, \ldots, b_k$ be the sizes of the blocks in* $\texttt{krew}(P)$. *Then there are*

$$b_1^{b_1-2} b_2^{b_2-2} \cdots b_k^{b_k-2} \binom{b-1-k}{b_1-1, b_2-1, \ldots, b_k-1}$$

maximal chains starting from P and ending at the maximal element of $NC(b-1)$.

Proof. Fix a block B in $\texttt{krew}(P)$, and suppose it contains elements $a_1, a_2, \ldots, a_m$. Then $a_1, a_2, \ldots, a_m$ are the possible labels corresponding to merges of blocks in P. In particular, if we merge the block of P containing a_i with the block containing $a_j > a_i$ then the Stanley label of that edge would be a_i. From the point of view of Stanley labelings, $a_1, \ldots, a_m$ can be viewed as singletons, corresponding to the minimal element of $P_{m,m+1}$. There are m^{m-2} possible labelings of the minimal element, giving the $b_i^{b_i-2}$

terms. Finally, we need to combine the labelings that come from each of the blocks. Since $\texttt{krew}(P)$ has k blocks, P must have $b-k$ blocks. Thus, labelings are of length $b-1-k$. The multinomial coefficient gives the number of ways of combining the labelings corresponding to each block of $\texttt{krew}(P)$. □

This chapter contains material from "Rational Noncrossing Partitions for all Coprime Pairs", to appear in *Journal of Combinatorics*, 2018. The dissertation author was the primary investigator and author of this paper.

Chapter 3

Rotation, Rank Sequences, and Reflection

3.1 The Rotation Operator

The classical set of noncrossing partitions of $[n]$ is closed under rotation, which maps i to $i+1$ modulo n, and reflection, which maps i to $n-i+1$. Visually it is clear that noncrossing partitions of n have this dihedral symmetry: one can rotate or flip a noncrossing partition without disturbing any crossings. Moreover, this rotation action exhibits the cyclic sieving phenomenon [28]. However, it is unclear that this dihedral symmetry is preserved in the rational case, especially given the fact that it is not obvious what rotation or reflection means at the level of Dyck paths. Our goal now is to define a rotation operator $\texttt{rot}'$ on a, b-Dyck paths that commutes with π and yields cyclic sieving results. In other words, if $\pi(D) = (P, Q)$, then $\pi(\texttt{rot}'(D)) = \texttt{rot}^{-1}(\pi(D))$ where $\texttt{rot}$

is the map acting componentwise on P and Q sending i to $i+1$, modulo $b-1$, which preserves ranks. Later, we will also show that $NC(a,b)$ is closed under reflection.

Definition 3.1.1. *Let $D = N^{i_1}E^{j_1}\cdots N^{i_m}E^{j_m}$ be the decomposition of D into nonempty vertical and horizontal runs. We define the* rotation operator $\texttt{rot}'$ *as follows:*

1. *If $m = 1$, so that $D = N^aE^b$, we set*

$$\texttt{rot}'(D) = N^aE^b = D.$$

2. *If $m, j_1 > 1$, we set*

$$\texttt{rot}'(D) = N^{i_1}E^{j_1-1}N^{i_2}E^{j_2}\cdots N^{i_m}E^{j_m+1}.$$

3. *If $m > 1$ and $j_1 = 1$, let $p = (1, i_1)$ be the westernmost valley of D. The laser $\ell(p)$ fired from p hits D on a horizontal run E^{j_k} for some $2 < k < m$. Suppose that $\ell(p)$ hits the horizontal run E^{j_k} on step r, where $1 \leq r \leq j_k$. There are two cases to consider:*

   If $r = 1$, we set

$$\texttt{rot}'(D) = N^{i_2}E^{j_2}\cdots N^{i_{k-1}}E^{j_{k-1}}N^{i_1}E^{j_k}N^{i_{k+1}}E^{j_{k+1}}\cdots N^{i_m}E^{j_m}N^{i_k}E.$$

   If $r > 1$, we set

$$\texttt{rot}'(D) = N^{i_2}E^{j_2}\cdots N^{i_k}E^{r-1}N^{i_1}E^{j_k-r+1}N^{i_{k+1}}E^{j_{k+1}}\cdots N^{i_m}E^{j_m+1}.$$

This definition is consistent with, but more general than, the one given in [13, Section 3.1]. The $r = 1$ case in (3) will never occur if $a < b$ but can if $a > b$, so this new definition is necessary. The next proposition shows that $\texttt{rot}'$ is the path analog of $\texttt{rot}^{-1}$ on set partitions.

Proposition 3.1.2. *The operator* $\texttt{rot}'$ *defined above gives a well-defined operator on the set of* a, b*-Dyck paths. Furthermore, for any Dyck path* D*, if* $\pi(D) = (P, Q)$*, then* $\pi(\texttt{rot}'(D)) = \texttt{rot}^{-1}(\pi(D))$.

Proof. First we must check that for any a, b-Dyck path D, $\texttt{rot}'(D)$ does in fact stay above the line $y = \frac{a}{b}x$. This is clear in case 1. In case 2, the portion of the path which lies east of the first east step is translated a single unit west, and that first east step is placed at the end of the path, so it is clear that this new path cannot cross $y = \frac{a}{b}x$. We now check case 3, when $r = 1$. It is easiest to explain what happens visually. In Figure 3.1 we break the generic Dyck path at the diagonal slashes into 5 pieces. The segment labeled 1 is the initial vertical run. Segment 2 is the single east step which follows. Segment 3 is the portion of the path between segment 2 and the Q-rise preceeding the east step hit by $\ell(1)$. Segment 4 is the aforementioned Q-rise. Segment 5 is the remainder of the path. The labeled path on the right shows how the inverse rotation operator shifts these segments.

Since segment 3 stays above a laser fired in D, segment 3 in $\texttt{rot}'(D)$ must stay above the line $y = \frac{a}{b}x$. Since the segment 4 is a Q-rise in D, we know that the segment 4 of $\texttt{rot}'(D)$ has length at most $\lfloor a/b \rfloor$, so the segments 4 and 2 of $\texttt{rot}'(D)$ stay above the line $y = \frac{a}{b}x$. Since segment 5 stays above the line in D, it is clear that it stays above the line in $\texttt{rot}'(D)$ as well. Finally, since segment 1 is a single vertical run, it cannot cross the line. Thus, the path $\texttt{rot}'(D)$ stays above the line $y = \frac{a}{b}x$ so it is a valid Dyck path. Next we need to argue that $\pi(\texttt{rot}'(D)) = \texttt{rot}^{-1}(\pi(D))$. To do this, we simply

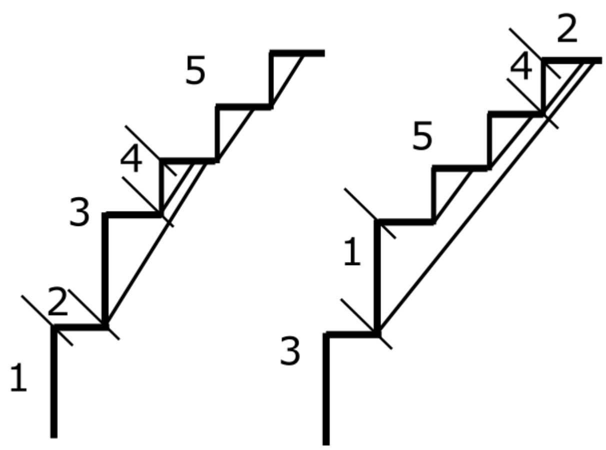


Figure 3.1: The Dyck path on the right is the rotated version of the path on the left

consider how the lasers change from D to $\texttt{rot}'(D)$.

1. The lasers fired from points in segment 5 of D are identical to the lasers fired in segment 5 of $\texttt{rot}'(D)$, shifted one unit west.

2. The lasers fired within segment 3 which hit just west of a label s in D hit just left of the label $s-1$ in $\texttt{rot}'(D)$.

3. The laser from the point labeled 1 in D is replaced by the laser fired from the end of segment 3 in $\texttt{rot}'(D)$, so the rotated block includes $b-1$ instead in the rotation as desired.

4. Let t be the label at the base of segment 4 in D. Then t and 1 are in the same block of Q in $\pi(D)$. In $\texttt{rot}'(D)$, this laser is fired from $t-1$, and as described in (3) it hits the terminal east step. Since segment 4 is translated to be the vertical

run immediately preceding the terminal east step, the laser fired from $b-1$ in $\texttt{rot}'(D)$ also hits the terminal east step, so $t-1$ and $b-1$ are in the same block of Q in $\pi(\texttt{rot}'(D))$, completing the proof that the blocks of $\pi(\texttt{rot}'(D))$ rotate as desired. When $r > 1$, the argument is equivalent to the one just given, but we treat segment 4 as being empty.

□

It now makes sense to define $\texttt{rot}(D) = \texttt{rot}'^{-1}(D)$. In other words, $\texttt{rot}(D)$ is such that $\pi(\texttt{rot}(D)) = (\texttt{rot}(P), \texttt{rot}(Q))$.

Given an a, b-Dyck path D, one can obtain a b, a-Dyck path $\tau(D)$ by applying the transposition operator τ which reflects a path about the line $y = -x$, then shifts it such that its southern-most point is at the origin. One might hope that transposition would commute with rotation in the sense that $\tau(\texttt{rot}(D)) = \texttt{rot}(\tau(D))$; however, this is not the case, which can be seen immediately from an example. Let $D = NNNNENENENNE$. If we first transpose, we obtain the path $NEENENEEEE$ which corresponds to the partition $A = \{\{1,2\},\{3,6\},\{4,5\}\}$. However, if we first rotate D, then transpose, we obtain the partition $B = \{\{1,6\},\{2,3\},\{4,5\}\}$, which is not obtainable from A via any rotation. Since the relevant information of a noncrossing partition is read off from the vertical runs of its associated Dyck path rather than the horizontal runs, which are not preserved under rotation, this is not surprising.

3.2 Rank Sequences

It will be enumeratively useful to encode lattice paths based on the sequence formed by their vertical run lengths. This will help us both with the classification of rational noncrossing partitions and the counting of fixed point sets under rotation. Let D be a Dyck path such that $\pi(D)$ is the labeled pair of noncrossing partitions (P, Q). If B is a block of P, we define $\text{rank}_P^D(B)$ to be the length of the vertical run preceding $\min(B)$ in D. If B is a block of Q, we define $\text{rank}_Q^D(B)$ to be the length of the vertical run above $\max(B)$ in D. Since the underlying Dyck path D is almost always clear from context, we will often simply write $\text{rank}_P(B)$ and $\text{rank}_Q(B)$. Given an a, b-Dyck path D such that $\pi(D) = (P, Q) \in NC(a, b)$, we define the associated P *and* Q *rank sequences*, denoted S_P and S_Q as follows:

$$S_P := (p_1, p_2, \ldots, p_{b-1})$$

where

$$p_i = \begin{cases} \text{rank}_P(B) \text{ if } i = \min(B) \text{ for some } B \in P \\ 0 \text{ otherwise.} \end{cases}$$

$$S_Q := (q_1, q_2, \ldots, q_{b-1})$$

where

$$q_i = \begin{cases} \text{rank}_Q(B) \text{ if } i = \max(B) \text{ for some } B \in Q \\ 0 \text{ otherwise.} \end{cases}$$

To solidify the connection to Dyck paths, observe that given $(P, Q) \in NC(a, b)$ we have $\pi^{-1}(P, Q) = D$ where

$$D = N^{p_1} E N^{\max(p_2, q_1)} E \cdots N^{\max(p_{b-1}, q_{b-2})} E N^{q_{b-1}} E.$$

More generally, we will simply define the *rank sequence* of (P, Q) to be the sequence given by

$$R(P, Q) := (p_1, \max(p_2, q_1), \cdots, \max(p_{b-1}, q_{b-2}), q_{b-1}).$$

This is precisely the sequence of vertical run lenghts of the Dyck path which gives rise to (P, Q). In particular, we have bijections between a, b-Dyck paths, associated rank sequence pairs (S_P, S_Q), and elements of $NC(a, b)$. Note that when $a < b$ we have that $S_Q = (0, 0, \ldots, 0)$.

For an example, consider the path and corresponding partitions shown in Figure 3.2. We have $S_P = (3, 0, 2, 3, 0, 0)$, $S_Q = (0, 0, 0, 0, 1, 1)$, and $R(P, Q) = (3, 0, 2, 3, 0, 1, 1)$.

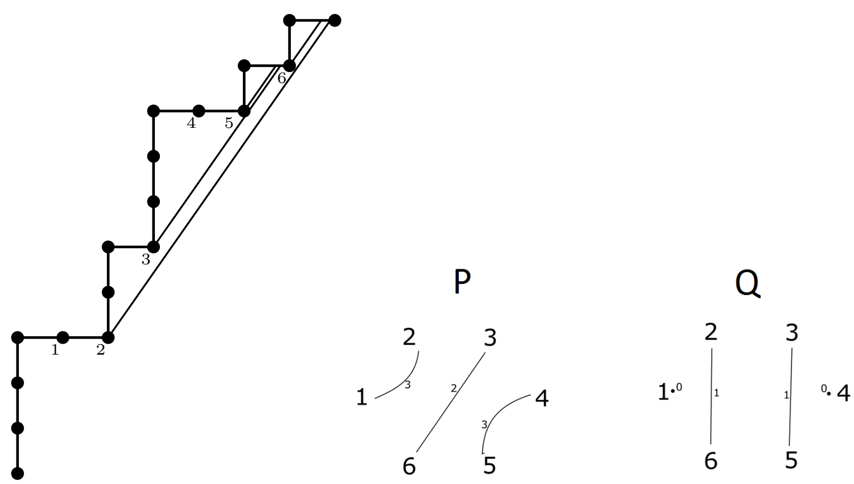


Figure 3.2: A 10,7-Dyck path with corresponding labeled partitions

Proposition 3.2.1. *Let a and b be coprime, D be an a,b-Dyck path, and $\pi(D) = (P,Q) \in NC(a,b)$. If B is a block of P, then*

$$\mathrm{rank}_P^D(B) = \mathrm{rank}_{\mathtt{rot}(P)}^{\mathtt{rot}(D)}(\mathtt{rot}(B)).$$

If B is a block of Q, then

$$\mathrm{rank}_Q^D(B) = \mathrm{rank}_{\mathtt{rot}(Q)}^{\mathtt{rot}(D)}(\mathtt{rot}(B)).$$

Proof. It will suffice to consider instead the inverse rotation operator $\mathtt{rot}'$ defined for a,b-Dyck paths. This operator preserves vertical run lengths and the underlying block structure of both P and Q. Preservation of rank is clear unless B contains 1, since $\mathtt{rot}'$ just subtracts 1 from every index modulo $b-1$. If B is in P and contains 1, then by the definition of $\mathtt{rot}'$, we translate the entire initial vertical run sequence so it immediately precedes the next element in B, after $\mathtt{rot}'$ is applied, so the rank is preserved. If B is in Q and contains 1, then the Q-rise preceding the maximal element in B is translated to the vertical run preceding the terminal east step in the path. Thus, the $\mathrm{rank}_Q^D(B) = \mathrm{rank}_{\mathtt{rot}(Q)}^{\mathtt{rot}(D)}(B')$ where B' is the block in $\mathtt{rot}(Q)$ coming from $\mathtt{rot}(D)$ which contains $b-1$. By Proposition 3.1.2, we have $B = B'$, so the rank is again preserved. □

Now we show how block ranks respect cardinality under the operation of merging blocks. In the case where $a < b$, there are no nontrivial Q blocks and merging P blocks of $(P,Q) \in NC(a,b)$ always yields another a,b-noncrossing partition. When $a > b$, the merging of blocks of P results in the splitting of blocks of Q, and we need to be careful about how we assign ranks to these split blocks. This is made precise in the following proposition. An example follows the end of the proof, which will help clarify the merging operation defined below.

Lemma 3.2.2. *Let a and b be coprime positive integers, D be an a,b-Dyck path such that $\pi(D) = (P,Q) \in NC(a,b)$, and B and B' be two blocks of P. Let P' be the result of replacing B and B' in P by $B \cup B'$. If P' is a noncrossing partition, then $(P',Q') \in NC(a,b)$ where $Q' = \texttt{krew}(P')$ and $\operatorname{rank}_{P'}(B \cup B') = \operatorname{rank}_P(B) + \operatorname{rank}_P(B')$. For any block $C' \in Q'$, if $\max(C') = \max(C)$ for some $C \in Q$, then $\operatorname{rank}_{Q'}(C') = \operatorname{rank}_Q(C)$. Otherwise $\operatorname{rank}_{Q'}(C') = 0$.*

Proof. Without loss of generality assume $\min(B) < \min(B')$. The Dyck path operation which merges B and B' consists of removing the vertical run of length $\operatorname{rank}(B')$ atop $\min(B')-1$ and adding $\operatorname{rank}(B')$ north steps to the vertical run atop $\min(B)-1$. We will now verify that this indeed gives the desired result. Let D' denote the Dyck path which results from applying this operation to D. The only lasers $\ell(p)$ which are potentially affected by this operation are those such that $\min(B) - 1 \leq p \leq \min(B') - 1$. For now, assume $p \neq \min(B) - 1$. If $\ell(p)$ hits west of $\min(B') - 1$ in D then it is unchanged in D', so we need only consider the case where it hits east of $\min(B') - 1$. Observe that the horizontal distance from $\ell(\min(B') - 1)$ and $\ell(P)$ is at most 1. To see this, suppose it were greater than 1. Then there would exist a label $q > \max(B')$ such that $\ell(\min(B')-1)$ hits D west of q and $\ell(p)$ hits D east of q. Let B'' be the block containing q. Then $\min(B') \leq \min(B'') < \max(B') < q \leq \max(B'')$ which contradicts the fact that P is noncrossing. This implies that in D', all such lasers hit the east step hit by $\ell(\min(B')-1)$. Each of these lasers is translated vertically by $\text{rank}(B')$ units, so the block structure and ranks of other blocks of P remain unchanged.

Now consider the case where $p = \min(B)-1$. If $\ell(p)$ hits D east of $\ell(\min(B')-1)$ then $\ell(p)$ is the same laser in D and D'. Since $\ell(\min(B')-1)$ disappears, all labels of B' become visible to labels of B, so the blocks union and the ranks sum, as desired. Now suppose $\ell(p)$ hits D west of $\ell(\min(B')-1)$. Let C denote the block containing $\min(B')-1$. Then we must have $\min(C) \leq \min(B)-1 < \min(B')-1$, which implies that merging B and B' would create a crossing, a contradiction.

By Lemma 2.3.3 we must have $Q' = \texttt{krew}(P')$. Let $C' \in Q'$ and suppose $\max(C') = \max(C)$ for some $C \in Q$. Since the merge operation preserves all vertical run lengths except two, each of which is a P-rise, we know that the rank of C must be preserved. On the other hand, the merge operation removes some lasers from D, so some elements which were originally in C will no longer fire lasers, forcing them to be in their own block of rank 0 in Q'. □

For example, consider once again the 10,7-Dyck path from Figure 3.2, along with its associated noncrossing partitions P and Q. Suppose we would like to merge the blocks $B = \{1,2\}$ and $B' = \{3,6\}$ in P. Doing so gives the partition $P' = \{\{1,2,3,6\},\{4,5\}\}$ which is indeed noncrossing, so $(P',Q') \in NC(a,b)$. We have $Q' = \texttt{krew}(P')$. Since 6 was the maximal element of the Q block $\{2,6\}$ with rank 1, the $\text{rank}_{Q'}(\{6\}) = 1$. Since 2 was not a maximal element of a block in Q, its rank is now 0. All other blocks and ranks are preserved.

Next let's examine how we need to modify D to obtain D', where $\pi(D') = (P', Q')$. We remove the vertical run of length $2 = \text{rank}(\{3, 6\})$ above 2 and adding north steps to the vertical run atop $0 = \min(\{1, 2\}) - 1$. The old path D and the new path D', along with P' and Q', each with rank labels shown, are shown below in Figure 3.3.

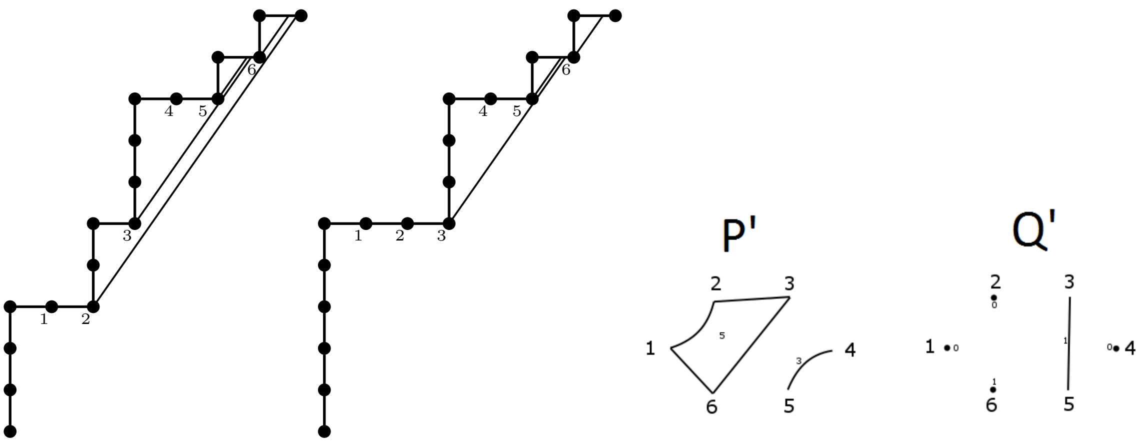


Figure 3.3: The Dyck paths D and D', along with P' and Q'

We now discuss the problem of determining whether an arbitrary labeled pair of noncrossing partitions is in fact a member of $NC(a, b)$. First, we will define a partial order $\preceq$ on the blocks of any pair (P, Q) of noncrossing partitions by

$$B' \preceq B \text{ if } \begin{cases} B, B' \in P \text{ and } [\min(B'), \max(B')] \subset [\min(B), \max(B)] \\ B' \in Q, B \in P, \text{ and } \max(B') \in [\min(B), \max(B)] \\ B' = B \text{ and } B \in Q \end{cases}$$

This partial order will tell us when we can absorb the rank of a block of Q into the rank of a block of P to obtain a new element of $NC(a, b)$.

Lemma 3.2.3. *Let $(P, Q) \in NC(a, b)$, $B \in P$, $B' \in Q$, and suppose B' is covered by B under $\preceq$. Define a pair (P', Q') as follows: P' is obtained from P by simply increasing the rank of B by* $\operatorname{rank}(B')$*. $Q' = \texttt{krew}(P')$ and B' is assigned rank 0. Then $(P', Q') \in NC(a, b)$.*

Proof. Let D denote the Dyck path such that $\pi(D) = (P, Q)$. The Dyck path operation which performs the desired merge moves the Q-rise from above $\max(B')$ to the P-rise above $\min(B) - 1$. This clearly increases the rank of B by $\operatorname{rank}(B')$. Furthermore, the east step hit by $\ell(p)$ for all $p \in B'$ is preceded by a Q-rise of length 0, so the block rank becomes 0. To see that all other blocks and ranks are fixed by this process it is enough to consider how the lasers are affected. The laser $\ell(p)$ is translated vertically by $\operatorname{rank}(B')$ units if and only if $\min(B) \leq p \leq \max(B')$. However, the portion of D which lies between these labels is also translated vertically by $\operatorname{rank}(B')$, so no changes can take place unless $\ell(p)$ hits D east of $\max(B')$.

Without loss of generality, assume p is the largest label such that $\ell(p)$ hits east of $\max(B')$, and suppose that $\ell(\max(B'))$ and $\ell(p)$ fail to hit the same east step. Then there must exist a label q which lies between $\ell(\max(B'))$ and $\ell(p)$. Let C be the block of P containg q. Then

$$\min(B) \leq p < \min(C) \leq \max(B') \leq \max(C) \leq \max(B),$$

where the last inequality follows since P is noncrossing. Thus, $B' \prec C \prec B$, contradicting the fact that B covers B'. Thus, we may assume that $\ell(\max(B'))$ and $\ell(p)$ hit the same east step. We know that $\ell(\max(B'))$ hits the east step immediately following the label max(B'), so $\ell(p)$ must as well. In the modified Dyck path, this step is translated down so the lasers will so the points where they make contact with the east step will shift west. Since the westernmost laser which hits this east step is $\ell(\max(B'))$, no laser will hit further west than the point labeled max(B'). Thus, $\ell(p)$ will still hit the same east step it originally did. Thus, the block structure is preserved and B' now has rank 0. □

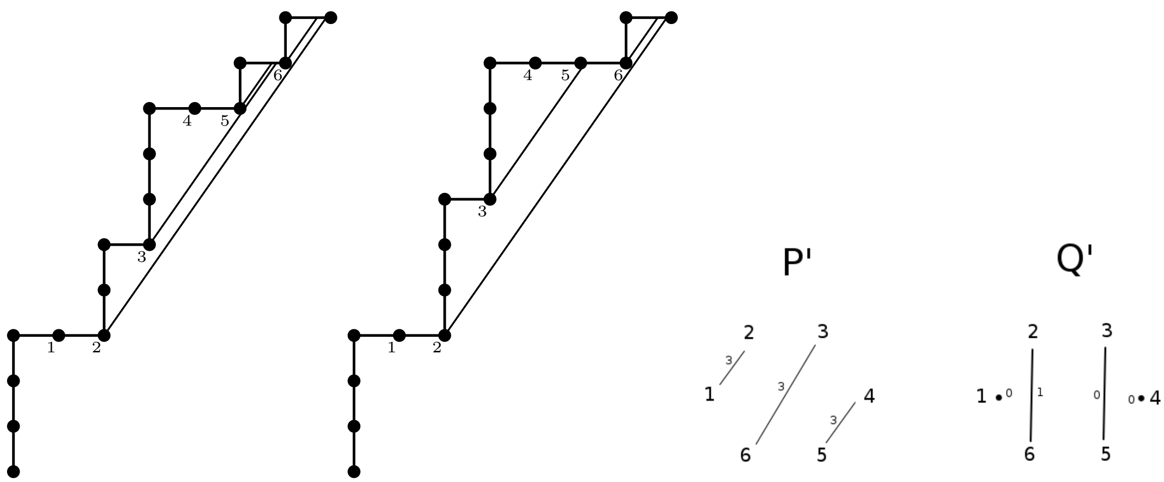


Figure 3.4: The original path, and the path and partitions obtained from the merge

For example, consider again the 10,7-Dyck path from Figure 3.2. Suppose we wish to merge the Q-block $\{3, 5\}$ of rank 1 with the P-block $\{3, 6\}$ of rank 2. The Dyck path operation removes the Q-rise of length 1 from above 5 and places it into the vertical run above 2. On the level of partitions, we obtain P' by increasing the rank of $\{3, 6\}$ by 1. We obtain Q' by changing the rank of $\{3, 5\}$ to 0. Figure 3.4 shows the original path, the resulting path, and the resulting partitions P' and Q'.

Unlike in the case where $a < b$, ranks are no longer uniquely determined by the partition structure. However, slope considerations do limit which ranks can possibly be assigned to a given block.

Definition 3.2.4. *Let a and b be coprime positive integers and (P, Q) be a pair of labeled, mutually noncrossing partitions of $[b-1]$. We say that a block B of P satiesfies the* rank condition *if*

$$(\max(B) - \min(B) + 1)\frac{a}{b} \leq \sum_{B' \preceq B} \operatorname{rank}(B') \leq (\max(B) - \min(B) + 1)\frac{a}{b} + \frac{a}{b}.$$

Note that here we use rank to indicate the label of the block B' rather than anything having to do with vertical run lengths. Such an inequality must hold for $(P, Q) \in NC(a, b)$. The argument is essentially identical to that given in Proposition 3.8 of [13] when one considers the fact that Q block ranks also contribute vertical runs. When $a < b$ the lower and upper bounds necessarily agree and uniquely determine the rank of each block, which is why labels on the partition were unnecessary in that case.

The following theorem characterizes precisely when a pair (P, Q) belongs to $NC(a, b)$. This is a generalization of Theorem 3.15 in [13], which provides such a characterization only when $a < b$.

Theorem 3.2.5. *Let (P, Q) be a pair of labeled mutually noncrossing partitions and a and b be fixed, coprime positive integers. Then $(P, Q) \in NC(a, b)$ if and only if the following conditions hold:*

1. $\sum_{B \in P} \operatorname{rank}(B) + \sum_{B' \in Q} \operatorname{rank}(B') = a$
2. *We have* $\operatorname{rank}(B) < a/b$ *for all* $B \in Q$

3. $Q = \texttt{krew}(P)$

4. The rank condition holds for all blocks in $\texttt{rot}^m(P)$ *coming from* $\texttt{rot}^m(P,Q)$ *for all* $1 \leq m \leq b-1$.

Proof. First suppose that $(P,Q) \in NC(a,b)$. Then there exists a Dyck path D such that $\pi(D) = (P,Q)$ and the vertical sequence of D comes from the ranks of blocks in P and Q, so they must sum to a. The second condition follows immediately from slope considerations. By Lemma 2.3.3, the Kreweras complement uniquely determines Q. Finally, Proposition 3.2.1 implies that condition (4) must hold.

Now suppose that we're given a pair (P,Q) which satisfies each of the conditions $(1)-(4)$. In the case where $a < b$, Theorem 3.2.5 reduces to Proposition 3.5 of [13], so we will only consider the case $a > b$ here. Although we previously defined rank sequences only for $(P,Q) \in NC(a,b)$, it makes sense to think of them for any labeled pair of noncrossing partitions. Let $D_{(P,Q)} = N^{p_1}EN^{\max(p_2,q_1)}E \cdots N^{\max(p_{b-1},q_{b-2})}EN^{q_{b-1}}E$. By condition (1), $D_{(P,Q)}$ will actually have height a so it is indeed an a,b-lattice path. By Lemma 2.3.2 we can immediately read off from $\texttt{krew}(P)$ what the laser set of D must be in order to have $\pi(D) = (P,Q)$.

For example, consider the pair $P = \{\{1,3\},\{2\}\}$ with ranks 5 and 1 respectively, and $Q = \{\{1,2\},\{3\}\}$ with ranks 1 and 0 respectively. The pair (P,Q) is not in $NC(7,4)$. It satisfies conditions (1) - (3), and each block of P satisfies the rank condition. By Lemma 2.3.2, $L(D) = \{(1,2),(2,2)\}$. Now let's examine the rank sequences of (P,Q). We have $S_Q = (5,1,0)$ and $S_Q = (0,1,0)$. Thus, the Dyck path which would have to give rise to (P, Q) (if such a Dyck path exists) must look like $D_{(P,Q)} = NNNNNENENENEE$, shown in Figure 3.5.

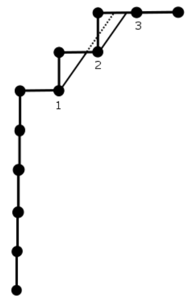


Figure 3.5: The candidate Dyck path for the pair (P, Q).

We immediately see that the laser set is $\{(1, 1), (2, 2)\}$. However, if we extend $\ell(1)$ further, along the dashed line shown in Figure 3.5, we see that it was certainly set up to hit in the appropriate spot, if only another part of D had not gotten in the way. More generally, this is true of any laser fired from the bottom of a P-rise since each block of P satisfies the rank condition. Moreover, this is the only thing which can go wrong since a laser fired from the bottom of a Q-rise will always hit the next east step by condition (2).

This is a special feature of the case $a > b$. Namely, the interval

$$\left[\left\lceil m\frac{a}{b}\right\rceil, \left\lfloor (m+1)\frac{a}{b}\right\rfloor\right]$$

is nonempty for any $m \in [b-1]$, so the rank condition is just another way of saying that the laser which cuts out a block can and will hit on the appropriate east step. However, the proof we give here for $a > b$ differs from the one given in [13] for Proposition 3.5 since knowing p and which east step $\ell(p)$ hits no longer uniquely determines the height difference of p and that east step.

Let i and k be such that $(i, k-1)$ should be a laser according to $\texttt{krew}(P)$, but such that it fails to be a laser in $D = D_{(P,Q)}$ because it first hits a horizontal segment of D whose easternmost endpoint is labeled j. To simplify things, we can repeatedly apply Lemmas 3.2.2 and 3.2.3 to (P, Q) to obtain a pair (P', Q') which has the same problem when we consider $D' = D_{(P',Q')}$. Since $NC(a, b)$ is closed under the merge operations of these lemmas, it will suffice to derive a contradiction for D' coming from (P', Q') of the form shown in Figure 3.6. In particular, D' will contain exactly three vertical runs: the initial vertical run above the origin of length A, the vertical run atop i of length B, and the vertical run atop j of length C.

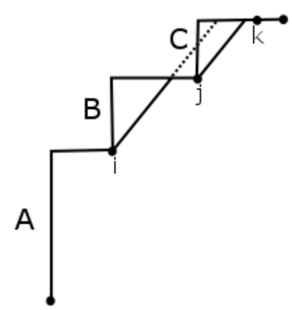


Figure 3.6: A simplified candidate Dyck path D'

Since each laser is of slope $\frac{a}{b}$, we must have

$$(j - i)a/b > B. \qquad (*)$$

Furthermore, $\ell(j)$ must hit D' on the east step between the labels $k-1$ and k. To see this, observe that if $\ell(j)$ hit west of $k-1$ then there would be no interference with $\ell(i)$. If $\ell(j)$ hit east of k then we would have one block containing j and $k+1$, and another block containing i and k, which would imply a crossing. There are now two cases to consider: either j is at the bottom of a P-rise or a Q-rise.

First, suppose j is at the bottom of a P-rise. Then we have $P' = \{B_1, B_2, B_3\}$ where $B_1 = [1, i] \cup [k, b-1]$, $B_2 = [i+1, j]$ and $B_3 = [j+1, k-1]$, and Q' consists of only trivial blocks of rank 0. By condition (4), every rotation of (P', Q') satisfies the rank condition for each rotation block of P', so we may as well assume P' is rotated so that $B_1 = [k-i, b-1]$, $B_2 = [1, j-i]$, and $B_3 = [j+1-i, k-1-i]$. Since B_2 satisfies the rank condition, we must have that $(j-i)a/b \leq \text{rank}(B_2) = B$. However, by $(*)$ we have $(j-i)a/b > B$, a contradiction.

Now suppose j is at the bottom of a Q-rise. Then we have $k = j+1$ and $P' = \{B_1, B_2\}$ where $B_1 = [1, i] \cup [j+1, b-1]$ and $B_2 = [i+1, j]$. The partition Q' consists of a single nontrivial block $B' = \{i, j\}$ of rank C. As before, it will suffice to consider rotations of (P', Q'), so we may now assume that $B_1 = [j+1-i, b-1]$, $B_2 = [1, j-i]$ and $B' = \{j-i, b-1\}$. Since it is no longer the case that $B' \preceq B_2$ and B_2 satisfies the rank condition, we must have $(j-i)a/b \leq \text{rank}(B_2) = B$. However, this contradicts $(*)$ which guarantees $B < (j-i)a/b$. In either case, we conclude that if (P, Q) satisfies conditions (1) through (4) then $(P, Q) \in NC(a, b)$. $\square$

3.3 Reflection

It was shown in [13] that $NC(a,b)$ is closed under the *reflection* operator, given by the permutation

$$\texttt{rfn} = \begin{pmatrix} 1 & 2 & \cdots & b-2 & b-1 \\ \\ b-1 & b-2 & \cdots & 2 & 1 \end{pmatrix}.$$

When $a > b$ we achieve closure under reflection provided that we choose the appropriate reflection operator on Q. Define $\texttt{rfn}'$ as follows:

$$\texttt{rfn}' = \begin{pmatrix} 1 & 2 & \cdots & b-2 & b-1 \\ \\ b-2 & b-3 & \cdots & 1 & b-1 \end{pmatrix}$$

To simplify notation, define a rotation operator $\texttt{rfn}''$ by

$$\texttt{rfn}''(B) = \begin{cases} \texttt{rfn}(B) & \text{if } B \in P \\ \\ \texttt{rfn}'(B) & \text{if } B \in Q \end{cases}$$

Proposition 3.3.1. *Let a and b be coprime. If $(P,Q) \in NC(a,b)$ then $(\texttt{rfn}(P), \texttt{rfn}'(Q)) \in NC(a,b)$, where block labels are preserved, meaning that*

$$\text{rank}(\texttt{rfn}''(B)) = \text{rank}(B)$$

for all blocks B.

Proof. Since $(P,Q) \in NC(a,b)$ it must satisfy conditions $(1) - (4)$ in Theorem 3.2.5. Since ranks are preserved, we have

$$\sum_{B \in \texttt{rfn}(P)} \text{rank}(B) + \sum_{B' \in \texttt{rfn}'(Q)} \text{rank}(B') = \sum_{B \in P} \text{rank}(B) + \sum_{B' \in Q} \text{rank}(B') = a$$

and $\text{rank}(B) < a/b$ for all $B \in Q$. By the way we have defined $\texttt{rfn}'$, we have that $\texttt{rfn}'(Q) = \texttt{krew}(\texttt{rfn}(P))$. Lastly, for all $1 \leq m \leq b-1$ if $B \in P$ we have $\text{rank}_{\texttt{rot}^m(\texttt{rfn}(P))}(\texttt{rot}^m(\texttt{rfn}(B))) = \text{rank}_P(B)$, so every block of $\texttt{rfn}(P)$ satisfies the rank condition. By Theorem 3.2.5, we have that $(\texttt{rfn}(P), \texttt{rfn}'(Q)) \in NC(a,b)$. □

Corollary 3.3.2. *Let a and b be coprime. The set $NC(a,b)$ of a,b noncrossing partitions is closed under the dihedral action $\langle \texttt{rot}, \texttt{rfn}'' \rangle$.*

It is worth noting that $NC(a,b)$ fails to be closed under other variants of reflection, for instance $(\texttt{rfn}(P), \texttt{rfn}(Q))$ or $(\texttt{rfn}'(P), \texttt{rfn}'(Q))$. The characterization of Q as the Kreweras complement of P and the fact that $\texttt{rfn}'(Q) = \texttt{krew}(\texttt{rfn}(P))$ is what makes our choice of reflection operators the only one that will work.

This chapter contains material from "Rational Noncrossing Partitions for all Coprime Pairs", to appear in *Journal of Combinatorics*, 2018. The dissertation author was the primary investigator and author of this paper.

Chapter 4

Cyclic Sieving

Let X be a finite set, $C = \langle c \rangle$ be a finite cyclic group acting on X, $X(q) \in \mathbb{N}[q]$ be a polynomial with nonnegative integer coefficents, and $\zeta \in \mathbb{C}$ be a root of unity with multiplicative order $|C|$. The triple $(X, C, X(q))$ exhibits the *cyclic sieving phenomenon* if for all $d \geq 0$ we have $X(\zeta^d) = \left|X^{c^d}\right| = \left|\{x \in X \mid c^d.x = x\}\right|$. Basic linear algebra tells us that such polynomials must exist, and a bit of representation theory can be used to show that we can always find one with nonnegative integer coefficients. What is surprising is that $X(q)$ often turns out to be very naturally associated with the set X, and tends to involve the theory of q-numbers. Explicitly, for $n \in \mathbb{N}$ define the *q-analog of n* by

$$[n]_q = 1 + q + q^2 + \cdots + q^{n-1}.$$

For $0 \leq k \leq n$, define the *q-binomial coefficients* by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}$$

where $[n]! = [n]_q[n-1]_q \cdots [2]_q[1]_q$. The following theorem of Reiner, Stanton, and White [28] provides a first example of this association.

Theorem 4.0.1. *[28, Reiner-Stanton-White 2004] Fix two positive integers $k \leq n$. Let X be the set of all subsets of $[n]$ having size k and let $C = \mathbb{Z}/nZ$ act on X via the long cycle $(1, 2, \ldots, n) \in \mathfrak{S}_n$. Then the triple $(X, C, X(q))$ exhibits the CSP, where $X(q) = \begin{bmatrix} n \\ k \end{bmatrix}_q$.*

We now set out to count the number of $(P, Q) \in NC(a, b)$ which are invariant under d-fold rotation, which will ultimately allow us to prove various instances of the cyclic sieving phenomenon. To do this, we introduce *d-modified rank sequences* of our pairs (P, Q). We will conclude by showing that these d-modified rank sequences are in bijective correspondence with those (P, Q) which are invariant under d-fold rotation. This will reduce our problem to counting these sequences.

4.1 d-modified Rank Sequences

Let $1 \leq d < n$ be such that $d|n$ and P be a noncrossing partition of $[n]$ which is invariant under $\mathtt{rot}^d$. Given a block B of P, we say B is a *central block* if $\mathtt{rot}^d(B) = B$. Clearly P can contain at most one central block. We say B is a *wrapping block* if B is not central and $[\min(B), \max(B)]$ contains every block in the $\langle \mathtt{rot}^d \rangle$-orbit of B. The $\langle \mathtt{rot}^d \rangle$-orbit of a block can contain at most one wrapping block.

Notation: For the remainder of this section, fix positive coprime integers a and b, and an integer $1 \leq d < b - 1$ such that $d|(b-1)$. Let $NC_d(a, b)$ denote the set of

$(P,Q) \in NC(a,b)$ *which are invariant under* $\texttt{rot}^d$*.*

Given $(P,Q) \in NC_d(a,b)$, we define the *d-modified P and Q rank sequences* as follows:

$$S_P^d := (p_1, \ldots, p_d) \text{ and } S_Q^d := (q_1, \ldots, q_d)$$

where

$$p_i := \begin{cases} \text{rank}_P(B) & \text{if } i = \min(B) \text{ for a noncentral, nonwrapping block} \\ & B \in P \\ 0 \text{ otherwise} \end{cases}$$

$$q_i := \begin{cases} \text{rank}_Q(B) & \text{if } b-1-d+i = \max(B) \text{ for a noncentral,} \\ & \text{nonwrapping block } B \in Q \\ 0 \text{ otherwise.} \end{cases}$$

It might seem surprising that in the definition of q_i we consider the largest d elements of $[b-1]$ rather than the smallest d elements, as we did for p_i. The reason comes from the fact that Q ranks are defined in terms of maximal block elements rather than minimal ones. In particular, $\{b-d, b-d+1, \ldots, b-1\}$ is guaranteed to contain at least one maximal element of a nonwrapping Q block in a $\langle \texttt{rot}^d \rangle$-orbit, whereas $\{1, 2, \ldots, d\}$ might not.

For example, consider the pair (P,Q) in $NC_3(10,7)$ given in Figure 4.1. We have

$S_P^3 = (3,0,0)$ since 1 is the minimal element of $\{1,2\}$ which has rank 3, 2 is not the minimal element of a block of P, and 3 is the minimal element of a central block of P. We also have $S_Q^3 = (0,1,0)$ since 4 is in a trivial Q block, 5 is the maximal element of a Q block of rank 1, and 6 is the maximal element of a wrapping block of Q. Had we instead only recorded Q ranks of 1, 2, and 3, we would have recorded $(0,0,0)$ and lost all information about the structure of Q.

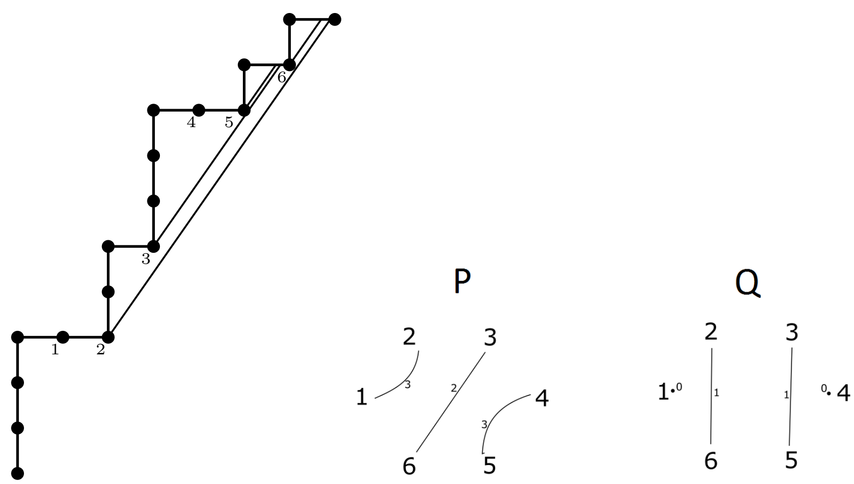


Figure 4.1: A 10,7-Dyck path with corresponding labeled partitions

Lemma 4.1.1. *Let $(P,Q) \in NC_d(a,b)$ and S_P^d and S_Q^d be the d-modified P and Q rank sequences of (P,Q). Then we have*

$$S_P^d(\texttt{rot}(P,Q)) = \texttt{rot}(S_P^d(P,Q))$$

and

$$S_Q^d(\texttt{rot}(P,Q)) = \texttt{rot}(S_Q^d(P,Q)),$$

where rotation of a sequence is given by $\texttt{rot}(a_1,\ldots,a_n) = (a_n,a_1,\ldots,a_{n-1})$.

Proof. We will make free use of Proposition 3.2.1, which implies that $\operatorname{rank}_P^D(B) = \operatorname{rank}_{\mathtt{rot}(P)}^{\mathtt{rot}(D)}(\mathtt{rot}(B))$ and $\operatorname{rank}_Q^D(B) = \operatorname{rank}_{\mathtt{rot}(Q)}^{\mathtt{rot}(D)}(\mathtt{rot}(B))$. For the first equality, let $S_P^d(\mathtt{rot}(P,Q)) = (p'_1, p'_2, \ldots, p'_d)$ be the d-modified P rank sequence of $\mathtt{rot}(P,Q)$ and let $1 \leq i \leq d$. We show that $p'_i = p_{i-1}$, where subscripts are interpreted modulo d.

Case 1: $2 \leq i \leq d$. Suppose $p_{i-1} > 0$. Then $i-1 = \min(B)$ for some non-central, non-wrapping block $B \in P$. Since B is non-central and non-wrapping and $1 \leq \min(B) \leq d-1$, we know that $\mathtt{rot}(B)$ is also non-central and non-wrapping with $\min(\mathtt{rot}(B)) = i$. We conclude that $p'_i = p_{i-1}$.

Suppose $p_{i-1} = 0$. If $i-1$ is not the minimum element of a block of P, then i is not the minimum element of a block of $\mathtt{rot}(P)$, so that $p'_i = 0$. If $i-1 = \min(B_0)$ for a central block $B_0 \in P$, then $\mathtt{rot}(B_0)$ is a central block in $\mathtt{rot}(P)$ with $i \in \mathtt{rot}(B_0)$, so that $p'_i = 0$. If $i-1 = \min(B)$ for a wrapping block $B \in P$, then the fact that $1 \leq \min(B) \leq d-1$ implies that either $i \neq \min(\mathtt{rot}(B))$ or $\mathtt{rot}(B)$ is wrapping with $i \in \mathtt{rot}(B)$. In either situation, we get that $p'_i = 0$.

Case 2: $i = 1$. Suppose $p_d > 0$. Then $d = \min(B)$ for some non-central, non-wrapping block $B \in P$. Let $t = \frac{b-1}{d}$. Recalling that $\mathtt{rot}^d(P) = P$, it follows that $\mathtt{rot}^{d(t-1)+1}(B)$ is a non-central, non-wrapping block of $\mathtt{rot}(P)$ containing 1. Thus, we get $p'_1 = \operatorname{rank}_{\mathtt{rot}(P)}(\mathtt{rot}(B)) = \operatorname{rank}_{\mathtt{rot}(P)}(\mathtt{rot}^{d(t-1)+1}(B)) = \operatorname{rank}_P(B) = p_d$.

Now suppose $p_d = 0$. If d is contained in a central block of P, then 1 is contained in a central block of $\mathtt{rot}(P)$ and $p'_1 = 0$. If d is contained in a wrapping block then $b-1$ must also be contained in that block, so that 1 is contained in a wrapping block of $\mathtt{rot}(P)$, making $p'_1 = 0$. If $d \in B$ for some block $B \in P$ which is non-central and non-wrapping, we must have that $d \neq \min(B)$. Thus, $\texttt{rot}^{-d+1}(B)$ is wrapping. Since P is noncrossing with $\texttt{rot}^d(P) = P$, it follows that $\texttt{rot}(P)$ has a wrapping block containing 1, so that $p'_1 = 0$.

Now we prove the second equality. Let

$$S^d_Q(\texttt{rot}(P, Q)) = (q'_1, q'_2, \ldots, q'_d)$$

be the d-modified Q rank sequence of $\texttt{rot}(P, Q)$ and $1 \leq i \leq d$. We will show that $q'_i = q_{i-1}$ where subscripts are interpreted modulo d.

Case 1: $2 \leq i \leq d$. If $q_{i-1} > 0$ then $i - 1 = \max(B)$ for some non-central, non-wrapping block $B \in Q$. Thus $i = \max(\texttt{rot}(B))$ and $\texttt{rot}(B)$ is non-central and non-wrapping so $q'_i = q_{i-1}$. Next suppose $q_{i-1} = 0$. If $i-1$ was not the maximal element of a block of Q then i is not the maximal element of $\texttt{rot}(Q)$, so $q'_i = 0$. If $i-1 = \max(B)$ for a wrapping block B then $\texttt{rot}(B)$ is wrapping and $i = \max(\texttt{rot}(B))$, so $q'_i = 0$. If $i-1 = \max(B)$ for a central block B then $\texttt{rot}(B)$ is central and $i = \max(\texttt{rot}(B))$ so q'_i is 0.

Case 2: $i = 1$. Suppose $b - 1 = \max(B)$ for some non-central, non-wrapping block B. By rotational symmetry, $\texttt{rot}^{b-1-d}(B)$ is a non-central, non-wrapping block of Q with max $b-1-d$ and rank q_d. Thus, $b-d$ is the max of a rotated block in $\texttt{rot}(Q)$, so we have $q'_1 = q_d$. Now suppose $b-1 \in B$ where B is central. Then $1 \in \texttt{rot}(B)$ which is also central, so $q'_1 = q_d = 0$. Lastly, suppose B is wrapping. If $b-1$ is the only element of B in $\{b-d, b-d+1, \ldots, b-1\}$ then 1 is not the maximal element of $\texttt{rot}(B)$ so by rotational symmetry, $b-d$ is not the maximal element of a Q block and we have $q'_1 = 0$.

On the other hand, if $b-1$ is not the only element of B in $\{b-d, b-d+1, \ldots, b-1\}$ then $\texttt{rot}(B)$ is still wrapping so $q_1' = 0$. □

Define the set of *good sequence pairs* to be the set of nonnegative integer sequence pairs of length d, $(S_P^d, S_Q^d) = ((p_1, \ldots, p_d), (q_1, \ldots, q_d))$, such that the following hold:

- $p_i = 0$ or $p_i > a/b$ for each $i \in [d]$
- $q_i < a/b$
- $\sum_{i=1}^d p_i + q_i \leq ad/(b-1)$, and
- there does not exist $i \in [d]$ such that both p_{i+1} and q_i are nonzero, where subscripts are interpreted modulo d.

When $a < b$, S_Q^d is always a sequence of all 0's and the sequences S_P^d are exactly the good sequences defined in [13]. Our goal is to show that the set of $\texttt{rot}^d$-invariant pairs of noncrossing partitions in $NC(a,b)$ are in bijective correspondence with the set of good sequence pairs. The next few pages will consist of a series of somewhat technical lemmas and propositions that will build up to a proof of this bijection.

We say $(P,Q) \in NC_d(a,b)$ is *noble* if the following conditions hold:

1. neither P nor Q contains any wrapping blocks
2. if P contains a central block B then $1 \in B$
3. if Q contains a central block B' then $b-1 \in B'$.

Observe that since P and Q are mutually noncrossing, there can be at most one central block in total.

Lemma 4.1.2. *Suppose* $(P, Q) \in NC_d(a, b)$ *and that* Q *contains a central block* B*. Then either* $b-1 \in B$ *or* P *contains a wrapping block, but not both.*

Proof. Let $(P, Q) \in NC_d(a, b)$ and suppose $b - 1 \notin B$. Since Q is a central block its minimal element must be contained in $[1, d]$ and its maximal element must be contained in $[b-d, b-1]$. We'll make use of the fact that $Q = \texttt{krew}(P)$, and that Q and P must be mutually noncrossing. In particular, there must be a block in P which contains both an element in $[\max(B), b-1]$ and $[1, \min(B)-1]$ in order to separate the block containing $b-1$ from B. This block is wrapping. On the other hand, suppose $b-1 \in B$ and that C is a wrapping block of P. If $\min(C) \leq \min(B)$ then the fact that C is wrapping implies that $\min(C) \leq \min(B) < \max(C) \leq \max(B)$, contradicting Proposition 2.3.5. If $\min(B) < \min(C)$ then we have $\min(B) < \min(C) < \min(B) + d \leq \max(C)$, which again would contradict Proposition 2.3.5. Note that $\min(B) + d$ is again an element of B since $\texttt{rot}^d(B) = B$. □

Lemma 4.1.3. *Suppose* $(P, Q) \in NC_d(a, b)$ *and* P *contains a central block containing 1. Then* Q *can contain no wrapping blocks.*

Proof. Let $(P, Q) \in NC_d(a, b)$ where P contains a central block C with 1, and suppose toward a contradiction that Q contains a wrapping block B. If $\max(C) \leq \max(B)$ then we have $1 \leq \min(B) < \max(C) \leq B$, which violates Proposition 2.3.5. On the other hand, if $\max(C) > \max(B)$ then we have $1 \leq \min(B) \leq d+1 \leq \max(B)$, which again implies a crossing since 1 and $d+1$ must both be elements of B. □

Proposition 4.1.4. *Every* `rot`*-orbit of* $(P,Q) \in NC_d(a,b)$ *contains at least one noble partition.*

Proof. If P contains a central block, rotate it so that it contains 1. Since P itself is noncrossing, there are no wrapping blocks in P. By Lemma 4.1.3, there can be no wrapping Q blocks. If Q contains a central block, rotate it so that it contains $b-1$. As before, since Q is noncrossing, it cannot contain any wrapping blocks. By Lemma 4.1.2, there can be no wrapping P blocks. Now assume that there is no central block, and suppose that either P or Q contains a wrapping block B. Rotate (P,Q) until the first time B is no longer wrapping. The result of this rotation cannot introduce any new wrapping blocks, so we have decreased the total number of wrapping blocks by at least 1. Continue in this way until no wrapping blocks in either P or Q remain. □

There may be many noble partitions in the rotation orbit of (P,Q). Consider, for instance, the partition in $NC_3(9,10)$ given by $P = \{1,2\},\{3\},\{4,5\},\{6\},\{7,8\},\{9\}$ with ranks 2,1,2,1,2,1 respectively and $Q = \{1\},\{2,3\},\{4\},\{5,6\},\{7\},\{8,9\}$ which are all trivial blocks. Then (P,Q) is noble and $\texttt{rot}^2(P,Q)$.

Let (S_P^d, S_Q^d) be a good sequence pair. Let $s = \sum_{i=1}^{d} p_i + q_i$ and $c = a - s(b-1)/d$. We call (S_P^d, S_Q^d) *very good* if $c = 0$, if $p_1 = 0$ and $c > a/b$, or if $q_d = 0$ and $0 < c < a/b$. Define a map

$$L : \{\text{very good sequences}\} \to \{\text{lattice paths from (0,0) to } (b,a)\}$$

as follows. If (S_P^d, S_Q^d) is a very good sequence pair, let $L(S_P^d, S_Q^d)$ be determined as follows.

Case 1: If $c = 0$,

If $p_1 = 0$, set $L(S_P^d, S_Q^d) = (N^{\max(p_2,q_1)}E \cdots N^{\max(p_d,q_{d-1})}EN^{q_d}E)^{(b-1)/d}E$.

If $q_d = 0$, set $L(S_P^d, S_Q^d) = (N^{p_1}EN^{\max(p_2,q_1)}E \cdots N^{\max(p_d,q_{d-1})}E)^{(b-1)/d}E$.

Case 2: If $c > a/b$, set

$$L(S_P^d, S_Q^d) = N^cE(N^{\max(p_2,q_1)}E \cdots N^{\max(p_d,q_{d-1})}EN^{q_d}E)^{(b-1)/d}$$

Case 3: If $0 < c < a/b$, set

$$L(S_P^d, S_Q^d) = (N^{p_1}EN^{\max(p_2,q_1)}E \cdots N^{\max(p_d,q_{d-1})}E)^{(b-1)/d}N^cE$$

We define a very good sequence pair (S_P^d, S_Q^d) to be *noble* if $L(S_P^d, S_Q^d)$ is an a, b-Dyck path.

Lemma 4.1.5. *Every good sequence pair is* `rot`*-conjugate to at least one noble sequence.*

Proof. Let (S_P^d, S_Q^d) be a good sequence pair and $S_{P,Q}^d = (s_1, \ldots, s_d)$ be such that $s_i = \max(p_i, q_{i-1})$ where we interpret q_0 as q_d. It will be convenient to also have a map γ which reverses this as follows:

$$\gamma(s_1, \ldots, s_d) = (S_P^d, S_Q^d)$$

where $p_i = s_i$ if $s_i > a/b$ and 0 otherwise, and $q_i = s_{i+1}$ if $s_{i+1} < a/b$ and 0 otherwise, interpreting s_{d+1} as s_1. We will use an argument similar to that given in the proof of the Cycle Lemma [23, Lemma 10.4.6].

Case 1: $c = a$.

In this case $s_1, \ldots, s_d$ is the zero sequence $(0, 0, \ldots, 0)$ and $L(S_P^d, S_Q^d)$ is the valid Dyck path $N^a E^b$.

Case 2: $a/b < c < a$.

Let L be the lattice path which starts at the origin and ends at $(2d, 2(s_1 + \cdots + s_d))$ given by

$$L = N^{s_1} E N^{s_2} E \cdots N^{s_d} E N^{s_1} E N^{s_2} E \cdots N^{s_d} E.$$

Label the lattice points P on L with integers $w(P)$ as follows: Label the origin 0. If P and P' are consecutive lattice points, set $w(P') = w(P) - a$ if P' is connected to P by an E-step, and $w(P') = w(P) + b$ if P' is connected to P with an N-step.

By coprimality, there exists a unique lattice point on L of minimal weight, P_0. Observe that by minimality, and the fact that $(s_1, \ldots, s_d)$ is not the zero sequence, P_0 must be immediately followed by a vertical run N^{s_i} for some $1 \leq i \leq d$. Note: If P_0 is the terminal point of L then we interpert the vertical run to be N^{s_1}.

If $i = 1$, then the entire path stays above the line $y = \frac{a}{b}x$ and it is clear that $L(S_P^d, S_Q^d)$ is a valid Dyck path. Now suppose $i > 1$ and let

$$S = (s_{i-1}, s_i, \ldots, s_d, s_1, \ldots, s_{i-2}).$$

The vertical run $N^{s_{i-1}}$ over the point A_0, immediately preceding P_0, must have height at most a/b. Otherwise A_0 would have smaller weight, contradicting minimality. Thus $\gamma(S)$ is a very good sequence pair, so that $L(\gamma(S))$ makes sense. We claim that $L(\gamma(S))$ is in fact a valid Dyck path so that $\gamma(S)$ is a noble sequence pair. Consider the segmentation $L(\gamma(S)) = L_1 \cdots L_{(b-1)/d}E$ where L_i contains d E steps. Since each segment is progressively further east, it will suffice to show that the final segment stays west of the line $y = \frac{a}{b}x$. Since (S_P^d, S_Q^d) is a good sequence pair, the copy of P_0 in L_q stays west of the line $y = \frac{a}{b}x$. Since P_0 is minimal, no other point to the east of P_0 can cross the line $y = \frac{a}{b}x$. Finally, since the vertical run immediately preceding P_0 has height at most a/b, we conclude that all of L_q stays west of the line $\frac{a}{b}x$.

Case 3: $0 \leq c < a/b$.

Define L as in Case 2, letting P_0 denote the lattice point of minimal weight. P_0 is beneath a vertical run N^{s_i} where $s_i > a/b$ since otherwise the point immediately following P_0 would be of smaller weight. Let $S = (s_i, s_{i+1}, \ldots, s_d, s_1, \ldots, s_{i-1})$. Since $s_i > a/b$, $\gamma(S)$ is a very good sequence pair so $L(\gamma(S))$ makes sense. We claim that $L(\gamma(S))$ is a a valid Dyck path. To see this, consider the segmentation $L(\gamma(S)) = L_1 \cdots L_{(b-1)/d}N^cE$. Since $c < a/b$, the point labeled $b-1$ stays west of the line $y = \frac{a}{b}x$. Each segment L_i is progressively further east, so it will again suffice to show that L_q remains west of the line $y = \frac{a}{b}x$. Since (S_P^d, S_Q^d) is a good sequence pair, the copy of P_0 in L_q stays west of the line $y = \frac{a}{b}x$, and since P_0 is minimal, no other point east of P_0 can cross the line $y = \frac{a}{b}x$.

For example, consider the good sequence pair $S_P^3 = (0, 3, 0)$ and $S_Q^3 = (0, 1, 1)$. Then we have $S_{P,Q}^3 = (1, 3, 1)$. The path on the left in Figure 4.2 shows the corresponding lattice path L from $(0, 0)$ to $(2 \cdot 3, 2 \cdot 5)$ with weight labels. The point of minimal weight is labeled -4. This appears at the bottom of a vertical run of length $3 > 11/7$. Thus, $S = (3, 1, 1)$. The path $L(\gamma(S))$ is shown on the right in Figure 4.2. The slashes indicate

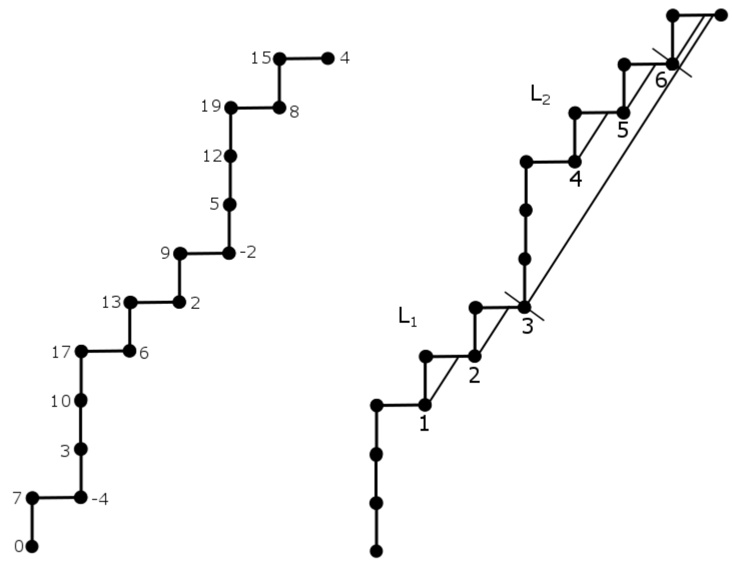


Figure 4.2: An aid in the proof of Lemma 4.1.5

the segmentation into L_1 and L_2. The final vertical run is $N^c = N^1$, and the result is a valid Dyck path. $\square$

Lemma 4.1.6. *Let* (S_P^d, S_Q^d) *is a noble sequence pair. Then* $\pi := \pi(L(S_P^d, S_Q^d)) \in NC_d(a, b)$ *is noble and the d-modified* P *and* Q *rank sequences of* π *are* S_P^d *and* S_Q^d*.*

Proof. Define $S_{P,Q}^d = (s_1, \ldots, s_d)$ as in the proof of Lemma 4.1.5 and let $c = a - q(s_1 + \cdots + s_d)$. The argument splits into cases:

Case 1: $c > a/b$.

Let $L := L(S_P^d, S_Q^d) = L_1 L_2 \cdots L_q E$ where $L_1 = N^c E N^{s_2} E \cdots N^{s_d} E$ and $L_i = N^{s_1} E \cdots N^{s_d} E$ for $2 \leq i \leq q$. Since (S_P^d, S_Q^d) is very good and $c > a/b$ we must have that the first entry of S_P^d is 0. Fix any index $1 \leq i \leq d$ such that $s_i > 0$ and any other index $1 \leq j \leq q - 1$. Both segments L_j and L_{j+1} of $L(s)$ contain a copy of the nonempty vertical run N^{s_i}. First suppose $s_i > a/b$ and let P_0 and P_1 denote the points at the bottom of these vertical runs. We have that $\ell(P_0)$ and $\ell(P_1)$ are rigid translations of one another, so the P block visible from the copy of N^{s_i} in L_{j+1} is the image of the block visible from the copy of N^{s_i} in L_j under the operator $\texttt{rot}^d$. Now suppose $s_i < a/b$ and let E_0 and E_1 denote the east steps immediately following the vertical runs in L_j and L_{j+1}. The collection of lasers which hit E_1 are a rigid translation of the lasers which hit E_0, which implies that the Q block determined by the lasers hitting E_1 is the image of the Q block which results from lasers hitting E_0 in L_j under the operator $\texttt{rot}^d$.

Since the first entry of S_P^d is 0 none of these blocks contain 1, so the set of blocks not containing 1 is stable under $\texttt{rot}^d$, which implies that the block containing 1 must be central, π is invariant under $\texttt{rot}^d$, and π has no wrapping blocks. Thus, π is noble and the d-modified P and Q rank sequences of π are S_P^d and S_Q^d.

Case 2: $0 \leq c < a/b$.

As before, consider the segmentation $L(S_P^d, S_Q^d) = L_1 \cdots L_q N^c E$ where $L_i = N^{s_1} E \cdots N^{s_d} E$. Since (S_P^d, S_Q^d) is very good and $0 < c < a/b$ we must have that the last entry of S_Q^d is 0. In this case, with the extra east step at the end of the path, lasers fired from the points at the bottom of consecutive copies of the vertical run N^{s_i} in L_j and L_{j+1} are either

1. translates of each other or

2. they both hit L on its terminal east step.

As described in case 1, $\texttt{rot}^d$ invariance is guaranteed for all P and Q blocks determined by (1) and the fact that every such laser pair satisfies (1) or (2) implies there can be no wrapping blocks. If $c = 0$ then there is no central block so we conclude π is noble. If $0 < c < a/b$ then all points P such that $\ell(P)$ hits L on its terminal east step are in a central Q block containing $b-1$ so π is noble in this case as well, and the d-modified P and Q rank sequences of π are S_P^d and S_Q^d. $\square$

Lemma 4.1.7. *Suppose that $(P,Q) \in NC_d(a,b)$. Then (P,Q) is noble if and only if (S_P^d, S_Q^d) is noble.*

Proof. First suppose (P,Q) is noble. Let $S_{P,Q}^d = (s_1, \ldots, s_d)$, $c = a - \frac{b-1}{d}(s_1 + \cdots + s_d)$, $S_P = (p_1, \ldots, p_{b-1})$, and $S_Q = (q_1, \ldots, q_{b-1})$. Since (P,Q) contains no wrapping blocks,

$$R(P,Q) = \begin{cases} (s_1, s_2, \ldots, s_d, s_1, \ldots, s_d, \ldots, s_1, \ldots, s_d, c) & \text{if } c < a/b \\ (c, s_2, \ldots, s_d, s_1, \ldots, s_d, \ldots, s_1, \ldots, s_d, s_1) & \text{if } c > a/b. \end{cases}$$

If $c = 0$ then (S_P^d, S_Q^d) is automatically very good. If $0 < c < a/b$ then the nobility of (P,Q) implies $b-1$ is contained in the central block of (P,Q). Thus $q_d = 0$ so (S_P^d, S_Q^d) is very good. On the other hand, if $c > a/b$ then the nobility of (P,Q) implies that 1 is in the central block so that $p_1 = 0$ and (S_P^d, S_Q^d) is very good. In both cases the vertical runs of $L(S_P^d, S_Q^d)$ agree with the rank sequence $R(P,Q)$, so (S_P^d, S_Q^d) is noble.

Now suppose that (P,Q) is not noble. The argument in the proof of Lemma 4.8 in [13] tells us that P contains no wrapping blocks, and if it has a central block then it must contain 1. If P contains a central block with 1 then by Lemma 4.1.3 Q cannot contain a wrapping block. Thus, we may assume P contains no central or wrapping blocks, which means $p_1 \neq 0$. Since (S_P, S_Q) is noble this means that $L := L(S_{P,Q}^d)$ takes the form $L_1 L_2 \cdots L_q N^c E$ where $L_i = (N^{p_1} E N^{\max(p_2,q_1)} E \cdots N^{\max(p_d,q_{d-1})} E$. Suppose Q contains a wrapping block B. Let $f = \min(B)$ and $g = \max(B)$. Since B is wrapping, we have $1 \leq f \leq d$ and $b - d \leq g \leq b - 1$. Let B' denote the inverse d-fold rotation of B. Then $\max(B') = b - d + f$ so that the f^{th} entry of S_Q^d is rank(B). Let g' be the copy of g contained in L_1 and f' denote the copy of f contained in L_2. Then $\ell(g')$ hits the east step immediately following the vertical run above f'. Let P_0 be the first point of L_2. Since $g' \leq d \leq P_0$ and $\ell(g')$ hits an east step in L_2, this implies that $\ell(P_0)$ hits an east step in L_2 as well. However, L_1 is a copy of L_2, so if we fire a laser from the initial point of L_1, the origin, then it must hit an east step of L_1, contradicting the fact that L stays above the line $y = \frac{a}{b}x$.

Finally, suppose Q contains a central block which does not contain $b - 1$. By Lemma 4.1.2 this implies that P contains a wrapping block so that (S_P^d, S_Q^d) is not noble, a contradiction. □

Theorem 4.1.8. *The map* $S^d : NC_d(a, b) \to \{$*good sequence pairs* $(S_P^d, S_Q^d)\}$ *is a bijection which commutes with the action of rotation.*

Proof. By Lemma 4.1.1, S^d commutes with rotation. Now suppose (S_P^d, S_Q^d) is a good sequence pair. By Lemma 4.1.5 it is conjugate to a noble sequence pair $(S_{P'}^d, S_{Q'}^d)$. By Lemma 4.1.6 there exists $(P', Q') \in NC_d(a, b)$ such that (P', Q') is noble and $S^d(P', Q') = (S_{P'}^d, S_{Q'}^d)$. As described in the proof of Lemma 4.1.7, this completely determines the rank sequence, and hence the vertical run sequence, of (P', Q'), which uniquely determines the partition. Therefore (P', Q') is unique. Since S^d commutes with rotation, there must be a unique rotated partition pair which is the inverse image of (S_P^d, S_Q^d), proving that S^d is a bijection. □

Now we can enumerate the good sequence pairs. To do this, begin by combining the pairs of sequences into a single sequence $(s_1, \ldots, s_d)$ of length d as follows:

$$s_i = \begin{cases} \max(p_{i+1}, q_i) \text{ if } 1 \leq i \leq d-1 \\ \\ \max(p_1, q_d) \text{ if } i = d \end{cases}$$

This transformation is a bijection onto the set of nonnegative integer sequences of length d whose entries sum to at most $ad/(b-1)$, which are counted by

$$\binom{\lfloor ad/(b-1) \rfloor + d}{d}.$$

Corollary 4.1.9. *Let a and b be coprime positive integers and $d|(b-1)$. Then*

$$|NC_d(a,b)| = \binom{\lfloor ad/(b-1) \rfloor + d}{d}.$$

Corollary 4.1.10. *Let a and b be coprime positive integers and $d|(b-1)$. Let p be a nonnegative integer such that $\frac{b-1}{d}p \leq a$. The number of $(P,Q) \in NC_d(a,b)$ with a central block in either P or Q and p orbits of non-central blocks under the action of $\texttt{rot}^d$ is*

$$\begin{cases} \binom{d}{p}\binom{\lfloor \frac{ad}{b-1} \rfloor}{p} & \textit{if } \frac{b-1}{d} \not| a \\ \\ \binom{d}{p}\binom{\lfloor \frac{ad}{b-1} \rfloor - 1}{p} & \textit{if } \frac{b-1}{d} | a. \end{cases}$$

The number of $(P,Q) \in NC_d(a,b)$ with no central block and p orbits of noncentral *blocks under the action of* $\texttt{rot}^d$ *is*

$$\begin{cases} \binom{d}{p}\binom{\lfloor \frac{ad}{b-1} \rfloor - 1}{p-1} & \text{if } \frac{b-1}{d} | a \\ 0 & \text{if } \frac{b-1}{d} \not| a. \end{cases}$$

Corollary 4.1.11. *Let a and b be coprime positive integers and $d|(b-1)$. Let $m_1, \ldots, m_a$ be nonnegative integers which satisfy $\frac{b-1}{d}(m_1 + 2m_2 + \cdots + am_a) \leq a$. The number of $(P, Q) \in NC(a, b)$ which are invariant under $\texttt{rot}^d$ and have m_i orbits of noncentral blocks of rank i under the action of $\texttt{rot}^d$ is*

$$\binom{d}{m_1, m_2, \ldots, m_a, d-m}$$

where $m = m_1 + m_2 + \cdots + m_a$.

4.2 Cyclic Sieving

Let X be a finite set, $C = \langle c \rangle$ be a finite cyclic group acting on X, $X(q) \in \mathbb{N}[q]$ be a polynomial with nonnegative integer coefficents, and $\zeta \in \mathbb{C}$ be a root of unity with multiplicative order $|C|$. The triple $(X, C, X(q))$ exhibits the *cyclic sieving phenomenon* if for all $d \geq 0$ we have $X(\zeta^d) = \left|X^{c^d}\right| = \left|\{x \in X \mid c^d.x = x\}\right|$. We are now ready to prove cyclic sieving results for $NC(a, b)$ under $\texttt{rot}$. Our proofs will be 'brute force' and use direct root-of-unity evaluations of q-analogs. We will make frequent use of the following fact: If $x \equiv y \pmod{z}$, then

$$\lim_{q \to e^{2\pi i/z}} \frac{[x]_q}{[y]_q} = \begin{cases} \frac{x}{y} & \text{if } y \equiv 0 \bmod z, \\ \\ 0 & \text{otherwise.} \end{cases}$$

From this, we get the useful fact that

$$\lim_{q \to e^{2\pi i/z}} \frac{[nz]_q!}{[kz]_q!} = \binom{n}{k} [nz - kz]_q!|_{q=e^{2\pi i/z}}.$$

Let a and b be coprime and $\mathbf{r} = (r_1, r_2, \ldots, r_a)$ be sequence of nonnegative integers satisfying $r_1 + 2r_2 + \cdots + ar_a = a$. Set $k = \sum_{i=1}^{a} r_i$. The *q-Kreweras numbers* are defined by

$$\mathsf{Krew}_q(a, b, \mathbf{r}) := \frac{[b-1]!_q}{[r_1]!_q \cdots [r_a]!_q [b-k]!_q}.$$

Reiner and Sommers proved that the q-Kreweras number $\mathsf{Krew}_q(a, b, \mathbf{r})$ is a polynomial in q with nonnegative integer coefficients using algebraic techniques [27]. No combinatorial proof of the polynomiality or the positivity of $\mathsf{Krew}_q(a, b, \mathbf{r})$ is known. That is, there is no statistic on a, b-noncrossing partitions of type $\mathbf{r}$ whose generating function is $\mathsf{Krew}_q(a, b, \mathbf{r})$.

On the other hand, we have the following elementary proof that $\mathsf{Krew}_q(a, b, \mathbf{r})$ is a polynomial in q with nonnegative integer coefficients which was generously pointed out by an anonymous referee for [13], which we will reproduce here.

Lemma 4.2.1. *[13] The q-Kreweras number* $\mathsf{Krew}_q(a, b, \mathbf{r})$ *is a polynomial in q with nonnegative integer coefficients.*

Proof. We start by showing that the rational expression

$$\mathsf{Krew}_q(a,b,\mathbf{r}) = \frac{1}{[k]_q} \begin{bmatrix} r_1+r_2+\cdots+r_a \\ r_1,r_2,\ldots,r_a \end{bmatrix}_q \begin{bmatrix} b-1 \\ k-1 \end{bmatrix}_q$$

is a polynomial in q. For any positive integer n we have

$$q^n - 1 = \prod_{d|n} \Phi_d(q),$$

where

$$\Phi_d(q) = \prod_{\substack{1\le m\le d \\ \gcd(m,d)=1}} \left(q - e^{2\pi i \frac{m}{d}}\right)$$

is the d^{th} cyclotomic polynomial in q. We'll also use the fact that

$$\prod_{p=1}^{r} \prod_{d|p} \Phi_d(q) = \prod_{d=1}^{r} \prod_{j=1}^{\lfloor \frac{r}{d} \rfloor} \Phi_d(q),$$

where r is a positive integer. Combining the above equations gives

$$\mathsf{Krew}_q(a,b,\mathbf{r}) = \prod_{d\ge 2} \Phi_d(q)^{e_d},$$

where

$$e_d = -\sum_{i=1}^{a} \left\lfloor \frac{r_i}{d} \right\rfloor + \left\lfloor \frac{b-1}{d} \right\rfloor - \left\lfloor \frac{b-k}{d} \right\rfloor.$$

To prove that $\mathsf{Krew}_q(a,b,\mathbf{r})$ is a polynomial in q, we must show that $e_d \ge 0$ for all d. To see that $e_d \ge 0$, we observe that the equation defining e_d can be written in two different ways. We have

$$e_d = -\chi(d|k) + \left(\left\lfloor \frac{k}{d} \right\rfloor - \sum_{i=1}^{a} \left\lfloor \frac{r_i}{d} \right\rfloor \right) + \left(\left\lfloor \frac{b-1}{d} \right\rfloor - \left\lfloor \frac{b-k}{d} \right\rfloor - \left\lfloor \frac{k-1}{d} \right\rfloor \right) \tag{4.1}$$

$$= -\chi(d|b) + \left(\left\lfloor \frac{k}{d} \right\rfloor - \sum_{i=1}^{a} \left\lfloor \frac{r_i}{d} \right\rfloor \right) + \left(\left\lfloor \frac{b}{d} \right\rfloor - \left\lfloor \frac{b-k}{d} \right\rfloor - \left\lfloor \frac{k}{d} \right\rfloor \right), \tag{4.2}$$

where for any statement $\mathcal{S}$ we set $\chi(\mathcal{S}) = 1$ if $\mathcal{S}$ is true and $\chi(\mathcal{S}) = 0$ if $\mathcal{S}$ is false. The terms in the second parentheses of both right-hand sides are obviously nonnegative. Since $r_1 + r_2 + \cdots + r_a = k$, the terms in the first parentheses of both right-hand sides are also nonnegative. Thus, unless $d|k$ and $d|b$, we have $e_d \geq 0$.

We must therefore assume that $d|k$ and $d|b$. If $d|r_i$ for all $1 \leq i \leq a$, then the relation $r_1 + 2r_2 + \cdots + ar_a = a$ forces $d|a$, which contradicts the coprimality of a and b. Thus, there exists $1 \leq i_0 \leq a$ such that $d \not| r_{i_0}$, meaning that $\frac{r_{i_0}}{d} > \lfloor \frac{r_{i_0}}{d} \rfloor$, and hence

$$\left\lfloor \frac{k}{d} \right\rfloor - \sum_{i=1}^{a} \left\lfloor \frac{r_i}{d} \right\rfloor \geq 1.$$

Either of the right hand sides of (4.1) or (4.2) implies that

$$e_d \geq -1 + 1 + 0 = 0,$$

as desired. We conclude that $\mathsf{Krew}_q(a, b, \mathbf{r})$ is a polynomial in q with integer coefficients.

It remains to show that the polynomial $\mathsf{Krew}_q(a, b, \mathbf{r})$ has nonnegative coefficients. To see this, observe that the product

$$[k]_q \times \mathsf{Krew}_q(a, b, \mathbf{r}) = \begin{bmatrix} r_1 + r_2 + \cdots + r_a \\ r_1, r_2, \ldots, r_a \end{bmatrix}_q \begin{bmatrix} b-1 \\ k-1 \end{bmatrix}_q$$

is a unimodal and reciprocal polynomial with nonnegative coefficients (since it can be written as a product of q-binomial coefficients). Since we know that $\mathsf{Krew}_q(a, b, \mathbf{r})$ is a polynomial in q with integer coefficients, [2, Proposition 10.1 (iii)] applies and $\mathsf{Krew}_q(a, b, \mathbf{r})$ has nonnegative coefficients (see also the discussion before [28, Corollary 10.4]). $\square$

Theorem 4.2.2. *Let a and b be coprime and $\mathbf{r} = (r_1, r_2, \ldots, r_a)$ be sequence of nonnegative integers satisfying $r_1 + 2r_2 + \cdots + ar_a = a$. Set $k = \sum_{i=1}^{a} r_i$. Let X be the set of $(P, Q) \in NC(a, b)$ with r_i blocks of rank i, where a block may come from either P or Q. Then the triple $(X, C, X(q))$ exhibits the cyclic sieving phenomenon, where $C = \mathbb{Z}_{b-1}$ acts on X by rotation and*

$$X(q) = \mathsf{Krew}_q(a, b, \mathbf{r}) = \frac{[b-1]!_q}{[r_1]!_q \cdots [r_a]!_q [b-k]!_q}$$

is the q-rational Kreweras number.

Proof. By Lemma 4.2.1 we have that $\mathsf{Krew}_q(a, b, \mathbf{r})$ is a polynomial in q with nonnegative integer coefficients. Let $\zeta = e^{\frac{2\pi i}{b-1}}$ and let $d|(b-1)$ with $1 \leq d < b-1$. Write $t = \frac{b-1}{d}$. We have that $X(\zeta^d) = 0$ unless $t|r_i$ for all but at most one $1 \leq i \leq a$, and that $r_{i_0} \equiv 1 (\mathrm{mod}\ t)$ if $t \not| r_{i_0}$. If the sequence $\mathbf{r}$ satisfies the condition of the last sentence, define a new sequence $(m_1, \ldots, m_a)$ by $m_i = \left\lfloor \frac{r_i}{t} \right\rfloor$ for $1 \leq i \leq a$. Let $m = m_1 + \cdots + m_a$. Write $r_{i_0} = c_{i_0} t + s_{i_0}$ for $s_{i_0} \in \{0, 1\}$ and assume $t|r_i$ for all $i \neq i_0$. We have

$$\begin{aligned}
\lim_{q \to \zeta^d} X(q) &= \binom{d}{m_1}\binom{d-m_1}{m_2} \cdots \binom{d-(m-m_a)}{m_a} \lim_{q \to \zeta^d} \frac{[b-1-mt]!_q [r_{i_0} - s_{i_0}]!_q}{[r_{i_0}]!_q [b-k]!_q} \\
&= \binom{d}{m_1, \ldots, m_a, d-m} \lim_{q \to \zeta^d} \frac{[b-1-(k-s_{i_0})]!_q [r_{i_0} - s_{i_0}]!_q}{[r_{i_0}]!_q [b-k]!_q} b \\
&= \begin{cases} \binom{d}{m_1, \ldots, m_a, d-m} \lim_{q \to \zeta^d} \frac{1}{[b-k]_q} & s_{i_0} = 0 \\ \binom{d}{m_1, \ldots, m_a, d-m} \lim_{q \to \zeta^d} \frac{1}{[r_{i_0}]_q} & s_{i_0} = 1 \end{cases} \\
&= \binom{d}{m_1, \ldots, m_a, d-m}.
\end{aligned}$$

By Corollary 4.1.11 we have $X(\zeta^d) = |X^{\mathtt{rot}^d}|$. □

Theorem 4.2.3. *Let a and b be coprime, $1 \leq k \leq a$, and X be the set of $(P,Q) \in NC(a,b)$ with k blocks in total. The triple $(X, C, X(q))$ exhibits the cyclic sieving phenomenon where $C = \mathbb{Z}_{b-1}$ acts on X by rotation and*

$$X(q) = \mathsf{Nar}_q(a,b,k) = \frac{1}{[a]_q} \begin{bmatrix} a \\ k \end{bmatrix}_q \begin{bmatrix} b-1 \\ k-1 \end{bmatrix}_q$$

is the q-rational Narayana number.

Proof. Reiner and Sommers proved that the q-Narayana numbers $\mathsf{Nar}_q(a,b,k)$ are polynomials in q with nonnegative integer coefficients [27]. As in the Kreweras case, no combinatorial proof of this fact is known. However, polynomiality and nonnegativity of $\mathsf{Nar}_q(a,b,k)$ may be proven by a direct argument similar to that in the Kreweras case [13]. By the same reasoning as in the proof of Lemma 4.2.1, we have that

$$\mathsf{Nar}_q(a,b,k) = \prod_{d \geq 2} \Phi_d(q)^{f_d},$$

where

$$f_d = -\chi(d|a) + \left(\left\lfloor \frac{a}{d} \right\rfloor - \left\lfloor \frac{k}{d} \right\rfloor - \left\lfloor \frac{a-k}{d} \right\rfloor \right) + \left(\left\lfloor \frac{b-1}{d} \right\rfloor - \left\lfloor \frac{k-1}{d} \right\rfloor - \left\lfloor \frac{b-k}{d} \right\rfloor \right) \quad (4.3)$$

$$= -\chi(d|k) + \left(\left\lfloor \frac{a-1}{d} \right\rfloor - \left\lfloor \frac{k-1}{d} \right\rfloor - \left\lfloor \frac{a-k}{d} \right\rfloor \right) + \left(\left\lfloor \frac{b-1}{d} \right\rfloor - \left\lfloor \frac{k-1}{d} \right\rfloor - \left\lfloor \frac{b-k}{d} \right\rfloor \right). \quad (4.4)$$

The terms in the parentheses of either of the right-hand sides above are clearly nonnegative. It follows that $f_d \geq 0$ whenever $d \not| a$ or $d \not| k$.

Suppose $d|a$ and $d|k$. Since a and b are coprime we conclude that $d \not| b$ and $\left\lfloor \frac{b-1}{d} \right\rfloor = \left\lfloor \frac{b}{d} \right\rfloor$. This means that the term in the second parentheses of the right-hand sides of equations (4.3) and (4.4) can be expressed as

$$\begin{aligned}\left(\left\lfloor\frac{b-1}{d}\right\rfloor-\left\lfloor\frac{k-1}{d}\right\rfloor-\left\lfloor\frac{b-k}{d}\right\rfloor\right)&=\left(\left\lfloor\frac{b}{d}\right\rfloor-\left\lfloor\frac{k-1}{d}\right\rfloor-\left\lfloor\frac{b-k}{d}\right\rfloor\right)\\&=\left(\left\lfloor\frac{b}{d}\right\rfloor-\left(\frac{k}{d}-1\right)-\left(\left\lfloor\frac{b}{d}\right\rfloor-\frac{k}{d}\right)\right)\\&=1,\end{aligned}$$

where the second equality used the fact that $d|k$. We conclude that

$$f_d \geq -1+0+1=0,$$

as desired, and conclude that $\mathsf{Nar}_q(a,b,k)$ is a polynomial in q with integer coefficients.

To complete the proof that $\mathsf{Nar}_q(a,b,k)$ is a polynomial with *nonnegative* integer coefficients, simply observe that

$$[a]_q \times \mathsf{Nar}_q(a,b,k) = \begin{bmatrix} a \\ k \end{bmatrix}_q \begin{bmatrix} b-1 \\ k-1 \end{bmatrix}_q$$

is a product of two q-binomial coefficients, and hence a unimodal and reciprocal polynomial in q with nonnegative coefficents. By [2, Proposition 10.1(iii)] and the fact that $\mathsf{Nar}_q(a,b,k)$ is a polynomial with integer coefficients, we get that the coefficients of $\mathsf{Nar}_q(a,b,k)$ are nonnegative.

Finally, let $\zeta = e^{\frac{2\pi i}{b-1}}$ and let $d|(b-1)$ with $1 \leq d < b-1$. Let $p = \left\lfloor \frac{kd}{b-1} \right\rfloor$, which gives the number of orbits of non-central blocks under the action of $\texttt{rot}^d$. Using the same techniques as in the proof of Theorem 4.2.2 we have

$$\lim_{q\to\zeta^d} X(q) = \begin{cases} \binom{d}{p}\binom{\lfloor\frac{ad}{b-1}\rfloor-1}{p-1} & \text{if } k\equiv 0(\text{mod } q), \\ \binom{d}{p}\binom{\lfloor\frac{ad}{b-1}\rfloor}{p} & \text{if } k\equiv 1(\text{mod } q) \text{ and } a\not\equiv 0(\text{mod } q) \\ \binom{d}{p}\binom{\lfloor\frac{ad}{b-1}\rfloor-1}{p} & \text{if } k\equiv 1(\text{mod } q) \text{ and } a\equiv 0(\text{mod } q) \\ 0 & \text{otherwise.} \end{cases}$$

By Corollary 4.1.10 we have $X(\zeta^d) = |X^{\mathtt{rot}^d}|$.

□

Theorem 4.2.4. *Let a and b be coprime, X be the set of $(P,Q)\in NC(a,b)$ and*

$$X(q) = \mathsf{Cat}_q(a,b) = \frac{1}{[a+b]_q}\begin{bmatrix} a+b \\ a,b \end{bmatrix}_q$$

be the q-rational Catalan number. Then the triple $(X, C, X(q))$ exhibits the cyclic sieving phenomenon, where $C = \mathbb{Z}_{b-1}$ acts by rotation.

Proof. An argument similar to those above, given fully in [13], shows that $\mathsf{Cat}_q(a,b)$ is a polynomial in q with nonnegative integer coefficients. Evaluating $\lim_{q\to\zeta^d}\mathsf{Cat}_q(a,b)$ gives the expression given in Corollary 4.1.9. □

The special case of $(a,b) = (n+1,n)$ was considered by Thiel in [36], and this instance of the cyclic sieving phenomenon was proven for this case. A simple bijection relates $n+1,n$-Dyck paths (and therefore elements of $NC(n+1,n)$) to the noncrossing $(1,2)$-configurations, a variant of one of the hundreds of Catalan objects listed in Stanley's Catalan addendum. [34]. For convenience, we reprint the relevant definitions here:

Definition 4.2.5. *(Thiel) Call a subset of* $[m]$ *a* ball *if it has cardinality 1 and an* arc *if it has cardinality 2. Define a* (1,2)-configuration *on* $[m]$ *as a set of pairwise disjoint balls and arcs. Say that a (1,2)-configuration* F *has a* crossing *if it contains arcs* $\{i_1, i_2\}$ *and* $\{j_1, j_2\}$ *with* $i_1 < j_1 < i_2 < j_2$*. If* F *has no crossing, it is called* noncrossing*. Define* X_n *to be the set of noncrossing (1,2)-configurations on* $[n-1]$*.*

Proposition 4.2.6. *There is a bijection* τ *between* $n+1, n$*-Dyck paths and* X_n *that commutes with the action of rotation.*

Proof. Given an $n+1, n$-Dyck path D, define $\tau(D)$ as follows: Read the labels from 1 to $n-1$. If the point labeled i does not fire a laser, leave it unmarked in the $(1,2)$ configuration. Otherwise, it fires a laser which hits an east step with left endpoint whose x-coordinate is j. If $i = j$, decorate i with a dot. Otherwise, draw an arc from i to j. For the reverse map, note that there is a unique way to fire a laser from i in such a way that it hits an east step with left endpoint having x-coordinate j. To see that this commutes with rotation, it will be easiest to think in terms of noncrossing partitions. In particular, given $(P, Q) \in NC(n+1, n)$ we obtain its corresponding $(1,2)$ configuration as follows: For each block B of P, draw an arc from $\min(B) - 1$ to $\max(B)$. If $\min(B) = 1$, draw an arc from $n-1$ to $\max(B)$. Each nontrivial block of Q will be a singleton $\{i\}$ of rank 1. Draw a ball at i. For an $n+1, n$-Dyck path D, this construction bijects $\pi(D)$ to $\tau(D)$. It follows that rotation of (P, Q) simply rotates its associated $(1,2)$ configuration. □

Hence, Theorem 4.2.4 specializes to Thiel's result when $(a, b) = (n+1, n)$. Figure 4.3 shows an example of a 7,6-Dyck path and its corresponding noncrossing (1,2) configuration.

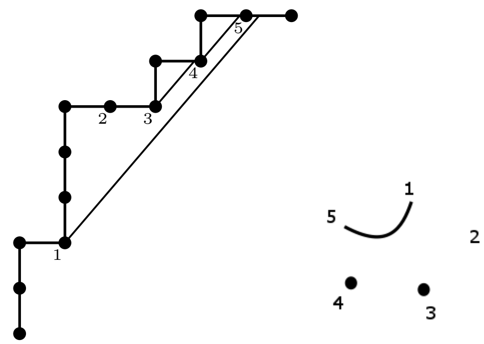


Figure 4.3: A 7,6-Dyck path and its corresponding noncrossing (1,2) configuration in X_6

This chapter contains material from “Rational Noncrossing Partitions for all Coprime Pairs”, to appear in *Journal of Combinatorics*, 2018. The dissertation author was the primary investigator and author of this paper.

This chapter contains material from “Cyclic Sieving and Rational Catalan Theory”, *Electronic Journal of Combinatorics*, v.23, 2016. The dissertation author and Brendon Rhoades were co-authors of this paper.

Chapter 5

Parking Functions

A classical *parking function* of length n is a map $f : [n] \to \mathbb{N}$ such that the increasing rearrangement $(b_1 \leq b_2 \leq \cdots \leq b_n)$ of the sequence $(f(1), f(2), \ldots, f(n))$ satisfies $b_i \leq i$. Let Park_n denote the set of parking functions of length n. The term parking function comes from a combinatorial interpretation [22] of such sequences. Namely, suppose n cars want to park in n linearly ordered parking spaces and that car i tries to park in space $f(i)$. If the spot is already occupied then car i parks in the first available spot after $f(i)$. If no such spot exists then i leaves the parking lot. The function f is called a parking function if every car is able to park.

Observe that Park_n carries an action of the symmetric group via $(w.f)(i) := f(w^{-1}(i))$, where $w \in \mathfrak{S}_n$. We have that $|\mathsf{Park}_n| = (n+1)^{n-1}$ and the number of $\mathfrak{S}_n$-orbits are counted by the Catalan number $\frac{1}{n+1}\binom{2n}{n}$. We may think of associating a particular orbit to an *increasing parking function*.

For example, Figure 5.1 show the $(3+1)^{3-1} = 16$ elements of Park_3 grouped into its $\mathsf{Cat}_3 = 5$ $\mathfrak{S}_3$-orbits, with the increasing parking functions listed first.

111					
112	121	212			
113	131	313			
122	212	221			
123	132	213	231	312	321

Figure 5.1: Park_3 grouped by $\mathfrak{S}_3$-orbits

Let W be an irreducible real reflection group with Coxeter number h. Armstrong, Reiner, and Rhoades defined a $W \times \mathbb{Z}_h$-set Park_w^{NC} called the set of W-*noncrossing parking functions* [5], a generalization of classical type A parking functions. In [30], Rhoades defines $k - W$-parking spaces, a Fuss analog of their work. In [13], the author and Rhoades provide a rational extension $\mathsf{Park}^{NC}(a, b)$ of [5] and [30] when W is the symmetric group $\mathfrak{S}_a$ and $a < b$. Though rational parking functions have been studied elsewhere in the literature [9], the action of $\mathfrak{S}_a \times \mathbb{Z}_{b-1}$ on parking functions had previously only been known in the case $a < b$. Here, we generalize to all coprime a and b, which gives evidence that $NC(a, b)$ gives the 'correct' definition of a rational noncrossing partition. Extending these results to other reflection groups remains an open problem.

For all coprime a and b, we define an a, b-*noncrossing parking function* as a pair $((P, Q), f)$ where $(P, Q) \in NC(a, b)$ and $f : \{B \mid B \in P \text{ or } B \in Q\} \to 2^{[a]}$ is a labeling of blocks of P and Q such that the following holds:

- $[a] = \bigsqcup_{B \in P \text{ or } B \in Q} f(B)$

- for all blocks B we have

$$|f(B)| = \begin{cases} \text{rank}_P(B) & \text{if } B \in P \\ \\ \text{rank}_Q(B) & \text{if } B \in Q. \end{cases}$$

Alternatively, we can view this as a labeling of the N steps of an a, b Dyck path by the numbers 1 through a, where the labels increase as one moves up a vertical run. We will refer to the set of all a, b-noncrossing parking functions as $\mathsf{Park}^{NC}(a, b)$. See Example 5.0.3 for a complete example.

Proposition 5.0.1. $\mathsf{Park}^{NC}(a, b)$ *carries an action of* $\mathfrak{S}_a \times \mathbb{Z}_{b-1}$ *where* $\mathfrak{S}_a$ *permutes block labels and* $\mathbb{Z}_{b-1}$ *rotates blocks.*

Proof. Rotation preserves vertical lengths, and thus ranks, by the definition of `rot` and Proposition 3.1.2. □

We would like to state a character formula for the action described in Proposition 5.0.1. The map $\phi : \mathfrak{S}_a \to \mathbb{C}$ by $\phi(w) = b^{\dim V^w}$ defines a class function for any pair of positive integers a and b, however it was proven by Ito and Okada [25] that ϕ is a character of $\mathfrak{S}_a$ if and only if $\gcd(b, a) = 1$. The author and Rhoades stated and proved a character formula in the case where $a < b$ and here we will extend to all coprime a and b, taking the generalization as far as it can possibly go in this setting.

Let $V = \mathbb{C}^a / \langle (1, \ldots, 1) \rangle$ be the reflection representation of $\mathfrak{S}_a$ and $\zeta = e^{\frac{2\pi i}{b-1}}$. Given $w \in \mathfrak{S}_a$ and $d \geq 0$, let $\text{mult}_w(\zeta^d)$ be the multiplicity of ζ^d as an eigenvalue in the action of w on V. With this notation, we have the following for the character χ:

Theorem 5.0.2. *Let $w \in \mathfrak{S}_a$ and g be a generator of $\mathbb{Z}_{b-1}$. Then we have*

$$\chi(w, g^d) = b^{\mathrm{mult}_w(\zeta^d)} \tag{1}$$

for all $w \in \mathfrak{S}_a$ and $d \geq 0$, where $\zeta = e^{\frac{2\pi i}{b-1}}$.

Proof. If $d|(b-1)$ we have

$$\mathrm{mult}_w(\zeta^d) = \begin{cases} \#(\text{cycles of } w) - 1 & \text{if } q = 1 \\ \#(\text{cycles of } w \text{ of length divisible by } q) & \text{otherwise} \end{cases} \tag{2}$$

where $q = \frac{b-1}{d}$. To see this, first suppose $q = 1$. Then $d = b - 1$ so $\zeta^d = 1$. The vectors which are fixed by the permutation matrix of w are precisely those which are constant on cycles of w. Deleting the all 1's vector leaves us with $\#(\text{cycles of } w) - 1$ such linearly independent vectors. Now suppose $q > 1$. Then the vectors which increase by a factor of 0 or ζ^d along cycles of length divisible by q, and which are 0 along cycles of length not divisible by q, are the eigenvectors with eigenvalue of ζ^d. Each cycle of length divisible by q contributes one such eigenvector.

We are now ready to count the number of a, b-noncrossing parking functions which are fixed under the action of (w, g^d). We will handle the cases $q = 1$ and $q > 1$ separately.

Case 1: $q = 1$. In this case, $g^d = g^{b-1} = 1$ so we can ignore the action of $\mathbb{Z}_{b-1}$ and just consider elements of $\mathsf{Park}^{NC}(a, b)$ which are fixed by $w \in \mathfrak{S}_a$. To do this, we will construct an equivariant bijection $f : \mathsf{Park}^{NC}(a, b) \to S$ and show that S has the desired character.

Let $\mathsf{Park}_{a,b}$ be the set of sequences $(p_1, \ldots, p_a)$ of positive integers whose nondecreasing rearrangement $(p'_1 \leq p'_2 \leq \cdots \leq p'_a)$ satisfies $p'_i \leq \frac{b}{a}(i-1)+1$. These are called *rational slope parking functions.* Our choice of S will be $\mathsf{Park}_{a,b}$. Then $\mathfrak{S}_a$ acts on $\mathsf{Park}_{a,b}$ by

$$w.(p_1, \ldots, p_a) = (p_{w(1)}, \ldots, p_{w(a)}).$$

In particular, w fixes precisely those parking functions which are constant on cycles of w. Let c_w denote the number of cycles of w and $\chi(w)$ denote the character of the action. There are b^{c_w} sequences of length a which are constant on cycles of w. By the cycle lemma, exactly one cyclic rotation of each of these will be a valid rational slope parking function, so we have

$$\chi(w) = b^{c_w - 1} = b^{\mathrm{mult}_w(1)}. \tag{3}$$

We are now left to build our equivariant bijection $\phi : \mathsf{Park}^{NC}(a,b) \to \mathsf{Park}_{a,b}$. Let $((P,Q),f)$ be an a,b-noncrossing parking function. Define $\phi((P,Q),f) = (p_1, \ldots, p_a)$ by

$$p_i = \begin{cases} \min(B) & \text{if } B \in P \text{ and } i \in f(B) \\ \\ \max(B)+1 & \text{if } B \in Q \text{ and } i \in f(B). \end{cases} \tag{4}$$

Equivalently, one can think of the pair $((P,Q),f)$ as an a,b-Dyck path where the north steps are labeled by the numbers 1 through a and each vertical run has increasing labels. The underlying dyck path D is such that $\pi(D) = (P,Q)$, and the labels on a particular vertical run that determine the rank of a block B are given by $f(B)$.

Example 5.0.3. *Consider the labeled 9,4-Dyck path shown in Figure 5.2. This corresponds to the partitions $P = \{\{1,2,3\}\}$ with rank 3 and $Q = \{\{1\},\{2\},\{3\}\}$ each* *with rank 2, and the function f defined by $f(\{1,2,3\}) = \{3,5,6\}$, $f(\{1\}) = \{1,8\}$, $f(\{2\}) = \{4,9\}$, and $f(\{3\}) = \{2,7\}$. The associated rational slope parking function $(p_1,\ldots,p_a)$ is $(2,4,1,3,1,1,4,2,3)$. This can be read off from $((P,Q),f)$ via equation 4 or by setting p_i equal to 1 greater than the x coordinate of the north step labeled by i. From this point of view, and the fact that D must stay above the line $y = \frac{a}{b}x$, we see that for all i, we must have $\frac{i-1}{p_i'-1} \geq \frac{a}{b}$ which is equivalent to the condition that $p_i' \leq \frac{b}{a}(i-1)+1$. In other words, $(p_1,\ldots,p_a)$ is indeed a sequence in $\mathsf{Park}_{a,b}$.*

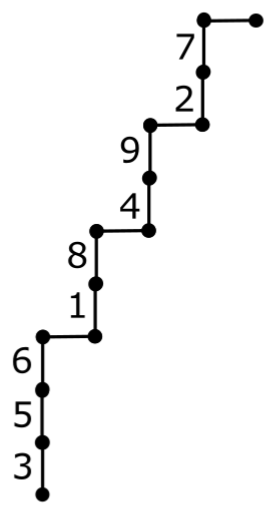


Figure 5.2: A 9,4-noncrossing parking function

Conversely, suppose we are given $(p_1,\ldots,p_a)$. Let n_i denote the number of entries of $(p_1,\ldots,p_a)$ which are equal to i. Then we can recover the labeled Dyck path D by setting

$$D = N^{n_1}EN^{n_2}E\cdots N^{n_a}E$$

and labeling the vertical run with x-coordinate $i-1$ by the numbers in $\{i \mid p_i = 1\}$ in increasing order. Since permuting labels i and j on the Dyck path corresponds to swapping p_i and p_j, we conclude that ϕ is in fact an equivariant bijection. Since ϕ preserves character formulas, equation 3 implies that $\chi(w, g^d) = b^{\text{mult}_w(\zeta^d)}$ when $q = 1$.

Case 2: $q > 1$. Let $r_q(w)$ denote the number of cycles of w having length divisible by q. We will show that

$$|\mathsf{Park}^{NC}(a,b)^{(w,g^d)}| = |\{p \in \mathsf{Park}^{NC}(a,b) | (w,g^d).p = p\}| = b^{r_q(w)}. \quad (5)$$

To do this, we will show that both sides count a certain set of functions. First, define an action of g on $[b-1] \cup \{0\}$ by the permutation $(1, 2, \ldots, b-1)(0)$. We say a function $e : [a] \to [b-1] \cup \{0\}$ is (w, g^d)-equivariant if

$$e(w(j)) = g^d e(j) \quad (6)$$

for all $1 \leq j \leq a$. To count such functions, we first consider what happens on cycles of w. By equation 6, if $e(k) \neq 0$ then we have $e(w(k)) = e(k) + d$, where addition is performed modulo $b - 1$. Thus, the values e takes on a cycle are completely determined by the value taken on one element of that cycle. Further, if a cycle has length not divisible by q then equation 6 forces $e(k) = 0$ for any k in that cycle. Thus, the number of (w, g^d)-equivariant functions is $b^{r_q(w)}$.

For example, let $(a, b) = (14, 13)$, $q = 3$, and consider

$$w = (5, 1, 8)(2, 3, 6, 7, 9, 10)(4, 11)(12, 13, 14),$$

written in cycle notation. Let e be the function defined by

$$(e(1), e(2), \ldots e(14)) = (9, 1, 5, 0, 5, 9, 1, 1, 5, 9, 0, 2, 6, 10).$$

Note, for instance, how $w(5) = 1$ and $9 = e(1) = 5 + 4 = e(5) + d$. In this example, e is indeed a (w, g^d)-equivariant function.

Next, we count equivariant functions according to their fiber structure. We say a set partition $\sigma = \{B_1, B_2, \ldots\}$ of $[a]$ is (w, q)-*admissible* if the following conditions hold:

1. σ is w-stable. ie, $w(\sigma) = \{w(B_1), w(B_2), \ldots\} = \sigma$

2. There is at most one block B_{i_0} such that $w(B_{i_0}) = B_{i_0}$

3. For any block B_i which is not w-stable, the blocks

$$B_i, w(B_i), \ldots, w^{q-1}(B_i)$$

are pairwise distinct, and we have $w^q(B_i) = B_i$.

Given a (w, g^d)-equivariant function e, define a set partition by σ by $i \sim j$ if and only if $e(i) = e(j)$. For instance, consider the example given above. Then we have

$$\sigma = \{\{4, 11\}, \{2, 7, 8\}, \{12\}, \{3, 5, 9\}, \{13\}, \{1, 6, 10\}, \{14\}\}.$$

In general, σ is w-stable because if $i \neq 0$ and $B = e^{-1}(i)$ then $w(B) = e^{-1}(i+d)$, and if $i = 0$ then $w(B) = B$. Furthermore, $e^{-1}(0)$ is the only block which is fixed, so (2) is satisfied. Lastly, since $d, 2d, \ldots, (q-1)d$ are distinct, this means $e^{-1}(i), w(e^{-1}(i)), \ldots,$ and $w^{q-1}(e^{-1}(i))$ are distinct, and $w^q(e^{-1}(i)) = e^{-1}(i) + qd = e^{-1}(i)$ since arithmetic is performed modulo $b - 1$.

Each w-stable orbit is of size q or size 1, depending on whether its blocks come from the inverse image of nonzero numbers or not. Given a particular fiber structure and

w, consider how many (w, g^d)-equivariant functions could give rise to such a structure. There are $b-1$ choices for how to map some element of the first orbit. In our example, given the orbit containing $\{2,7,8\}$, $\{3,5,9\}$, and $\{1,6,10\}$, we have $b-1$ ways to assign $e(2)$, which then forces $e(7) = e(2)$, $e(8) = e(2)$, $e(3) = e(2) + d$, and so on. Once this choice is made, the value of e is determined for all elements in the orbit. Since orbits are of size q, this eliminates q possible assignments from the next orbit we consider. In our example, this would give us $b - 1 - q = 9$ choices for $e(12)$ in the orbit $\{12\}, \{13\}, \{14\}$. More generally, if we let t_σ denote the number of w-orbits in σ of size q then we have

$$(b-1)(b-1-q)\cdots(b-1-(t_\sigma-1)q)$$

(w, g^d)-equivariant functions corresponding to a (w, q)-admissible set partition σ. Thus, there are

$$\sum_{\substack{\sigma \text{ a } (w,q)\text{–admissible} \\ \text{partition}}} (b-1)(b-1-q)\cdots(b-1-(t_\sigma-1)q) = b^{r_q(w)} \tag{7}$$

(w, g^d)-equivariant functions.

To relate this back to parking functions, fix $((P,Q), f) \in \mathsf{Park}^{NC}(a,b)$. Let $\tau((P,Q), f)$ be the set partition of $[a]$ defined by $i \sim j$ if and only if $i, j \in f(B)$. In Example 5.0.3 we recover the set partition $\tau((P,Q), f) = \{\{3,5,6\},\{1,8\},\{4,9\},\{2,7\}\}$. More generally, if $((P,Q), f)$ is an element of $\mathsf{Park}^{NC}(a,b)^{(w,g^d)}$ then $\tau(\pi, f)$ is a (w, q)-admissible set partition. It is w-stable because if $i \sim j$, then $i \sim j$ after we apply g^d, so w must also keep i and j in the same block. There is at most one central block in (P,Q) so at most one B such that $w(B) = B$. Finally, if $((P,Q), f) \in \mathsf{Park}^{NC}(a,b)^{(w,g^d)}$ then

w behaves like $\texttt{rot}^{-d}$ on (P, Q), which proves that $\tau((P, Q), f)$ is (w, q)-admissible.

Given a (w, q)-admissible partition σ of $[a]$, we will count how many $((P, Q), f) \in \mathsf{Park}^{NC}(a, b)^{(w,g^d)}$ are such that $\tau((P, Q), f) = \sigma$. We begin by constructing the underlying a, b-noncrossing partition pair (P, Q). If σ has m_i non-singleton w-orbits of blocks of size i, then (P, Q) must have m_i $\texttt{rot}^d$-orbits of non-central blocks of rank i. By Corollary 4.1.11, there are

$$\binom{d}{m_1, \ldots, m_a, d - t_\sigma}$$

such $(P, Q) \in NC_d(a, b)$. It now only remains to define f. The $\texttt{rot}^d$-orbits of noncentral blocks of (P, Q) of rank i must be paired with nonsingleton w-orbits of blocks of σ of size i. For each i, there are $m_i!$ ways to perform this matching. Each orbit has size q, so there are q ways to choose which block determines labeling of the first blocks in a noncentral $\texttt{rot}^d$ orbit. Thus, the number of $((P, Q), f) \in \mathsf{Park}^{NC}(a, b)^{(w,g^d)}$ such that $\tau((P, Q), f) = \sigma$ is given by

$$\begin{aligned}
q^{m_1} q^{m_2} \cdots q^{m_a} m_1! m_2! \cdots m_a! &\binom{d}{m_1, m_2, \ldots, d - t_\sigma} \\
&= q^{m_1} q^{m_2} \cdots q^{m_a} \frac{d!}{(d - t_\sigma)!} \\
&= q^{t_\sigma} d(d-1)(d-2) \cdots (d - (t_\sigma - 1)) \\
&= (b-1)(b-1-q) \cdots (b-1-(t_\sigma - 1)q).
\end{aligned}$$

Summing over all (w, q)-admissible partitions gives equation 7, so we conclude that 5 holds as desired. □

Theorem 5.0.2 can be used to generalize Theorem 6.3 in [13] to all coprime a and

b. In particular, we obtain a rational analog of the Generic Strong Conjecture of [31] in type A for any coprime pair (a, b). Following the definitions and terminology given in [31], the following holds:

Theorem 5.0.4. *Let* $\mathcal{R} \subset \mathsf{Hom}_{\mathbb{C}[\mathfrak{S}_a]}(V^*, \mathbb{C}[V]_b)$ *denote the set of* $\Theta \in \mathsf{Hom}_{\mathbb{C}[\mathfrak{S}_a]}(V^*, \mathbb{C}[V]_b)$ *such that the parking locus* $V^\Theta(b) \subset V$ *cut out by the ideal*

$$\langle \Theta(x_1) - x_1, \ldots, \Theta(x_{a-1}) - x_{a-1} \rangle \subset \mathbb{C}[V]$$

is reduced, where $x_1, \ldots, x_{a-1}$ *is any basis of* V^*. *For any* $\Theta \in \mathcal{R}$, *there exists an equivariant bijection of* $\mathfrak{S}_a \times \mathbb{Z}_{b-1}$*-sets*

$$V^\Theta(b) \simeq_{\mathfrak{S}_a \times \mathbb{Z}_{b-1}} \mathsf{Park}^{NC}(a, b).$$

There also exists a nonempty Zariski open subset $\mathcal{U} \subseteq \mathsf{Hom}_{\mathbb{C}[\mathfrak{S}_a]}(V^*, \mathbb{C}[V]_b)$ *such that* $\mathcal{U} \subseteq \mathcal{R}$.

The proof is again a recreation of sections [31, Sections 4, 5]. The only difference now is that we replace the reference to the proof of [30, Lemma 8.5] in the proof of [31, Lemma 4.6] with the corresponding argument in the proof of Theorem 5.0.2.

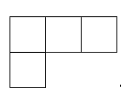
This chapter contains material from "Rational Noncrossing Partitions for all Coprime Pairs", to appear in *Journal of Combinatorics*, 2018. The dissertation author was the primary investigator and author of this paper.

Chapter 6

Generalizations

6.1 A bijection

We'll now proceed to give a group theoretic construction of rational Catalan objects in type A. Fix positive coprime integers a and b. A Ferrers diagram $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, written using English notation, is a set of boxes which are left and top justified such that the i^{th} row contains λ_i boxes, and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$. Let $\lambda(a, b)$ be the largest Ferrers diagram which fits inside an $a \times b$ rectangle and such that each of its boxes stays entirely above the line $y = \frac{a}{b}x$. For example,

$\lambda(3,5) =$  .

For a permutation $\sigma \in \mathfrak{S}_n$, the *Rothe diagram* of σ is the set of boxes inside the

n-staircase $(n-1, n-2, \ldots, 2, 1)$ given by

$$D(\sigma) = \{(i, \sigma_j) \mid i < j \text{ and } \sigma_i > \sigma_j\}.$$

For a given λ, there is a unique permutation, called $\sigma(\lambda) \in \mathfrak{S}_n$, such that $D(\sigma(\lambda)) = \lambda$, and n is minimal among n-staircases containing λ. To construct $\sigma(\lambda)$, let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$. Let c_j be the product of transpositions

$$c_j = (j, j+1)(j-1, j) \cdots (2,3)(1,2)$$

and let $c_j + m$ denote the shifted product

$$c_j + m = (j+m, j+m+1)(j+m-1, j+m) \cdots (2+m, 3+m)(1+m, 2+m).$$

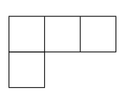
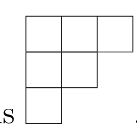
Then we have

$$\sigma(\lambda) = c_{\lambda_1}(c_{\lambda_2}+1)(c_{\lambda_3}+2) \cdots (c_{\lambda_{k-1}}+k-2)(c_{\lambda_k}+k-1).$$

For example, we have $\sigma(\lambda(3,5)) = 4213 \in \mathfrak{S}_4$ since

$$D(4213) = \{(1,2), (1,1), (1,3), (2,1)\}$$

and $\lambda(3,5)$ fits inside the staircase $(3,2,1)$.

$\lambda(3,5) =$  and the 4-staircase is  .

Let $\sigma'(a,b) = \sigma(\lambda(a,b)) = \sigma'_1\sigma'_2 \cdots \sigma'_n$ in one-line notation, and define

$$\sigma(a,b) = \sigma_1\sigma_2 \cdots \sigma_{n+1}$$

where $\sigma_1 = 1$ and $\sigma_i = \sigma'_{i-1} + 1$. In our example, $\sigma(3,5) = 15324$.

Given $\sigma \in \mathfrak{S}_n$, define the set of compatible sequnces of σ, $\mathrm{CS}(\sigma)$, to be all $2 \times \ell$ arrays with positive integer entries

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_\ell \\ b_1 & b_2 & \cdots & b_\ell \end{bmatrix}$$

such that

1. $a_1 \leq a_2 \leq \cdots \leq a_\ell$
2. If $a_i = a_{i+1}$ then $b_i > b_{i+1}$
3. $b_1 b_2 \cdots b_\ell$ is a reduced word for σ, where i denotes the adjacent transposition $(i, i+1)$
4. $a_i \leq b_i$ for all $1 \leq i \leq \ell$.

Proposition 6.1.1. *We have* $|CS(\sigma(a,b))| = \mathsf{Cat}(a,b)$.

Proof. In [32] a bijection is given from $CS(\sigma(a,b))$ to the set of 1-flagged tableaux of shape $\lambda(a,b)$. Subtracting i from the i^{th} row of a 1-flagged tableau gives a 0-1 tableau. The zeros indicate a Ferrers diagram $\mu \subseteq \lambda(a,b)$. Since each a,b-Dyck path is uniquely determined by the Ferrers diagram inside $\lambda(a,b)$ it cuts out from the northwest corner of the $a \times b$ rectangle which encloses it, this proves the result. □

We will now explain the explicit bijection between $CS(\sigma(a,b))$ and a,b-Dyck paths [32]. To understand the bijection, we will first need pipe dreams, flagged tableaux, and Edelman-Greene insertion [18]. A *pipe dream* of size n is a filling of the boxes staircase shape $(n-1, n-2, \ldots, 2, 1)$ with the symbols $+$ and ╯╭. Given a pipe dream D, we may consider its associated permutation $\pi(D)$. To determine $\pi(D)$, write the numbers 1 to $n-1$ across the top of the Ferrers diagram. Next, for each number i written, follow the pipe which starts at i until where it ends, which will occur at the left edge of a box in the first column of the diagram. Label the left side of that box i as well. Then the one line notation for $\pi(D)$ is obtained by reading the numbers to the left of the Ferrers diagram from top to bottom. We call a pipe dream *reduced* if any two of its pipes cross at most once. Reduced pipe dreams also appear under the name rc-graphs [19, 8]. Note that a single pipe may cross many other pipes, but it will never cross a particular other pipe twice. Let $\mathcal{RP}(\pi)$ denote the set of reduced pipe dreams for a permutation π.

Next, define a flagged tableau to be a semistandard tableau in which the entries in row i are all either i or $i+1$. Let $\mathcal{FT}(\lambda)$ denote the set of flagged tableaux of shape λ. Let D be a reduced pipe dream for $\sigma \in \mathfrak{S}_n$. Define the *reading biword* of σ to be the $2 \times \ell$ array created as follows: Read D from right to left and top to bottom. When a $+$ is encountered at position (i, j) in D, add the column $\begin{bmatrix} i \\ i+j-1 \end{bmatrix}$ to the reading word. Thus, ℓ is the number of crosses in D, or equivalently the inversion number of σ. It is known that this map is in fact a bijection from $\mathcal{RP}(\sigma)$ to $\mathrm{CS}(\sigma)$ [11].

We'll perform Edelman-Greene (column) insertion on the second row of the reading biword. Recall that Edelman-Greene insertion works as follows: Let T =

$C_1, C_2, \ldots, C_k$ be a tableau with columns $C_1, C_2, \ldots, C_k$. To insert m into T:

1. If T is the empty tableau, append m to C_1.

2. If n is larger than the largest entry in C_1, append n to C_1.

3. Otherwise, let c_i denote the smallest and uppermost entry in C_1 such that $m < c_i$

   (a) If C_1 contains both m and $m+1$, Edelman-Greene insert $m+1$ into

   $$T' = C_2, C_3, \ldots, C_k.$$

   (b) Otherwise, replace c_i by m and Edelman-Greene insert m into

   $$T' = C_2, C_3, \ldots, C_k.$$

As with the usual RSK algorithm, one records in a recording tableau the position of the i^{th} box added as a result of inserting the i^{th} integer into an empty tableau. Let D be a reduced pipe dream and

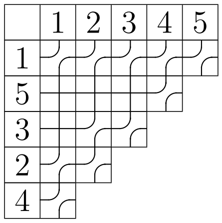
$$\begin{bmatrix} a_1 & a_2 & \cdots & a_\ell \\ b_1 & b_2 & \cdots & b_\ell \end{bmatrix}$$

be the reading biword for D. Edelman-Greene insert $b_1, b_2, \ldots, b_\ell$ into the empty tableau, and use the entries $a_1, a_2, \ldots, a_\ell$ to fill the recording tableau. Let $P(D)$ denote the insertion tableau and $Q(D)$ denote the recording tableau. By [32] $Q(D)$ is a flagged tableau. Serrano and Stump proved that the map sending $D \in \mathcal{RP}(\sigma)$ to the recording tableau of the reading biword of D is a bijection between $\mathcal{RP}(\sigma(\lambda))$ and $\mathcal{FT}(\lambda)$.

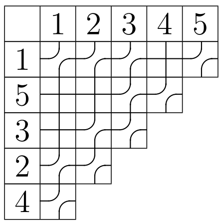
Given an element of $T \in \mathcal{FT}(\lambda)$, subtract i from the i^{th} row of T. Then T is a 0-1 semistandard tableau. Superimposing T onto an $a \times b$ rectangle, there is a unique Dyck path which stays above every box containing 1 and below every box containing 0. Since every rational Dyck path is determined by the boxes it cuts out of $\lambda(a, b)$, we must have that $|\mathcal{RP}(\sigma(a, b))| = |\mathrm{CS}(\sigma(a, b))| = \mathsf{Cat}(a, b)$.

We give here all reduced pipe dreams for $\sigma(3, 5)$, along with their associated reading biwords, insertion tableaux, and recording tableaux.

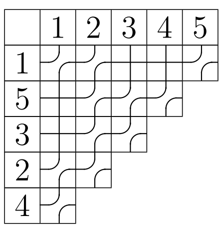
Reduced Pipe Dream	Compatible Sequence	Insertion Tableau	Recording Tableau



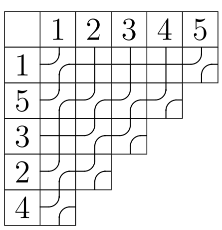
$$\begin{bmatrix} 2 & 2 & 2 & 3 \\ 4 & 3 & 2 & 3 \end{bmatrix} \qquad \begin{array}{|c|c|c|}\hline 2 & 3 & 4 \\ \hline 3 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline 2 & 2 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$$



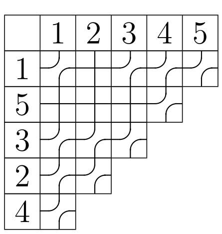
$$\begin{bmatrix} 1 & 2 & 2 & 3 \\ 4 & 3 & 2 & 3 \end{bmatrix} \qquad \begin{array}{|c|c|c|}\hline 2 & 3 & 4 \\ \hline 3 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$$



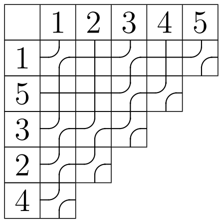
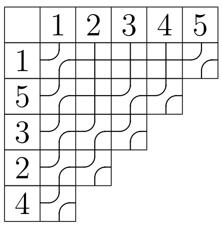
$$\begin{bmatrix} 1 & 1 & 2 & 3 \\ 4 & 3 & 2 & 3 \end{bmatrix} \qquad \begin{array}{|c|c|c|}\hline 2 & 3 & 4 \\ \hline 3 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$$



$$\begin{bmatrix} 1 & 1 & 1 & 3 \\ 4 & 3 & 2 & 3 \end{bmatrix} \qquad \begin{array}{|c|c|c|}\hline 2 & 3 & 4 \\ \hline 3 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array}$$



$$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 3 & 2 \end{bmatrix} \qquad \begin{array}{|c|c|c|}\hline 2 & 3 & 4 \\ \hline 3 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}$$

Reduced Pipe Dream	Compatible Sequence	Insertion Tableau	Recording Tableau
	$\begin{bmatrix} 1 & 1 & 2 & 2 \\ 4 & 2 & 3 & 2 \end{bmatrix}$	$\begin{array}{\|c\|c\|c\|}\hline 2 & 3 & 4 \\ \hline 3 \\ \cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|c\|}\hline 1 & 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}$
	$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 4 & 3 & 2 & 3 \end{bmatrix}$	$\begin{array}{\|c\|c\|c\|}\hline 2 & 3 & 4 \\ \hline 3 \\ \cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|c\|}\hline 1 & 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}$

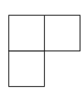
To map the recording tableau

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}$$

to a Dyck path, we first subtract i from the i^{th} row. This gives the $0-1$ tableau

$$\mu = \begin{array}{|c|c|c|}\hline 0 & 0 & 1 \\ \hline 0 \\ \cline{1-1}\end{array}.$$

The $(3,5)$ Dyck path which cuts out the tableau



formed by the zeros of μ is shown in Figure 6.1.

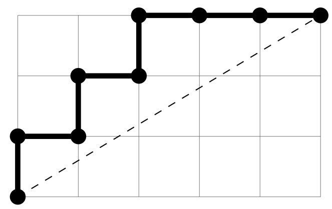


Figure 6.1: 3,5-Dyck Path which cuts out μ

6.2 Future Work

The next step in this research is to generalize the results given here to other reflection groups. Given a reflection group W, Reiner [26] defined a W-noncrossing partition which reduces precisely to our notion of noncrossing partition when $W = \mathfrak{S}_a$ and $(a, b) = (a, a+1)$. Explicitly, let $\text{Abs}(W)$ denote the poset of W under the absolute order. Recall that the poset of noncrossing partitions of W is

$$NC(W, c) = [1, c]$$

where $c \in W$ is a Coxeter element. Since $[1, c] \cong [1, c']$ for any choice of Coxeter elements c and c', we may simply write $NC(W)$. When $W = \mathfrak{S}_a$, we have that $NC(\mathfrak{S}_a)$ is just the usual poset of noncrossing partitions of $[a]$, ordered by refinement. When $(a, b) = (n, kn + 1)$ for some positive integer k, $NC(a, b)$ consists of noncrossing partitions with block sizes divisible by k. Armstrong [3] studied a Fuss-Catalan version of these k-divisible noncrossing partitions which made sense for any reflection group W by considering k-multichains in the lattice of noncrossing partitions $NC(W)$. It is then natural to ask whether the results obtained here for rational noncrossing partitions may be generalized to other reflection groups. In type B, where W is the group of signed permutations, the combinatorial model [5, Section 6] for noncrossing partitions is the centrally symmetric partitions of $\pm[n]$, those for which at most one block is sent to itself by n-fold rotation. Central symmetry is a concept that makes sense even for rational noncrossing partitions, so there is hope to extend these results to the rational case in type B. However, it is less clear what to do for the other Coxeter groups, and would be nice to have a uniform approach for defining and working with rational noncrossing partitions for any Coxeter group.

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