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Generating Functions for Composition/Word Statistics

A dissertation submitted in partial satisfaction of the
requirements for the degree
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by

Evan Fuller

Committee in charge:

Professor Jeffrey Remmel, Chair
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2009

The dissertation of Evan Fuller is approved, and it is acceptable in quality and form for publication on microfilm:

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Chair

University of California, San Diego

2009

DEDICATION

I dedicate this dissertation to my wife and parents for their love and support through the difficult times I faced while writing it.

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ACKNOWLEDGEMENTS

I want to acknowledge Professor Jeffrey Remmel for his support as my advisor. His consistent guidance, both personal and professional, made this dissertation possible. He helped maintain my love of mathematics, and his patience and willingness to fully explore arguments helped me immensely.

Chapter 3 contains some material as it will appear in Annals of Combinatorics, Fuller, E; Remmel, J; 2009. The dissertation author was a coauthor of this paper.

Chapter 4 has been submitted for publication of the material as it may appear in Discrete Mathematics, Fuller, E; Remmel, J; 2009. The dissertation author was a coauthor of this paper.

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ABSTRACT OF THE DISSERTATION

Generating Functions for Composition/Word Statistics

by

Evan Fuller
Doctor of Philosophy in Mathematics

University of California San Diego, 2009

Professor Jeffrey Remmel, Chair

The statistics $\mathrm{des}, \mathrm{inv}, \mathrm{maj}$ are well-known statistics on S_n. A central theme of this dissertation is to extend these statistics and others to compositions. A composition, or word on $\mathbb{P}$, the set of positive integers, is simply a sequence of positive integers $\gamma_1, \gamma_2, \ldots, \gamma_n$. In Chapter 3, we derive generating functions for basic composition statistics such as $\mathrm{des}, \mathrm{inv}, \mathrm{maj}$, as well as statistics unique to compositions such as lev and levmaj, defined by

$$\begin{aligned} Lev(\gamma) &= \{i : \gamma_i = \gamma_{i+1}\}, \\ \mathrm{lev}(\gamma) &= |Lev(\gamma)|, \text{ and} \\ \mathrm{levmaj}(\gamma) &= \sum_{i \in Lev(\gamma)} i. \end{aligned}$$

A permutation $\sigma \in S_n$ is called *up-down* if $\sigma_1 < \sigma_2 > \sigma_3 < \cdots$. When we consider analogues for up-down words, we find that there are four classes to consider: strict or weak increase, followed by strict or weak decrease. We derive generating functions for all four classes in Chapter 4, and we also generalize previous results to words that have a weakly/strictly increasing block of length s followed by a weak/strict decrease. In order to handle all classes, we use an involution that reduces the original classes considered to ones that are easier to count. In Chapter 5, we generalize these results by forcing the final letter in each block of length s to be in some set $X \subset \mathbb{P}$.

In Chapter 6, we apply an alternate method to find the generating functions for certain classes of alternating words on alphabet $\mathbb{P}$. In addition, we use the results of previous chapters to find generating functions for statistics defined by

$$\begin{aligned}\text{altdes}(w) &= |\{2i : w_{2i} > w_{2i+1}\} \cup \{2i+1 : w_{2i+1} < w_{2i+2}\}|,\\ \text{waltdes}(w) &= |\{2i : w_{2i} \geq w_{2i+1}\} \cup \{2i+1 : w_{2i+1} \leq w_{2i+2}\}|,\\ \text{altmaj}(w) &= \sum_{i \in Altdes(w)} i, \text{ and}\\ \text{waltmaj}(w) &= \sum_{i \in Waltdes(w)} i.\end{aligned}$$

Finally, in Chapter 7 we find generating functions for additional composition patterns. For instance, we can partition a composition into blocks of fixed length and count levels between the maxima of these blocks.

Chapter 1

Introduction

1.1 General introduction

A *permutation statistic* is a function mapping permutations to nonnegative integers. The modern analysis of such objects began in the early twentieth century with the work of Percy MacMahon [31]. He popularized the "classic" notions of the descents, rises, inversions, and major index statistics. Here if $\sigma = \sigma_1 \cdots \sigma_n$ is an element of the symmetric group S_n written in one line notation, then

$$\begin{array}{ll} \mathrm{des}(\sigma) = |\{i : \sigma_i > \sigma_{i+1}\})| & \mathrm{rise}(\sigma) = |\{i : \sigma_i < \sigma_{i+1}\}| \\ \mathrm{inv}(\sigma) = \sum_{i<j} \chi(\sigma_i > \sigma_j) & \mathrm{coinv}(\sigma) = \sum_{i<j} \chi(\sigma_i < \sigma_j) \\ \mathrm{maj}(\sigma) = \sum_{i=1}^{n-1} i\chi(\sigma_i > \sigma_{i+1}) & \mathrm{comdes}(\sigma, \tau) = |\{i : \sigma_i > \sigma_{i+1} \ \& \ \tau_i > \tau_{i+1}\}|, \end{array}$$

where for any statement A, $\chi(A)$ is 1 if A is true and 0 if A is false. These definitions make sense if $\sigma = \sigma_1 \ldots \sigma_n$ is any sequence of natural numbers, not just a permutation.

There has been a long line of research, [27], [28], [35], [37], [43], [33], showing that a large number of generating functions for permutation statistics can be obtained by applying homomorphisms defined on the ring of symmetric functions Λ in infinitely many variables $x_1, x_2, \ldots$ to simple symmetric function identities. For example, the nth elementary symmetric function, e_n, and the nth homogeneous symmetric function, h_n, are defined by the generating functions

$$E(t) = \sum_{n\geq 0} e_n t^n = \prod_i (1 + x_i t) \tag{1.1.1}$$

and

$$H(t) = \sum_{n\geq 0} h_n t^n = \prod_i \frac{1}{1 - x_i t}. \tag{1.1.2}$$

We let $P(t) = \sum_{n\geq 0} p_n t^n$, where $p_n = \sum_i x_i^n$ is the n-th power symmetric function. For any partition $\mu = (\mu_1, \ldots, \mu_\ell)$, we let $h_\mu = \prod_{i=1}^{\ell} h_{\mu_i}$, $e_\mu = \prod_{i=1}^{\ell} e_{\mu_i}$, and $p_\mu = \prod_{i=1}^{\ell} p_{\mu_i}$. It is well known that

$$H(t) = 1/E(-t) \tag{1.1.3}$$

and

$$P(t) = \frac{\sum_{n\geq 1} (-1)^{n-1} n e_n t^n}{E(-t)}. \tag{1.1.4}$$

A surprisingly large number of results on generating functions for various permutation statistics in the literature and a large number of new generating functions can be derived by applying homomorphisms on Λ to simple identities such as (1.1.3) and (1.1.4).

We shall use standard notation for q analogues: $[n]_q = 1 + q + \cdots + q^{n-1} = \frac{1-q^n}{1-q}$ and $[n]_q! = [n]_q [n-1]_q \cdots [1]_q$. In addition, let $(x;q)_0 = 1$ and $(x;q)_n = (1-x)(1-xq)\cdots(1-xq^{n-1})$. Then all of the following results can be proved by applying a suitable homomorphism to the identity (1.1.3).

1) $\sum_{n=0}^{\infty} \frac{u^n}{n!} \sum_{\sigma\in S_n} x^{\mathrm{des}(\sigma)} = \frac{1-x}{-x+e^{u(x-1)}}$.

2) (Stanley 1976) [41]
$\sum_{n=0}^{\infty} \frac{u^n}{[n]_q!} \sum_{\sigma\in S_n} x^{\mathrm{des}(\sigma)} q^{\mathrm{inv}(\sigma)} = \frac{1-x}{-x+e_q(u(x-1))}$.

3) (Stanley 1976) [41]
$\sum_{n=0}^{\infty} \frac{u^n}{[n]_q!} \sum_{\sigma\in S_n} x^{\mathrm{des}(\sigma)} q^{\mathrm{coinv}(\sigma)} = \frac{1-x}{-x+E_q(u(x-1))}$.

4) (Fedou and Rawlings 1995) [16]

$\sum_{n=0}^{\infty} \frac{u^n}{[n]_q![n]_p!} \sum_{(\sigma,\tau)\in S_n\times S_n} x^{\mathrm{comdes}(\sigma,\tau)} q^{\mathrm{inv}(\sigma)} p^{\mathrm{inv}(\tau)} = \frac{1-x}{-x+J_{q,p}(u(x-1))}$.

5) (Garsia and Gessel 1979) [18]
$\sum_{n\geq 0} \frac{t^n}{[n]_q!(x;r)_{n+1}} \sum_{\sigma\in S_n} x^{\mathrm{des}(\sigma)} r^{\mathrm{maj}(\sigma)} q^{\mathrm{inv}(\sigma)} = \sum_{k\geq 0} \frac{x^k}{e_q(-tr^0)\cdots e_q(-tr^k)}$,

where $e_q(u) = \sum_{n=0}^{\infty} \frac{u^n}{[n]_q!} q^{\binom{n}{2}}$, $E_q(u) = \sum_{n=0}^{\infty} \frac{u^n}{[n]_q!}$, and $J_{q,p}(u) = \sum_{n=0}^{\infty} \frac{u^n}{[n]_q![n]_p!}$.

One of the main goals of this dissertation is to extend these types of results to more general sequences: compositions. Here a composition γ is a sequence of positive integers $\gamma = (\gamma_1, \ldots, \gamma_k)$. We call the γ_i's the parts of γ and let $\ell(\gamma)$ denote the number of parts of γ. We let $|\gamma| = \gamma_1 + \cdots + \gamma_k$ and x^γ be the monomial $x_{\gamma_1} \cdots x_{\gamma_k}$.

Brenti [9] used a ring homomorphism to find the generating function for

$$\sum_{n\geq 0} \frac{t^n}{n!} \sum_{\sigma\in S_n} y^{\mathrm{des}(\sigma)}.$$

Gessel gave a generating function for

$$\sum_{\sigma\in S_n} x^{\mathrm{des}(\sigma)} u^{\mathrm{maj}(\sigma)} q^{\mathrm{inv}(\sigma)}$$

both in his thesis and in a paper coauthored with Garsia [18, 19]. This function was rederived by Mendes and Remmel [34] using a ring homomorphism.

Since compositions can have repeated entries, it is natural to have analogues of des and maj where we replace $>$ by $\geq$ or $=$ in the definition of des and maj. That is, if $\gamma = \gamma_1 \ldots \gamma_n$ is a composition, then we let

$$\begin{aligned} Des(\gamma) &= \{i : \gamma_i > \gamma_{i+1}\}, \\ WDes(\gamma) &= \{i : \gamma_i \geq \gamma_{i+1}\}, \text{and} \\ Lev(\gamma) &= \{i : \gamma_i = \gamma_{i+1}\}. \end{aligned}$$

Then we define

$$\begin{aligned}\mathrm{des}(\gamma) &= |Des(\gamma)|,\\ \mathrm{wdes}(\gamma) &= |WDes(\gamma)|, \text{and}\\ \mathrm{lev}(\gamma) &= |Lev(\gamma)|\end{aligned}$$

and

$$\begin{aligned}\mathrm{maj}(\gamma) &= \sum_{i\in Des(\gamma)} i,\\ \mathrm{wmaj}(\gamma) &= \sum_{i\in WDes(\gamma)} i, \text{ and}\\ \mathrm{levmaj}(\gamma) &= \sum_{i\in Lev(\gamma)} i.\end{aligned}$$

Notice that there are three analogues for des and maj, respectively. It turns out that similar methods to those used to find the generating functions for permutations can be extended to compositions. We will use these methods to find the following generating functions:

$$\sum_{n\geq 0} t^n \sum_{\gamma\in\mathbb{P}^n} y^{\mathrm{des}(\gamma)} x^{\gamma},$$

$$\sum_{n\geq 0} t^n \sum_{\gamma\in\mathbb{P}^n} y^{\mathrm{wdes}(\gamma)} x^{\gamma},$$

$$\sum_{n\geq 0} t^n \sum_{\gamma\in\mathbb{P}^n} y^{\mathrm{lev}(\gamma)} x^{\gamma},$$

$$\sum_{n\geq 0} \frac{t^n}{(y;u)_{n+1}} \sum_{\gamma\in\mathbb{P}^n} x^{\gamma} y^{\mathrm{des}(\gamma)} u^{\mathrm{maj}(\gamma)},$$

$$\sum_{n\geq 0} \frac{t^n}{(y;u)_{n+1}} \sum_{\gamma\in\mathbb{P}^n} x^{\gamma} y^{\mathrm{wdes}(\gamma)} u^{\mathrm{wmaj}(\gamma)}, \text{ and}$$

$$\sum_{n\geq 0} \frac{t^n}{(y;u)_{n+1}} \sum_{\gamma\in\mathbb{P}^n} x^{\gamma} y^{\mathrm{lev}(\gamma)} u^{\mathrm{levmaj}(\gamma)}.$$

We say that σ is an *up-down permutation* if

$$\sigma_1 < \sigma_2 > \sigma_3 < \sigma_4 > \sigma_5 \cdots .$$

André [1, 2] found the following simple generating functions for UD_n, the number of up-down permutations in S_n.

$$1 + \sum_{n \in \mathbb{E}} UD_n \frac{t^n}{n!} = \sec(t) \text{ and}$$
$$\sum_{n \in \mathbb{O}} UD_n \frac{t^n}{n!} = \tan(t),$$

where $\mathbb{E}$ is the set of even positive integers and $\mathbb{O}$ is the set of odd positive integers. When we consider analogues for up-down words, we find that there are four classes to consider: strict or weak increase, followed by strict or weak decrease. We derive generating functions for all four classes, and we also generalize previous results to words that have a weakly/strictly increasing block of length s followed by a weak/strict decrease. In order to handle all classes, we use an involution that reduces the original classes considered to ones that are easier to count. In addition, we can generalize these results by forcing the final letter in each block of length s to be in some set $X \subset \mathbb{P}$.

Chebikin [14] used up-down permutations to introduce the notion of *alternating descents* for permutations, defined by

$$\hat{d}(\sigma) = |\{2i : \sigma_{2i} < \sigma_{2i+1}\} \cup \{2i+1 : \sigma_{2i+1} > \sigma_{2i+2}\}|.$$

He also found the generating function for *alternating Eulerian polynomials*, defined as $\hat{A}_n(t) = \sum_{\sigma \in S_n} t^{\hat{d}(\sigma)+1}$. In addition, Remmel [39] introduced the notion of alternating major index, defined by

$$\text{altmaj}(\sigma) = \sum_{i \in Altdes(\sigma)} i.$$

Remmel then found the following extension of Chebikin's generating function:

$$\sum_{n \geq 0} \frac{t^n}{n!} \frac{\sum_{\sigma \in S_n} x^{\text{altdes}(\sigma)} q^{\text{altmaj}(\sigma)}}{(1-x)(1-xq)\cdots(1-xq^n)}.$$

When we consider analogues for words, we can apply both strong and weak versions of these statistics. Chebikin and Remmel defined alternating descents as places where σ *deviates* from an up-down pattern, but we find it more natural to define alternating descents as places where σ *follows* an up-down pattern. That is, we will use the following definitions.

$$\begin{aligned}
Altdes(w) &= \{2i : w_{2i} > w_{2i+1}\} \cup \{2i+1 : w_{2i+1} < w_{2i+2}\} \\
&= (\mathbb{E} \cap Des(w)) \cup (\mathbb{O} \cap Ris(w)), \\
\text{altdes}(w) &= |Altdes(w)|, \\
Waltdes(w) &= \{2i : w_{2i} \geq w_{2i+1}\} \cup \{2i+1 : w_{2i+1} \leq w_{2i+2}\} \\
&= (\mathbb{E} \cap WDes(w)) \cup (\mathbb{O} \cap WRis(w)), \\
\text{waltdes}(w) &= |Waltdes(w)|, \\
\text{altmaj}(w) &= \sum_{i \in Altdes(w)} i, \text{ and} \\
\text{waltmaj}(w) &= \sum_{i \in Waltdes(w)} i.
\end{aligned}$$

We will find the following generating functions:

$$\begin{aligned}
&\sum_{n\geq 0} t^n \sum_{w\in[m]^n} x^{\text{altdes}(w)} \\
&\sum_{n\geq 0} t^n \sum_{w\in[m]^n} x^{\text{waltdes}(w)} \\
&\sum_{n\geq 0} \frac{t^n}{(y;u)_{n+1}} \sum_{w\in[m]^n} y^{\text{altdes}(w)} u^{\text{altmaj}(w)} \text{ and} \\
&\sum_{n\geq 0} \frac{t^n}{(y;u)_{n+1}} \sum_{w\in[m]^n} y^{\text{waltdes}(w)} u^{\text{waltmaj}(w)},
\end{aligned}$$

where $[m] = \{1, 2, \ldots, m\}$.

In Chapter 2, we develop background needed to prove results in the remaining chapters. In Chapter 3, we examine analogues for the number of descents and major index of a composition. In Chapter 4, we extend existing work on up-down permutations and words to obtain four different analogues of *generalized Euler numbers* for words. That is, for any $s \geq 2$, we consider classes of words that can be divided up into an initial set of blocks of size s followed by a block of size

j where $0 \leq j \leq s-1$. We then consider the classes of such words where all the blocks are strictly increasing (weakly increasing) and there are strict (weak) decreases between blocks. We show that the weight generating functions of such words $w = w_1 \ldots w_m$, where the weight of a word is $\prod_{i=1}^{m} z_{w_i}$, is always the quotient of sums of quasi-symmetric functions. Moreover, we give a direct combinatorial proof of our results via simple involutions. In Chapter 5, we generalize the results of Chapter 4 by considering the same classes of words with the added condition that all entries at the end of a block lie in some set $X \subset \mathbb{P}$. Chapters 4 and 5 only examine words over a finite alphabet. In Chapter 6, we use a different method to obtain generating functions for two of the classes of words from Chapter 4 over an infinite alphabet. We also consider a variation on the block condition: words with equal entries within each block, but inequalities between blocks. In addition, we use the results of the previous chapters to find generating functions for the number of alternating descents and the alternating major index of a word. Finally, in Chapter 7, we again examine words that can be partitioned into blocks of fixed length, but we consider other conditions on entries within the blocks. We introduce these results in more detail in the next subsections.

1.2 Introduction to Chapter 3

The main goal of Chapter 3 is to develop generating functions for the number of descents and major index of a composition. Gessel gave a generating function for

$$\sum_{\sigma \in S_n} x^{\mathrm{des}(\sigma)} u^{\mathrm{maj}(\sigma)} q^{\mathrm{inv}(\sigma)}$$

both in his thesis and in a paper coauthored with Garsia [18, 19]. Later, Mendes and Remmel showed how Gessel's result could be derived by applying a homomorphism defined on the ring of symmetric functions. In particular, Mendes and Remmel proved the following formula, which is easily derived from the Garsia-

Gessel formula for the generating function of des(σ), maj(σ) and inv(σ):

$$\sum_{n\geq 0}\frac{t^n}{[n]_{p,q}!(x,y;u,v)_{n+1}}\sum_{\sigma\in S_n} x^{\mathrm{des}(\sigma)}y^{\mathrm{ris}(\sigma)}u^{\mathrm{maj}(\sigma)}v^{\mathrm{comaj}(\sigma)}q^{\mathrm{inv}(\sigma)}p^{\mathrm{coinv}(\sigma)}$$
$$= \sum_{k\geq 0}\frac{x^k}{y^{k+1}\mathbf{e}_{p,q}^{-t(u/v)^0}\cdots\mathbf{e}_{p,q}^{-t(u/v)^k}}.$$

Here we use standard notation from hypergeometric function theory. For $n \geq 1$ and $\lambda \vdash n$, let

$$[n]_{p,q} = \frac{p^n - q^n}{p-q} = p^{n-1}q^0 + \cdots + p^0q^{n-1},$$

and

$$[n]_{p,q}! = [n]_{p,q}\cdots[1]_{p,q},$$

be the p,q-analogues of n and $n!$. By convention, let $[0]_{p,q} = 0$ and $[0]_{p,q}! = 1$. We let $(x;q)_0 = 1$ and

$$(x;q)_n = (1-x)(1-xq)\cdots(1-xq^{n-1}).$$

In addition, let $(x,y;p,q)_0 = 1$ and

$$(x,y;p,q)_n = (x-y)(xp-yq)\cdots(xp^{n-1}-yq^{n-1}).$$

Finally, $\mathbf{e}_{p,q}^t$ is a p,q-analog for the exponential function defined by

$$\mathbf{e}_{p,q}^t = \sum_{n\geq 0}\frac{t^n}{[n]_{p,q}!}q^{\binom{n}{2}}.$$

Since compositions can have repeated entries, it is natural to have analogues of des and maj where we replace $>$ by $\geq$ or $=$ in the definition of des and maj. That is, if $\gamma = \gamma_1\ldots\gamma_n$ is a composition, then we let

$$\begin{aligned} Des(\gamma) &= \{i : \gamma_i > \gamma_{i+1}\}, \\ WDes(\gamma) &= \{i : \gamma_i \geq \gamma_{i+1}\}, \text{and} \\ Lev(\gamma) &= \{i : \gamma_i = \gamma_{i+1}\}. \end{aligned}$$

Then we define

$$\begin{aligned}
\operatorname{des}(\gamma) &= |Des(\gamma)|, \\
\operatorname{wdes}(\gamma) &= |WDes(\gamma)|, \text{and} \\
\operatorname{lev}(\gamma) &= |Lev(\gamma)|
\end{aligned}$$

and

$$\begin{aligned}
\operatorname{maj}(\gamma) &= \sum_{i \in Des(\gamma)} i, \\
\operatorname{wmaj}(\gamma) &= \sum_{i \in WDes(\gamma)} i, \text{ and} \\
\operatorname{levmaj}(\gamma) &= \sum_{i \in Lev(\gamma)} i.
\end{aligned}$$

Let $\mathbb{P}$ denote the set of positive integers, and let x^γ denote $\prod_{i=1}^n x_{\gamma_i}$. We will prove the following theorems.

Theorem 1.2.1.

$$\sum_{n \geq 0} t^n \sum_{\gamma \in \mathbb{P}^n} y^{\operatorname{des}(\gamma)} x^\gamma = \frac{1-y}{-y + \prod_{j \geq 1}(1 + t(y-1)x_j)}.$$

Theorem 1.2.2.

$$\sum_{n \geq 0} t^n \sum_{\gamma \in \mathbb{P}^n} y^{\operatorname{wdes}(\gamma)} x^\gamma = \frac{1-y}{-y + \prod_{j \geq 1} \frac{1}{1-t(y-1)x_j}}.$$

Theorem 1.2.3.

$$\sum_{n \geq 0} t^n \sum_{\gamma \in \mathbb{P}^n} y^{\operatorname{lev}(\gamma)} x^\gamma = \frac{1}{1 - \sum_{j \geq 1} \frac{t x_j}{1-t(y-1)x_j}}.$$

Theorem 1.2.4.

$$\begin{aligned}
&\sum_{n \geq 0} \frac{t^n}{(y;u)_{n+1}} \sum_{\gamma \in \mathbb{P}^n} x^\gamma y^{\operatorname{des}(\gamma)} u^{\operatorname{maj}(\gamma)} \\
&= \sum_{k \geq 0} \frac{y^k}{\prod_{i \geq 1}(x_i t; u)_{k+1}},
\end{aligned}$$

where $(x;q)_n = (1-x)(1-xq)\cdots(1-xq^{n-1})$.

Theorem 1.2.5.

$$\sum_{n\geq 0} \frac{t^n}{(y;u)_{n+1}} \sum_{\gamma\in\mathbb{P}^n} x^{\gamma} y^{\mathrm{wdes}(\gamma)} u^{\mathrm{wmaj}(\gamma)} = \sum_{k\geq 0} y^k \prod_{i\geq 1} (-x_i t;u)_{k+1}.$$

Theorem 1.2.6.

$$\sum_{n\geq 0} \frac{t^n}{(y;u)_{n+1}} \sum_{\gamma\in\mathbb{P}^n} x^{\gamma} y^{\mathrm{lev}(\gamma)} u^{\mathrm{levmaj}(\gamma)} = \sum_{k\geq 0} \frac{y^k}{\prod_{j=0}^{k}(\sum_{n\geq 0} p_n(-u^j t)^n)},$$

where $p_n = p_n(x_1, x_2, \ldots) = \sum_{i\geq 1} x_i^n$ is the nth power symmetric function. Each of these theorems can be easily extended to compositions with parts in some set $S \subset \mathbb{P}$.

It should be noted that there has been considerable work on enumerating compositions by the number of occurrences of certain patterns in a composition. For example, if $\gamma = (\gamma_1, \ldots, \gamma_k)$ is a composition and we define $\mathrm{ris}(\gamma) = |\{s : \gamma_s < \gamma_{s+1}\}|$, then Carlitz [13] proved that

$$\sum_{\gamma\in\mathbb{P}^*} u^{\ell(\gamma)} q^{|\gamma|} x^{\mathrm{ris}(\gamma)} y^{\mathrm{des}(\gamma)} z^{\mathrm{lev}(\gamma)} = \frac{e(qu(z-y),q) - e(qu(z-x),q)}{xe(qu(z-x),q) - ye(qu(z-y),q)}$$

where

$$e(x,q) = \sum_{n=0}^{\infty} \frac{x^n}{(q)_n} = \prod_{n=0}^{\infty} \frac{1}{1-q^n x}$$

and $(q)_0 = 1$ and $(q)_n = (1-q)(1-q^2)\cdots(1-q^n)$ for $n \geq 1$. Similarly, Heubach and Mansour [25] found generating functions of compositions according to the number of occurrences of various 3 letter patterns, and Mansour and Sirhan [32] extended the work of Heubach and Mansour by finding generating functions of compositions according to the number of occurrences of various l letter patterns. Enumerating various types of compositions according to other types of patterns can be found in [24], [23], and [26].

1.3 Introduction to Chapter 4

The main goal of Chapter 4 is to develop 4 different analogues of alternating permutations, proving that the weight generating functions of such words are always quotients of sums of quasi-symmetric functions.

Let $\mathbb{P} = \{1, 2, 3, \ldots\}$ denote the set of positive integers, $\mathbb{E} = \{2, 4, 6, \ldots\}$ denote the set of even integers in $\mathbb{P}$, and $\mathbb{O} = \{1, 3, 5, \ldots\}$ denote the set of odd integers in $\mathbb{P}$. Let $\mathbb{P}_n = \{1, \ldots, n\}$, $\mathbb{E}_n = \mathbb{E} \cap \mathbb{P}_n$, and $\mathbb{O}_n = \mathbb{O} \cap \mathbb{P}_n$. Let S_n denote the set of all permutations of $\mathbb{P}_n$. Then if $\sigma = \sigma_1\sigma_2 \ldots \sigma_n \in S_n$, we define $Des(\sigma) = \{i : \sigma_i > \sigma_{i+1}\}$ and $Ris(\sigma) = \{i : \sigma_i < \sigma_{i+1}\}$. We say that σ is an *up-down permutation* if

$$\sigma_1 < \sigma_2 > \sigma_3 < \sigma_4 > \sigma_5 \cdots ,$$

or, equivalently, if $Des(\sigma) = \mathbb{E}_{n-1}$ and $Ris(\sigma) = \mathbb{O}_{n-1}$. Let UD_n denote the number of up-down permutations in S_n. Then André [1, 2] proved the following.

$$\begin{aligned} \sec(t) &= 1 + \sum_{n \in \mathbb{E}} UD_n \frac{t^n}{n!} \text{ and} \\ \tan(t) &= \sum_{n \in \mathbb{O}} UD_n \frac{t^n}{n!}. \end{aligned}$$

If $s \geq 2$ and $1 \leq j \leq s-1$, let $s\mathbb{P} = \{s, 2s, 3s, \ldots\}$ and $j+s\mathbb{P} = \{j, s+j, 2s+j, \ldots\}$. For any $n > 0$, let $(s\mathbb{P})_n = s\mathbb{P} \cap \mathbb{P}_n$ and $(j + s\mathbb{P})_n = (j + s\mathbb{P}) \cap \mathbb{P}_n$. Let $E_{n,s}$ denote the number of permutations $\sigma \in S_n$ such that $Des(\sigma) = (s\mathbb{P})_{n-1}$. The $E_{n,s}$'s are called generalized Euler numbers [29]. There are well-known generating functions for q-analogues of the generalized Euler numbers; see Stanley's book [42], page 148. Various divisibility properties of the q-Euler numbers have been studied in [4, 5, 17], and properties of the generalized q-Euler numbers were studied in [20, 40]. More general generating functions for statistics on permutations $\sigma \in S_n$ such that $Des(\sigma) = (j + s\mathbb{P})_{n-1}$ were given by Mendes, Remmel, and Riehl [36].

We extend the idea of up-down permutations to words by defining the following four classes.

Definition 1.3.1. Let $s \geq 2$, $WRis(w) = \{i : w_i \leq w_{i+1}\}$,
and $WDes(w) = \{i : w_i \geq w_{i+1}\}$.

1. $SU^{s-1}SD_{n,m}$ is the set of all words $w \in \mathbb{P}_n^+$ of length m such that $Des(w) = (s\mathbb{P})_{m-1}$ and $Ris(w) = \mathbb{P}_{m-1} - (s\mathbb{P})_{m-1}$. We let $SU^{s-1}SD_n = \bigcup_{m\geq 0} SU^{s-1}SD_{n,m}$.

2. $WU^{s-1}SD_{n,m}$ is the set of all words $w \in \mathbb{P}_n^+$ of length m such that $Des(w) = (s\mathbb{P})_{m-1}$ and $WRis(w) = \mathbb{P}_{m-1} - (s\mathbb{P})_{m-1}$. We let $WU^{s-1}SD_n = \bigcup_{m\geq 0} WU^{s-1}SD_{n,m}$.

3. $SU^{s-1}WD_{n,m}$ is the set of all words $w \in \mathbb{P}_n^+$ of length m such that $WDes(w) = (s\mathbb{P})_{m-1}$ and $Ris(w) = \mathbb{P}_{m-1} - (s\mathbb{P})_{m-1}$. We let $SU^{s-1}WD_n = \bigcup_{m\geq 0} SU^{s-1}WD_{n,m}$.

4. $WU^{s-1}WD_{n,m}$ is the set of all words $w \in \mathbb{P}_n^+$ of length m such that $WDes(w) = (s\mathbb{P})_{m-1}$ and $WRis(w) = \mathbb{P}_{m-1} - (s\mathbb{P})_{m-1}$. We let $WU^{s-1}WD_n = \bigcup_{m\geq 0} WU^{s-1}WD_{n,m}$.

Carlitz [12, 11] proved analogues of André's formulas for strict up-down words. In particular, he used recursions to prove the following formulas:

$$1 + \sum_{m\in\mathbb{E}} |SU^1SD_{n,m}| z^m = \frac{1}{Q_n(z)}$$

and

$$\sum_{m\in\mathbb{O}} |SU^1SD_{n,m}| z^m = \frac{P_n(z)}{Q_n(z)},$$

where

$$\begin{aligned} P_n(z) &= \sum_{k=0}^{n} (-1)^k \binom{n+k}{2k+1} z^{2k+1} \text{ and} \\ Q_n(z) &= \sum_{k=0}^{n} (-1)^k \binom{n+k-1}{2k} z^{2k}. \end{aligned}$$

Rawlings [38] developed more general recursions, a special case of which can prove the following formulas:

$$1 + \sum_{m\in\mathbb{E}} \sum_{w\in WU^1WD_{n,m}} q^{|w|} z^{\ell(w)} = \frac{1}{B_n(q,z)}$$

and

$$\sum_{m\in\mathbb{O}}\sum_{w\in WU^1WD_{n,m}} q^{|w|}z^{\ell(w)} = \frac{A_n(q,z)}{B_n(q,z)},$$

where

$$\begin{aligned} B_n(q,z) &= \sum_{k\geq 0}(-1)^k q^{k(k+1)}\begin{bmatrix} n+k \\ 2k \end{bmatrix}_q z^{2k} \text{ and} \\ A_n(q,z) &= \sum_{k\geq 0}(-1)^k q^{k^2+2k+1}\begin{bmatrix} n+k \\ 2k+1 \end{bmatrix}_q z^{2k+1}. \end{aligned}$$

We define the following generating functions for any $s \geq 2$:

$$\begin{aligned} H_{n,s,0}^{SU^{s-1}SD}(z_1,\ldots,z_n) &= 1+\sum_{m\in s\mathbb{P}}\sum_{w\in SU^{s-1}SD_{n,m}} z(w) \text{ and} \\ H_{n,s,j}^{SU^{s-1}SD}(z_1,\ldots,z_n) &= \sum_{m\in j+s\mathbb{P}}\sum_{w\in SU^{s-1}SD_{n,m}} z(w) \text{ for } j=1,\ldots,s-1. \end{aligned}$$

We define $H_{n,s,j}^{WU^{s-1}SD}(z_1,\ldots,z_n)$, $H_{n,s,j}^{SU^{s-1}WD}(z_1,\ldots,z_n)$, and $H_{n,s,j}^{WU^{s-1}WD}(z_1,\ldots,z_n)$ for $j=0,\ldots,s-1$ similarly.

We will find explicit expressions for each of these generating functions in terms of Gessel quasi-symmetric functions [21]. Our expressions can then be specialized to explicit formulas from the literature. Let

$$Set(\gamma) = \{\gamma_1, \gamma_1+\gamma_2, \ldots, \gamma_1+\gamma_2+\cdots+\gamma_{t-1}\}.$$

For example, if $\gamma = (2,3,1,1,2)$, then $|\gamma| = 9$ and $Set(\gamma) = \{2,5,6,7\}$. Gessel [21] defined the quasi-symmetric function

$$Q_\gamma(z_1,\ldots,z_n) = \sum_{\substack{1\leq i_1\leq\cdots\leq i_{|\gamma|}\leq n \\ i_j<i_{j+1} \text{ if } j\in Set(\gamma)}} z_{i_1}z_{i_2}\cdots z_{i_{|\gamma|}}.$$

Using a simple involution, we will prove the following theorems:

Theorem 1.3.2. *Let $s \geq 2$. Then*

$$\begin{aligned} H_{n,s,0}^{SU^{s-1}SD}(z_1,\ldots,z_n) &= 1+\sum_{m\in s\mathbb{P}}\sum_{w\in SU^{s-1}SD_{n,m}} z(w) \\ &= \frac{1}{1+\sum_{k\geq 1}(-1)^k Q_{1(1^{s-2}2)^{k-1}1^{s-1}}(z_1,\ldots,z_n)}, \end{aligned}$$

$$H_{n,s,0}^{WU^{s-1}SD}(z_1,\ldots,z_n) = 1+\sum_{m\in s\mathbb{P}}\sum_{w\in WU^{s-1}SD_{n,m}} z(w) = \frac{1}{1+\sum_{k\geq 1}(-1)^k Q_{(ks)}(z_1,\ldots,z_n)},$$

$$H_{n,s,0}^{SU^{s-1}WD}(z_1,\ldots,z_n) = 1+\sum_{m\in s\mathbb{P}}\sum_{w\in SU^{s-1}WD_{n,m}} z(w) = \frac{1}{1+\sum_{k\geq 1}(-1)^k Q_{(1^{ks})}(z_1,\ldots,z_n)},$$

and

$$H_{n,s,0}^{WU^{s-1}WD}(z_1,\ldots,z_n) = 1+\sum_{m\in s\mathbb{P}}\sum_{w\in WU^{s-1}WD_{n,m}} z(w) = \frac{1}{1+\sum_{k\geq 1}(-1)^k Q_{(s^k)}(z_1,\ldots,z_n)}.$$

Theorem 1.3.3. *Let $s \geq 2$ and $1 \leq j \leq s-1$. Then*

$$H_{n,s,j}^{SU^{s-1}SD}(z_1,\ldots,z_n) = \sum_{m\in j+s\mathbb{P}}\sum_{w\in SU^{s-1}SD_{n,m}} z(w) = \frac{\sum_{k\geq 0}(-1)^k Q_{1(1^{s-2}2)^k 1^{j-1}}(z_1,\ldots,z_n)}{1+\sum_{k\geq 1}(-1)^k Q_{1(1^{s-2}2)^{k-1}1^{s-1}}(z_1,\ldots,z_n)},$$

$$H_{n,s,j}^{WU^{s-1}SD}(z_1,\ldots,z_n) = \sum_{m\in j+s\mathbb{P}}\sum_{w\in WU^{s-1}SD_{n,m}} z(w) = \frac{\sum_{k\geq 0}(-1)^k Q_{(ks+j)}(z_1,\ldots,z_n)}{1+\sum_{k\geq 1}(-1)^k Q_{(ks)}(z_1,\ldots,z_n)},$$

$$H_{n,s,j}^{SU^{s-1}WD}(z_1,\ldots,z_n) = \sum_{m\in j+s\mathbb{P}}\sum_{w\in SU^{s-1}WD_{n,m}} z(w) = \frac{\sum_{k\geq 0}(-1)^k Q_{(1^{ks+j})}(z_1,\ldots,z_n)}{1+\sum_{k\geq 1}(-1)^k Q_{(1^{ks})}(z_1,\ldots,z_n)},$$

and

$$\begin{aligned} H_{n,s,j}^{WU^{s-1}WD}(z_1,\ldots,z_n) &= \sum_{m\in s\mathbb{P}} \sum_{w\in WU^{s-1}WD_{n,m}} z(w) \\ &= \frac{\sum_{k\geq 0}(-1)^k Q_{(s^k j)}(z_1,\ldots,z_n)}{1+\sum_{k\geq 1}(-1)^k Q_{(s^k)}(z_1,\ldots,z_n)}. \end{aligned}$$

1.4 Introduction to Chapter 5

In Chapter 4, we were able to enumerate four classes of up-down words via a simple involution. In this chapter, we will enumerate these same classes of up-down words with the added condition that all peaks–entries at the end of a block–are in a certain set $X \subset \mathbb{P}$. We will show that the same involution applies, although the results can no longer be expressed in terms of quasi-symmetric functions.

Let $s \geq 2$. We extend our definition from Chapter 4 as follows.

Definition 1.4.1. $SU^{s-1}SD_{n,X,m}$ is the set of all words $w \in \mathbb{P}_n^+$ of length m such that $w_{si} \in X$ $\forall i$, $Des(w) = (s\mathbb{P})_{m-1}$ and $Ris(w) = \mathbb{P}_{m-1} - (s\mathbb{P})_{m-1}$. We let $SU^{s-1}SD_{n,X} = \bigcup_{m\geq 0} SU^{s-1}SD_{n,m}$.

We define $WU^{s-1}SD_{n,X,m}$, $WU^{s-1}SD_{n,X}$, $SU^{s-1}WD_{n,X,m}$, $SU^{s-1}WD_{n,X}$, $WU^{s-1}WD_{n,X,m}$, $WU^{s-1}WD_{n,X}$, $SU^{s-1}WU_{n,X,m}$, $SU^{s-1}WU_{n,X}$, $WU^{s-1}WU_{n,X,m}$, $WU^{s-1}WU_{n,X}$, $SU^{s-1}SU_{n,X,m}$, $SU^{s-1}SU_{n,X}$, $WU^{s-1}SU_{n,X,m}$, and $WU^{s-1}SU_{n,X}$ similarly.

Also, define the following generating functions for any $s \geq 2$:

$$\begin{aligned} H_{n,X,s,0}^{SU^{s-1}SD}(z_1,\ldots,z_n) &= 1+\sum_{m\in s\mathbb{P}} \sum_{w\in SU^{s-1}SD_{n,X,m}} z(w) \text{ and} \\ H_{n,X,s,j}^{SU^{s-1}SD}(z_1,\ldots,z_n) &= \sum_{m\in j+s\mathbb{P}} \sum_{w\in SU^{s-1}SD_{n,X,m}} z(w) \text{ for } j=1,\ldots,s-1. \end{aligned}$$

We define $H_{n,X,s,j}^{WU^{s-1}SD}(z_1,\ldots,z_n)$, $H_{n,X,s,j}^{SU^{s-1}WD}(z_1,\ldots,z_n)$, and $H_{n,X,s,j}^{WU^{s-1}WD}(z_1,\ldots,z_n)$ for $j = 0,\ldots,s-1$ similarly.

We wish to define the following additional generating functions for $s \geq 2$:

$$P_{n,X,s,0}^{SU^{s-1}WU}(z_1,\ldots,z_n) = 1+\sum_{k\geq 1}(-1)^k \sum_{w\in SU^{s-1}WU_{n,X,ks}} z(w) \text{ and}$$
$$P_{n,X,s,j}^{SU^{s-1}WU}(z_1,\ldots,z_n) = \sum_{k\geq 0}(-1)^k \sum_{w\in SU^{s-1}WU_{n,X,ks+j}} z(w) \text{ for } j=1,\ldots,s-1.$$

We define $P_{n,X,s,j}^{WU^{s-1}WU}(z_1,\ldots,z_n)$, $P_{n,,X,s,j}^{SU^{s-1}SU}(z_1,\ldots,z_n)$, and $P_{n,X,s,j}^{WU^{s-1}SU}(z_1,\ldots,z_n)$ for $j=0,\ldots,s-1$ similarly. The same involution from Chapter 4 will give, for example, the following theorem:

Theorem 1.4.2.

$$H_{n,X,s,0}^{SU^{s-1}SD}(z_1,\ldots,z_n) = \frac{1}{P_{n,X,s,0}^{SU^{s-1}WU}(z_1,\ldots,z_n)},$$

and

$$H_{n,X,s,j}^{SU^{s-1}SD}(z_1,\ldots,z_n) = \frac{P_{n,X,s,j}^{SU^{s-1}WU}(z_1,\ldots,z_n)}{P_{n,s,0}^{SU^{s-1}WU}(z_1,\ldots,z_n)}.$$

We then consider the special case $s=2$ and $X=\mathbb{E}$ or $X=\mathbb{O}$. Let

$$\begin{aligned} EV_{n,0}^{SUSU}(z,q) &= P_{n,\mathbb{E},2,0}^{SUSU}(z_1,\ldots,z_n)|_{z_i=q^iz} \\ &= 1+\sum_{k\geq 1}(-1)^k \sum_{w\in SUSU_{n,\mathbb{E},2k}} z^{\ell(w)}q^{|w|}, \end{aligned}$$

and

$$\begin{aligned} EV_{n,1}^{SUSU}(z,q) &= P_{n,\mathbb{E},2,1}^{SUSU}(z_1,\ldots,z_n)|_{z_i=q^iz} \\ &= \sum_{k\geq 0}(-1)^k \sum_{w\in SUSU_{n,\mathbb{E},2k+1}} z^{\ell(w)}q^{|w|}. \end{aligned}$$

Then we have theorems such as the following.

Theorem 1.4.3.

$$EV_{2n,0}^{SUSU}(z,q) = \sum_{k=0}^{n}(-1)^k z^{2k} \sum_{j=0}^{k} q^{2j^2-j+4k^2+2k-4kj} \begin{bmatrix} n+j \\ 2k \end{bmatrix}_{q^2} \begin{bmatrix} k \\ j \end{bmatrix}_{q^4}$$

and

$$EV_{2n,1}^{SUSU}(z,q) = \sum_{k=0}^{n}(-1)^k z^{2k+1}([2]_{1/q}) \sum_{j=0}^{k} q^{2j^2-3j+4k^2+6k-4kj+2} \begin{bmatrix} n+j \\ 2k+1 \end{bmatrix}_{q^2} \begin{bmatrix} k \\ j \end{bmatrix}_{q^4}.$$

As the reader can see, the condition that all peaks must be in a set X forces us to use more subtle reasoning to arrive at the right generating functions. Similar theorems are proved for the other classes of words. In handling some of the cases, we also exhibit a bijection between certain classes of words.

1.5 Introduction to Chapter 6

In this chapter, we will apply another method that allows us to count two of the classes of up-down words from Chapter 4 with an infinite alphabet. Define $SU^{s-1}WD_{\infty,n} = \{w \in \mathbb{P}^n : WDes(w) = (s\mathbb{P})_{n-1}\}$ and $WU^{s-1}SD_{\infty,n} = \{w \in \mathbb{P}^n : Des(w) = (s\mathbb{P})_{n-1}\}$. We will prove the following theorem:

Theorem 1.5.1. *Let $s \geq 2$ and $1 \leq J < s$. Then*

$$\sum_{n\geq 0} t^{sn} \sum_{w \in SU^{s-1}WD_{\infty,sn}} z(w) = \left(\frac{1}{s}\sum_{i=1}^{s}\prod_{k\geq 1}(1+\zeta_i t z_k)\right)^{-1},$$

$$\sum_{n\geq 1} t^{sn-J} \sum_{w \in SU^{s-1}WD_{\infty,sn-J}} z(w) = \frac{\sum_{i=1}^{s} \zeta_i^{-J} \prod_{k\geq 1}(1+\zeta_i t z_k)}{-\sum_{i=1}^{s}\prod_{k\geq 1}(1+\zeta_i t z_k)},$$

$$\sum_{n\geq 0} t^{sn} \sum_{w \in WU^{s-1}SD_{\infty,sn}} z(w) = \left(\frac{1}{s}\sum_{i=1}^{s}\prod_{k\geq 1}\frac{1}{1-\zeta_i t z_k}\right)^{-1},$$

and

$$\sum_{n\geq 1} t^{sn-J} \sum_{w \in WU^{s-1}SD_{\infty,sn-J}} z(w) = \frac{\sum_{i=1}^{s} \zeta_i^{-J} \prod_{k\geq 1}\frac{1}{1-\zeta_i t z_k}}{-\sum_{i=1}^{s}\prod_{k\geq 1}\frac{1}{1-\zeta_i t z_k}},$$

where $\zeta_1, \ldots, \zeta_s$ are all sth roots of -1.

We will also use results from previous chapters to enumerate words by *alternating descents* and *alternating major index*. Chebikin [14] first introduced the notion of *alternating descents* for permutations, defined by

$$\hat{d}(\sigma) = |\{2i : \sigma_{2i} < \sigma_{2i+1}\} \cup \{2i+1 : \sigma_{2i+1} > \sigma_{2i+2}\}|.$$

He also found the generating function for *alternating Eulerian polynomials*, defined as $\hat{A}_n(t) = \sum_{\sigma \in S_n} t^{\hat{d}(\sigma)+1}$. That is, he showed that

$$\sum_{n \geq 1} \hat{A}_n(t) \frac{u^n}{n!} = \frac{t(1 - h(u(t-1)))}{h(u(t-1)) - t},$$

where $h(x) = \tan(x) + \sec(x)$. In addition, Remmel [39] introduced the notion of alternating major index, defined by

$$\mathrm{altmaj}(\sigma) = \sum_{i \in Altdes(\sigma)} i.$$

Remmel then extended Chebikin's generating function to the following:

$$\sum_{n \geq 0} \frac{t^n}{n!} \frac{\sum_{\sigma \in S_n} x^{\mathrm{altdes}(\sigma)} q^{\mathrm{altmaj}(\sigma)}}{(1-x)(1-xq)\cdots(1-xq^n)} =$$
$$\sum_{k \geq 0} \frac{x^k}{(\sec(-t) + \tan(-t))(\sec(-tq) + \tan(-tq)) \cdots (\sec(-tq^{k-1}) + \tan(-tq^{k-1}))}.$$

Remmel also obtained similar formulas for common alternating descents and major index, as well as for the hyperoctahedral group B_n and its subgroup D_n.

When we consider analogues for words, we can apply both strong and weak versions of these statistics. Chebikin and Remmel defined alternating descents as places where σ *deviates* from an up-down pattern, but we find it more natural to define alternating descents as places where σ *follows* an up-down pattern. That is, we will use the following definitions.

$$\begin{aligned}
Altdes(w) &= \{2i : w_{2i} > w_{2i+1}\} \cup \{2i+1 : w_{2i+1} < w_{2i+2}\} \\
&= (\mathbb{E} \cap Des(w)) \cup (\mathbb{O} \cap Ris(w)), \\
\text{altdes}(w) &= |Altdes(w)|, \\
Waltdes(w) &= \{2i : w_{2i} \geq w_{2i+1}\} \cup \{2i+1 : w_{2i+1} \leq w_{2i+2}\} \\
&= (\mathbb{E} \cap WDes(w)) \cup (\mathbb{O} \cap WRis(w)), \\
\text{waltdes}(w) &= |Waltdes(w)|, \\
\text{altmaj}(w) &= \sum_{i \in Altdes(w)} i, \text{ and} \\
\text{waltmaj}(w) &= \sum_{i \in Waltdes(w)} i.
\end{aligned}$$

Then we will prove the following theorems:

Theorem 1.5.2.

$$\sum_{n\geq 0} t^n \sum_{w\in[m]^n} x^{\text{altdes}(w)} = (1-x)\left[-x + \frac{1+\sum_{k=0}^{m}(-1)^k\binom{m+k}{2k+1}(t[x-1])^{2k+1}}{\sum_{k=0}^{m}(-1)^k\binom{m+k-1}{2k}(t[x-1])^{2k}}\right]^{-1}$$

and

$$\sum_{n\geq 0} t^n \sum_{w\in[m]^n} x^{\text{waltdes}(w)} = (1-x)\left[-x + \frac{1+\sum_{k=0}^{m}(-1)^k\binom{m+k}{2k+1}(t[x-1])^{2k+1}}{\sum_{k=0}^{m}(-1)^k\binom{m+k}{2k}(t[x-1])^{2k}}\right]^{-1}.$$

Theorem 1.5.3.

$$\sum_{n\geq 0} \frac{t^n}{(y;u)_{n+1}} \sum_{w\in[m]^n} y^{\text{altdes}(w)} u^{\text{altmaj}(w)} = \sum_{p\geq 0} \frac{y^p}{\prod_{j=0}^{p} \frac{1+\sum_{k=0}^{m}(-1)^{k+1}\binom{m+k}{2k+1}(tu^j)^{2k+1}}{\sum_{k=0}^{m}(-1)^k\binom{m+k-1}{2k}(tu^j)^{2k}}}$$

and

$$\sum_{n\geq 0} \frac{t^n}{(y;u)_{n+1}} \sum_{w\in[m]^n} y^{\text{waltdes}(w)} u^{\text{waltmaj}(w)} = \sum_{p\geq 0} \frac{y^p}{\prod_{j=0}^{p} \frac{1+\sum_{k=0}^{m}(-1)^{k+1}\binom{m+k}{2k+1}(tu^j)^{2k+1}}{\sum_{k=0}^{m}(-1)^k\binom{m+k}{2k}(tu^j)^{2k}}}.$$

1.6 Introduction to Chapter 7

This chapter builds on Chapter 4, where we considered words that could be partitioned into blocks of fixed length so that, within each block, the entries were strictly or weakly increasing and there were strict or weak increases between blocks. In this chapter, we still consider words that can be partitioned into blocks of fixed length, but we examine more general patterns within the blocks. We will consider blocks where the only condition is that the first element of each block is the (unique) maximum of the block, as well as blocks with a fixed number of rises followed by a fixed number of descents. Also, we will consider blocks with a fixed number of levels followed by a descent. We then apply the statistics des, wdes, and lev from Chapter 3 to these blocks, where we will sometimes compare maximal entries within each block and sometimes compare the final entry of one block with the first entry of the following block.

We first consider a relatively weak condition: each block has a strong maximum at a particular place in the block (say, the first). Let

$$BlockMax(K, Kn) = \{w \in \mathbb{P}^{Kn} : w_{iK+1} > w_j \text{ for } j = iK + 2, \ldots, (i + 1)K\}.$$

For words in this class, we will be interested in block levels, or places in which maxima with the same value. Let

$$\text{levKmax}(w) = |\{i : \max_{j=iK+1}^{(i+1)K} w_j = \max_{j=(i+1)K+1}^{(i+2)K} w_j\}|.$$

For example, when $K = 4$, the word $w = 6\ 3\ 5\ 4|7\ 1\ 4\ 2|7\ 5\ 6\ 3 \in BlockMax(4, 12)$ has $\text{levKmax}(w) = 1$, coming from the repeated maximal element 7 ("|" indicates separations between blocks). We will prove the following theorem.

Theorem 1.6.1.

$$\sum_{n\geq 0} t^{Kn} \sum_{w\in BlockMax(K,Kn)} x^{\text{levKmax}(w)} q^{|w|}$$
$$= \left(1 - \sum_{j\geq 1} \frac{t^K q^{(j+K)}([j]_q)^{(K-1)}}{1 - t^K(x-1)q^{(j+K)}([j]_q)^{(K-1)}}\right)^{-1}$$

We will also consider the condition that each block has r (strong) rises followed by d (strong) descents. Let $K = r + d + 1$, and let $SU^rSD^d(n)$ be the set of words $w \in \mathbb{P}^n$ with this pattern. For example, one element of $SU^2SD^3(12)$ is given by 1 3 7 6 2 1|2 4 8 5 4 3. We will prove the following theorem and corollary.

Theorem 1.6.2.

$$\sum_{n\geq 0} t^{Kn} \sum_{w\in SU^rSD^d(Kn)} x^{\mathrm{levKmax}(w)} q^{|w|} = \left(1 - \sum_{j\geq 1} \frac{t^K q^{j+K+\binom{r}{2}+\binom{d}{2}} \begin{bmatrix} j \\ r \end{bmatrix}_q \begin{bmatrix} j \\ d \end{bmatrix}_q}{1-(x-1)t^K q^{j+K+\binom{r}{2}+\binom{d}{2}} \begin{bmatrix} j \\ r \end{bmatrix}_q \begin{bmatrix} j \\ d \end{bmatrix}_q}\right)^{-1}.$$

Corollary 1.6.3.

$$\sum_{n\geq 0} t^{Kn} \sum_{w\in SU^rSD^d(Kn)} q^{|w|} = \left(1 - t^K \sum_{j\geq 0} q^{j+K+\binom{r}{2}+\binom{d}{2}} \begin{bmatrix} j \\ r \end{bmatrix}_q \begin{bmatrix} j \\ d \end{bmatrix}_q\right)^{-1}$$

For any $K \geq 3$, let

$$LSD(K,m,n) = \{w \in [m]^n : \ \forall i, w_{iK+1} = w_{iK+2} = \cdots = w_{iK+K-1} > w_{iK+K}\};$$

i.e. the set of words with a $K-2$ level followed by a drop (in each block of length K, we have $K-1$ equal entries followed by a smaller entry). For example, one element of $LSD(4,9,8)$ is given by 5 5 5 2|9 9 9 3. Define

$$\mathrm{blockKwdes}(w) = |\{i : w_{iK} \geq w_{iK+1}\}| \text{ and } \mathrm{blockKdes}(w) = |\{i : w_{iK} > w_{iK+1}\}|.$$

Then we will prove the following theorem:

Theorem 1.6.4. *Let $K \geq 3$. Then*

$$\sum_{n\geq 0} t^{Kn} \sum_{w\in LSD(K,m,n)} x^{\mathrm{blockKwdes}(w)} = \left(1 - \frac{1}{x-1} \sum_{n\geq 1} [t^K(x-1)]^n \binom{m+n-1}{2n}\right)^{-1}$$

and

$$\sum_{n\geq 0} t^{Kn} \sum_{w\in LSD(K,m,n)} x^{\mathrm{blockKdes}(w)} = \left(1 - \frac{1}{x-1} \sum_{n\geq 1} [t^K(x-1)]^n \binom{m}{2n}\right)^{-1}$$
$$= \frac{1-x}{1-x+(1+\sqrt{t^K(x-1)})^m + (1-\sqrt{t^K(x-1)})^m}.$$

Next, we will consider words that can be partitioned into blocks of length 3, where each block has the pattern strict increase, strict decrease; and there are weak increases between blocks. Let $SUSDWU(m,n)$ be the set of such words on alphabet $[m]$ of length n. For example, one element of $SUSDWU(7,6)$ is given by 1 6 3 3 7 5. Although we cannot find a simplified expression for the generating function of $SUSDWU$ in general, we will prove the following theorem

Theorem 1.6.5.

$$\sum_{n\geq 0} |SUSDWU(m,n)| t^n = \frac{P_m(t)}{Q_m(t)},$$

where P_m and Q_m are polynomials.

Our method for proving this theorem will reduce the problem to counting

$$SUSDSD(m,n) = \{w \in [m]^n : \; w_{3i-2} < w_{3i-1} > w_{3i} > w_{3i+1} \forall i\},$$

which we accomplish by recursion, working out several example cases. For instance,

$$|SUSDSD(5,3n)| = \left(\frac{30 + 27x + 10x^2 + x^3}{1 - 10x + x^2} \right) \Bigg|_{x^{n-1}},$$

so that

$$\sum_{n\geq 0} |SUSDWU(5,3n)| t^{3n} = \frac{1 + 10t^3 + t^6}{1 - 19t^3 + 38t^6 - 9t^9 + t^{12}}.$$

Chapter 2

Background

2.1 Permutation statistics

Let S_n denote the symmetric group, and consider a permutation $\sigma \in S_n$ written in one-line notation: $\sigma = \sigma_1 \cdots \sigma_n$. Writing σ in this way, it is natural to ask about the distributions arising from patterns of rises and descents when reading σ left-to-right. The following statistics on S_n are well-known:

$$\begin{array}{ll} Des(\sigma) = \{i : \sigma_i > \sigma_{i+1}\} & Ris(\sigma) = \{i : \sigma_i < \sigma_{i+1}\} \\ \mathrm{des}(\sigma) = |Des(\sigma)| & \mathrm{ris}(\sigma) = |Ris(\sigma)| \\ \mathrm{inv}(\sigma) = \sum_{i<j} \chi(\sigma_i > \sigma_j) & \mathrm{coinv}(\sigma) = \sum_{i<j} \chi(\sigma_i < \sigma_j) \\ \mathrm{maj}(\sigma) = \sum_{i \in Des(\sigma)} i, & \end{array}$$

where for any statement A, $\chi(A) = 1$ if A is true and 0 if A is false. These statistics count the *descents*, *rises*, and *inversions* of σ. The last statistic, called the *major index* of σ, is a weighted sum of the descents of σ. For example, suppose $\sigma \in S_{12}$ is given by

$$\sigma = 4\ 6\ 9\ 1\ 12\ 7\ 10\ 3\ 5\ 2\ 8\ 11.$$

Then $Des(\sigma) = \{3, 5, 7, 9\}$, so that $\mathrm{des}(\sigma) = 4$.
$Ris(\sigma) = \{1, 2, 4, 6, 8, 10, 11\}$, so that $\mathrm{ris}(\sigma) = 7$.
$\mathrm{maj}(\sigma) = 3 + 5 + 7 + 9 = 24$.

$\mathrm{inv}(\sigma) = 3+8+6+0+5+0+2+3+0+1+1 = 29$, since there are 3 elements larger than 1 to its left, 8 elements larger than 2 to its left, etc.

2.2 q-binomial coefficients

We use standard notation for q-analogues. For $n \geq 1$, let

$$[n]_q = \frac{1-q}{1-q^n} = 1 + q + q^2 + \cdots + q^{n-1},$$
$$[n]_q! = [n]_q[n-1]_q \cdots [1]_q,$$

and

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

By convention, let $[0]_q = 0$ and $[0]_q! = 1$. For a set S of finite sequences, we will use the term q-count S to mean finding a simplified expression for

$$\sum_{s \in S} q^{|s|},$$

where $|s| = s_1 + s_2 + \ldots$.

It is well known that

$$\sum_{0 \leq a_1 \leq a_2 \leq \cdots \leq a_k \leq n} q^{a_1+a_2+\cdots+a_k} = \begin{bmatrix} n+k \\ k \end{bmatrix}_q$$

and

$$\sum_{1 \leq a_1 < a_2 < \cdots < a_k \leq n} q^{a_1+a_2+\cdots+a_k} = q^{\binom{k+1}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q.$$

Equivalently, we have the following theorems:

q-binomial theorem

$$\prod_{j=1}^{n} (1 + zq^j) = \sum_{k \geq 0} q^{\binom{k+1}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q z^k.$$

q-binomial series

$$\prod_{j=1}^{n} \frac{1}{1 - zq^j} = \sum_{k \geq 0} q^k \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q z^k.$$

2.3 Partitions

A partition of n, written $\lambda \vdash n$, is an increasing sequence of positive integers $\lambda = (\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_\ell)$ such that $n = \sum_{i=1}^{\ell} \lambda_i$. In such a situation, we write $|\lambda| = n$ and $\ell(\lambda) = \ell$. Let $x^\lambda = x_1^{\lambda_1} \cdots x_n^{\lambda_n}$. We will use several well known generating functions for partitions, see [3]:

$$1 + \sum_{n \geq 1} \sum_{\lambda \vdash n} q^{|\lambda|} t^{\text{parts}(\lambda)} = \prod_{i \geq 1} \frac{1}{1 - tq^i}$$

and

$$1 + \sum_{n \geq 1} \sum_{\lambda \vdash n} x^\lambda t^{\text{parts}(\lambda)} = \prod_{i \geq 1} \frac{1}{1 - tx_i},$$

as well as

$$1 + \sum_{n \geq 1} \sum_{\lambda \vdash n,\ \lambda \text{ has distinct parts}} q^{|\lambda|} t^{\text{parts}(\lambda)} = \prod_{i \geq 1} (1 + tq^i)$$

and

$$1 + \sum_{n \geq 1} \sum_{\lambda \vdash n,\ \lambda \text{ has distinct parts}} x^\lambda t^{\text{parts}(\lambda)} = \prod_{i \geq 1} (1 + tx_i).$$

More generally, for any set $S \subseteq \mathbb{P}$, let $Ptn_n(S)$ denote the set of partitions of n with parts from S. Then

$$1 + \sum_{n \geq 1} \sum_{\lambda \in Ptn_n(S)} x^\lambda t^{\text{parts}(\lambda)} = \prod_{i \in S} \frac{1}{1 - tx_i},$$

and

$$1 + \sum_{n \geq 1} \sum_{\lambda \in Ptn_n(S),\ \lambda \text{ has distinct parts}} x^\lambda t^{\text{parts}(\lambda)} = \prod_{i \in S} (1 + tx_i).$$

2.4 Generating functions

The generating function for a sequence of integers $a_0, a_1, \ldots$ is the formal power series $\sum_{i \geq 0} a_i t^i \in \mathbb{Z}[[t]]$. More generally, we can let the a_i themselves be power series. Let $f = \sum_{i \geq 0} a_i t^i$. Define $f \mid_{t^n}$ to be the coefficient of t^n in f; i.e. $f \mid_{t^n} = a_n$.

We will wish to apply this idea to partition generating functions. For example,

$$\left(\prod_{i\geq 1}\frac{1}{1-tq^i}\right)|_{t^n}$$

can be interpreted as enumerating all partition with n parts, weighted by size:

$$\sum_{\lambda=(\lambda_1,\ldots,\lambda_n)} q^{|\lambda|}.$$

We will also make use of the following simple observation. Let f be any formal power series in $x_1, x_2, \ldots$ and suppose $g = \sum_{i\geq 0} t^i(f\ |_{x_1^i})$.
Then $g = f(t, x_2, \ldots)$.

2.5 Symmetric functions

The idea of extracting information about permutation statistics through symmetric function theory has been used for decades, but the method of this dissertation—defining a homomorphism on the elementary symmetric functions and evaluating it on the homogeneous symmetric functions—was first given by Francesco Brenti [10, 9]. Desiree Beck and Jeff Remmel reproved his results combinatorially [6, 8, 7]. It is this approach which is closest to our own.

A symmetric polynomial p in the variables $x_1, \ldots, x_N$ is a polynomial over a field F of characteristic 0 with the property that $p(x_1, \ldots, x_N) = p(x_{\sigma_1}, \ldots, x_{\sigma_N})$ for all $\sigma = \sigma_1 \cdots \sigma_N \in S_N$. A symmetric function in the variables $x_1, x_2, \ldots$ may be thought of as a symmetric polynomial in an infinite number of variables. Let Λ be the ring of all symmetric functions (a more formal definition of Λ may be found in [30]). The previously defined elementary symmetric functions e_n and the homogeneous symmetric functions h_n are both elements of Λ. We can also define these functions in an equivalent but more intuitive way:

$$\begin{aligned} h_n(\overline{x}) &= \sum_{1 \le i_1 \le \cdots \le i_n} x_{i_1} \cdots x_{i_n}, \\ e_n(\overline{x}) &= \sum_{1 \le i_1 < \cdots < i_n} x_{i_1} \cdots x_{i_n}, \text{ and} \\ p_n(\overline{x}) &= \sum_i x_i^n. \end{aligned}$$

For example,

$$\begin{aligned} &h_3(x_1, x_2, x_3) \\ &\quad = x_1^3 + x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + x_1 x_2 x_3 + x_1 x_3^2 + x_2^3 + x_2^2 x_3 + x_2 x_3^2 + x_3^3, \\ &e_3(x_1, x_2, x_3, x_4) = x_1 x_2 x_3 + x_1 x_2 x_4 + x_1 x_3 x_4 + x_2 x_3 x_4, \text{ and} \\ &p_3(x_1, x_2, x_3, x_4) = x_1^3 + x_2^3 + x_3^3 + x_4^3. \end{aligned}$$

If $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ is an integer partition, we let $e_\lambda = e_{\lambda_1} \cdots e_{\lambda_\ell}$. The well-known fundamental theorem of symmetric functions says that
$\{e_\lambda : \lambda \text{ is a partition}\}$ is a basis for Λ. Similarly, if we define $h_\lambda = h_{\lambda_1} \cdots h_{\lambda_\ell}$ and $p_\lambda = p_{\lambda_1} \cdots p_{\lambda_\ell}$, then $\{h_\lambda : \lambda \text{ is a partition}\}$ and $\{p_\lambda : \lambda \text{ is a partition}\}$ are also bases for Λ.

2.6 Transition matrices

In this subsection, we shall present the combinatorics of the transition matrices between various bases of symmetric functions that will be needed for our methods. Since the elementary symmetric functions e_λ and the homogeneous symmetric functions h_λ are both bases for Λ, it makes sense to talk about the coefficient of the homogeneous symmetric functions when written in terms of the elementary symmetric function basis. This coefficient has been shown to equal the size of a certain set of combinatorial objects. A rectangle of height 1 and length n chopped into "bricks" of lengths found in the partition λ is known as a *brick tabloid of shape* (n) *and type* λ, or a λ-brick tabloid for short. One brick tabloid of shape (12) and type $(1, 1, 2, 3, 5)$ is displayed below.

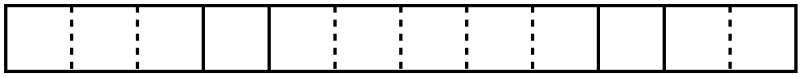

Figure 2.1: A brick tabloid of shape (12) and type $(1,1,2,3,5)$

A λ-brick tabloid can be viewed as a sequence of brick lengths $(b_1, b_2, \ldots b_{\ell(\lambda)})$, where the b_i are a rearrangement of the parts of λ. For instance, the brick lengths in Figure 2.1 are $(3,1,5,1,2)$. Let $\mathcal{B}_{\lambda,n}$ denote the set of all λ-brick tabloids of shape (n) and let $B_{\lambda,n} = |\mathcal{B}_{\lambda,n}|$. Through simple recursions stemming from (1.1.3), Eğecioğlu and Remmel proved in [15] that

$$h_n = \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} B_{\lambda,n} e_\lambda. \tag{2.6.1}$$

More generally, suppose that R is a ring and we are given any sequence $\vec{u} = (u_1, u_2, \ldots)$ of elements of R. Then for any brick tabloid $T \in \mathcal{B}_{\lambda,n}$, we set $w_{\vec{u}}(T) = u_{b_k}$, where b_k is the length of final brick in T. We then set $w_{\vec{u}}(B_{\lambda,n}) = \sum_{T \in \mathcal{B}_{\lambda,n}} w_{\vec{u}}(T)$. For example if $u = (1,2,3,\ldots)$, then $w_{\vec{u}}(T) = w(T)$ is just the length of the final brick of T. We have given $w(T)$ for each of the brick tabloids in Figure 2.2.

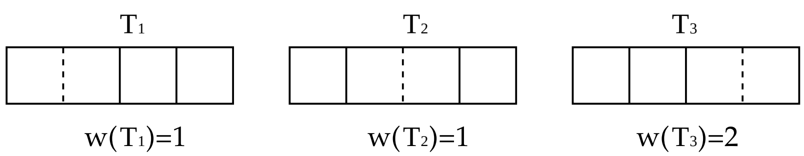

Figure 2.2: $w(\mathcal{B}_{\lambda,4})$ for $\lambda = (1,1,2)$

This given, we can define a new family of symmetric functions $p_{\lambda,\vec{u}}$ as follows. First we let $p_{0,\vec{u}} = 1$ and

$$p_{n,\vec{u}} = \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} w_{\vec{u}}(B_{\lambda,n}) e_\lambda \tag{2.6.2}$$

for $n \geq 1$. Finally if $\mu = (\mu_1, \ldots, \mu_k)$ is a partition, we set $p_{\mu,\vec{u}} = p_{\mu_1,\vec{u}} \cdots p_{\mu_k,\vec{u}}$. The functions $p_{n,\vec{u}}$ were first introduced in [28] and [35]. It follows from the results of Eğecioğlu and Remmel [15] that if $u = (1, 2, 3, \ldots)$, then $p_{n,\vec{u}}$ is just the usual power symmetric function p_n. Thus we call $p_{n,\vec{u}}$ a *generalized power symmetric function*.

Mendes and Remmel [35] proved the following:

$$\sum_{n\geq 1} p_{n,\vec{u}} t^n = \frac{\sum_{n\geq 1}(-1)^{n-1} u_n e_n t^n}{E(-t)} \text{ and } \tag{2.6.3}$$

$$1 + \sum_{n\geq 1} p_{n,\vec{u}} t^n = \frac{1 + \sum_{n\geq 1}(-1)^{n}(e_n - u_n e_n) t^n}{E(-t)}. \tag{2.6.4}$$

Note that if we take $\vec{u} = (1, 1, \ldots)$, then (2.6.3) becomes

$$1 + \sum_{n\geq 1} p_{n,\vec{u}} t^n = 1 + \frac{\sum_{n\geq 1}(-1)^{n-1} e_n t^n}{\sum_{n\geq 0}(-1)^{n} e_n t^n} = \frac{1}{\sum_{n\geq 0}(-1)^{n} e_n t^n} = 1 + \sum_{n\geq 1} h_n t^n,$$

which implies $p_{n,(1,1,\ldots)} = h_n$. Other special cases for $\vec{u}$ give well-known generating functions. For example, by taking $u_n = (-1)^k \chi(n \geq k+1)$ for some $k \geq 1$, $p_{n,\vec{u}}$ is the Schur function corresponding to the partition $(1^k, n)$.

2.7 Quasi-symmetric functions

Gessel [21] introduced quasi-symmetric functions to enumerate *P-partitions*. We let P be a partial order on $[n]$, and we use $<_p$ for the partial order P, $<$ for the usual total order.

A P-partition is a function $f : [n] \rightarrow \mathbb{P}$ s.t.

1. $i <_p j$ implies $f(i) \leq f(j)$

2. $i <_p j$ and $i > j$ implies $f(i) < f(j)$.

For example, one can view column-strict fillings of tableax as a special case of P-partitions.

Symmetric functions are typically indexed by partitions, whereas quasi-symmetric functions are indexed by compositions. Let $\gamma = (\gamma_1, \ldots, \gamma_t)$ be a composition, i.e. a sequence of nonnegative integers. Then we let $|\gamma| = \gamma_1 + \cdots + \gamma_t$ and

$$Set(\gamma) = \{\gamma_1, \gamma_1 + \gamma_2, \ldots, \gamma_1 + \gamma_2 + \cdots + \gamma_{t-1}\}.$$

For example, if $\gamma = (2, 3, 1, 1, 2)$, $|\gamma| = 9$ and $Set(\gamma) = \{2, 5, 6, 7\}$. Then Gessel [21] defined the quasi-symmetric function

$$Q_\gamma(z_1, \ldots, z_n) = \sum_{\substack{1 \le i_1 \le \cdots \le i_{|\gamma|} \\ i_j < i_{j+1} \text{ if } j \in Set(\gamma)}} z_{i_1} z_{i_2} \cdots z_{i_{|\gamma|}}. \tag{2.7.1}$$

Thus, for example, if $\gamma = (2, 3, 1, 1, 2)$, then

$$Q_\gamma(z_1, \ldots, z_n) = \sum_{1 \le i_1 \le i_2 < i_3 \le i_4 \le i_5 < i_6 < i_7 < i_8 \le i_9} \prod_{j=1}^{9} z_{i_j}.$$

Q_γ is not symmetric unless $\gamma = 1^n$ or $\gamma = n$. However, it does have the property that if $x_1 < x_2 < \cdots < x_m$ and $y_1 < y_2 < \cdots < y_m$, then the coefficient of $x_1^{i_1} \cdots x_m^{i_m}$ is equal to the coefficient of $y_1^{i_1} \cdots y_m^{i_m}$. Gessel called power series in $\mathbb{Z}[[X]]$ with this property quasi-symmetric and showed that $\{Q_\gamma : \gamma \text{ is a composition of } n\}$ is a basis for $Qsym_n$, the homogeneous quasi-symmetric functions of degree n.

Chapter 3

Basic results on composition statistics

A permutation statistic is a function mapping permutations to nonnegative integers. The modern analysis of such objects began in the early twentieth century with the work of Percy MacMahon [31]. He popularized the “classic” notions of the descents, rises, inversions, coinversions, major index and comajor index statistics. Here if $\sigma = \sigma_1 \cdots \sigma_n$ is an element of the symmetric group S_n written in one line notation, then

$$\begin{aligned} \mathrm{des}(\sigma) &= \sum_{i=1}^{n-1} \chi(\sigma_i > \sigma_{i+1}) \qquad \mathrm{ris}(\sigma) = 1 + \sum_{i=1}^{n-1} \chi(\sigma_i < \sigma_{i+1}), \\ \mathrm{inv}(\sigma) &= \sum_{1 \le i < j \le n} \chi(\sigma_i > \sigma_j) \quad \mathrm{coinv}(\sigma) = \sum_{1 \le i < j \le n} \chi(\sigma_i < \sigma_j), \\ \mathrm{maj}(\sigma) &= \sum_{i=1}^{n-1} i\, \chi(\sigma_i > \sigma_{i+1}) \quad \mathrm{comaj}(\sigma) = \sum_{i=1}^{n-1} i\, \chi(\sigma_i < \sigma_{i+1}), \end{aligned}$$

where for any statement A, $\chi(A)$ is 1 if A is true and 0 if A is false. These definitions make sense if $\sigma = \sigma_1 \ldots \sigma_n$ is any sequence of natural numbers.

The study of the properties of these statistics and subsequent generalizations of these statistics to other groups and sequences remains an active area of research today. In this chapter, we shall find analogues of the joint distribution of $\mathrm{des}(\sigma)$, maj(σ), and inv(σ). That is, Gessel gave a generating function for

$$\sum_{\sigma \in S_n} x^{\mathrm{des}(\sigma)} u^{\mathrm{maj}(\sigma)} q^{\mathrm{inv}(\sigma)} \tag{3.0.1}$$

both in his thesis and in a paper coauthored with Garsia [18, 19]. Later, Mendes and Remmel showed how Gessel's result could be derived by applying a homomorphism defined on the ring of symmetric functions Λ in infinitely many variables $x_1, x_2, \ldots$ to the simple symmetric function identity

$$H(t) = \frac{1}{E(-t)} \tag{3.0.2}$$

where $H(t)$ is the generating function for the homogeneous symmetric functions $h_n = h_n(x_1, x_2, \ldots)$ and $E(t)$ is the generating function for the elementary symmetric functions $e_n = e_n(x_1, x_2, \ldots)$. That is,

$$H(t) = \sum_{n \geq 0} h_n t^n = \prod_{n \geq 1} \frac{1}{1 - x_i t} \tag{3.0.3}$$

and

$$E(t) = \sum_{n \geq 0} e_n t^n = \prod_{n \geq 1} (1 + x_i t). \tag{3.0.4}$$

In particular, Mendes and Remmel proved the following formula, which is easily derived from the Garsia-Gessel formula for the generating function of des(σ), maj(σ) and inv(σ),

$$\sum_{n \geq 0} \frac{t^n}{[n]_{p,q}!(x, y; u, v)_{n+1}} \sum_{\sigma \in S_n} x^{\mathrm{des}(\sigma)} y^{\mathrm{ris}(\sigma)} u^{\mathrm{maj}(\sigma)} v^{\mathrm{comaj}(\sigma)} q^{\mathrm{inv}(\sigma)} p^{\mathrm{coinv}(\sigma)}$$
$$= \sum_{k \geq 0} \frac{x^k}{y^{k+1} \mathbf{e}_{p,q}^{-t(u/v)^0} \cdots \mathbf{e}_{p,q}^{-t(u/v)^k}}.$$

Here we use standard notation from hypergeometric function theory. For $n \geq 1$ and $\lambda \vdash n$, let

$$[n]_{p,q} = \frac{p^n - q^n}{p - q} = p^{n-1} q^0 + \cdots + p^0 q^{n-1},$$

and

$$[n]_{p,q}! = [n]_{p,q} \cdots [1]_{p,q},$$

be the p, q-analogues of n and $n!$. By convention, let $[0]_{p,q} = 0$ and $[0]_{p,q}! = 1$. We let $(x;q)_0 = 1$ and

$$(x;q)_n = (1-x)(1-xq)\cdots(1-xq^{n-1}).$$

In addition, let $(x,y;p,q)_0 = 1$ and

$$(x,y;p,q)_n = (x-y)(xp-yq)\cdots(xp^{n-1}-yq^{n-1}).$$

Finally, $\mathbf{e}^t_{p,q}$ is a p, q-analog for the exponential function defined by

$$\mathbf{e}^t_{p,q} = \sum_{n\geq 0} \frac{t^n}{[n]_{p,q}!} q^{\binom{n}{2}}.$$

Mendes and Remmel also showed how their methods can be used to extend such results to the hyperoctahedral group B_n and its subgroup D_n.

The main goal of this chapter is to show how the methods of Mendes and Remmel can prove similar results for compositions. Here a composition γ is a sequence of positive integers $\gamma = (\gamma_1, \ldots, \gamma_k)$. We call the γ_i's the parts of γ and let $\ell(\gamma)$ denote the number of parts of γ. We let $|\gamma| = \gamma_1 + \cdots + \gamma_k$ and x^γ be the monomial $x_{\gamma_1} \cdots x_{\gamma_n}$. Since compositions can have repeated entries, it is natural to have analogues of des and maj where we replace $>$ by $\geq$ or $=$ in the definition of des and maj. That is, if $\gamma = \gamma_1 \ldots \gamma_n$ is a composition, then we let

$$\begin{aligned} Des(\gamma) &= \{i : \gamma_i > \gamma_{i+1}\}, \\ WDes(\gamma) &= \{i : \gamma_i \geq \gamma_{i+1}\}, \text{and} \\ Lev(\gamma) &= \{i : \gamma_i = \gamma_{i+1}\}. \end{aligned}$$

Then we define

$$\begin{aligned} \text{des}(\gamma) &= |Des(\gamma)|, \\ \text{wdes} &= |WDes(\gamma)|, \text{and} \\ \text{lev} &= |Lev(\gamma)| \end{aligned}$$

and

$$\begin{aligned} \text{maj}(\gamma) &= \sum_{i \in Des(\gamma)} i, \\ \text{wmaj}(\gamma) &= \sum_{i \in WDes(\gamma)} i, \text{ and} \\ \text{levmaj}(\gamma) &= \sum_{i \in Lev(\gamma)} i. \end{aligned}$$

3.1 Descents and weak descents

Brenti [9] showed the following. Define a ring homomorphism
$\xi : \Lambda$, the ring of symmetric functions $\rightarrow \mathbb{Q}[y]$ by setting

$$\xi(e_k) = \frac{(1-y)^{k-1}}{k!},$$

where e_k is the k-th elementary symmetric function, and $\xi(e_0) = 1$. Then:

$$n!\xi(h_n) = \sum_{\sigma \in S_n} y^{\text{des}(\sigma)}$$

and

$$\sum_{n \geq 0} \frac{t^n}{n!} \sum_{\sigma \in S_n} y^{\text{des}(\sigma)} = \frac{1-y}{-y + e^{t(y-1)}}.$$

We can readily extend this result to compositions, so that we obtain the following theorems:

Theorem 3.1.1.

$$\sum_{n \geq 0} t^n \sum_{\gamma \in \mathbb{P}^n} y^{\text{des}(\gamma)} x^{\gamma} = \frac{1-y}{-y + \prod_{j \geq 1}(1 + t(y-1)x_j)}.$$

Theorem 3.1.2.

$$\sum_{n \geq 0} t^n \sum_{\gamma \in \mathbb{P}^n} y^{\text{wdes}(\gamma)} x^{\gamma} = \frac{1-y}{-y + \prod_{j \geq 1} \frac{1}{1-t(y-1)x_j}}.$$

Before proving these theorems, we will outline the general method beyond this type of result, which will be used repeatedly throughout this dissertation. The basic steps are as follows:

1. Define a homomorphism on the ring of symmetric functions by specifying its action on the elementary symmetric functions e_n.
2. Apply this homomorphism to h_n (or another class of symmetric functions) and interpret the result in terms of *labeled filled brick tabloids*.
3. Perform an *involution* on the set of all possible labeled filled brick tabloids, and characterize the fixed points of the involution.
4. Find a nice generating function using the relationship between h_n (or other symmetric functions) and e_n.

We will now use these steps to prove Theorem 3.1.2. Define $\Theta_1 : \Lambda \to \mathbb{Q}[[y, x_1, x_2, \dots]]$ by

$$\Theta_1(e_n) = (-1)^{n-1}(y-1)^{n-1} \left(\prod_{j \geq 1} \frac{1}{1 - tx_j} \right) |_{t^n}$$

for $n \geq 1$, and $\Theta_1(e_0) = 1$.

Claim:

$$\Theta_1(h_n) = \sum_{\gamma \in \mathbb{P}^n} y^{\mathrm{wdes}(\gamma)} x^{\gamma}$$

We saw in Chapter 2 that

$$h_n = \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} B_{\lambda,n} e_{\lambda}, \tag{3.1.1}$$

where $B_{\lambda,n}$ is the number of λ-brick tabloids of shape n. Thus,

$$\begin{aligned}\Theta_1(h_n) &= \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} B_{\lambda,n} \Theta_1(e_\lambda) \\ &= \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} B_{\lambda,n} \prod_{i=1}^{\ell(\lambda)} (-1)^{\lambda_i - 1}(1-y)^{\lambda_i - 1} \left(\prod_{j \geq 1} \frac{1}{1-tx_j} \right) |_{t^{\lambda_i}} \\ &= \sum_{\lambda \vdash n} B_{\lambda,n} \prod_{i=1}^{\ell(\lambda)} (y-1)^{\lambda_i - 1} \left(\prod_{j \geq 1} \frac{1}{1-tx_j} \right) |_{t^{\lambda_i}} \ . \qquad (3.1.2)\end{aligned}$$

Our goal is to interpret $\Theta_1(h_n)$ as a sum of weighted combinatorial objects. We interpret $\sum_{\lambda \vdash n} B_{\lambda,n}$ as letting us choose some $\lambda \vdash n$ and create a brick tabloid $T = (b_1, \ldots, b_{\ell(\lambda)})$ of shape n and type λ. For instance, Figure 3.1 shows a brick tabloid $T = (3, 4, 2, 2)$ of shape 12 and type $(2, 2, 3, 4)$.

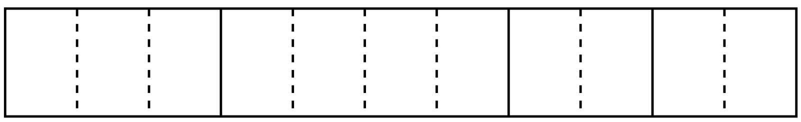

Figure 3.1: A brick tabloid of shape 12 and type $(2, 2, 3, 4)$

Recall from section 2.3 that $\left(\prod_{j \geq 1} \frac{1}{1-tx_j} \right) |_{t^{\lambda_i}}$ is the sum over all possible partitions μ with λ_i parts, weighted by x^μ [3]. Thus, we can interpret the term $\prod_{i=1}^{\ell(\lambda)} \left(\prod_{j \geq 1} \frac{1}{1-tx_j} \right) |_{t^{\lambda_i}}$ as letting us choose a partition μ^i with λ_i parts for each brick, weighted by $x^{\mu^1} \cdots x^{\mu^{\ell(\lambda)}}$. Within each brick, we write the chosen partition in weakly decreasing order, with one part per cell. Figure 3.2 shows a possible filling of the brick tabloid from Figure 3.1.

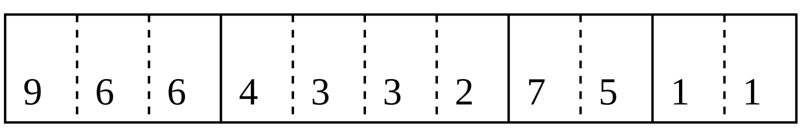

Figure 3.2: A filled brick tabloid of shape 12 and type $(2, 2, 3, 4)$ coming from Equation 3.1.2

Next, the term $\prod_{i=1}^{\ell(\lambda)} (y-1)^{\lambda_i - 1}$ lets us leave the last cell of every brick alone, and label every other cell of each brick with either a y or -1. We call the result a *filled labeled brick tabloid* (also, a *decorated* brick tabloid). Figure 3.3 shows an example of a filled labeled brick tabloid.

y	−1		y	−1	y		y		−1	
9	6	6	4	3	3	2	7	5	1	1

Figure 3.3: A filled labeled brick tabloid of shape 12 and type $(2, 2, 3, 5)$ coming from Equation 3.1.2

Let $\mathcal{T}_{\Theta_1}(n)$ be the set of filled labeled brick tabloids obtained by interpreting every term in this sum. Thus, a $C \in \mathcal{T}_{\Theta_1}(n)$ consists of a brick tabloid T, a composition $\gamma \in \mathbb{P}^n$, and a labeling L of the cells of T with elements from $\{y, -1\}$ such that

1. γ is strictly decreasing within each brick, and
2. each cell which is not a final cell of a brick is labeled with y or -1.

We then define the weight $W(C)$ of C to be the monomial $x^\gamma = x_{\gamma_1} x_{\gamma_2} \cdots x_{\gamma_n}$ times the product of all the y labels in L and the sign $sgn(C)$ of C to be the product of all the -1 labels in L. For example, the weight of the object in Figure 3.3 is $x_1^2 x_2 x_3^2 x_4 x_5 x_6^2 x_7 x_9 y^4$ and its sign is -1. Then $\Theta_1(h_n) = \sum_{C \in \mathcal{T}_{\Theta_1}(n)} sgn(C) W(C)$.

Next, we wish to get rid of all objects with negative sign. To this end, we define a weight-preserving, sign-reversing involution $I : \mathcal{T}_{\Theta_1}(n) \to \mathcal{T}_{\Theta_1}(n)$. To define $I(C)$, we scan the cells of $C = (T, \gamma, L)$ from left to right, looking for the leftmost cell a such that either (i) a is labeled with -1 or (ii) a is at the end of a brick b_j and the brick b_{j+1} immediately following b_j has the property that γ is weakly decreasing in all the cells corresponding to b_j and b_{j+1}.
In case (i), $I(C) = (T', \gamma', L')$, where

1. T' is the result of replacing the brick b in T containing a by two bricks b^* and b^{**}, where $b*$ contains the cell a plus all the cells in b to the left of a and b^{**} contains all the cells in b to the right of a;

2. $\gamma' = \gamma$; and

3. L' is the labeling that results from L by removing the -1 label of cell a.

In case (ii), $I(C) = (T', \gamma', L')$, where

1. T' is the result of replacing the bricks b_j and b_{j+1} in T by a single brick b;

2. $\gamma' = \gamma$; and

3. L' is the labeling that results from L by inserting a -1 label for cell a.

For instance, if C is the element of $\mathcal{T}_{\Theta_1}(11)$ pictured in Figure 3.3, then $I(C)$ is given in Figure 3.4. In this case, $a = 2$, since the second cell of T has a -1.

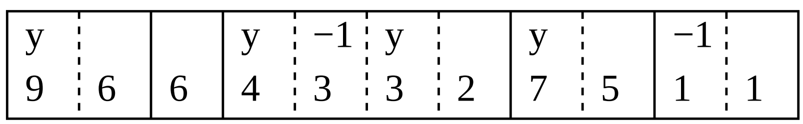

Figure 3.4: The image $I(C)$ for C in Figure 3.3

It is easy to see that I is a weight-preserving, sign-reversing involution. Hence, I shows that

$$\Theta_1(h_n) = \sum_{C \in \mathcal{T}_{\Theta_1}(n): I(C) = C} sgn(C) W(C)$$

Thus, we must examine the fixed points $C = (T, \gamma, L)$ of I. First, there can be no -1 labels in L, so that $sgn(C) = 1$. Moreover, if b_j and b_{j+1} are two consecutive bricks in T and a is the last cell of b_j, then it cannot be the case that $\gamma_a \geq \gamma_{a+1}$; otherwise, we could combine b_j and b_{j+1}. For any such fixed point, we associate the composition γ. γ must weakly decrease within each brick and strictly increase between bricks. For example, Figure 3.5 is a fixed point corresponding to the composition 9 6 6 4 3 3 2 7 5 1 1.

It follows that every cell with a weak decrease is labeled with y, and those with increases are not, so that for a fixed point $C = (T, \gamma, L)$, $sgn(C)W(C) =$

y	y	y	y	y	y		y	y	y	
9	6	6	4	3	3	2	7	5	1	1

Figure 3.5: A fixed point of I coming from Equation 3.1.2

$y^{\mathrm{wdes}(\gamma)}x^{\gamma}$. On the other hand, given any composition γ, we can create a fixed point $C = (T, \gamma, L)$ by having the bricks in T end at every cell a where $a \notin Wdes(\gamma)$. Therefore, we have shown that

$$\Theta_1(h_n) = \sum_{\gamma \in \mathbb{P}^n} y^{\mathrm{wdes}(\gamma)} x^{\gamma}$$

as desired.

It remains only to use our identity relating the e_n and h_n to obtain a generating function:

$$\begin{aligned}
&\sum_{n\geq 0} t^n \sum_{\gamma \in \mathbb{P}^n} y^{\mathrm{wdes}(\gamma)} x^{\gamma} = \sum_{n \geq 0} t^n \Theta_1(h_n) \\
&= \Theta_1\left(\sum_{n\geq 0} h_n t^n\right) = \Theta_1\left(\sum_{n \geq 0} e_n(-t)^n\right)^{-1} \\
&= \left(1 + \sum_{n\geq 1} (-t)^n (-1)^{n-1} (y-1)^{n-1} \left(\prod_{j\geq 1} \frac{1}{1-tx_j}\right)\Big|_{t^n}\right)^{-1} \\
&= \left(1 - \frac{1}{y-1} \sum_{n \geq 1} [t(y-1)]^n \left(\prod_{j \geq 1} \frac{1}{1-tx_j}\right)\Big|_{t^n}\right)^{-1} \\
&= \frac{1-y}{1 - y + \sum_{n\geq 1}[t(y-1)]^n \left(\prod_{j\geq 1} \frac{1}{1-tx_j}\right)\big|_{t^n}} \\
&= \frac{1-y}{-y + \sum_{n\geq 0}[t(y-1)]^n \left(\prod_{j\geq 1} \frac{1}{1-tx_j}\right)\big|_{t^n}} \\
&= \frac{1-y}{-y + \prod_{j \geq 1} \frac{1}{1-t(y-1)x_j}},
\end{aligned}$$

proving Theorem 3.1.2.

We will repeatedly use this method of defining an appropriate homomorphism, interpreting the homomorphism applied to one symmetric function basis in terms of filled labeled brick tabloids, and characterizing the fixed points. Our interpretations will typically be similar to that of Equation 3.1.2. To avoid burdening the reader with detail, we will usually describe our interpretations less technically than we did in this case. It is easier to process the filled labeled brick tabloids if they are described as a set of choices. In addition, we collapse the weight and sign into a single weight function. Nevertheless, each interpretation and involution can be made fully formal, just as we did above.

The proof of Theorem 3.1.1 is so similar to the proof of Theorem 3.1.2 that we omit even the detailed sketch. We use a related homomorphism $\Theta_2 : \Lambda \to \mathbb{Q}[[y, x_1, x_2, \ldots]]$ defined by

$$\Theta_2(e_n) = (-1)^{n-1}(y-1)^{n-1} \left(\prod_{j \geq 1} (1 + tx_j) \right) |_{t^n}$$

for $n \geq 1$ and $\Theta_2(e_0) = 1$. Our interpretation of $\Theta_2(h_n)$ is the same as that for $\Theta_1(h_n)$, except that we fill in each brick using a partition with *distinct* parts.

3.2 Levels

Theorem 3.2.1.

$$\sum_{n \geq 0} t^n \sum_{\gamma \in \mathbb{P}^n} y^{\mathrm{lev}(\gamma)} x^{\gamma} = \left(1 - \sum_{j \geq 1} \frac{tx_j}{1 - t(y-1)x_j} \right)^{-1}$$

To prove Theorem 3.2.1, we define a homomorphism on the ring of symmetric functions by

$$\Theta_3(e_n) = (-1)^{n-1}(y-1)^{n-1} \sum_{j \geq 1} x_j^n$$

for $n \geq 1$, and $\Theta_3(e_0) = 1$.

Claim: $\Theta_3(h_n) = \sum_{\gamma \in \mathbb{P}^n} y^{\mathrm{lev}(\gamma)} x^{\gamma}$.

Expanding h_n in terms of e_n, we get

$$\begin{aligned}\Theta_3(h_n) &= \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} B_{\lambda,n} \Theta_3(e_\lambda) \\ &= \sum_{\lambda \vdash n} B_{\lambda,n} \prod_{i=1}^{\ell(\lambda)} (y-1)^{\lambda_i - 1} \sum_{j \geq 1} x_j^{\lambda_i} \end{aligned} \tag{3.2.1}$$

Again, we interpret each term in this sum as creating a filled labeled brick tabloid in stages, where the only difference is in the partitions we use to fill in each brick. $\sum_{\lambda \vdash n} B_{\lambda,n}$ lets us choose some $\lambda \vdash n$ and create a brick tabloid of shape n and type λ. The term $\prod_{i=1}^{\ell(\lambda)} \sum_{j\geq 1} x_j^{\lambda_i}$ lets us choose a partition with identical parts for each brick (i.e. choose a number j for each brick and fill every cell of the brick with it), weighting the brick by $x_j^{\lambda_i}$. Next, the term $\prod_{i=1}^{\ell(\lambda)} (y-1)^{\lambda_i - 1}$ lets us leave the last cell of every brick alone, and label every other cell with either a y or -1. For instance, Figure 3.6 is one such object.

−1	−1		y	y	−1	y	y		−1	
5	5	5	7	7	7	7	7	7	1	1

Figure 3.6: A filled labeled brick tabloid coming from Equation 3.2.1

Let $\mathcal{T}_{\Theta_3}(n)$ denote the set of all such filled labeled brick tabloids. We define the weight $W(C)$ of $C \in \mathcal{T}_{\Theta_3}(n)$ to be the monomial $x^\gamma = x_{\gamma_1} x_{\gamma_2} \cdots x_{\gamma_n}$ times the product of all the y labels and the sign $sgn(C)$ of C to be the product of all the -1 labels. For example, the weight of the object given in Figure 3.6 is $x_1^2 x_5^3 x_7^6 y^4$ and its sign is 1. Then $\Theta_3(h_n) = \sum_{T \in \mathcal{T}_{\Theta_3}(n)} sgn(C) W(C)$.

We define an involution as follows to get rid of all objects with negative weight; the involution is very similar to that used in the previous section. Scan left to right for a -1 or two consecutive bricks with a level between them (last entry of the first brick is the same as first entry in the second brick). If a -1 is found, break the brick in two after that cell and remove the -1 label. If a level between bricks is found, insert a -1 label for the last cell in the first brick and combine the bricks. For example, the image of Figure 3.6 is depicted in Figure 3.7.

	−1		y	y	−1	y	y		−1	
5	5	5	7	7	7	7	7	7	1	1

Figure 3.7: The image of Figure 3.6

The fixed points can be read as compositions that have identical entries within each brick, but unequal entries between bricks, and only y labels. A fixed point is displayed in Figure 3.8. $\sum_{T \in \mathcal{T}_{\Theta_3}(n)} sgn(C)W(C)$ reduces to a sum over fixed points, which is exactly given by $\sum_{\gamma \in \mathbb{P}^n} y^{\text{lev}(\gamma)} x^{\gamma}$.

y	y		y	y	y	y	y		y	
5	5	5	7	7	7	7	7	7	1	1

Figure 3.8: A fixed point coming from Equation 3.2.1 when $n = 11$

Thus,

$$\begin{aligned}
\sum_{n\geq 0} t^n \sum_{\gamma \in \mathbb{P}^n} y^{\text{lev}(\gamma)} x^{\gamma} &= \Theta_3\left(\sum_{n\geq 0} h_n t^n\right) = \Theta_3\left(\sum_{n\geq 0} e_n(-t)^n\right)^{-1} \\
&= \left(1 + \sum_{n\geq 1} (-t)^n (-1)^{n-1} (y-1)^{n-1} \sum_{j\geq 1} x_j^n\right)^{-1} \\
&= \left(1 - \sum_{j\geq 1}\sum_{n\geq 1} t^n (y-1)^{n-1} x_j^n\right)^{-1} \\
&= \left(1 - \sum_{j\geq 1} \frac{t x_j}{1 - t(y-1)x_j}\right)^{-1},
\end{aligned}$$

proving Theorem 3.2.1.

3.3 Major index

Let $\mathbb{P}$ denote the set of positive integers. We shall prove the following three theorems.

Theorem 3.3.1.

$$\sum_{n\geq 0}\frac{t^n}{(y;u)_{n+1}}\sum_{\gamma\in\mathbb{P}^n} x^\gamma y^{\mathrm{des}(\gamma)}u^{\mathrm{maj}(\gamma)} = \sum_{k\geq 0}\frac{y^k}{\prod_{i\geq 1}(x_it;u)_{k+1}}.$$

Theorem 3.3.2.

$$\sum_{n\geq 0}\frac{t^n}{(y;u)_{n+1}}\sum_{\gamma\in\mathbb{P}^n} x^\gamma y^{\mathrm{wdes}}u^{\mathrm{wmaj}(\gamma)} = \sum_{k\geq 0}y^k\prod_{i\geq 1}(-x_it;u)_{k+1}.$$

Theorem 3.3.3.

$$\sum_{n\geq 0}\frac{t^n}{(y;u)_{n+1}}\sum_{\gamma\in\mathbb{P}^n} x^\gamma y^{\mathrm{lev}}u^{\mathrm{levmaj}(\gamma)} = \sum_{k\geq 0}\frac{y^k}{\prod_{j=0}^k(\sum_{n\geq 0}p_n(-u^jt)^n)},$$

where $p_n = p_n(x_1, x_2, \ldots) = \sum_{i\geq 1} x_i^n$ is the power symmetric function.

It should be noted that there has been considerable work on enumerating compositions by the number of occurrences of certain patterns in a composition. For example, if $\gamma = (\gamma_1, \ldots, \gamma_k)$ is a composition and we define $\mathrm{ris}(\gamma) = |\{s : \gamma_s < \gamma_{s+1}\}|$, then Carlitz [13] proved that

$$\sum_{\gamma\in\mathbb{P}^*} u^{\ell(\gamma)}q^{|\gamma|}x^{\mathrm{ris}(\gamma)}y^{\mathrm{des}(\gamma)}z^{\mathrm{lev}} = \frac{e(qu(z-y),q)-e(qu(z-x),q)}{xe(qu(z-x),q)-ye(qu(z-y),q)} \tag{3.3.1}$$

where

$$e(x,q) = \sum_{n=0}^{\infty}\frac{x^n}{(q)_n} = \prod_{n=0}^{\infty}\frac{1}{1-q^nx}$$

and $(q)_0 = 1$ and $(q)_n = (1-q)(1-q^2)\cdots(1-q^n)$ for $n \geq 1$. Similarly, Heubach and Mansour [25] found generating functions of compositions according to the number of occurrences of various 3 letter patterns and Mansour and Sirhan [32] extended the work of Heubach and Mansour by finding generating functions of compositions according to the number of occurrences of various l letter patterns. Enumerating various types of compositions according to other types of patterns can be found in [24], [23], and [26]. In each case, one can find such generating functions by applying the transfer matrix method, see [42], section 4.7 or [22]. The basic idea is the following. Suppose you want to find the generating function

$$C(u,v,x) = \sum_{\gamma \in \mathbb{P}^*} u^{\ell(\gamma)} v^{|\gamma|} x^{\mathrm{des}(\gamma)}. \tag{3.3.2}$$

Then one can define

$$C(i;u,v,x) = \sum_{\gamma \in \mathbb{P}^+, \gamma_1 = i} u^{\ell(\gamma)} v^{|\gamma|} x^{\mathrm{des}(\gamma)}$$

and we have simple recursions

$$C(i;u,v,x) = uv^i + uv^i (\sum_{j<i} xC(j;u,v,x) + \sum_{j \geq i} C(j;u,v,x)) \tag{3.3.3}$$

for all $i \geq 1$. Thus, if U and V are the infinite vectors $U = [uv^1, uv^2, \ldots]$ and $V = [C(1;u,v,x), C(2;u,v,x), \ldots]$, we can write down an invertible matrix M such that

$$U^T = MV^T$$

and, hence, we can solve for V^T as

$$V^T = M^{-1}U^T.$$

Then, at least in some cases, one can simplify the expression for $1 + \sum_{i \geq 1} C(i;u,v,x)$ to obtain nice formulas for the desired generating function. Of course, this method is more straightforward if we restrict ourselves to finite alphabets, but it can still work over infinite alphabets, as Carlitz basically showed in [13]. However, when we try the same thing while adding a variable q recording the major index, we cannot derive such an equation. That is, define

$$C(i;u,v,x,q) = \sum_{\gamma \in \mathbb{P}^+, \gamma_1 = i} u^{\ell(\gamma)} v^{|\gamma|} x^{\mathrm{des}(\gamma)} q^{\mathrm{maj}(\gamma)}. \tag{3.3.4}$$

When we consider a composition $\gamma = (\gamma_1, \ldots, \gamma_k)$ where $\gamma_1 = j$ and add i to the front of γ to obtain the composition $\delta = (i, \gamma_1, \ldots, \gamma_k)$, then a descent $\gamma_s \geq \gamma_{s+1}$ which contributes s to $\text{maj}(\gamma)$ will contribute $1+s$ to $\text{maj}(\delta)$ since that descent will occur at position $s+1$ in δ. Thus $\text{maj}(\delta) = 1 + \text{des}(\gamma) + \text{maj}(\gamma)$ if $j < i$ and $\text{maj}(\delta) = \text{des}(\gamma) + \text{maj}(\gamma)$ if $i \leq j$. Hence, in this case we obtain the recursion

$$C(i; u, v, x, q) = uv^i + uv^i\Big(\sum_{j<i} qxC(j; u, v, qx, q) + \sum_{j \geq i} C(j; u, v, qx, q)\Big). \quad (3.3.5)$$

The fact that $C(j; u, v, qx, q)$ appears on the RHS of (3.3.5) as opposed to the $C(j; u, v, x, q)$ which appear on the RHS of (3.3.3) means that we cannot solve directly for V^T in this case. Instead, if

$$V = V(u, v, x, q) = (C(1; u, v, qx, q), C(2; u, v, qx, q), C(3; u, v, qx, q), \ldots)$$

and $A = (uv, uv^2, uv^3, \ldots)$, then we end up with an equation of the form

$$V(u, v, x, q)^T = A^T + B(u, v, x, q)V(u, v, xq, q)^T \quad (3.3.6)$$

where $B(u, v, x, q)$ is a matrix. We can iterate (3.3.6) to obtain an expression for $V(u, v, x, q)^T$ of the form

$$A^T + B(u, v, x, q)A^T + B(u, v, x, q)B(u, v, xq, q)A^T + \\ B(u, v, x, q)B(u, v, xq, q)B(u, v, xq^2, q)A^T + \cdots .$$

However, in this case, even when we restrict ourselves to finite alphabets $\{1, \ldots, n\}$ so that the matrix $B(u, v, x, q)$ is finite, this leads to a complicated expression for $V(u, v, x, q)^T$. We were unable to see how we could simplify these expressions for $V(u, v, x, q)^T$ or $1 + \sum_{i=1} C(i; u, v, x, q)$ to obtain anything as simple as the formula in Theorem 3.3.1.

It should be noted, however, that various specializations easily follow from Theorems 3.3.1, 3.3.2, and 3.3.3. That is, by setting the variables $x_i = 0$ for certain i, we can obtain formulas for an arbitrary alphabet $A \subseteq \mathbb{P}$. By setting $x_i = uv^i$ for all i, we can also obtain generating functions like

$$C(u, v, x, q) = \sum_{\gamma \in \mathbb{P}^*} u^{\ell(\gamma)} v^{|\gamma|} x^{\text{des}(\gamma)} q^{\text{maj}(\gamma)}.$$

To prove Theorem 3.3.1, define a ring homomorphism $\Theta^{(k)}$ by defining it on the elementary symmetric function e_n so that

$$\Theta^{(k)}(e_n) = \sum_{\substack{i_0,\ldots,i_k \geq 0 \\ i_0+\cdots+i_k=n}} u^{0i_0+\cdots+ki_k} \left[\prod_{j=0}^{k} \prod_{i \geq 1} (1+x_i z_j) \right] \Bigg|_{z_0^{i_0}\cdots z_k^{i_k}},$$

where $expression|_{t^k}$ means to take the coefficient of t^k in $expression$.

First we apply $\Theta^{(k)}$ to h_n. We have

$$\begin{aligned} &\Theta^{(k)}(h_n) \qquad (3.3.7) \\ &= \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} B_{\lambda,n} \Theta^{(k)}(e_\lambda) \\ &= \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} B_{\lambda,n} \prod_{m=1}^{\ell(\lambda)} \sum_{\substack{i_0,\ldots,i_k \geq 0 \\ i_0+\cdots+i_k=\lambda_m}} u^{0i_0+\cdots+ki_k} \left[\prod_{j=0}^{k} \prod_{i \geq 1} (1+x_i z_j) \right] \Bigg|_{z_0^{i_0}\cdots z_k^{i_k}}. \end{aligned}$$

Our goal is to interpret $\Theta^{(k)}(h_n)$ as a sum of weighted combinatorial objects. We interpret the sum $\sum_{\lambda \vdash n} B_{\lambda,n}$ as all ways of picking a brick tabloid T of shape (n). Then the factor $(-1)^{n-\ell(\lambda)}$ allows us to place a -1 in each non-terminal cell of a brick in T and place a 1 at the terminal cell of each brick in T. Next, for each brick in T, choose nonnegative integers $i_0, \ldots, i_k$ that sum to the total length of the brick. This accounts for the product and second sum in (3.3.7). Using powers of u, these choices for $i_0, \ldots, i_k$ can be recorded in T. In each brick, place a power of u in each cell such that the powers weakly increase from left to right and the number of occurrences of u^j is i_j. At this point, we have constructed an object which may look something like Figure 3.9 below.

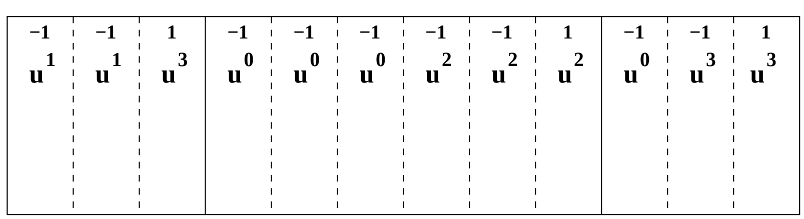

Figure 3.9: One possible object when $k = 3$ and $n = 12$.

Now, the term $\left[\prod_{j=0}^{k}\prod_{i\geq 1}(1+x_i z_j)\right]\Big|_{z_0^{i_0}\cdots z_k^{i_k}}$ lets us choose $k+1$ partitions with distinct parts, $\pi^{(0)},\ldots,\pi^{(k)}$ where $\ell(\pi^{(j)}) = i_j$ for $j = 0,\ldots,k$, which we write in strictly decreasing order. Each i that occurs in such a configuration is weighted with x_i, so that we write these factors in the bottom row of each configuration. Figure 3.10 gives one example of such an object created in this manner. The weight of such a composite object is the product of the signs at the top of the configuration times the product of the x_i's that appear in the bottom row of the configuration times the products of the u^j's in the second row of the configuration. Thus, the weight of the object in Figure 3.10 is $-x_1x_2^3x_3^4x_4^2x_5x_6u^{17}$.

−1	−1	1	−1	−1	−1	−1	−1	1	−1	−1	1
u^1	u^1	u^3	u^0	u^0	u^0	u^2	u^2	u^2	u^0	u^3	u^3
6	1	3	5	3	2	4	3	2	2	4	3
x_6	x_1	x_3	x_5	x_3	x_2	x_4	x_3	x_2	x_2	x_4	x_3

Figure 3.10: An object coming from (3.3.7) when $k = 3$ and $n = 12$.

These decorated brick tabloids of shape (n) and type λ for some $\lambda \vdash n$ have the following properties:

1. the cells in each brick contain -1 except for the final cell, which contains 1,

2. each cell contains a power of u such that the powers weakly increase within each brick and the largest possible power of u is u^k, and

3. T contains a composition of n which must strictly decrease between consecutive cells within a brick if the cells are marked with the same power of u.

In addition, each entry i in the composition is weighted by x_i. In this way, $\Theta^{(k)}(h_n)$ is the weighted sum over all possible decorated brick tabloids of shape (n).

Next, we define a sign-reversing involution I which will allows us to cancel all the terms T with a negative weight. To define I, scan the cells from left to right looking for either a cell containing -1 or two consecutive bricks which may be combined to preserve the properties of this collection of objects. If a -1 is scanned first, break the brick containing the -1 into two immediately after the violation and change the -1 to 1. If the second situation is scanned first, glue the brick together and change the 1 in the first brick to -1. For example, the image of Figure 3.10 is displayed in Figure 3.11.

1	−1	1	−1	−1	−1	−1	−1	1	−1	−1	1
u^1	u^1	u^3	u^0	u^0	u^0	u^2	u^2	u^2	u^0	u^3	u^3
6	1	3	5	3	2	4	3	2	2	4	3
x_6	x_1	x_3	x_5	x_3	x_2	x_4	x_3	x_2	x_2	x_4	x_3

Figure 3.11: The image under I of Figure 3.10.

It is easy to see that I is a sign-reversing, weight-preserving involution. Thus, I shows that $\Theta^{(k)}(h_n)$ is equal to the sum of the weights of all the fixed points of I.

Let us consider the fixed points of I. First, there can be no -1's, so every brick must be of size 1. Next, it cannot be the case that the power of u strictly increases as we move from brick i to brick $i+1$, since then we could combine these two bricks and still satisfy properties (1), (2), and (3). Thus, the powers of u must weakly decrease as we read from left to right. Let $\gamma = (\gamma_1, \ldots, \gamma_n)$ denote the underlying composition. We note that if the power of u is the same on brick i and $i+1$, then it must be the case that $\gamma_i \leq \gamma_{i+1}$: otherwise, we could combine brick i and brick $i+1$. One example of a fixed point may be found in Figure 3.12.

1	1	1	1	1	1	1	1	1	1	1	1
u^3	u^3	u^3	u^3	u^3	u^2	u^2	u^2	u^1	u^1	u^0	u^0
1	2	3	3	5	3	3	6	2	2	3	4
x_1	x_2	x_3	x_3	x_5	x_3	x_3	x_6	x_2	x_2	x_3	x_4

Figure 3.12: A fixed point when $k = 3$ and $n = 12$.

We now turn our attention to counting fixed points. Suppose that the powers of u in a fixed point are $r_1, \ldots, r_n$ when read from left to right. It must be the case that $k \geq r_1 \geq \cdots \geq r_n$. Define nonnegative integers a_i by $a_i = r_i - r_{i+1}$ for $i = 1, \ldots, n-1$ and let $a_n = r_n$. It follows that $r_1 + \cdots + r_n = a_1 + 2a_2 + \cdots + na_n$, $a_1 + \cdots + a_n = r_1 \leq k$. Now suppose that γ is the composition in a fixed point. Then if $\gamma_i > \gamma_{i+1}$, it cannot be that $r_i = r_{i+1}$ because that would violate our conditions for fixed points. Thus, it must be the case that $a_i \geq \chi(\gamma_i > \gamma_{i+1})$. Let x^γ denote $\prod_{i=1}^n x_{\gamma_i}$. In this way, the sum of the weights of all fixed points of I equals

$$\begin{aligned} &\sum_{\gamma \in \mathbb{P}^n} x^\gamma \sum_{\substack{a_1+\cdots+a_n \leq k \\ a_i \geq \chi(i \in Des(\gamma))}} u^{a_1+2a_2+\cdots+na_n} \\ &= \sum_{\gamma \in \mathbb{P}^n} x^\gamma \sum_{a_1 \geq \chi(1 \in Des(\gamma))} \cdots \sum_{a_n \geq \chi(n \in Des(\gamma))} y^{a_1+\cdots+a_n} u^{a_1+2a_2+\cdots+na_n} \Bigg|_{y \leq k}, \end{aligned}$$

where $expression|_{t \leq k}$ means to sum the coefficients of t^j for $j = 0, \ldots, k$ in $expression$. Rewriting the above equation, we have

$$\begin{aligned} &\sum_{\gamma \in \mathbb{P}^n} x^\gamma \sum_{a_1 \geq \chi(1 \in Des(\gamma))} (yu)^{a_1} \cdots \sum_{a_n \geq \chi(n \in Des(\gamma)} (yu^n)^{a_n} \Bigg|_{y \leq k} \\ &= \sum_{\gamma \in \mathbb{P}^n} \frac{x^\gamma (yu)^{\chi(1 \in Des(\gamma))} (yu^2)^{\chi(2 \in Des(\gamma))} \cdots (yu^n)^{\chi(n \in Des(\gamma))}}{(1-yu)(1-yu^2)\cdots(1-yu^n)} \Bigg|_{y \leq k} \\ &= \sum_{\gamma \in \mathbb{P}^n} \frac{x^\gamma y^{\mathrm{des}(\gamma)} u^{\mathrm{maj}(\gamma)}}{(1-yu)(1-yu^2)\cdots(1-yu^n)} \Bigg|_{y \leq k}. \end{aligned}$$

Dividing by $(1-y)$ allows the above expression to be rewritten as

$$\sum_{\gamma \in \mathbb{P}^n} \frac{x^\gamma y^{\mathrm{des}(\gamma)} u^{\mathrm{maj}(\gamma)}}{(1-y)(1-yu)\cdots(1-yu^n)} \Bigg|_{y^k}.$$

Therefore, we have

$$\begin{aligned}
&\sum_{n\geq 0}\frac{t^n}{(y;u)_{n+1}}\sum_{\gamma\in\mathbb{P}^n}x^\gamma y^{\mathrm{des}(\gamma)}u^{\mathrm{maj}(\gamma)}\\
&= \sum_{k\geq 0}y^k\Theta^{(k)}\left(\sum_{n\geq 0}t^n h_n\right)\\
&= \sum_{k\geq 0}\frac{y^k}{\left(\sum_{n\geq 0}(-t)^n\Theta^{(k)}(e_n)\right)}\\
&= \sum_{k\geq 0}\frac{y^k}{\left(\sum_{n\geq 0}(-t)^n\sum_{\substack{i_0,\ldots,i_k\geq 0\\ i_0+\cdots+i_k=n}}u^{0i_0+\cdots+ki_k}\prod_{j=0}^{k}\prod_{i\geq 1}(1+x_iz_j)|_{z_0^{i_0}\cdots z_k^{i_k}}\right)}.
\end{aligned}$$

However,

$$\begin{aligned}
&\sum_{n\geq 0}(-t)^n\sum_{\substack{i_0,\ldots,i_k\geq 0\\ i_0+\cdots+i_k=n}}u^{0i_0+\cdots+ki_k}\prod_{j=0}^{k}\prod_{i\geq 1}(1+x_iz_j)|_{z_0^{i_0}\cdots z_k^{i_k}} =\\
&\sum_{n\geq 0}(-t)^n\prod_{j=0}^{k}\prod_{i\geq 1}(1+u^jx_iz)|z^n =\\
&\prod_{j=0}^{k}\prod_{i\geq 1}(1-x_iu^jt) =\\
&\prod_{i\geq 1}(x_it;u)_{k+1}.
\end{aligned}$$

Thus, we have shown that

$$\begin{aligned}
&\sum_{n\geq 0}\frac{t^n}{(y;u)_{n+1}}\sum_{\gamma\in\mathbb{P}^n}x^\gamma y^{\mathrm{des}(\gamma)}u^{\mathrm{maj}(\gamma)} =\\
&\sum_{k\geq 0}\frac{y^k}{\prod_{i\geq 1}(x_it;u)_{k+1}},
\end{aligned}$$

which proves Theorem 3.3.1.

To prove Theorem 3.3.2, we define a homomorphism $\Theta_w^{(k)}$ on Λ by defining

$$\Theta_w^{(k)}(e_n) = \sum_{\substack{i_0,\ldots,i_k\geq 0\\ i_0+\cdots+i_k=n}}u^{0i_0+\cdots+ki_k}\left[\prod_{j=0}^{k}\prod_{i\geq 1}\frac{1}{1-x_iz_j}\right]\Bigg|_{z_0^{i_0}\cdots z_k^{i_k}}.$$

Again we apply $\Theta_w^{(k)}$ to h_n. We have

$$\begin{aligned}
&\Theta_w^{(k)}(h_n) \qquad\qquad (3.3.8)\\
&= \sum_{\lambda\vdash n}(-1)^{n-\ell(\lambda)}B_{\lambda,n}\Theta_w^{(k)}(e_\lambda)\\
&= \sum_{\lambda\vdash n}(-1)^{n-\ell(\lambda)}B_{\lambda,n}\prod_{m=1}^{\ell(\lambda)}\sum_{\substack{i_0,\ldots,i_k\geq 0\\ i_0+\cdots+i_k=\lambda_m}} u^{0i_0+\cdots+ki_k}\left[\prod_{j=0}^{k}\prod_{i\geq 1}\frac{1}{1-x_iz_j}\right]\Bigg|_{z_0^{i_0}\cdots z_k^{i_k}}.
\end{aligned}$$

Again we interpret $\Theta_w^{(k)}(h_n)$ as a sum of weighted combinatorial objects. Everything is the same as before except that the term $\left[\prod_{j=0}^{k}\prod_{i\geq 1}\frac{1}{1-x_iz_j}\right]\Big|_{z_0^{i_0}\cdots z_k^{i_k}}$ lets us choose $k+1$ partitions, $\pi^{(0)},\ldots,\pi^{(k)}$ where $\ell(\pi^{(j)}) = i_j$ for $j=0,\ldots,k$, which we write in weakly decreasing order. Each i that occurs in such a configuration is weighted with x_i, so we write these factors in the bottom row of each configuration. Figure 3.13 gives one example of such an object created in this manner. The weight of such a composite object is the product of the signs at the top of the configuration times the product of the x_i's that appear in the bottom row of the configuration times the products of the u^j's in the second row of the configuration. Thus, the weight of the object in Figure 3.13 is $-x_1x_2^3x_3^4x_4^3x_6u^{17}$.

−1	−1	1	−1	−1	−1	−1	−1	1	−1	−1	1
u^1	u^1	u^3	u^0	u^0	u^0	u^2	u^2	u^2	u^0	u^3	u^3
6	1	3	3	3	2	4	3	2	2	4	4
x_6	x_1	x_3	x_3	x_3	x_2	x_4	x_3	x_2	x_2	x_4	x_4

Figure 3.13: An object coming from (3.3.8) when $k=3$ and $n=12$.

These decorated brick tabloids of shape (n) and type λ for some $\lambda\vdash n$ have the following properties:

1. the cells in each brick contain -1 except for the final cell, which contains 1,
2. each cell contains a power of u such that the powers weakly increase within each brick and the largest possible power of u is u^k, and

3. T contains a composition of n which must weakly decrease between consecutive cells within a brick if the cells are marked with the same power of u.

In addition, each entry i in the composition is weighted by x_i. In this way, $\Theta_w^{(k)}(h_n)$ is the weighted sum over all possible decorated brick tabloids.

We define a sign-reversing involution I exactly as before. That is, we scan the cells from left to right looking for either a cell containing -1 or two consecutive bricks which may be combined to preserve the properties of this collection of objects. If a -1 is scanned first, break the brick containing the -1 into two immediately after the violation and change the -1 to 1. If the second situation is scanned first, glue the brick together and change the 1 in the first brick to -1. Thus, I shows that $\Theta_w^{(k)}(h_n)$ is equal to the sum of the weights of all the fixed points of I.

Again, let us consider the fixed points of I. First, there can be no -1's, so every brick must be of size 1. Next, it cannot be the case that the power of u strictly increases as we move from brick i to brick $i+1$, since then we could combine these two bricks and still satisfy properties (1), (2), and (3). Thus, the powers of u must weakly decrease as we read from left to right. Let $\gamma = (\gamma_1, \ldots, \gamma_n)$ denote the underlying composition. We note that if the power of u is the same on brick i and $i+1$, then it must be the case that $\gamma_i < \gamma_{i+1}$: otherwise, we could combine brick i and brick $i+1$. One example of a fixed point may be found in Figure 3.14.

1	1	1	1	1	1	1	1	1	1	1	1
u^3	u^3	u^3	u^3	u^3	u^2	u^2	u^2	u^1	u^1	u^0	u^0
1	2	3	4	6	3	5	6	2	5	3	4
x_1	x_2	x_3	x_4	x_6	x_3	x_5	x_6	x_2	x_5	x_3	x_4

Figure 3.14: A fixed point when $k = 3$ and $n = 12$.

We can then count the fixed points as before. That is, suppose that the powers of u in a fixed point are $r_1, \ldots, r_n$ when read from left to right. It must be the case that $k \geq r_1 \geq \cdots \geq r_n$. Define nonnegative integers a_i by $a_i = r_i - r_{i+1}$ for $i = 1, \ldots, n-1$ and let $a_n = r_n$. It follows that $r_1 + \cdots + r_n = a_1 + 2a_2 + \cdots + na_n$, $a_1 + \cdots + a_n = r_1 \leq k$. Now, suppose that γ is the composition in a fixed point. Then if $\gamma_i \geq \gamma_{i+1}$, it cannot be that $r_i = r_{i+1}$ because that would violate our conditions for fixed points. Thus, it must be the case that $a_i \geq \chi(\gamma_i \geq \gamma_{i+1})$. Let x^γ denote $\prod_{i=1}^n x_{\gamma_i}$. In this way, the sum the weights of all fixed points of I equals

$$\begin{aligned}
&\sum_{\gamma \in \mathbb{P}^n} x^\gamma \sum_{\substack{a_1 + \cdots + a_n \leq k \\ a_i \geq \chi(i \in WDes(\gamma))}} u^{a_1 + 2a_2 + \cdots + na_n} \\
&= \sum_{\gamma \in \mathbb{P}^n} x^\gamma \sum_{a_1 \geq \chi(1 \in WDes(\gamma))} \cdots \sum_{a_n \geq \chi(n \in WDes(\gamma))} y^{a_1 + \cdots + a_n} u^{a_1 + 2a_2 + \cdots + na_n} \Bigg|_{y^{\leq k}} .
\end{aligned}$$

Rewriting the above equation, we have

$$\begin{aligned}
&\sum_{\gamma \in \mathbb{P}^n} x^\gamma \sum_{a_1 \geq \chi(1 \in WDes(\gamma))} (yu)^{a_1} \cdots \sum_{a_n \geq \chi(n \in WDes(\gamma))} (yu^n)^{a_n} \Bigg|_{y^{\leq k}} \\
&= \sum_{\gamma \in \mathbb{P}^n} \frac{x^\gamma (yu)^{\chi(1 \in WDes(\gamma))} (yu^2)^{\chi(2 \in WDes(\gamma))} \cdots (yu^n)^{\chi(n \in WDes(\sigma))}}{(1-yu)(1-yu^2)\cdots(1-yu^n)} \Bigg|_{y^{\leq k}} \\
&= \sum_{\gamma \in \mathbb{P}^n} \frac{x^\gamma y^{\mathrm{wdes}} u^{\mathrm{wmaj}(\gamma)}}{(1-yu)(1-yu^2)\cdots(1-yu^n)} \Bigg|_{y^{\leq k}} .
\end{aligned}$$

Dividing by $(1-y)$ allows the above expression to be rewritten as

$$\sum_{\gamma \in \mathbb{P}^n} \frac{x^\gamma y^{\mathrm{wdes}} u^{\mathrm{wmaj}(\gamma)}}{(1-y)(1-yu)\cdots(1-yu^n)} \Bigg|_{y^k} .$$

Therefore, we have

$$\begin{aligned}
&\sum_{n\geq 0}\frac{t^n}{(y;u)_{n+1}}\sum_{\gamma\in\mathbb{P}^n}x^\gamma y^{\mathrm{wdes}}q^{\mathrm{wmaj}(\gamma)}\\
&= \sum_{k\geq 0}y^k\Theta_w^{(k)}\left(\sum_{n\geq 0}t^nh_n\right)\\
&= \sum_{k\geq 0}\frac{y^k}{\left(\sum_{n\geq 0}(-t)^n\Theta_w^{(k)}(e_n)\right)}\\
&= \sum_{k\geq 0}\frac{y^k}{\left(\sum_{n\geq 0}(-t)^n\sum_{\substack{i_0,\ldots,i_k\geq 0\\ i_0+\cdots+i_k=n}}u^{0i_0+\cdots+ki_k}\prod_{j=0}^k\prod_{i\geq 1}\frac{1}{1-x_iz_j}|_{z_0^{i_0}\cdots z_k^{i_k}}\right)}.
\end{aligned}$$

However,

$$\begin{aligned}
&\sum_{n\geq 0}(-t)^n\sum_{\substack{i_0,\ldots,i_k\geq 0\\ i_0+\cdots+i_k=n}}u^{0i_0+\cdots+ki_k}\prod_{j=0}^k\prod_{i\geq 1}\frac{1}{1-x_iz_j}|_{z_0^{i_0}\cdots z_k^{i_k}}=\\
&\sum_{n\geq 0}(-t)^n\prod_{j=0}^k\prod_{i\geq 1}\frac{1}{1-u^jx_iz}|z^n=\\
&\prod_{j=0}^k\prod_{i\geq 1}\frac{1}{1+x_iu^jt}=\\
&\prod_{i\geq 1}\frac{1}{(-x_it;u)_{k+1}}.
\end{aligned}$$

Thus, we have shown that

$$\begin{aligned}
&\sum_{n\geq 0}\frac{t^n}{(y;n)_{n+1}}\sum_{\gamma\in\mathbb{P}^n}x^\gamma y^{\mathrm{wdes}}u^{\mathrm{wmaj}(\gamma)}=\\
&\sum_{k\geq 0}y^k\prod_{i\geq 1}(-x_it;u)_{k+1},
\end{aligned}$$

which proves Theorem 3.3.2.

To prove Theorem 3.3.3, we define a homomorphism $\Theta_\ell^{(k)}(e_n)$ on Λ by setting

$$\Theta_\ell^{(k)}(e_n)=\sum_{\substack{i_0,\ldots,i_k\geq 0\\ i_0+\cdots+i_k=n}}u^{0i_0+\cdots+ki_k}\prod_{j=0}^k p_{i_j}.$$

where p_n is the n-th power symmetric function.

As before, we apply $\Theta_\ell^{(k)}$ to h_n. We have

$$\begin{aligned}
&\Theta_\ell^{(k)}(h_n) \qquad (3.3.9)\\
&= \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} B_{\lambda,n} \Theta^{(k)}(e_\lambda)\\
&= \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} B_{\lambda,n} \prod_{m=1}^{\ell(\lambda)} \sum_{\substack{i_0,\ldots,i_k \geq 0 \\ i_0+\cdots+i_k=\lambda_m}} u^{0i_0+\cdots+ki_k} p_{i_0} \cdots p_{i_k}.
\end{aligned}$$

Again we interpret $\Theta_\ell^{(k)}(h_n)$ as a sum of weighted combinatorial objects. Everything is the same as before except that the term $p_{i_0} \cdots p_{i_k}$ lets us choose $k+1$ partitions, $\pi^{(0)}, \ldots, \pi^{(k)}$ where $\pi^{(j)} = (n_j^{i_j})$ for some n_j for $j = 0, \ldots, k$. Each i that occurs in such a configuration is weighted with x_i so that we write these factors in the bottom row of each configuration. Figure 3.15 gives one example of such an object created in this manner. The weight of such a composite object is the product of the signs at the top of the configuration times the product of the x_i's that appear in the bottom row of the configuration times the products of the u^j's in the second row of the configuration. Thus, the weight of the object in Figure 3.15 is $-x_2x_3^6x_4^3x_6^2u^{17}$.

−1	−1	1	−1	−1	−1	−1	−1	1	−1	−1	1
u^1	u^1	u^3	u^0	u^0	u^0	u^2	u^2	u^2	u^0	u^3	u^3
6	6	3	3	3	3	4	4	4	2	3	3
x_6	x_6	x_3	x_3	x_3	x_3	x_4	x_4	x_4	x_2	x_3	x_3

Figure 3.15: An object coming from (3.3.9) when $k = 3$ and $n = 12$.

These decorated brick tabloids of shape (n) and type λ for some $\lambda \vdash n$ have the following properties:

1. the cells in each brick contain -1 except for the final cell, which contains 1,

2. each cell contains a power of u such that the powers weakly increase within each brick and the largest possible power of u is u^k, and

3. T contains a composition of n whose entries must be equal for any two consecutive cells within a brick if the cells are marked with the same power of u.

In addition, each entry i in the composition is weighted by x_i. In this way, $\Theta_\ell^{(k)}(h_n)$ is the weighted sum over all possible decorated brick tabloids.

We define a sign-reversing involution I exactly as before. That is, we scan the cells from left to right looking for either a cell containing -1 or two consecutive bricks which may be combined to preserve the properties of this collection of objects. If a -1 is scanned first, break the brick containing the -1 into two immediately after the violation and change the -1 to 1. If the second situation is scanned first, glue the brick together and change the 1 in the first brick to -1. Thus, I shows that $\Theta_\ell^{(k)}(h_n)$ is equal to the sum of the weights of all the fixed points of I.

Again, let us consider the fixed points of I. First, there can be no -1's, so every brick must be of size 1. Next, it cannot be the case that the power of u strictly increases as we move from brick i to brick $i+1$, since then we could combine these two bricks and still satisfy properties (1), (2), and (3). Thus, the powers of u must weakly decrease as we read from left to right. Let $\gamma = (\gamma_1, \ldots, \gamma_n)$ denote the underlying composition. We note that if the power of u is the same on brick i and $i+1$, then it must be the case that $\gamma_i \neq \gamma_{i+1}$: otherwise, we could combine brick i and brick $i+1$. One example of a fixed point may be found in Figure 3.16.

1	1	1	1	1	1	1	1	1	1	1	1
u^3	u^3	u^3	u^3	u^3	u^2	u^2	u^2	u^1	u^1	u^0	u^0
3	2	1	3	5	3	2	6	6	2	3	4
x_3	x_2	x_1	x_3	x_5	x_3	x_2	x_6	x_6	x_2	x_3	x_4

Figure 3.16: A fixed point when $k = 3$ and $n = 12$.

We can then count the fixed points as before. That is, suppose that the powers of u in a fixed point are $r_1, \ldots, r_n$ when read from left to right. It must be the case that $k \geq r_1 \geq \cdots \geq r_n$. Define nonnegative integers a_i by $a_i = r_i - r_{i+1}$ for $i = 1, \ldots, n-1$ and let $a_n = r_n$. It follows that $r_1 + \cdots + r_n = a_1 + 2a_2 + \cdots + na_n$, $a_1 + \cdots + a_n = r_1 \leq k$. Now suppose that γ is the composition in a fixed point. Then if $\gamma_i = \gamma_{i+1}$, then it cannot be that $r_i = r_{i+1}$ because that would violate our conditions for fixed points. Thus, it must be the case that $a_i \geq \chi(\gamma_i = \gamma_{i+1})$. Let x^γ denote $\prod_{i=1}^n x_{\gamma_i}$. In this way, the sum the weights of all fixed points of I equals

$$\begin{aligned}
&\sum_{\gamma \in \mathbb{P}^n} x^\gamma \sum_{\substack{a_1 + \cdots + a_n \leq k \\ a_i \geq \chi(i \in Lev(\gamma))}} u^{a_1 + 2a_2 + \cdots + na_n} \\
&= \sum_{\gamma \in \mathbb{P}^n} x^\gamma \sum_{a_1 \geq \chi(1 \in Lev(\gamma))} \cdots \sum_{a_n \geq \chi(n \in Lev(\gamma))} y^{a_1 + \cdots + a_n} u^{a_1 + 2a_2 + \cdots + na_n} \Bigg|_{y^{\leq k}}.
\end{aligned}$$

Rewriting the above equation, we have

$$\begin{aligned}
&\sum_{\gamma \in \mathbb{P}^n} x^\gamma \sum_{a_1 \geq \chi(1 \in Lev(\gamma))} (yu)^{a_1} \cdots \sum_{a_n \geq \chi(n \in Lev(\gamma))} (yu^n)^{a_n} \Bigg|_{y^{\leq k}} \\
&= \sum_{\gamma \in \mathbb{P}^n} \frac{x^\gamma (yu)^{\chi(1 \in Lev(\gamma))} (yu^2)^{\chi(2 \in Lev(\gamma))} \cdots (yu^n)^{\chi(n \in Lev(\gamma))}}{(1-yu)(1-yu^2)\cdots(1-yu^n)} \Bigg|_{y^{\leq k}} \\
&= \sum_{\gamma \in \mathbb{P}^n} \frac{x^\gamma y^{\mathrm{lev}} u^{\mathrm{levmaj}(\gamma)}}{(1-yu)(1-yu^2)\cdots(1-yu^n)} \Bigg|_{y^{\leq k}}.
\end{aligned}$$

Dividing by $(1-y)$ allows the above expression to be rewritten as

$$\sum_{\gamma \in \mathbb{P}^n} \frac{x^\gamma y^{\mathrm{lev}} u^{\mathrm{levmaj}(\gamma)}}{(1-y)(1-yu)\cdots(1-yu^n)} \Bigg|_{y^k}.$$

Therefore, we have

$$\begin{aligned}
&\sum_{n\geq 0}\frac{t^n}{(y;u)_{n+1}}\sum_{\gamma\in\mathbb{P}^n}x^{\gamma}y^{\text{lev}}u^{\text{levmaj}(\gamma)}\\
&= \sum_{k\geq 0}y^k\Theta_\ell^{(k)}\left(\sum_{n\geq 0}t^n h_n\right)\\
&= \sum_{k\geq 0}\frac{y^k}{\left(\sum_{n\geq 0}(-t)^n\Theta_\ell^{(k)}(e_n)\right)}\\
&= \sum_{k\geq 0}\frac{y^k}{\left(\sum_{n\geq 0}(-t)^n\sum_{\substack{i_0,\ldots,i_k\geq 0\\ i_0+\cdots+i_k=n}}u^{0i_0+\cdots+ki_k}p_{i_0}\cdots p_{i_j}\right)}.
\end{aligned}$$

However,

$$\begin{aligned}
&\sum_{n\geq 0}(-t)^n\sum_{\substack{i_0,\ldots,i_k\geq 0\\ i_0+\cdots+i_k=n}}u^{0i_0+\cdots+ki_k}p_{i_0}\cdots p_{i_j} =\\
&\sum_{n\geq 0}(-t)^n\prod_{j=0}^{k}(\sum_{m\geq 0}p_m(u^jz)^m)|z^n =\\
&\prod_{j=0}^{k}(\sum_{m\geq 0}p_m(-u^jt)^m).
\end{aligned}$$

Thus, we have shown that

$$\begin{aligned}
&\sum_{n\geq 0}\frac{t^n}{(y;u)_{n+1}}\sum_{\gamma\in\mathbb{P}^n}x^{\gamma}y^{\text{lev}}u^{\text{levmaj}(\gamma)} =\\
&\sum_{k\geq 0}\frac{y^k}{\prod_{j=0}^{k}(\sum_{0\geq 1}p_m(-u^jt)^m)},
\end{aligned}$$

which proves Theorem 3.3.3.

Now suppose that S is a subset of $\mathbb{P}$. Then we can restrict to compositions with parts from S by simply setting $x_i = 0$ for all $i \notin S$. Thus, we immediately have the following corollaries:

Corollary 3.3.4.

$$\sum_{n\geq 0}t^n\sum_{\gamma\in S^n}y^{\text{des}(\gamma)}x^{\gamma} = \frac{1-y}{-y+\prod_{j\in S}(1+t(y-1)x_j)}$$

Corollary 3.3.5.

$$\sum_{n\geq 0} t^n \sum_{\gamma\in S^n} y^{\mathrm{wdes}(\gamma)} x^\gamma = \frac{1-y}{-y+\prod_{j\in S}\frac{1}{1-t(y-1)x_j}}$$

Corollary 3.3.6.

$$\sum_{n\geq 0} t^n \sum_{\gamma\in S^n} y^{\mathrm{lev}(\gamma)} x^\gamma = \frac{1}{1-\sum_{j\in S}\frac{tx_j}{1-t(y-1)x_j}}$$

Corollary 3.3.7.

$$\begin{aligned}\sum_{n\geq 0} \frac{t^n}{(y;u)_{n+1}} \sum_{\gamma\in S^n} x^\gamma y^{\mathrm{des}(\gamma)} u^{\mathrm{maj}(\gamma)} \\ = \sum_{k\geq 0} \frac{y^k}{\prod_{i\in S}(x_i t;u)_{k+1}}.\end{aligned}$$

Corollary 3.3.8.

$$\begin{aligned}\sum_{n\geq 0} \frac{t^n}{(y;u)_{n+1}} \sum_{\gamma\in S^n} x^\gamma y^{\mathrm{wdes}} u^{\mathrm{wmaj}(\gamma)} \\ = \sum_{k\geq 0} y^k \prod_{i\in S}(-x_i t;u)_{k+1}.\end{aligned}$$

Corollary 3.3.9.

$$\begin{aligned}\sum_{n\geq 0} \frac{t^n}{(y;u)_{n+1}} \sum_{\gamma\in S^n} x^\gamma y^{\mathrm{lev}} u^{\mathrm{levmaj}(\gamma)} \\ = \sum_{k\geq 0} \frac{y^k}{\prod_{j=0}^{k}(\sum_{n\geq 0} p_{n,S}(-u^j t)^n)},\end{aligned}$$

where $p_{n,S} = \sum_{i\in S} x_i^n$.

In addition, we can replace x_j by q^j in order to keep track of $q^{|\gamma|} = q^{\gamma_1+\cdots+\gamma_n}$. For example, we will have

Corollary 3.3.10.

$$\sum_{n\geq 0} t^n \sum_{\gamma\in S^n} y^{\mathrm{des}(\gamma)} q^{|\gamma|} = \frac{1-y}{-y+\prod_{j\in S}(1+t(y-1)q^j)}$$

We can also derive analogues of our results for other partial orders on $\mathbb{P}$ by specializing our results. For instance, suppose that $\preccurlyeq$ is the partial order where all the odd numbers are incomparable, every even number is larger than every odd number, and the even numbers are ordered as in the standard universe. In this case, we define for any composition $\gamma = (\gamma_1, \ldots, \gamma_n)$,

$$\begin{aligned}\overline{Des}(\gamma) &= \{i : \gamma_i \succ \gamma_{i+1}\}, \\ \overline{\mathrm{des}}(\gamma) &= |\overline{Des}(\gamma)|, \text{ and} \\ \overline{\mathrm{maj}}(\gamma) &= \sum_{i \in \overline{Des}(\gamma)} i.\end{aligned}$$

Then it easy to see that the generating function

$$\sum_{n\geq 0} \frac{t^n}{(y;u)_{n+1}} \sum_{\gamma \in \mathbb{P}^n} x^\gamma y^{\overline{\mathrm{des}}(\gamma)} u^{\overline{\mathrm{maj}}(\gamma)}$$

arises by taking the generating function

$$\sum_{n\geq 0} \frac{t^n}{(y;u)_{n+1}} \sum_{\gamma \in \mathbb{P}^n} x^\gamma y^{\mathrm{des}(\gamma)} u^{\mathrm{maj}(\gamma)}$$

and setting $x_1 = \frac{1}{1-\sum_{n\geq 0} x_{2n+1}}$ and setting $x_{2i+1} = 0$ for $i \geq 1$. Thus, we have the following corollary.

Corollary 3.3.11.

$$\begin{aligned}&\sum_{n\geq 0} \frac{t^n}{(y;u)_{n+1}} \sum_{\gamma \in \mathbb{P}^n} x^\gamma y^{\overline{\mathrm{des}}(\gamma)} u^{\overline{\mathrm{maj}}(\gamma)} \\ &= \sum_{k\geq 0} \frac{y^k}{\prod_{j=0}^{k}\left(1 - \frac{u^j t}{1-\sum_{n\geq 0} x_{2n+1}}\right) \prod_{i\geq 1} (x_{2i}t;u)_{k+1}}.\end{aligned}$$

3.4 Common descents or levels

We can also generalize to looking at common descents or levels in m-tuples of compositions. Define $\mathrm{comlev}(\gamma^1, \gamma^2, \ldots, \gamma^m)$ to be number of places where each of $\gamma^1, \ldots, \gamma^m$ has a level. Then we have the following theorem.

Theorem 3.4.1.

$$\sum_{n\geq 0} t^n \sum_{\gamma^1,\ldots,\gamma^m \in \mathbb{P}^n} y^{\text{comlev}(\gamma^1,\ldots,\gamma^m)} q_1^{|\gamma^1|} \ldots q_m^{|\gamma^m|} = \frac{1}{1 - \sum_{a_1,\ldots,a_m\geq 1} \frac{t(y-1)q_1^{a_1}\cdots q_m^{a_m}}{1-t(y-1)q_1^{a_1}\cdots q_m^{a_m}}}$$

The proof follows the same structure: apply the homomorphism

$$\Theta_3^m(e_n) = (-1)^{n-1}(y-1)^{n-1} \prod_{i=1}^{m} \sum_{j\geq 1} q_i^{jn}$$

to h_n to obtain a brick tabloid in which we fill in m compositions and decorate in the same manner as before. For example, one such object is displayed in Figure 3.17.

	−1		y	y	−1	y	y		−1	
5	5	5	7	7	7	7	7	7	1	1
6	6	6	7	7	7	7	7	7	5	5

Figure 3.17: An object coming from $\Theta_3^2(h_{11})$

Next, we perform an involution on the resulting objects as follows. Scan left to right for a -1 or two consecutive bricks with a level between them in each of the m compositions. If a -1 is found, break the brick in two after that cell and remove the -1 label. If a level between bricks for every composition is found, insert a -1 label for the last cell in the first brick and combine the bricks. For example, the image of Figure 3.17 is displayed in Figure 3.18.

−1	−1		y	y	−1	y	y		−1	
5	5	5	7	7	7	7	7	7	1	1
6	6	6	7	7	7	7	7	7	5	5

Figure 3.18: The image of Figure 3.17

Thus, we will have

$$\Theta_3^{(m)}(h_n) = \sum_{\gamma^1,\ldots,\gamma^m \in \mathbb{P}^n} y^{\mathrm{comlev}(\gamma^1,\ldots,\gamma^m)} q_1^{|\gamma^1|} \ldots q_m^{|\gamma^m|}.$$

Therefore,

$$\begin{aligned}
&\sum_{n\geq 0} t^n \sum_{\gamma^1,\ldots,\gamma^m \in \mathbb{P}^n} y^{\mathrm{comlev}(\gamma^1,\ldots,\gamma^m)} q_1^{|\gamma^1|} \ldots q_m^{|\gamma^m|} \\
&\qquad = \left(1 + \sum_{n\geq 1} (-t)^n (1-y)^{n-1} \prod_{i=1}^{m} \sum_{j\geq 1} q_i^{jn}\right)^{-1} \\
&\qquad = \left(1 + \sum_{n\geq 1} (-t)^n (1-y)^{n-1} \sum_{a_1,\ldots,a_k \geq 1} (q_1^{a_1} \cdots q_k^{a_k})^n\right)^{-1} \\
&\qquad = \left(1 + \frac{1}{1-y} \sum_{n\geq 1} \sum_{a_1,\ldots,a_k \geq 1} (t(y-1) q_1^{a_1} \cdots q_k^{a_k})^n\right)^{-1} \\
&\qquad = \left(1 + \frac{1}{1-y} \sum_{a_1,\ldots,a_k \geq 1} \sum_{n\geq 1} (t(y-1) q_1^{a_1} \cdots q_k^{a_k})^n\right)^{-1} \\
&\qquad = \left(1 + \frac{1}{1-y} \sum_{a_1,\ldots,a_k \geq 1} \frac{t(y-1) q_1^{a_1} \cdots q_k^{a_k}}{1 - t(y-1) q_1^{a_1} \cdots q_k^{a_k}}\right)^{-1}
\end{aligned}$$

Similarly, define $\mathrm{comdes}(\gamma^1, \gamma^2, \ldots, \gamma^m)$ to be number of places where each of $\gamma^1, \ldots, \gamma^m$ has a descent and $\mathrm{comwdes}(\gamma^1, \gamma^2, \ldots, \gamma^m)$ to be number of places where each of $\gamma^1, \ldots, \gamma^m$ has a weak descent. Unfortunately, the generating functions for common descents are less nice than those for common levels as they contain *Hadamard products* that cannot be simplified. We can define the homomorphisms:

$$\Theta_1^{(m)}(e_n) = (-1)^{n-1} (y-1)^{n-1} \prod_{j=1}^{m} \left[\left(\prod_{j\geq 1} \frac{1}{1 - t q_j^i} \right) |_{t^n} \right]$$

and

$$\Theta_2^{(m)}(e_n) = (-1)^{n-1} (y-1)^{n-1} \prod_{j=1}^{m} \left[\left(\prod_{j\geq 1} (1 + t q_j^i) \right) |_{t^n} \right].$$

The same extended involutions will give us fixed points with the appropriate weighting for the objects we want. However, the final results remain as

$$\sum_{n\geq 0} t^n \sum_{\gamma^1,\ldots,\gamma^m\in\mathbb{P}^n} y^{\mathrm{comdes}(\gamma)} q_1^{|\gamma^1|}\cdots q_n^{|\gamma^m|}$$
$$= \frac{1}{1+\frac{1}{1-y}\sum_{n\geq 1}[t(y-1)]^n \prod_{j=1}^m \left[\left(\prod_{j\geq 1}\frac{1}{1-tq_j^i}\right)|_{t^n}\right]}$$

and

$$\sum_{n\geq 0} t^n \sum_{\gamma^1,\ldots,\gamma^m\in\mathbb{P}^n} y^{\mathrm{comwdes}(\gamma)} q_1^{|\gamma^1|}\cdots q_n^{|\gamma^m|}$$
$$= \frac{1}{1+\frac{1}{1-y}\sum_{n\geq 1}[t(y-1)]^n \prod_{j=1}^m \left[\left(\prod_{j\geq 1}(1+tq_j^i)\right)|_{t^n}\right]},$$

which provide little insight.

3.5 j-levels

We can look at j-levels, but we pay a price for it. The method of the previous sections does not work well for statistics that look at more than 2 adjacent entries. Thus, we must give up keeping track of the monomial x^γ, which also leads us to restrict to a finite alphabet $[m]=\{1,2,\ldots,m\}$. For any word $\gamma\in[m]^n$, define the number of j-levels by $\mathrm{jlev}(\gamma)=|\{i:\gamma_i=\gamma_{i+1}=\cdots=\gamma_{i+j}\}|$, i.e. the number of places a letter is repeated $j+1$ times. Then we will prove the following theorem.

Theorem 3.5.1.

$$\sum_{n\geq 0} t^n \sum_{\gamma\in[m]^n} y^{\mathrm{jlev}(\gamma)} = \frac{1+\frac{t-t^j}{1-t}+\frac{t^j}{1-ty}}{1-(m-1)[\frac{t-t^j}{1-t}+\frac{t^j}{1-ty}]}.$$

To prove Theorem 3.5.1, we define a homomorphism on the ring of symmetric functions by $\phi_j(e_0)=1$ and, for $n\geq 1$,

$$\phi_j(e_n) = \begin{cases} (-1)^{n-1}(m-1) & n<j \\ (-1)^{n-1}(m-1)y^{n-j} & n\geq j \end{cases}$$
$$= (-1)^{n-1}(m-1)y^{\max(0,n-j)}$$

Then

$$\begin{aligned}\frac{m}{m-1}\phi_j(h_n) &= \frac{m}{m-1}\sum_{\lambda\vdash n}(-1)^{n-l(\lambda)}B_{\lambda,n}\prod_{i=1}^{l(\lambda)}\phi_j(e_{\lambda_i}) \\ &= \frac{m}{m-1}\sum_{\lambda\vdash n}(-1)^{n-l(\lambda)}B_{\lambda,n}\prod_{i=1}^{l(\lambda)}(-1)^{\lambda_i-1}(m-1)y^{\max(0,\lambda_i-j)} \\ &= \frac{m}{m-1}\sum_{\lambda\vdash n}B_{\lambda,n}\prod_{i=1}^{l(\lambda)}(m-1)y^{\max(0,\lambda_i-j)} \qquad (3.5.1)\end{aligned}$$

We still interpret each term as a filled labeled brick tabloid. The term $\sum_{\lambda\vdash n} B_{\lambda,n}$ lets us choose a brick tabloid of shape λ. For the first brick, the factor of $\frac{m}{m-1}(m-1)$ lets us choose any an entry $\in [m]$, which we use to fill every cell of the brick. For every other brick, the factor of $\prod_{i=1}^{l(\lambda)}(m-1)$ lets us choose any entry $\in [m]$ except the label of the previous brick, which we use to fill every cell. The factor of $\prod_{i=1}^{l(\lambda)} y^{\max(0,\lambda_i-j)}$ does nothing to bricks of length $\leq j$, and weights bricks of length $> j$ by y^{λ_i-j}. We can think of this as labeling the first $\lambda_i - j$ cells with y.

For instance, one filled labeled brick tabloid for $j = 2$ is displayed in Figure 3.19. The first brick has the first $3-2=1$ cells labeled with y, the second brick has the first $6-2=4$ cells labeled with y, and the third brick has no cells labeled with y since its length is not greater than 2.

y			y	y	y	y				
5	5	5	7	7	7	7	7	7	1	1

Figure 3.19: A filled labeled brick tabloid coming from Equation 3.5.1 with $j = 2$ and $n = 11$

Notice that there are no signs in our filled labeled brick tabloid. In this case, we do not need to perform an involution in order to get the desired objects–we already have them. Let $\mathcal{T}_{\phi_j}(n)$ be the set of filled labeled brick tabloids that arise in this way. Any $C \in \mathcal{T}_{\phi_j}(n)$ has equal entries within each brick, while adjacent bricks cannot have equal entries. Moreover, there is a y at the beginning of every sequence of $j+1$ equal entries. On the other hand, for any $\gamma \in [m]^n$, we can create a filled labeled brick tabloid by breaking a brick every time adjacent entries are different and, within a brick b, labeling the first $j - b$ cells with a y.

Therefore, for $n \geq 1$,

$$\frac{m}{m-1}\phi_j(h_n) = \sum_{\gamma \in [m]^n} y^{\text{jlev}(\gamma)}.$$

Thus

$$\begin{aligned}
\sum_{n\geq 0} t^n \sum_{\gamma\in[m]^n} y^{\text{jlev}(\gamma)} &= 1 + \frac{m}{m-1}\left(\frac{1}{1+\sum_{n\geq 1}(-t)^n \phi_j(e_n)} - 1\right) \\
&= 1 + \frac{m}{m-1}\left(\frac{1}{1+\sum_{n\geq 1}(-t)^n(-1)^{n-1}(m-1)y^{\max(0,n-j)}} - 1\right) \\
&= 1 + \frac{m}{m-1}\left(\frac{1}{1-(m-1)\sum_{n\geq 1} t^n y^{\max(0,n-j)}} - 1\right) \\
&= 1 + \frac{m}{m-1}\left(\frac{1}{1-(m-1)[\frac{t-t^j}{1-t} + \frac{t^j}{1-ty}]} - 1\right) \\
&= \frac{1 + \frac{t-t^j}{1-t} + \frac{t^j}{1-ty}}{1-(m-1)[\frac{t-t^j}{1-t} + \frac{t^j}{1-ty}]}
\end{aligned}$$

Of course, when $j = 1$ this reduces to the regular level generating function with each $x_i = 1$ and parts from $[m]$.

Chapter 4

Enumerating up-down words

Let $\mathbb{P} = \{1, 2, 3, \ldots\}$ denote the set of positive integers, $\mathbb{E} = \{2, 4, 6, \ldots\}$ denote the set of even integers in $\mathbb{P}$, and $\mathbb{O} = \{1, 3, 5, \ldots\}$ denote the set of odd integers in $\mathbb{P}$. Let $\mathbb{P}_n = \{1, \ldots, n\}$, $\mathbb{E}_n = \mathbb{E} \cap \mathbb{P}_n$, and $\mathbb{O}_n = \mathbb{O} \cap \mathbb{P}_n$. Let S_n denote the set of all permutations of $\mathbb{P}_n$. Then if $\sigma = \sigma_1\sigma_2 \ldots \sigma_n \in S_n$, we define $Des(\sigma) = \{i : \sigma_i > \sigma_{i+1}\}$ and $Ris(\sigma) = \{i : \sigma_i < \sigma_{i+1}\}$. We say that σ is an *up-down permutation* if

$$\sigma_1 < \sigma_2 > \sigma_3 < \sigma_4 > \sigma_5 \cdots ,$$

or, equivalently, if $Des(\sigma) = \mathbb{E}_{n-1}$ and $Ris(\sigma) = \mathbb{O}_{n-1}$. Similarly, we say that σ is a *down-up permutation* if

$$\sigma_1 > \sigma_2 < \sigma_3 > \sigma_4 < \sigma_5 \cdots ,$$

or, equivalently, if $Ris(\sigma) = \mathbb{E}_{n-1}$ and $Des(\sigma) = \mathbb{O}_{n-1}$. Clearly, if $\sigma = \sigma_1\sigma_2 \ldots \sigma_n \in S_n$ is an up-down permutation, then the complement of σ,

$$\sigma^c = (n + 1 - \sigma_1)(n + 1 - \sigma_2) \ldots (n + 1 - \sigma_n)$$

is a down-up permutation. Thus, the number of up-down permutations in S_n is equal to the number of down-up permutations in S_n. Let UD_n denote the number of up-down permutations in S_n. Then André [1, 2] proved the following.

$$\sec(t) = 1 + \sum_{n\in\mathbb{E}} UD_n \frac{t^n}{n!} \text{ and} \quad (4.0.1)$$

$$\tan(t) = \sum_{n\in\mathbb{O}} UD_n \frac{t^n}{n!}. \quad (4.0.2)$$

If $s \geq 2$ and $1 \leq j \leq s-1$, let $s\mathbb{P} = \{s, 2s, 3s, \ldots\}$ and $j+s\mathbb{P} = \{j, s+j, 2s+j, \ldots\}$. For any $n > 0$, let $(s\mathbb{P})_n = s\mathbb{P} \cap \mathbb{P}_n$ and $(j+s\mathbb{P})_n = (j+s\mathbb{P}) \cap \mathbb{P}_n$. Let $E_{n,s}$ denote the number of permutations $\sigma \in S_n$ such that $Des(\sigma) = (s\mathbb{P})_{n-1}$. The $E_{n,s}$'s are called generalized Euler numbers [29]. There are well-known generating functions for q-analogues of the generalized Euler numbers; see Stanley's book [42], page 148. Various divisibility properties of the q-Euler numbers have been studied in [4, 5, 17], and properties of the generalized q-Euler numbers were studied in [20, 40]. More general generating functions for statistics on permutations $\sigma \in S_n$ such that $Des(\sigma) = (j+s\mathbb{P})_{n-1}$ were given by Mendes, Remmel, and Riehl [36].

Carlitz [12] and Rawlings [38] proved two different analogues of André's results for words. To state their results, we first need to introduce some more notation. Let $\mathbb{P}^*$ denote the set of all words over the alphabet $\mathbb{P}$ and $\mathbb{P}^+$ denote the set of all non-empty words in $\mathbb{P}^*$. We let ϵ denote the empty word. For any $w = w_1w_2\ldots w_m \in \mathbb{P}^+$, we let $\ell(w) = m$ denote the length of w, $|w| = \sum_{i=1}^m w_i$, and $z(w) = \prod_{i=1}^m z_{w_i}$. For example, if $w = 1\ 2\ 1\ 3\ 2\ 4\ 5\ 4$, then $\ell(w) = 8$, $|w| = 22$, and $z(w) = z_1^2 z_2^2 z_3 z_4^2 z_5$. Given $w = w_1w_2\ldots w_n \in \mathbb{P}^+$, we define the descent set $Des(w)$, the weak descent set $WDes(w)$, the rise set $Ris(w)$, and the weak rise set $WRis(w)$ as follows:

$$Des(w) = \{i : w_i > w_{i+1}\}, \quad (4.0.3)$$

$$WDes(w) = \{i : w_i \geq w_{i+1}\}, \quad (4.0.4)$$

$$Ris(w) = \{i : w_i < w_{i+1}\}, \text{ and} \quad (4.0.5)$$

$$WRis(w) = \{i : w_i \leq w_{i+1}\}. \quad (4.0.6)$$

Definition 4.0.2. Let $w = w_1w_2\ldots w_m \in \mathbb{P}^+$.

1. We say that w is a *strict up-down* word if $w_1 < w_2 > w_3 < w_4 > w_5 \cdots$, or, equivalently if $Ris(w) = \mathbb{O}_{m-1}$ and $Des(w) = \mathbb{E}_{m-1}$.

2. We say that w is a *strict down-up* word if $w_1 > w_2 < w_3 > w_4 < w_5 \cdots$, or, equivalently if $Des(w) = \mathbb{O}_{m-1}$ and $Ris(w) = \mathbb{E}_{m-1}$.

3. We say that w is a *weak up-down* word if $w_1 \leq w_2 \geq w_3 \leq w_4 \geq w_5 \cdots$, or, equivalently if $WRis(w) = \mathbb{O}_{m-1}$ and $WDes(w) = \mathbb{E}_{m-1}$.

4. We say that w is a *weak down-up* word if $w_1 \geq w_2 \leq w_3 \geq w_4 \leq w_5 \cdots$, or, equivalently if $WDes(w) = \mathbb{O}_{m-1}$ and $WRis(w) = \mathbb{E}_{m-1}$.

By convention, the empty word ϵ and one letter word w_1 are considered to be (all of) strict up-down words, strict down-up words, weak up-down words, and weak down-up words. We let SUD_n, SDU_n, WUD_n, and WDU_n denote the sets of all words in $\mathbb{P}_n^*$ which are strict up-down, strict down-up, weak up-down, and weak down-up, respectively. Clearly, if $w = w_1 w_2 \ldots w_m \in \mathbb{P}_n^*$, then $w \in SUD_n(WUD_n)$ if and only if the complement of w relative to n,

$$w^{c,n} = (n+1-w_1)(n+1-w_2)\ldots(n+1-w_m) \in SDU_n(WDU_n).$$

We let $SUD_{n,m}$, $SDU_{n,m}$, $WUD_{n,m}$, and $WDU_{n,m}$ denote the sets of all words in $\mathbb{P}_n^*$ of length m which are strict up-down, strict down-up, weak up-down, and weak down-up, respectively.

Carlitz [12, 11] proved analogues of André's formulas for strict up-down words. In particular, Carlitz [12] considered the following generating functions.

$$F_n(z_1, \ldots, z_n) = \sum_{m \in \mathbb{O}} \sum_{w \in SUD_{n,m}} z(w), \tag{4.0.7}$$

$$G_n(z_1, \ldots, z_n) = 1 + \sum_{m \in \mathbb{E}} \sum_{w \in SUD_{n,m}} z(w), \tag{4.0.8}$$

$$F_n(z) = \sum_{m \in \mathbb{O}} |SUD_{n,m}| z^m, \text{ and} \tag{4.0.9}$$

$$G_n(z) = 1 + \sum_{m \in \mathbb{E}} |SUD_{n,m}| z^m. \tag{4.0.10}$$

For example, if $n = 2$, then clearly $SUD_{2,1} = \{1, 2\}$ and $SUD_{2,2m} = \{(1\ 2)^m\}$ and $SUD_{2,2m+1} = \{(1\ 2)^m 1\}$ for $m \geq 1$. Thus

$$\begin{aligned} G_2(z_1, z_2) &= \frac{1}{1 - z_1 z_2}, \\ G_2(z) &= \frac{1}{1 - z^2}, \\ F_2(z_1, z_2) &= z_2 + \frac{z_1}{1 - z_1 z_2} = \frac{z_1 + z_2 - z_1 z_2^2}{1 - z_1 z_2}, \text{ and} \\ F_2(z) &= \frac{2z - z^3}{1 - z^2}. \end{aligned}$$

In general, Carlitz [12] proved that

$$G_n(z_1, \ldots, z_n) = \frac{1}{Q_n(z_1, \ldots, z_n)} \text{ and} \tag{4.0.11}$$

$$F_n(z_1, \ldots, z_n) = \frac{P_n(z_1, \ldots, z_n)}{Q_n(z_1, \ldots, z_n)} \tag{4.0.12}$$

where

$$P_{n+1}(z_1, \ldots, z_{n+1}) = (1 - z_{n+1}^2) P_n(z_1, \ldots, z_n) + z_{n+1} Q_n(z_1, \ldots, z_n) \tag{4.0.13}$$

and

$$Q_{n+1}(z_1, \ldots, z_{n+1}) = -z_{n+1} P_n(z_1, \ldots, z_n) + Q_n(z_1, \ldots, z_n). \tag{4.0.14}$$

In particular, he used these recursions to prove the following formulas:

$$G_n(z) = \frac{1}{Q_n(z)} \tag{4.0.15}$$

and

$$F_n(z) = \frac{P_n(z)}{Q_n(z)} \tag{4.0.16}$$

where

$$P_n(z) = \sum_{k=0}^{n} (-1)^k \binom{n+k}{2k+1} z^{2k+1} \text{ and} \tag{4.0.17}$$

$$Q_n(z) = \sum_{k=0}^{n} (-1)^k \binom{n+k-1}{2k} z^{2k}. \tag{4.0.18}$$

Rawlings proved q-analogues of (4.0.15) and (4.0.16) for weak down-up words. That is, let $[n] = 1 + q + \cdots q^{n-1} = \frac{1-q^n}{1-q}$, $[n]_q! = [n]_q[n-1]_q \cdots [1]_q$, and $\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}$. Let

$$B_n(q,z) = \sum_{k\geq 0} (-1)^k q^{k(k+1)} \begin{bmatrix} n+k \\ 2k \end{bmatrix}_q z^{2k} \text{ and} \tag{4.0.19}$$

$$A_n(q,z) = \sum_{k\geq 0} (-1)^k q^{k^2+3k+1} \begin{bmatrix} n+k \\ 2k+1 \end{bmatrix}_q z^{2k+1}. \tag{4.0.20}$$

Then Rawlings [38] proved that

$$1 + \sum_{m\in\mathbb{E}} \sum_{w\in WDU_{n,m}} q^{|w|} z^{\ell(w)} = \frac{1}{B_n(q,z)} \tag{4.0.21}$$

and

$$\sum_{m\in\mathbb{O}} \sum_{w\in WDU_{n,m}} q^{|w|} z^{\ell(w)} = \frac{A_n(q,z)}{B_n(q,z)}. \tag{4.0.22}$$

This chapter was motivated by our attempt to give direct proofs via involutions of the formulas of Carlitz and Rawlings described above. That is, Carlitz [12] proved (4.0.15) and (4.0.16) by recursions. Rawlings [38] developed much more general recursions for generating functions of words and proved (4.0.21) and (4.0.22) as special cases of these recursions. The main goal of this chapter is to show that all of the formulas of Carlitz and Rawlings described above can be proved directly by simple involutions. In fact, we shall give direct combinatorial proofs of generalizations of these formulas. That is, we shall prove formulas for the analogues of generalized Euler numbers for words. To this end, we define the following classes of words.

Definition 4.0.3. Let $s \geq 2$.

1. $SU^{s-1}SD_{n,m}$ is the set of all words $w \in \mathbb{P}_n^+$ of length m such that $Des(w) = (s\mathbb{P})_{m-1}$ and $Ris(w) = \mathbb{P}_{m-1} - (s\mathbb{P})_{m-1}$.
   We let $SU^{s-1}SD_n = \bigcup_{m\geq 0} SU^{s-1}SD_{n,m}$.

2. $WU^{s-1}SD_{n,m}$ is the set of all words $w \in \mathbb{P}_n^+$ of length m such that $Des(w) = (s\mathbb{P})_{m-1}$ and $WRis(w) = \mathbb{P}_{m-1} - (s\mathbb{P})_{m-1}$.
   We let $WU^{s-1}SD_n = \bigcup_{m\geq 0} WU^{s-1}SD_{n,m}$.

3. $SU^{s-1}WD_{n,m}$ is the set of all words $w \in \mathbb{P}_n^+$ of length m such that $WDes(w) = (s\mathbb{P})_{m-1}$ and $Ris(w) = \mathbb{P}_{m-1} - (s\mathbb{P})_{m-1}$.
   We let $SU^{s-1}WD_n = \bigcup_{m\geq 0} SU^{s-1}WD_{n,m}$.

4. $WU^{s-1}WD_{n,m}$ is the set of all words $w \in \mathbb{P}_n^+$ of length m such that $WDes(w) = (s\mathbb{P})_{m-1}$ and $WRis(w) = \mathbb{P}_{m-1} - (s\mathbb{P})_{m-1}$.
   We let $WU^{s-1}WD_n = \bigcup_{m\geq 0} WU^{s-1}WD_{n,m}$.

For example, $SU^{s-1}SD_n$ consists of all words that start out with $s-1$ strict increases followed by a strict decrease, then another sequence of $s-1$ strict increases followed by a strict decrease, etc. For example, we can describe SU^2SD_n as the set of all words $w = w_1 \dots w_m \in \mathbb{P}_n^*$ such that $w_i > w_{i+1}$ if $i \equiv 0 \mod 3$ and $w_i < w_{i+1}$ if $i \not\equiv 0 \mod 3$ or, alternatively, SU^2SD_n consists of all words in $w = w_1 \dots w_m \in \mathbb{P}_n$ such that

$$w_1 < w_2 < w_3 > w_4 < w_5 < w_6 > w_7 < w_8 < w_9 > w_{10} \cdots .$$

Similarly, WU^2SD_n consists of all words $w = w_1 \dots w_m \in \mathbb{P}_n^*$ such that $w_i > w_{i+1}$ if $i \equiv 0 \mod 3$ and $w_i \leq w_{i+1}$ if $i \not\equiv 0 \mod 3$. That is, WU^2SD_n denotes the set of all words in $w = w_1 \dots w_m \in \mathbb{P}_n$ such that

$$w_1 \leq w_2 \leq w_3 > w_4 \leq w_5 \leq w_6 > w_7 \leq w_8 \leq w_9 > w_{10} \cdots .$$

It will be useful for later developments to have a pictorial representation of these classes of words. The idea is that we are interested in words w that we can partition into an initial sequence of blocks of size s and ending in a block of size j where $0 \leq j \leq s-1$. The letters in any given block are either strictly increasing if we pick SU^{s-1} or weakly increasing if we pick WU^{s-1}. Then, either we have strict decreases between blocks as pictured in the top of Figure 4.1 if we are considering either $SU^{s-1}SD$ or $WU^{s-1}SD$ or we have weak decreases between blocks as pictured at the bottom of Figure 4.1 if we are considering either $SU^{s-1}WD$ or $WU^{s-1}WD$.

It is then easy to see that the collection of words studied by Carlitz [12] is $SUD_n = SU^1SD_n$ and the collection of words studied by Rawlings [38] is

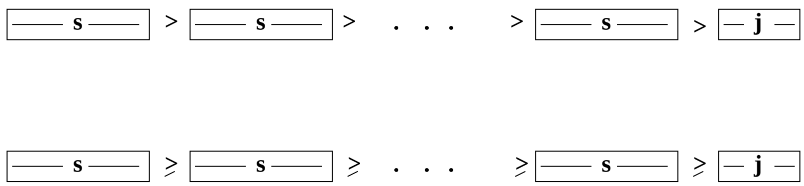

Figure 4.1: Pictorial representation of words in $SU^{s-1}SD$, $WU^{s-1}SD$, $SU^{s-1}WD$, and $WU^{s-1}WD$.

$WUD_n = WU^1WD_n$. This given, we define the following generating functions for any $s \geq 2$:

$$\begin{aligned} H^{SU^{s-1}SD}_{n,s,0}(z_1,\ldots,z_n) &= 1+\sum_{m\in s\mathbb{P}}\ \sum_{w\in SU^{s-1}SD_{n,m}} z(w) \text{ and} \\ H^{SU^{s-1}SD}_{n,s,j}(z_1,\ldots,z_n) &= \sum_{m\in j+s\mathbb{P}}\ \sum_{w\in SU^{s-1}SD_{n,m}} z(w) \text{ for } j=1,\ldots,s-1. \end{aligned}$$

We define $H^{WU^{s-1}SD}_{n,s,j}(z_1,\ldots,z_n)$, $H^{SU^{s-1}WD}_{n,s,j}(z_1,\ldots,z_n)$, and $H^{WU^{s-1}WD}_{n,s,j}(z_1,\ldots,z_n)$ for $j=0,\ldots,s-1$ similarly. We shall give an explicit expression for each of these generating functions in terms of Gessel quasi-symmetric functions [21]. Our expressions can then be specialized to explicit formulas like (4.0.15), (4.0.16), (4.0.21), and (4.0.22).

The outline of this chapter is as follows. In section 4.1, we shall define the Gessel quasi-symmetric functions and some additional classes of words that can be defined in terms of quasi-symmetric functions that we will need for our proofs. In section 4.2, we state and prove our generating functions for $H^{SU^{s-1}SD}_{n,s,j}(z_1,\ldots,z_n)$, $H^{SU^{s-1}WD}_{n,s,j}(z_1,\ldots,z_n)$, $H^{WU^{s-1}SD}_{n,s,j}(z_1,\ldots,z_n)$, and $H^{WU^{s-1}WD}_{n,s,j}(z_1,\ldots,z_n)$ and give some specializations. Finally, in section 4.3, we shall end with a brief discussion about some extensions of our work.

4.1 Quasi-symmetric functions

Let $\gamma = (\gamma_1, \ldots, \gamma_t)$ be a composition, i.e. a sequence of positive integers. Then we let $|\gamma| = \gamma_1 + \cdots + \gamma_t$ and

$$Set(\gamma) = \{\gamma_1, \gamma_1 + \gamma_2, \ldots, \gamma_1 + \gamma_2 + \cdots + \gamma_{t-1}\}.$$

For example, if $\gamma = (2,3,1,1,2)$, $|\gamma| = 9$ and $Set(\gamma) = \{2,5,6,7\}$. Then Gessel [21] defined the quasi-symmetric function

$$Q_\gamma(z_1, \ldots, z_n) = \sum_{\substack{1 \le i_1 \le \cdots \le i_{|\gamma|} \le n \\ i_j < i_{j+1} \text{ if } j \in Set(\gamma)}} z_{i_1} z_{i_2} \cdots z_{i_{|\gamma|}}. \tag{4.1.1}$$

Thus, for example, if $\gamma = (2,3,1,1,2)$, then

$$Q_\gamma(z_1, \ldots, z_n) = \sum_{1 \le i_1 \le i_2 < i_3 \le i_4 \le i_5 < i_6 < i_7 < i_8 \le i_9 \le n} \prod_{j=1}^{9} z_{i_j}.$$

We shall also need explicit expressions for the specializations

$$Q_\gamma(z_1, \ldots, z_n)|_{z_i \to z} \text{ and } Q_\gamma(z_1, \ldots, z_n)|_{z_i \to q^i z}.$$

Lemma 4.1.1.

$$Q_\gamma(z_1, \ldots, z_n)|_{z_i \to z} = \binom{n + |\gamma| - \ell(\gamma)}{|\gamma|} z^{|\gamma|} \tag{4.1.2}$$

and

$$Q_\gamma(z_1, \ldots, z_n)|_{z_i \to q^i z} = q^{\sum_i i\gamma_i} \begin{bmatrix} n + |\gamma| - \ell(\gamma) \\ |\gamma| \end{bmatrix}_q z^{|\gamma|}. \tag{4.1.3}$$

Proof. For the specialization, $Q_\gamma(z_1, \ldots, z_n)|_{z_i \to z}$, we must count the number of sequences $1 \le i_1 \le i_2 \le \cdots \le i_{|\gamma|} \le n$ such that $i_j < i_{j+1}$ if $j \in Set(\gamma)$. Let $\vec{s}(\gamma) = a_1 \ldots a_{|\gamma|}$ where $a_1 = 1$ and $a_{i+1} = a_i$ if $i \notin Set(\gamma)$ and $a_{i+1} = a_i + 1$ if $i \in Set(\gamma)$; thus, $\vec{s}(\gamma)$ is the minimal sequence of this type. For example, if $\gamma = (2,3,1,1,2)$, then $\vec{s}(\gamma) = 112223455$. Now if $1 \le i_1 \le \ldots \le i_{|\gamma|} \le n$ is a sequence such that $i_j < i_{j+1}$ if $j \in Set(\gamma)$, then it easy to see that we have designed $\vec{s}(\gamma) = a_1 \ldots a_{|\gamma|}$ so that if $b_j = i_j - a_j$ for $j = 1, \ldots, |\gamma|$, then

$0 \leq b_1 \leq b_2 \leq \cdots \leq b_{|\gamma|} \leq n-1-|Set(\gamma)|$. Note that $|Set(\gamma)| = \ell(\gamma)-1$. Thus, the number of such sequences $b_1 \ldots b_{|\gamma|}$ is the number of partitions contained in the $|\gamma| \times (n-\ell(\gamma))$ rectangle, which is well known to be $\binom{n+|\gamma|-\ell(\gamma)}{|\gamma|}$. Thus, the number of sequences $1 \leq i_1 \leq i_2 \leq \cdots \leq i_{|\gamma|} \leq n$ such that $i_j < i_{j+1}$ if $j \in Set(\gamma)$ equals $\binom{n+|\gamma|-\ell(\gamma)}{|\gamma|}$, which yields (4.1.2).

For the specialization $Q_\gamma(z_1,\ldots,z_n)|_{z_i \to q^i z}$, note that

$$\sum_{0 \leq b_1 \leq b_2 \leq \cdots \leq b_{|\gamma|} \leq n-\ell(\gamma)} q^{b_1+\cdots+b_{|\gamma|}} = \begin{bmatrix} n+|\gamma|-\ell(\gamma) \\ |\gamma| \end{bmatrix}_q .$$

Thus

$$\begin{aligned} \sum_{\substack{1 \leq i_1 \leq i_2 \leq \cdots \leq i_{|\gamma|} \leq n \\ i_j < i_{j+1} \text{ if } j \in Set(\gamma)}} q^{i_1+\cdots+i_{|\gamma|}} &= q^{|\vec{s}(\gamma)|} \begin{bmatrix} n+|\gamma|-\ell(\gamma) \\ |\gamma| \end{bmatrix}_q \\ &= q^{\sum_i i\gamma_i} \begin{bmatrix} n+|\gamma|-\ell(\gamma) \\ |\gamma| \end{bmatrix}_q . \end{aligned}$$

□

Next, we define several more classes of words. In particular, we are interested in words w that we can partition into blocks of size s and ending in a block of size j where $0 \leq j \leq s-1$ like those considered for the classes in $SU^{s-1}SD$, $WU^{s-1}SD$, $SU^{s-1}WD$, and $WU^{s-1}WD$. That is, letters in a given block are either strictly increasing or weakly increasing, but this time we want either weak increases or strict increases between the blocks. In pictures, we want to consider words as pictured in Figure 4.2.

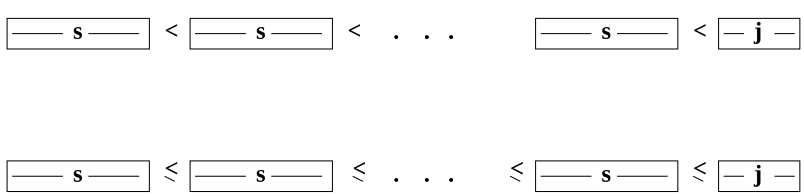

Figure 4.2: Pictorial representation of words in $SU^{s-1}WU$, $WU^{s-1}WU$, $SU^{s-1}SU$, and $WU^{s-1}SU$.

Formally, we consider the following sets of words.

Definition 4.1.2. Let $s \geq 2$.

1. $SU^{s-1}WU_{n,m}$ is the set of all words $w = w_1 \dots w_m \in \mathbb{P}_n^+$ of length m such that $w_i \leq w_{i+1}$ if $i \in s\mathbb{P}$ and $w_i < w_{i+1}$ if $i \notin s\mathbb{P}$.
   We let $SU^{s-1}WU_n = \bigcup_{m \geq 0} SU^{s-1}WU_{n,m}$.

2. $WU^{s-1}WU_{n,m}$ is the set of all words $w = w_1 \dots w_m \in \mathbb{P}_n^+$ of length m such that $w_i \leq w_{i+1}$ if $i \in s\mathbb{P}$ and $w_i \leq w_{i+1}$ if $i \notin s\mathbb{P}$.
   We let $WU^{s-1}WU_n = \bigcup_{m \geq 0} WU^{s-1}WU_{n,m}$. Thus $WU^{s-1}WU_n$ is just the set of all weakly increasing words in $\mathbb{P}_n^+$.

3. $SU^{s-1}SU_{n,m}$ is the set of all words $w = w_1 \dots w_m \in \mathbb{P}_n^+$ of length m such that $w_i < w_{i+1}$ if $i \in s\mathbb{P}$ and $w_i < w_{i+1}$ if $i \notin s\mathbb{P}$.
   We let $SU^{s-1}SU_n = \bigcup_{m \geq 0} SU^{s-1}SU_{n,m}$. Thus $SU^{s-1}SU_n$ is just the set of all strictly increasing words in $\mathbb{P}_n^+$.

4. $WU^{s-1}SU_{n,m}$ is the set of all words $w = w_1 \dots w_m \in \mathbb{P}_n^+$ of length m such that $w_i < w_{i+1}$ if $i \in s\mathbb{P}$ and $w_i \leq w_{i+1}$ if $i \notin s\mathbb{P}$.
   We let $SU^{s-1}WU_n = \bigcup_{m \geq 0} SU^{s-1}WU_{n,m}$.

We then define the following generating functions for any $s \geq 2$:

$$\begin{aligned} P_{n,s,0}^{SU^{s-1}WU}(z_1, \dots, z_n) &= 1 + \sum_{k \geq 1} (-1)^k \sum_{w \in SU^{s-1}WU_{n,ks}} z(w) \text{ and} \\ P_{n,s,j}^{SU^{s-1}WU}(z_1, \dots, z_n) &= \sum_{k \geq 0} (-1)^k \sum_{w \in SU^{s-1}WU_{n,ks+j}} z(w) \text{ for } j = 1, \dots, s-1. \end{aligned}$$

We define $P_{n,s,j}^{WU^{s-1}WU}(z_1, \dots, z_n)$, $P_{n,s,j}^{SU^{s-1}SU}(z_1, \dots, z_n)$, and $P_{n,s,j}^{WU^{s-1}SU}(z_1, \dots, z_n)$ for $j = 0, \dots, s-1$ similarly. We can express each of these generating functions in terms of quasi-symmetric functions. That is, for any $s \geq 2$,

$$P_{n,s,0}^{SU^{s-1}WU}(z_1, \dots, z_n) = 1 + \sum_{k \geq 1} (-1)^k Q_{1(1^{s-2}2)^{k-1}1^{s-1}}(z_1, \dots, z_n) \text{ and}$$

$$P_{n,s,j}^{SU^{s-1}WU}(z_1, \dots, z_n) = \sum_{k \geq 0} (-1)^k Q_{1(1^{s-2}2)^k 1^{j-1}}(z_1, \dots, z_n) \text{ for } j = 1, \dots, s-1.$$

It then follows from Lemma 4.1.1 that for $s \geq 2$ and $j = 1, \ldots, s-1$,

$$P^{SU^{s-1}WU}_{n,s,0}(z_1, \ldots, z_n)|_{z_i=q^i z} = 1 + \sum_{k\geq 1} (-1)^k q^{\binom{k(s-1)+1}{2} + (s-1)\binom{k+1}{2}} \begin{bmatrix} n+k-1 \\ ks \end{bmatrix}_q z^{ks}$$

and

$$P^{SU^{s-1}WU}_{n,s,j}(z_1, \ldots, z_n)|_{z_i=q^i z} = \sum_{k\geq 0} (-1)^k q^{\binom{k(s-1)+j+1}{2} + (s-1)\binom{k+1}{2}} \begin{bmatrix} n+k \\ ks+j \end{bmatrix}_q z^{ks+j}$$

(note that these are finite sums as $\begin{bmatrix} n+k \\ ks \end{bmatrix}_q = 0$ for $k > \frac{n}{s} + 1$).

Similarly, for $s \geq 2$ and $j = 1, \ldots, s-1$,

$$\begin{aligned} P^{WU^{s-1}WU}_{n,s,0}(z_1, \ldots, z_n) &= 1 + \sum_{k\geq 1} (-1)^k Q_{(ks)}(z_1, \ldots, z_n) \text{ and} \\ P^{WU^{s-1}WU}_{n,s,j}(z_1, \ldots, z_n) &= \sum_{k\geq 0} (-1)^k Q_{(ks+j)}(z_1, \ldots, z_n) \end{aligned}$$

and with the specializations

$$\begin{aligned} P^{WU^{s-1}WU}_{n,s,0}(z_1, \ldots, z_n)|_{z_i=q^i z} &= 1 + \sum_{k\geq 1} (-1)^k q^{ks} \begin{bmatrix} n+ks-1 \\ ks \end{bmatrix}_q z^{ks}, \text{ and} \\ P^{WU^{s-1}WU}_{n,s,j}(z_1, \ldots, z_n)|_{z_i=q^i z} &= \sum_{k\geq 0} (-1)^k q^{ks+j} \begin{bmatrix} n+ks+j-1 \\ ks+j \end{bmatrix}_q z^{ks+j} \end{aligned}$$

(sums are truly infinite).

We also have, for any $s \geq 2$ and $1 \leq j \leq s-1$,

$$\begin{aligned} P^{SU^{s-1}SU}_{n,s,0}(z_1, \ldots, z_n) &= 1 + \sum_{k\geq 1} (-1)^k Q_{(1^{ks})}(z_1, \ldots, z_n) \text{ and} \\ P^{SU^{s-1}SU}_{n,s,j}(z_1, \ldots, z_n) &= \sum_{k\geq 0} (-1)^k Q_{(1^{ks+j})}(z_1, \ldots, z_n) \end{aligned}$$

with the specializations

$$\begin{aligned} P^{SU^{s-1}SU}_{n,s,0}(z_1, \ldots, z_n)|_{z_i=q^i z} &= 1 + \sum_{k\geq 1} (-1)^k q^{\binom{ks+1}{2}} \begin{bmatrix} n \\ ks \end{bmatrix}_q z^{ks}, \text{ and} \\ P^{SU^{s-1}SU}_{n,s,j}(z_1, \ldots, z_n)|_{z_i=q^i z} &= \sum_{k\geq 0} (-1)^k q^{\binom{ks+j+1}{2}} \begin{bmatrix} n \\ ks+j \end{bmatrix}_q z^{ks+j} \end{aligned}$$

(nonzero terms when $ks+j \le n$).

Finally, for any $s \ge 2$ and $j = 1, \ldots, s-1$,

$$\begin{aligned} P^{WU^{s-1}SU}_{n,s,0}(z_1,\ldots,z_n) &= 1 + \sum_{k\ge 1} (-1)^k Q_{(s^k)}(z_1,\ldots,z_n) \text{ and} \\ P^{WU^{s-1}SU}_{n,s,j}(z_1,\ldots,z_n) &= \sum_{k\ge 0} (-1)^k Q_{(s^k j)}(z_1,\ldots,z_n) \end{aligned}$$

with the specializations

$$P^{WU^{s-1}SU}_{n,s,0}(z_1,\ldots,z_n)|_{z_i=q^i z} = 1 + \sum_{k\ge 1} (-1)^k q^{s\binom{k+1}{2}} \begin{bmatrix} n+k(s-1) \\ ks \end{bmatrix}_q z^{ks}$$

and

$$\begin{aligned} &P^{WU^{s-1}SU}_{n,s,j}(z_1,\ldots,z_n)|_{z_i=q^i z} \\ &\quad = \sum_{k\ge 0} (-1)^k q^{s\binom{k+1}{2}+j(k+1)} \begin{bmatrix} n+k(s-1)+j-1 \\ ks+j \end{bmatrix}_q z^{ks+j} \end{aligned}$$

(nonzero terms when $k < n$).

4.2 Main results

In this section, we shall prove our desired formulas. Our first theorem is the following.

Theorem 4.2.1. *Let $s \ge 2$. Then*

$$\begin{aligned} H^{SU^{s-1}SD}_{n,s,0}(z_1,\ldots,z_n) &= 1 + \sum_{m\in s\mathbb{P}} \sum_{w\in SU^{s-1}SD_{n,m}} z(w) \\ &= \frac{1}{P^{SU^{s-1}WU}_{n,s,0}(z_1,\ldots,z_n)} \\ &= \frac{1}{1+\sum_{k\ge 1}(-1)^k Q_{1(1^{s-2}2)^{k-1}1^{s-1}}(z_1,\ldots,z_n)}, \end{aligned} \tag{4.2.1}$$

$$\begin{aligned} H_{n,s,0}^{WU^{s-1}SD}(z_1,\ldots,z_n) &= 1+\sum_{m\in s\mathbb{P}}\sum_{w\in WU^{s-1}SD_{n,m}} z(w) \\ &= \frac{1}{P_{n,s,0}^{WU^{s-1}WU}(z_1,\ldots,z_n)} \\ &= \frac{1}{1+\sum_{k\geq 1}(-1)^k Q_{(ks)}(z_1,\ldots,z_n)}, \end{aligned} \tag{4.2.2}$$

$$\begin{aligned} H_{n,s,0}^{SU^{s-1}WD}(z_1,\ldots,z_n) &= 1+\sum_{m\in s\mathbb{P}}\sum_{w\in SU^{s-1}WD_{n,m}} z(w) \\ &= \frac{1}{P_{n,s,0}^{SU^{s-1}SU}(z_1,\ldots,z_n)} \\ &= \frac{1}{1+\sum_{k\geq 1}(-1)^k Q_{(1^{ks})}(z_1,\ldots,z_n)}, \end{aligned} \tag{4.2.3}$$

and

$$\begin{aligned} H_{n,s,0}^{WU^{s-1}WD}(z_1,\ldots,z_n) &= 1+\sum_{m\in s\mathbb{P}}\sum_{w\in WU^{s-1}WD_{n,m}} z(w) \\ &= \frac{1}{P_{n,s,0}^{WU^{s-1}SU}(z_1,\ldots,z_n)} \\ &= \frac{1}{1+\sum_{k\geq 1}(-1)^k Q_{(s^k)}(z_1,\ldots,z_n)}. \end{aligned} \tag{4.2.4}$$

Proof. We start by proving (4.2.1). We must show that

$$H_{n,s,0}^{SU^{s-1}SD}(z_1,\ldots,z_n)\cdot P_{n,s,0}^{SU^{s-1}WU}(z_1,\ldots,z_n)=1. \tag{4.2.5}$$

Now we can interpret the LHS of (4.2.5) as

$$\sum_{(a,b)\in T} z(a)z(b)(-1)^{\ell(b)/s} \tag{4.2.6}$$

where T is the set of all pairs of words (a,b) such that
$a\in\{\epsilon\}\cup\bigcup_{m\in s\mathbb{P}} SU^{s-1}SD_{n,m}$ and $b\in\{\epsilon\}\cup\bigcup_{m\in s\mathbb{P}} SU^{s-1}WU_{n,m}$.
The empty word ϵ accounts for the leading 1 in the series of $H_{n,s,0}^{SU^{s-1}SD}(z_1,\ldots,z_n)$

and $P_{n,s,0}^{SU^{s-1}WU}(z_1,\dots,z_n)$. Thus, in general, a consists of a number of strictly increasing blocks of size s where there are strict decreases between blocks and b consists of a number of strictly increasing blocks of size s where there are weak increases between blocks. We will define a sign-reversing, weight-preserving involution I_1 on the collection of all such pairs of words (a,b). The definition of I_1 proceeds in 4 cases.

Case 1. The last block of a is $a_{ks+1} < \ldots < a_{ks+s}$ and the first block of b is $b_1 < \cdots < b_s$.
If $a_{ks+s} > b_1$, then $I_1(a,b) = (\bar{a},\bar{b})$, where $\bar{a}$ is the result of inserting the first block of b at the end of a and $\bar{b}$ is the result of removing the first block of b from b. Clearly $(\bar{a},\bar{b})$ is again a pair in T. However if $a_{ks+s} \leq b_1$, then we let $I_1(a,b) = (\bar{\bar{a}},\bar{\bar{b}})$ where $\bar{\bar{a}}$ is the result of removing the last block of a from a and $\bar{\bar{b}}$ is the result of inserting the last block of a at the start of b. Clearly $(\bar{\bar{a}},\bar{\bar{b}})$ is again a pair in T.

Case 2. The first block of b is $b_1 < \cdots < b_s$ and $a = \epsilon$.
Then $I_1(a,b) = (\bar{a},\bar{b})$, where $\bar{a} = b_1 \ldots b_s$ and $\bar{b}$ is the result of removing the first block of b from b. Clearly $(\bar{a},\bar{b})$ is again a pair in T.

Case 3. The last block of a is $a_{ks+1} < \ldots < a_{ks+s}$ and $b = \epsilon$.
Then $I_1(a,b) = (\bar{\bar{a}},\bar{\bar{b}})$, where $\bar{\bar{a}}$ is the result of removing the last block of a from a and $\bar{\bar{b}} = a_{ks+1} \ldots a_{ks+s}$.

Case 4. $a = b = \epsilon$.
Then $I(a,b) = (a,b)$.

It is easy to see that I_1 is a sign-reversing, weight-preserving involution with trivial fixed point (ϵ,ϵ), so that I_1 proves (4.2.5).

The exact same involution will prove that

$$H_{n,s,0}^{WU^{s-1}SD}(z_1,\dots,z_n) \cdot P_{n,s,0}^{WU^{s-1}WU}(z_1,\dots,z_n) = 1, \tag{4.2.7}$$

since the only difference in this case is that the blocks are weakly increasing.

The same proof, with minor modifications, will also prove

$$H^{SU^{s-1}WD}_{n,s,0}(z_1,\ldots,z_n)\cdot P^{SU^{s-1}SU}_{n,s,0}(z_1,\ldots,z_n)=1 \tag{4.2.8}$$

and

$$H^{WU^{s-1}WD}_{n,s,0}(z_1,\ldots,z_n)\cdot P^{WU^{s-1}SU}_{n,s,0}(z_1,\ldots,z_n)=1. \tag{4.2.9}$$

That is, we can interpret the LHS of (4.2.8) as

$$\sum_{(a,b)\in U} z(a)z(b)(-1)^{\ell(b)/s} \tag{4.2.10}$$

where U is the set of all pairs of words (a,b) such that
$a\in\{\epsilon\}\cup\bigcup_{m\in s\mathbb{P}} SU^{s-1}WD_{n,m}$ and $b\in\{\epsilon\}\cup\bigcup_{m\in s\mathbb{P}} SU^{s-1}SU_{n,m}$.
Thus, in general, a consists of a number of strictly increasing blocks of size s where there are weak decreases between blocks and b consists of a number of strictly increasing blocks of size s where there are strict increases between blocks. Again, we define a sign-reversing, weight-preserving involution I_2 on the collection of all such pairs of words (a,b). The definition of I_2 proceeds in 4 cases just like the definition of I_1, where only Case 1 has to change. That is, I_2 is defined as follows.

Case 1. The last block of a is $a_{ks+1}<\ldots<a_{ks+s}$ and the first block of b is $b_1<\cdots<b_s$.
If $a_{ks+s}\geq b_1$, then $I_2(a,b)=(\bar{a},\bar{b})$, where $\bar{a}$ is the result of inserting the first block of b at the end of a and $\bar{b}$ is the result of removing the first block of b from b. Clearly $(\bar{a},\bar{b})$ is again a pair in T. However, if $a_{ks+s}<b_1$, then we let $I_2(a,b)=(\bar{\bar{a}},\bar{\bar{b}})$, where $\bar{\bar{a}}$ is the result of removing the last block of a from a and $\bar{\bar{b}}$ is the result of inserting the last block of a at the start of b. Clearly $(\bar{\bar{a}},\bar{\bar{b}})$ is again a pair in U.

Case 2. The first block of b is $b_1<\cdots<b_s$ and $a=\epsilon$.
Then $I_2(a,b)=(\bar{a},\bar{b})$, where $\bar{a}=b_1\ldots b_s$ and $\bar{b}$ is the result of removing the first block of b from b. Clearly $(\bar{a},\bar{b})$ is again a pair in U.

Case 3. The last block of a is $a_{ks+1} < \ldots < a_{ks+s}$ and $b = \epsilon$.
Then $I_2(a,b) = (\bar{\bar{a}}, \bar{\bar{b}})$, where $\bar{\bar{a}}$ is the result of removing the last block of a from a and $\bar{\bar{b}} = a_{ks+1} \ldots a_{ks+s}$.

Case 4. $a = b = \epsilon$.
Then we let $I_2(a,b) = (a,b)$.

Clearly I_2 proves the LHS of (4.2.8). Essentially the same involution will also prove (4.2.9) since the only difference in that case is that the blocks are weakly increasing.

□

Using Lemma 4.1.1, we immediately have the following corollaries.

Corollary 4.2.2. *Let $s \geq 2$. Then*

$$
\begin{aligned}
1 + \sum_{m \in s\mathbb{P}} \sum_{w \in SU^{s-1}SD_{n,m}} q^{|w|} z^{\ell(w)} &= \frac{1}{1 + \sum_{k \geq 1} (-1)^k q^{\binom{k(s-1)+1}{2} + (s-1)\binom{k+1}{2}} \begin{bmatrix} n+k \\ ks \end{bmatrix}_q z^{ks}}, \\
1 + \sum_{m \in s\mathbb{P}} \sum_{w \in WU^{s-1}SD_{n,m}} q^{|w|} z^{\ell(w)} &= \frac{1}{1 + \sum_{k \geq 1} (-1)^k q^{ks} \begin{bmatrix} n+ks-1 \\ ks \end{bmatrix}_q z^{ks}}, \\
1 + \sum_{m \in s\mathbb{P}} \sum_{w \in SU^{s-1}WD_{n,m}} q^{|w|} z^{\ell(w)} &= \frac{1}{1 + \sum_{k \geq 1} (-1)^k q^{\binom{ks+1}{2}} \begin{bmatrix} n \\ ks \end{bmatrix}_q z^{ks}}, \quad \textit{and} \\
1 + \sum_{m \in s\mathbb{P}} \sum_{w \in WU^{s-1}WD_{n,m}} q^{|w|} z^{\ell(w)} &= \frac{1}{1 + \sum_{k \geq 1} (-1)^k q^{s\binom{k+1}{2}} \begin{bmatrix} n+k(s-1) \\ ks \end{bmatrix}_q z^{ks}}.
\end{aligned}
$$

Our next theorem will give the other generating functions mentioned in the introduction.

Theorem 4.2.3. *Let $s \geq 2$ and $1 \leq j \leq s-1$. Then*

$$\begin{aligned}
H_{n,s,j}^{SU^{s-1}SD}(z_1,\ldots,z_n) &= \sum_{m \in j+s\mathbb{P}} \sum_{w \in SU^{s-1}SD_{n,m}} z(w) \\
&= \frac{P_{n,s,j}^{SU^{s-1}WU}(z_1,\ldots,z_n)}{P_{n,s,0}^{SU^{s-1}WU}(z_1,\ldots,z_n)} \qquad (4.2.11)\\
&= \frac{\sum_{k\geq 0}(-1)^k Q_{1(1^{s-2}2)^k 1^{j-1}}(z_1,\ldots,z_n)}{1+\sum_{k\geq 1}(-1)^k Q_{1(1^{s-2}2)^{k-1}1^{s-1}}(z_1,\ldots,z_n)},
\end{aligned}$$

$$\begin{aligned}
H_{n,s,j}^{WU^{s-1}SD}(z_1,\ldots,z_n) &= \sum_{m \in j+s\mathbb{P}} \sum_{w \in WU^{s-1}SD_{n,m}} z(w) \\
&= \frac{P_{n,s,j}^{WU^{s-1}WU}(z_1,\ldots,z_n)}{P_{n,s,0}^{WU^{s-1}WU}(z_1,\ldots,z_n)} \qquad (4.2.12)\\
&= \frac{\sum_{k\geq 0}(-1)^k Q_{(ks+j)}(z_1,\ldots,z_n)}{1+\sum_{k\geq 1}(-1)^k Q_{(ks)}(z_1,\ldots,z_n)},
\end{aligned}$$

$$\begin{aligned}
H_{n,s,j}^{SU^{s-1}WD}(z_1,\ldots,z_n) &= \sum_{m \in j+s\mathbb{P}} \sum_{w \in SU^{s-1}WD_{n,m}} z(w) \\
&= \frac{P_{n,s,j}^{SU^{s-1}SU}(z_1,\ldots,z_n)}{P_{n,s,0}^{SU^{s-1}SU}(z_1,\ldots,z_n)} \qquad (4.2.13)\\
&= \frac{\sum_{k\geq 0}(-1)^k Q_{(1^{ks+j})}(z_1,\ldots,z_n)}{1+\sum_{k\geq 1}(-1)^k Q_{(1^{ks})}(z_1,\ldots,z_n)},
\end{aligned}$$

and

$$\begin{aligned}
H_{n,s,j}^{WU^{s-1}WD}(z_1,\ldots,z_n) &= \sum_{m \in s\mathbb{P}} \sum_{w \in WU^{s-1}WD_{n,m}} z(w) \\
&= \frac{P_{n,s,j}^{WU^{s-1}SU}(z_1,\ldots,z_n)}{P_{n,s,0}^{WU^{s-1}SU}(z_1,\ldots,z_n)} \qquad (4.2.14)\\
&= \frac{\sum_{k\geq 0}(-1)^k Q_{(s^k j)}(z_1,\ldots,z_n)}{1+\sum_{k\geq 1}(-1)^k Q_{(s^k)}(z_1,\ldots,z_n)}.
\end{aligned}$$

Proof. We start by proving (4.2.11). Since we know that

$$H_{n,s,0}^{SU^{s-1}SD}(z_1,\ldots,z_n) = \frac{1}{P_{n,s,0}^{SU^{s-1}WU}(z_1,\ldots,z_n)},$$

we must show that

$$H_{n,s,0}^{SU^{s-1}SD}(z_1,\ldots,z_n)\cdot P_{n,s,j}^{SU^{s-1}WU}(z_1,\ldots,z_n)=H_{n,s,j}^{SU^{s-1}SD}(z_1,\ldots,z_n). \quad (4.2.15)$$

Now we can interpret the LHS of (4.2.15) as

$$\sum_{(a,b)\in V} z(a)z(b)(-1)^{(\ell(b)-j)/s} \quad (4.2.16)$$

where V is the set of all pairs of words (a,b) such that $a\in\{\epsilon\}\cup\bigcup_{m\in s\mathbb{P}} SU^{s-1}SD_{n,m}$ and $b\in\bigcup_{m\in j+s\mathbb{P}} SU^{s-1}WU_{n,m}$. Thus, in general, a consists of a number of strictly increasing blocks of size s where there are strict decreases between blocks and b consists of a number of strictly increasing blocks of size s followed by a strictly increasing block of size j where there are weak increases between blocks. We will define a sign-reversing weight preserving involution I_3 on the collection of all such pairs of words (a,b). The definition of I_3 proceeds in 4 cases.

Case 1. The last block of a is $a_{ks+1}<\ldots<a_{ks+s}$ and the first block of b is $b_1<\cdots<b_s$.
If $a_{ks+s}>b_1$, then $I_3(a,b)=(\bar{a},\bar{b})$ where $\bar{a}$ is the result of inserting the first block of b at the end of a and $\bar{b}$ is the result of removing the first block of b from b. Clearly $(\bar{a},\bar{b})$ is again a pair in V. However if $a_{ks+s}\le b_1$, then we let $I_3(a,b)=(\bar{\bar{a}},\bar{\bar{b}})$ where $\bar{\bar{a}}$ is the result of removing the last block of a from a and $\bar{\bar{b}}$ is the result of inserting the last block of a at the start of b. Clearly $(\bar{\bar{a}},\bar{\bar{b}})$ is again a pair in V.

Case 2. The first block of b is $b_1<\cdots<b_s$ and $a=\epsilon$.
Then $I_3(a,b)=(\bar{a},\bar{b})$ where $\bar{a}=b_1\ldots b_s$ and $\bar{b}$ is the result of removing the first block of b from b. Clearly $(\bar{a},\bar{b})$ is again a pair in V.

Case 3. The last block of a is $a_{ks+1}<\ldots<a_{ks+s}$ and $b=b_1<\cdots<b_j$ where $a_{ks+s}\le b_1$.
Then $I_3(a,b)=(\bar{\bar{a}},\bar{\bar{b}})$ where $\bar{\bar{a}}$ is the result of removing the last block of a from a and $\bar{\bar{b}}$ is the result of inserting the last block of a to the start of b.

Case 4. The last block of a is $a_{ks+1} < \ldots < a_{ks+s}$ and $b = b_1 < \cdots < b_j$ where $a_{ks+s} > b_1$.
Then $I_3(a,b) = (a,b)$.

It is easy to see that I_3 is a sign-reversing, weight-preserving involution, so that I_3 proves that the LHS of (4.2.15) reduces to summing the weights of the pairs of words (a,b) in Case 4. To do this, first observe that the signs of all the pairs of words in Case 4 are positive. Moreover, it is easy to see that if we insert b at the end of a, we will create a word in $\bigcup_{m\in j+s\mathbb{P}} SU^{s-1}SD_{n,m}$ and that all words in $\bigcup_{m\in j+s\mathbb{P}} SU^{s-1}SD_{n,m}$ arise from the pairs of words in Case 4 in this way. Thus, the sum of the weights in Case 4 is equal to $H_{n,s,j}^{SU^{s-1}SD}(z_1,\ldots,z_n)$ as desired.

The exact same involution will prove that

$$H_{n,s,0}^{WU^{s-1}SD}(z_1,\ldots,z_n) \cdot P_{n,s,0}^{WU^{s-1}WU}(z_1,\ldots,z_n) = H_{n,s,j}^{WU^{s-1}SD}(z_1,\ldots,z_n) \quad (4.2.17)$$

since the only difference in this case is that the blocks are weakly increasing.

It is also the case that we can make the same type of modifications to the involution as we did in Theorem 4.2.1 to prove

$$H_{n,s,0}^{SU^{s-1}WD}(z_1,\ldots,z_n) \cdot P_{n,s,j}^{SU^{s-1}SU}(z_1,\ldots,z_n) = H_{n,s,j}^{SU^{s-1}WD}(z_1,\ldots,z_n) \quad (4.2.18)$$

and

$$H_{n,s,0}^{WU^{s-1}WD}(z_1,\ldots,z_n) \cdot P_{n,s,j}^{WU^{s-1}SU}(z_1,\ldots,z_n) = H_{n,s,j}^{WU^{s-1}WD}(z_1,\ldots,z_n). \quad (4.2.19)$$

□

Using Lemma 4.1.1, we immediately have the following corollaries.

Corollary 4.2.4. *Let $s \geq 2$ and $1 \leq j \leq s-1$. Then*

$$\sum_{m \in j+s\mathbb{P}} \sum_{w \in SU^{s-1}SD_{n,m}} q^{|w|} z^{\ell(w)} = \frac{\sum_{k\geq 0}(-1)^k q^{\binom{k(s-1)+j+1}{2}+(s-1)\binom{k+1}{2}} \begin{bmatrix} n+k \\ ks+j \end{bmatrix}_q z^{ks+j}}{1+\sum_{k\geq 1}(-1)^k q^{\binom{k(s-1)+1}{2}+(s-1)\binom{k+1}{2}} \begin{bmatrix} n+k \\ ks \end{bmatrix}_q z^{ks}},$$

$$\sum_{m \in j+s\mathbb{P}} \sum_{w \in WU^{s-1}SD_{n,m}} q^{|w|} z^{\ell(w)} = \frac{\sum_{k\geq 0}(-1)^k q^{ks+j} \begin{bmatrix} n+ks+j-1 \\ ks+j \end{bmatrix}_q z^{ks+j}}{1+\sum_{k\geq 1}(-1)^k q^{ks} \begin{bmatrix} n+ks-1 \\ ks \end{bmatrix}_q z^{ks}},$$

$$\sum_{m \in j+s\mathbb{P}} \sum_{w \in SU^{s-1}WD_{n,m}} q^{|w|} z^{\ell(w)} = \frac{\sum_{k\geq 0}(-1)^k q^{\binom{ks+j+1}{2}} \begin{bmatrix} n \\ ks+j \end{bmatrix}_q z^{ks+j}}{1+\sum_{k\geq 1}(-1)^k q^{\binom{ks+1}{2}} \begin{bmatrix} n \\ ks \end{bmatrix}_q z^{ks}}, \text{ and}$$

$$\sum_{m \in j+s\mathbb{P}} \sum_{w \in WU^{s-1}WD_{n,m}} q^{|w|} z^{\ell(w)} = \frac{\sum_{k\geq 0}(-1)^k q^{s\binom{k+1}{2}+j(k+1)} \begin{bmatrix} n+k(s-1)+j-1 \\ ks+j \end{bmatrix}_q z^{ks+j}}{1+\sum_{k\geq 1}(-1)^k q^{s\binom{k+1}{2}} \begin{bmatrix} n+k(s-1) \\ ks \end{bmatrix}_q z^{ks}}.$$

4.3 Extensions

It should be clear from our definitions of the involutions in section 3 that they did not depend on the nature of what was in the blocks. We only needed that the blocks in the pairs of words (a, b) are of the same type. Thus, the same type of theorems will hold for any type of block conditions. For example, suppose that we consider a block condition $a_1 \dots a_s$ where we require that the $a_{i+1} - a_i \geq r$ for $i = 1, \dots, s-1$. That is fix $s \geq 2$ and $r \geq 1$. We then define the following classes of words.

1. $S^rU^{s-1}SD_{n,m}$ is the set of all words $w = w_1 \dots w_m \in \mathbb{P}_n^+$ of length m such that $w_i > w_{i+1}$ if $i \in s\mathbb{P}$ and $r + w_i \leq w_{i+1}$ if $i \notin s\mathbb{P}$.
   We let $S^rU^{s-1}SD_n = \bigcup_{m\geq 0} S^rU^sSD_{n,m}$.

2. $S^rU^{s-1}WU_{n,m}$ is the set of all words $w = w_1 \dots w_m \in \mathbb{P}_n^+$ of length m such that $w_i \leq w_{i+1}$ if $i \in s\mathbb{P}$ and $r + w_i \leq w_{i+1}$ if $i \notin s\mathbb{P}$.
   We let $S^rU^{s-1}WU_n = \bigcup_{m\geq 0} S^rU^sWU_{n,m}$.

We can also define the following set of generating functions.

$$\begin{aligned}
H_{n,s,r,0}^{S^rU^{s-1}SD}(z_1,\ldots,z_n) &= 1+\sum_{k\geq 1}\sum_{w\in S^rU^{s-1}SD_{n,ks}} z(w),\\
H_{n,s,r,j}^{S^rU^{s-1}SD}(z_1,\ldots,z_n) &= \sum_{k\geq 0}\sum_{w\in S^rU^{s-1}SD_{n,ks+j}} z(w) \text{ for } j=1,\ldots,s-1,\\
P_{n,s,r,0}^{S^rU^{s-1}WU}(z_1,\ldots,z_n) &= 1+\sum_{k\geq 1}(-1)^k\sum_{w\in S^rU^{s-1}WU_{n,ks}} z(w) \text{ and}\\
P_{n,s,r,j}^{S^rU^{s-1}WU}(z_1,\ldots,z_n) &= \sum_{k\geq 0}(-1)^k\sum_{w\in S^rU^{s-1}WU_{n,ks+j}} z(w) \text{ for } j=1,\ldots,s-1.
\end{aligned}$$

Then we can use the same proofs as in Theorems 4.2.1 and 4.2.3 to prove that

$$H_{n,s,r,0}^{S^rU^{s-1}SD}(z_1,\ldots,z_n)=\frac{1}{P_{n,s,r,0}^{S^rU^{s-1}WU}(z_1,\ldots,z_n)}, \tag{4.3.1}$$

and

$$H_{n,s,r,j}^{S^rU^{s-1}SD}(z_1,\ldots,z_n)=\frac{P_{n,s,r,j}^{S^rU^{s-1}WU}(z_1,\ldots,z_n)}{P_{n,s,r,0}^{S^rU^{s-1}WU}(z_1,\ldots,z_n)}, \tag{4.3.2}$$

In this case, we cannot express the $P_{n,s,r,j}^{S^rU^{s-1}WU}(z_1,\ldots,z_n)$ as a sum of quasi-symmetric functions, but we can still give explicit expressions for the specializations where we replace z_i by q^iz for $i=1,\ldots,n$. That is, suppose that $m=ks+j$ where $0\leq j\leq s-1$, and we are given a word $a_1\ldots a_m\in S^rU^{s-1}WU_{n,m}$. Then, let $b=b_1\ldots b_{ks+j}$ be such that $b_1=1$ and $b_{i+1}-b_i=r$ if $i\notin s\mathbb{P}$ and $b_{i+1}=b_i$ if $i\in s\mathbb{P}$. For example if $s=3$, $r=2$, and $m=10$, then $b_1\ldots b_{10}=1\ 3\ 5\ 5\ 7\ 9\ 9\ 11\ 13\ 13$. Note that the largest letter in b is $b_{ks+j}=1+r((s-1)k+[j-1]^+)$, where $[j-1]^+=\max(j-1,0)$, and that

$$\begin{aligned}
|b| &= \sum_{i=0}^{k(s-1)+j-1}(1+ir)+\sum_{i=1}^{k} i(s-1)r+1\\
&= ks+j+r\left(\sum_{i=0}^{k(s-1)+j-1} i\right)+r(s-1)\sum_{i=1}^{k} i\\
&= ks+j+r\binom{k(s-1)+j}{2}+r(s-1)\binom{k+1}{2}.
\end{aligned}$$

It is then easy to see that we have designed b so that if $c_i = a_i - b_i$, then $0 \leq c_1 \leq c_2 \leq \cdots \leq c_{ks+j} \leq n - (1 + r(k(s-1) + [j-1]^+))$. Thus, the sequences $c = c_1 \ldots c_{ks+j}$ that arise in this way are just the partitions that lie in the $(ks + j) \times (n - (1 + r(k(s-1) + [j-1]^+)))$ rectangle. Since

$$\sum_{0 \leq c_1 \leq c_2 \leq \cdots \leq c_{ks+j} \leq n-(1+r(k(s-1)+[j-1]^+))} q^{c_1+\cdots+c_{ks+j}} = \begin{bmatrix} n + ks + j - (1 + r(k(s-1) + [j-1]^+)) \\ ks + j \end{bmatrix}_q,$$

it follows that

$$\sum_{a_1 \ldots a_{ks+j} \in S^r U^{s-1} W U_{n,ks+j}} q^{a_1+\cdots+a_{ks+j}} = q^{ks+j+r\binom{k(s-1)+j}{2}+r(s-1)\binom{k+1}{2}} \begin{bmatrix} n + ks + j - (1 + r(k(s-1) + [j-1]^+)) \\ ks + j \end{bmatrix}_q.$$

Thus, for $s \geq 2$, $r \geq 1$, and $j = 1, \ldots, s-1$,

$$P^{S^r U^{s-1} W U}_{n,s,r,0}(z_1, \ldots, z_n)|_{z_i \to q^i z} = 1 + \sum_{k \geq 1} (-1)^k q^{ks + r\binom{k(s-1)}{2} + r(s-1)\binom{k+1}{2}} \begin{bmatrix} n + kr - (r-1)ks - 1 \\ ks \end{bmatrix}_q z^{ks} \text{ and}$$

$$P^{S^r U^{s-1} W U}_{n,s,r,j}(z_1, \ldots, z_n) = \sum_{k \geq 0} (-1)^k q^{ks+j+r\binom{k(s-1)+j}{2}+r(s-1)\binom{k+1}{2}} \begin{bmatrix} n + kr - (r-1)(ks + j - 1) \\ ks + j \end{bmatrix}_q z^{ks+j}.$$

Thus, we have the following theorem.

Theorem 4.3.1. *For $s \geq 2$, $r \geq 1$, and $j = 1, \ldots, s-1$,*

$$1 + \sum_{m \in s\mathbb{P}} \sum_{w \in S^r U^{s-1} SD_{n,m}} q^{|w|} z^{\ell(w)} = \frac{1}{1 + \sum_{k \geq 1} (-1)^k q^{ks + r\binom{k(s-1)}{2} + r(s-1)\binom{k+1}{2}} \begin{bmatrix} n+kr-(r-1)ks-1 \\ ks \end{bmatrix}_q z^{ks}}$$

and

$$\sum_{m\in j+s\mathbb{P}}\sum_{w\in S^rU^{s-1}SD_{n,m}} q^{|w|}z^{\ell(w)} = \frac{q^{ks+j+r\binom{k(s-1)+j}{2}+r(s-1)\binom{k+1}{2}}\begin{bmatrix}n+kr-(r-1)(ks+j-1)\\ks+j\end{bmatrix}_q z^{ks+j}}{1+\sum_{k\geq 1}(-1)^k q^{ks+r\binom{k(s-1)}{2}+r(s-1)\binom{k+1}{2}}\begin{bmatrix}n+kr-(r-1)ks-1\\ks\end{bmatrix}_q z^{ks}}.$$

Chapter 5

Enumerating up-down words with peak conditions

In Chapter 4, we were able to enumerate 4 classes of up-down words via a simple involution. In this chapter, we will enumerate these same classes of up-down words with the added condition that all peaks–entries at the end of a block–are in a certain set $X \subset \mathbb{P}$. We will see that the same involution applies, although the resulting generating functions can no longer be expressed in terms of quasi-symmetric functions.

Definition 5.0.2. Let $s \geq 2$.

1. $SU^{s-1}SD_{n,X,m}$ is the set of all words $w \in \mathbb{P}_n^+$ of length m such that $w_{si} \in X\ \forall i$, $Des(w) = (s\mathbb{P})_{m-1}$ and $Ris(w) = \mathbb{P}_{m-1} - (s\mathbb{P})_{m-1}$.
   We let $SU^{s-1}SD_{n,X} = \bigcup_{m\geq 0} SU^{s-1}SD_{n,X,m}$.

2. $WU^{s-1}SD_{n,X,m}$ is the set of all words $w \in \mathbb{P}_n^+$ of length m such that $w_{si} \in X\ \forall i$, $Des(w) = (s\mathbb{P})_{m-1}$ and $WRis(w) = \mathbb{P}_{m-1} - (s\mathbb{P})_{m-1}$.
   We let $WU^{s-1}SD_{n,X} = \bigcup_{m\geq 0} WU^{s-1}SD_{n,X,m}$.

3. $SU^{s-1}WD_{n,X,m}$ is the set of all words $w \in \mathbb{P}_n^+$ of length m such that $w_{si} \in X\ \forall i$, $WDes(w) = (s\mathbb{P})_{m-1}$ and $Ris(w) = \mathbb{P}_{m-1} - (s\mathbb{P})_{m-1}$.
   We let $SU^{s-1}WD_{n,X} = \bigcup_{m\geq 0} SU^{s-1}WD_{n,X,m}$.

4. $WU^{s-1}WD_{n,X,m}$ is the set of all words $w \in \mathbb{P}_n^+$ of length m such that $w_{si} \in X\ \forall i$, $WDes(w) = (s\mathbb{P})_{m-1}$ and $WRis(w) = \mathbb{P}_{m-1} - (s\mathbb{P})_{m-1}$. We let $WU^{s-1}WD_{n,X} = \bigcup_{m\geq 0} WU^{s-1}WD_{n,X,m}$.

We define $SU^{s-1}WU_{n,X,m}$, $SU^{s-1}WU_{n,X}$, $WU^{s-1}WU_{n,X,m}$, $WU^{s-1}WU_{n,X}$, $SU^{s-1}SU_{n,X,m}$, $SU^{s-1}SU_{n,X}$, $WU^{s-1}SU_{n,X,m}$, and $WU^{s-1}SU_{n,X}$ similarly.

Also, define the following generating functions for any $s \geq 2$:

$$\begin{aligned} H_{n,X,s,0}^{SU^{s-1}SD}(z_1,\ldots,z_n) &= 1 + \sum_{m\in s\mathbb{P}} \sum_{w\in SU^{s-1}SD_{n,X,m}} z(w) \text{ and} \\ H_{n,X,s,j}^{SU^{s-1}SD}(z_1,\ldots,z_n) &= \sum_{m\in j+s\mathbb{P}} \sum_{w\in SU^{s-1}SD_{n,X,m}} z(w) \text{ for } j = 1,\ldots,s-1. \end{aligned}$$

We define $H_{n,X,s,j}^{WU^{s-1}SD}(z_1,\ldots,z_n)$, $H_{n,X,s,j}^{SU^{s-1}WD}(z_1,\ldots,z_n)$, and $H_{n,X,s,j}^{WU^{s-1}WD}(z_1,\ldots,z_n)$ for $j = 0,\ldots,s-1$ similarly.

We wish to find simple expressions for each of these generating functions. The results from the previous chapter can be viewed as the special case when $X = \mathbb{P}$. The proof technique we use to find the above generating functions will be identical to that from Chapter 4. Thus, we wish to define the following additional generating functions for $s \geq 2$:

$$\begin{aligned} P_{n,X,s,0}^{SU^{s-1}WU}(z_1,\ldots,z_n) &= 1 + \sum_{k\geq 1}(-1)^k \sum_{w\in SU^{s-1}WU_{n,X,ks}} z(w) \text{ and} \\ P_{n,X,s,j}^{SU^{s-1}WU}(z_1,\ldots,z_n) &= \sum_{k\geq 0}(-1)^k \sum_{w\in SU^{s-1}WU_{n,X,ks+j}} z(w) \text{ for } j = 1,\ldots,s-1. \end{aligned}$$

We define $P_{n,X,s,j}^{WU^{s-1}WU}(z_1,\ldots,z_n)$, $P_{n,,X,s,j}^{SU^{s-1}SU}(z_1,\ldots,z_n)$, and $P_{n,X,s,j}^{WU^{s-1}SU}(z_1,\ldots,z_n)$ for $j = 0,\ldots,s-1$ similarly.

5.1 Involution

We have the following theorem.

Theorem 5.1.1. *Let $s \geq 2$. Then*

$$H_{n,X,s,0}^{SU^{s-1}SD}(z_1,\ldots,z_n) = \frac{1}{P_{n,X,s,0}^{SU^{s-1}WU}(z_1,\ldots,z_n)}, \tag{5.1.1}$$

$$H_{n,X,s,0}^{WU^{s-1}SD}(z_1,\ldots,z_n) = \frac{1}{P_{n,X,s,0}^{WU^{s-1}WU}(z_1,\ldots,z_n)}, \tag{5.1.2}$$

$$H_{n,X,s,0}^{SU^{s-1}WD}(z_1,\ldots,z_n) = \frac{1}{P_{n,X,s,0}^{SU^{s-1}SU}(z_1,\ldots,z_n)}, \tag{5.1.3}$$

and

$$H_{n,X,s,0}^{WU^{s-1}WD}(z_1,\ldots,z_n) = \frac{1}{P_{n,X,s,0}^{WU^{s-1}SU}(z_1,\ldots,z_n)}. \tag{5.1.4}$$

We start by proving (5.1.1). We must show that

$$H_{n,X,s,0}^{SU^{s-1}SD}(z_1,\ldots,z_n) \cdot P_{n,X,s,0}^{SU^{s-1}WU}(z_1,\ldots,z_n) = 1. \tag{5.1.5}$$

If the reader reflects on the proofs given in Chapter 4, she will realize that they will carry through regardless of any condition on the peaks (or any entries, for that matter). Thus, this proof is essentially the same as the proof of (4.2.1).

We can interpret the LHS of (5.1.5) as

$$\sum_{(a,b)\in T} z(a)z(b)(-1)^{\ell(b)/s}, \tag{5.1.6}$$

where T is the set of all pairs of words (a,b) such that
$a \in \{\epsilon\} \cup \bigcup_{m\in s\mathbb{P}} SU^{s-1}SD_{n,X,m}$ and
$b \in \{\epsilon\} \cup \bigcup_{m\in s\mathbb{P}} SU^{s-1}WU_{n,X,m}$, where ϵ denotes the empty word.
The empty word ϵ accounts for the leading 1 in the series of $H_{n,X,s,0}^{SU^{s-1}SD}(z_1,\ldots,z_n)$ and $P_{n,X,s,0}^{SU^{s-1}WU}(z_1,\ldots,z_n)$. Thus, in general, a consists of a number of strictly increasing blocks of size s where there are strict decreases between blocks and b consists of a number of strictly increasing blocks of size s where there are weak increases between blocks. We will define a sign-reversing, weight-preserving involution I_1 on the collection of all such pairs of words (a,b). The definition of I_1 proceeds in 4 cases.

Case 1. The last block of a is $a_{ks+1} < \ldots < a_{ks+s}$ and the first block of b is $b_1 < \cdots < b_s$.

If $a_{ks+s} > b_1$, then $I_1(a,b) = (\bar{a},\bar{b})$, where $\bar{a}$ is the result of inserting the first block of b at the end of a and $\bar{b}$ is the result of removing the first block of b from b. Clearly $(\bar{a},\bar{b})$ is again a pair in T. However if $a_{ks+s} \le b_1$, then we let $I_1(a,b) = (\bar{\bar{a}},\bar{\bar{b}})$ where $\bar{\bar{a}}$ is the result of removing the last block of a from a and $\bar{\bar{b}}$ is the result of inserting the last block of a at the start of b. Clearly $(\bar{\bar{a}},\bar{\bar{b}})$ is again a pair in T.

Case 2. The first block of b is $b_1 < \cdots < b_s$ and $a = \epsilon$.
Then $I_1(a,b) = (\bar{a},\bar{b})$, where $\bar{a} = b_1 \ldots b_s$ and $\bar{b}$ is the result of removing the first block of b from b. Clearly $(\bar{a},\bar{b})$ is again a pair in T.

Case 3. The last block of a is $a_{ks+1} < \ldots < a_{ks+s}$ and $b = \epsilon$.
Then $I_1(a,b) = (\bar{\bar{a}},\bar{\bar{b}})$, where $\bar{\bar{a}}$ is the result of removing the last block of a from a and $\bar{\bar{b}} = a_{ks+1} \ldots a_{ks+s}$.

Case 4. $a = b = \epsilon$.
Then $I_1(a,b) = (a,b)$.

It is easy to see that I_1 is a sign-reversing, weight-preserving involution with trivial fixed point (ϵ,ϵ), so that I_1 proves (5.1.5).

The analogous proofs will carry over to show 5.1.2, 5.1.3, and 5.1.4. Moreover, analogous proofs will also give us results with a final block of length j. Thus, we have the following theorem.

Theorem 5.1.2. *Let $s \ge 2$ and $1 \le j \le s-1$. Then*

$$H^{SU^{s-1}SD}_{n,X,s,j}(z_1,\ldots,z_n) = \frac{P^{SU^{s-1}WU}_{n,X,s,j}(z_1,\ldots,z_n)}{P^{SU^{s-1}WU}_{n,X,s,0}(z_1,\ldots,z_n)}, \tag{5.1.7}$$

$$H^{WU^{s-1}SD}_{n,X,s,j}(z_1,\ldots,z_n) = \frac{P^{WU^{s-1}WU}_{n,X,s,j}(z_1,\ldots,z_n)}{P^{WU^{s-1}WU}_{n,X,s,0}(z_1,\ldots,z_n)}, \tag{5.1.8}$$

$$H^{SU^{s-1}WD}_{n,X,s,j}(z_1,\ldots,z_n) = \frac{P^{SU^{s-1}SU}_{n,X,s,j}(z_1,\ldots,z_n)}{P^{SU^{s-1}SU}_{n,X,s,0}(z_1,\ldots,z_n)}, \tag{5.1.9}$$

and

$$H^{WU^{s-1}WD}_{n,X,s,j}(z_1,\dots,z_n) = \frac{P^{WU^{s-1}SU}_{n,X,s,j}(z_1,\dots,z_n)}{P^{WU^{s-1}SU}_{n,X,s,0}(z_1,\dots,z_n)}. \tag{5.1.10}$$

Thus, we have reduced our original task to finding the generating functions $P^{SU^{s-1}WU}_{n,X,s,j}(z_1,\dots,z_n)$, $P^{WU^{s-1}WU}_{n,X,s,j}(z_1,\dots,z_n)$, $P^{SU^{s-1}SU}_{n,X,s,j}(z_1,\dots,z_n)$, and $P^{WU^{s-1}SU}_{n,X,s,j}(z_1,\dots,z_n)$ for $j=0,\dots,s-1$.

Unfortunately, there does not seem to be any direct way to find compact expressions for these generating functions for arbitrary X and s. One can develop recursions for such generating functions, but they are not easy to solve in general. However, in the special case where $X=\mathbb{E}$ or $X=\mathbb{O}$, $s=2$, and $z_i=q^iz$, we can find compact expressions for these generating functions. This will be the subject of our next section. Future work could extend these results to more general values s and sets X.

5.2 Special case: $s=2$ and $X=\mathbb{E}$ or $\mathbb{O}$

Define

$$\begin{aligned} EV^{WUSU}_{n,0}(z,q) &= P^{WUSU}_{n,\mathbb{E},2,0}(z_1,\dots,z_n)|_{z_i=q^iz} \\ &= 1+\sum_{k\geq 1}(-1)^k \sum_{w\in WUSU_{n,\mathbb{E},2k}} z^{\ell(w)}q^{|w|}, \end{aligned}$$

$$\begin{aligned} EV^{WUSU}_{n,1}(z,q) &= P^{WUSU}_{n,\mathbb{E},2,1}(z_1,\dots,z_n)|_{z_i=q^iz} \\ &= \sum_{k\geq 0}(-1)^k \sum_{w\in WUSU_{n,\mathbb{E},2k+1}} z^{\ell(w)}q^{|w|}, \end{aligned}$$

$$\begin{aligned} OD^{WUSU}_{n,0}(z,q) &= P^{WUSU}_{n,\mathbb{O},2,0}(z_1,\dots,z_n)|_{z_i=q^iz} \\ &= 1+\sum_{k\geq 1}(-1)^k \sum_{w\in WUSU_{n,\mathbb{O},2k}} z^{\ell(w)}q^{|w|}, \end{aligned}$$

and

$$OD_{n,1}^{WUSU}(z,q) = P_{n,\mathbb{O},2,1}^{WUSU}(z_1,\ldots,z_n)|_{z_i=q^i z} = \sum_{k\geq 0}(-1)^k \sum_{w\in WUSU_{n,\mathbb{O},2k+1}} z^{\ell(w)}q^{|w|}.$$

Similarly define
$EV_{n,0}^{SUWU}(z,q)$, $EV_{n,1}^{SUWU}(z,q)$, $EV_{n,0}^{WUWU}(z,q)$,
$EV_{n,1}^{WUWU}(z,q)$, $EV_{n,0}^{SUSU}(z,q)$, $EV_{n,1}^{SUSU}(z,q)$,
and
$OD_{n,0}^{SUWU}(z,q)$, $OD_{n,1}^{SUWU}(z,q)$, $OD_{n,0}^{WUWU}(z,q)$,
$OD_{n,1}^{WUWU}(z,q)$, $OD_{n,0}^{SUSU}(z,q)$, $OD_{n,1}^{SUSU}(z,q)$.

The following observation reduces our work slightly. Consider, for example, $SUSD_{2n+1,\mathbb{E}}$. No word in $SUSD_{2n+1,\mathbb{E}}$ can contain $2n+1$ except the singleton word $w_1 = 2n+1$, because any other word must have some even peak above that entry. Similarly, no word in $SUSD_{2n,\mathbb{O}}$ can contain $2n$ except the singleton word $w_1 = 2n$. The same reasoning applies to $SUWD$, $WUSD$, and $WUWD$. Thus, we get the following lemma.

Lemma 5.2.1. *Let $n \geq 1$. Then*

$$\begin{aligned}
H_{2n+1,\mathbb{E},2,0}^{WUWD}(z_1,\ldots,z_n) &= H_{2n,\mathbb{E},2,0}^{WUWD}(z_1,\ldots,z_n),\\
H_{2n+1,\mathbb{E},2,1}^{WUWD}(z_1,\ldots,z_n) &= H_{2n,\mathbb{E},2,1}^{WUWD}(z_1,\ldots,z_n) + z_{2n+1},\\
H_{2n+1,\mathbb{E},2,0}^{SUWD}(z_1,\ldots,z_n) &= H_{2n,\mathbb{E},2,0}^{SUWD}(z_1,\ldots,z_n),\\
H_{2n+1,\mathbb{E},2,1}^{SUWD}(z_1,\ldots,z_n) &= H_{2n,\mathbb{E},2,1}^{SUWD}(z_1,\ldots,z_n) + z_{2n+1},\\
H_{2n+1,\mathbb{E},2,0}^{SUSD}(z_1,\ldots,z_n) &= H_{2n,\mathbb{E},2,0}^{SUSD}(z_1,\ldots,z_n),\\
H_{2n+1,\mathbb{E},2,1}^{SUSD}(z_1,\ldots,z_n) &= H_{2n,\mathbb{E},2,1}^{SUSD}(z_1,\ldots,z_n) + z_{2n+1},\\
H_{2n+1,\mathbb{E},2,0}^{WUSD}(z_1,\ldots,z_n) &= H_{2n,\mathbb{E},2,0}^{WUSD}(z_1,\ldots,z_n),\\
H_{2n+1,\mathbb{E},2,1}^{WUSD}(z_1,\ldots,z_n) &= H_{2n,\mathbb{E},2,1}^{WUSD}(z_1,\ldots,z_n) + z_{2n+1},
\end{aligned}$$

and

$$\begin{aligned}
H^{WUWD}_{2n,\mathbb{O},2,0}(z_1,\dots,z_n) &= H^{WUWD}_{2n-1,\mathbb{O},2,0}(z_1,\dots,z_n),\\
H^{WUWD}_{2n,\mathbb{O},2,1}(z_1,\dots,z_n) &= H^{WUWD}_{2n-1,\mathbb{O},2,1}(z_1,\dots,z_n)+z_{2n},\\
H^{SUWD}_{2n,\mathbb{O},2,0}(z_1,\dots,z_n) &= H^{SUWD}_{2n-1,\mathbb{O},2,0}(z_1,\dots,z_n),\\
H^{SUWD}_{2n,\mathbb{O},2,1}(z_1,\dots,z_n) &= H^{SUWD}_{2n-1,\mathbb{O},2,1}(z_1,\dots,z_n)+z_{2n},\\
H^{SUSD}_{2n,\mathbb{O},2,0}(z_1,\dots,z_n) &= H^{SUSD}_{2n-1,\mathbb{O},2,0}(z_1,\dots,z_n),\\
H^{SUSD}_{2n,\mathbb{O},2,1}(z_1,\dots,z_n) &= H^{SUSD}_{2n-1,\mathbb{O},2,1}(z_1,\dots,z_n)+z_{2n},\\
H^{WUSD}_{2n,\mathbb{O},2,0}(z_1,\dots,z_n) &= H^{WUSD}_{2n-1,\mathbb{O},2,0}(z_1,\dots,z_n),\\
H^{WUSD}_{2n,\mathbb{O},2,1}(z_1,\dots,z_n) &= H^{WUSD}_{2n-1,\mathbb{O},2,1}(z_1,\dots,z_n)+z_{2n}.
\end{aligned}$$

Based on this lemma, in order to find all the H generating functions under the specialization $z_i = q^i z$, it suffices to find the generating functions $EV_{2n,j}$ and $OD_{2n-1,j}$ for $j \in \{0,1\}$. In the following subsections, we will find compact expressions for the generating functions $EV_{2n,j}$ and $OD_{2n-1,j}$ for $j \in \{0,1\}$, sketching the proof for each by directly counting the desired objects. Although the bijection from Chapter 4 is the same, finding the generating functions for the classes of words that we reduce to is different in each case and cannot be handled with a general lemma. We treat the first case more carefully, illustrating both the subtle reasoning involved and the simplification steps. We present other cases in slightly less detail. It will be useful to note that, when q-counting words, the power of q in an expression is equal to the sum of the letters in the minimal possible word of the type considered. This provides a check for our reasoning.

5.2.1 SUSU

Theorem 5.2.2.

$$OD^{SUSU}_{2n-1,0}(z,q) = \sum_{k=0}^{n-1}(-1)^k z^{2k}[2]_q \sum_{j=0}^{k-1} q^{2j^2+4k^2+3j-4jk} \begin{bmatrix} n+j \\ 2k \end{bmatrix}_{q^2} \begin{bmatrix} k-1 \\ j \end{bmatrix}_{q^4}$$

and

$$OD^{SUSU}_{2n-1,1}(z,q) = \sum_{k=0}^{n-1}(-1)^k z^{2k+1}([2]_q)^2 \sum_{j=0}^{k-1} q^{j+2j^2+4k-4jk+4k^2} \begin{bmatrix} n+j \\ 2k+1 \end{bmatrix}_{q^2} \begin{bmatrix} k-1 \\ j \end{bmatrix}_{q^4}.$$

We shall classify the words $w = w_1 \dots w_{2k}$ in $SUSU_{2n-1,\mathbb{O},2k}$ by the number of odd positions $2t+1 > 1$ such that w_{2t+1} is even. First, label the odd positions > 1 from left to right with $1, 2, \dots, k-1$. Thus, position 3 gets label 1, position 5 gets label 2, and so on. Let $1 \le i_1 < i_2 < \cdots < i_j \le k-1$ be the labels of the odd positions $2t+1 > 1$ such that w_{2t+1} is even. To arrive at a possible word w, we first choose some sequence

$$0 \le a_1 \le a_2 \le \cdots \le a_{2k} \le n+j-2k.$$

The set of such sequences is q-counted by $\begin{bmatrix} n+j \\ 2k \end{bmatrix}_q$. Next, we consider the sequence b defined by $b_m = 2a_m + 1$, so that our q-count becomes $q^{2k}\begin{bmatrix} n+j \\ 2k \end{bmatrix}_{q^2}$. We will have $b_{2k} \le 2(n+2-2k)+1$. Now, we want to force $<$ everywhere except at the specified locations $i_1, i_2, \dots, i_j$. Thus, we add $0, 2, 4, 6$, etc to our sequence entries, except at the specified locations, where we add the same number again. For instance, suppose $j = 2$, $i_1 = 1$, and $i_2 = 3$. After we have our sequence b, then we choose a new sequence c, where

$$c_1 = b_1, c_2 = b_2 + 2, c_3 = b_3 + 2, c_4 = b_4 + 4, c_5 = b_5 + 6, c_6 = b_6 + 8, c_7 = b_7 + 8, \dots$$

What do we add to the place corresponding to i_m? It turns out we need to add $4i_m - 2m$. For instance, in our example above, we added $4(3) - 2(2) = 8$ to b_7, which corresponds to $i_2 = 3$. Thus, moving from sequence b to c multiplies our q-count by a factor of $q^{2(\binom{2k-j}{2}+(2i_1-1)+(2i_2-2)+\cdots+(2i_j-j))}$. We will have $c_{2k} \le (2n+2j-4k+1) + 2(2k-j-1) = 2n-1$, as needed. Next, we add 1 to each of the locations specified by $i_1, i_2, \dots, i_j$ to obtain a new sequence $d = d_1 \dots d_{2k}$. This multiplies our q-count by an additional factor of q^j. Overall, the q-count of our word d is given by:

$$\sum_{j=0}^{k-1} q^{2k} \begin{bmatrix} n+j \\ 2k \end{bmatrix}_{q^2} q^{2\binom{2k-j}{2}+j} \sum_{1 \le i_1 < i_2 < \cdots < i_j \le k-1} q^{2((2i_1-1)+(2i_2-2)+\cdots+(2i_j-j))}. \qquad (5.2.1)$$

The largest j can be is $k-1$, since that's how many odd places >1 there are. Equation 5.2.1 simplifies to

$$\begin{aligned}
&\sum_{j=0}^{k-1} q^{j+2k+2\binom{2k-j}{2}} \begin{bmatrix} n+j \\ 2k \end{bmatrix}_{q^2} \sum_{1\le i_1<i_2<\cdots<i_j\le k-1} q^{-2\binom{j+1}{2}} (q^4)^{i_1+i_2+\cdots+i_j} \\
&= \sum_{j=0}^{k-1} q^{j+2k+2(\binom{2k-j}{2}-\binom{j+1}{2})} \begin{bmatrix} n+j \\ 2k \end{bmatrix}_{q^2} \sum_{1\le i_1<i_2<\cdots<i_j\le k-1} (q^4)^{i_1+i_2+\cdots+i_j} \\
&= \sum_{j=0}^{k-1} q^{j+2k+2(\binom{2k-j}{2}+\binom{j+1}{2})} \begin{bmatrix} n+j \\ 2k \end{bmatrix}_{q^2} \begin{bmatrix} k-1 \\ j \end{bmatrix}_{q^4}.
\end{aligned}$$

We can obtain our final word w by either leaving d_1 alone or adding 1 to d_1. This multiplies our q-count by an additional factor of $(1+q)$, so we get (after simplifying):

$$\sum_{w\in SUSU_{2n-1,\mathbb{O},2k}} q^{|w|} = (1+q)\sum_{j=0}^{k-1} q^{2j^2+4k^2+j(3-4k)} \begin{bmatrix} n+j \\ 2k \end{bmatrix}_{q^2} \begin{bmatrix} k-1 \\ j \end{bmatrix}_{q^4},$$

which proves the first part of Theorem 5.2.2.

Recall that we can also ascertain the smallest power of q present in our q-count by summing the entries in the minimal word of the desired type. For a given j, the minimal word will be

$$1,3,3,5,5,\ldots,2j+1,2j+1,2j+3,\ldots 2(2k-j-2)+3$$

This gives $2(j+1)^2-1+(2k-j)^2-(j+1)^2+j = 2j^2+j(3-4k)+4k^2$, confirming that our power of q is correct.

Our reasoning for the second part of Theorem 5.2.2 is similar. To q-count $SUSU_{2n-1,\mathbb{O},2k+1}$, we classify the words $w_1w_2\ldots w_{2k+1}$ by the number of odd positions $2t+1$ with $1<2t+1<2k+1$ such that w_{2t+1} is even. First, label the odd positions $1<2t+1<2k+1$ from left to right with $1,2,\ldots,k-1$. Thus, position 3 gets label 1, position 5 gets label 2, and so on. Let $1\le i_1<i_2<\cdots<i_j\le k-1$ be the labels of the odd positions $1<2t+1<2k+1$ such that w_{2t+1} is even. To arrive at a possible word w, we first choose some sequence

$$0\le a_1\le a_2\le\cdots\le a_{2k}\le a_{2k+1}\le n+j-2k-1.$$

The set of such sequences is q-counted by $\begin{bmatrix} n+j \\ 2k+1 \end{bmatrix}_q$. Next, we consider the sequence b defined by $b_m = 2a_m + 1$, so that our q-count becomes $q^{2k+1}\begin{bmatrix} n+j \\ 2k+1 \end{bmatrix}_{q^2}$. Now, we want to force $<$ everywhere except at the specified locations $i_1, i_2, \ldots, i_j$ *and* before the last entry b_{2k+1}. Thus, we add $0, 2, 4, 6$, etc to our sequence entries, except at the specified locations and the last entry, where we add the same number again. For instance, suppose $j = 2$, $i_1 = 1$, and $i_2 = 3$. After we have our sequence b, then we choose a new sequence c, where

$$c_1 = b_1, c_2 = b_2 + 2, c_3 = b_3 + 2, c_4 = b_4 + 4, c_5 = b_5 + 6, c_6 = b_6 + 8, c_7 = b_7 + 8, \ldots$$

We then add 1 to the specified locations to obtain a sequence $d = d_1 \ldots d_{2k+1}$ that is q-counted by

$$\sum_{j=0}^{k-1} q^{2k+1}\begin{bmatrix} n+j \\ 2k+1 \end{bmatrix}_{q^2} q^{2\binom{2k-j}{2}+2(2k-j-1)+j} \sum_{1\le i_1<i_2<\cdots<i_j\le k-1} q^{2((2i_1-1)+(2i_2-2)+\cdots+(2i_j-j))}.$$

Finally, we have 2 choices to make in order to extend d to our final word w. We can increase d_1 by 1 or not, and increase d_{2k+1} by 1 or 2. Thus, we will have $w_{2k+1} \le 2(n+j-2k-1)+1+2(2k-j-1)+2 = 2n-1$, as needed. These multiply our q-count by a factor of $(1+q)(q+q^2)$, so that

$$\sum_{w\in SUSU_{2n-1,\mathbb{O},2k+1}} q^{|w|} = (1+q)(q+q^2)\sum_{j=0}^{k-1} q^{2k+1}\begin{bmatrix} n+j \\ 2k+1 \end{bmatrix}_{q^2} q^{2\binom{2k-j}{2}+2(2k-j-1)+j} \times \sum_{1\le i_1<i_2<\cdots<i_j\le k-1} q^{2((2i_1-1)+(2i_2-2)+\cdots+(2i_j-j))},$$

which simplifies to give the second part of Theorem 5.2.2.

We can again check the power of q using the minimal word of this type. For a given j, the minimal word will be

$$1, 3, 3, 5, 5, \ldots, 2j+1, 2j+1, 2j+3, \ldots 2(2k-j-2+1)+3$$

with the last entry reduced by 1. This gives $2(j+1)^2 - 1 + (2k-j+1)^2 - (j+1)^2 + j - 1 = 2j^2 + j - 4jk + 4k^2$.

Theorem 5.2.3.

$$EV_{2n,0}^{SUSU}(z,q)=\sum_{k=0}^{n}(-1)^k z^{2k}\sum_{j=0}^{k} q^{2j^2-j+4k^2+2k-4kj}\begin{bmatrix} n+j \\ 2k\end{bmatrix}_{q^2}\begin{bmatrix} k \\ j\end{bmatrix}_{q^4}$$

and

$$EV_{2n,1}^{SUSU}(z,q)=\sum_{k=0}^{n}(-1)^k z^{2k+1}([2]_{1/q})\sum_{j=0}^{k} q^{2j^2-3j+4k^2+6k-4kj+2}\begin{bmatrix} n+j \\ 2k+1\end{bmatrix}_{q^2}\begin{bmatrix} k \\ j\end{bmatrix}_{q^4}.$$

Our reasoning for even peaks is similar to that used in Theorem 5.2.2. To q-count $SUSU_{2n,\mathbb{E},2k}$, we classify the words $w_1w_2\dots w_{2k}\in SUSU_{2n,\mathbb{E},2k}$ by the number of odd positions $2t+1$ such that w_{2t+1} is odd. First, label the odd positions $2t+1$ from left to right with $1,2,\dots,k$. Thus, position 1 gets label 1, position 3 gets label 2, and so on. Let $1\le i_1<i_2<\cdots<i_j\le k$ be the labels of the odd positions $2t+1$ such that w_{2t+1} is odd. To arrive at a possible word w, we first choose some sequence

$$0\le a_1\le a_2\le\cdots\le a_{2k}\le n+j-2k.$$

The set of such sequences is q-counted by $\begin{bmatrix} n+j \\ 2k\end{bmatrix}_q$. Next, we consider the sequence b defined by $b_m=2a_m+2$, so that our q-count becomes $q^{4k}\begin{bmatrix} n+j \\ 2k\end{bmatrix}_{q^2}$. Now, we want to force $<$ everywhere except at the specified locations $i_1,i_2,\dots,i_j$. Thus, we add $0,2,4,6$, etc to our sequence entries, except at the specified locations, where we add the same number again. Thus, $c_{2k}\le 2(n+j-2k)+2+2(2k-j-1)=2n$, as needed. We will end up adding $4a_m-2(m+1)$ to the place with label i_m. For instance, suppose $j=2$, $i_1=1$, and $i_2=3$. After we have our sequence b, then we choose a new sequence c, where

$$c_1=b_1,c_2=b_2,c_3=b_3+2,c_4=b_4+4,c_5=b_5+6,c_6=b_6+6,c_7=b_7+6,\dots$$

We then subtract 1 from the specified locations to obtain the final sequence w, which is q-counted by

$$\sum_{j=0}^{k} q^{4k}\begin{bmatrix} n+j \\ 2k+1\end{bmatrix}_{q^2} q^{2\binom{2k-j}{2}-j}\sum_{1\le i_1<i_2<\cdots<i_j\le k} q^{2((2i_1-2)+(2i_2-3)+\cdots+(2i_j-j-1))},$$

which simplifies to give the first part of Theorem 5.2.3.

We can again check the power of q using the minimal word of this type. For a given j, the minimal word (before reduction) will be

$$2, 2, \dots, 2j, 2j, 2j+2, 2j+4, \dots, 2(2(k-j)+j+1)$$

This gives $4\binom{j+1}{2} + 2\left[\binom{2k-j+2}{2} - \binom{j+1}{2}\right] - j = 2j^2 - j + 4k^2 + 2k - 4kj$.

To q-count $SUSU_{2n,\mathbb{E},2k+1}$, we classify the words $w_1w_2\dots w_{2k+1}$ by the number of odd positions $2t+1$ with $2t+1 < 2k+1$ such that w_{2t+1} is odd. First, label the odd positions $2t+1 < 2k+1$ from left to right with $1, 2, \dots, k$. Thus, position 1 gets label 1, position 3 gets label 2, and so on. Let $1 \le i_1 < i_2 < \cdots < i_j \le k$ be the labels of the odd positions $2t+1 < 2k+1$ such that w_{2t+1} is odd. To arrive at a possible word w, we first choose some sequence

$$0 \le a_1 \le a_2 \le \cdots \le a_{2k} \le a_{2k+1} \le n+j-2k-1.$$

The set of such sequences is q-counted by $\begin{bmatrix} n+j \\ 2k+1 \end{bmatrix}_q$. Next, we consider the sequence b defined by $b_m = 2a_m + 2$, so that our q-count becomes $q^{4k+2}\begin{bmatrix} n+j \\ 2k+1 \end{bmatrix}_{q^2}$. Now, we want to force $<$ everywhere except at the specified locations $i_1, i_2, \dots, i_j$.Thus, we add $0, 2, 4, 6$, etc to our sequence entries, except at the specified locations and the last entry, where we add the same number again. Thus, we will have $c_{2k+1} \le 2(n+j-2k-1) + 2 + 2(2k-j) = 2n$, as needed. We will end up adding $4a_m - 2(m+1)$ to the place with label i_m. For instance, suppose $j = 2$, $i_1 = 1$, and $i_2 = 3$. After we have our sequence b, then we choose a new sequence c, where

$$c_1 = b_1, c_2 = b_2, c_3 = b_3 + 2, c_4 = b_4 + 4, c_5 = b_5 + 6, c_6 = b_6 + 6, c_7 = b_7 + 6, \dots$$

We then subtract 1 from the specified locations to obtain a sequence $d = d_1 \dots d_{2k+1}$ that is q-counted by

$$\sum_{j=0}^{k} q^{4k+2} \begin{bmatrix} n+j \\ 2k+1 \end{bmatrix}_{q^2} q^{2\binom{2k-j+1}{2}-j} \sum_{1 \le i_1 < i_2 < \cdots < i_j \le k} q^{2((2i_1-2)+(2i_2-3)+\cdots+(2i_j-j-1))}.$$

We can obtain our final word w by either leaving d_{2k+1} alone or subtracting 1 from d_{2k+1}. This multiplies our q-count by a factor of $(1+1/q)$, so that

$$\sum_{w\in SUSU_{2n,\mathbb{E},2k+1}} q^{|w|} = (1+1/q)\sum_{j=0}^{k} q^{4k+2} \begin{bmatrix} n+j \\ 2k+1 \end{bmatrix}_{q^2} q^{2\binom{2k-j+1}{2}-j} \times \sum_{1\le i_1<i_2<\cdots<i_j\le k} q^{2((2i_1-2)+(2i_2-3)+\cdots+(2i_j-j-1))},$$

which simplifies to give the second part of Theorem 5.2.3.

We can again check the power of q using the minimal word of this type. For a given j, the minimal word (before reduction) will be

$$2,2,\ldots,2j,2j,2j+2,2j+4,\ldots,2(2(k-j)+j+1).$$

This gives $4\binom{j+1}{2}+2\left[\binom{2k-j+2}{2}-\binom{j+1}{2}\right]-j = 4k^2-4jk+6k+2j^2-3j+2$.

5.2.2 WUSU

Theorem 5.2.4.

$$EV_{2n,0}^{WUSU}(z,q) = \sum_{k=0}^{n}(-1)^k z^{2k}([2]_{1/q})^k q^{2k^2+2k} \begin{bmatrix} n+k \\ 2k \end{bmatrix}_{q^2}$$

and

$$EV_{2n,1}^{WUSU}(z,q) = \sum_{k=0}^{n}(-1)^k z^{2k+1}([2]_{1/q})^{k+1} q^{2k^2+4k+2} \begin{bmatrix} n+k \\ 2k+1 \end{bmatrix}_{q^2}.$$

To obtain a word $w \in WUSU_{2n,\mathbb{E},2k}$, we first choose some sequence

$$0 \le a_1 \le a_2 \le a_3 \cdots \le a_{2k} \le n-k$$

The set of such sequences is q-counted by $\begin{bmatrix} n+k \\ 2k \end{bmatrix}_q$. Next, we consider the sequence b defined by $b_m = 2a_m + 2$, so that our q-count becomes $q^{4k}\begin{bmatrix} n+k \\ 2k \end{bmatrix}_{q^2}$. Then, we want to force $<$ every other place, so generate a new sequence c by adding 0 to b_1 and b_2, 2 to b_3 and b_4, 4 to b_5 and b_6, and so on, ending by adding $2(k-1)$ to b_{2k-1} and b_{2k}. We will have $c_{2k} \le 2(n-k)+2+2(k-1) = 2n$, as needed. This increases our q-count by $2+2+4+4+\cdots+2(k-1)+2(k-1) = 4(1+\cdots+k-1) = 4\binom{k}{2}$, so that we have $q^{4k+4\binom{k}{2}}\begin{bmatrix} n+k \\ 2k \end{bmatrix}_{q^2}$. We obtain our final word w from the sequence c by choosing whether or not to subtract 1 from each odd place $c_1, c_3, \ldots, c_{2k-1}$. This multiplies our q-count by a factor of $(1+1/q)^k$, so that we have.

$$\sum_{w \in WUSU_{2n,\mathbb{E},2k}} q^{|w|} = (1+1/q)^k q^{4k+4\binom{k}{2}} \begin{bmatrix} n+k \\ 2k \end{bmatrix}_{q^2},$$

which simplifies to give the first part of Theorem 5.2.4.

To obtain a word $w \in WUSU_{2n,\mathbb{E},2k+1}$, we first choose some sequence

$$0 \leq a_1 \leq a_2 \leq a_3 \cdots \leq a_{2k+1} \leq n-k-1$$

The set of such sequences is q-counted by $\begin{bmatrix} n+k \\ 2k+1 \end{bmatrix}_q$. Next, we consider the sequence b defined by $b_m = 2a_m + 2$, so that our q-count becomes $q^{4k+2}\begin{bmatrix} n+k \\ 2k+1 \end{bmatrix}_{q^2}$. Then, we want to force $<$ every other place, so generate a new sequence c by adding 0 to b_1 and b_2, 2 to b_3 and b_4, 4 to b_5 and b_6, and so on, ending by adding $2k$ to b_{2k+1}. Thus, we will have $c_{2k+1} \leq 2(n-k-1)+2+2k = 2n$, as needed. This increases our q-count by $2+2+4+4+\cdots+2(k-1)+2(k-1)+2k = 4(1+\cdots+k-1)+2k = 4\binom{k}{2}+2k$, so that we have $q^{4k+2+4\binom{k}{2}+2k}\begin{bmatrix} n+k \\ 2k+1 \end{bmatrix}_{q^2}$.
We obtain our final word w from the sequence c by choosing whether or not to subtract 1 from each odd place $c_1, c_3, \ldots, c_{2k+1}$. This multiplies our q-count by a factor of $(1+1/q)^{k+1}$, so that we have

$$\sum_{w \in WUSU_{2n,\mathbb{E},2k+1}} q^{|w|} = (1+1/q)^{k+1} q^{4k+2+4\binom{k}{2}+2k} \begin{bmatrix} n+k \\ 2k+1 \end{bmatrix}_{q^2},$$

which simplifies to give the second part of Theorem 5.2.4.

Theorem 5.2.5.

$$OD_{2n-1,0}^{WUSU}(z,q) = \sum_{k=0}^{n} (-1)^k z^{2k} ([2]_{1/q})^{k-1} \times \left(q^{2k^2} \begin{bmatrix} n+k-1 \\ 2k-1 \end{bmatrix}_{q^2} + [2]_{1/q} \sum_{j=1}^{n-k-1} q^{2k^2+4jk} \begin{bmatrix} n-j+k-1 \\ 2k-1 \end{bmatrix}_{q^2} \right)$$

and

$$OD^{WUSU}_{2n-1,1}(z,q) = \sum_{k=0}^{n}(-1)^k z^{2k+1}([2]_{1/q})^k \Big(q^{2k^2+2k+1}\begin{bmatrix} n+k-1 \\ 2k \end{bmatrix}_{q^2} + [2]_{1/q}\sum_{j=1}^{n-k-2} q^{2k^2+4jk+2k+2j+1}\begin{bmatrix} n-j+k-1 \\ 2k \end{bmatrix}_{q^2} \Big).$$

To q-count $WUSU_{2n-1,\mathbb{O},2k}$, we classify words $w_1w_2\ldots w_{2k} \in WUSU_{2n-1,\mathbb{O},2k}$ by the first letter w_1. If $w_1 = 1$, we obtain a word w as follows. First, choose a sequence

$$0 \le a_2 \le a_3 \le a_4 \le \cdots \le a_{2k} \le n-k,$$

which is q-counted by $\begin{bmatrix} n+k-1 \\ 2k-1 \end{bmatrix}_q$. Next, we consider the sequence b defined by $b_1 = 1$ and $b_m = 2a_m + 1$ for $m > 1$, so that our q-count becomes $q^{2k}\begin{bmatrix} n+k-1 \\ 2k-1 \end{bmatrix}_{q^2}$. Then, we want to force $<$ every other place, so generate a new sequence c by adding 0 to b_1 and b_2, 2 to b_3 and b_4, 4 to b_5 and b_6, and so on, ending by adding $2(k-1)$ to b_{2k-1} and b_{2k}. Thus, we will have $c_{2k} \le 2(n-k)+1+2(k-1) = 2n-1$, as needed. This increases our q-count by $2+2+4+4+\cdots+2(k-1)+2(k-1) = 4(1+\cdots+k-1) = 4\binom{k}{2}$, so that we have $q^{2k+4\binom{k}{2}}\begin{bmatrix} n+k-1 \\ 2k-1 \end{bmatrix}_{q^2}$. We then obtain our final word w from the sequence c by choosing whether or not to subtract 1 from each odd place except the first: $c_3, c_5, c_7, \ldots, c_{2k-1}$. This multiplies our q-count by a factor of $(1+1/q)^{k-1}$, so that we have

$$\sum_{\substack{w \in WUSU_{2n-1,\mathbb{O},2k} \\ w_1 = 1}} q^{|w|} = (1+1/q)^{k-1} q^{2k+4\binom{k}{2}}\begin{bmatrix} n+k-1 \\ 2k-1 \end{bmatrix}_{q^2}.$$

If $w_1 = 2j$ or $w_1 = 2j+1$, we obtain a word w as follows. First, note that since we must have $k-1$ strict increases after w_1, the largest j can be is $n-k-1$. We choose a sequence

$$0 \le a_2 \le a_3 \le a_4 \le \cdots \le a_{2k} \le n-j-k,$$

which is q-counted by $\begin{bmatrix} n-j+k-1 \\ 2k-1 \end{bmatrix}_q$. Next, we consider the sequence b defined by $b_1 = 2j+1$ and $b_m = 2a_m + 2j + 1$ for $m > 1$, so that our q-count becomes

$q^{(2j+1)(2k)}\begin{bmatrix}n+k-1\\2k-1\end{bmatrix}_{q^2}$. Then, we want to force $<$ every other place, so generate a new sequence c by adding 0 to b_1 and b_2, 2 to b_3 and b_4, 4 to b_5 and b_6, and so on, ending by adding $2(k-1)$ to b_{2k-1} and b_{2k}. Thus, we will have $c_{2k} \leq 2(n-j-k)+2j+1+2(k-1) = 2n-1$, as needed. This increases our q-count by $2+2+4+4+\cdots+2(k-1)+2(k-1) = 4(1+\cdots+k-1) = 4\binom{k}{2}$, so that we have $q^{(2j+1)(2k)+4\binom{k}{2}}\begin{bmatrix}n+k-1\\2k-1\end{bmatrix}_{q^2}$. We then obtain our final word w from the sequence c by choosing whether or not to subtract 1 from each odd place: $c_1, c_3, c_5, c_7, \ldots, c_{2k-1}$. This multiplies our q-count by a factor of $(1+1/q)^k$, so that we have

$$\sum_{\substack{w\in WUSU_{2n-1,\mathbb{O},2k}\\ w_1\in\{2j,2j+1\}}} q^{|w|} = (1+1/q)^k q^{(2j+1)(2k)+4\binom{k}{2}}\begin{bmatrix}n+k-1\\2k-1\end{bmatrix}_{q^2}.$$

Simplifying

$$\sum_{\substack{w\in WUSU_{2n-1,\mathbb{O},2k}\\ w_1=1}} q^{|w|} + \sum_{j=1}^{n-k-1}\sum_{\substack{w\in WUSU_{2n-1,\mathbb{O},2k}\\ w_1\in\{2j,2j+1\}}} q^{|w|}$$

yields the first part of Theorem 5.2.5.

We can again check the power of q using the minimal word of this type. If $w_1 = 1$, the minimal word before reducing will be $1, 1, 3, 3 \ldots 2k-1, 2k-1$. It is twice the sum of the first k odd numbers, so $2k^2$.

If $w_1 = 2j+1$, the minimal word (after reductions) will be $2j, 2j+1, \ldots, 2j+2k-1$, which gives $2j + \binom{2j+2k}{2} - \binom{2j+1}{2} + k - 1 = 2k^2 + 4jk$.

To q-count $WUSU_{2n-1,\mathbb{O},2k+1}$, we use essentially the same reasoning. We classify words $w_1w_2\ldots w_{2k+1} \in WUSU_{2n-1,\mathbb{O},2k+1}$ by the first letter w_1. If $w_1 = 1$, we obtain a word w as follows. First, choose a sequence

$$0 \leq a_2 \leq a_3 \leq a_4 \leq \cdots \leq a_{2k+1} \leq n-k-1,$$

which is q-counted by $\begin{bmatrix}n+k-1\\2k\end{bmatrix}_q$. Next, we consider the sequence b defined by $b_1 = 1$ and $b_m = 2a_m+1$ for $m > 1$, so that our q-count becomes $q^{2k+1}\begin{bmatrix}n+k-1\\2k\end{bmatrix}_{q^2}$. Then, we want to force $<$ every other place, so generate a new sequence c by adding 0 to b_1 and b_2, 2 to b_3 and b_4, 4 to b_5 and b_6, and so on, ending by adding $2k$ to b_{2k+1}. Thus, we will have $c_{2k+1} \leq 2(n-k-1)+1+2k = 2n-1$, as needed. This increases our q-count by $2+2+4+4+\cdots+2(k-1)+2(k-1)+2k = 4(1+\cdots+k-1)+2k = 4\binom{k}{2}+2k$, so that we have $q^{2k+1+4\binom{k}{2}+2k}\begin{bmatrix} n+k-1 \\ 2k \end{bmatrix}_{q^2}$. We then obtain our final word w from the sequence c by choosing whether or not to subtract 1 from each odd place except the first: $c_3, c_5, c_7, \ldots, c_{2k+1}$. This multiplies our q-count by a factor of $(1+1/q)^k$,so that we have

$$\sum_{\substack{w \in WUSU_{2n-1,\mathbb{O},2k+1} \\ w_1=1}} q^{|w|} = (1+1/q)^k q^{4k+1+4\binom{k}{2}} \begin{bmatrix} n+k-1 \\ 2k \end{bmatrix}_{q^2}.$$

If $w_1 = 2j$ or $w_1 = 2j+1$, we obtain a word w as follows. First, note that since we must have k strict increases after w_1, the largest j can be is $n-k-2$. We choose a sequence

$$0 \leq a_2 \leq a_3 \leq a_4 \leq \cdots \leq a_{2k+1} \leq n-j-k-1,$$

which is q-counted by $\begin{bmatrix} n-j+k-1 \\ 2k-1 \end{bmatrix}_q$. Next, we consider the sequence b defined by $b_1 = 2j+1$ and $b_m = 2a_m + 2j + 1$ for $m > 1$, so that our q-count becomes $q^{(2j+1)(2k+1)}\begin{bmatrix} n+k-1 \\ 2k \end{bmatrix}_{q^2}$. Then, we want to force $<$ every other place, so generate a new sequence c by adding 0 to b_1 and b_2, 2 to b_3 and b_4, 4 to b_5 and b_6, and so on, ending by adding $2k$ to b_{2k+1}. Thus, we will have $c_{2k+1} \leq 2(n-j-k-1)+2j+1+2k = 2n-1$, as needed.. This increases our q-count by $2+2+4+4+\cdots+2(k-1)+2(k-1)+2k = 4(1+\cdots+k-1)+2k = 4\binom{k}{2}+2k$, so that we have $q^{(2j+1)(2k+1)+4\binom{k}{2}+2k}\begin{bmatrix} n+k-1 \\ 2k \end{bmatrix}_{q^2}$. We then obtain our final word w from the sequence c by choosing whether or not to subtract 1 from each odd place: $c_1, c_3, c_5, c_7, \ldots, c_{2k+1}$. This multiplies our q-count by a factor of $(1+1/q)^{k+1}$,so that we have

$$\sum_{\substack{w \in WUSU_{2n-1,\mathbb{O},2k+1} \\ w_1 \in \{2j,2j+1\}}} q^{|w|} = (1+1/q)^{k+1} q^{(2j+1)(2k+1)+4\binom{k}{2}+2k} \begin{bmatrix} n+k-1 \\ 2k \end{bmatrix}_{q^2}.$$

Simplifying

$$\sum_{\substack{w \in WUSU_{2n-1,\mathbb{O},2k+1} \\ w_1=1}} q^{|w|} + \sum_{j=1}^{n-k-2} \sum_{\substack{w \in WUSU_{2n-1,\mathbb{O},2k+1} \\ w_1 \in \{2j,2j+1\}}} q^{|w|}$$

yields the second part of Theorem 5.2.5.

We can again check the power of q using the minimal word of this type. If $w_1 = 1$, the minimal word before reducing is

$$1, 1, 3, 3 \ldots 2k-1, 2k-1, 2k+1,$$

which gives $2k^2 + 2k + 1$.
If $w_1 = 2j+1$, the minimal word before reducing is

$$2j+1, 2j+1, 2j+3, 2j+3, \ldots, 2j+2k, 2j+2k, 2j+2k+1,$$

which gives $2(j+k)^2 - 2j^2 + 2j + 2k + 1 = 2k^2 + 4jk + 2k + 2j + 1$.

5.2.3 SUWU

Theorem 5.2.6.

$$\begin{aligned} &EV^{SUWU}_{2n,0}(z,q) \\ &= \sum_{k=0}^{n} (-1)^k z^{2k} ([2]_q)^{k-1} \Bigg(\sum_{j=1}^{n-k+1} q^{2k^2+4kj-4k+1} \begin{bmatrix} n-j+k-1 \\ 2(k-1) \end{bmatrix}_{q^2} \\ &\qquad + [2]_{1/q} q^{2k^2+4k} \begin{bmatrix} n+k-1 \\ 2k \end{bmatrix}_{q^2} \Bigg) \end{aligned}$$

and

$$\begin{aligned} &EV^{SUWU}_{2n,1}(z,q) \\ &= \sum_{k=0}^{n} (-1)^k z^{2k+1} \Bigg[q^{2n} ([2]_q)^{k-1} \Bigg(\sum_{j=1}^{n-k+1} q^{1+2k^2+4kj-4k} \begin{bmatrix} n+k-j-1 \\ 2(k-1) \end{bmatrix}_{q^2} \\ &\qquad + ([2]_{1/q}) q^{2k^2+4k} \begin{bmatrix} n+k-1 \\ 2k \end{bmatrix}_{q^2} \Bigg) \\ &\qquad + ([2]_q)^k \Bigg(\sum_{j=1}^{n-k} q^{2j-2k+4jk+2k^2-1} \begin{bmatrix} n-j+k-1 \\ 2k-1 \end{bmatrix}_{q^2} \\ &\qquad + [2]_{1/q} q^{2k^2+6k+2} \begin{bmatrix} n+k-1 \\ 2k+1 \end{bmatrix}_{q^2} \Bigg) \Bigg]. \end{aligned}$$

To q-count $SUWU_{2n,\mathbb{E},2k}$, we classify words $w_1w_2\dots w_{2k}$ by the difference $w_2 - w_1$. If $w_2 - w_1 = 1$, then we let $w_2 = 2j, w_1 = 2j-1$ and obtain a word w as follows. First, note that the largest j can be is $n-k+1$, since w_2 is followed by $k-1$ strict increases. We choose some sequence

$$0 \leq a_3 \leq a_4 \leq a_5 \leq \cdots \leq a_{2k} \leq n+1-j-k,$$

which is q-counted by $\begin{bmatrix} n-j+k-1 \\ 2(k-1) \end{bmatrix}_q$. Next, we consider the sequence b defined by $b_1 = 2j-1$, $b_2 = 2j$, and $b_m = 2a_m + 2j$ for $m > 2$, so that our q-count becomes $q^{2k(2j)-1}\begin{bmatrix} n-j+k-1 \\ 2(k-1) \end{bmatrix}_{q^2}$. Then, we want to force $<$ every other place, so generate a new sequence c by adding 2 to b_4 and b_5, 4 to b_6 and b_7, and so on, ending by adding $2(k-1)$ to b_{2k}. Thus, we will have $c_{2k} \leq 2(n+1-j-k)+2j+2(k-1) = 2n$, as needed. This increases our q-count by $2+2+4+4+\cdots+2(k-2)+2(k-2)+2(k-1) = 4(1+\cdots+k-2)+2k-2 = 4\binom{k-1}{2}+2k-2$, so that we have $q^{2k(2j)-1+4\binom{k-1}{2}+2k-2}\begin{bmatrix} n-j+k-1 \\ 2(k-1) \end{bmatrix}_{q^2}$. We then obtain our final word w from the sequence c by choosing whether or not to add 1 to each odd place except the first: $c_3, c_5, c_7, \dots, c_{2k-1}$. This multiplies our q-count by a factor of $(1+q)^{k-1}$,so that we have

$$\sum_{\substack{w \in SUWU_{2n,\mathbb{E},2k} \\ w_2 = w_1+1 = 2j}} q^{|w|} = (1+q)^{k-1} q^{2k(2j)-1+4\binom{k-1}{2}+2k-2} \begin{bmatrix} n-j+k-1 \\ 2(k-1) \end{bmatrix}_{q^2}.$$

On the other hand, if $w_2 - w_1 > 1$, we obtain a word w as follows. We choose a sequence

$$0 \leq a_1 \leq a_2 \leq a_3 \leq a_4 \leq \cdots \leq a_{2k} \leq n-1-k,$$

which is q-counted by $\begin{bmatrix} n+k-1 \\ 2k \end{bmatrix}_q$. Next, we consider the sequence b defined by $b_m = 2a_m + 2$, so that our q-count becomes $q^{4k}\begin{bmatrix} n+k-1 \\ 2k \end{bmatrix}_{q^2}$. Then, we want to force $<$ every other place, so generate a new sequence c by adding 0 to b_1, 2 to b_2 and b_3, 4 to b_4 and b_5, and so on, ending by adding $2k$ to b_{2k}. Thus, we will have $c_{2k} \leq 2(n-1-k)+2+2k = 2n$, as needed. This increases our q-count by $2+2+4+4+\cdots+2(k-1)+2(k-1)+2k = 4(1+\cdots+k-1)+2k = 4\binom{k}{2}+2k$, so that we have $q^{6k+4\binom{k}{2}}\begin{bmatrix} n+k-1 \\ 2k \end{bmatrix}_{q^2}$. We then obtain our final word w from the sequence c by choosing whether or not to add 1 to each odd place except the first and choosing whether or not to subtract 1 from c_1. This multiplies our q-count by a factor of $(1+1/q)(1+q)^{k-1}$,so that we have

$$\sum_{\substack{w\in SUWU_{2n,\mathbb{E},2k}\\ w_2-w_1>1}} q^{|w|} = (1+1/q)(1+q)^{k-1}q^{6k+4\binom{k}{2}}\begin{bmatrix} n+k-1 \\ 2k \end{bmatrix}_{q^2}.$$

Simplifying

$$\sum_{j=1}^{n-k+1} \sum_{\substack{w\in SUWU_{2n,\mathbb{E},2k}\\ w_2=w_1+1=2j}} q^{|w|} + \sum_{\substack{w\in SUWU_{2n,\mathbb{E},2k}\\ w_2-w_1>1}} q^{|w|}$$

yields the first part of Theorem 5.2.6.

We can again check the power of q using the minimal words of this type. If $w_2 = w_1 + 1 = 2j$, the minimal word (before increasing) is

$$2j-1, 2j, 2j, \ldots, 2(j+k-2), 2(j+k-2), 2(j+k-1),$$

which gives $4\left[\binom{j+k}{2} - \binom{j}{2}\right] + 2j - 1 - 2(j+k-1) = 2k^2 + 4jk - 4k + 1$.
If $w_2 - w_1 > 1$, the minimal word (before increasing) is

$$2, 4, 4, 6, 6, \ldots, 2k, 2k, 2k+2,$$

which gives $4\binom{k+1}{2} + 2k = 2k^2 + 4k$.

To q-count $SUWU_{2n,\mathbb{E},2k+1}$, we classify words $w_1w_2\ldots w_{2k+1}$ by both the value of w_{2k+1} and the difference $w_2 - w_1$. If $w_{2k+1} = 2n$, then we obtain w by taking any element of $SUWU_{2n,\mathbb{E},2k}$ and inserting the letter $2n$ at the end. Thus, such words are q-counted by $zq^{2n}EV_{2n,0}^{SUWU}(z,q)$.

If $w_{2k+1} < 2n$ and $w_2 - w_1 = 1$, then we let $w_2 = 2j, w_1 = 2j-1$ and obtain a word w as follows. First, note that the largest j can be is $n-k$, since w_2 is followed by $k-1$ strict increases. We choose some sequence

$$0 \le a_3 \le a_4 \le a_5 \le \cdots \le a_{2k+1} \le n-j-k,$$

which is q-counted by $\begin{bmatrix} n-j+k-1 \\ 2k-1 \end{bmatrix}_q$. Next, we consider the sequence b defined by $b_1 = 2j-1$, $b_2 = 2j$, and $b_m = 2a_m + 2j$ for $m > 2$, so that our q-count becomes

$q^{2j(2k+1)-1}\begin{bmatrix} n-j+k-1 \\ 2k-1 \end{bmatrix}_{q^2}$. Then, we want to force $<$ every other place, so generate a new sequence c by adding 2 to b_4 and b_5, 4 to b_6 and b_7, and so on, ending by adding $2(k-1)$ to b_{2k} and b_{2k+1}. Thus, we will have $c_{2k+1} \leq 2(n-j-k)+2j+2(k-1) = 2n-2$, as needed (since $w_{2k+1} < 2n$ in this case). This increases our q-count by $2+2+4+4+\cdots+2(k-1)+2(k-1) = 4(1+\cdots+k-1) = 4\binom{k}{2}$, so that we have $q^{2j(2k+1)-1+4\binom{k}{2}}\begin{bmatrix} n-j+k-1 \\ 2k-1 \end{bmatrix}_{q^2}$. We then obtain our final word w from the sequence c by choosing whether or not to add 1 to each odd place except the first: $c_3, c_5, c_7, \ldots, c_{2k+1}$. This multiplies our q-count by a factor of $(1+q)^k$, so that we have

$$\sum_{\substack{w \in SUWU_{2n,\mathbb{E},2k+1} \\ w_{2k+1}<2n,\ w_2=w_1+1=2j}} q^{|w|} = (1+q)^k q^{2j(2k+1)-1+4\binom{k}{2}} \begin{bmatrix} n-j+k-1 \\ 2k-1 \end{bmatrix}_{q^2}.$$

If $w_{2k+1} < 2n$ and $w_2 - w_1 > 1$, we obtain a word w as follows. We choose a sequence

$$0 \leq a_1 \leq a_2 \leq a_3 \leq a_4 \leq \cdots \leq a_{2k+1} \leq n-2-k,$$

which is q-counted by $\begin{bmatrix} n+k-1 \\ 2k+1 \end{bmatrix}_q$. Next, we consider the sequence b defined by $b_m = 2a_m + 2$, so that our q-count becomes $q^{4k+2}\begin{bmatrix} n+k-1 \\ 2k+1 \end{bmatrix}_{q^2}$. Then, we want to force $<$ every other place, so generate a new sequence c by adding 0 to b_1, 2 to b_2 and b_3, 4 to b_4 and b_5, and so on, ending by adding $2k$ to b_{2k} and b_{2k+1}. Thus, we will have $c_{2k+1} \leq 2(n-2-k)+2+2k = 2n-2$, as needed. This increases our q-count by $2+2+4+4+\cdots+2k+2k = 4(1+\cdots+k) = 4\binom{k+1}{2}$, so that we have $q^{4k+2+4\binom{k+1}{2}}\begin{bmatrix} n+k-1 \\ 2k+1 \end{bmatrix}_{q^2}$. We then obtain our final word w from the sequence c by choosing whether or not to add 1 to each odd place except the first and choosing whether or not to subtract 1 from c_1. This multiplies our q-count by a factor of $(1+1/q)(1+q)^k$, so that we have

$$\sum_{\substack{w \in SUWU_{2n,\mathbb{E},2k+1} \\ w_{2k+1}<2n,\ w_2-w_1>1}} q^{|w|} = (1+1/q)(1+q)^k q^{4k+2+4\binom{k+1}{2}} \begin{bmatrix} n+k-1 \\ 2k+1 \end{bmatrix}_{q^2}.$$

The second part of Theorem 5.2.6 follows by combining

$$zq^{2n} EV^{SUWU}_{2n,0}(z,q)$$

with

$$\sum_{j=0}^{n-k} \sum_{\substack{w \in SUWU_{2n,\mathbb{E},2k+1} \\ w_{2k+1}<2n,\ w_2=w_1+1=2j}} q^{|w|} + \sum_{\substack{w \in SUWU_{2n,\mathbb{E},2k+1} \\ w_{2k+1}<2n,\ w_2-w_1>1}} q^{|w|}.$$

We can again check the power of q using the minimal word of this type. For $w_2 = 2j, w_1 = 2j-1$, the minimal word is

$$2j-1, 2j, 2j, \ldots, 2(j+k-1), 2(j+k-1),$$

which gives $2j-1+4(j+(j+1)+\cdots+j+k-1) = 2j-1+4\left[\binom{j+k}{2}-\binom{j}{2}\right] = 2j-2k+4jk+2k^2-1$.

For $w_1 < w_2 - 1$, the minimal word is

$$2, 4, 4, 6, 6, \ldots, 2k, 2k, 2k+2, 2k+2,$$

which gives $4(1+2+\cdots+k+1)-2 = 4\binom{k+2}{2}-2 = 2k^2+6k+2$.

Theorem 5.2.7.

$$OD^{SUWU}_{2n-1,0}(z,q) = \sum_{k=0}^{n} (-1)^k z^{2k} ([2]_q)^k q^{2k^2+2k} \begin{bmatrix} n+k-1 \\ 2k \end{bmatrix}_{q^2}$$

$$OD^{SUWU}_{2n-1,1}(z,q) = \sum_{k=0}^{n} (-1)^k z^{2k+1} \Bigg(([2]_q)^k q^{2k^2+2k+2n-1} \begin{bmatrix} n+k-1 \\ 2k \end{bmatrix}_{q^2} + ([2]_q)^{k+1} q^{2k^2+4k+1} \begin{bmatrix} n+k-1 \\ 2k+1 \end{bmatrix}_{q^2} \Bigg).$$

To obtain a word $w \in SUWU_{2n-1,\mathbb{O},2k}$, we first choose some sequence

$$0 \le a_1 \le a_2 \le a_3 \cdots \le a_{2k} \le n-k-1$$

The set of such sequences is q-counted by $\begin{bmatrix} n+k-1 \\ 2k \end{bmatrix}_q$. Next, we consider the sequence b defined by $b_m = 2a_m + 1$, so that our q-count becomes $q^{2k} \begin{bmatrix} n+k-1 \\ 2k \end{bmatrix}_{q^2}$. Then, we want to force $<$ every other place, so generate a new sequence c by adding 0 to b_1, 2 to b_2 and b_3, 4 to b_4 and b_5, and so on, adding $2k$ to b_{2k}. Thus, we will have $c_{2k} \le 2(n-k-1)+1+2k = 2n-1$, as needed. This increases our q-count by

$2+2+4+4+\cdots+2(k-1)+2(k-1)+2k = 4(1+\cdots+k-1)+2k = 4\binom{k}{2}+2k$, so that we have $q^{4k+4\binom{k}{2}}\begin{bmatrix}n+k-1\\2k\end{bmatrix}_{q^2}$. We obtain our final word w from the sequence c by choosing whether or not to add 1 to each odd place $c_1, c_3, \ldots, c_{2k-1}$. This multiplies our q-count by a factor of $(1+q)^k$, so that we have

$$\sum_{w\in SUWU_{2n-1,\mathbb{O},2k}} q^{|w|} = (1+q)^k q^{4k+4\binom{k}{2}}\begin{bmatrix}n+k-1\\2k\end{bmatrix}_{q^2},$$

which simplifies to give the first part of Theorem 5.2.7.

To q-count $SUWU_{2n-1,\mathbb{O},2k+1}$, we classify words $w_1w_2\ldots w_{2k+1}$ by the value of w_{2k+1}. If $w_{2k+1} = 2n-1$, we obtain w by taking any element of $SUWU_{2n-1,\mathbb{O},2k}$ and inserting the letter $2n-1$ at the end. Thus, such words are q-counted by $zq^{2n-1}OD^{SUWU}_{2n-1,0}(z,q)$.

If $w_{2k+1} < 2n-1$, we obtain w as follows. First, we first choose some sequence

$$0 \le a_1 \le a_2 \le a_3 \cdots \le a_{2k+1} \le n-2-k$$

The set of such sequences is q-counted by $\begin{bmatrix}n+k-1\\2k+1\end{bmatrix}_q$. Next, we consider the sequence b defined by $b_m = 2a_m+1$, so that our q-count becomes $q^{2k+1}\begin{bmatrix}n+k-1\\2k+1\end{bmatrix}_{q^2}$. Then, we want to force $<$ every other place, so generate a new sequence c by adding 0 to b_1, 2 to b_2 and b_3, 4 to b_4 and b_5, and so on, ending by adding $2k$ to b_{2k} and b_{2k+1}. Thus, we will have $c_{2k+1} \le 2(n-2-k)+1+2k = 2n-3$, as needed (since $w_{2k+1} < 2n-1$). This increases our q-count by $2+2+4+4+\cdots+2k+2k = 4(1+\cdots+k) = 4\binom{k+1}{2}$, so that we have $q^{2k+1+4\binom{k+1}{2}}\begin{bmatrix}n+k\\2k+1\end{bmatrix}_{q^2}$.

We obtain our final word w from the sequence c by choosing whether or not to subtract 1 from each odd place $c_1, c_3, \ldots, c_{2k+1}$. This multiplies our q-count by a factor of $(1+1/q)^{k+1}$, so that we have

$$\sum_{w\in WUSU_{2n,\mathbb{E},2k+1}} q^{|w|} = (1+1/q)^{k+1} q^{2k+1+4\binom{k+1}{2}}\begin{bmatrix}n+k\\2k+1\end{bmatrix}_{q^2}.$$

Combining this with

$$zq^{2n-1}OD^{SUWU}_{2n-1,0}(z,q)$$

simplifies to give the second part of Theorem 5.2.7.

We can again check the power of q using the minimal word of this type. If $w_{2k+1} < 2n-1$, the minimal word is

$$1, 3, 3, 5, 5, \ldots, 2k-1, 2k-1, 2k+1, 2k+1,$$

which gives $2(1+3+\cdots+2k+1)-1 = 2(k+1)^2-1 = 2k^2+4k+1$.

5.2.4 WUWU

The results in this subsection are all based on the following lemma.

Lemma 5.2.8. *Define level weak-up words by*

$$LWU_{n,X,2i} = \{1 \le a_1 = a_2 \le a_3 = a_4 \le a_5 \cdots \le a_{2i-1} = a_{2i} \le n : a_{2p} \in X \forall p\}.$$

Then

$$\bigcup_{i=0}^{\lfloor k/2 \rfloor} LWU_{n,X,2i} \times SUSU_{n,X,k-2i} \cong WUWU_{n,X,k}.$$

The bijection proceeds as follows. Given any pair (a, w) in $\bigcup_{i=0}^{\lfloor k/2 \rfloor} LWU_{n,X,2i} \times SUSU_{n,X,k-2i}$, we send (a, w) to the word obtained by inserting each element of a into w so as to keep weak increases between entries. This is clearly well-defined and reversible, thus a bijection for any peak condition X.
For example, the image of
$(11113377, 1379)$ is
(111113337779).

Thus, we wish to q-count $LWU_{n,X,2i}$. Define

$$EV_{n,0}^{LWU}(z,q) = 1 + \sum_{k \ge 1} (-1)^k \sum_{w \in LWU_{n,\mathbb{E},2k}} z^{\ell(w)} q^{|w|} \text{ and}$$
$$OD_{n,0}^{LWU}(z,q) = 1 + \sum_{k \ge 1} (-1)^k \sum_{w \in LWU_{n,\mathbb{O},2k}} z^{\ell(w)} q^{|w|}.$$

Theorem 5.2.9.

$$EV_{2n,0}^{LWU}(z,q) = \sum_{k \ge 0} (-1)^k z^{2k} q^{4k} \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_{q^4} = \prod_{i=1}^{n} \frac{1}{1+z^2 q^{4i}} \qquad (5.2.2)$$

and

$$OD^{LWU}_{2n-1,0}(z,q) = \sum_{k\geq 0}(-1)^k z^{2k} q^{2k} \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_{q^4} = \prod_{i=1}^{n} \frac{1}{1+z^2q^{2i}}. \qquad (5.2.3)$$

To q-count $LWU_{2n,\mathbb{E},2k}$, we first choose a sequence $1 \leq a_1 \leq a_2 \cdots \leq a_k \leq n$, which is counted by $q^n \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q$. Then, double and repeat each entry to get a new sequence b given by

$$b_1 = b_2 = 2a_1, b_3 = b_4 = 2a_2, \ldots b_{2k-1} - b_{2k} = 2a_k$$

This affects our q-count by replacing q with q^4. Applying q-binomial series, we get (5.2.2).

To q-count $LWU_{2n-1,\mathbb{O},2k}$, we can take any element of $LWU_{2n,\mathbb{E},2k}$ and subtract one from each entry. Thus, we reduce the power of q by $2k$, which yields (5.2.3). Therefore, we immediately get the following corollary:

Corollary 5.2.10.

$$\begin{aligned} EV^{WUWU}_{2n,0}(z,q) &= EV^{LWU}_{2n,0}(z,q)EV^{SUSU}_{2n,0}(z,q) \\ &= \left(\prod_{i=1}^{n} \frac{1}{1+z^2q^{4i}}\right) \sum_{k=0}^{n}(-1)^k z^{2k} \sum_{j=0}^{k} q^{2j^2-j+4k^2+2k-4kj} \begin{bmatrix} n+j \\ 2k \end{bmatrix}_{q^2} \begin{bmatrix} k \\ j \end{bmatrix}_{q^4}, \end{aligned}$$

$$\begin{aligned} EV^{WUWU}_{2n,1}(z,q) &= EV^{LWU}_{2n,0}(z,q)EV^{SUSU}_{2n,1}(z,q) \\ &= \left(\prod_{i=1}^{n} \frac{1}{1+z^2q^{4i}}\right) \sum_{k=0}^{n}(-1)^k z^{2k+1}[2]_{1/q} \sum_{j=0}^{k} q^{2j^2-3j+4k^2+6k-4kj+2} \begin{bmatrix} n+j \\ 2k \end{bmatrix}_{q^2} \begin{bmatrix} k \\ j \end{bmatrix}_{q^4}, \end{aligned}$$

$$\begin{aligned} OD^{WUWU}_{2n-1,0}(z,q) &= OD^{LWU}_{2n-1,0}(z,q)OD^{SUSU}_{2n-1,0}(z,q) \\ &= \left(\prod_{i=1}^{n} \frac{1}{1+z^2q^{2i}}\right) \sum_{k=0}^{n}(-1)^k z^{2k}[2]_q \sum_{j=0}^{k-1} q^{2j^2+4k^2+3j-4kj} \begin{bmatrix} n+1+j \\ 2k \end{bmatrix}_{q^2} \begin{bmatrix} k-1 \\ j \end{bmatrix}_{q^4}, \end{aligned}$$

and

$$
\begin{aligned}
OD^{WUWU}_{2n-1,1}(z,q) =& OD^{LWU}_{2n-1,0}(z,q)OD^{SUSU}_{2n-1,1}(z,q)\\
=& \left(\prod_{i=1}^{n}\frac{1}{1+z^2q^{2i}}\right)\sum_{k=0}^{n}(-1)^k z^{2k+1}([2]_q)^2\\
&\times\sum_{j=0}^{k-1}q^{j+2j^2+4k-4jk+4k^2}\begin{bmatrix}n+j\\2k+1\end{bmatrix}_{q^2}\begin{bmatrix}k-1\\j\end{bmatrix}_{q^4}.
\end{aligned}
$$

5.3 Further specializations

A few of the formulas from the previous subsections simplify significantly if we set $q = 1$. Thus, we get the following corollaries:

Corollary 5.3.1.

$$
OD^{WUSU}_{2n-1,0}(z,1) = \sum_{k=0}^{n}(-1)^k z^{2k}2^{k-1}\left[\binom{n+k-1}{2k}+\binom{n+k}{2k}\right]
$$

and

$$
OD^{WUSU}_{2n-1,1}(z,1) = \sum_{k=0}^{n}(-1)^k z^{2k+1}2^{k}\left[\binom{n+k-1}{2k+1}+\binom{n+k}{2k+1}\right].
$$

Corollary 5.3.2.

$$
EV^{SUWU}_{2n,0}(z,1) = \sum_{k=0}^{n}(-1)^k z^{2k}\left[2^{k-1}\binom{n+k-1}{2k-1}+2^{k}\binom{n+k-1}{2k}\right]
$$

and

$$
EV^{SUWU}_{2n,1}(z,1) = \sum_{k=0}^{n}(-1)^k z^{2k+1}\left[2^{k-1}\binom{n+k-1}{2k-1}+2^{k+1}\binom{n+k}{2k+1}\right].
$$

Now that we have successfully found generating functions for up-down words with peaks in $\mathbb{E}$ or $\mathbb{O}$, we can ask how this might relate to results on up-down permutations. We note that the analogous condition for permutations would be trivial, as there would be at most one alternating permutation with even peaks: when n is even, this would be $n-1, n, n-3, n-2, n-5, n-4, \ldots 3, 4, 1, 2$. Thus, enumerating up-down permutations with peaks in some specialized set is not likely to be an interesting problem in general. A related issue is how to find the distribution of peaks from some set X over up-down permutations or words. Addressing this issue is beyond the scope of this dissertation.

5.4 Extensions

The reasoning used in the previous sections can be extended to count our classes of words with peaks in $u\mathbb{P}$ or $j + u\mathbb{P}$ (i.e. congruent to $j \mod u$). For example, we have the following theorem.

Theorem 5.4.1. *Let $u \geq 2$. Then*

$$P^{WUSU}_{un,u\mathbb{P},2,0}(z_1,\ldots,z_n)|_{z_i=q^iz} = \sum_{k=0}^{n}(-1)^k z^{2k}([u]_{1/q})^k q^{uk^2+uk}\begin{bmatrix} n+k \\ 2k \end{bmatrix}_{q^u}.$$

The reasoning is essentially identical to that for $WUSU_{2n,\mathbb{E},2k}$. To obtain a word $w \in WUSU_{2n,u\mathbb{P},2k}$, we first choose some sequence

$$0 \leq a_1 \leq a_2 \leq a_3 \cdots \leq a_{2k} \leq n-k$$

The set of such sequences is q-counted by $\begin{bmatrix} n+k \\ 2k \end{bmatrix}_q$. Next, we consider the sequence b defined by $b_m = ua_m + u$, so that our q-count becomes $q^{2ku}\begin{bmatrix} n+k \\ 2k \end{bmatrix}_{q^u}$. Then, we want to force $<$ every other place, so generate a new sequence c by adding 0 to b_1 and b_2, u to b_3 and b_4, $2u$ to b_5 and b_6, and so on, ending by adding $u(k-1)$ to b_{2k-1} and b_{2k}. We will have $c_{2k} \leq u(n-k)+u+u(k-1) = un$, as needed. This increases our q-count by $u+u+2u+2u+\cdots+u(k-1)+u(k-1) = 2u(1+\cdots+k-1) = 2u\binom{k}{2}$, so that we have $q^{2ku+2u\binom{k}{2}}\begin{bmatrix} n+k \\ 2k \end{bmatrix}_{q^u}$. We obtain our final word w from the sequence c by choosing to reduce each odd place $c_1, c_3, \ldots, c_{2k-1}$ by some number v with $0 \leq v < u$. This multiplies our q-count by a factor of $([u]_{1/q})^k$, so that we have.

$$\sum_{w\in WUSU_{2n,u\mathbb{P},2k}} q^{|w|} = ([u]_{1/q})^k q^{2uk+2u\binom{k}{2}}\begin{bmatrix} n+k \\ 2k \end{bmatrix}_{q^u},$$

which simplifies to give Theorem 5.4.1. Counting other classes of words with peaks in $u\mathbb{P}$ for $u \geq 2$ requires more subtle modifications but can be accomplished with similar reasoning.

The reader may also wonder about the difficulty in extending these results to other values of s. To illustrate why this is more difficult, consider $WU^2SU_{2n,\mathbb{E},6} = \{w : 1 \leq w_1 \leq w_2 \leq w_3 < w_4 \leq w_5 \leq w_6 \leq 2n;\ w_3, w_6 \in \mathbb{E}\}$ (so that $s = 3$). The key to our reasoning for $WUSU_{2n,\mathbb{E},2k}$ was that, given any sequence of even numbers $2 \leq c_1 \leq c_2 < c_3 \leq \cdots \leq c_{2k} \leq 2n$, we could always subtract one from any set of odd positions and still obtain an element of $WUSU_{2n,\mathbb{E},2k}$.

If we try to apply the same reasoning to $WU^2SU_{2n,\mathbb{E},6}$, we would need to reach an intermediate stage involving some sequence of even numbers $2 \leq c_1 \leq c_2 \leq c_3 < c_4 \leq c_5 \leq c_6 \leq 2n$. We would then obtain a word $w \in WU^2SU_{2n,\mathbb{E},6}$ by subtracting appropriate amounts from entries in this sequence. The difficulty is that what we are allowed to subtract from c_1 and c_2, as well as c_4 and c_5, depends on the particular sequence c we chose. For example, if we have $c_1 = c_2 = 2$, then we can extend c to a word w by having (w_1, w_2) equal $(1, 1)$, $(1, 2)$, or $(2, 2)$. This gives 3 possibilities for extending our sequence to an element of $WU^2SU_{2n,\mathbb{E},6}$. On the other hand, if we initially chose $c_1 = 4, c_2 = 6$, then we can extend c to a word w by having (w_1, w_2) equal $(4, 5)$, $(3, 6)$, $(3, 5)$, or $(4, 6)$. This gives 4 possibilities for extending our sequence c to an element of $WU^2SU_{2n,\mathbb{E},6}$. Thus, even when $q = 1$ we have a problem in trying to uniformly alter our intermediate sequences. Things only get worse when we factor in the possibilities for w_4 and w_5. Thus, in order to proceed, we would need to create a complicated function based on the distribution of letters within our starting sequences. Such an approach is beyond the scope of this dissertation.

Chapter 6

Enumerating up-down words on an infinite alphabet

In this chapter, we will extend the results of Chapter 4 by enumerating two of the classes of up-down words with an infinite alphabet. That is, we will obtain generating functions for $SU^{s-1}WD_{\infty,n} = \{w \in \mathbb{P}^n : \; WDes(w) = (s\mathbb{P})_{n-1}\}$ and $WU^{s-1}SD_{\infty,n} = \{w \in \mathbb{P}^n : \; Des(w) = (s\mathbb{P})_{n-1}\}$ by counting *multiple rises*. In addition, we will enumerate compositions by *alternating descents* and *alternating major index*.

6.1 Up-down words on $\mathbb{P}$ with $s = 2$

We first consider $SU^{s-1}WD_{\infty,n}$ and $WU^{s-1}SD_{\infty,n}$ with $s = 2$. Define *double rises* and *weak double rises* as follows:

$$\text{2-ris}(w) = |\{i : w_{2i+1} < w_{2i+2} < w_{2i+3} < w_{2i+4}\}|$$

and

$$\text{w2-ris}(w) = |\{i : w_{2i+1} \leq w_{2i+2} \leq w_{2i+3} \leq w_{2i+4}\}|.$$

6.1.1 WUSD

Theorem 6.1.1. *Let $z(w)$ denote the monomial $z_{w_1} z_{w_2} \cdots z_{w_{\ell(w)}}$. Then*

$$\sum_{n\geq 0} t^{2n} \sum_{w\in\mathbb{P}^{2n}:\ w_{2j-1}\leq w_{2j}\forall j} x^{\text{w2-ris}(w)} z(w) = \frac{1-x}{-x+\frac{1}{2}\left[\prod_{k\geq 1}\frac{1}{1-t\sqrt{x-1}z_k}+\prod_{k\geq 1}\frac{1}{1+t\sqrt{x-1}z_k}\right]}.$$

Corollary 6.1.2.

$$\sum_{n=0}^{\infty} t^{2n} \sum_{w\in WUSD_{\infty,2n}} z(w) = 2\left(\prod_{k\geq 1}\frac{1}{1-itq^k}+\prod_{k\geq 1}\frac{1}{1+itz_k}\right)^{-1},$$

where $i=\sqrt{-1}$.

Corollary 6.1.2 follows by setting $x = 0$ in Theorem 6.1.1. This comes from the fact that weak-up, strict-down words can be counted by looking at words with weak increases in the appropriate places and no weak double rises.
Note that the combinatorial interpretation of these expressions ensures that all coefficients are positive integers, which would not be at all obvious out of context.

To prove Theorem 6.1.1, define a function on nonnegative integers by

$$f(n) = \begin{cases} 0 & n \text{ is odd} \\ 1 & n = 0 \\ -(x-1)^{k-1} & n = 2k > 0 \end{cases}$$

and define a homomorphism on the ring of symmetric functions by

$$\Theta_{WUSD}(e_n) = f(n)\left(\prod_{k\geq 1}\frac{1}{1-tz_k}\right)|_{t^n}\ .$$

Claim:

$$\Theta_{WUSD}(h_n) = \sum_{w\in\mathbb{P}^{2n}:\ w_{2j-1}\leq w_{2j}\forall j} x^{\text{w2-ris}(w)} z(w).$$

To see this, we interpret each term in

$$\Theta_{WUSD}(h_n) = \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} |B_{\lambda,n}| \prod_{j=1}^{\ell(\lambda)} f(\lambda_j) \left(\prod_{k \geq 1} \frac{1}{1 - tz_k} \right) |_{t^{\lambda_j}} . \qquad (6.1.1)$$

By the definition of f, this sum will contribute nothing unless each part of λ is even, which also implies that n must be even. Thus, a term in (6.1.1) corresponds to a brick tabloid with even bricks. Then $(-1)^{n-\ell(\lambda)} \prod_{j=1}^{\ell(\lambda)} f(\lambda_j)$ reduces to $\prod_{j=1}^{\ell(\lambda)} (x-1)^{\lambda_j/2}$, which weights each brick with a factor of x or -1 in every other nonterminal cell. The term $\prod_{j=1}^{\ell(\lambda)} \left(\prod_{k \geq 1} \frac{1}{1-tz_k} \right) |_{t^{\lambda_j}}$ lets us choose a partition π^j with λ_j parts for each brick, where we write the partition in weakly increasing order and weight by $z(\pi^j)$. We define the weight of a filled labeled brick tabloid created in this manner to be the product of the x and -1 labels times the monomial $z(w)$, where w denotes the underlying word. For example, the weight of Figure 6.1 is given by $xz_1z_2^3z_3^2z_4z_5$.

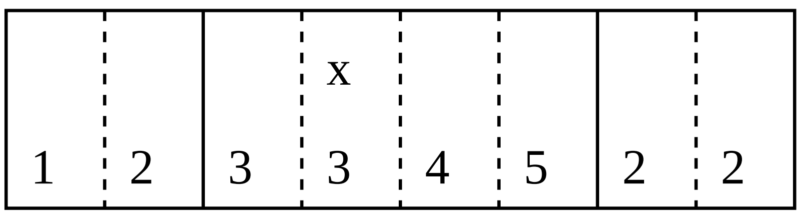

Figure 6.1: A brick tabloid coming from Equation 6.1.1 with $n = 8$

Thus, Equation 6.1.1 corresponds to a weighted sum over all such brick tabloids. We perform the usual involution on these brick tabloids, where bricks are scanned from left to right for the first occurrence of either a -1 or a weak increase between bricks. If a -1 is encountered first, we break the brick after it and remove the -1. If a weak increase is found first, we combine the bricks and insert a -1. For instance, the image of Figure 6.1 is given in Figure 6.2. Thus, Equation 6.1.1 can be reduced to summing over fixed points.

Fixed points must have bricks of even length and decreases between bricks. Since we never break a brick at an odd place, we will always have $w_{2j-1} \leq w_{2j} \forall j$. In addition, the power of x will register the number of weak double rises, as desired.

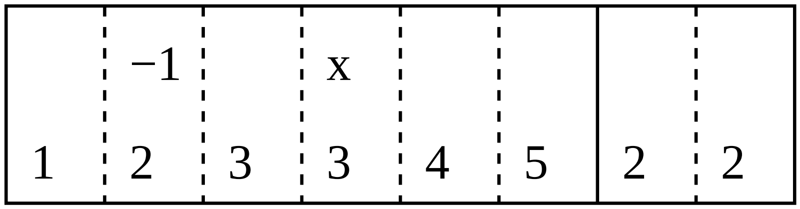

Figure 6.2: The image of Figure 6.1

For example, the fixed point in Figure 6.3 has weight $x^2 z_1 z_2^3 z_3^2 z_4 z_5$. Thus, the sum over all fixed points of Equation 6.1.1 is given by

$$\sum_{w \in \mathbb{P}^{2n}:\ w_{2j-1} \leq w_{2j} \forall j} x^{\text{w2-ris}(w)} z(w).$$

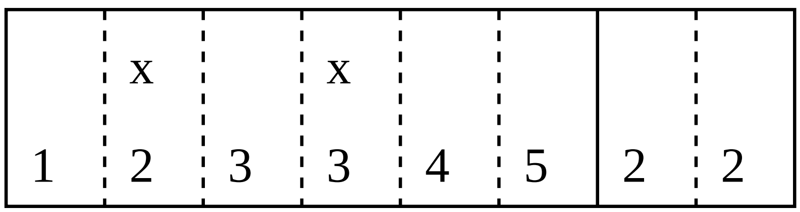

Figure 6.3: A fixed point coming from Equation 6.1.1 when $n = 8$

Thus, we can use Equation 6.1.1 to obtain:

$$\begin{aligned}
&\sum_{n \geq 0} t^{2n} \sum_{w \in \mathbb{P}^{2n}:\ w_{2j-1} \leq w_{2j}} x^{\text{w2-ris}(w)} z(w) \\
&= 1 + \sum_{n=1}^{\infty} t^n \Theta_{WUSD}(h_n) = \Theta_{WUSD}\left(\sum_{n=0}^{\infty} (-t)^n e_n\right)^{-1} \\
&= \left(1 + \sum_{n=1}^{\infty} -(x-1)^{n-1} t^{2n} \left(\prod_{k \geq 1} \frac{1}{1 - t z_k}\right) |_{t^{2n}}\right)^{-1} \\
&= \frac{1 - x}{1 - x + \sum_{n \geq 1} [t\sqrt{x-1}]^{2n} \left(\prod_{k \geq 1} \frac{1}{1 - t z_k}\right) |_{t^{2n}}} \\
&= \frac{1 - x}{-x + \frac{1}{2}\left[\prod_{k \geq 1} \frac{1}{1 - t\sqrt{x-1} z_k} + \prod_{k \geq 1} \frac{1}{1 + t\sqrt{x-1} z_k}\right]},
\end{aligned}$$

which proves Theorem 6.1.1.

6.1.2 SUWD

Theorem 6.1.3.

$$\sum_{n\geq 0} t^{2n} \sum_{w\in\mathbb{P}^{2n}:\ w_{2j-1}<w_{2j}\forall j} x^{2\text{-ris}(w)} z(w)$$
$$= \frac{1-x}{-x+\frac{1}{2}\left[\prod_{k\geq 1}(1+t\sqrt{x-1}z_k)+\prod_{k\geq 1}(1-t\sqrt{x-1}z_k)\right]} \tag{6.1.2}$$

and

$$\sum_{n\geq 0} t^{2n+1} \sum_{w\in\mathbb{P}^{2n+1}:\ w_{2j-1}<w_{2j}\forall j} x^{2\text{-ris}(w)} z(w)$$
$$= \frac{\sqrt{x-1}\left[\prod_{k\geq 1}(1+t\sqrt{x-1}z_k)-\prod_{k\geq 1}(1-t\sqrt{x-1}z_k)\right]}{2x-\left[\prod_{k\geq 1}(1+t\sqrt{x-1}z_k)+\prod_{k\geq 1}(1-t\sqrt{x-1}z_k)\right]} \tag{6.1.3}$$

Corollary 6.1.4.

$$\sum_{n=0}^{\infty} t^{2n} \sum_{w\in SUWD_{\infty,2n}} q^{|w|} = 2\left(\prod_{k\geq 1}(1+itz_k)+\prod_{k\geq 1}(1-itz_k)\right)^{-1}, \tag{6.1.4}$$

and

$$\sum_{n=0}^{\infty} t^{2n+1} \sum_{w\in SUWD_{\infty,2n+1}} z(w) =$$
$$\sum_{n=0}^{\infty} t^{2n+1} \sum_{w\in WUSD_{\infty,2n+1}} z(w) =$$
$$\frac{-i\left[\prod_{k\geq 1}(1+itz_k)-\prod_{k\geq 1}(1-itz_k)\right]}{\prod_{k\geq 1}(1+itz_k)+\prod_{k\geq 1}(1-itz_k)}, \tag{6.1.5}$$

where $i=\sqrt{-1}$.

Corollary 6.1.4 follows from setting $x=0$ in Equations 6.1.2 and 6.1.3 and from the following lemma.

Lemma 6.1.5. $|SUWD_{m,2n+1}| = |WUSD_{m,2n+1}|$ *for any* n, m *(including* $m=\infty$*).*

To prove this lemma, we define a weight-preserving bijection
$\rho : SUWD_{m,2n+1} \to WUSD_{m,2n+1}$ by
$\rho(w_1, w_2, \ldots, w_{2n+1}) = (w_{2n+1}, w_{2n}, \ldots, w_1)$.
That is, ρ simply reverses the order of letters in w. Since $w_1 < w_2 \geq w_3 \ldots$ implies that $w_{2n+1} \leq w_{2n} > w_{2n-1} \ldots$, this gives the desired bijection. Note that no such result extends to words of even length or with increasing blocks of length s.

We now turn our attention to proving Theorem 6.1.3. Define a homomorphism on the ring of symmetric functions by

$$\Theta_{SUWD}(e_n) = f(n) \left(\prod_{k\geq 1} (1 + tz_k) \right) |_{t^n} \ .$$

Claim:

$$\Theta_{SUWD}(h_n) = \sum_{w \in \mathbb{P}^{2n}:\ w_{2j-1} < w_{2j} \forall j} x^{\text{2-ris}(w)} z(w).$$

The proof for this claim is identical to our proof that

$$\Theta_{WUSD}(h_n) = \sum_{w \in \mathbb{P}^{2n}:\ w_{2j-1} \leq w_{2j} \forall j} x^{\text{w2-ris}(w)} z(w),$$

except that we fill each brick with a partition with distinct parts, written in strictly increasing order. From this, it is a straightforward simplification to obtain the first part of Theorem 6.1.3.

To prove the second part of Theorem 6.1.3, we define a weighting function for the brick tabloids by $\nu(2n+1) = 0$ and

$$\nu(2n) = \frac{\left(\prod_{k\geq 1}(1+tz_k)\right) |_{t^{2n-1}}}{\left(\prod_{k\geq 1}(1+tz_k)\right) |_{t^{2n}}}.$$

We claim that, for all $n > 0$,

$$\Theta_{SUWD}(p_{2n,\nu}) = \sum_{w \in \mathbb{P}^{2n-1}:\ w_{2j-1} < w_{2j} \forall j} x^{\text{2-ris}(w)} z(w).$$

To see this, we interpret each term in

$$\Theta_{SUWD}(p_{2n,\nu}) = \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} w_\nu(B_{\lambda,n}) \prod_{j=1}^{\ell(\lambda)} f(\lambda_j) \prod_{k\geq 1} (1+tz_k) \ |_{t^{\lambda_j}} \ . \tag{6.1.6}$$

By the definition of f, this sum will contribute nothing unless each part of λ is even, which also implies that n must be even. Thus, a term in the sum corresponds to a brick tabloid with even bricks. Then $(-1)^{n-\ell(\lambda)} \prod_{j=1}^{\ell(\lambda)} f(\lambda_j)$ reduces to $\prod_{j=1}^{\ell(\lambda)} (x-1)^{\lambda_j/2}$, which weights each brick with a factor of x or -1 in every other nonterminal cell. The term $\prod_{j=1}^{\ell(\lambda)} \prod_{k\geq 1}(1+tz_k) \mid_{t^{\lambda_j}}$ lets us choose a partition π^j with λ_j distinct parts for each brick, where we write π^j in strictly increasing order. The main difference in our interpretation here as compared with that of Equation 6.1.1 is in the last brick, where the weight ν replaces the partition $\pi^{\ell(\lambda)}$ in the last brick with a strictly increasing partition of length one less than the length of the brick, leaving the last cell empty. This means that we end up with a word of length $n-1$ (which is odd) rather than a word of length n. We define the weight of a filled labeled brick tabloid created in this manner to be the product of the x and -1 labels times times the monomial $z(w)$, where w denotes the underlying word. For example, Figure 6.4 is one such object with weight $-xz_1^2 z_2 z_4 z_6 z_8 z_9$.

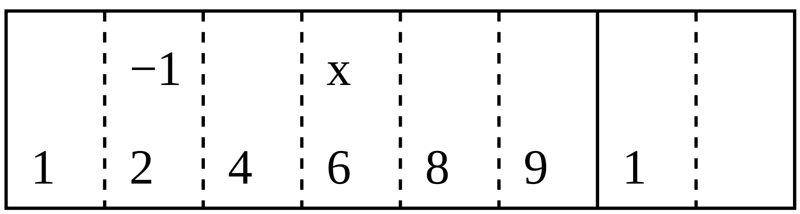

Figure 6.4: An object coming from Equation 6.1.6 when $n = 8$

Thus, Equation 6.1.6 corresponds to a weighted sum over all such brick tabloids. We perform the usual involution, where bricks are scanned from left to right for the first occurrence of either a -1 or a strict increase between bricks. For the former, we break bricks and remove the -1; for the latter, we combine bricks and insert a -1. For example, the image of Figure 6.4 is displayed in Figure 6.5.

Fixed points must have bricks of even length and weak decreases between bricks. The power of x will register the number of double rises, where we specify by convention that there is a rise for the last entry in the last brick. Figure 6.6 displays one fixed point. Since we never break a brick at an odd place, we will always have

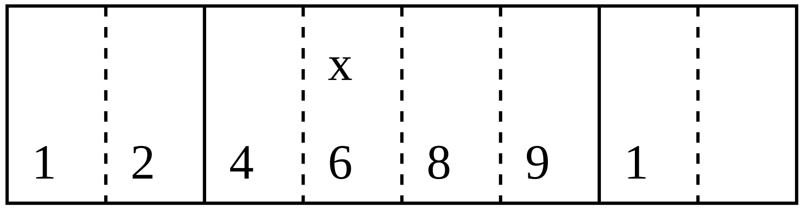

Figure 6.5: The image of Figure 6.4

$w_{2j-1} < w_{2j}$, where w denotes the underlying word. Thus, the weighted sum of the fixed points is exactly given by

$$\sum_{w \in \mathbb{P}^{2n-1}:\ w_{2j-1} < w_{2j} \forall j} x^{2\text{-ris}(w)} z(w).$$

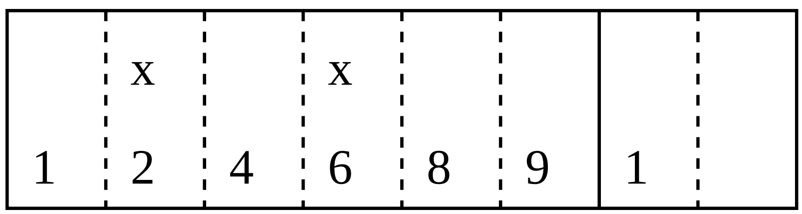

Figure 6.6: A fixed point coming from Equation 6.1.6 when $n = 8$

Thus,

$$
\begin{aligned}
\sum_{n\geq 1} t^{2n-1} & \sum_{w\in\mathbb{P}^{2n-1}:\ w_{2j-1}<w_{2j}\forall j} x^{2\text{-ris}(w)} z(w) \\
&= \frac{1}{t}\sum_{n=1}^{\infty} t^n \Theta_{SUWD}(p_{n,\nu}) \\
&= \frac{1}{t}\Theta_{SUWD}\frac{\sum_{n\geq 1}(-1)^{n-1}\nu(n)e_n t^n}{\sum_{n=0}^{\infty}(-t)^n e_n} \\
&= \frac{1}{t}\frac{\sum_{n\geq 1}(-1)^{n-1}\nu(n)f(n)\left[\prod_{k\geq 1} 1+tz_k\right]|_{t^n}\ t^n}{1+\sum_{n\geq 1}(-t)^n f(n)\left[\prod_{k\geq 1} 1+tz_k\right]|_{t^n}} \\
&= \frac{1}{t}\frac{\sum_{n\geq 1}\left[\prod_{k\geq 1} 1+tz_k\right]|_{t^{2n-1}}\ (x-1)^{n-1}t^{2n}}{1-\sum_{n\geq 1} t^{2n}(x-1)^{n-1}\left[\prod_{k\geq 1} 1+tz_k\right]|_{t^{2n}}} \\
&= \frac{\sqrt{x-1}\sum_{n\geq 1}\left[\prod_{k\geq 1} 1+tz_k\right]|_{t^{2n-1}}\ [t\sqrt{x-1}]^{2n-1}}{x-1-\sum_{n\geq 1}\left[\prod_{k\geq 1} 1+tz_k\right]|_{t^{2n}}\ [t\sqrt{x-1}]^{2n}} \\
&= \frac{\frac{\sqrt{x-1}}{2}\left[\prod_{k\geq 1}(1+t\sqrt{x-1}z_k)-\prod_{k\geq 1}(1-t\sqrt{x-1}z_k)\right]}{x-\frac{1}{2}\left[\prod_{k\geq 1}(1+t\sqrt{x-1}z_k)+\prod_{k\geq 1}(1-t\sqrt{x-1}z_k)\right]} \\
&= \frac{\sqrt{x-1}\left[\prod_{k\geq 1}(1+t\sqrt{x-1}z_k)-\prod_{k\geq 1}(1-t\sqrt{x-1}z_k)\right]}{2x-\left[\prod_{k\geq 1}(1+t\sqrt{x-1}z_k)+\prod_{k\geq 1}(1-t\sqrt{x-1}z_k)\right]},
\end{aligned}
$$

which proves the second part of Theorem 6.1.3.

6.2 General up-down words on $\mathbb{P}$

To generalize beyond the case $s = 2$, we define

$$\text{s-ris}(w) = |\{i : w_{si+1} < w_{si+2} < \cdots < w_{s(i+2)}\}|$$

and

$$\text{ws-ris}(w) = |\{i : w_{si+1} \leq w_{si+2} \leq \cdots \leq w_{s(i+2)}\}|,$$

which are the number of places with a block of length $2s$ consisting of strict increases or weak increases.

6.2.1 $SU^{s-1}WD$

Theorem 6.2.1. *Let $s \geq 2$ and $1 \leq J < s$. Then*

$$\sum_{n\geq 0} t^{sn} \sum_{w\in\mathbb{P}^{sn}:\ WDes(w)\subseteq(s\mathbb{P})_{sn-1}} x^{\text{s-ris}(w)} z(w) = \frac{1-x}{-x+\frac{1}{s}\sum_{i=1}^{s}\prod_{k\geq 1}(1+\zeta_i t\sqrt[s]{1-x}z_k)} \tag{6.2.1}$$

and

$$\sum_{n\geq 1} t^{sn-J} \sum_{w\in\mathbb{P}^{sn-J}:\ WDes(w)\subseteq(s\mathbb{P})_{sn-J-1}} x^{\text{s-ris}\, w} z(w) = \frac{(\sqrt[s]{1-x})^J \sum_{i=1}^{s}\zeta_i^{-J}\prod_{k\geq 1}(1+\zeta_i t\sqrt[s]{1-x}z_k)}{sx-\sum_{i=1}^{s}\prod_{k\geq 1}(1+\zeta_i t\sqrt[s]{1-x}z_k)}, \tag{6.2.2}$$

where $\zeta_1,\ldots,\zeta_s$ are all sth roots of -1.

Corollary 6.2.2. *Let $s \geq 2$ and $1 \leq J < s$. Then*

$$\sum_{n\geq 0} t^{sn} \sum_{w\in SU^{s-1}WD_{\infty,sn}} z(w) = \left(\frac{1}{s}\sum_{i=1}^{s}\prod_{k\geq 1}(1+\zeta_i t z_k)\right)^{-1} \tag{6.2.3}$$

and

$$\sum_{n\geq 1} t^{sn-J} \sum_{w\in SU^{s-1}WD_{\infty,sn-J}} z(w) = \frac{\sum_{i=1}^{s}\zeta_i^{-J}\prod_{k\geq 1}(1+\zeta_i t z_k)}{-\sum_{i=1}^{s}\prod_{k\geq 1}(1+\zeta_i t z_k)}, \tag{6.2.4}$$

where $\zeta_1,\ldots,\zeta_s$ are all sth roots of -1

As before, Corollary 6.2.2 follows by setting $x = 0$ in Theorem 6.2.1. To prove Theorem 6.2.1, define a function on nonnegative integers:

$$g_s(n) = \begin{cases} 1 & n = 0 \\ (-1)^{n-1}(x-1)^{(n/s)-1} & n \equiv 0 \mod s \\ 0 & \text{otherwise} \end{cases}$$

and a homomorphism

$$\Theta_{SU^{s-1}WD}(e_n) = g_s(n)\left(\prod_{k\geq 1}(1+tz_k)\right)\Big|_{t^n}.$$

Claim:

$$\Theta_{SU^{s-1}WD}(h_{sn}) = \sum_{w \in \mathbb{P}^{sn}:\ WDes(w) \subseteq (s\mathbb{P})_{sn-1}} x^{\text{s-ris}(w)} z(w). \tag{6.2.5}$$

To see this, we interpret each term in

$$\begin{aligned} \Theta_{SU^{s-1}WD}(h_{sn}) &= \sum_{\mu \vdash sn} (-1)^{sn-\ell(\mu)} B_{\mu,sn} \prod_{j=1}^{\ell(\mu)} \Theta_{SU^{s-1}WD}(e_{\mu_j}) \\ &= \sum_{s\lambda \vdash sn} (-1)^{sn-\ell(\lambda)} B_{s\lambda,sn} \prod_{j=1}^{\ell(\lambda)} g_s(s\lambda_j) \left(\prod_{k \geq 1} (1 + tz_k) \right) |_{t^{s\lambda_j}} \\ &= \sum_{\lambda \vdash n} B_{\lambda,n} \prod_{j=1}^{\ell(\lambda)} (x-1)^{n-1} \left(\prod_{k \geq 1} (1 + tz_k) \right) |_{t^{s\lambda_j}} . \end{aligned} \tag{6.2.6}$$

By the definition of g_s, this sum will contribute nothing unless each part of μ is divisible by s, so we can obtain μ by taking any $\lambda \vdash n$ and multiplying each part of λ by s. Thus, we interpret the term $\sum_{\lambda \vdash n} B_{\lambda,n}$ as creating a brick tabloid with bricks whose lengths are multiples of s. We interpret the term $\prod_{j=1}^{\ell(\lambda)} (x-1)^{n-1}$ as labeling each brick with a factor of x or -1 in every s'th nonterminal cell. Finally, we interpret the term $\prod_{j=1}^{\ell(\lambda)} \left(\prod_{k \geq 1} (1+tz_k) \right) |_{t^{s\lambda_j}}$ as choosing a partition π^j with $s\lambda_j$ distinct parts to fill each brick, where we write π^j in strictly increasing order and weight by the monomial $z(\pi^j)$. For example, Figure 6.7 depicts a brick tabloid with $s = 3$ and $n = 9$.

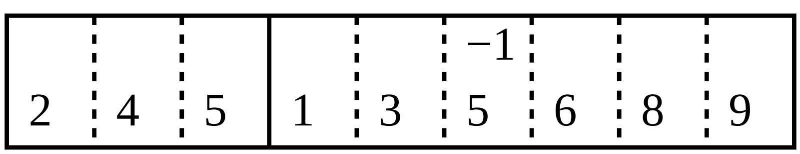

Figure 6.7: An object coming from Equation 6.2.6 when $s = 3$ and $n = 9$

The weight is given by the product of the monomial weights for each brick times the product of the x and -1 labels. For example, the weight of the object in Figure 6.7 is $-z_1 z_2 z_3 z_4 z_5^2 z_6 z_8 z_9$. Thus, Equation 6.2.6 above corresponds to a weighted sum over all such filled labeled brick tabloids.

We perform the usual involution, where bricks are scanned from left to right for the first occurrence of either a -1 or a strict increase between bricks; we break bricks and remove the -1 for the former, combine bricks and insert a -1 for the latter. For example, the image of Figure 6.7 is given in Figure 6.8. Thus, Equation 6.2.6 reduces to summing the weights of all fixed points.

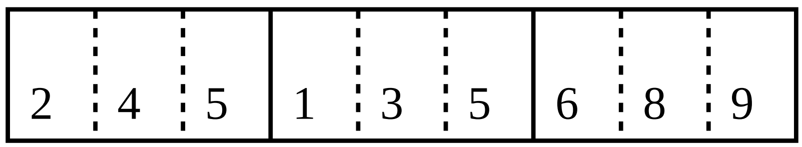

Figure 6.8: The image of Figure 6.7

Fixed points must have bricks whose lengths are multiples of s and weak decreases between bricks. The power of x will register the number of s-rises, as desired. For example, a fixed point is given in Figure 6.9. Since entries increase within bricks and we can never break a brick other than at a multiple of s, weak descents can only occur at multiples of s; i.e. $WDes(w) \subseteq (s\mathbb{P})_{sn-1}$. Thus, the weighted sum of fixed points is exactly counted by

$$\sum_{n\geq 0} t^{sn} \sum_{w\in\mathbb{P}^{sn}:\ WDes(w)\subseteq(s\mathbb{P})_{sn-1}} x^{\text{s-ris}(w)} z(w).$$

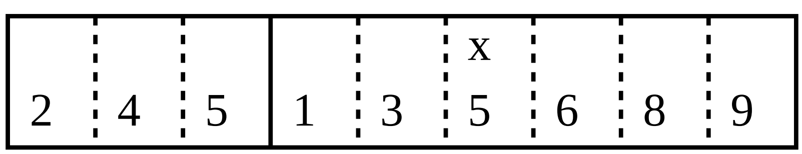

Figure 6.9: A fixed point of Equation 6.2.6 when $s = 3$ and $n = 9$

Thus,

$$
\begin{aligned}
&\sum_{n\geq 0} t^{sn} \sum_{w\in\mathbb{P}^{sn}:\ WDes(w)\subseteq(s\mathbb{P})_{sn-1}} x^{\text{s-ris}(w)} z(w) \\
&= \sum_{n=0}^{\infty} t^n \Theta_{SU^{s-1}WD}(h_n) = \Theta_{SU^{s-1}WD}\left(\sum_{n=0}^{\infty}(-t)^n e_n\right)^{-1} \\
&= \left(1+\sum_{n=1}^{\infty}(-t)^{sn}(-1)^{sn-1}(x-1)^{n-1}\left(\prod_{k\geq 1}(1+tz_k)\right)|_{t^{sn}}\right)^{-1} \\
&= \frac{1-x}{1-x+\sum_{n\geq 1}(-1)^n[t\sqrt[s]{1-x}]^{sn}\left(\prod_{k\geq 1}(1+tz_k)\right)|_{t^{sn}}} \\
&= \frac{1-x}{-x+\frac{1}{s}\sum_{i=1}^{s}\prod_{k\geq 1}(1+\zeta_i t\sqrt[s]{1-x}z_k)},
\end{aligned}
$$

where $\zeta_1,\ldots,\zeta_s$ are all sth roots of -1 (for details on how these play a role, see section 6.6).

We now turn our attention to proving the 2nd part of Theorem 6.2.1. To this end, define a weighting function $\nu_{s,J}$ by $\nu_{s,J}(n)=0$ unless $n\equiv 0 \mod s$ and

$$\nu_{s,J}(sn) = \frac{\left(\prod_{k\geq 1}(1+tz_k)\right)|_{t^{sn-J}}}{\left(\prod_{k\geq 1}(1+tz_k)\right)|_{t^{sn}}}$$

With this weighting function, we can see that

$$\Theta_{SU^{s-1}WD}(p_{sn,\nu_{s,J}}) = \sum_{w\in\mathbb{P}^{sn-J}:\ WDes(w)\subseteq(s\mathbb{P})_{sn-J-1}} x^{\text{s-ris}(w)} z(w) \tag{6.2.7}$$

by the same reasoning as that for Equation 6.2.5, where our interpretation in terms of brick tabloids is identical except that the weighting function $\nu_{s,J}$ will leave the last J cells of the final brick empty (by convention, we count these as strict increases. Therefore,

$$\begin{aligned}
&\sum_{n\geq 1} t^{sn-J} \sum_{w\in\mathbb{P}^{sn-J}:\ WDes(w)\subseteq(s\mathbb{P})_{sn-J-1}} x^{\text{s-ris}\,w} z(w) \\
&= \frac{1}{t^J}\sum_{n=0}^{\infty} t^n \Theta_{SU^{s-1}WD}(p_{n,\nu_{s,J}}) = \frac{1}{t^J}\Theta_{SU^{s-1}WD}\left(\frac{\sum_{n\geq 1}(-1)^{n-1}\nu(n)e_n t^n}{\sum_{n=0}^{\infty} e_n(-t)^n}\right) \\
&= \frac{1}{t^J}\frac{\sum_{n\geq 1}(-1)^{n-1}\nu_{s,J}(n)g_s(n)\left[\prod_{k\geq 1} 1+tz_k\right]|_{t^n}\ t^n}{1+\sum_{n=1}^{\infty} g_s(n)\left[\prod_{k\geq 1}(1+tz_k)\right]|_{t^n}\ (-t)^n} \\
&= \frac{1}{t^J}\frac{\sum_{n\geq 1}\left[\prod_{k\geq 1}(1+tz_k)\right]|_{t^{sn-J}}\ (x-1)^{n-1}t^{sn}}{1-\sum_{n\geq 1}\left[\prod_{k\geq 1}(1+tz_k)\right]|_{t^{sn}}\ t^{sn}(x-1)^{n-1}} \\
&= \frac{1}{t^J}\frac{\sum_{n\geq 1}\left[\prod_{k\geq 1}(1+tz_k)\right]|_{t^{sn-J}}\ (-1)^n[t\sqrt[s]{1-x}]^{sn}}{x-1-\sum_{n\geq 1}\left[\prod_{k\geq 1}(1+tz_k)\right]|_{t^{Kn}}\ [t\sqrt[K]{1-x}]^{Kn}} \\
&= \frac{(\sqrt[s]{1-x})^J\sum_{i=1}^{s}\zeta_i^{-J}\prod_{k\geq 1}(1+\zeta_i t\sqrt[s]{1-x}z_k)}{sx-\sum_{i=1}^{s}\prod_{k\geq 1}(1+\zeta_i t\sqrt[s]{1-x}z_k)},
\end{aligned}$$

which proves the second part of Theorem 6.2.1.

6.2.2 $WU^{s-1}SD$

Theorem 6.2.3. *Let $s \geq 2$ and $1 \leq J < s$. Then*

$$\sum_{n\geq 0} t^{sn} \sum_{w\in\mathbb{P}^{sn}:\ Des(w)\subseteq(s\mathbb{P})_{sn-1}} x^{\text{ws-ris}(w)} z(w) = \frac{1-x}{-x+\frac{1}{s}\sum_{i=1}^{s}\prod_{k\geq 1}\frac{1}{1+\zeta_i t\sqrt[s]{1-x}z_k}}, \tag{6.2.8}$$

and

$$\sum_{n\geq 0} t^{sn-J} \sum_{w\in\mathbb{P}^{sn}:\ Des(w)\subseteq(s\mathbb{P})_{sn-J-1}} x^{\text{ws-ris}\,w} z(w) = \frac{(\sqrt[s]{1-x})^J\sum_{i=1}^{s}\zeta_i^{-J}\prod_{k\geq 1}\frac{1}{1-\zeta_i t\sqrt[s]{1-x}z_k}}{sx-\sum_{i=1}^{s}\prod_{k\geq 1}\frac{1}{1-\zeta_i t\sqrt[s]{1-x}z_k}}, \tag{6.2.9}$$

where $\zeta_1, \ldots, \zeta_s$ are all sth roots of -1.

Corollary 6.2.4. *Let $s \geq 2$ and $1 \leq J < s$. Then*

$$\sum_{n\geq 0} t^{sn} \sum_{w\in WU^{s-1}SD_{\infty,sn}} z(w) = \left(\frac{1}{s}\sum_{i=1}^{s}\prod_{k\geq 1}\frac{1}{1-\zeta_i t z_k}\right)^{-1} \tag{6.2.10}$$

and

$$\sum_{n\geq 1} t^{sn-J} \sum_{w\in WU^{s-1}SD_{\infty,sn-J}} z(w) = \frac{\sum_{i=1}^{s}\zeta_i^{-J}\prod_{k\geq 1}\frac{1}{1-\zeta_i t z_k}}{-\sum_{i=1}^{s}\prod_{k\geq 1}\frac{1}{1-\zeta_i t z_k}}, \tag{6.2.11}$$

where $\zeta_1, \ldots, \zeta_s$ are all sth roots of -1.

The proofs of Theorem 6.2.3 and Corollary 6.2.4 are essentially the same as those of Theorem 6.2.1 and Corollary 6.2.2, except that they use the homomorphism

$$\Theta_{WU^{s-1}SD}(e_n) = g_s(n)\left(\prod_{k\geq 1}\frac{1}{1+tz_k}\right)|_{t^n}$$

and the weighting function defined by $\nu'_{s,J}(n) = 0$ unless $n \equiv 0 \mod s$ and

$$\nu'_{s,J}(sn) = \frac{\left(\prod_{k\geq 1}\frac{1}{1+tz_k}\right)|_{t^{sn-J}}}{\left(\prod_{k\geq 1}\frac{1}{1+tz_k}\right)|_{t^{sn}}}.$$

For brevity, we omit the full details.

6.3 Level alternating words

We define level-alternating words as having the pattern $=, \neq$; i.e., $LNL_{\infty,n} = \{w \in \mathbb{P}^n : \ w_1 = w_2 \neq w_3 = w_4 \ldots\}$ or the set of words s.t. $Lev(w) = \mathbb{O}_{n-1}$. Also, define 2-lev$(w) = |\{i : w_{2i+1} = w_{2i+2} = w_{2i+3} = w_{2i+4}\}|$. Then we have the following theorem and corollary:

Theorem 6.3.1.

$$\sum_{n\geq 0} t^{2n} \sum_{w\in\mathbb{P}^{2n}:\ w_{2j-1}=w_{2j}\forall j} x^{\text{2-lev}(w)} z(w) = \left(1 - \sum_{i\geq 1}\frac{t^2 z_i^2}{1-t^2(x-1)z_i^2}\right)^{-1}. \tag{6.3.1}$$

Corollary 6.3.2.

$$\sum_{n\geq 0} t^{2n} \sum_{w\in LNL_{\infty,2n}} z(w) = \left(1 - \sum_{i\geq 1} \frac{t^2 z_i^2}{1+t^2 z_i^2}\right)^{-1}. \tag{6.3.2}$$

As before, Corollary 6.3.2 follows from setting $x = 0$ in Theorem 6.3.1. To prove Theorem 6.3.1, define a homomorphism by

$$\Theta_{LNL}(e_n) = f(n) p_n(z_1, z_2, \dots) = f(n) \sum_{i\geq 1} z_i^n.$$

Claim:

$$\Theta_{LNL}(h_{2n}) = \sum_{w\in \mathbb{P}^{2n}:\ w_{2j-1}=w_{2j}\forall j} x^{\text{2-lev}(w)} z(w).$$

To see this, we interpret each term in

$$\Theta_{LNL}(h_n) = \sum_{\lambda\vdash n} (-1)^{n-\ell(\lambda)} |B_{\lambda,n}| \prod_{j=1}^{\ell(\lambda)} f(\lambda_j) \sum_{i\geq 1} z_i^{\lambda_j}. \tag{6.3.3}$$

By the definition of f, this sum will contribute nothing unless each part of λ is even, which also implies that n must be even. Thus, a term in the sum corresponds to a brick tabloid with bricks of even length. Then $(-1)^{n-\ell(\lambda)} \prod_{i=1}^{\ell(\lambda)} f(\lambda_j)$ reduces to $\prod_{j=1}^{\ell(\lambda)} (x-1)^{\lambda_j/2}$, which weights each brick with a factor of x or -1 in every other nonterminal cell. The term $\prod_{j=1}^{\ell(\lambda)} \sum_{i\geq 1} z_i^{\lambda_j}$ lets us choose a number i for each brick, which we use to fill every cell of the brick and weight by $z_i^{\lambda_j}$. The weight is given by the product of the monomial weights for each brick times the product of the x and -1 labels. For example, Figure 6.10 is one such object with weight $x z_1^2 z_6^6$.

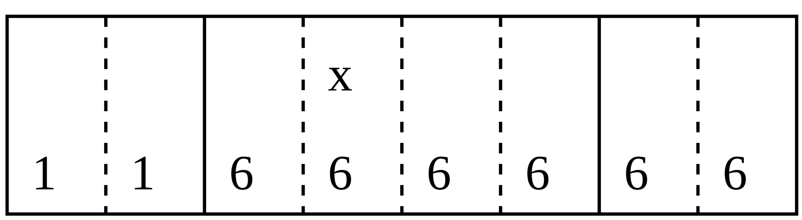

Figure 6.10: An object coming from Equation 6.3.3 with $n = 8$

Thus, Equation 6.3.3 corresponds to a weighted sum over all such brick tabloids. We perform the usual involution on these brick tabloids, where bricks are scanned from left to right for the first occurrence of either a -1 or the same entry in adjacent bricks. If a -1 is encountered first, we break the brick after it and remove the -1. If the same entry is found first, we combine the bricks and insert a -1. For instance, the image of Figure 6.10 is given in Figure 6.11. Thus, Equation 6.3.3 can be reduced to summing over fixed points.

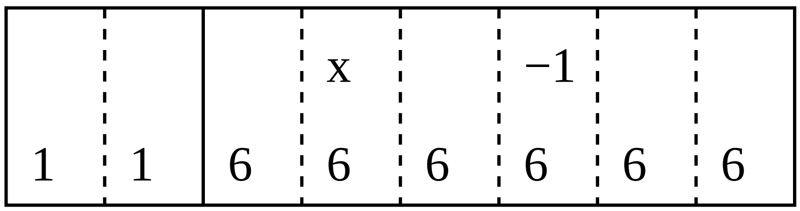

Figure 6.11: The image of Figure 6.10

Fixed points must have bricks of even length, equality within a brick, and inequalities between bricks. The power of x will register the number of double levels, as desired. Since all brick lengths are even, we will always have $w_{2j-1} = w_{2j}$, where w denotes the underlying word. For example, the fixed point in Figure 6.12 has weight $x^2 z_1^2 z_6^6$. The sum of the weights of all fixed points is exactly given by

$$\sum_{w \in \mathbb{P}^{2n}:\ w_{2j-1}=w_{2j} \forall j} x^{2\text{-lev}(w)} z(w).$$

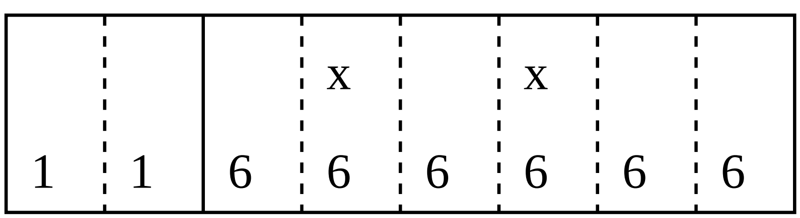

Figure 6.12: A fixed point coming from Equation 6.3.3 when $n = 8$

Thus, we can use Equation 6.3.3 to obtain:

$$\begin{aligned}
&\sum_{n\geq 0} t^{2n} \sum_{w\in\mathbb{P}^{2n}:\ w_{2j-1}=w_{2j}\forall j} x^{2\text{-lev}(w)} z(w) \\
&= \sum_{n=0}^{\infty} t^n \Theta_{LNL}(h_n) = \Theta_{LNL}\left(\sum_{n=0}^{\infty}(-t)^n e_n\right)^{-1} \\
&= \left(1 + \sum_{n=1}^{\infty} -(x-1)^{n-1} t^{2n} \sum_{i\geq 1} z_i^{2n}\right)^{-1} \\
&= \left(1 - \sum_{i\geq 1}\sum_{n=1}^{\infty}(x-1)^{n-1}(tz_i)^{2n}\right)^{-1} \\
&= \left(1 - \sum_{i\geq 1}\frac{t^2 x_i^2}{1 - t^2(x-1)z_i^2}\right)^{-1},
\end{aligned}$$

which proves Theorem 6.3.1.

In this case, Corollary 6.3.2 could also have been derived by replacing tx_i with $t^2 z_i^2$ and setting $y = 0$ in Theorem 3.2.1. This should be obvious, since a level-alternating word can be reduced to a word with no levels by replacing each repeated letter by a single occurrence. For example, the level-alternating word 2 2 4 4 3 3 reduces to 2 4 3. In fact, the same reduction will work for s-level-alternating words, which yields the following corollary:

Corollary 6.3.3.

$$\sum_{n\geq 0} t^{sn} \sum_{w\in L^{s-1}NL_{\infty,sn}} z(w) = \left(1 - \sum_{i\geq 1}\frac{t^s x_i^s}{1 + t^s z_i^s}\right)^{-1}, \tag{6.3.4}$$

where $L^{s-1}NL_{\infty,sn} = \{w \in \mathbb{P}^{sn} : \ Lev(w) = \mathbb{P}_{n-1} - s\mathbb{P}_{n-1}\}$.

6.4 Alternating descents

Chebikin [14] first introduced the notion of *alternating descents* for permutations, defined by

$$\hat{d}(\sigma) = |\{2i : \sigma_{2i} < \sigma_{2i+1}\} \cup \{2i+1 : \sigma_{2i+1} > \sigma_{2i+2}\}|.$$

He also found the generating function for *alternating Eulerian polynomials*, defined as $\hat{A}_n(t) = \sum_{\sigma \in S_n} t^{\hat{d}(\sigma)+1}$. That is, he showed that

$$\sum_{n \geq 1} \hat{A}_n(t) \frac{u^n}{n!} = \frac{t(1 - h(u(t-1)))}{h(u(t-1)) - t}, \tag{6.4.1}$$

where $h(x) = \tan(x) + \sec(x)$. In addition, Remmel [39] introduced the notion of alternating major index, defined by

$$\mathrm{altmaj}(\sigma) = \sum_{i \in Altdes(\sigma)} i.$$

Remmel then extended Chebikin's generating function to the following:

$$\sum_{n \geq 0} \frac{t^n}{n!} \frac{\sum_{\sigma \in S_n} x^{\mathrm{altdes}(\sigma)} q^{\mathrm{altmaj}(\sigma)}}{(1-x)(1-xq)\cdots(1-xq^n)} = \tag{6.4.2}$$
$$\sum_{k \geq 0} \frac{x^k}{(\sec(-t)+\tan(-t))(\sec(-tq)+\tan(-tq))\cdots(\sec(-tq^{k-1})+\tan(-tq^{k-1}))}.$$

Remmel also obtained similar formulas for common alternating descents and major index, as well as for the hyperoctahedral group B_n and its subgroup D_n.

The central goal of the next two sections is to develop analogues of (6.4.1) and (6.4.2) for words instead of permutations. When we consider analogues for words, we can apply both strong and weak versions of these statistics. Chebikin and Remmel defined alternating descents as places where σ *deviates* from an up-down pattern, but we find it more natural to define alternating descents as places where σ *follows* an up-down pattern (the two statistics are equidistributed over words on a finite alphabet, so it makes no difference). That is, we will use the following definitions.

$$\begin{aligned} Altdes(w) &= \{2i : w_{2i} > w_{2i+1}\} \cup \{2i+1 : w_{2i+1} < w_{2i+2}\} \\ &= (\mathbb{E} \cap Des(w)) \cup (\mathbb{O} \cap Ris(w)), \text{ and} \\ \mathrm{altdes}(w) &= |Altdes(w)|. \end{aligned}$$

Similarly, define

$$\begin{aligned} Waltdes(w) &= \{2i : w_{2i} \geq w_{2i+1}\} \cup \{2i+1 : w_{2i+1} \leq w_{2i+2}\} \\ &= (\mathbb{E} \cap WDes(w)) \cup (\mathbb{O} \cap WRis(w)), \text{ and} \\ \text{waltdes}(w) &= |Waltdes(w)|. \end{aligned}$$

Also define

$$\text{altmaj}(w) = \sum_{i \in Altdes(w)} i \text{ and}$$

$$\text{waltmaj}(w) = \sum_{i \in Waltdes(w)} i.$$

Again, altdes measures how often a word matches the up-down pattern, and waltdes measures how often a word matches the weak up-down pattern. Since an up-down word is also weak up-down, $\text{altdes}(w) \leq \text{waltdes}(w)$ for any w. We will prove the following theorem.

Theorem 6.4.1.

$$\sum_{n\geq 0} t^n \sum_{w\in[m]^n} x^{\text{altdes}(w)} = (1-x)\left[-x + \frac{1+\sum_{k=0}^{m}(-1)^k\binom{m+k}{2k+1}(t[x-1])^{2k+1}}{\sum_{k=0}^{m}(-1)^k\binom{m+k-1}{2k}(t[x-1])^{2k}}\right]^{-1}$$

and

$$\sum_{n\geq 0} t^n \sum_{w\in[m]^n} x^{\text{waltdes}(w)} = (1-x)\left[-x + \frac{1+\sum_{k=0}^{m}(-1)^k\binom{m+k}{2k+1}(t[x-1])^{2k+1}}{\sum_{k=0}^{m}(-1)^k\binom{m+k}{2k}(t[x-1])^{2k}}\right]^{-1}$$

Our method for proving these theorems will be to use the generating functions for up-down words to define a homomorphism on the ring of symmetric functions. Recall from Chapter 4 that Carlitz [13] showed that

$$\sum_{m\in\mathbb{O}} |SUD_{n,m}| z^m = \frac{1}{Q_n(z)}$$

and

$$1 + \sum_{m\in\mathbb{E}} |SUD_{n,m}| z^m = \frac{P_n(z)}{Q_n(z)}$$

where

$$\begin{aligned} P_n(z) &= \sum_{k=0}^{n}(-1)^k\binom{n+k}{2k+1}z^{2k+1} \text{ and} \\ Q_n(z) &= \sum_{k=0}^{n}(-1)^k\binom{n+k-1}{2k}z^{2k}. \end{aligned}$$

Also, Rawlings [38] showed that

$$\sum_{m\in\mathbb{O}}|WUD_{n,m}|z^m = \frac{1}{B_n(z)}$$

and

$$1+\sum_{m\in\mathbb{E}}|WUD_{n,m}|z^m = \frac{A_n(z)}{B_n(z)}$$

where

$$\begin{aligned} A_n(z) &= \sum_{k=0}^{n}(-1)^k\binom{n+k}{2k+1}z^{2k+1} \text{ and} \\ B_n(z) &= \sum_{k=0}^{n}(-1)^k\binom{n+k}{2k}z^{2k}. \end{aligned}$$

We will also make use of the fact (noted in Chapter 4) that up-down words and down-up words over a finite alphabet are equinumerous: $|SUD_{m,n}| = |SDU_{m,n}|$ and $|WUD_{m,n}| = |WDU_{m,n}|$. This observation is key for proving Theorem 6.4.1.

To prove the first part of Theorem 6.4.1, we fix some alphabet $[m]$ and define a homomorphism on the ring of symmetric functions by $\Theta_{altdes}(e_0) = 1$ and, for $n \geq 1$,

$$\begin{aligned} \Theta_{altdes}(e_n) &= (-1)^{n-1}(x-1)^{n-1}|SUD_{m,n}| \\ &= (-1)^{n-1}(x-1)^{n-1}\frac{1+\sum_{k=0}^{m}(-1)^k\binom{m+k}{2k+1}z^{2k+1}}{\sum_{k=0}^{m}(-1)^k\binom{m+k-1}{2k}z^{2k}}\ \Big|_{z^n} \end{aligned}$$

Claim:

$$\Theta_{altdes}(h_n) = \sum_{w\in[m]^n} x^{\text{altdes}(w)} \tag{6.4.3}$$

To see this, we interpret each term in

$$\Theta_{altdes}(h_n) = \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} B_{\lambda,n} \prod_{i=1}^{\ell(\lambda)} (-1)^{\lambda_i - 1}(x-1)^{\lambda_i - 1} |SUD_{m,\lambda_i}|$$

$$= \sum_{\lambda \vdash n} B_{\lambda,n} \prod_{i=1}^{\ell(\lambda)} (x-1)^{\lambda_i - 1} \frac{1 + \sum_{k=0}^{m} (-1)^k \binom{m+k}{2k+1} z^{2k+1}}{\sum_{k=0}^{m} (-1)^k \binom{m+k-1}{2k} z^{2k}} \,|_{z^{\lambda_i}} \, .$$

The term $\sum_{\lambda \vdash n} B_{\lambda,n}$ lets us choose some $\lambda \vdash n$ and create a brick tabloid of shape n and type λ. The term $\prod_{i=1}^{\ell(\lambda)} \frac{1+\sum_{k=0}^{m}(-1)^k \binom{m+k}{2k+1} z^{2k+1}}{\sum_{k=0}^{m}(-1)^k \binom{m+k-1}{2k} z^{2k}} \,|_{z^{\lambda_i}}$ lets us fill in each brick with either an up-down or a down-up word. If the brick starts at an odd place, we fill it with an up-down; if it starts at an even place, we fill it with a down-up. This is the step that requires up-down and down-up words to be equinumerous, and which prevents us from tracking more information (the bijection between does not preserve the sum of the entries, so we cannot q-count). Finally, the term $\prod_{i=1}^{\ell(\lambda)} (x-1)^{\lambda_i - 1}$ lets us leave the last cell of every brick alone, and label every other cell with either an x or -1. We define the weight of a filled labeled brick tabloid created in this manner to be the product of the x and -1 labels. For example, Figure 6.13 displays one such object with weight x^4. The first brick contains an up-down word, while the second and third bricks contain down-up words, since they begin at even places.

x	−1		x		−1	x	x	
2	4	3	6	3	8	2	6	3

Figure 6.13: An object coming from $\Theta_{altdes}(h_9)$

We perform an involution on the decorated brick tabloids that result, breaking a brick at the first -1 encountered or combining bricks if this will preserve their up-down or down-up nature. The image of Figure 6.13 under this involution is depicted in Figure 6.14.

We now consider the fixed points of $\Theta_{altdes}(h_n)$. Fixed points will thus have no -1 weights and up-down or down-up behavior within each brick, but not between.

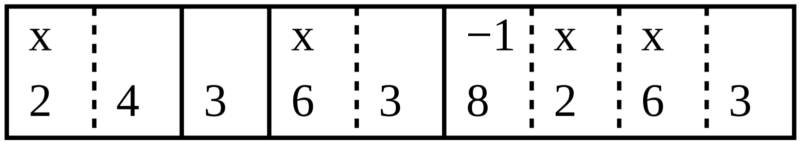

Figure 6.14: The image of Figure 6.13

The factors of x will thus give us exactly altdes(w), which verifies Equation 6.4.3. One fixed point is depicted in Figure 6.15.

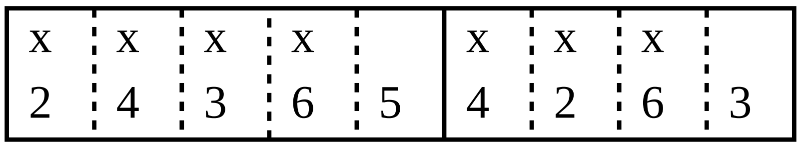

Figure 6.15: A fixed point of $\Theta_{altdes}(h_9)$

Thus,

$$\sum_{n\geq 0} t^n \sum_{w\in[m]^n} x^{\mathrm{altdes}(w)} = \left(1+\sum_{n\geq 1}(-t)^n \Theta_{altdes}(e_n)\right)^{-1}$$
$$= \frac{1-x}{1-x+\sum_{n\geq 1}[t(x-1)]^n \frac{1+\sum_{k=0}^m (-1)^k \binom{m+k}{2k+1} z^{2k+1}}{\sum_{k=0}^m (-1)^k \binom{n+k-1}{2k} z^{2k}} \,|_{z^n}}$$
$$= (1-x)\left[-x+\frac{1+\sum_{k=0}^m (-1)^k \binom{m+k}{2k+1}(t[x-1])^{2k+1}}{\sum_{k=0}^m (-1)^k \binom{m+k-1}{2k}(t[x-1])^{2k}}\right]^{-1}$$

The second part of Theorem 6.4.1 is proved in a similar manner, interpreting as brick tabloids and using the generating function for WUD (WDU) to fill in bricks. For brevity, we omit the full details.

6.5 Alternating major index

In this section, we will prove two analogues of (6.4.2) for compositions. Define *alternating major index* and *weak alternating major index* as follows.

$$\text{altmaj}(w) = \sum_{i \in Altdes(w)} i \text{ and}$$

$$\text{waltmaj}(w) = \sum_{i \in Waltdes(w)} i.$$

Then we have the following theorems.

Theorem 6.5.1.

$$\sum_{n\geq 0} \frac{t^n}{(y;u)_{n+1}} \quad \sum_{w\in[m]^n} y^{\text{altdes}(w)} u^{\text{altmaj}(w)} = \sum_{p\geq 0} \frac{y^p}{\prod_{j=0}^{p} \frac{1+\sum_{k=0}^{m}(-1)^{k+1}\binom{m+k}{2k+1}(tu^j)^{2k+1}}{\sum_{k=0}^{m}(-1)^k\binom{m+k-1}{2k}(tu^j)^{2k}}}.$$

Theorem 6.5.2.

$$\sum_{n\geq 0} \frac{t^n}{(y;u)_{n+1}} \quad \sum_{w\in[m]^n} y^{\text{waltdes}(w)} u^{\text{waltmaj}(w)} = \sum_{p\geq 0} \frac{y^p}{\prod_{j=0}^{p} \frac{1+\sum_{k=0}^{m}(-1)^{k+1}\binom{m+k}{2k+1}(tu^j)^{2k+1}}{\sum_{k=0}^{m}(-1)^k\binom{m+k}{2k}(tu^j)^{2k}}}.$$

In order to prove these theorems, we will combine the methods of the previous section and section 3.3. The idea behind these theorems is similar to Theorem 3.3.1, except that we use the generating function for up-down words to fill in our brick tabloid with up-down (down-up) words, instead of simply partitions.

To prove Theorem 6.5.1, define a ring homomorphism $\Theta^{(p)}$ by defining it on the elementary symmetric function e_n so that

$$\begin{aligned}\Theta^{(p)}(e_n) &= \sum_{\substack{i_0,\ldots,i_p\geq 0\\ i_0+\cdots+i_p=n}} u^{0i_0+\cdots+pi_p} \prod_{j=0}^{p} |SUD_{m,i_j}| \\ &= \sum_{\substack{i_0,\ldots,i_p\geq 0\\ i_0+\cdots+i_p=n}} u^{0i_0+\cdots+pi_p} \left[\prod_{j=0}^{p} \frac{1+\sum_{k=0}^{m}(-1)^k\binom{m+k}{2k+1}z_j^{2k+1}}{\sum_{k=0}^{m}(-1)^k\binom{m+k-1}{2k}z_j^{2k}} \right] \Bigg|_{z_0^{i_0}\cdots z_p^{i_p}},\end{aligned}$$

where $expression|_{t^p}$ means to take the coefficient of t^p in $expression$.

First we apply $\Theta^{(p)}$ to h_n. We have

$$\begin{aligned}\Theta^{(p)}(h_n) &= \sum_{\lambda\vdash n}(-1)^{n-\ell(\lambda)}B_{\lambda,n}\Theta^{(p)}(e_\lambda)\\ &= \sum_{\lambda\vdash n}(-1)^{n-\ell(\lambda)}B_{\lambda,n}\prod_{r=1}^{\ell(\lambda)}\sum_{\substack{i_0,\ldots,i_p\geq 0\\ i_0+\cdots+i_p=\lambda_r}} u^{0i_0+\cdots+pi_p} \qquad (6.5.1)\\ &\qquad \times\left[\prod_{j=0}^{p}\frac{1+\sum_{k=0}^{m}(-1)^k\binom{m+k}{2k+1}z_j^{2k+1}}{\sum_{k=0}^{m}(-1)^k\binom{m+k-1}{2k}z_j^{2k}}\right]\Bigg|_{z_0^{i_0}\cdots z_p^{i_p}}.\end{aligned}$$

Our goal is to interpret $\Theta^{(p)}(h_n)$ as a sum of weighted combinatorial objects. We interpret the sum $\sum_{\lambda\vdash n}B_{\lambda,n}$ as all ways of picking a brick tabloid T of shape (n). Then the factor $(-1)^{n-\ell(\lambda)}$ allows us to place a -1 in each non-terminal cell of a brick in T and place a 1 at the terminal cell of each brick in T. Next, for each brick in T, choose nonnegative integers $i_0,\ldots,i_p$ that sum to the total length of the brick. This accounts for the product and second sum in (6.5.1). Using powers of u, these choices for $i_0,\ldots,i_p$ can be recorded in T. In each brick, place a power of u in each cell such that the powers weakly increase from left to right and the number of occurrences of u^j is i_j. At this point, we have constructed an object which may look something like Figure 6.16 below.

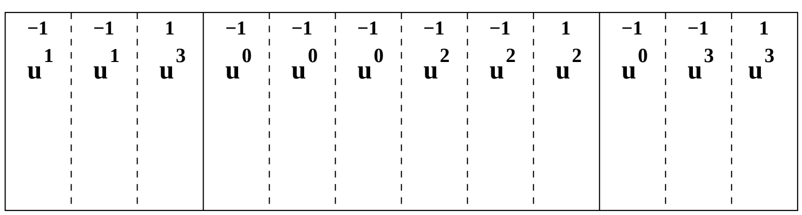

Figure 6.16: A partial object coming from Equation 6.5.1

Now, the term $\left[\prod_{j=0}^{p}\frac{1+\sum_{k=0}^{m}(-1)^k\binom{m+k}{2k+1}z_j^{2k+1}}{\sum_{k=0}^{m}(-1)^k\binom{m+k-1}{2k}z_j^{2k}}\right]\Big|_{z_0^{i_0}\cdots z_p^{i_p}}$ lets us choose $p+1$ up-down (or down-up) words $w^{(0)},\ldots,w^{(p)}$ where $\ell(w^{(j)})=i_j$ for $j=0,\ldots,p$. We write these words in the order chosen, where we insert an up-down word if the starting cell is odd (e.g. it is the 5th cell in the overall tabloid) and a down-up word if the starting cell is even. Figure 6.17 gives one example of an object created in this manner. We call these objects filled labeled brick tabloids. The weight of such a

−1	−1	1	−1	−1	−1	−1	−1	1	−1	−1	1
u^1	u^1	u^3	u^0	u^0	u^0	u^2	u^2	u^2	u^0	u^3	u^3
2	6	5	4	3	4	2	3	1	6	4	5

Figure 6.17: An object coming from Equation 6.5.1 when $n = 12$

composite object is the product of the signs at the top of the configuration times the products of the u^j's in the second row of the configuration. Thus, the weight of the object in Figure 6.17 is $-u^{17}$.

These filled labeled brick tabloids of shape (n) and type λ for some $\lambda \vdash n$ have the following properties:

1. the cells in each brick contain -1 except for the final cell, which contains 1,

2. each cell contains a power of u such that the powers weakly increase within each brick and the largest possible power of u is u^p, and

3. T contains a composition of n which must

   strictly decrease between consecutive cells within a brick if the cells are marked with the same power of u and the first cell is even, and must

   strictly increase between consecutive cells within a brick if the cells are marked with the same power of u and the first cell is odd.

In this way, $\Theta^{(p)}(h_n)$ is the weighted sum over all possible filled labeled brick tabloids of shape (n).

Next, we define a sign-reversing involution I which will allows us to cancel all the terms T with a negative weight. To define I, scan the cells from left to right looking for either a cell containing -1 or two consecutive bricks which may be combined to preserve the properties of this collection of objects. If a -1 is scanned first, break the brick containing the -1 into two immediately after the violation and change the -1 to 1. If the second situation is scanned first, glue the brick together and change the 1 in the first brick to -1. For example, the image of Figure 6.17 is displayed in Figure 6.18.

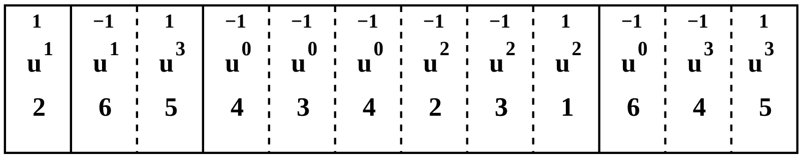

Figure 6.18: The image under I of Figure 6.17.

It is easy to see that I is a sign-reversing, weight-preserving involution. Thus, I shows that $\Theta^{(p)}(h_n)$ is equal to the sum of the weights of all the fixed points of I.

Let us consider the fixed points of I. First, there can be no -1's, so every brick must be of size 1. Next, it cannot be the case that the power of u strictly increases as we move from brick i to brick $i+1$, since then we could combine these two bricks and still satisfy properties (1), (2), and (3). Thus, the powers of u must weakly decrease as we read from left to right. Let $w = (w_1, \ldots, w_n)$ denote the underlying composition. We note that if the power of u is the same on brick i and $i+1$, then it must be the case that

1. $w_i \geq w_{i+1}$ if i is odd, and

2. $w_i \leq w_{i+1}$ if i is even.

Otherwise, we could combine brick i and brick $i+1$. One example of a fixed point may be found in Figure 6.19.

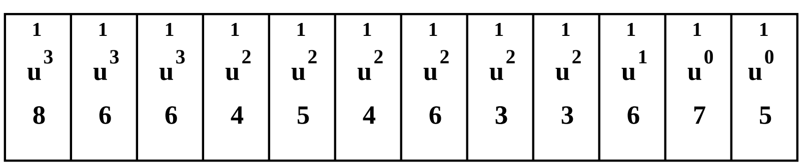

Figure 6.19: A fixed point coming from Equation 6.5.1

We now turn our attention to counting fixed points. Suppose that the powers of u in a fixed point are $r_1, \ldots, r_n$ when read from left to right. It must be the case that $p \geq r_1 \geq \cdots \geq r_n$. Define nonnegative integers a_i by $a_i = r_i - r_{i+1}$ for $i = 1, \ldots, n-1$ and let $a_n = r_n$. It follows that $r_1 + \cdots + r_n = a_1 + 2a_2 + \cdots + na_n$, $a_1 + \cdots + a_n = r_1 \leq p$. Now suppose that w is the composition in a fixed point. Then if $w_i > w_{i+1}$ and i is even or $w_i < w_{i+1}$ and i is odd, it cannot be that $r_i = r_{i+1}$ because that would violate our conditions for fixed points. Thus, it must be the case that

$$a_i \geq \chi(w_i > w_{i+1})\chi(i \in \mathbb{E}) + \chi(w_i < w_{i+1})\chi(i \in \mathbb{O}) = \chi(i \in Altdes(w)).$$

In this way, the sum of the weights of all fixed points of I equals

$$\sum_{w \in [m]^n} \sum_{\substack{a_1 + \cdots + a_n \leq p \\ a_i \geq \chi(i \in Altdes(w))}} u^{a_1 + 2a_2 + \cdots + na_n}$$

$$= \sum_{w \in [m]^n} \sum_{a_1 \geq \chi(1 \in Altdes(w))} \cdots \sum_{a_n \geq \chi(n \in Altdes(w))} y^{a_1 + \cdots + a_n} u^{a_1 + 2a_2 + \cdots + na_n} \Big|_{y^{\leq p}},$$

where $expression|_{t^{\leq k}}$ means to sum the coefficients of t^j for $j = 0, \ldots, k$ in $expression$. Rewriting the above equation, we have

$$\sum_{w \in [m]^n} \sum_{a_1 \geq \chi(1 \in Altdes(w))} (yu)^{a_1} \cdots \sum_{a_n \geq \chi(n \in Altdes(w))} (yu^n)^{a_n} \Big|_{y^{\leq p}}$$

$$= \sum_{w \in [m]^n} \frac{(yu)^{\chi(1 \in Altdes(w))} (yu^2)^{\chi(2 \in Altdes(w))} \cdots (yu^n)^{\chi(n \in Altdes(w))}}{(1 - yu)(1 - yu^2) \cdots (1 - yu^n)} \Big|_{y^{\leq p}}$$

$$= \sum_{w \in [m]^n} \frac{y^{\mathrm{altdes}(w)} u^{\mathrm{altmaj}(w)}}{(1 - yu)(1 - yu^2) \cdots (1 - yu^n)} \Big|_{y^{\leq p}}.$$

Dividing by $(1 - y)$ allows the above expression to be rewritten as

$$\sum_{w \in [m]^n} \frac{y^{\mathrm{des}(w)} u^{\mathrm{altmaj}(w)}}{(1 - y)(1 - yu) \cdots (1 - yu^n)} \Big|_{y^p}.$$

Therefore, we have

$$\begin{aligned}
&\sum_{n\geq 0}\frac{t^n}{(y;u)_{n+1}}\sum_{w\in[m]^n} y^{\text{altdes}(w)}u^{\text{altmaj}(w)}\\
&= \sum_{p\geq 0} y^p\Theta^{(p)}\left(\sum_{n\geq 0} t^n h_n\right)\\
&= \sum_{p\geq 0}\frac{y^p}{\left(\sum_{n\geq 0}(-t)^n\Theta^{(p)}(e_n)\right)}\\
&= \sum_{p\geq 0}\frac{y^p}{\sum_{n\geq 0}(-t)^n\sum_{\substack{i_0,\dots,i_p\geq 0\\ i_0+\dots+i_p=n}} u^{0i_0+\dots+pi_p}\left[\prod_{j=0}^{p}\frac{1+\sum_{k=0}^{m}(-1)^k\binom{m+k}{2k+1}z_j^{2k+1}}{\sum_{k=0}^{m}(-1)^k\binom{m+k-1}{2k}z_j^{2k}}\right]\Bigg|_{z_0^{i_0}\cdots z_p^{i_p}}}.
\end{aligned}$$

However,

$$\begin{aligned}
&\sum_{n\geq 0}(-t)^n\sum_{\substack{i_0,\dots,i_p\geq 0\\ i_0+\dots+i_p=n}} u^{0i_0+\dots+pi_p}\left[\prod_{j=0}^{p}\frac{1+\sum_{k=0}^{m}(-1)^k\binom{m+k}{2k+1}z_j^{2k+1}}{\sum_{k=0}^{m}(-1)^k\binom{m+k-1}{2k}z_j^{2k}}\right]\Bigg|_{z_0^{i_0}\cdots z_p^{i_p}} =\\
&\sum_{n\geq 0}(-t)^n\prod_{j=0}^{p}\frac{1+\sum_{k=0}^{m}(-1)^k\binom{m+k}{2k+1}(u^jz)^{2k+1}}{\sum_{k=0}^{m}(-1)^k\binom{m+k-1}{2k}(u^jz)^{2k}}\Bigg|_{z^n} =\\
&\prod_{j=0}^{p}\frac{1+\sum_{k=0}^{m}(-1)^k\binom{m+k}{2k+1}(-tu^j)^{2k+1}}{\sum_{k=0}^{m}(-1)^k\binom{m+k-1}{2k}(-tu^j)^{2k}} =\\
&\prod_{j=0}^{p}\frac{1+\sum_{k=0}^{m}(-1)^{k+1}\binom{m+k}{2k+1}(tu^j)^{2k+1}}{\sum_{k=0}^{m}(-1)^k\binom{m+k-1}{2k}(tu^j)^{2k}}
\end{aligned}$$

Thus, we have shown that

$$\sum_{n\geq 0}\frac{t^n}{(y;u)_{n+1}}\sum_{w\in[m]^n} y^{\text{altdes}(w)}u^{\text{altmaj}(w)} = \sum_{p\geq 0}\frac{y^p}{\prod_{j=0}^{p}\frac{1+\sum_{k=0}^{m}(-1)^{k+1}\binom{m+k}{2k+1}(tu^j)^{2k+1}}{\sum_{k=0}^{m}(-1)^k\binom{m+k-1}{2k}(tu^j)^{2k}}}$$

which proves Theorem 6.5.1.

The proof of Theorem 6.5.2 is very similar and will be omitted; it uses a homomorphism defined by

$$\phi^{(p)}(e_n) = \sum_{\substack{i_0,\dots,i_p \geq 0 \\ i_0+\cdots+i_p=n}} u^{0i_0+\cdots+pi_p} \prod_{j=0}^{p} |WUD_{m,i_j}|$$

$$= \sum_{\substack{i_0,\dots,i_p \geq 0 \\ i_0+\cdots+i_p=n}} u^{0i_0+\cdots+pi_p} \left[\prod_{j=0}^{p} \frac{1+\sum_{k=0}^{m}(-1)^k \binom{m+k}{2k+1} z_j^{2k+1}}{\sum_{k=0}^{m}(-1)^k \binom{m+k}{2k} z_j^{2k}} \right] \Bigg|_{z_0^{i_0}\cdots z_p^{i_p}}.$$

Notice that the only difference in the homomorphisms–and thus the theorems–is the binomial coefficient in the denominators: $\binom{m+k}{2k}$ instead of $\binom{m+k-1}{2k}$.

6.6 Appendix: roots of -1

In section 6.2, our generating functions turned out to involve roots of -1. In this section, we provide a general theorem that explains their origin.

Let $\zeta_1, \dots, \zeta_k$ be all of the kth roots of -1.

Theorem 6.6.1. *For any sequence a,*

$$\sum_{n\geq 0}(-1)^n a_{kn} t^{kn} = \frac{1}{k}\sum_{j=1}^{k}\sum_{n\geq 0} a_n(\zeta_j t)^n. \tag{6.6.1}$$

Proof. It suffices to show that, for any n,

$$\sum_{j=1}^{k} \zeta_j^n = \begin{cases} (-1)^m k & n = km \\ 0 & k \nmid n \end{cases} \tag{6.6.2}$$

The first case is obvious: if $n = km$, then

$$\sum_{j=1}^{k} \zeta_j^n = \sum_{j=1}^{k} (\zeta_j^k)^m = \sum_{j=1}^{k} (-1)^m = (-1)^m k.$$

For the second case, we wish to write down the roots of -1 explicitly:

$\zeta_j = e^{\frac{\pi i}{k}} e^{\frac{2\pi(j-1)}{k}}$. Then we have

$$\begin{aligned}
&\sum_{j=0}^{k-1} (e^{\frac{\pi i}{k}} e^{\frac{2\pi j}{k}})^n \\
&= e^{\frac{n\pi i}{k}} \sum_{j=0}^{k-1} (e^{\frac{2\pi n}{k}})^j \\
&= e^{\frac{n\pi i}{k}} \frac{1 - (e^{\frac{2\pi n}{k}})^k}{1 - e^{\frac{2\pi n}{k}}} \\
&= e^{\frac{n\pi i}{k}} \frac{1 - 1}{1 - e^{\frac{2\pi n}{k}}}
\end{aligned}$$

If $k \nmid n$, then the denominator is not 0, whereas the numerator is, so we get the desired result. □

We can shift the start of the summation and our sequence entries to obtain the following corollary.

Corollary 6.6.2. *Let $J < k$. Then*

$$\sum_{n \geq N_0} (-1)^n a_{kn-J} t^{kn-J} = \frac{1}{k} \sum_{j=1}^{k} \zeta_j^{-J} [\sum_{n \geq N_0} a_n (\zeta_j t)^n].$$

Chapter 7

Results on other composition patterns

This chapter builds on Chapter 4, where we considered words that could be partitioned into blocks of fixed length so that, within each block, the entries were strictly or weakly increasing and there were strict or weak increases between blocks. In this chapter, we still consider words that can be partitioned into blocks of fixed length, but we examine more general patterns within the blocks. We will consider blocks where the only condition is that the first element of each block is the (unique) maximum of the block, as well as blocks with a fixed number of rises followed by a fixed number of descents. Also, we will consider blocks with a fixed number of levels followed by a descent. We then apply the statistics des, wdes, and lev from Chapter 3 to these blocks, where we will sometimes compare maximal entries within each block and sometimes compare the final entry of one block with the first entry of the following block.

7.1 Block maxima

In this section, we will consider words that are made up of blocks of size K, where each block has a strong maximum at a particular place in the block. Without loss of generality, we can let this be the first place in the block. That is, let

$$BlockMax(K,Kn) = \{w \in \mathbb{P}^{Kn} : w_{iK+1} > w_j \text{ for } j = iK+2, \dots, (i+1)K\}.$$

For words in this class, we will be interested in block levels, or places in which adjacent block maxima have the same value. Let

$$\text{levKmax}(w) = |\{i : \max_{j=iK+1}^{(i+1)K} w_j = \max_{j=(i+1)K+1}^{(i+2)K} w_j\}|.$$

For $w \in BlockMax(K,n)$, $\text{levKmax}(w) = |\{i : w_{iK+1} = w_{(i+1)K+1}\}|$. For example, when $K = 4$, the word $w = 6\ 3\ 5\ 4|7\ 1\ 4\ 2|7\ 5\ 6\ 3 \in BlockMax(4,12)$ has $\text{levKmax}(w) = 1$, coming from the repeated maximal element 7 ("|" indicates separations between blocks). Then we have the following theorem and corollary.

Theorem 7.1.1.

$$\sum_{n\geq 0} t^{Kn} \sum_{w \in BlockMax(K,Kn)} x^{\text{levKmax}(w)} q^{|w|} = \left(1 - \sum_{j\geq 1} \frac{t^K q^{(j+K)}([j]_q)^{(K-1)}}{1 - t^K(x-1)q^{(j+K)}([j]_q)^{(K-1)}}\right)^{-1}.$$

Corollary 7.1.2.

$$\sum_{n\geq 0} t^{Kn} \sum_{w \in BlockMax(K,Kn)} q^{|w|} = \left(1 - \sum_{j\geq 1} t^K q^{(j+K)}([j]_q)^{K-1}\right)^{-1}.$$

To prove Theorem 7.1.1, we define a homomorphism to choose a value j for the maximum spots, then fill in the rest of each block with entries $< j$. That is, we let $\Theta_{levKmax}(e_N) = 0$ if $K \nmid N$, $\Theta_{levKmax}(e_0) = 1$, and for $n \geq 1$

$$\begin{aligned}
\Theta_{levKmax}(e_{Kn}) &= (-1)^{Kn-1}(x-1)^{n-1} \sum_{j\geq 2} q^{jn}(q + q^2 + \cdots + q^{j-1})^{(K-1)n} \\
&= (-1)^{Kn-1}(x-1)^{n-1} \sum_{j\geq 2} q^{jn} q^{(K-1)n}([j-1]_q)^{(K-1)n} \\
&= (-1)^{Kn-1}(x-1)^{n-1} \sum_{j\geq 1} q^{(j+K)n}([j]_q)^{(K-1)n}.
\end{aligned}$$

Claim:

$$\Theta_{levKmax}(h_{Kn}) = \sum_{w \in BlockMax(K,Kn)} x^{\text{levKmax}(w)} q^{|w|} \tag{7.1.1}$$

To see this, we interpret each term in

$$\begin{aligned}
\Theta_{levKmax}(h_{Kn}) &= \sum_{\mu \vdash Kn} (-1)^{Kn-\ell(\mu)} B_{\mu,Kn} \prod_{i=1}^{\ell(\mu)} \Theta_{levKmax}(e_{\mu_i}) \\
&= \sum_{\lambda \vdash n} (-1)^{Kn-\ell(\lambda)} B_{\lambda,n} \prod_{i=1}^{\ell(\lambda)} \Theta_{levKmax}(e_{K\lambda_i}) \\
&= \sum_{\lambda \vdash n} (-1)^{Kn-\ell(\lambda)} B_{\lambda,n} \prod_{i=1}^{\ell(\lambda)} (-1)^{K\lambda_i - 1} (x-1)^{\lambda_i - 1} \sum_{j \geq 1} q^{(j+K)\lambda_i} ([j]_q)^{(K-1)\lambda_i} \\
&= \sum_{\lambda \vdash n} B_{\lambda,n} \prod_{i=1}^{\ell(\lambda)} (x-1)^{\lambda_i - 1} \sum_{j \geq 1} q^{(j+K)\lambda_i} ([j]_q)^{(K-1)\lambda_i}
\end{aligned}$$

By the definition of $\Theta_{levKmax}$, this sum will contribute nothing unless each part of μ is divisible by K, so we can obtain $\mu \vdash Kn$ by taking some $\lambda \vdash n$ and multiplying each part of λ by K. Thus, the term $\sum_{\lambda \vdash n} B_{\lambda,n}$ can be interpreted as creating a brick tabloid with bricks whose lengths are $K\lambda_1, K\lambda_2, \ldots, K\lambda_{\ell(\lambda)}$. Within a brick, we fill in each block of size K with a word having first entry j and other entries all smaller. We also label each nonterminal block of size K with an x or -1. We define the weight of a filled labeled brick tabloid created in this manner to be the product of the x and -1 labels times q raised to the sum of the entries. For example, the object depicted in Figure 7.1 has blocks of size 4 and weight $-q^{66}$.

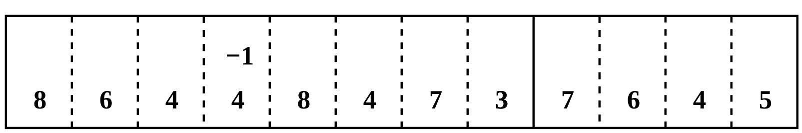

Figure 7.1: A filled labeled brick tabloid coming from $\Theta_{levKmax}(h_{12})$ with $K = 4$

We perform an involution on the set of filled label brick tabloids that results, breaking a brick at the first -1 encountered or combining bricks if the first entries of adjacent blocks are equal. For example, the image of Figure 7.1 is depicted in Figure 7.2.

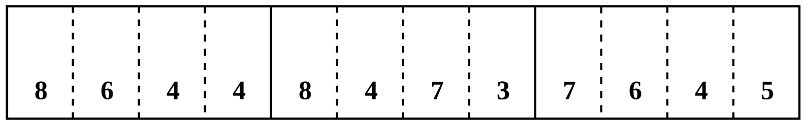

Figure 7.2: The image of Figure 7.1

Fixed points will thus have no -1 labels, and blocks of K will have equal maximum (first) entries within each brick, but not between bricks. The factors of x will thus give us exactly $x^{\text{levKmax}(w)}$, which verifies Equation 7.1.1. One fixed point is depicted in Figure 7.3.

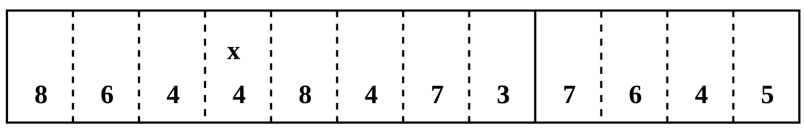

Figure 7.3: A fixed point coming from $\Theta_{levKmax}(h_{12})$ with $K = 4$

Thus,

$$
\begin{aligned}
&\sum_{n\geq 0} t^{Kn} \sum_{w\in BlockMax(K,Kn)} x^{\text{levKmax}(w)} q^{|w|} = \sum_{n\geq 0} t^{Kn}\Theta_{levKmax}(h_{Kn}) \\
&= \left(1 + \sum_{n\geq 1} (-t)^{Kn}(-1)^{Kn-1}(x-1)^{n-1} \sum_{j\geq 1} q^{(j+K)n}([j]_q)^{(K-1)n}\right)^{-1} \qquad (7.1.2) \\
&= \left(1 - \frac{1}{x-1}\sum_{n\geq 1} t^{Kn}(x-1)^n \sum_{j\geq 1} q^{(j+K)n}([j]_q)^{(K-1)n}\right)^{-1} \\
&= \left(1 - \frac{1}{x-1}\sum_{j\geq 1}\sum_{n\geq 1} \left[t^K(x-1)q^{(j+K)}([j]_q)^{(K-1)}\right]^n\right)^{-1} \\
&= \left(1 - \frac{1}{x-1}\sum_{j\geq 1} \frac{t^K(x-1)q^{(j+K)}([j]_q)^{(K-1)}}{1 - t^K(x-1)q^{(j+K)}([j]_q)^{(K-1)}}\right)^{-1} \\
&= \left(1 - \sum_{j\geq 1} \frac{t^K q^{(j+K)}([j]_q)^{(K-1)}}{1 - t^K(x-1)q^{(j+K)}([j]_q)^{(K-1)}}\right)^{-1},
\end{aligned}
$$

proving Theorem 7.1.1.

Corollary 7.1.2 follows by setting $x = 1$ in Equation 7.1.2. Thus, all terms vanish except when $n = 1$, which simplifies into

$$\left(1 - \sum_{j\geq 1} t^K q^{(j+K)}([j]_q)^{K-1}\right)^{-1}.$$

7.2 SU^rSD^d

In this subsection, we will consider the condition that each block has r strict increases followed by d strict decreases. Let $K = r + d + 1$, and let $SU^rSD^d(n)$ be the set of words $w \in \mathbb{P}^n$ with this pattern. For example, one element of $SU^2SD^3(12)$ is given by 1 3 7 6 2 1|2 4 8 5 4 3. Then we have the following theorem and corollary.

Theorem 7.2.1.

$$\sum_{n\geq 0} t^{Kn} \sum_{w\in SU^rSD^d(Kn)} x^{\text{levKmax}(w)} q^{|w|} =$$
$$\left(1 - \sum_{j\geq 1} \frac{t^K q^{j+K+\binom{r}{2}+\binom{d}{2}} \begin{bmatrix} j \\ r \end{bmatrix}_q \begin{bmatrix} j \\ d \end{bmatrix}_q}{1-(x-1)t^K q^{j+K+\binom{r}{2}+\binom{d}{2}} \begin{bmatrix} j \\ r \end{bmatrix}_q \begin{bmatrix} j \\ d \end{bmatrix}_q}\right)^{-1}.$$

Corollary 7.2.2.

$$\sum_{n\geq 0} t^{Kn} \sum_{w\in SU^rSD^d(Kn)} q^{|w|} = \left(1 - t^K \sum_{j\geq 0} q^{j+K+\binom{r}{2}+\binom{d}{2}} \begin{bmatrix} j \\ r \end{bmatrix}_q \begin{bmatrix} j \\ d \end{bmatrix}_q\right)^{-1}.$$

To prove Theorem 7.2.1, we define a homomorphism that chooses a value j for the block maxima, then independently selects r distinct numbers less than j and d distinct numbers less than j. That is, let

$\phi_{SU^rSD^d}(e_0) = 1$, $\phi_{SU^rSD^d}(e_N) = 0$ for $N \neq 0 \mod K$ and, for $n \geq 1$,

$$\begin{aligned}
\phi_{SU^rSD^d}(e_{Kn}) &= (-1)^{Kn-1}(x-1)^{n-1} \sum_{j\geq 2} \left(q^{j+\binom{r+1}{2}+\binom{d+1}{2}} \begin{bmatrix} j-1 \\ r \end{bmatrix}_q \begin{bmatrix} j-1 \\ d \end{bmatrix}_q \right)^n \\
&= (-1)^{Kn-1}(x-1)^{n-1} \sum_{j\geq 1} \left(q^{j+1+\binom{r+1}{2}+\binom{d+1}{2}} \begin{bmatrix} j \\ r \end{bmatrix}_q \begin{bmatrix} j \\ d \end{bmatrix}_q \right)^n \\
&= (-1)^{Kn-1}(x-1)^{n-1} \sum_{j\geq 1} \left(q^{j+K+\binom{r}{2}+\binom{d}{2}} \begin{bmatrix} j \\ r \end{bmatrix}_q \begin{bmatrix} j \\ d \end{bmatrix}_q \right)^n,
\end{aligned}$$

where the last step follows from the fact that

$$j+1+\binom{r+1}{2}+\binom{d+1}{2} = j+(1+r+d)+\binom{r}{2}+\binom{d}{2} = j+K+\binom{r}{2}+\binom{d}{2}.$$

Claim:

$$\phi_{SU^rSD^d}(h_{Kn}) = \sum_{w\in SU^rSD^d(Kn)} x^{\text{levKmax}(w)} q^{|w|}.$$

To see this, we interpret each term in:

$$\begin{aligned}
\phi_{SU^rSD^d}(h_{Kn}) &= \sum_{\mu\vdash Kn} (-1)^{Kn-\ell(\mu)} B_{\mu,Kn} \prod_{i=1}^{\ell(\mu)} \phi_K(e_{\mu_i}) \\
&= \sum_{\lambda\vdash n} (-1)^{Kn-\ell(\lambda)} B_{\lambda,n} \prod_{i=1}^{\ell(\lambda)} \phi_K(e_{K\lambda_i}) \\
&= \sum_{\lambda\vdash n} B_{\lambda,n} \prod_{i=1}^{\ell(\lambda)} (x-1)^{\lambda_i-1} q^{\binom{r}{2}+\binom{d}{2}} \sum_{j\geq 1} q^{\lambda_i(j+1)} \begin{bmatrix} j \\ r \end{bmatrix}_q^{\lambda_i} \begin{bmatrix} j \\ d \end{bmatrix}_q^{\lambda_i}. \qquad (7.2.1)
\end{aligned}$$

By the definition of $\phi_{SU^rSD^d}$, this sum will contribute nothing unless each part of μ is divisible by K, so we can obtain $\mu \vdash Kn$ by taking some $\lambda \vdash n$ and multiplying each part of λ by K. Thus, the term $\sum_{\lambda\vdash n} B_{\lambda,n}$ can be interpreted as creating a brick tabloid with bricks whose lengths are $K\lambda_1, K\lambda_2, \ldots, K\lambda_{\ell(\lambda)}$. The term $\prod_{i=1}^{\ell(\lambda)}(x-1)^{\lambda_i-1}$ labels each non-terminal block of K with either x or -1. The term $\prod_{i=1}^{\ell(\lambda)} q^{\binom{r}{2}+\binom{d}{2}} \sum_{j\geq 1} q^{\lambda_i(j+1)} \begin{bmatrix} j \\ r \end{bmatrix}_q^{\lambda_i} \begin{bmatrix} j \\ d \end{bmatrix}_q^{\lambda_i}$ selects some $j \geq 2$ to be the maximum for each brick and fills in each block of the bricks with a sequence $w_1 < w_2 < \cdots < w_r < j > w_{r+2} > \cdots > w_{r+d+1}$, weighted by $q^{|w|}$. We then define the weight of such a filled labeled brick tabloid to be the product of the -1 and x labels times $q^{|w|}$, where w denotes the underlying word. Figure 7.4 depicts one such filled labeled brick tabloid.

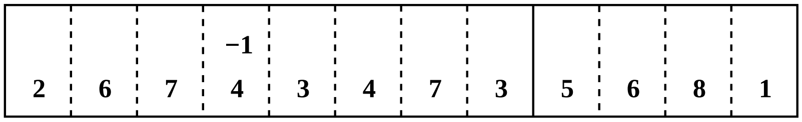

Figure 7.4: An object coming from Equation 7.2.1 with $K = 4, n = 3$

Thus, Equation 7.2.1 above corresponds to a weighted sum over all such brick tabloids. We perform an involution on these tabloids to cancel in pairs. The involution proceeds as follows. Scan left to right looking for the first occurrence of either 2 bricks with adjacent blocks of size K having same maxima or a -1. If a -1 is scanned first, break the brick after the -1 and remove it. If 2 adjacent bricks have blocks with the same maxima j, we combine the bricks and insert a -1 into the final block of the first brick. For instance, the image of Figure 7.4 is given in Figure 7.5.

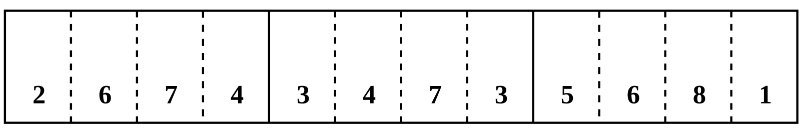

Figure 7.5: The image of Figure 7.4

Therefore, Equation 7.2.1 reduces to summing over the fixed points. A fixed point is displayed in Figure 7.6. Fixed points will have no -1's and no adjacent bricks with the blocks having the same maxima. Thus, they must have an x corresponding to each pair of adjacent blocks with the same maxima, which is exactly counted by

$$\sum_{w \in SU^r SD^d(Kn)} x^{\text{levKmax}(w)} q^{|w|}.$$

			x								
2	**6**	**7**	**4**	**3**	**4**	**7**	**3**	**5**	**6**	**8**	**1**

Figure 7.6: A fixed point of Equation 7.2.1

Therefore,

$$\begin{aligned}
&\sum_{n\geq 0} t^{Kn} \sum_{w\in SU^rSD^d(Kn)} x^{\mathrm{levKmax}(w)} q^{|w|} = \sum_{n\geq 0} t^{Kn}\phi_{SU^rSD^d}(h_{Kn}) \\
&= \left(1+\sum_{n\geq 1}(-t)^{Kn}(-1)^{Kn-1}(x-1)^{n-1}\sum_{j\geq 1}\left(q^{j+K+\binom{r}{2}+\binom{d}{2}}\begin{bmatrix} j\\ r\end{bmatrix}_q \begin{bmatrix} j\\ d\end{bmatrix}_q\right)^n\right)^{-1} \\
&= \left(1-\frac{1}{x-1}\sum_{n\geq 1}\left((x-1)t^K\right)^n\sum_{j\geq 1}\left(q^{j+K+\binom{r}{2}+\binom{d}{2}}\begin{bmatrix} j\\ r\end{bmatrix}_q \begin{bmatrix} j\\ d\end{bmatrix}_q\right)^n\right)^{-1} \\
&= \left(1-\frac{1}{x-1}\sum_{j\geq 1}\sum_{n\geq 1}\left((x-1)t^K q^{j+K+\binom{r}{2}+\binom{d}{2}}\begin{bmatrix} j\\ r\end{bmatrix}_q \begin{bmatrix} j\\ d\end{bmatrix}_q\right)^n\right)^{-1} \qquad (7.2.2) \\
&= \left(1-\frac{1}{x-1}\sum_{j\geq 1}\frac{(x-1)t^K q^{j+K+\binom{r}{2}+\binom{d}{2}}\begin{bmatrix} j\\ r\end{bmatrix}_q \begin{bmatrix} j\\ d\end{bmatrix}_q}{1-(x-1)t^K q^{j+K+\binom{r}{2}+\binom{d}{2}}\begin{bmatrix} j\\ r\end{bmatrix}_q \begin{bmatrix} j\\ d\end{bmatrix}_q}\right)^{-1} \\
&= \left(1-\sum_{j\geq 1}\frac{t^K q^{j+K+\binom{r}{2}+\binom{d}{2}}\begin{bmatrix} j\\ r\end{bmatrix}_q \begin{bmatrix} j\\ d\end{bmatrix}_q}{1-(x-1)t^K q^{j+K+\binom{r}{2}+\binom{d}{2}}\begin{bmatrix} j\\ r\end{bmatrix}_q \begin{bmatrix} j\\ d\end{bmatrix}_q}\right)^{-1},
\end{aligned}$$

which proves Theorem 7.2.1.

To prove Corollary 7.2.2, we wish to set $x = 1$ in Equation 7.2.2. Thus, all terms vanish except when $n = 1$, which gives

$$\sum_{n\geq 0} t^{Kn} \sum_{w\in SU^rSD^d(Kn)} q^{|w|} = \left(1+(-t)^K(-1)^{K-1}\sum_{j\geq 1} q^{j+K+\binom{r}{2}+\binom{d}{2}}\begin{bmatrix} j\\ r\end{bmatrix}_q \begin{bmatrix} j\\ d\end{bmatrix}_q\right)^{-1},$$

which simplifies to Corollary 7.2.2.

7.3 LSD

In this section, we consider words in which each block has the condition that all entries are equal, except for the last entry in the block, which is smaller. For any $K \geq 3$, let
$LSD(K, m, n) = \{w \in [m]^n : \forall i, w_{iK+1} = w_{iK+2} = \cdots = w_{iK+K-1} > w_{iK+K}\}$; i.e. the set of words with $K-2$ levels followed by a strict decrease (in each block of length K, we have $K-1$ equal entries followed by a smaller entry). For example, one element of $LSD(4, 9, 8)$ is given by 5 5 5 2|9 9 9 3. Define
$\text{blockKwdes}(w) = |\{i : w_{iK} \geq w_{iK+1}\}|$ and $\text{blockKdes}(w) = |\{i : w_{iK} > w_{iK+1}\}|$. Then we have the following theorem:

Theorem 7.3.1. *Let $K \geq 3$. Then*

$$\sum_{n\geq 0} t^{Kn} \sum_{w\in LSD(K,m,n)} x^{\text{blockKwdes}(w)} = \left(1 - \frac{1}{x-1}\sum_{n\geq 1}[t^K(x-1)]^n \binom{m+n-1}{2n}\right)^{-1}$$

and

$$\sum_{n\geq 0} t^{Kn} \sum_{w\in LSD(K,m,n)} x^{\text{blockKdes}(w)} = \left(1 - \frac{1}{x-1}\sum_{n\geq 1}[t^K(x-1)]^n \binom{m}{2n}\right)^{-1}$$
$$= \frac{1-x}{1-x+(1+\sqrt{t^K(x-1)})^m + (1-\sqrt{t^K(x-1)})^m}.$$

To prove the first part of Theorem 7.3.1, we can define a homomorphism on Λ by $\Theta_{LSD}(e_0) = 1$, $\Theta_{LSD}(e_N) = 0$ for $N \neq 0 \mod K$, and, for $n \geq 1$,

$$\Theta_{LSD}(e_{Kn}) = (-1)^{Kn-1}(x-1)^{n-1}\binom{m+n-1}{2n}.$$

Lemma 7.3.2.

$$\binom{m+n-1}{2n} = |\{w \in LSD(K, m, Kn) : \text{blockKwdes}(w) = n-1\}|.$$

To see this, we interpret $\binom{m+n-1}{2n}$ as first choosing a sequence $m+n-1 \geq a_1 > a_2 > \cdots > a_{2n-1} > a_{2n} \geq 1$. Next, we obtain a new sequence b by subtracting $n-1$ from a_1 and a_2, $n-2$ from a_3 and a_4, and so on, leaving a_{2n-1} and a_{2n}

alone. We will have $m \geq b_1 > b_2 \geq b_3 > \cdots \geq b_{2n-1} > b_{2n} \geq 1$. Then the word $w = w_1 w_2 \ldots w_{Kn} \in LSD(K, m, Kn)$ that we obtain is given by

$$\begin{aligned}
&w_1 = w_2 = \cdots = w_{K-1} = b_1 \\
&w_K = b_2 \\
&w_{K+1} = w_{K+2} = \cdots = w_{2K-1} = b_3 \\
&w_{2K} = b_4 \\
&\ldots \\
&w_{(n-1)K+1} = w_{(n-1)K+2} = \cdots = w_{nK-1} = b_{2n-1} \\
&w_{nK} = b_{2n}
\end{aligned}$$

Notice that we have forced weak descents between blocks at every possible place, so that $\text{blockKwdes}(w) = n - 1$. As the reader can see, this lemma does not allow us to keep track of the sum of the entries in the word.

Claim:

$$\Theta_{LSD}(h_{Kn}) = \sum_{w \in LSD(K,m,n)} x^{\text{blockKwdes}(w)}.$$

To see this, we interpret each term in

$$\begin{aligned}
\Theta_{LSD}(h_{Kn}) &= \sum_{\mu \vdash Kn} (-1)^{Kn-\ell(\mu)} B_{\mu,Kn} \prod_{i=1}^{\ell(\mu)} \Theta_{LSD}(e_{\mu_i}) \\
&= \sum_{\lambda \vdash n} (-1)^{Kn-\ell(\lambda)} B_{\lambda,n} \prod_{i=1}^{\ell(\lambda)} \Theta_{LSD}(e_{K\lambda_i}) \\
&= \sum_{\lambda \vdash n} B_{\lambda,n} \prod_{i=1}^{\ell(\lambda)} (x-1)^{\lambda_i - 1} \binom{m + \lambda_i - 1}{2\lambda_i}. \qquad (7.3.1)
\end{aligned}$$

By the definition of Θ_{LSD}, this sum will contribute nothing unless each part of μ is divisible by K, so we can obtain $\mu \vdash Kn$ by taking some $\lambda \vdash n$ and multiplying each part of λ by K. Thus, the term $\sum_{\lambda \vdash n} B_{\lambda,n}$ can be interpreted as creating a brick tabloid with bricks whose lengths are $K\lambda_1, K\lambda_2, \ldots, K\lambda_{\ell(\lambda)}$. By Lemma 7.3.2, we interpret $\prod_{i=1}^{\ell(\lambda)} \binom{m+\lambda_i-1}{2\lambda_i}$ as filling in a word with $K-1$ equal entries followed by a smaller entry for each block of length K, and forcing weak decreases between blocks within the same brick. In addition, we interpret the term $\prod_{i=1}^{\ell(\lambda)}(x-1)^{\lambda_i-1}$ as labeling each nonterminal block with either x or -1. The weight of a brick tabloid is given by the product of the x and -1 labels. One such filled labeled brick tabloid is depicted in Figure 7.7.

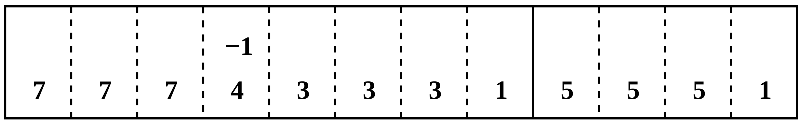

Figure 7.7: An object coming from Equation 7.3.1 with $K = 4, n = 3$

Thus, Equation 7.3.1 above corresponds to a weighted sum over all such brick tabloids. We perform an involution on these tabloids to cancel in pairs. The involution proceeds as follows. Scan left to right looking for the first occurrence of either 2 bricks with a weak decrease between adjacent blocks, or a -1. If a -1 is scanned first, break the brick after the -1 and remove it. If 2 adjacent bricks have a weak decrease between blocks, we combine the bricks and insert a -1 into the final block of the first brick. For instance, the image of Figure 7.7 is given in Figure 7.8.

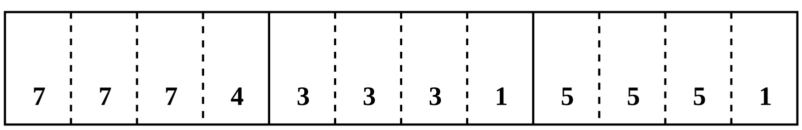

Figure 7.8: The image of Figure 7.7

Therefore, Equation 7.3.1 reduces to summing over the fixed points. One fixed point is displayed in Figure 7.9. Fixed points will have no -1's and no adjacent bricks with weak decreases between blocks. Thus, they must have an x corresponding to each pair of adjacent blocks with a weak decrease, which is exactly counted by

$$\sum_{w \in LSD(K,m,n)} x^{\text{blockKwdes}(w)}.$$

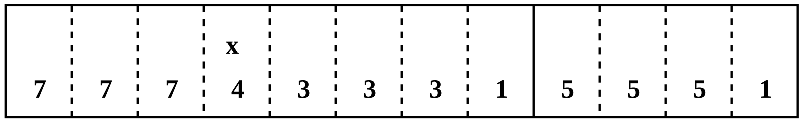

Figure 7.9: A fixed point of Equation 7.3.1

Thus

$$\sum_{n\geq 0} t^{Kn} \sum_{w\in LSD(K,m,n)} x^{\text{blockKwdes}(w)} = \sum_{n\geq 0} t^{Kn}\Theta_{LSD}(h_{Kn})$$

$$= \left(1+\sum_{n\geq 1}(-t)^{Kn}(-1)^{Kn-1}(x-1)^{n-1}\binom{m+n-1}{2n}\right)^{-1}$$

$$= \left(1-\frac{1}{x-1}\sum_{n\geq 1}[t^K(x-1)]^n\binom{m+n-1}{2n}\right)^{-1},$$

proving the first part of Theorem 7.3.1. The second part of Theorem 7.3.1 is proved in a similar manner using the homomorphism $\phi_{LSD}(e_0)=1$, $\phi_{LSD}(e_N)=0$ for $N\neq 0 \mod K$ and, for $n\geq 1$,

$$\phi_{LSD}(e_{Kn}) = (-1)^{Kn-1}(x-1)^{n-1}\binom{m}{2n}.$$

We make a few notes here. First of all we could have defined level-alternating words to have j-levels in each position:

$$\{w: \; w_1 = w_2 = \cdots = w_j < w_{j+1} = w_{j+2} = \ldots w_{2j} > w_{2j+1} = \cdots = w_{3j} < \ldots\}.$$

However, these would have been isomorphic to up-down words of shorter length (as we noted in Corollary 6.3.3). Also, we could have applied the same method from Chapter 4 to LSD; that is, we can relate level strict-down words to level weak-up words via an involution. However, since level weak-up words are no easier to count than level strict-down, this method does not yield additional insight. For this reason, we used the alternative method of defining an appropriate homomorphism to count weak descents between blocks.

7.4 SUSDWU

As we mentioned in Section 3.3, our method of defining a homomorphism on the ring of symmetric functions complements the usual technique of writing down recursions for the desired objects (often in terms of the starting letter) and using the transfer matrix method (see [42], section 4.7 or [22]). This section is a prime example of this interplay, where the homomorphism method reduces our original task to one that can be easily accomplished through solving recursions.

Suppose we consider words that can be partitioned into blocks of length 3, where each block has the pattern strict increase, strict decrease; and there are weak increases between blocks. Let $SUSDWU(m,n)$ be the set of such words on alphabet $[m]$ of length n. For example, one element of $SUSDWU(7,6)$ is given by 1 6 3 3 7 5. Then we have the following theorem.

Theorem 7.4.1.

$$\sum_{n\geq 0} |SUSDWU(m,n)|t^n = \frac{P_m(t)}{Q_m(t)},$$

where P_m and Q_m are polynomials.

To prove Theorem 7.4.1, we will first enumerate a more general class of words by *block descents*. Let

$$SUSDA(m,n) = \{w \in [m]^n : \ w_{3i-2} < w_{3i-1} > w_{3i} \forall i\}$$

(the acronym coming from strict-up, strict-down, anything). We will continue to use our block descent statistic from the previous section, where our blocks are now of length 3: $\text{block3des}(w) = |\{i : \ w_{3i} > w_{3i+1}\}|$. In addition, we use a third class of words in order to define our homomorphism. Let

$$SUSDSD(m,n) = \{w \in [m]^n : \ w_{3i-2} < w_{3i-1} > w_{3i} > w_{3i+1} \forall i\}.$$

We wish to define a homomorphism by $\Theta_{block}(0) = 1$, $\Theta_{block}(e_j) = 0$ for $j \neq 0 \mod 3$ and, for $n \geq 1$,

$$\Theta_{block}(e_{3n}) = (-1)^{3n-1}(x-1)^{n-1}|SUSDSD(m,3n)|. \tag{7.4.1}$$

Claim:

$$\Theta_{block}(h_{3n}) = \sum_{w \in SUSDA(m,3n)} x^{\text{block3des}(w)}.$$

To see this, we interpret each term in

$$\begin{aligned}
\Theta_{block}(h_{3n}) &= \sum_{\mu \vdash 3n} (-1)^{3n-\ell(\mu)} B_{\mu,3n} \prod_{i=1}^{\ell(\mu)} \Theta_{block}(e_{\mu_i}) \\
&= \sum_{\lambda \vdash n} (-1)^{3n-\ell(\lambda)} B_{\lambda,n} \prod_{i=1}^{\ell(\lambda)} \Theta_{block}(e_{3\lambda_i}) \\
&= \sum_{\lambda \vdash n} B_{\lambda,n} \prod_{i=1}^{\ell(\lambda)} (x-1)^{\lambda_i - 1} |SUSDSD(m, 3\lambda_i)|. \qquad (7.4.2)
\end{aligned}$$

By the definition of Θ_{block}, this sum will contribute nothing unless each part of μ is divisible by 3, so we can obtain $\mu \vdash 3n$ by taking some $\lambda \vdash n$ and multiplying each part of λ by 3. Thus, the term $\sum_{\lambda \vdash n} B_{\lambda,n}$ can be interpreted as creating a brick tabloid with bricks whose lengths are $3\lambda_1, 3\lambda_2, \ldots, 3\lambda_{\ell(\lambda)}$. We fill every brick with a $SUSDSD$ word, and we label every third cell–except the final one in a brick–with x or -1. Thus, Equation 7.4.2 can be interpreted as a sum over all

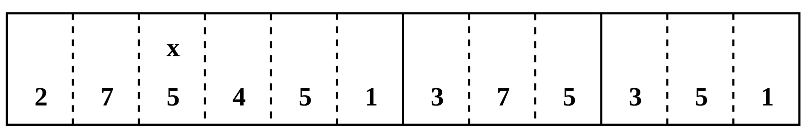

Figure 7.10: An object coming from Equation 7.4.2

such filled labeled brick tabloids. We perform an involution to cancel out negative terms in this sum. The involution proceeds as follows. Scan from left to right looking for the first occurrence of either 2 bricks with a strict decrease between them, or a -1. If a -1 is scanned first, break the brick after the -1 and remove it. If 2 adjacent bricks have a strict decrease between them, we combine the bricks and insert a -1 into the final cell of the first brick. The image of Figure 7.10 is given in Figure 7.11.

Therefore, Equation 7.4.2 reduces to summing over the fixed points. A fixed point is depicted in Figure 7.12. Fixed points will have no -1's and no adjacent

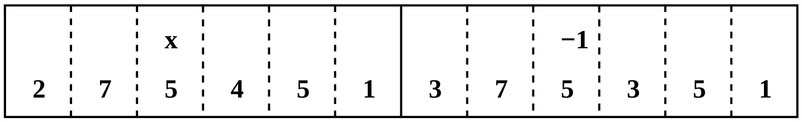

Figure 7.11: the image of Figure 7.10

bricks with a decrease between them. Thus, they must have an x corresponding to each block descent. This is exactly counted by

$$\sum_{w \in SUSDA(m,3n)} x^{\text{block3des}(w)}.$$

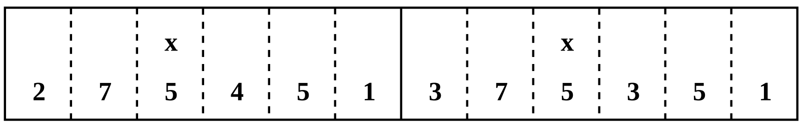

Figure 7.12: A fixed point of Equation 7.4.2

Setting $x = 0$ in this expression eliminates any terms with block descents, forcing a weak rise between blocks. Thus, we can obtain $\sum_{n \geq 0} |SUSDWU(m, 3n)| t^{3n}$ by taking $\sum_{n \geq 0} t^n \Theta_{block}(h_n) \mid_{x=0}$, so that:

$$\begin{aligned}\sum_{n \geq 0} |SUSDWU(m, 3n)| t^{3n} &= \left[\sum_{n \geq 0} (-t)^n \Theta_{block}(e_n)\right]^{-1} \Bigg|_{x=0} \\ &= \left[\sum_{n \geq 0} (-t)^{3n} (-1)^{3n-1} (x-1)^{n-1} |SUSDSD(m, 3n)|\right]^{-1} \Bigg|_{x=0} \\ &= \left[\sum_{n \geq 0} t^{3n} (-1)^n |SUSDSD(m, 3n)|\right]^{-1}\end{aligned}$$

The reasoning for words of length other than a multiple of three will be similar; we can use the same homomorphism along with a weighting, which will yield a polynomial numerator. Thus, we have reduced Theorem 7.4.1 to finding a rational polynomial expression for $\sum_{n \geq 0} t^{3n} (-1)^n |SUSDSD(m, 3n)|$.

We now turn our attention to counting $SUSDSD(m, 3n)$, which we will accomplish via recursions. Fix some m and let $B_{k,j} = \{w \in SUSDSD(m, 3k) : \ w_1 = j\}$, the set of patterns containing k blocks of this type, where the first block begins with j and there are strict decreases between blocks. Let $b_{k,j} = |B_{k,j}|$. We can enumerate $B_{1,j}$ for each j. For example, when $m = 3$, $B_{1,1} = \{132, 121, 131\}$ and $B_{1,2} = \{232, 231\}$.

It is easy to see that $b_{k,j}$ satisfy the recurrence:

$$b_{k,j} = \sum_{r=1}^{m-1} a_{j,r} b_{k-1,r},$$

where $a_{j,r}$ counts the number of 1-block patterns starting with j and ending with an entry $> r$. Moreover, we can find a simple formula for the $a_{j,r}$. For a block to start with j and end with something greater than r, the middle entry must be larger than both j and $r+1$. The number of such entries is $m - \max(j, r+1)$. For each such entry, we can end with anything between it and $r+1$. It is useful to separate cases, so that we get:

$$a_{j,r} = \begin{cases} \frac{(m-j)(m+j-2r-1)}{2} & r < j \\ \binom{m-r}{2} & r \geq j \end{cases}.$$

The case $r \geq j$ is obvious: we simply choose $r < w_2 < w_3 \leq m$. When $r < j$, we consider first choosing $w_3 \leq j$, which gives $m - j$ choices for w_2 and $j - r$ choices for w_3. On the other hand, if $w_3 > j$, we choose $j < w_2 < w_3 \leq m$. Then we have

$$a_{j,r} = (m-j)(j-r) + \binom{m-j}{2} = \frac{(m-j)(m+j-2r-1)}{2}.$$

Thus, we can write

$$b_{k,j} = \sum_{r=1}^{j-1} \frac{(m-j)(m+j-2r-1)}{2} b_{m-1,r} + \sum_{r=j}^{m-2} \binom{m-r}{2} b_{k-1,r}.$$

Imagining a block of 0s following a single block (so that $r = 0$), we also obtain

$$b_{1,j} = \frac{(m-j)(m+j-1)}{2}.$$

These recursions can be solved for any particular value of m. We will illustrate the first few cases here.

When $m = 3$, we have already noted that

$$b_{1,1} = 3 \text{ and}$$
$$b_{1,2} = 2.$$

For $n > 1$, we get the following recursions:

$$b_{n,1} = \sum_{r=1}^{2} \binom{3-r}{2} b_{n-1,r} = b_{n-1,1} \text{ and}$$
$$b_{n,2} = \frac{(3-2)(3+2-2-1)}{2} b_{n-1,1} = b_{n-1,1}.$$

Thus, for $k \geq 2$, $|SUSDSD(3, 3k)| = 3 + 3 = 6$. Then

$$\sum_{n\geq 2} t^{3n}(-1)^n |SUSDSD(3, 3n)| = 6\frac{t^6}{1+t^3}.$$

Let $m = 4$. Then we have:

$$B_{1,1} = \{132, 121, 131, 142, 141, 143\}, \qquad b_{1,1} = 6.$$
$$B_{1,2} = \{232, 231, 242, 241, 243\}, \qquad b_{1,2} = 5.$$
$$B_{2,3} = \{341, 342, 343\}, \qquad b_{1,3} = 3.$$

For $n > 1$, we obtain the recursions:

$$b_{n,1} = 3b_{n-1,1} + b_{n-1,2},$$
$$b_{n,2} = 3b_{n-1,1} + b_{n-1,2}, \text{ and}$$
$$b_{n,3} = 2b_{n-1,1} + b_{n-1,2}.$$

Let

$$C_1(x) = \sum_{n\geq 1} b_{n,1} x^{n-1},$$
$$C_2(x) = \sum_{n\geq 1} b_{n,2} x^{n-1}, \text{ and}$$
$$C_3(x) = \sum_{n\geq 1} b_{n,3} x^{n-1}.$$

Applying our recursion, we get:

$$\begin{aligned} C_1(x) &= 6 + \sum_{n>1}(3b_{n-1,1} + b_{n-1,2})x^{n-1} \\ &= 6 + 3\sum_{n\geq 2} b_{n-1,1}x^{n-1} + \sum_{n\geq 2} b_{n-1,2}x^{n-1} \\ &= 6 + 3xC_1(x)(x) + xC_2(x) \end{aligned} \tag{7.4.3}$$

It is clear from the recursions and initial conditions that $C_1(x) = 1 + C_2(x)$, so Equation 7.4.3 becomes

$$\begin{aligned} 1 + C_2(x) &= 6 + 3x(1 + C_2(x)) + xC_2(x) \\ (1 - 4x)C_2(x) &= 5 + 3x \\ C_2(x) &= \frac{5 + 3x}{1 - 4x} \end{aligned}$$

Therefore,

$$\begin{aligned} C_1(x) &= 1 + \frac{5 + 3x}{1 - 4x} \\ &= \frac{6 - x}{1 - 4x} \end{aligned}$$

Similarly,

$$\begin{aligned} C_3(x) &= 3 + 2x(C_1(x)) + xC_2(x) \\ &= \frac{(3 - 12x) + (12x - 2x^2) + (5x + 3x^2)}{1 - 4x} \\ &= \frac{3 + 5x + x^2}{1 - 4x} \end{aligned}$$

From the nice linear form of our denominators (or solving the recursions more directly), we find that:

$$\begin{aligned} &b_{n,1} = b_{n,2} = 23 \cdot 4^{n-2} \ (n \geq 2) \text{ and} \\ &b_{n,3} = 69 \cdot 4^{n-3} \ (n \geq 3), \end{aligned}$$

while $b_{2,3} = 17$. Thus, $|SUSDSD(4, 3n)| = 253 \cdot 4^{n-3}$ for $n \geq 3$, so that

$$\sum_{n\geq 3} t^{3n}(-1)^n|SUSDSD(4, 3n)| = 253\frac{-t^9}{1 + 4t^3}.$$

Let $m = 5$. We have $b_{1,1} = 10, b_{2,2} = 9, b_{1,3} = 7, b_{1,4} = 4$.
Our recursions will be

$$b_{n,1} = b_{n,2} = 6b_{n-1,1} + b_{n-1,2} + b_{n-1,3},$$
$$b_{n,3} = 5b_{n-1,1} + b_{n-1,2} + b_{n-1,3}, \text{ and}$$
$$b_{n,4} = 3b_{n-1,1} + 2b_{n-1,2} + b_{n-1,3}.$$

Let

$$D_1(x) = \sum_{n\geq 1} b_{n,1}x^{n-1},$$
$$D_2(x) = \sum_{n\geq 1} b_{n,2}x^{n-1},$$
$$D_3(x) = \sum_{n\geq 1} b_{n,3}x^{n-1}, \text{ and}$$
$$D_4(x) = \sum_{n\geq 1} b_{n,4}x^{n-1}.$$

As before, $D_1(x) = 1 + D_2(x)$. Using this, our recursions become:

$$D_1(x) = 10 + 6xD_1(x) + 3x(D_1(x) - 1) + xD_3(x),$$
$$D_3(x) = 7 + 5xD_1(x) + 3x(D_1(x) - 1) + xD_3(x), \text{ and}$$
$$D_4(x) = 4 + 3xD_1(x) + 2x(D_1(x) - 1) + xD_3(x).$$

We can solve for $D_3(x)$ in terms of $D_1(x)$:
$(1 - x)D_3(x) = 7 - 3x + 8xD_1(x)$, so $D_3(x) = \frac{7-3x+8xD_1(x)}{1-x}$.
Then

$$\begin{aligned} D_1(x) &= 10 + 6xD_1(x) + 3x(D_1(x) - 1) + x\frac{7 - 3x + 8xD_1(x)}{1 - x} \\ (1 - x)D_1(x) &= 10 - 10x + 9x(1 - x)D_1(x) - 3x + 3x^2 + 7x - 3x^2 + 8x^2D_1(x) \\ &= 10 - 6x - x^2D_1(x) + 9xD_1(x) \\ D_1(x) &= \frac{10 - 6x}{1 - 10x + x^2} \end{aligned}$$

and

$$\begin{aligned}
D_3(x) &= \frac{7-3x+8x\frac{10-6x}{1-10x+x^2}}{1-x} \\
&= \frac{7-70x+7x^2-3x+30x^2-3x^3+80x-48x^2}{(1-10x+x^2)(1-x)} \\
&= \frac{7+7x-11x^2-3x^3}{(1-10x+x^2)(1-x)} \\
&= \frac{7+14x+3x^2}{1-10x+x^2}
\end{aligned}$$

and

$$\begin{aligned}
D_4(x) &= 4-2x+5xD_1(x)+xD_3(x) \\
&= 4-2x+5x\frac{10-6x}{1-10x+x^2}+x\frac{7+14x+3x^2}{1-10x+x^2} \\
&= \frac{4-40x+4x^2-2x+20x^2-2x^3+50x-30x^2+7x+14x^2+3x^3}{1-10x+x^2} \\
&= \frac{4+15x+8x^2+x^3}{1-10x+x^2}.
\end{aligned}$$

Although we do not obtain a nice formula for the coefficients of these generating functions, we can conclude that

$$\begin{aligned}
|SUSDSD(5,3n)| &= [D_1(x)+D_2(x)+D_3(x)+D_4(x)]|_{x^{n-1}} \\
&= \left[\frac{30+27x+10x^2+x^3}{1-10x+x^2}\right]\Bigg|_{x^{n-1}},
\end{aligned}$$

so that

$$\begin{aligned}
&\sum_{n\geq 0} t^{3n}(-1)^n|SUSDSD(5,3n)| \\
&\quad = 1+\sum_{n\geq 1} t^{3n}(-1)^n \left[\frac{30+27x+10x^2+x^3}{1-10x+x^2}\right]\Bigg|_{x^{n-1}} \\
&\quad = 1-t^3\sum_{n\geq 1}(-t^3)^{n-1} \left[\frac{30+27x+10x^2+x^3}{1-10x+x^2}\right]\Bigg|_{x^{n-1}} \\
&\quad = 1-t^3\left[\frac{30+27(-t^3)+10(-t^3)^2+(-t^3)^3}{1-10(-t^3)+(-t^3)^2}-1\right] \\
&\quad = \frac{1-19t^3+38t^6-9t^9+t^{12}}{1+10t^3+t^6}.
\end{aligned}$$

As the reader can see, similar reasoning will continue to work for larger values of m, so that we still get rational expressions for $\sum_{n\geq 0} t^{3n}(-1)^n|SUSDSD(m,3n)|$.

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