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Rigidification of Algebras Over Algebraic Theories in Diagram Categories

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Rigidification of Algebras Over Algebraic Theories in Diagram Categories

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by

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ABSTRACT OF THE DISSERTATION

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by

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Doctor of Philosophy, Graduate Program in Mathematics
University of California, Riverside, June 2018
Professor Julie Bergner, Co-Chairperson
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The notion of an algebraic theory, which is able to describe many algebraic structures, has been used extensively since its introduction by Lawvere in 1963. This perspective has been very fruitful for understanding in a wide variety of algebraic structures, including rigidification results for simplicial algebras over algebraic theories by Badzioch and Bergner. In this thesis, we extend the rigidification results to algebras over a larger class of categories, which includes bisimplicial sets. In particular, we prove the rigidification result is true in any diagram category $\mathcal{SS}ets^{\mathcal{C}^{op}}$ for a small category $\mathcal{C}$.

Contents

Chapter 1

Introduction

In 1963, Lawvere introduced the notion of an algebraic theory in [13]. The development of these algebraic theories provided a syntax to describe a variety of algebraic structures by focusing on the structure maps of these objects as opposed to the objects themselves. A benefit of utilizing this equivalent interpretation of algebraic categories is that it provides a functorial description of the objects in these categories rather than focusing on particular generating sets and relations. This approach provides a much more categorical interpretation of such categories and facilitates the study of these structures in the context of category theory. In Lawvere's paper, he proves that for categories for which an algebraic theory exists, an algebraic category $\mathcal{C}$ is equivalent to the category of product-preserving functors from the theory into the category of sets. Many of the categories we would expect to be "algebraic", such as the category of groups, are indeed algebraic categories.

Additionally, one can build these algebraic objects in categories other than that of sets. For instance, functors from the algebraic theory of groups into the category of spaces are models for group objects in the category of spaces, i.e. topological groups. However, there are many natural structures, such as loop spaces or H-spaces, which are not topological groups, but whose structure holds as a group "up to homotopy". In [3] and [6], Badzioch and Bergner proved that an algebraic object defined up to homotopy is weakly equivalent to an algebraic object whose maps are strict. That is to say, given a weakly defined structure on a topological space X, we can pass to a space X' equipped with a strict structure that is homotopy equivalent to X.

In this paper, we generalize this rigidification result to a larger class of categories. In particular, we will prove that it holds for diagram categories of the form $\mathcal{M} = \mathcal{SS}ets^{\mathcal{C}^{op}}$, where $\mathcal{C}$ is any small category. Recall that the category of simplicial sets, denoted $\mathcal{SS}ets$, provides a combinatorial analogue of topological spaces.

For some motivation for this paper, we consider the work of Bergner and Rezk in [8], as they proved a rigidification result for algebras in a specific collection of diagram categories, $\mathcal{SS}ets^{\Theta_n^{op}}$. Here Θ_n is an iterated wreath product of the simplex category Δ [5]. When $n = 1$, we have $\Theta_1 = \Delta$, which corresponds to the category of simplicial spaces, i.e. the category of bisimplicial sets.

1.1 Organization

In Section 2, we discuss many of the definitions and foundational prerequisites to provide the contextual background of our problem. We first introduce the necessary definitions for model categories, followed by a discussion of the previous rigidification results of Badzioch and Bergner. In Section 3, we show that the diagram categories $\mathcal{M} = \mathcal{SS}ets^{\mathcal{C}^{op}}$ satisfy the necessary conditions to support the rigidification result. In Section 4, we discuss the model structures that we must impose on the categories of strict as well as homotopy algebras over our algebraic theories. Finally in Section 5, we develop the main result, namely to determine that the two model categories described in Section 4 are in fact Quillen equivalent.

1.2 Future Work

While the result found in the thesis expands the rigidification result to a wide class of categories, there are many categories for which that question is still uncertain. One such collection would be to extend this rigidification result to the category of all small categories, where a notion of a strong and weak algebraic structure is already understood and well studied.

Chapter 2

Background

2.1 Overview of model categories

In this section, we give a brief overview of the necessary tools for model categories that we need. A more extensive treatment of model categories can be found in [11] or [12]. We begin by introducing the definition of a model category.

Definition 2.1. *[11, 7.1.3] A* model category *is a category $\mathcal{M}$ with three distinguished classes of maps, labeled* weak equivalences*,* fibrations*, and* cofibrations*. Each of these classes is closed under composition and contains all identity maps. A map which is both a fibration (resp. cofibration) and a weak equivalence is called a* acyclic fibration *(resp.* acyclic cofibration*). Additionally, we require the following axioms to hold.*

MC1 *(Limit axiom) The category $\mathcal{M}$ admits all small limits and colimits.*

MC2 *(Two-out-of-three axiom) If f and g are maps in $\mathcal{M}$ such that gf is well-defined and if two of the three maps f, g, gf are weak equivalences, then so is the third.*

MC3 *(Retract axiom) If f is a retract of g and g is a fibration, cofibration, or a weak equivalence, then so is f.*

MC4 *(Lifting axiom) Given a commutative diagram*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "g"'] \arrow[ur, dashed, "h"] & Y \end{tikzcd}$$

a lift $h : B \to X$ exists in the diagram in either of the following situations:

1. *the map i is a cofibration and p is an acyclic fibration, or*
2. *the map i is an acyclic cofibration and p is a fibration.*

MC5 *(Factorization axiom) Any map f can be factored in two ways:*

1. *$f = pi$, where i is a cofibration and p is an acyclic fibration, or*
2. *$f = pi$, where i is an acyclic cofibration and p is a fibration.*

We say that an object X is fibrant *if the unique map from X into the terminal object is a fibration. Likewise, an object Y is said to be* cofibrant *if the unique map from the initial object into Y is a cofibration.*

At an intuitive level, we can understand that imposing a model category structure on a category allows us to consider homotopy theory in a context more general than that of topological spaces. Notice that by the lifting axiom, if any two of the three classes of maps are provided, then the remaining class of maps can be described by the lifting property.

In addition to the model structure, the underlying category may also be equipped with a monoidal structure. In the case where the monoidal product is given by the cartesian product, we can rephrase the definition of a closed monoidal category [7, 1.1.36] in the following manner.

Definition 2.2. *A category* $\mathcal{C}$ *is said to be* cartesian closed *if it has finite products and, for any two objects* X *and* Y *of* $\mathcal{C}$*, an internal function object* Y^X*, together with a bijection*

$$\mathrm{Hom}_{\mathcal{C}}(Z, Y^X) \cong \mathrm{Hom}_{\mathcal{C}}(Z \times X, Y)$$

for any third object Z *of* $\mathcal{C}$*.*

When a category that is cartesian closed also has a model category structure imposed on it, it is important to consider the compatibility of these two structures, which we have as follows.

Definition 2.3. *A model category* $\mathcal{M}$ *is* cartesian *if its underlying category is cartesian closed, its terminal object is cofibrant, and the following equivalent conditions hold.*

1. *If* $f\colon A \to A'$ *and* $g\colon B \to B'$ *are cofibrations in* $\mathcal{M}$*, then the induced map*

$$h\colon A \times B' \coprod_{A \times B} A' \times B \to A' \times B'$$

is a cofibration. If either f *or* g *is a weak equivalence, then so is* h*.*

2. *If $f\colon A \to A'$ is a cofibration and $p\colon X \to X'$ is a fibration in $\mathcal{M}$, then the induced map*

$$q\colon (X')^{A'} \to (X')^{A} \times_{X^A} X^{A'}$$

is a fibration. If either f or g is a weak equivalence, then so is h.

Of particular importance for the purposes of this paper is the localization of these cartesian model categories with respect to a collection of maps P. In order for such a localization to result in a suitable model category, or even to ensure the localization exists, we first require that the model categories satisfy the following condition.

Definition 2.4. *A model category is said to be* cofibrantly generated *if*

1. *there exists a set I of maps (called a set of* generating cofibrations*) that permits the small object argument [9, 7.12] and such that a map is an acyclic fibration if and only if it has the right lifting property with respect to every element of I, and*

2. *there exists a set J of maps (called a set of* generating acyclic cofibrations*) that permits the small object argument and such that a map is a fibration if and only if it has the right lifting property with respect to every element of J.*

Additionally, to make sure that the lifting axioms of the model category carry over to the localization, it is necessary to have the model category be left proper.

Definition 2.5. *A model category $\mathcal{M}$ is said to be* left proper *if every pushout of a weak equivalence along a cofibration is a weak equivalence.*

Given a left proper cofibrantly generated model structure, there are two sufficient conditions that ensure that the localized model structure is indeed a model structure. The key feature of each of these additional structures is that they both provide the necessary control over the generating cofibrations in the model category structure. The first sufficient condition is that the model structure be combinatorial, and the second is that the model category be cellular.

Definition 2.6. *A model category $\mathcal{M}$ is* combinatorial *if:*

1. *it is cofibrantly genererated,*
2. *it admits all small colimits, and*
3. *there is a set S of small objects of $\mathcal{M}$ such that any object of $\mathcal{M}$ can be obtained as a colimit of a small diagram with objects in S.*

Definition 2.7. *A* celluar model category *is a cofibrantly generated model category $\mathcal{M}$ for which there are a set I of generating cofibrations and a set J of generating acyclic cofibrations such that*

1. *both the domains and the codomains of the elements of I are compact [11, 11.4.1],*
2. *the domains of the elements of J are small relative to I [11, 10.5.12], and*
3. *the cofibrations are effective monomorphisms [11, 10.9.1].*

2.2 Overview of algebraic theories

In this section we discuss the necessary background needed in the study of algebraic theories within the context of this paper. We begin by providing the definition of an algebraic theory.

Definition 2.8. *For a given set S, we say an S-*sorted algebraic theory *(or a* multi-sorted theory*) $\mathcal{T}$ is a small category with objects $T_{\underline{\alpha}^n}$ with $n \geq 0$, $\underline{\alpha}^n = \langle \alpha_1, \ldots, \alpha_n \rangle$ for $\alpha_i \in S$, and for each $T_{\underline{\alpha}^n}$, an isomorphism*

$$T_{\underline{\alpha}^n} \cong \prod_{i=1}^{n} T_{\alpha_i}.$$

The entries α_i may repeat for some $\underline{\alpha}^n$, but they are not ordered. There exists a terminal object T_0, which corresponding to the empty subset of S.

For the remainder of the paper, we refer to any multi-sorted theory as an algebraic theory, or more simply as a theory. Note that an algebraic theory with one sort coincides with the definition found in [3, 1.1]. To define algebras over these theories as in [3] and [6], we must define a simplicial set. Recall that the category Δ has finite ordered sets as objects and weakly-order preserving functions as morphisms .

Definition 2.9. *A* simplicial set *is a functor from $\Delta^{op} \to \mathcal{S}ets$.*

The category of simplicial sets, denoted by $\mathcal{SS}ets$, has a model category structure where the weak equivalences are the maps $f : X \to Y$ such that the induced map $|f| : |X| \to |Y|$, which takes a simplicial set X to its geometric realization $|X|$, is a weak equivalence of topological spaces, the cofibrations are monomorphisms, and the fibrations have the right lifting property with respect to acyclic cofibrations [10, 11.3]. With this model structure in mind, Badzioch and Bergner defined a collection of functors out of an algebraic theory and into the category of simplicial sets that preserved the product structure.

Definition 2.10. *Given an S-sorted theory $\mathcal{T}$, a* strict $\mathcal{T}$-algebra *is a product-preserving functor $A : \mathcal{T} \to \mathcal{S}Sets$, where product-preserving in this setting means that the map*

$$A(T_{\underline{\alpha}^n}) \to \prod_{i=1}^{n} A(T_{\alpha_i}),$$

induced by the projections $T_{\underline{\alpha}^n} \to T_{\alpha_i}$ for all $1 \le i \le n$, is an isomorphism in $SSets$.

The collection of all $\mathcal{T}$-algebras forms a small category, which is denoted by $\mathcal{A}lg^{\mathcal{T}}$. The other type of functors that are important throughout the paper are those functors that preserve the structure of the algebraic theory, but only up to homotopy.

Definition 2.11. *Given an S-sorted theory $\mathcal{T}$, a* homotopy $\mathcal{T}$-algebra *is a functor X : $\mathcal{T} \to \mathcal{S}Sets$ which preserves products up to homotopy, i.e. for any $\alpha \in S^n$, the canonical map*

$$X(T_{\underline{\alpha}^n}) \to \prod_{i=1}^{n} X(T_{\alpha_i})$$

induced by the projection maps $T_{\underline{\alpha}^n} \to T_{\alpha_i}$ (for each $1 \le i \le n$) is a weak equivalence in $SSets$.

Unlike strict $\mathcal{T}$-algebras, the collection of homotopy $\mathcal{T}$-algebras does not admit all small limits and colimits, so there is no model category structure of homotopy $\mathcal{T}$-algebras. However, there is a model category structure for homotopy $\mathcal{T}$-algebras.

Theorem 2.12. *[3, 5.4], [6, 4.9] There is a model category structure on the category of all diagrams from an algebraic theory $\mathcal{T} \to SSets$ for which the fibrant objects are homotopy $\mathcal{T}$-algebras.*

With the appropriate model structures defined for each type of $\mathcal{T}$-algebra, Badzioch and Bergner proved their respective rigidification results.

Theorem 2.13. *[3, 6.4], [6, 5.13] There is an equivalence of model categories between the model category of strict algebras and the model category for homotopy algebras.*

Chapter 3

Properties of $\mathcal{M} = \mathcal{SS}ets^{\mathcal{C}^{op}}$

For the remainder of this paper, we let $\mathcal{C}$ be any small category, and define $\mathcal{M}$ to be the functor category $\mathcal{M} = \mathcal{SS}ets^{\mathcal{C}^{op}}$. The objects in this category are functors $X : \mathcal{C}^{op} \to \mathcal{SS}ets$ and the morphisms are natural transformations between the functors. In order to use this category $\mathcal{M}$, we must first verify that $\mathcal{M}$ possesses the properties necessary to define strict and homotopy algebras, specifically that $\mathcal{M}$ is a model category. In this section, we will work out the necessary background results of Badzioch and Bergner found in the previous section in our more general context.

First, we would like to equip $\mathcal{M}$ with a model structure, building it from the model structure on $\mathcal{SS}ets$. Inspired from the model structure on simplicial sets, we choose the weak equivalences to be taken to be levelwise; that is to say a map $f : X \to Y$ is a weak equivalence in $\mathcal{M}$ if for any c in $\mathrm{Ob}(\mathcal{C})$, we have $f(c) : X(c) \to Y(c)$ is a weak equivalence in $\mathcal{SS}ets$.

We can then define two different model category structures on $\mathcal{M}$, both of which are cofibrantly generated, where either the cofibrations or the fibrations are chosen to be defined levelwise as well.

Theorem 3.1. *[10, VIII 2.4] There is a cofibrantly generated model category on $\mathcal{M}$, denoted by $\mathcal{M}_c$, where the weak equivalences and cofibrations are taken levelwise, and the fibrations are the maps with the right lifting property with respect to maps that are both weak equivalences and cofibrations. This model category structure is referred to as the injective model structure.*

Theorem 3.2. *[10, IX 1.4] There is a cofibrantly generated model category on $\mathcal{M}$, denoted by $\mathcal{M}_f$, where the weak equivalences and fibrations are taken levelwise, and the cofibrations are the maps with the left lifting property with respect to maps that are both weak equivalences and fibrations. This model category structure is referred to as the projective model structure.*

In addition to the structure provided by the model category structure, it is also necessary to check that the category $\mathcal{M}$ is also cartesian in order to ensure that there is an internal function object, which will be important as we enter the upcoming sections.

Theorem 3.3. *The categories $\mathcal{M}_c$ and $\mathcal{M}_f$ are cartesian model categories.*

The internal function object Y^X for both cartesian structures is described by

$$Y^X(c) = \mathrm{Map}(\mathrm{Hom}_{\mathcal{C}}(c, -), Y^X) = \mathrm{Map}(X \times \mathrm{Hom}_{\mathcal{C}}(c, -), Y).$$

Now by the Yoneda Lemma [11, 11.5.8], it is sufficient to only consider representable functors in $\mathcal{M}$. These representable functors have the form $\mathrm{Hom}_{\mathcal{C}}(c,-)$, which is to be taken as a constant diagram in $\mathcal{M}$. Hence with the projective model structure, $\mathcal{M}_f$, in addition to the cartesian model structure given, we determine the generating cofibrations as follows:

Let $X \to Y$ in $\mathcal{M}$ be an acyclic fibration and consider the diagram

$$\begin{tikzcd} \partial\Delta[n] \arrow[r] \arrow[d] & X(c) \arrow[d] \\ \Delta[n] \arrow[r] \arrow[ur, dashed] & Y(c) \end{tikzcd}$$

where $X(c) \to Y(c)$ is an acyclic fibration in $\mathcal{SS}ets$. Note using the isomorphism $X(c) = \mathrm{Hom}_{\mathcal{M}}(\mathrm{Hom}_{\mathcal{C}}(c,-),X)$ given by the Yoneda Embedding [7, 1.1.34], we can rewrite the diagram as

$$\begin{tikzcd} \partial\Delta[n] \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathcal{M}}(\mathrm{Hom}_{\mathcal{C}}(c,-),X) \arrow[d] \\ \Delta[n] \arrow[r] \arrow[ur, dashed] & \mathrm{Hom}_{\mathcal{M}}(\mathrm{Hom}_{\mathcal{C}}(c,-),Y). \end{tikzcd}$$

Focusing on the horizontal arrows, we notice that the top map is an morphism in

$$\mathrm{Hom}_{\mathcal{M}}(\partial\Delta[n], \mathrm{Hom}_{\mathcal{M}}(\mathrm{Hom}_{\mathcal{C}}(c,-),X) \cong \mathrm{Hom}_{\mathcal{M}}(\partial\Delta[n] \times \mathrm{Hom}_{\mathcal{C}}(c,-),X).$$

Thus by the adjuction given in a cartesian category, we can display the lift in the diagram above as:

$$\begin{tikzcd} \partial\Delta[n] \times \mathrm{Hom}_{\mathcal{C}}(c,-) \arrow[r] \arrow[d] & X \arrow[d] \\ \Delta[n] \times \mathrm{Hom}_{\mathcal{C}}(c,-) \arrow[r] \arrow[ur, dashed] & Y. \end{tikzcd}$$

Hence for the model category $\mathcal{M}$, one can describe the generating cofibrations as

$$I = \{\partial\Delta[n] \times \mathrm{Hom}_{\mathcal{C}}(c,-) \to \Delta[n] \times \mathrm{Hom}_{\mathcal{C}}(c,-) \mid n \geq 0, c \in \mathrm{Ob}(\mathcal{C})\}$$

and the generating acyclic cofibrations, using a similar style of proof, are

$$J = \{V[n,k] \times \mathrm{Hom}_{\mathcal{C}}(c,-) \to \Delta[n] \times \mathrm{Hom}_{\mathcal{C}}(c,-) \mid n \geq 1, 0 \leq k \leq n, c \in \mathrm{Ob}(\mathcal{C})\}$$

With these cofibrantly generated cartesian model categories defined on $\mathcal{M}$, we now verify that these structures are left proper, cellular and combinatorial to ensure that a left Bousfield localization is well-defined. We already know that $\mathcal{SS}ets$ are left proper [11, 13.1.13], cellular [11, 12.1.4] and combinatorial, and these properties are passed onto the diagram category as well.

Theorem 3.4. *[11, 12.1.5; 13.1.14] The category $\mathcal{M}$ is a left proper cellular model category.*

Theorem 3.5. *[4, 2.14] The category $\mathcal{M}$ is a combinatorial model category.*

With $\mathcal{M}$ satisfying all of the necessary conditions, we can now define strict $\mathcal{T}$-algebras as well as homotopy $\mathcal{T}$-algebras over $\mathcal{M}$.

Definition 3.6. *Let $\mathcal{M}$ be any model category. Given an S-sorted theory $\mathcal{T}$, a* strict $\mathcal{T}$-algebra *is a product-preserving functor $A : \mathcal{T} \to \mathcal{M}$, where product-preserving means that the map*

$$A(T_{\underline{\alpha}^n}) \to \prod_{i=1}^{n} A(T_{\alpha_i}),$$

induced by the projections $T_{\underline{\alpha}^n} \to T_{\alpha_i}$ for all $1 \leq i \leq n$, is an isomorphism of $\mathcal{M}$.

The collection of all $\mathcal{T}$-algebras forms a category, and we denote this category by $\mathcal{A}lg^{\mathcal{T}}_{\mathcal{M}}$ to indicate the model category $\mathcal{M}$ that we are mapping into.

Definition 3.7. *Let $\mathcal{M}$ be any model category. Given an S-sorted theory $\mathcal{T}$, a* homotopy $\mathcal{T}$-algebra *is a functor $X : \mathcal{T} \to \mathcal{M}$ which preserves products up to homotopy, i.e., for any $\alpha \in S^n$, the canonical map*

$$X(T_{\underline{\alpha}^n}) \to \prod_{i=1}^{n} X(T_{\alpha_i})$$

induced by the projection maps $T_{\underline{\alpha}^n} \to T_{\alpha_i}$ (for each $1 \leq i \leq n$) is a weak equivalence in $\mathcal{M}$.

Chapter 4

Model Structures for Categories of Algebras

Having shown that our category $\mathcal{M}$ satisfies the necessary conditions, in this section we equip the category of all product-preserving functors with the structure of a cofibrantly generated model category. However, before doing so, we must present the following definitions to understand the next theorem, proven by Kan, which we use to verify that such a model structure even exists.

Definition 4.1. *[11, 10.5.2] Let I be a set of maps in a category $\mathcal{M}$.*

1. *A map is an I-*fibration *if it has the right lifting property with respect to all maps in I.*
2. *A map is an I-*cofibration *if it has the left lifting property with respect to all the I-fibrations.*

Definition 4.2. *[11, 10.5.8] Let $\mathcal{M}$ be a category that admits all small colimits, and I be a set of maps in $\mathcal{M}$. Then*

1. *the subcategory of* relative I-cell complexes *is the subcategory of maps that can be constructed as a transfinite composition of pushouts of elements of I, and*

2. *an object is an I*-cell complex *if the map to it from the initial object of $\mathcal{M}$ is a relative I-cell complex.*

Theorem 4.3. *[11, 11.3.1] Let $\mathcal{M}$ be a category that admits all small limits and colimits and let W be a class of maps in $\mathcal{M}$ that is closed under retracts and satisfies the "two out of three" axiom. If I and J are sets of maps in $\mathcal{M}$ such that*

1. *both the sets I and J permit the small object argument,*

2. *every I-fibration is both a J-fibration and an element of W,*

3. *every J-cofibration is both an I-cofibration and an element of W, and*

4. *one of the following two conditions hold:*

   (a) a map that is both an I-cofibration and an element of W is a J-cofibration, or

   (b) a map that is both a J-fibration and an element of W is an I-fibration.

then there is a cofibrantly generated model category structure on $\mathcal{M}$ in which W is the class of weak equivalences, I is a set of generating cofibrations, and J is the set of generating acyclic cofibrations.

Given a cofibrantly generated model structure on a category $\mathcal{M}$ and an adjunction $F : \mathcal{M} \rightleftarrows \mathcal{N} : U$, one would like be able to define a model structure for $\mathcal{N}$. For sufficiently nice categories $\mathcal{N}$, such a statement holds as in the theorem below.

Theorem 4.4. *[11, 11.3.2] Let $\mathcal{M}$ be a cofibrantly generated model category with generating cofibrations I and generating acyclic cofibrations J. Let $\mathcal{N}$ be a category that is closed under small limits and colimits, and let $F\colon \mathcal{M} \rightleftarrows \mathcal{N}\colon U$ be a pair of adjoint functors. If we let $FI = \{Fu \mid u \in I\}$ and $FJ = \{Fv \mid v \in J\}$ and if*

1. *both the sets FI and FJ permit the small object argument and*

2. *U takes relative FJ-cell complexes to weak equivalences,*

then there is a cofibrantly generated model category structure on $\mathcal{N}$ in which FI is a set of generating cofibrations, FJ is a set of generating acyclic cofibrations, and the weak equivalences are the maps that U takes into a weak equivalence in $\mathcal{M}$. Furthermore, with respect to this model category structure, (F, U) is a Quillen pair.

In our situation we have an adjoint pair $F_\alpha : \mathcal{M} \rightleftarrows \mathcal{A}lg^{\mathcal{T}}_{\mathcal{M}} : U_\alpha$ for each $\alpha \in S$. Here, U_α is the evaluation functor $U_\alpha(A) = A(T_\alpha)$ for any strict algebra A. Hence for our purposes, we will need to use a generalized version of Theorem 4.4 to allow for multisorted theories.

Theorem 4.5. *Let $\mathcal{M}$ be a cofibrantly generated model category with generating cofibrations I and generating acyclic cofibrations J. Let $\mathcal{N}$ be a category that is closed under small limits and colimits, and let $F_\alpha\colon \mathcal{M} \rightleftarrows \mathcal{N}\colon U_\alpha$ be a collection of pairs of adjoint functors for $\alpha \in S$. If we let $F_\alpha I = \{F_\alpha u \mid u \in I\}$ and $F_\alpha J = \{F_\alpha v \mid v \in J\}$ and if*

1. *the sets $F_\alpha I$ and $F_\alpha J$ permit the small object argument for each $\alpha \in S$ and*

2. *U_α takes relative $F_\alpha J$-cell complexes to weak equivalences for each $\alpha \in S$,*

then there is a cofibrantly generated model category structure on $\mathcal{N}$ in which $FI = \{F_\alpha u \mid u \in I,\ \alpha \in S\}$ is a set of generating cofibrations, $FJ = \{F_\alpha v \mid v \in J,\ \alpha \in S\}$ is a set of generating acyclic cofibrations, and the weak equivalences are the maps that U takes to weak equivalences in $\mathcal{M}$. Furthermore, with respect to this model structure, (F_α, U_α) is a Quillen pair for each $\alpha \in S$.

Proof. To prove this theorem, we use Theorem 4.3 on the category $\mathcal{N}$. Note for a fixed $\alpha \in S$, $F_\alpha I$ and $F_\alpha J$ satisfy the conditions of Theorem 4.3. Let FW be the collection of weak equivalences in $\mathcal{N}$. Note that this class of maps in $\mathcal{N}$ is closed under retracts and satisfies the "two-out-of-three" axiom required in Theorem 4.4.

1. For each $\alpha \in S$, $F_\alpha I$ and $F_\alpha J$ satisfy the small object argument, so FI and FJ satisfy the small object argument.

2. Given an FI-fibration f, we have that f has the right lifting property with respect to any element of FI. By Theorem 4.4, it follows that for any α in S, f is an $F_\alpha J$-fibration as well as an element of $F_\alpha W$. Thus f is an FJ-fibration as well as an element of FW.

3. Given an FJ-cofibration p, we have that p has the left lifting property with respect to any element of FI. By Theorem 4.4, it follows that for any α in S, p is an $F_\alpha I$-cofibration as well as an element of $F_\alpha W$. Thus p must be an FI-cofibration as well as an element of FW.

4. Suppose g is a map that is both an FI-cofibration and and element of FW. Then g is an $F_\alpha I$-cofibration and an element of $F_\alpha W$ for every α in S. Hence by Theorem 4.4, g is an $F_\alpha J$-cofibration for every α in S, and thus a FJ-cofibration, as desired.

□

Using the preceding theorem, we can now show that $\mathcal{A}lg^{\mathcal{T}}_{\mathcal{M}}$ is a cofibrantly generated model category.

Theorem 4.6. *Let $\mathcal{T}$ be a multi-sorted algebraic theory, and let $\mathcal{M} = \mathcal{SS}ets^{\mathcal{C}^{op}}$ for a small category $\mathcal{C}$. Define $U_\alpha : \mathcal{A}lg^{\mathcal{T}}_{\mathcal{M}} \to \mathcal{M}$ to be the evaluation functor for each sort $\alpha \in S$, and define F_α to be its left adjoint. Then there is a cofibrantly generated model category structure on $\mathcal{A}lg^{\mathcal{T}}_{\mathcal{M}}$ in which a weak equivalence is a map which induces a weak equivalence in the category $\mathcal{M}$ after applying the inclusion functor U_α for any $\alpha \in S$. Similarly, we define a fibration to be a map which induces a fibration in the category $\mathcal{M}$ after applying the evaluation functor U_α. Finally, we define a cofibration to be a map with the left lifting property with respect to the acyclic fibrations.*

Proof. Let I and J be sets of generating cofibrations and generating acyclic cofibrations, respectively, as part of the cofibrantly generated model structure of $\mathcal{M}$. We use Theorem 4.5 with the adjoint pairs $F_\alpha \colon \mathcal{M} \rightleftarrows \mathcal{A}lg^{\mathcal{T}}_{\mathcal{M}} \colon U_\alpha$ for all $\alpha \in S$ and the cofibrantly generated model structure defined on $\mathcal{M}$. First, note that the existence of all limits and colimits in $\mathcal{A}lg^{\mathcal{T}}_{\mathcal{M}}$ follows from [1, 1.2]. Next, we must show $F_\alpha I$ permits the small object argument, that is to say we must show every domain element in $F_\alpha I$ is small relative to $F_\alpha I$. So suppose $F_\alpha(\partial\Delta[n] \times \mathrm{Hom}_{\mathcal{C}}(c, -))$ is a domain element of $F_\alpha I$ and there is some $\mathcal{T}$-algebra A that can be written as a directed colimit $\mathrm{colim}_m(A_m)$. Then

$$
\begin{aligned}
&\operatorname{Hom}_{\mathcal{A}lg^{\mathcal{T}}_{\mathcal{M}}}(F_\alpha(\partial\Delta[n] \times \operatorname{Hom}_{\mathcal{C}}(c,-)), \operatorname*{colim}_m(A_m)) \\
&= \operatorname{Hom}_{\mathcal{M}}(\partial\Delta[n] \times \operatorname{Hom}_{\mathcal{C}}(c,-), U_\alpha \operatorname*{colim}_m(A_m)) \\
&= \operatorname{Hom}_{\mathcal{M}}(\partial\Delta[n] \times \operatorname{Hom}_{\mathcal{C}}(c,-), \operatorname*{colim}_m(U_\alpha A_m)) \\
&= \operatorname*{colim}_m \operatorname{Hom}_{\mathcal{M}}(\partial\Delta[n] \times \operatorname{Hom}_{\mathcal{C}}(c,-), U_\alpha A_m) \\
&= \operatorname*{colim}_m \operatorname{Hom}_{\mathcal{A}lg^{\mathcal{T}}_{\mathcal{M}}}(F_\alpha(\partial\Delta[n] \times \operatorname{Hom}_{\mathcal{C}}(c,-)), A_m).
\end{aligned}
\tag{4.0.1}
$$

The first equality is given by adjointness, while the second equality is true because we can compute colimts objectwise, so

$$
\begin{aligned}
U_\alpha \operatorname*{colim}_m(A_m) &= [\operatorname*{colim}_m(A_m)](T_\alpha) \\
&= \operatorname*{colim}_m[A_m(T_\alpha)] \\
&= \operatorname*{colim}_m(U_\alpha A_m).
\end{aligned}
\tag{4.0.2}
$$

The third equality in (4.0.1) is because $\partial\Delta[n] \times \operatorname{Hom}_{\mathcal{C}}(c,-)$ is small relative to I and I is closed under transfinite pushouts, and the final equality of (4.0.1) is again given by adjointness. A similar proof shows that any domain element $F_\alpha(V[n,k] \times \operatorname{Hom}_{\mathcal{C}}(c,-))$ is small relative to $F_\alpha J$.

To show that U_α takes colimits of pushouts along maps of $F_\alpha J$ to weak equivalences, we note that because weak equivalences are taken levelwise, and the colimit of weak equivalences is still a weak equivalence, it suffices to show the result holds for a single pushout. So consider a map $F_\alpha(V[n,k] \times \operatorname{Hom}_{\mathcal{C}}(c,-)) \to F_\alpha(\delta[n] \times \operatorname{Hom}_{\mathcal{C}}(c,-))$ which is a map in $F_\alpha J$ and suppose there is a map from $F_\alpha(V[n,k] \times \operatorname{Hom}_{\mathcal{C}}(c,-))$ to some object A in $\mathcal{A}lg^{\mathcal{T}}_{\mathcal{M}}$.

Define B to be the pushout of the diagram

$$\begin{tikzcd}
F_\alpha(V[n,k] \times \mathrm{Hom}_{\mathcal{C}}(c,-)) \arrow[r] \arrow[d] & A \arrow[d] \\
F_\alpha(\Delta[n] \times \mathrm{Hom}_{\mathcal{C}}(c,-) \arrow[r] & B.
\end{tikzcd}$$

We need to show that the map from A to B is a weak equivalence. If $P \to Q$ is a map in $\mathcal{A}lg^{\mathcal{T}}_{\mathcal{M}}$ with the right lifting property with respect to maps in $F_\alpha J$, then we note that this map $P \to Q$ must be a fibration in $\mathcal{A}lg^{\mathcal{T}}_{\mathcal{M}}$. Thus we have a diagram

$$\begin{tikzcd}
F_\alpha(V[n,k] \times \mathrm{Hom}_{\mathcal{C}}(c,-)) \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & P \arrow[d] \\
F_\alpha(\Delta[n] \times \mathrm{Hom}_{\mathcal{C}}(c,-)) \arrow[r] \arrow[urr, dashed] & B \arrow[r] & Q
\end{tikzcd}$$

and note that a lift $F_\alpha(\Delta[n] \times \mathrm{Hom}_{\mathcal{C}}(c,-)) \to P$ exists since fibrations in $\mathcal{A}lg^{\mathcal{T}}_{\mathcal{M}}$ have the right lifting property with respect to maps in FJ, which implies since B is a pushout of the left diagram that there is also a lift $B \to P$. Applying the functor U_α to the this diagram, we get the diagram

$$\begin{tikzcd}
V[n,k] \times \mathrm{Hom}_{\mathcal{C}}(c,-) \arrow[r] \arrow[d] & U_\alpha A \arrow[r] \arrow[d] & U_\alpha P \arrow[d] \\
\Delta[n] \times \mathrm{Hom}_{\mathcal{C}}(c,-) \arrow[r] \arrow[urr, dashed] & U_\alpha B \arrow[r] & U_\alpha Q
\end{tikzcd}$$

and note that by our lift $B \to P$, we have a lift $U_\alpha B \to U_\alpha P$. Since $U_\alpha P \to U_\alpha Q$ is a fibration in M, and the fibration $U_\alpha P \to U_\alpha Q$ has the right lifting property with respect to the map $U_\alpha A \to U_\alpha B$, then we have that $U_\alpha A \to U_\alpha B$ is an acyclic cofibration in $\mathcal{M}$. Therefore, we can conclude $A \to B$ is an acyclic cofibration in $\mathcal{A}lg^{\mathcal{T}}_{\mathcal{M}}$, and is thus a weak equivalence, as desired.

$\square$

Next, we turn our attention to the category of homotopy $\mathcal{T}$-algebras, as we wish to impose a model structure on this category. However, such a structure is not possible to construct, as the category of homotopy $\mathcal{T}$-algebras do not admit all small limits and colimits. To work around this issue, we instead focus on creating a model structure the category of all functors from our theory $\mathcal{T} \to \mathcal{M}$ so that the fibrant objects in that model category are homotopy $\mathcal{T}$-algebras. To do this, we first provide several definitions that will be necessary as we move forward.

Definition 4.7. *[11, 9.1.6] A* simplicial model category $\mathcal{M}$ *is a model category $\mathcal{M}$ that is also a simplicial category such that the following two axioms hold:*

- *(SM6) For every two objects X and Y of $\mathcal{M}$ and every simplicial set K, there are objects $X \otimes K$ and Y^K in $\mathcal{M}$ such that there are isomorphisms of simplicial sets*

$$\mathrm{Map}(X \otimes K, Y) \cong \mathrm{Map}(K, \mathrm{Map}(X, Y)) \cong \mathrm{Map}(X, Y^K)$$

  that are natural in X, Y, and K.

- *(SM7) If $i : A \to B$ is a cofibration in $\mathcal{M}$ and $p : X \to Y$ is a fibration in $\mathcal{M}$, then the map of simplicial sets*

$$i^* \times p_* : \mathrm{Map}(B, X) \to \mathrm{Map}(A, X) \times_{\mathrm{Map}(A,Y)} \mathrm{Map}(B, Y)$$

  is a fibration which is an acyclic fibration if either i or p is a weak equivalence.

Definition 4.8. *For any objects X and Y in a simplicial category $\mathcal{M}$, the* function complex *is the simplicial set* $\mathrm{Map}(X, Y)$.

Note that a function complex in a simplicial model category is homotopy invariant only when X is cofibrant and Y is fibrant. So in general, we must use the following definitions:

Definition 4.9. *[7, 1.4.6] Let X be an object of a model category. A* cofibrant replacement *for X is a cofibrant object $\widetilde{X}$ together with a weak equivalence $\widetilde{X} \xrightarrow{\simeq} X$. A* fibrant replacement *for X is a fibrant object $\widehat{X}$ together with a weak equivalence $X \xrightarrow{\simeq} \widehat{X}$.*

Definition 4.10. *[11, 17.3.1] A* homotopy function complex $\mathrm{Map}^h(X, Y)$ *in a simplicial model category $\mathcal{M}$ is the simplicial set* $\mathrm{Map}(\widetilde{X}, \widehat{Y})$ *where $\widetilde{X}$ is a cofibrant replacement of X in $\mathcal{M}$ and $\widehat{Y}$ is a fibrant replacement for Y.*

Some of the model structures we use are created after localizing a given model category structure with respect to a collection of maps. Let P be a set of maps that one uses to localize a model category $\mathcal{M}$.

Definition 4.11. *A P-*local *object W is a fibrant object of $\mathcal{M}$ such that for any $f : A \to B$ in P, the induced map on homotopy function complexes*

$$f^* : \mathrm{Map}^h(B, W) \to \mathrm{Map}^h(A, W)$$

is a weak equivalence of simplicial sets.

*A map $g : X \to Y$ in $\mathcal{M}$ is then a P-*local equivalence *if for every local object W, the induced map on homotopy function complexes*

$$g^* : \mathrm{Map}^h(Y, W) \to \mathrm{Map}^h(X, W)$$

is a weak equivalence of simplicial sets.

We now describe two model category structures we can impose on all diagrams from $\mathcal{T} \to \mathcal{M}$.

Theorem 4.12. *[10, VIII 2.4] The category of all diagrams $\mathcal{T} \to \mathcal{M}$ has a cofibrantly generated model category structure, denoted $\mathcal{M}_c^{\mathcal{T}}$, where cofibrations and weak equivalences are taken objectwise, and fibrations are maps with the left lifting property with respect to maps that are both weak equivalences and cofibrations.*

Theorem 4.13. *[10, IX 1.4] The category of all diagrams $\mathcal{T} \to \mathcal{M}$ has a cofibrantly generated model category structure, denoted $\mathcal{M}_f^{\mathcal{T}}$, where fibrations and weak equivalences are taken objectwise, and cofibrations are maps with the right lifting property with respect to maps that are both weak equivalences and fibrations.*

In fact, since the two preceding model structures are cofibrantly generated, one can determine explicitly the generating cofibrations and generating acyclic cofibrations of $\mathcal{M}_f^{\mathcal{T}}$ in a manner similar to that found in the discussion after Theorem 3.2. We will now localize the model category structure on $\mathcal{M}_f^{\mathcal{T}}$, utilizing the following theorem.

Theorem 4.14. *[11, 4.1.1] Let $\mathcal{N}$ be a left proper cellular model category and P a set of morphisms of $\mathcal{N}$. There is a model category structure $\mathcal{L}_P\mathcal{N}$ on the underlying category of $\mathcal{N}$ such that:*

1. *The weak equivalences are the P-local equivalences.*
2. *The cofibrations are the cofibrations of $\mathcal{N}$.*
3. *The fibrations are the maps which have the right lifting property with respect to the maps which are both cofibrations and P-local equivalences.*
4. *The fibrant objects are the P-local objects.*
5. *If $\mathcal{N}$ is a simplicial model category, then that simplicial structure gives $\mathcal{L}_P\mathcal{N}$ the structure of a simplicial model category.*

The set of maps P that we localize with respect to will be constructed in a similar manner those done by Badzioch and Bergner. They used free diagrams which were corepresented by the objects of the theory $\mathcal{T}$. Hence the maps are built on the projections maps $T_{\underline{\alpha}^n} \to T_{\alpha_i}$ for each $\underline{\alpha}^n = \langle \alpha_1, \ldots, \alpha_n \rangle$ and $1 \leq i \leq n$. These projections induce the maps

$$\mathrm{Hom}_{\mathcal{T}}(T_{\alpha_i}, -) \to \mathrm{Hom}_{\mathcal{T}}(T_{\underline{\alpha}^n}, -),$$

which we view as maps in $\mathcal{M}^{\mathcal{T}}$ with trivial structure maps. Taking the coproduct of all such maps, we establish

$$P = \{p_{\underline{\alpha}^n} : \coprod_{i=1}^{n} \mathrm{Hom}_{\mathcal{T}}(T_{\alpha_i}, -) \to \mathrm{Hom}_{\mathcal{T}}(T_{\underline{\alpha}^n}, -) |\ \underline{\alpha}^n \in S^n, n \geq 0\}.$$

Theorem 4.15. *There is a model category structure $\mathcal{LM}^{\mathcal{T}}$ on the category $\mathcal{M}^{\mathcal{T}}$ with weak equivalences the P-local equivalences, cofibrations as in $\mathcal{M}^{\mathcal{T}}_f$, and fibrations the maps which have the right lifting property with respect to the maps which are cofibrations and weak equivalences.*

By localizing with respect to the maps in P, the P-local objects will be exactly those functors $X : \mathcal{T} \to \mathcal{M}$ such that

$$\mathrm{Map}(\mathrm{Hom}_{\mathcal{T}}(T_{\underline{\alpha}^n}, -), X) \xrightarrow{\simeq} \mathrm{Map}(\coprod_{i=1}^{n} \mathrm{Hom}_{\mathcal{T}}(T_{\alpha_i}, -), X).$$

But as for any $n \geq 0$,

$$\mathrm{Map}(\mathrm{Hom}_{\mathcal{T}}(T_{\underline{\alpha}^n}, -), X) \cong X(T_{\underline{\alpha}^n})$$

as well as

$$\mathrm{Map}(\coprod_{i=1}^{n} \mathrm{Hom}_{\mathcal{T}}(T_{\alpha_i}, -), X) \cong \prod_{i=1}^{n} X(T_{\alpha_i}),$$

then we have

$$X(T_{\underline{\alpha}^n}) \xrightarrow{\simeq} \prod_{i=1}^{n} X(T_{\alpha_i}),$$

which is exactly the condition to be a homotopy $\mathcal{T}$ algebra. So with this model category structure on $\mathcal{M}^{\mathcal{T}}_f$, we can now describe any fibrant objects of the model structure as a homotopy $\mathcal{T}$-algebras.

Proposition 4.16. *An object X in $\mathcal{LM}^{\mathcal{T}}$ is fibrant if and only if it is a homotopy $\mathcal{T}$-algebra that is fibrant as an object of $\mathcal{M}^{\mathcal{T}}_f$.*

Proposition 4.17. *A map $f : X \to Y$ is a P-local equivalence if and only if for any $\mathcal{T}$-algebra Z which is fibrant in $\mathcal{M}_c^{\mathcal{T}}$, the induced map of function complexes*

$$f^* : \mathrm{Map}(Y, Z) \to \mathrm{Map}(X, Z)$$

is a weak equivalence in $\mathcal{M}$.

In fact the above results are just a special case of a more general statement concerning fibrant objects in a localized model category; more details can be found in [11].

Chapter 5

Quillen Equivalence of $\mathcal{A}lg^{\mathcal{T}}_{\mathcal{M}}$ and $\mathcal{LM}^{\mathcal{T}}$

Now that we have constructed the two model categories necessary, we can now prove the main result of the paper. Many of the arguments are similar to those found in [6]. To do so, we must first define a Quillen pair between our categories $\mathcal{A}lg^{\mathcal{T}}_{\mathcal{M}}$ and $\mathcal{LM}^{\mathcal{T}}$. Let the functor

$$J^{\mathcal{T}}_{\mathcal{M}} : \mathcal{A}lg^{\mathcal{T}}_{\mathcal{M}} \to \mathcal{LM}^{\mathcal{T}}$$

be the inclusion functor. To show that this functor has a left adjoint, we begin by introducing the following definitions.

Definition 5.1. *Let $\mathcal{D}$ be a small category and $\mathcal{M}^{\mathcal{D}}$ the category of functors $\mathcal{D} \to \mathcal{M}$. Let P be a set of morphisms in $\mathcal{M}^{\mathcal{D}}$. An object Y in $\mathcal{M}^{\mathcal{D}}$ is* strictly P-local *if for every* *morphism $f : A \to B$ in P, the induced map on function complexes*

$$f^* : \mathrm{Map}(B, Y) \to \mathrm{Map}(A, Y)$$

is an isomorphism in $\mathcal{M}$.

A map $g : M \to N$ in $\mathcal{M}^{\mathcal{D}}$ is a strict P-local equivalence *if for every strictly P-local object Y in $\mathcal{M}^{\mathcal{D}}$, the induced map*

$$g^* : \mathrm{Map}(N, Y) \to \mathrm{Map}(M, Y)$$

is an isomorphism in $\mathcal{M}$.

Now we will show that category of all diagrams does indeed have a right adjoint with respect to the inclusion functor.

Lemma 5.2. *Consider two categories, the category of all diagrams $X : \mathcal{D} \to \mathcal{M}$ and the category of strictly local diagrams with respect to the set of maps $P = \{f : A \to B\}$. Then the forgetful functor from the category of strictly local diagrams to the category of all diagrams has a left adjoint.*

Proof. Without loss of generality, we can suppose there is one map f in P, since otherwise we can just replace f by $\coprod_\alpha f_\alpha$. Given an arbitrary diagram X, we would like to construct a strictly local diagram from X. So, suppose that X is not strictly local, i.e., the map

$$f^* : \mathrm{Map}(B, X) \to \mathrm{Map}(A, X)$$

is not an isomorphism. To make f^* is surjective, we create an object X' as the pushout in the following diagram:

$$\begin{tikzcd} \coprod_{n\geq 0}\coprod_{A\times\Delta[n]\to X} A\times\Delta[n] \arrow[r] \arrow[d] & X \arrow[d] \\ \coprod_{n\geq 0}\coprod_{A\times\Delta[n]\to X} B\times\Delta[n] \arrow[r] & X' \end{tikzcd} .$$

If f^* is not injective, then we make f^* injective by creating an object X'' by taking the pushout

$$\begin{tikzcd} \coprod_{n\geq 0}\coprod(B\coprod_A B)\times\Delta[n] \arrow[r] \arrow[d] & X' \arrow[d] \\ \coprod_{n\geq 0}\coprod B\times\Delta[n] \arrow[r] & X'' \end{tikzcd} ,$$

where the map $B\coprod B\to B$ is the fold map.

In the construction of X', given a strictly local object Y, we can create a pullback diagram

$$\begin{tikzcd} \mathrm{Map}(X',Y) \arrow[r] \arrow[d] & \mathrm{Map}(\coprod B,Y) \arrow[d] \\ \mathrm{Map}(X,Y) \arrow[r] & \mathrm{Map}(\coprod A,Y) \end{tikzcd}$$

and since $f : A\to B$ is a strict local equivalence, then the map $X\to X'$ is a strict local equivalence as well.

In the construction of X'', we can again obtain a pullback diagram

$$\begin{tikzcd} \mathrm{Map}(X'',Y) \arrow[r] \arrow[d] & \mathrm{Map}(\coprod B,Y) \arrow[d] \\ \mathrm{Map}(X',Y) \arrow[r] & \mathrm{Map}(\coprod(B\coprod_A B),Y) \end{tikzcd} .$$

As before, to see that $X' \to X''$ is a strict local equivalence, it is sufficient to show that the right-hand vertical arrow is an isomorphism.

Since $B \coprod_A B$ is defined as the pushout in the diagram

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ B & \longrightarrow & B \coprod_A B \end{array}$$

we can look at the pullback diagram

$$\begin{array}{ccc} \mathrm{Map}(B \coprod_A B, Y) & \longrightarrow & \mathrm{Map}(B, Y) \\ \downarrow & & \downarrow \\ \mathrm{Map}(B, Y) & \longrightarrow & \mathrm{Map}(A, Y). \end{array}$$

Hence the map

$$B \to B \coprod_A B$$

is a strict local equivalence. But, this map fits into

$$B \longrightarrow B \coprod_A B \longrightarrow B$$

where the composite map is the identity map on B. Since the identity map is a strict local equivalence, it follows that the map

$$B \coprod_A B \to B$$

is a strict local equivalence, since it can be shown that the strictly local equivalences satisfy model category axiom MC2.

Thus, we have created a map $X \to X''$ which is a strict local equivalence. However, this construction does not guarantee that the map

$$\mathrm{Map}(B, X'') \to \mathrm{Map}(A, X'')$$

is an isomorphism. By attaching n-cells in the construction of X'', we cannot be certain that the map f^* remains surjective onto X''. So we must repeat this process, taking a potentially transfinite colimit to obtain a strictly local object $\widetilde{X}$ such that there is a local equivalence $X \to \widetilde{X}$.

What we must show is that the functor that takes X to $\widetilde{X}$ is the functor that is left adjoint to the forgetful functor. That is to say, we must show that

$$\mathrm{Map}(X, Y) = \mathrm{Map}(X, JY) \cong \mathrm{Map}(KX, Y) = \mathrm{Map}(\widetilde{X}, Y)$$

where X is any diagram, Y is any strict local diagram, J is the forgetful functor from the category of strictly local diagrams to the category of all diagrams, and K is the functor that sends a diagram X to a strictly local diagram $\widetilde{X}$. However, note that the outer function complexes $\mathrm{Map}(X, Y)$ and $\mathrm{Map}(\widetilde{X}, Y)$ have been proven already to be isomorphic at every step of the procedure to construction of $\widetilde{X}$, and thus holds for the colimit as well. Thus we do indeed have that

$$\mathrm{Map}(X, Y) \cong \mathrm{Map}(\widetilde{X}, Y)$$

and restricting our focus to the 0-simplices of each object yields

$$\mathrm{Hom}(KX, Y) \to \mathrm{Hom}(X, JY)$$

which is exactly the isomorphism desired.

□

Now to use the preceding lemma in our situation, we must first show that the category of strictly local diagrams with respect to P and $\mathcal{A}lg^{\mathcal{T}}_{\mathcal{M}}$ are in fact the one and the same category. To see this we must prove the following lemma, which is the Yoneda Lemma in our setting.

Lemma 5.3. *Let $T_{\underline{\alpha}}$ be an object in an algebraic theory $\mathcal{T}$ and A a strict $\mathcal{T}$-algebra. Then there is an isomorphism in $\mathcal{M}$*

$$\mathrm{Map}_{\mathcal{M}^{\mathcal{T}}_{f}}(\mathrm{Hom}_{\mathcal{T}}(T_{\underline{\alpha}}, -), A) \cong A(T_{\underline{\alpha}}).$$

Proof. As the functor A is an object in $\mathcal{M}^{\mathcal{T}}_{f}$, we can also view A as a functor from $\mathcal{T} \times \mathcal{C}^{op} \to \mathcal{SS}ets$. Thus we can consider A to be a diagram in $\mathcal{SS}ets^{\mathcal{T}}$ in the shape of $\mathcal{C}$, and hence for any object c in $\mathcal{C}$, $A(c)$ is an object in $\mathcal{SS}ets^{\mathcal{T}}$. Then by [6, 5.7], a homotopical extension of the classical Yoneda Lemma, we have a natural isomorphism

$$\mathrm{Map}_{\mathcal{SS}ets^{\mathcal{T}}}(\mathrm{Hom}_{\mathcal{T}}(T_{\underline{\alpha}}, -), A(c)) \cong A(T_{\underline{\alpha}})(c),$$

for any c in $\mathcal{C}$. Taking objects to be constant diagrams in their respective categories as needed, we have that

$$\begin{aligned}
\mathrm{Map}_{\mathcal{SS}ets^{\mathcal{T}}}(\mathrm{Hom}_{\mathcal{T}}(T_{\underline{\alpha}}, -), A(c)) &\cong \mathrm{Map}_{\mathcal{M}^{\mathcal{T}}_{f}}(\mathrm{Hom}_{\mathcal{T}}(T_{\underline{\alpha}}, -), \mathrm{Map}_{\mathcal{M}^{\mathcal{T}}_{f}}(\mathrm{Hom}_{\mathcal{C}}(c, -), A(-))) \\
&\cong \mathrm{Map}_{\mathcal{M}^{\mathcal{T}}_{f}}(\mathrm{Hom}_{\mathcal{T}}(T_{\underline{\alpha}}, -) \times \mathrm{Hom}_{\mathcal{C}}(c, -), A(-)) \\
&\cong \mathrm{Map}_{\mathcal{M}^{\mathcal{T}}_{f}}(\mathrm{Hom}_{\mathcal{T}}(T_{\underline{\alpha}}, -), A)(c).
\end{aligned}$$

Since each of the maps above are natural, we obtain a natural functor

$$\mathrm{Map}_{\mathcal{M}^{\mathcal{T}}_{f}}(\mathrm{Hom}_{\mathcal{T}}(T_{\underline{\alpha}}, -), A) \to A(T_{\underline{\alpha}})$$

which is an isomorphism since it is an isomorphism at each level. ☐

Now we the Yoneda Lemma proven in our context, we can now show that the category of strictly local diagrams with respect to P and $\mathcal{A}lg^{\mathcal{T}}_{\mathcal{M}}$ are equivalent.

Lemma 5.4. *A diagram $A : \mathcal{T} \to \mathcal{M}$ is a strict $\mathcal{T}$-algebra if and only if the diagram A is strictly local with respect to the maps in P.*

Proof. Recall that a diagram A is a strict $\mathcal{T}$-algebra if and only if for each $\underline{\alpha}^n = \langle \alpha_1, \ldots, \alpha_n \rangle$, we have a natural isomorphism

$$A(T_{\underline{\alpha}}) \cong \prod_{i=1}^{n} A(T_{\alpha_i})$$

induced by projection maps in $\mathcal{T}$. By Lemma 5.3, the above isomorphism is equivalent to having an isomorphism

$$\begin{aligned}\mathrm{Map}_{\mathcal{M}^{\mathcal{T}}_f}(\mathrm{Hom}_{\mathcal{T}}(T_{\underline{\alpha}}, -), A) &\cong \prod_{i=1}^{n} \mathrm{Map}_{\mathcal{M}^{\mathcal{T}}_f}(\mathrm{Hom}_{\mathcal{T}}(T_{\alpha_i}, -), A) \\ &\cong \mathrm{Map}_{\mathcal{M}^{\mathcal{T}}_f}(\coprod_{i=1}^{n} \mathrm{Hom}_{\mathcal{T}}(T_{\alpha_i}, -), A).\end{aligned}$$

Since these isomorphisms are all induced by the projection maps, it follows that all strict $\mathcal{T}-$algebras are strictly local with respect to maps in P.

Conversely, given a diagram A that is strictly local with respect to maps in P, we have that

$$\mathrm{Map}_{\mathcal{M}^{\mathcal{T}}_f}(\mathrm{Hom}_{\mathcal{T}}(T_{\underline{\alpha}}, -), A) \cong \mathrm{Map}_{\mathcal{M}^{\mathcal{T}}_f}(\coprod_{i=1}^{n} \mathrm{Hom}_{\mathcal{T}}(T_{\alpha_i}, -), A)$$

which is equivalent to

$$\mathrm{Map}_{\mathcal{M}^{\mathcal{T}}_f}(\mathrm{Hom}_{\mathcal{T}}(T_{\underline{\alpha}}, -), A) \cong \prod_{i=1}^{n} \mathrm{Map}_{\mathcal{M}^{\mathcal{T}}_f}(\mathrm{Hom}_{\mathcal{T}}(T_{\alpha_i}, -), A).$$

By Lemma 5.3, we have

$$A(T_{\underline{\alpha}}) \cong \prod_{i=1}^{n} A(T_{\alpha_i})$$

and thus A must be a strict $\mathcal{T}$-algebra. □

This theorem shows that $\mathcal{A}lg^{\mathcal{T}}_{\mathcal{M}}$ is in fact the category of strictly local diagrams, and so by Lemma 5.2, we have that the forgetful functor $J^{\mathcal{T}}_{\mathcal{M}} : \mathcal{A}lg^{\mathcal{T}}_{\mathcal{M}} \to \mathcal{M}^{\mathcal{T}}_{f}$, has a left adjoint, denoted $K^{\mathcal{T}}_{\mathcal{M}} : \mathcal{M}^{\mathcal{T}}_{f} \to \mathcal{A}lg^{\mathcal{T}}_{\mathcal{M}}$.

Proposition 5.5. *The adjoint pair of functors*

$$K^{\mathcal{T}}_{\mathcal{M}} : \mathcal{M}^{\mathcal{T}}_{f} \rightleftarrows \mathcal{A}lg^{\mathcal{T}}_{\mathcal{M}} : J^{\mathcal{T}}_{\mathcal{M}}$$

is a Quillen pair.

Proof. Using Lemma 5.4, we see that $\mathcal{A}lg^{\mathcal{T}}_{\mathcal{M}}$ is a subcategory of $\mathcal{M}^{\mathcal{T}}_{f}$ by the inclusion functor $J^{\mathcal{T}}_{\mathcal{M}}$. Since the fibrations and weak equivalences are defined objectwise, and $J^{\mathcal{T}}_{\mathcal{M}}$ is a right adjoint, then it must preserves fibrations and acyclic fibrations. □

Lemma 5.6. *Each map $K^{\mathcal{T}}_{\mathcal{M}}(p_{\underline{\alpha}^n})$ is a weak equivalence in $\mathcal{A}lg^{\mathcal{T}}_{\mathcal{M}}$.*

Proof. Recall that the functor $\mathrm{Hom}_{\mathcal{T}}(T_{\underline{\alpha}}, -)$ is a strict $\mathcal{T}$-algebra and that $J_{\mathcal{T}}A = A$ for any strict $\mathcal{T}$-algebra A. Then, for each map in P, we have by Lemma 5.3 and adjointness the following composite isomorphism.

$$
\begin{aligned}
\mathrm{Map}_{\mathcal{A}lg^{\mathcal{T}}_{\mathcal{M}}}(K^{\mathcal{T}}_{\mathcal{M}}(\mathrm{Hom}_{\mathcal{T}}(T_{\underline{\alpha}}, -)), A) &\cong \mathrm{Map}_{\mathcal{M}^{\mathcal{T}}_f}(\mathrm{Hom}_{\mathcal{T}}(T_{\underline{\alpha}}, -), A) \\
&\cong A(T_{\underline{\alpha}}) \\
&\cong \prod_{i=1}^{n} A(T_{\alpha_i}) \\
&\cong \prod_{i=1}^{n} \mathrm{Map}_{\mathcal{M}^{\mathcal{T}}_f}(\mathrm{Hom}_{\mathcal{T}}(T_{\alpha_i}, -), A) \\
&\cong \mathrm{Map}_{\mathcal{M}^{\mathcal{T}}_f}(\coprod_{i=1}^{n} \mathrm{Hom}_{\mathcal{T}}(T_{\alpha_i}, -), A) \\
&\cong \mathrm{Map}_{\mathcal{A}lg^{\mathcal{T}}_{\mathcal{M}}}(K^{\mathcal{T}}_{\mathcal{M}}(\coprod_{i=1}^{n} \mathrm{Hom}_{\mathcal{T}}(T_{\alpha_i}, -)), A).
\end{aligned}
$$

Since all the isomorphisms are naturally induced, it follows that $K^{\mathcal{T}}_{\mathcal{M}}(p_{\underline{\alpha}})$ is a strict local equivalence in $\mathcal{M}^{\mathcal{T}}$, and by the preservation of weak equivalences over $K^{\mathcal{T}}_{\mathcal{M}}$, it is also a weak equivalence in $\mathcal{A}lg^{\mathcal{T}}_{\mathcal{M}}$. □

Having established that the adjoint pair of functors

$$K^{\mathcal{T}}_{\mathcal{M}} : \mathcal{M}^{\mathcal{T}}_f \rightleftarrows \mathcal{A}lg^{\mathcal{T}}_{\mathcal{M}} : J^{\mathcal{T}}_{\mathcal{M}}$$

is a Quillen pair, we must now show that this adjoint pair in fact remains a Quillen pair when the $\mathcal{M}^{\mathcal{T}}$ is replaced by the category $\mathcal{L}\mathcal{M}^{\mathcal{T}}$. To show this, we utilize the following proposition.

Proposition 5.7. *[11, 3.3.18] Let $\mathcal{M}$ be a model category and let P be a class of maps in $\mathcal{M}$. If $L_P\mathcal{M}$ is the left Bousfield localization of $\mathcal{M}$ with respect to P, $\mathcal{N}$ is a model category, and $F\colon \mathcal{M} \to \mathcal{N}$ is a left Quillen functor that takes every cofibrant replacement to an element of P into a weak equivalence in $\mathcal{N}$, then F is a left Quillen functor when considered as a functor $L_P\mathcal{M} \to \mathcal{N}$.*

With this proposition, we now prove the pair of functors remains a Quillen pair.

Proposition 5.8. *The adjoint pair*

$$K^{\mathcal{T}}_{\mathcal{M}} : \mathcal{LM}^{\mathcal{T}} \rightleftarrows \mathcal{A}lg^{\mathcal{T}}_{\mathcal{M}} : J^{\mathcal{T}}_{\mathcal{M}}$$

is a Quillen pair.

Proof. Consider again the set of maps used to localize the model category $\mathcal{M}^{\mathcal{T}}$, $P = \{p_{\underline{\alpha}^n} : \coprod_i \mathrm{Hom}_{\mathcal{T}}(T_{\alpha_i}, -) \to \mathrm{Hom}_{\mathcal{T}}(T_{\underline{\alpha}}, -)\}$. Since all of these maps free diagrams corepresented by the objects in the theory $\mathcal{T}$, then these maps are all cofibrant in $\mathcal{M}^{\mathcal{T}}_f$. The model category structure $\mathcal{LM}^{\mathcal{T}}$ is defined by localizing with respect to these maps in P. By Lemma 5.6, we know that each map in P is sent by $K^{\mathcal{T}}_{\mathcal{M}}$ to a weak equivalence in $\mathcal{A}lg^{\mathcal{T}}_{\mathcal{M}}$. Hence, by Proposition 5.7, the pair of adjoints also forms a Quillen pair on $\mathcal{LM}^{\mathcal{T}}$ and $\mathcal{A}lg^{\mathcal{T}}_{\mathcal{M}}$. □

Now to prove that the Quillen pair above is indeed the desired Quillen equivalence, we must first prove the following result concerning the unit map in $\mathcal{LM}^{\mathcal{T}}$.

Lemma 5.9. *If X is cofibrant in $\mathcal{LM}^{\mathcal{T}}$, then the unit map η is a weak equivalence in $\mathcal{LM}^{\mathcal{T}}$.*

Proof. First, we note that any cofibrant object X in $\mathcal{LM}^{\mathcal{T}}$ can be expressed as a homotopy colimit of representable functors in $\mathcal{LM}^{\mathcal{T}}$ [4, 2.5]. To examine the general case, let us first consider the special case where the cofibrant object $X = \coprod_{\underline{\alpha}} \mathrm{Hom}_{\mathcal{T}}(T_{\underline{\alpha}}, -)$, with the coproduct taken over elements $\underline{\alpha}$ from the sorts S^n. Then for any P-local object Y in $\mathcal{LM}^{\mathcal{T}}$,

$$
\begin{aligned}
\mathrm{Map}_{\mathcal{LM}^{\mathcal{T}}}(X,Y) &= \mathrm{Map}_{\mathcal{LM}^{\mathcal{T}}}(\coprod_{\underline{\alpha}} \mathrm{Hom}_{\mathcal{T}}(T_{\underline{\alpha}},-),Y) \\
&\simeq \prod_{\underline{\alpha}} \mathrm{Map}_{\mathcal{LM}^{\mathcal{T}}}(\mathrm{Hom}_{\mathcal{T}}(T_{\underline{\alpha}},-),Y) \\
&\simeq \prod_{\underline{\alpha}} \mathrm{Map}_{\mathcal{A}lg^{\mathcal{T}}}(K^{\mathcal{T}}_{\mathcal{M}}(\mathrm{Hom}_{\mathcal{T}}(T_{\underline{\alpha}},-)),Y) \\
&\simeq \mathrm{Map}_{\mathcal{A}lg^{\mathcal{T}}}(\coprod_{\underline{\alpha}} K^{\mathcal{T}}_{\mathcal{M}}(\mathrm{Hom}_{\mathcal{T}}(T_{\underline{\alpha}},-)),Y) \\
&\simeq \mathrm{Map}_{\mathcal{A}lg^{\mathcal{T}}}(K^{\mathcal{T}}_{\mathcal{M}}(\coprod_{\underline{\alpha}} \mathrm{Hom}_{\mathcal{T}}(T_{\underline{\alpha}},-)),Y) \\
&\simeq \mathrm{Map}_{\mathcal{LM}^{\mathcal{T}}}(J^{\mathcal{T}}_{\mathcal{M}}K^{\mathcal{T}}_{\mathcal{M}}(\coprod_{\underline{\alpha}} \mathrm{Hom}_{\mathcal{T}}(T_{\underline{\alpha}},-)),Y) \\
&\simeq \mathrm{Map}_{\mathcal{LM}^{\mathcal{T}}}(J^{\mathcal{T}}_{\mathcal{M}}K^{\mathcal{T}}_{\mathcal{M}}(X),Y).
\end{aligned}
$$

Hence the unit map η is a P-local equivalence, and thus a weak equivalence, in $\mathcal{LM}^{\mathcal{T}}$.

Now for any cofibrant object X in $\mathcal{LM}^{\mathcal{T}}$, we can write $X \simeq \mathrm{hocolim}_{\mathcal{C}^{op}\times\mathbf{\Delta}^{op}} X_i$, with each X_i as described in the special case above. Using Proposition 4.17 as well as the previous case, for any P-local object Y, we have the following:

$$
\begin{aligned}
\mathrm{Map}(X,Y) &\simeq \mathrm{Map}_{\mathcal{LM}^{\mathcal{T}}}(\mathrm{hocolim}\, X_i, Y) \\
&\simeq \mathrm{holim}\ \mathrm{Map}_{\mathcal{LM}^{\mathcal{T}}}(X_i,Y) \\
&\simeq \mathrm{holim}\ \mathrm{Map}_{\mathcal{LM}^{\mathcal{T}}}(K^{\mathcal{T}}_{\mathcal{M}}X_i,Y) \\
&\simeq \mathrm{Map}_{\mathcal{LM}^{\mathcal{T}}}(\mathrm{hocolim}\, K^{\mathcal{T}}_{\mathcal{M}}X_i,Y) \\
&\simeq \mathrm{Map}_{\mathcal{LM}^{\mathcal{T}}}(K^{\mathcal{T}}_{\mathcal{M}}X,Y).
\end{aligned}
$$

Thus for any cofibrant object X in $\mathcal{LM}^{\mathcal{T}}$, the unit map η is a weak equivalence in $\mathcal{LM}^{\mathcal{T}}$. $\square$

And now we are finally ready to prove the main result of the thesis.

Theorem 5.10. *The Quillen pair*

$$K_{\mathcal{M}}^{\mathcal{T}} : \mathcal{LM}^{\mathcal{T}} \rightleftarrows \mathcal{A}lg_{\mathcal{M}}^{\mathcal{T}} : J_{\mathcal{M}}^{\mathcal{T}}.$$

is a Quillen equivalence.

Proof. Suppose X is a cofibrant object in $\mathcal{LM}^{\mathcal{T}}$, Y a fibrant object in $\mathcal{A}lg_{\mathcal{M}}^{\mathcal{T}}$, and $f : X \to Y = J_{\mathcal{T}}Y$ a map in $\mathcal{LM}^{\mathcal{T}}$. To show that f is a weak equivalence in $\mathcal{LM}^{\mathcal{T}}$ if and only if its adjoint map $g : K_{\mathcal{M}}^{\mathcal{T}}X \to Y$ is a weak equivalence in $\mathcal{A}lg_{\mathcal{M}}^{\mathcal{T}}$, consider the commutative diagram

$$\begin{tikzcd} X \arrow[r, "\eta"] \arrow[d, "f"'] & K_{\mathcal{M}}^{\mathcal{T}}X \arrow[d, "g"] \\ J_{\mathcal{M}}^{\mathcal{T}}Y \arrow[r, "="] & Y \end{tikzcd}$$

If f is a weak equivalence in $\mathcal{LM}^{\mathcal{T}}$. Then g must also be a weak equivalence since by Lemma 5.9, η is a weak equivalence. However, g is a map in $\mathcal{A}lg_{\mathcal{M}}^{\mathcal{T}}$, and so a weak equivalence in $\mathcal{A}lg_{\mathcal{M}}^{\mathcal{T}}$.

Conversely, suppose that g is a weak equivalence in $\mathcal{A}lg_{\mathcal{M}}^{\mathcal{T}}$. Then $J_{\mathcal{M}}^{\mathcal{T}}g = g$ is a weak equivalence in $\mathcal{LM}^{\mathcal{T}}$. Hence, $f = g \circ \eta$ is also a weak equivalence in $\mathcal{LM}^{\mathcal{T}}$, and the result is proven. ∎

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