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Interference Alignment and Degrees of Freedom of the Two-User X Channel with an Instantaneous Relay

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Abstract

In this paper we investigate the sum degrees of freedom (DoF) of multiple unicasts in a wireless network. With 2 source nodes, 2 destination nodes, there are a total of 4 independent unicast sessions (messages), one from each source to each sink node (this setting is also known as an X network), and also there is a delay-free relay working in full-duplex mode helping the transmissions from the source to destination nodes. For such a channel setting, we prove that $5/3$ DoF is achievable almost surely for time-varying/frequency-selective channels, based on the ideas of aligned interference neutralization, linear forwarding and interference alignment. Also, the achievable scheme can be easily translated to the rational alignment scheme for the network with constant-values channel coefficients. In addition, we provide an intuition for the $5/3$ DoF result from the perspective of counting the number of linear equations and variables.

1 Introduction

In network information theory, exploring the multiple unicast capacity is one of the most important and fundamental problems. Due to the broadcast nature of the wireless transmission, multiple users in concurrent transmissions will cause interference to each other. Thus, efficient interference management is essential and key to improve the network throughput. The study of multiple unicast capacity gives rise to many powerful ideas such as interference alignment that has been shown to effectively achieve the capacity of many wireless networks. So far, interference alignment has been applied primarily to *single hop* wireless networks. If the network goes beyond one hop transmission, i.e., relay nodes are introduced to help the transmission from the source to the destination, the capacity or even the DoF of the network is of interest. For the two-source two-sink multihop networks, if the network has a *layered* structure¹, then it has been shown that the DoF can only take the values of 1, 3/2, 2 in the interference networks [2], and the values of 1, 4/3, 3/2, 5/3, 2 in the X networks [3]. However, if the network is *non-layered*, then the capacity also depend on the types of the relay. In this paper, we will investigate the achievable DoF of a two-user X channel with a delay-free relay helping the transmission from the two source nodes to the two sink nodes.

1.1 Prior Work

The relay nodes introduced into the networks may greatly impact the channel capacity. Usually we consider the following two types of relay:

- (a) Conventional relay: The transmitted signal of the relay only depends on its past received signals.
- (b) Delay-free relay: The transmitted signal of the relay depends on both its current and past received signals.

For interference networks, e.g., two user interference channel with a conventional relay, it is well known that if there is a direct link from one source to the unintended receiver, then the DoF of the channel would collapse to only one [19]. However, with the same channel model but replacing the conventional relay with an instantaneous relay, it is recently shown in [4] that 1.5 DoF is achievable almost surely, based on the idea of aligned interference neutralization recently introduced in [1]. It is a non-trivial result and somewhat surprising because a delay-free relay instead of a conventional relay is able to provide at least 50% improvement in the DoF sense. It is well known that orthogonal transmission is optimal for the one hop two user interference channel. However, adding a delay-free relay makes interference alignment possible.

1.2 The Problem and Contribution

Since an instantaneous relay is able to improve the capacity of a two-user interference channel, a natural question is what DoF can be achieved for the X channel setting. The channel model we consider in this paper consists of two source nodes, two destinations, and one instantaneous relay node. Each source wants to send one independent message to each destination, and thus there are a total of four messages in this network. The relay node instantaneous amplifies the received signals

¹The layered structure means if we denote the network with a graph, i.e., the connectivity between nodes is denoted as an edge, then we can label each node with a layer index such that each node hears only from the nodes in the previous layer and only talks to the nodes in the next layer.

and forwards them without delay to the destinations. Our aim is to characterize the sum degrees of freedom (DoF) of this network. Interestingly, for time-varying/frequency selective channels, we show that a total of $5/3$ DoF can be achievable almost surely, a 25% capacity increase compared to the $4/3$ DoF of the two user X channel. The achievable scheme is based on aligned interference neutralization, linear forwarding and interference alignment. Moreover, this scheme can also be easily translated to rational alignment scheme to obtain the same result for the constant-valued channels. In addition, we also provide another perspective to interpret the $5/3$ DoF result, by a two-step counting the number of linear equations and variables argument that is previously introduced in [5] for multiple unicasts capacity of the layered interference channel.

The rest of this paper is organized as follows. Section 2 describes the system model and definitions we use in this paper. In Section 3, we provide a linear scheme to achieve $5/3$ DoF. In Section 4, we also provide another perspective from the linear equations arguments to interpret the DoF results. Finally, we conclude this paper in Section 5.

2 System Model

The wireless X network we consider in this paper consists of two source nodes S_1, S_2 , two destination nodes D_1, D_2 and an intermediate relay node R working in full-duplex mode, each with only one antenna, as shown in Figure 1. The relay node operates in full-duplex mode, instantaneously demodulating and forwarding the currently received data symbols from the source to the destination nodes. There are a total of four independent messages in this network. Each source S_i wants to send one independent message W_{ij} to the each destination D_j where $i, j \in \{1, 2\}$, with the collaboration of the instantaneous relay node.

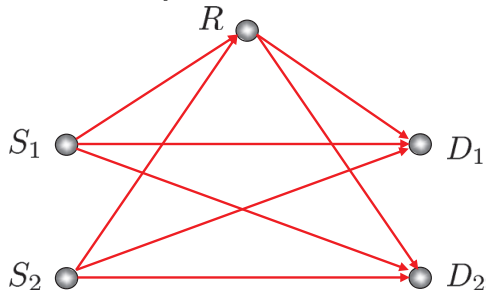


Figure 1: Two-User X Channel with an Instantaneous Relay

Given the channel model in Figure 1, the relationships of the channel input and output are given by:

$$y_R(t) = h_{R1}x_1(t) + h_{R2}x_2(t) + z_R(t) \quad (1)$$

$$y_1(t) = h_{11}x_1(t) + h_{12}x_2(t) + h_{1R}x_R(t) + z_1(t) \quad (2)$$

$$y_2(t) = h_{21}x_1(t) + h_{22}x_2(t) + h_{2R}x_R(t) + z_2(t) \quad (3)$$

where x_i, y_i denote the transmit and receive signals, respectively at the node i , h_{ji} stands for the channel coefficient from the node i to the node j where $i \in \{S_1, S_2, R\}$ and $j \in \{R, D_1, D_2\}$. z_j denotes the additive white Gaussian noise at the node j , which is a random variable drawn from an independent and identically distributed (i.i.d.) complex Gaussian distribution with zero mean and with unit variance. We also assume that the transmitted signals from the source S_1, S_2 and

the relay R satisfy the average power constraints

$$\frac{1}{T} \sum_{t=1}^T \mathbb{E}[|x_i(t)|^2] \leq P, \quad i \in \{S_1, S_2, R\} \quad (4)$$

for T channel uses. The rate $R_{ij}(P) = \log \frac{|W_{ij}|}{T}$ is achievable if the destination D_j can decode the message W_{ij} with arbitrarily small error probability. In this work we focus on the high SNR capacity, i.e., DoF, because not only it is more tractable than the general capacity problem, but also it yields the most significant rate improvement compared to the conventional schemes. The DoF corresponding to the message W_{ij} is defined as:

$$d_{ij} = \lim_{P \rightarrow \infty} \frac{R_{ij}(P)}{\log P}. \quad (5)$$

In this work, our aim is to characterize the sum achievable DoF of the four messages transmitted through the network we defined.

3 Achievability of 5/3 DoF in Time Varying/Frequency Selective Channels

In this section, we are considering time varying or frequency selective channels. We have the following DoF result.

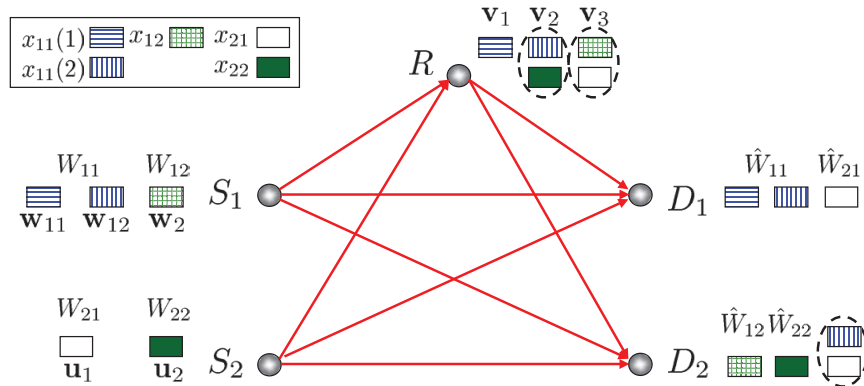


Figure 2: Achievability of 5/3 DoF

Theorem 1 *For the two-user X channel with an instantaneous relay defined in Section 2, a total of 5/3 DoF is achievable almost surely.*

We need to emphasize that the 5/3 DoF result is in the sense of *almost surely* since two DoF is achievable if the channel coefficients satisfy some constraints. For example, we denote the scalar α as the amplifying coefficient at the relay node. If the channel coefficients satisfy:

$$\alpha h_{1R} h_{R2} + h_{12} = 0 \quad (6)$$

$$\alpha h_{2R} h_{R1} + h_{21} = 0 \quad (7)$$

or equivalently

$$h_{1R}h_{R2}h_{21} - h_{2R}h_{R1}h_{12} = 0, \quad (8)$$

then it can be seen that the interference would be neutralized at each unintended destination, and thus two DoF is achievable. Since the channel coefficients satisfying the conditions above only constitute a subset with zero measure, we will investigate the DoF in the *almost surely* sense in what follows.

Proof: The achievable scheme is based on interference alignment in the linear space. In order to achieve 5/3 DoF, we will use three symbol extension to show that a total of 5 DoF is achievable. Because we use three symbol extension, each node is able to see a three dimensional space, and thus the original channel coefficient of each link is converted to a 3×3 diagonal channel matrix, and each diagonal entry is a i.i.d. random variable. In this new MIMO channel, therefore, the input-output relationships of the channel are given by:

$$\mathbf{y}_R = \mathbf{H}_{R1}\mathbf{x}_1 + \mathbf{H}_{R2}\mathbf{x}_2 + \mathbf{z}_R \quad (9)$$

$$\mathbf{y}_1 = \mathbf{H}_{11}\mathbf{x}_1 + \mathbf{H}_{12}\mathbf{x}_2 + \mathbf{H}_{1R}\mathbf{x}_R + \mathbf{z}_1 \quad (10)$$

$$\mathbf{y}_2 = \mathbf{H}_{21}\mathbf{x}_1 + \mathbf{H}_{22}\mathbf{x}_2 + \mathbf{H}_{2R}\mathbf{x}_R + \mathbf{z}_2 \quad (11)$$

where \mathbf{x}_i , \mathbf{y}_i denote the 3×1 transmit and receive signal vectors, respectively at the node i , \mathbf{H}_{ji} stands for the 3×3 diagonal channel matrices from the node i to the node j . \mathbf{z}_i denotes the 3×1 noise vector and it follows the distribution $\mathcal{CN}(\mathbf{0}, \mathbf{I})$. We want to send a total of 5 symbols in this network. Specifically, S_1 sends two symbols $x_{11}(1)$, $x_{11}(2)$ encoded with two linearly independent 3×1 beamforming vectors \mathbf{w}_{11} , \mathbf{w}_{12} , respectively to D_1 , one symbol x_{12} encoded with another linearly independent 3×1 beamforming vector \mathbf{w} to D_2 ; Similarly, S_2 sends x_{21} and x_{22} to D_1 and D_2 using two linearly independent 3×1 beamforming vectors \mathbf{u}_1 , \mathbf{u}_2 respectively. As what we will show in the achievable scheme, the beamforming vectors of each source will depend on each other. However, linear independencies among the beamforming vectors associated with each source are still guaranteed, because otherwise each destination can not decode all of its desired symbols. In what follows, we are going considering the transmission strategies from the source to the relay, and destinations respectively.

From the Source to the Relay: At the source side, we design the four beamforming vectors \mathbf{w}_{12} , \mathbf{w}_2 , \mathbf{u}_1 and \mathbf{u}_2 such that two symbols $x_{11}(2)$ and x_{22} align in one dimension, and two symbols x_{12} and x_{21} align in one dimension, at the relay node. Therefore, we obtain the following two alignment equations:

$$\mathbf{H}_{R1}\mathbf{w}_{12} = \mathbf{H}_{R2}\mathbf{u}_2 \quad (12)$$

$$\mathbf{H}_{R1}\mathbf{w}_2 = \mathbf{H}_{R2}\mathbf{u}_1. \quad (13)$$

At the Relay Node: As what we described above, x_{22} and x_{21} align with $x_{11}(2)$ and x_{12} respectively at the relay node. Since the three symbols from S_1 are carried by three linearly independent directions, the relay is able to demodulate the three symbols $x_{11}(1)$, $x_{11}(2) + x_{22}$ and $x_{12} + x_{21}$ in a three dimensional space, and then forwards them with other three beamforming vectors \mathbf{v}_1 , \mathbf{v}_2 and \mathbf{v}_3 .

At the Destinations: At the destination side, D_1 intends to decode three symbols $x_{11}(1)$, $x_{11}(2)$ and x_{21} . Thus, we need to protect a three dimensional space for the desired symbols. Since it can only see a three dimensional space, we need to neutralize two interference symbols x_{12} and x_{22} at

D_1 . Since each interference symbol can arrive at D_1 through two paths, i.e., one direct from the source and the other via the relay, we have the following two equations:

$$\mathbf{H}_{1R}\mathbf{v}_2 + \mathbf{H}_{12}\mathbf{v}_2 = \mathbf{0} \quad (14)$$

$$\mathbf{H}_{1R}\mathbf{v}_3 + \mathbf{H}_{11}\mathbf{w}_2 = \mathbf{0}. \quad (15)$$

At the other destination, D_2 wishes to decode x_{12} and x_{22} . Similarly D_2 has a three dimensional space, and thus only one dimensional space can be left for the interference symbol. Because there are three interference symbols, we neutralize one interference symbol $x_{11}(1)$,

$$\mathbf{H}_{2R}\mathbf{v}_1 + \mathbf{H}_{21}\mathbf{w}_{11} = \mathbf{0} \quad (16)$$

and align the other two interference symbols $x_{11}(2)$ and x_{21} in one dimension, each again transmitted from the source and the relay. That is,

$$\mathbf{H}_{2R}\mathbf{v}_3 + \mathbf{H}_{22}\mathbf{u}_1 = \mathbf{H}_{2R}\mathbf{v}_2 + \mathbf{H}_{21}\mathbf{w}_{12}. \quad (17)$$

So far we have obtained six equations from (14) to (17). Because we have a total of eight beamforming vectors that need specifying, i.e., three at S_1 , two at S_2 , and three at R , intuitively we can randomly pick two of them, and rewrite the other six beamforming vectors as a function of the two that we pick. Here let us pick \mathbf{w}_{11} and \mathbf{w}_2 randomly, and thus the remaining six beamforming vectors can be written as:

$$\mathbf{w}_{12} = \mathbf{H}_{R1}^{-1}\mathbf{H}_{R2}(\mathbf{H}_{21}\mathbf{H}_{R1}^{-1}\mathbf{H}_{R2} - \mathbf{H}_{2R}\mathbf{H}_{1R}^{-1}\mathbf{H}_{12})^{-1}(\mathbf{H}_{22}\mathbf{H}_{R2}^{-1}\mathbf{H}_{R1} - \mathbf{H}_{2R}\mathbf{H}_{1R}^{-1}\mathbf{H}_{11})\mathbf{w}_2 \quad (18)$$

$$\mathbf{u}_1 = \mathbf{H}_{R2}^{-1}\mathbf{H}_{R1}\mathbf{w}_2 \quad (19)$$

$$\mathbf{u}_2 = (\mathbf{H}_{21}\mathbf{H}_{R1}^{-1}\mathbf{H}_{R2} - \mathbf{H}_{2R}\mathbf{H}_{1R}^{-1}\mathbf{H}_{12})^{-1}(\mathbf{H}_{22}\mathbf{H}_{R2}^{-1}\mathbf{H}_{R1} - \mathbf{H}_{2R}\mathbf{H}_{1R}^{-1}\mathbf{H}_{11})\mathbf{w}_2 \quad (20)$$

$$\mathbf{v}_1 = -\mathbf{H}_{2R}^{-1}\mathbf{H}_{21}\mathbf{w}_{11} \quad (21)$$

$$\mathbf{v}_2 = -\mathbf{H}_{1R}^{-1}\mathbf{H}_{12}(\mathbf{H}_{21}\mathbf{H}_{R1}^{-1}\mathbf{H}_{R2} - \mathbf{H}_{2R}\mathbf{H}_{1R}^{-1}\mathbf{H}_{12})^{-1}(\mathbf{H}_{22}\mathbf{H}_{R2}^{-1}\mathbf{H}_{R1} - \mathbf{H}_{2R}\mathbf{H}_{1R}^{-1}\mathbf{H}_{11})\mathbf{w}_2 \quad (22)$$

$$\mathbf{v}_3 = -\mathbf{H}_{1R}^{-1}\mathbf{H}_{11}\mathbf{w}_2. \quad (23)$$

What remains to be shown is that with the beamforming vectors that we design, each destination node is able to decode its desired symbols. Note that this also guarantees the linear independencies among the beamforming vectors associated with each source. Let us consider each destination node respectively.

At the Destination D_1 : At D_1 , since the two interference symbols are neutralized, we only need to show that the three desired symbols arrive at the receiver 1 in three linearly independent directions. The received signal vector at D_1 is given by:

$$y_1 = (\mathbf{H}_{1R}\mathbf{v}_1 + \mathbf{H}_{11}\mathbf{w}_{11})x_{11}(1) + (\mathbf{H}_{1R}\mathbf{v}_2 + \mathbf{H}_{11}\mathbf{w}_{12})x_{11}(2) + (\mathbf{H}_{1R}\mathbf{v}_3 + \mathbf{H}_{12}\mathbf{u}_1)x_{21} + z_1. \quad (24)$$

Thus, we need to show the three vectors

$$\mathbf{f}_1 \triangleq \mathbf{H}_{1R}\mathbf{v}_1 + \mathbf{H}_{11}\mathbf{w}_{11} \quad (25)$$

$$\mathbf{f}_2 \triangleq \mathbf{H}_{1R}\mathbf{v}_2 + \mathbf{H}_{11}\mathbf{w}_{12} \quad (26)$$

$$\mathbf{f}_3 \triangleq \mathbf{H}_{1R}\mathbf{v}_3 + \mathbf{H}_{12}\mathbf{u}_1 \quad (27)$$

carrying the three symbols respectively, are linearly independent. Substituting the equations of (18) to (23), we obtain:

$$\mathbf{f}_1 = (\mathbf{H}_{11} - \mathbf{H}_{1R}\mathbf{H}_{2R}^{-1}\mathbf{H}_{21})\mathbf{w}_{11} \quad (28)$$

$$\mathbf{f}_3 = (\mathbf{H}_{12}\mathbf{H}_{R2}^{-1}\mathbf{H}_{R1} - \mathbf{H}_{11})\mathbf{w}_2 \quad (29)$$

$$\begin{aligned} \mathbf{f}_2 &= -(\mathbf{H}_{12}\mathbf{H}_{R2}^{-1}\mathbf{H}_{R1} - \mathbf{H}_{11})\mathbf{H}_{R1}^{-1}\mathbf{H}_{R2}(\mathbf{H}_{21}\mathbf{H}_{R1}^{-1}\mathbf{H}_{R2} - \mathbf{H}_{2R}\mathbf{H}_{1R}^{-1}\mathbf{H}_{12})^{-1}(\mathbf{H}_{22}\mathbf{H}_{R2}^{-1}\mathbf{H}_{R1} - \mathbf{H}_{2R}\mathbf{H}_{1R}^{-1}\mathbf{H}_{11})\mathbf{w}_2 \\ &= \underbrace{-\mathbf{H}_{R1}^{-1}\mathbf{H}_{R2}(\mathbf{H}_{21}\mathbf{H}_{R1}^{-1}\mathbf{H}_{R2} - \mathbf{H}_{2R}\mathbf{H}_{1R}^{-1}\mathbf{H}_{12})^{-1}(\mathbf{H}_{22}\mathbf{H}_{R2}^{-1}\mathbf{H}_{R1} - \mathbf{H}_{2R}\mathbf{H}_{1R}^{-1}\mathbf{H}_{11})}_{\mathbf{H}_f}\mathbf{f}_3. \end{aligned} \quad (30)$$

Since each matrix \mathbf{H}_{ji} is generated i.i.d., it turns out in (30) that the matrix \mathbf{H}_f is not a identity matrix almost surely. Thus, \mathbf{f}_2 and \mathbf{f}_3 , as functions of only \mathbf{w}_2 , are two linearly independent almost surely. In addition, because \mathbf{f}_1 is a function of only \mathbf{w}_{11} which is independently generated with \mathbf{w}_2 , the three vectors \mathbf{f}_1 , \mathbf{f}_2 and \mathbf{f}_3 are linearly independent. Therefore, D_1 is able to decode its three symbols successfully.

At the Destination D_2 : The sink D_2 wishes to decode the two symbols x_{12} and x_{22} . Since we neutralize one interference symbol, and align the other two interference symbols in one dimension, we only need to show that the two desired symbols x_{12} , x_{22} and the remaining interference (summation) symbol $x_{11}(2) + x_{21}$ arrive at D_2 in three linearly independent directions. The received signal vector at D_2 is given by:

$$y_2 = (\mathbf{H}_{2R}\mathbf{v}_3 + \mathbf{H}_{21}\mathbf{w}_2)x_{12} + (\mathbf{H}_{2R}\mathbf{v}_2 + \mathbf{H}_{22}\mathbf{u}_2)x_{22} + (\mathbf{H}_{2R}\mathbf{v}_3 + \mathbf{H}_{22}\mathbf{u}_1)(x_{12}(2) + x_{21}) + z_2. \quad (31)$$

Our aim is to show the three vectors

$$\mathbf{g}_1 \triangleq \mathbf{H}_{2R}\mathbf{v}_3 + \mathbf{H}_{21}\mathbf{w}_2 \quad (32)$$

$$\mathbf{g}_2 \triangleq \mathbf{H}_{2R}\mathbf{v}_2 + \mathbf{H}_{22}\mathbf{u}_2 \quad (33)$$

$$\mathbf{g}_3 \triangleq \mathbf{H}_{2R}\mathbf{v}_3 + \mathbf{H}_{22}\mathbf{u}_1 \quad (34)$$

are linearly independent. Again substituting the equations of (18) to (23), we obtain:

$$\mathbf{g}_1 = (\mathbf{H}_{21} - \mathbf{H}_{2R}\mathbf{H}_{1R}^{-1}\mathbf{H}_{11})\mathbf{w}_2 \quad (35)$$

$$\mathbf{g}_2 = (\mathbf{H}_{22} - \mathbf{H}_{2R}\mathbf{H}_{1R}^{-1}\mathbf{H}_{12})(\mathbf{H}_{21}\mathbf{H}_{R1}^{-1}\mathbf{H}_{R2} - \mathbf{H}_{2R}\mathbf{H}_{1R}^{-1}\mathbf{H}_{12})^{-1}(\mathbf{H}_{22}\mathbf{H}_{R2}^{-1}\mathbf{H}_{R1} - \mathbf{H}_{2R}\mathbf{H}_{1R}^{-1}\mathbf{H}_{11})\mathbf{w}_2 \quad (36)$$

$$\mathbf{g}_3 = (\mathbf{H}_{22}\mathbf{H}_{R2}^{-1}\mathbf{H}_{R1} - \mathbf{H}_{2R}\mathbf{H}_{1R}^{-1}\mathbf{H}_{11})\mathbf{w}_2. \quad (37)$$

It turns out that all these three vectors are functions of only \mathbf{w}_2 that we can pick freely. In order to show more brevity, let $\mathbf{w}_2 = (\mathbf{H}_{22}\mathbf{H}_{R2}^{-1}\mathbf{H}_{R1} - \mathbf{H}_{2R}\mathbf{H}_{1R}^{-1}\mathbf{H}_{11})^{-1}\mathbf{w}_0$ where we pick \mathbf{w}_0 randomly. Thus, the three vectors above can be rewritten as:

$$\mathbf{g}_1 = \underbrace{(\mathbf{H}_{21} - \mathbf{H}_{2R}\mathbf{H}_{1R}^{-1}\mathbf{H}_{11})(\mathbf{H}_{22}\mathbf{H}_{R2}^{-1}\mathbf{H}_{R1} - \mathbf{H}_{2R}\mathbf{H}_{1R}^{-1}\mathbf{H}_{11})^{-1}}_{\mathbf{H}_{g_1}}\mathbf{w}_0 \quad (38)$$

$$\mathbf{g}_2 = \underbrace{(\mathbf{H}_{22} - \mathbf{H}_{2R}\mathbf{H}_{1R}^{-1}\mathbf{H}_{12})(\mathbf{H}_{21}\mathbf{H}_{R1}^{-1}\mathbf{H}_{R2} - \mathbf{H}_{2R}\mathbf{H}_{1R}^{-1}\mathbf{H}_{12})^{-1}}_{\mathbf{H}_{g_2}}\mathbf{w}_0 \quad (39)$$

$$\mathbf{g}_3 = \mathbf{w}_0. \quad (40)$$

Notice that \mathbf{H}_{12} does not appear in \mathbf{H}_{g_1} but \mathbf{H}_{g_2} depends on \mathbf{H}_{12} , and also \mathbf{H}_{11} does not appear in \mathbf{H}_{g_2} but \mathbf{H}_{g_1} depends on \mathbf{H}_{11} . Thus, \mathbf{g}_1 and \mathbf{g}_2 are linearly independent almost surely. In addition,

$\mathbf{H}_{\mathbf{g}_1}$ and $\mathbf{H}_{\mathbf{g}_2}$ are both not identity matrices almost surely. Hence, \mathbf{g}_1 , \mathbf{g}_2 and \mathbf{g}_3 are linearly independent almost surely². The two desired symbols therefore can be decoded successfully.

Since five symbols are transmitted over three time slots, a total of 5/3 DoF is achievable almost surely. ■

Remark: If the channel coefficient is constant-valued over the time, we need the *rational alignment* scheme to achieve 5/3 DoF. Instead of in the linear space, the rational alignment scheme mimics the linear alignment scheme in rational space, and thus it can be easily translated from the linear alignment scheme.

4 Another Interpretation of Achieving 5/3 DoF

In this section, we provide another perspective to interpret the achievability of 5/3 DoF in the channel model defined in Section 2, by counting the number of equations and variables. The new perspective is based on interference neutralization and interference alignment in linear spaces.

Let us consider N symbol extension, and thus each node can see an N dimensional space. We assume that relay node only amplifies and forwards the received signals to the destinations, which means the relay node multiplies an $N \times N$ square matrix to the received signal vector and then forwards to the destinations. At the source side, we need to design beamforming vectors to encode symbols such that each destination is able to decode its desired symbols.

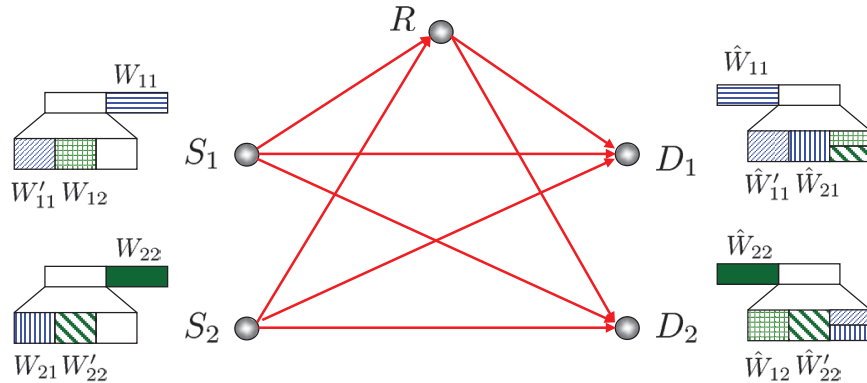


Figure 3: Another Interpretation of Achieving 5/3 DoF

We design the transmission scheme by two steps.

In the first step, each source S_i only talks to the destination D_i , i.e., it forms a two-user interference channel. In Figure 3, we explicate this interference channel by sending the message W_{ii} only intended to D_i from S_i . The symbols in the first step are transmitted using randomly picked beamforming vectors. Since the transmitted symbols can arrive at the unintended destination through two paths, i.e., $S_i - D_j$ and $S_i - R - D_j$ ($i \neq j$), we design the $N \times N$ amplifying matrix at the relay such that the interference can be neutralized. Specifically, we denote \mathbf{A} as the $N \times N$ amplifying matrix at the relay. Let the interference at D_1 and D_2 be neutralized, then we have:

$$(\mathbf{H}_{1R}\mathbf{A}\mathbf{H}_{R2} + \mathbf{H}_{12})\mathbf{x}_2 = 0 \quad (41)$$

$$(\mathbf{H}_{2R}\mathbf{A}\mathbf{H}_{R1} + \mathbf{H}_{21})\mathbf{x}_1 = 0. \quad (42)$$

²The rigorous proof of linear independencies among \mathbf{g}_1 , \mathbf{g}_2 and \mathbf{g}_3 is not difficult yet complicated, and thus we omit it here for brevity. We recommend readers to refer [14] for the rigorous proof.

where \mathbf{x}_i is the transmit signal vector of user i which carries the same number of symbols, and is given by:

$$\mathbf{x}_i = [\mathbf{w}_{i1} \ \mathbf{w}_{i2} \ \cdots \ \mathbf{w}_{iN_1}] [u_{i1} \ u_{i2} \ \cdots \ u_{iN_1}]^T. \quad (43)$$

where u_{i1}, \dots, u_{iN_1} are denote symbols of user i in the first step, and $\mathbf{w}_{i1}, \dots, \mathbf{w}_{iN_1}$ are their corresponding beamforming vectors. Since the beamforming vectors are picked randomly, and in order to satisfy the two neutralization equations, we have:

$$(\mathbf{H}_{1R}\mathbf{A}\mathbf{H}_{R2} + \mathbf{H}_{12}) [\mathbf{w}_{21} \ \mathbf{w}_{22} \ \cdots \ \mathbf{w}_{2N_1}] = \mathbf{O}_{N \times N_1} \quad (44)$$

$$(\mathbf{H}_{2R}\mathbf{A}\mathbf{H}_{R1} + \mathbf{H}_{21}) [\mathbf{w}_{11} \ \mathbf{w}_{12} \ \cdots \ \mathbf{w}_{1N_1}] = \mathbf{O}_{N \times N_1}. \quad (45)$$

Note that the six channel matrices \mathbf{H}_{1R} , \mathbf{H}_{2R} , \mathbf{H}_{R1} , \mathbf{H}_{R2} , \mathbf{H}_{12} , \mathbf{H}_{21} , and the beamforming vectors at the source are all independently generated. Our aim is to design the entries of the matrix \mathbf{A} to satisfy all linear equations in (44) and (45). In other words, we need to find non-zero solutions for all those linear equations. It is well known that if linear equations have non-zero solutions almost surely, then the necessary condition is that the number of variables should be larger than that of equations³. The entries of the matrix \mathbf{A} comprises of the variables, and thus we obtain $2NN_1 < N^2$. Here N_1 is the number of symbols that we can transmit at each source in the first step. Thus, the maximum number of the first category symbols is equal to

$$\max N_1 = N_1^* = \lfloor \frac{N^2 - 1}{2N} \rfloor \Rightarrow \lim_{N \rightarrow \infty} \frac{N_1^*}{N} = \frac{1}{2}. \quad (46)$$

It implies that we can achieve a normalized of 1/2 DoF per user in the first step design. Once we establish this, the amplifying matrix \mathbf{A} at the relay is fixed as well. Moreover, each destination receives the desired signal within one half of its linear space, leaving the other half clean.

In the second step, each source sends one independent message to each destination, i.e., it forms a 2×2 user X channel as shown in Figure 3. Since we have only one half of the linear space still clean for the X channel, the total number of DoF for this X channel is $\frac{1}{2} \times \frac{4}{3} = \frac{2}{3}$.

Adding up the number of DoF that we obtained from the two step shown above, we have the sum DoF = $\frac{1}{2} \times 2 + \frac{2}{3} = \frac{5}{3}$.

5 Conclusion

In this paper we study the achievable degrees of freedom (DoF) of the two user X channel with an instantaneous relay working in full-duplex mode. We show that 5/3 DoF is achievable almost surely, a 25% DoF improvement compared to the two user X channel that has 4/3 DoF. The achievable scheme incorporates aligned interference neutralization, linear forwarding and interference alignment. Moreover, we provide another perspective - counting the number of linear equations and variables - to interpret the new DoF result.

Several issues are worthy of further study. One issue is whether 5/3 DoF is optimal almost surely for the channel setting we concern. Specifically, the channel has two DoF if the channel coefficients satisfy some constraints even though they only constitute a subset with measure zero. In the non-zero measure sense, the outer bound of the two user X channel and even the interference channel with an instantaneous relay is still open. Another interesting issue is that what are the DoF of the interference or X networks if the relay works in half-duplex mode, i.e., the transmission in different paths may have different delays.

³In this work, it can also be easily shown that this is also the sufficient condition.

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